PROCEEDINGS OF THE 15TH SYMPOSIUM ON RING THEORY

HELD AT THE TAKARAZUKA-SÔ, TAKARAZUKA CITY

OCTOBER 29—31, 1982

EDITED BY

YUKIO TSUSHIMA

Osaka City University

1982

OKAYAMA, JAPAN

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PREFACE

The 15th Symposium on Ring Theory was held in Takarazuka City, Japan, on October 29 - 31, 1982. A number of new results were announced there, which will be found in these Proceedings. In addition to it, there was an expository lecture given by H. Sasaki (Hokkaido University) on the Auslander-Reiten quiver of a finite group.

The meeting and these Proceedings were financially supported by the Scientific Research Grant of the Educational Ministry of Japan through the arrangement by Professor H. Tachikawa (University of Tsukuba).

It has been fourteen years since the annual Symposium on Ring Theory was founded in 1968. I wish to take this opportunity to thank Algebra staffs of the Department of Mathematics, Okayama University, for their long standing assistance in publishing the Proceedings so far. Without it, they would not have existed as they do. I also wish to thank T. Sumioka for his best help in the organization of the meeting.

November 1982

Y. Tsushima

QF-3 RINGS WITH SEMI-PRIMARY QUOTIENT RING

Kanzo MASAIKE

In this paper we give an outline of results which will be appeared in [5].

Let R be a ring with identity. R is said to be left Qf-3, if R has a minimal faithful left R-module (cf. [10]). A theorem of Faith and Walker [2] states that if R is a ring such that every injective left R-module is projective, then R is left Aritinian (and hence quasi-Frobenius). In this paper we shall apply a generalization of this result to investigate those rings whose maximal left quotint rings are semi-primary left and right QF-3. Let (T, F) be a hereditary torsion theory with a left exact radical t, where T (resp. F) is the t-torsion class (resp. t-torsion free class). A submodule M of a left R-module N is said to be t-closed, if N/M is t-torsion free. Every t-torsion free left R-module becomes an R/t(R)-module canonically. In the following let us denote $E(_RM)$ the injective hull of a left R-module M. ACC (resp. DCC) means the ascending chain condition (resp. the descending chain condition).

Lemma 1. Let R be a ring with a hereditary torsion theory (T, F) such that R is the ring of quotient with respect to t. If $E(_RR)$ is projective and R has ACC on t-closed left ideals, R is semi-primary.

From this result we have the next

Theorem 1. Let (T, F) be a hereditary torsion theory of a ring R. If every t-torsion free injective left R-module is projective as an R/t(R)-module, then the ring of quotient of R with respect to t is semi-primary and R satisfies DCC on t-closed left ideals.

In [6] it is proved that if R has DCC on t-closed left ideals and every t-torsion free left R-module is embedded in a direct product of copies of R/t(R), every t-torsion free injective left R-module is R/t(R)-projective. A left R-module M is said to be $E({}_RR)$-tosionless, if M is embedded in a direct product of copies of $E({}_RR)$. Now, assume that (the torsion free class) F is the class of all $E({}_RR)$-torsionless modules. If $E({}_RR)$ is torsionless, a t-closed left ideal coincides with an annihilator left ideal of R. If R is a ring with ACC on annihilator left ideals and $E({}_RR)$ is projective, we can prove that R has DCC on annihilator left ideals. Therefore, we have

Corollary 1. The following conditions are equivalent, if $E({}_RR)$ is projective.

(i) $E({}_RR)$ is Σ-injective.

(ii) Every direct product of copies of $E({}_RR)$ is projective.

(iii) R has DCC on annihilator left ideals

If F is a class of all $E({}_RR)$-torsionless modules, the ring of quotient with respect to t is called the maximal left quotient ring.
The next Lemma was proved in [3].

Theorem 2. The following conditions are equivalent for a ring R.

(i) Every finitely generated submodule of $E({}_RR)$ is torsionless.

(ii) Every finitely generated $E({}_RR)$-torsionless left R-module is torsionless.

(iii) The maximal left quotient ring Q of R is embedded is a maximal right quotient ring and every finitely generated $E({}_QQ)$-torsionless left Q-module is torsionless.

From Lemma 1, Theorem 2 and a result of [1 , Theorem 1.3] we have the following

Theorem 3. The following conditions are equivalent for a ring R.

(i) R has a semi-primary left and right QF-3 maximal two-sided quotient ring .

(ii) R satisfies DCC on annihilator left ideals and every finitely generated submodule of $E({}_RR)$ is torsionless.

(iii) R satisfies DCC on annihilator left ideals and every finitely generated $E({}_RR)$-torsionless left R-module is embedded in a free left R-module.

(iv) R has a maximal two-sided quotient ring Q such that every $E({}_QQ)$-torsionless left Q-module is embedded in a free left Q-module.

Remark 1. Generalizing a result of Rutter, Sumioka [9] proved that a perfect ring R is left QF-3, if and only if every finitely generated submodule of $E({}_RR)$ is torsionless.

Corollary 2. If R has DCC on annihilator left ideals and every finitely generated submodule of $E({}_RR)$ is torsionless, then every finitely generated submodule of $E(R_R)$, the injective hull of R_R, is torsionless.

Lemma 2 (Masaike [4]). The following conditions are equivalent for a ring R with the maximal left quotient ring Q.

(i) R is left and right QF-3.

(ii) Q is left and right QF-3 and Q is torsionless as right and left R-modules.

(iii) Q is a left and right QF-3 maximal two-sided quotient ring of R and R has a minimal dense right ideal and a minimal dense left ideal.

(The proof of (i) $\Rightarrow$ (iii) is a immediate consequence of Rutter [8].)

Proposition 1. Let R be a ring with DCC on annihilator left ideals. Then, the following conditions are equivalent.

(i) R is left and right QF-3.

(ii) Every finitely generated submodule of $E({}_RR)$ is torsionless and R contains a minimal dense left ideal and a minimal dense right ideal.

(iii) R is left QF-3 and foer every minimal left ideal S_i there exists a local idempotent element f_i such that $S_i \cong Rf_i/Jf_i$, where J is the Jacobson radical of R.

Now, assume that R is a left QF-3 ring with ACC on annihilator left ideals. If R has an essential left socle and a minimal dense left ideal, we can see that $E({}_RR)$ is finitely generated projective. It follows that R satisfies DCC on annihilaltor left ideals. Since R has a minimal dense left ideal and a minimal dense right ideal, from Proposition 1 we have that R is QF-3 and we have the next

Theorem 4. Let R be a left QF-3 ring with ACC on annihilator left ideals. Then, the following conditions are equivalent.

(i) R is right QF-3.

(ii) R has an essential left socle and a minimal dense left ideal.

(iii) R has an essential left socle and for every minimal left ideal S_i there exists a local idempotent element f_i such that $S_i \cong Rf_i/Jf_i$.

Corollary 3. The following conditions are equivalent.

(i) Every $E({}_RR)$-torsionless left R-module is embedded in a free left R-module.

(ii) Every $E(R_R)$-torsionless right R-module is embedded in a free right R-module.

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WEAKLY DIVISIBLE, DIVISIBLE AND STRONGLY M-INJECTIVE MODULES

Shoji MORIMOTO

This note is an abstract of the author's papers [6] and [7], but some new results are added to them. In the first place of this paper, we characterize weakly divisible and divisible modules by using the concepts of coindependence and weakly coindependence. Next, we prove that Sato's pseudo-cohereditarity of idempotent preradicals coincides with Jirásko's one. Further, we show that the Goldie torsion radical G is pseudo-cohereditary if and only if every nonsingular module is injective. In the latter part, we study strongly M-injective modules.

1. Preliminaries

Throughout this note, R means a ring with identity and modules mean unitary left R-modules. Also we denote the category of all modules by R-mod. Recall that a preradical t of R-mod is a subfunctor of the identity functor. We shall say that a preradical t

- idempotent if $t(t(M)) = t(M)$ for all modules M,
- radical if $t(M/t(M)) = 0$ for all modules M,
- left exact if $t(N) = t(M) \cap N$ for all modules M and submodules N,
- cohereditary if $t(M/N) = (t(M) + N)/N$ for all modules M and submodules N,
- a cotorsion radical if t is idempotent and

cohereditary.

To each preradical t of R-mod, we put $T(t) = \{M \in R\text{-mod} \mid t(M) = M\}$ and $F(t) = \{M \in R\text{-mod} \mid t(M) = 0\}$. In general, T(t) is closed under homomorphic images and direct sums, while F(t) is closed under submodules and direct products.

2. Weakly divisible and divisible modules

Definition 2.1. For a preradical t, we call a module M weakly divisible (resp. divisible) with respect to t if the functor $\mathrm{Hom}_R(-,M)$ preserves the exactness of all exact sequences of modules $0 \to A \to B \to C \to 0$ with $B \in T(t)$ (resp. $C \in T(t)$).

When there is no confusion, we say simply that M is weakly divisible (resp. divisible). Clearly, every injective module is divisible and every divisible module is weakly divisible, but in general, the converses of these two implications are not true.

Lemma 2.2. Let t be a preradical and let M be a module and N its submodule. Then

(1) If M is weakly divisible and $N \supseteq t(M)$, then N is also weakly divisible.

(2) If t is idempotent, N is essential in M and is weakly divisible, then $N \supseteq t(M)$.

(3) If t is left exact and t(M) is weakly divisible, then M is weakly divisible.

Especially, if t is a preradical and M is weakly divisible, then t(M) is also weakly divisible. Furthermore, in case t is idempotent, M is weakly divisible if and only if t(M) = t(E(M)) (cf. [9, Theorem 2.1] and [10, Lemma 1.3], where E(M) denotes the injective hull of M.

Example 2.3. Let K be a field, R the ring of all 2 × 2 upper triangular matrices over K and $I = \begin{pmatrix} K & K \\ 0 & 0 \end{pmatrix}$. Then I is an idempotent two-sided ideal of R. Hence I determines a cotorsion radical t by t(M) = IM. Since $E(R) = \begin{pmatrix} K & K \\ K & K \end{pmatrix}$, t(E(R)) = I. By Lemma 2.2, I is weakly divisible and since t is a cotorsion radical, I is divisible. But I is not a direct summand of R, namely, I is not injective.

Example 2.4. Let R be as above. We put $J = \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}$ and $I = \begin{pmatrix} K & K \\ 0 & 0 \end{pmatrix}$. Then J is a two-sided ideal of R and $J^2 = 0$. Let t be the left exact preradical corresponding to the linear topology which has the smallest element J. Since $t(M) = \{m \in M \mid Jm = 0\}$, t(E(R)) = I. By Lemma 2.2, I is weakly divisible. Also $R/I \in T(t)$. If I is divisible, then I is a direct summand of R. This is a contradiction. Hence I is not divisible.

Definition 2.5. For a preradical t, we call an exact sequence $0 \to A \xrightarrow{j} B$ of modules coindependent (resp. weakly coindependent) with respect to t if B = j(A) + t(B) (resp. $B/(j(A) + t(B)) \in T(t)$).

Remark 2.6. Let t be a preradical and let $0 \to A \to B \to C \to 0$ be an exact sequence. Then

(1) If $0 \to A \to B$ is coindependent and $A \in T(t)$, then $B \in T(t)$.

(2) If t is an idempotent radical and $0 \to A \to B$ is weakly coindependent, then $C \in T(t)$.

Now we give examples of coindependent (resp. weakly coindependent) sequence $0 \to A \to B$ with $B \notin T(t)$ (resp. $B/A \notin T(t)$).

Example 2.7. Take $J = \begin{pmatrix} 0 & K \\ 0 & K \end{pmatrix}$ in Example 2.3. Then $0 \to J \to R$ is coindependent, but, since $t(R) = I$, $R \notin T(t)$.

Example 2.8. Let K be a field, R the ring of all 3×3 upper triangular matrices over K. If we put

$I = \begin{pmatrix} 0 & K & K \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $J = \begin{pmatrix} 0 & 0 & K \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then both I and J are

two-sided ideals in R and $J^2 = 0$. Let t be the left exact preradical corresponding to the linear topology having the smallest element I. Then, since $I \subseteq t(R)$, $R/t(R) \in T(t)$ and $0 \to J \to R$ is weakly coindependent, but $R/J \notin T(t)$ because $J \subsetneq I$.

Theorem 2.9. Let t be an idempotent preradical. Then the following conditions are equivalent for a module M:

(1) M is weakly divisible (resp. divisible).

(2) For every coindependent (resp. weakly coindependent) sequence $0 \to A \xrightarrow{j} B$ and every homomorphism $f: A \to M$, there exists a homomorphism $g: B \to M$ such that $g \circ j = f$.

(3) For every coindependent (resp. weakly coindependent) sequence $0 \to A \xrightarrow{j} B$ with $j(A)$ essential in B and every

homomorphism $f: A \to M$, there exists a homomorphism $g: B \to M$ such that $g \circ j = f$.

(4) Every coindependent (resp. weakly coindependent) sequence $0 \to M \overset{j}{\to} N$ splits.

(5) Every coindependent (resp. weakly coindependent) sequence $0 \to M \overset{j}{\to} N$ with $j(M)$ essential in N splits.

3. Pseudo-cohereditary preradicals

Definition 3.1. A preradical t is called pseudo-cohereditary in the sense of Sato if every weakly divisible module is divisible.

Definition 3.2. We call a preradical t pseudo-cohereditary in the sense of Jirásko if every homomorphic image of $M/(t(E(M)) \cap M)$ is in $F(t)$.

Theorem 3.3. Let t be an idempotent preradical. Then the following conditions are equivalent:

(1) t is pseudo-cohereditary in the sense of Sato.

(2) t is pseudo-cohereditary in the sense of Jirásko.

(3) For any weakly divisible module H, every homomorphic image of $H/t(H)$ is in $F(t)$.

(4) For every module M, an exact sequence $0 \to M \to D(M)$ is coindependent, where $D(M)$ is defined by $D(M)/M = t(E(M)/M)$.

Proof. Refer to [6] for the equivalence of (1), (2) and (3). (1)$\Longrightarrow$(4). We put $H(M) = M + t(E(M))$. Since $H(M)$ is weakly divisible by Lemma 2.2, it is divisible by the hypothesis. Also, since $D(M)/H(M) \in T(t)$, $H(M)$ is a direct summand of $D(M)$. Further $H(M)$ is essential in $D(M)$, namely, $H(M) = D(M)$. Hence $0 \to M \to D(M)$ is coindependent.

(4) $\Longrightarrow$ (1). Let M be a weakly divisible module. Since $0 \to M \to D(M)$ is coindependent, M is a direct summand of D(M) by Theorem 2.9. So M = D(M), namely, M is divisible. Therefore t is pseudo-cohereditary in the sense of Sato.

Here, we consider the following condition for a preradical t:

(*) Every weakly coindependent sequence is coindependent.

Theorem 3.4. Let t be an idempotent preradical. Then t satisfies the condition (*) if and only if t is a cotorsion radical.

Corollary 3.5. For a left exact preradical t, the following conditions are equivalent:

(1) t is an exact radical.

(2) t is pseudo-cohereditary in the sense of Sato.

(3) t is pseudo-cohereditary in the sense of Jirásko.

(4) t satisfies the condition (*).

Corollary 3.6. For the Goldie torsion radical G, the following conditions are equivalent:

(1) G is pseudo-cohereditary.

(2) Every nonsingular module is injective.

(3) $R = G(R) \oplus K$, where K is a semisimple artinian ring.

The equivalence of (2) and (3) is due to Armendariz [1, Theorem 3.2].

4. Strongly M-injective modules

In this section, we fix a module M and denote the left annihilator $l_R(M)$ of M by T. Also t means the left exact preradical corresponding to the linear topology which has the smallest element T. As is easily seen, a module X is in T(t) if and only if TX = 0.

Definition 4.1. A module Q is called strongly M-injective if every homomorphism of any submodule of M^J (J is any indexed set) into Q can be extended to a homomorphism of M^J into Q.

Clearly every injective module is strongly M-injective and every strongly M-injective module is M-injective. Also we can easily check that a direct product of modules is strongly M-injective if and only if each factor is strongly M-injective.

Theorem 4.2. For a module Q, the following conditions are equivalent:

(1) Q/TQ is strongly M-injective.

(2) Q/TQ is weakly divisible.

(3) Q/TQ is an injective R/T-module.

Corollary 4.3. For a module Q, the following conditions are equivalent:

(1) t(Q) is strongly M-injective module.

(2) t(Q) is weakly divisible.

(3) t(Q) is an injective R/T-module.

By Lemma 2.2, we obtain the following result.

Theorem 4.4. For a module Q, the following coditions are equivalent:

(1) Q is strongly M-injective.

(2) Q is weakly divisible.

(3) t(Q) is an injective R/T-module.

The following corollary corresponds to Theorem 14 of Azumaya [2].

Corollary 4.5. Let M be a faithful module. Then the following conditions are equivalent:

(1) Q is strongly M-injective.

(2) Q is weakly divisible.

(3) Q is injective.

Next, we investigate rings for which every module is strongly M-injective.

Theorem 4.6. The following conditions are equivalent:

(1) Every module is strongly M-injective.

(2) Every module in T(t) is strongly M-injective.

(3) Every module is weakly divisible.

(4) Every module in T(t) is weakly divisible.

(5) R/T is a semisimple artinian ring.

(6) M^J is completely reducible for any indexed set J.

(7) Every cyclic module is strongly M-projective.

(8) Every module is strongly M-projective.

The equivalence of (5), (6), (7) and (8) was given by Varadarajan [11, Theorem 2.8].

Corollary 4.7. Let M be a faithful module. Then the following conditions are equivalent:

(1) Every module is strongly M-injective.

(2) Every cyclic module is strongly M-projective.

(3) Every module is strongly M-projective.

(4) R is a semisimple artinian ring.

(5) M^J is completely reducible for any indexed set J.

In general, M-injective module is not strongly M-injective. To show this, it is enough to give a completely reducible module M for which M^J is not completely reducible for some indexed set J.

Example 4.8. Let M be a field and $R = M^J$ (J is any infinite indexed set). We can regard M as an R-module. Clearly, M is a simple R-module, but M^J is not completely reducible.

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ON NORMAL CLASSES AND WEAKLY SPECIAL CLASSES OF SEMIPRIME RINGS

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Introduction. Amitsur [1] showed that if (R,V,W,S) is a Morita context then $VN(S)W \subseteq N(R)$, where N(R) denotes the Baer lower radical, Levitzki radical or Jacobson radical of the ring R. Radicals with this property were called normal radicals by Jaegermann [11]. Amitsur also showed that if (R,V,W,S) is an S-faithful Morita context and if R is a prime ring or a primitive ring, then so is S [1]. In [14] and [15], Nicholson and Watters studied classes of rings possessing this property and called them normal classes, and showed that the radical determined by a normal class of prime rings is a normal and special radical (in the sense of Andrunakievič [2]). In this paper, we mainly study properties of normal classes of semiprime rings and extend the results obtained in [15] for normal classes of prime rings to those of semiprime rings.

1. Preliminaries. All the rings we consider will be associative rings not necessarily containing identity, and all the classes of rings are non-empty and closed under isomorphisms. A Morita context (R,V,W,S) consists of two rings R and S and two bimodules ${}_RV_S$ and ${}_SW_R$ together with mappings $V \times W \longrightarrow R$ and $W \times V \longrightarrow S$ (written multiplicatively) which induce bimodule homomorphisms $V \otimes_S W \longrightarrow R$ and $W \otimes_R V \longrightarrow S$ and which satisfy the associativity conditions $(vw)v' = v(wv')$ and $(wv)w' = w(vw')$ for $v, v' \in V$ and $w, w' \in W$ (see Amitsur [1] for details).

An equivalent formulation is that $\begin{pmatrix} R & V \\ W & S \end{pmatrix}$ is a ring with the usual matrix operation. An example we shall refer to is the standard Morita context (R,M,M^*,E) determined by a left R-module, where $M^* = \mathrm{Hom}_R(M,R)$ and $E = \mathrm{Hom}_R(M,M)$. In order to avoid constant repetition, a Morita context (R,V,W,S) will be called S-faithful if $S \neq 0$ and $VsW = 0$ $(s \in S)$ implies $s = 0$. In case (R,V,W,S) is a Morita context and P is an ideal of R, we write $V_P = \{v \in V \mid vW \subseteq P\}$, $W_P = \{w \in W \mid Vw \subseteq P\}$ and $S_P = \{s \in S \mid VsW \subseteq P\}$. Then it is known that $(R/P, V/V_P, W/W_P, S/S_P)$ is a Morita context, the products being defined in the natural manner.

In the next proposition, the classes of rings of interest to us are isolated.

Proposition 1 ([15, Proposition 1]). The following are equivalent for a class $\mathcal{P}$ of rings:

1) If (R,V,W,S) is a Morita context and P is an ideal of R such that $R/P \in \mathcal{P}$, then either $S_P = S$ or $S/S_P \in \mathcal{P}$.

2) If (R,V,W,S) is a Morita context and $R \in \mathcal{P}$, then either $S_0 = S$ or $S/S_0 \in \mathcal{P}$.

3) If (R,V,W,S) is an S-faithful Morita context, then $R \in \mathcal{P}$ implies $S \in \mathcal{P}$.

Following [15], a class $\mathcal{P}$ of rings is called normal if it satisfies one of the equivalent conditions in Proposition 1.

Let $\mathcal{P}$ be a class of semiprime rings. We consider the following conditions:

(WS1) Every non-zero ideal of R is in P whenever R is in P.

(WS2) Let A be a non-zero ideal of R. Then $R/A^{\perp}$ belongs to P whnever A is in P, where $A^{\perp} = \ell_R(A) \cap r_R(A)$.

(WS3) Let A be a non-zero ideal of a semiprime ring R such that A is essential in ${}_R R$ (and in R_R). Then R belongs to P whenever A is in P.

(WS4) Let A be a non-zero ideal of a semiprime ring R such that $A^{\perp} = 0$. Then R belongs to P whenever A is in P.

Proposition 2 ([9, Lemma 6]). Let P be a class of semiprime rings. Then the following are equivalent:

1) P satisfies (WS1) and (WS2).

2) P satisfies (WS1) and (WS3).

3) P satisfies (WS1) and (WS4).

Following Ju. Rjabuhin (see [9]), a class P of semiprime rings is called a weakly special class if P satisfies one of the equivalent conditions 1) - 3) in Proposition 2.

In order to obtain conditions analogous to the above conditions (WS1) and (WS4) which characterize normal classes, we need the following.

Proposition 3. Let (R,V,W,S) be a Morita context with $R \neq 0$, and write $C = \begin{pmatrix} R & V \\ W & S \end{pmatrix}$. Then C is a semiprime ring if and only if

1) R is a semiprime ring,

2) $Vw = 0$ ($w \in W$) implies $w = 0$,

3) $vW = 0$ $(v \in V)$ implies $v = 0$, and

4) $S = 0$ or S is a semiprime ring.

Proof. Observe that the lack of symmetry in 2) and 3) is only apparent. For example, $wV = 0$ implies $(Vw)^2 = 0$, so $w = 0$ by 1) and 2). Similarly, $Wv = 0$ implies $v = 0$. To see that C is a semiprime ring, suppose $cCc = 0$, where $c = \begin{pmatrix} r & v \\ w & s \end{pmatrix} \in C$. Since $0 = \begin{pmatrix} r & v \\ w & s \end{pmatrix}\begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} r & v \\ w & s \end{pmatrix} = \begin{pmatrix} rRr & rRv \\ wRr & wRv \end{pmatrix}$ and R is a semiprime ring, we have $r = 0$. And since $0 = \begin{pmatrix} 0 & v \\ w & s \end{pmatrix}\begin{pmatrix} 0 & 0 \\ W & 0 \end{pmatrix}\begin{pmatrix} 0 & v \\ w & s \end{pmatrix} = \begin{pmatrix} 0 & vWv \\ 0 & sWv \end{pmatrix}$, we have $vWRvW = 0$. By the semiprimeness of R, we have $vW = 0$ and $v = 0$. Similarly, we can obtain $w = 0$. Since $0 = \begin{pmatrix} 0 & 0 \\ 0 & s \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & s \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & sSs \end{pmatrix}$, we have $s = 0$ by 4). The converse is equally easy to check.

In case L is a left ideal of a ring R, we write $L \lhd_\ell R$. Similarly, $T <_r R$ indicates that T is a right ideal of R.

Theorem 1 (cf. [15, Theorem 1]). Let P be a class of semiprime rings. Then P is a normal class if and only if P satisfies the following conditions:

(N1) If $L \lhd_\ell T \lhd_r R$, $R \in P$ and L is a semiprime ring, then $L \in P$,

(N2) Let $L \lhd_\ell T \lhd_r R$, $\ell_T(L) \cap r_T(L) = 0$ and $\ell_R(T) \cap r_R(T) = 0$, where $\ell_T(L)$ (resp. $r_T(L)$) is the left (resp. right) annihilator of L in T. If $L \in P$ and R is a semiprime ring, then $R \in P$.

Proof. Suppose P satisfies (N1) and (N2). Let (R,V,W,S) be an S-faithful Morita context with $R \in P$. In the notation preceding Proposition 1 (with $P = 0$), we have a context $(R, V/V_0, W/W_0, S)$ which satisfies the conditions 1) - 3) in Proposition 3. Suppose that $sSs = 0$ $(s \in S)$. Then, we have $(VsW)^2 = 0$ by using $sWVs = 0$. Hence, $VsW = 0$, and so $s = 0$, that is S is a semiprime ring, proving 4). Hence, the ring $C = \begin{pmatrix} R & V/V_0 \\ W/W_0 & S \end{pmatrix}$ is semiprime by Proposition 3. Now, $R \in P$ and $R \simeq \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} \triangleleft_\ell \begin{pmatrix} R & V/V_0 \\ 0 & 0 \end{pmatrix} \triangleleft_r C$. Let $R' = \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix}$, and $T = \begin{pmatrix} R & V/V_0 \\ 0 & 0 \end{pmatrix}$. Since $\ell_T(R') \cap r_T(R') = 0$ and $\ell_C(T) \cap r_C(T) = 0$, we have $C \in P$ by (N2). Again, $S \simeq \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix} \triangleleft_\ell \begin{pmatrix} 0 & 0 \\ W/W_0 & S \end{pmatrix} \triangleleft_r C$ and S is a semiprime ring, so (N1) implies $S \in P$. Conversely, suppose P is a normal class and $L \triangleleft_\ell T \triangleleft_r R$. If $R \in P$ and L is a semiprime ring, then the context (R,RL,T,L) is L-faithful, and so $L \in P$. If $\ell_T(L) \cap r_T(L) = 0$, $\ell_R(T) \cap r_R(T) = 0$, $L \in P$ and R is a semiprime ring, then the context (L,T,RL,R) is R-faithful, and so $R \in P$ and (N2) is satisfied.

Corollary 1 ([10, Theorem 3.2]). Every normal class P of semiprime rings is weakly special.

Corollary 2. Let P be a normal class of semiprime rings. Let (R,V,W,S) be a Morita context with $R \in P$, and $C = \begin{pmatrix} R & V \\ W & S \end{pmatrix}$. Then the following are equivalent:

1) $C \in P$.
2) C is a semiprime ring.
3) (R,V,W,S) satisfies 2), 3) and 4) in Proposition 3.

Proof. By Proposition, it is clear that 1) $\Longrightarrow$ 2) and 2) $\Longrightarrow$ 3). Now, combining Proposition 3 and the proof of Theorem 1, we readily obtain 3) $\Longrightarrow$ 1).

2. Examples of normal classes and weakly special classes. Let R be a ring, and M a non-zero left R-module such that $RM \neq 0$. Following [3], M is called prime if one of the following equivalent conditions is satisfied:

(1) $aRm = 0$ ($m \in M$, $a \in R$) implies $m = 0$ or $aM = 0$;

(2) $\ell_R(M) = \ell_R(N)$ for all non-zero submodule N of M, where $\ell_R(X) = \{a \in R \mid aX = 0\}$ for any subset X of M.

We call M monoform if every non-zero partial endomorphism of M is a monomorphism. Furthermore, M is called strongly prime (abbreviated as SP) if for any m' and non-zero m in M there exists a finite set $\{a_1,\ldots,a_n\} \subseteq R$ such that $\bigcap_{i=1}^{n} \ell_R(a_i m) \subseteq \ell_R(m')$. Following [5], M is called compressible if M can be embedded in each of its non-zero submodules. Following [16], M is said to have property K if M is prime and $\ell_R(M/N) = \ell_R(M)$ for ${}_RN \subseteq {}_RM$ only when $N = 0$.

A ring R is called endoprimitive (resp. weakly primitive) if there exists a faithful SP left R-module (resp. a faithful, compressible and monoform left R-module) [6]. A ring R is called strongly prime (resp. strictly prime) if, given a in R, there exists a finite set $\{b_1,\ldots,b_n\} \subseteq R$ such that $\bigcap_{i=1}^{n} \ell_R(b_i a) = 0$ (resp. if for any a in R there exists $b \in R$ such that $\ell_R(ba) = 0$) [8]. A ring R is called K-primitive if there exists a faithful left R-module having property K [17]. If R is a non-singular ring with uniform left ideal, then R is called a prime Johnson ring [19]. If R is a prime ring with no locally nilpotent ideals, then R is called a prime Levitzki semi-simple ring [14].

Theorem 2. The following classes are normal:

(1) The semiprime rings [1, 10].

(2) The prime rings [1].

(3) The endoprimitive rings [6, 10].

(4) The weakly primitive rings [14].

(5) The K-primitive rings [16].

(6) The primitive rings [1].

(7) The primitive rings with non-zero socle [16].

(8) The prime Johnson rings [16].

(9) The prime Levitzki semi-simple rings [1].

(10) The prime subdirectly irreducible rings [15].

(11) The prime rings with no non-zero nil left (right, two-sided) ideals [14].

(12) The rings with a faithful uniform SP module [10].

(13) The rings with a faithful monoform SP module [10].

(14) The rings with a faithful compressible module [14].

(15) The rings with a faithful uniform compressible module [14].

Theorem 3. The following classes of rings are weakly special:

(1) The strongly prime rings [7].

(2) The strictly prime rings [10].

(3) The completely prime rings, i.e. the domains [10].

(4) The reduced rings, i.e. the rings with no non-zero nilpotent elements [10].

(5) The simple rings.

Remark. The weakly special classes of rings in Theorem 3 are not normal (cf. Corollary 1): Let V be an infinite dimensional vector space over a field F, and $E = Hom_F(V, V)$. Then, (F,V,V^*,E) is an E-faithful Morita context. But E is neither reduced nor strongly prime [10].

3. Properties of normal classes and weakly special classes. A ring R is called faithful if $\ell_R(R) = r_R(R) = 0$. We shall now study properties of normal classes.

Proposition 4 ([10, Proposition 3.2]). Let P be a normal class of rings.

(1) If $R \in P$, then $eRe \in P$ for each non-zero idempotent e of R.

(2) Let e be a non-zero idempotent of a ring R such that $eRe \cap RaR \neq 0$ for any non-zero $a \in R$. If $eRe \in P$, then $R \in P$.

(3) Let e be a non-zero idempotent of a semiprime ring R. If $eRe \in P$, then $R/(\ell_R(Re) \cap r_R(eR)) \in P$.

(4) Let R be a faithful ring. If $R \in P$, then the $n \times n$ matrix ring $(R)_n \in P$ for each positive integer n.

(5) Let R be a faithful ring. If $(R)_n \in P$ for some positive integer n, then $R \in P$.

As an immediate consequence of Proposition 4 (1) and (3), we obtain [12, Theorem 1] and [15, Corollary 2 to Proposition 5].

Corollary 3. Let P be a normal class of prime rings, and e a non-zero idempotent of a prime ring R. Then, $eRe \in P$ if and only if $R \in P$. Especially, eRe is a primitive ring if and only if R is a primitive ring.

The next generalizes [15, Proposition 5].

Proposition 5. Let P be a normal class of semiprime rings, and R a semiprime ring.

(1) If $L <_{\ell} T <_{r} R$, $\ell_T(L) \cap r_T(L) = 0$ and $\ell_R(T) \cap r_R(T) = 0$, then the following are equivalent:

1) $L \in P$.

2) L is a semiprime ring and $R \in P$.

3) $LaT = 0$ $(a \in L)$ implies $a = 0$, and $R \in P$.

(2) Let L be a non-zero left ideal of a semiprime ring R such that $\ell_R(L) \cap r_R(L) = 0$, and let T be a non-zero right ideal of R such that $\ell_R(T) \cap r_R(T) = 0$. Then, $L \cap T \in P$ if and only if $L \cap T$ is a semiprime ring and $R \in P$.

Proof. (1) Obviously, 2)$\Longrightarrow$3).

1)$\Longrightarrow$2) follows by the condition (N2) in Theorem 1.

3)$\Longrightarrow$1). Consider the Morita context (R,RL,T,L). It suffices to show that this context is L-faithful by the normality of P. If $RLaT = 0$ $(a \in L)$, then $LaT = 0$, and hence $a = 0$ by 3), that is the above context is L-faithful.

(2) Let $L \cap T \in P$. Consider the Morita context $(L \cap T, T, L, R)$. If $TaL = 0$ $(a \in R)$, then $aLTRaLT = 0$. Since R is a semiprime ring, we have $aLT = 0$. Hence, we have $aL \subseteq \ell_R(T) \cap r_R(T) = 0$. Similarly, we have $a \in \ell_R(L) \cap r_R(L) = 0$. Hence, the above context is R-faithful, and so $R \in P$. Conversely, let $L \cap T$ be a semiprime ring and $R \in P$. Since the Morita context $(R, L, T, L \cap T)$ is $(L \cap T)$-faithful, we have $L \cap T \in P$.

As immediate consequences of Proposition 5, we obtain the next corollary.

Corollary 4. Let P be a normal class of semiprime rings.

(1) Let A be a non-zero ideal of a semiprime ring R such that $\ell_R(A) \cap r_R(A) = 0$. Then, $A \in P$ if and only if $R \in P$.

(2) Let $R \in P$, and L a non-zero left ideal of R such that $\ell_R(L) \cap r_R(L) = 0$. Then, $L \in P$ if and only if $L \cap r_R(L) = 0$.

(3) Let R be a semiprime ring, and L a non-zero left ideal of R such that $\ell_R(L) \cap r_R(L) = 0$. Then, $L \in P$ if and only if $L \cap r_R(L) = 0$ and $R \in P$.

Proposition 6 ([10, Proposition 3.3]). Let P be a normal class of semiprime rings, and $R \in P$. If L is a non-zero left ideal of R and T is a non-zero right ideal of R, then the following are equivalent:

1) $L \cap T \in P$.

2) $L \cap T$ is a semiprime ring.

3) For any non-zero $x \in L \cap T$, $x(L \cap T) \neq 0$ and $(L \cap T)x \neq 0$.

4) For any non-zero $x \in L \cap T$, $LxT \neq 0$.

Corollary 5 ([10, Corollary 3.2]). Let P be a normal class of semiprime rings, and $R \in P$. Then, a left (resp. right) ideal L (resp. T) of R is in P if and only if $r_R(L) \cap L = 0$ (resp. $\ell_R(T) \cap T = 0$).

Proposition 7 ([10, Proposition 3.4]). Let P be a normal class of semiprime rings, and L a left ideal of a semiprime ring R with $r_R(L) = 0$. Then the following are equivalent:

1) $R \in P$.

2) $L \in P$.

3) Every subring of R containing L is in P.

4) Some subring of R containing L is in P.

According to Theorem 2, the class of (left) primitive rings, the class of prime rings, the class of weakly primitive rings, the class of endoprimitive rings, K-primitive rings, the class of prime Johnson rings and the class of prime Levitzki semi-simple rings are normal. Now, the next is an immediate consequence of Proposition 7.

Corollary 6 (cf. [12, Corollary to Theorem 2]). Let R be a semiprime ring, and L a left ideal of R with $r_R(L) = 0$. Then the following are equivalent:

1) L is a primitive (resp. prime, weakly primitive, endoprimitive, K-primitive, prime Johnson or prime Levitzki semi-simple) ring.

2) Every subring of R containing L is a primitive (resp. prime, weakly primitive, endoprimitive, K-primitive, prime Johnson or prime Levitzki semi-simple) ring.

3) Some subring of R containing L is a primitive (resp. prime, weakly primitive, endoprimitive, K-primitive, prime Johnson or prime Levitzki semi-simple) ring.

Following [22], a left R-module M is called semiprime in the sense of Zelmanowitz (abbreviated as ZsP) if $(m, M^*) = 0$ $(m \in M)$ implies $m = 0$, where $(m, f) = mf$ for

$m \in M$ and $f \in M^*$. And M is called torsionless if $(m,M^*) = 0$ $(m \in M)$ implies $m = 0$. The first result in the next proposition was proved for prime rings by Zelmanowitz [21], and for primitive rings by Posner [18] and extended by Amitsur [1] to normal classes.

Proposition 8 ([15, Proposition 8]). Let P be a normal class of rings, M a left R-module, and $E = \mathrm{Hom}_R(M,M)$.

(1) If M is torsionless, then $R \in P$ implies $E \in P$.

(2) Suppose that M is faithful and $M^*m = 0$ $(m \in M)$ implies $m = 0$. Then $E \in P$ implies $R \in P$.

(3) Let M be a faithful ZsP module. Then, $R \in P$ if and only if $E \in P$.

The possibility of embedding a ring in a class of rings P in a ring with identity in P is included in the following more general result which answers a question of Szasz [20, Problem 76].

Proposition 9 ([9, Theorem 1]). Let P be a weakly special class. If $R \in P$, then there exists a ring $S \in P$ such that $S \ni 1$ and R is isomorphic to an ideal of S.

As a combination of Theorems 2, 3 and Proposition 9, we readily obtain

Corollary 7. If R is a semiprime (resp. reduced) ring, then there exists a semiprime (resp. reduced) ring with identity such that R is isomorphic to its ideal. Every prime (resp. primitive) ring can be embedded in a prime (resp. primitive) ring with identity as an ideal. Every strongly prime (resp. strictly prime) ring can be embedded in a strongly prime (resp. strictly prime) ring with identity as an ideal. And every completely prime ring can be embedded in a completely prime ring with identity as an ideal (cf. [4, p. 101] and [13, p. 518]).

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NON-SOLVABLE MULTIPLICATIVE SUBGROUPS OF $M_2(D)$

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Let $M_2(D)$ be the full matrix algebra of degree 2 over a division algebra D of characteristic 0. In [7] we determined the simple groups which are homomorphic images of multiplicative subgroups G of $M_2(D)$. In this paper we will study non-solvable multiplicative subgroups G of $M_2(D)$. Our main result is as follows.

Theorem. Let G be a finite non-solvable multiplicative subgroup of $M_2(D)$ over a division algebra D of characteristic 0. Then there exists a non-trivial solvable normal subgroup N of G such that $Aut(T) \geqq G/N \geqq T$ and $T \cong PSL(2,5)$, $PSL(2,9)$ or $PSL(2,5) \times PSL(2,5)$.

Let K be a field contained in the center of D. Let G be a multiplicative subgroup of $M_2(D)$. We define $V_K(G) = \{\sum \alpha_i g_i \mid \alpha_i \in K, g_i \in G\}$ as a K-subalgebra of $M_2(D)$. Then we have

Lemma 1 ([6]). $V_K(G) \cong D_1$, $D_1 \oplus D_2$ or $M_2(D_1)$ where D_1 and D_2 are division algebras.

On p-subgroups of $M_2(D)$ we have

Lemma 2 ([5],[6]). Let P be a p-group which is a subgroup of $M_2(D)$.

(1) If P is abelian, then P is generated by at most 2 elements.

(2) If $p \neq 2$, then P is abelian.

(3) If $p = 2$, then $P/[P,P]$ is generated by at most 4 elements.

All simple groups whose Sylow 2-subgroups are generated by at most 4 elements have been determined in Gorenstein - Harada [4]. Using their theorem and Alperin-Brauer-Gorenstein's theorem [1] we have

Theorem 3 ([7]). Let S be a simple group. If there exist a division algebra D of characteristic 0, a finite multiplicative subgroup G of $M_2(D)$ and a normal subgroup N of G satisfying $G/N \cong S$, then S is isomorphic to $PSL(2,5)$ or $PSL(2,9)$ and $N \neq 1$.

Let G be a non-solvable subgroup of $M_2(D)$. Let $G = G_1 \supset G_2 \supset \dots \supset G_r = 1$ be a principal series for G. Since G is not solvable, there exists a factor group $G_i / G_{i+1} \cong S \times \dots \times S$ such that $S \cong PSL(2,5)$ or $PSL(2,9)$. A Sylow 3-group of $PSL(2,5)$ (resp. $PSL(2,9)$) is a cyclic group (resp. an abelian group of rank 2), which implies $G_i / G_{i+1} \cong PSL(2,5)$, $PSL(2,9)$ or $PSL(2,5) \times PSL(2,5)$, because a Sylow 3-subgroup of G_i is generated by at most 2 elements. Further we can prove that the number n of non-solvable factor groups G_i / G_{i+1} is $\leqq 2$ and that if $n = 2$ then $G_i / G_{i+1} \cong PSL(2,5)$.

From this we obtain our main theorem.

Theorem 4. Let G be a non-solvable subgroup of $M_2(D)$. Then $G \trianglerighteq N \neq 1$ such that $Aut(T) \gtrsim G/N \gtrsim T$ and $T \cong PSL(2,5)$, $PSL(2,9)$ or $PSL(2,5) \times PSL(2,5)$.

Put $\pi = \{2,3,5,7\}$. Since a solvable subgroup of $M_2(D)$ has a normal Hall π'-subgroup (see [6]), we have

Corollary 5. If G is a finite multiplicative subgroup of $M_2(D)$ for some division algebra D . Then G has a normal Hall π'-subgroup.

Let P be a Sylow 2-subgroup of G . Multiplicative subgroups of $M_2(D)$ with an abelian Sylow 2-subgroup have been studied in [6]. Here we assume that P is not abelian. Since the rational number field $\mathbb{Q}$ is a subfield of the center of D , we can define $V_{\mathbb{Q}}(P)$ and by lemma 1 $V_{\mathbb{Q}}(P) \cong D_1$, $D_1 \oplus D_2$ or $M_2(D_1)$. Let $H_o = \mathbb{Q} + \mathbb{Q}i + \mathbb{Q}j + \mathbb{Q}k$ be the ordinary quaternion algebra over $\mathbb{Q}$. Let ε_n be a primitive n-th root of unity. Then one of the following conditions is satisfied:

(1) P is a generalized quaternion group of order 2^n and $V_{\mathbb{Q}}(P) = H_o \otimes_{\mathbb{Q}} \mathbb{Q}(\varepsilon_{2^n} + \varepsilon_{2^n}^{-1})$.

(2) $V_{\mathbb{Q}}(P) \cong D_1 \oplus D_2$, $D_1 \cong H_o \otimes_{\mathbb{Q}} \mathbb{Q}(\varepsilon_{2^n} + \varepsilon_{2^n}^{-1})$, and $D_2 \cong$ a field or $H_o \otimes_{\mathbb{Q}} \mathbb{Q}(\varepsilon_{2^m} + \varepsilon_{2^m}^{-1})$.

(3) $V_{\mathbb{Q}}(P) \cong M_2(D_1)$, D_1 is a field.

(4) $V_{\mathbb{Q}}(P) \cong M_2(H_o \otimes_{\mathbb{Q}} \mathbb{Q}(\varepsilon_{2^n} + \varepsilon_{2^n}^{-1}))$.

In the case (2) or (4) for $n \geqq 3$, we have

Proposition 6. Let G be a subgroup of $M_2(D)$. Let P be a Sylow 2-subgroup of G. If P satisfies one of the follows conditions (i), (ii), then the Schur index of D over $\mathbb{Q}$ is equal to 2 and G is a subgroup of $GL(4,\mathbb{C})$:

(i) $V_{\mathbb{Q}}(P) = (H_o \otimes_{\mathbb{Q}} \mathbb{Q}(\varepsilon_{2^n} + \varepsilon_{2^n}^{-1}) \oplus D_2$, $n \geqq 3$, for some division algebra D_2.

(ii) $V_{\mathbb{Q}}(P) = M_2(H_o \otimes_{\mathbb{Q}} \mathbb{Q}(\varepsilon_{2^n} + \varepsilon_{2^n}^{-1}))$, $n \geqq 3$.

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ON ALGEBRAS OF SECOND LOCAL TYPE

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Introduction. Throughout the report, A denotes a (left and right) artinian ring with identity 1, J its Jacobson radical. Let n be any natural number. Then we say that A is of right n^{th} local type in case for every finitely generated indecomposable right A-module M, the n^{th} top $top^n M := M/MJ^n$ of M is indecomposable. For such a ring A, the question of indecomposability of finitely generated right A-modules can be reduced to the corresponding problem of modules over A/J^n. In [8] H. Tachikawa has studied the case n = 1 and obtain the necessary and sufficient condition for algebras (by algebra we always mean a finite dimensional algebra over a field k) to be of this type. Further the representation theory of algebras with square-zero radical is well known [4],[5],[6]. So in this note we examine the case n = 2 and give some necessary conditions for artinian rings with selfduality to be of this type. Further in particular for QF rings, we give necessary and sufficient conditions to be of this type. More precisely we show:

Theorem 1. Let A be with selfduality, of right 2^{nd} local type and e any primitive idempotent in A. Then

(1) J^2e is a uniserial waist in Ae if $J^2e \neq 0$ (see section 2 for definition of a waist),

(2) eJ^m is a direct sum of local modules for every $m \geqslant 2$,

(3) for each local direct summand L of eJ^2, LJ^2 is uniserial.

Further if A is an algebra we have

(4) Ae is uniserial if $h(Ae) \geqslant 5$.

In particular if the ground field k is in addition an algebraically closed field, then

(5) Ae is uniserial if $h(Ae) \geqslant 4$,

and then

(6) eJ^2 is a direct sum of uniserial modules.

<u>Theorem</u> 2. Let A be a QF ring. Then the following statements are aquivalent:

(1) A is of right 2^{nd} local type,

(2) A is of right 2^{nd} colocal type (see section 1 for definition),

(3) for any primitive idempotent e in A, eA is uniserial if $h(Ae) \geqslant 4$,

(4) A/J^t is QF for every $t \geqslant 3$,

(5) for each M_A indecomposable and $h(M) \geqslant 3$, there is a primitive idempotent e in A such that $M \cong eA/eJ^{h(M)}$,

(6) $A = A_1 \times A_2$ for some QF rings A_1 and A_2 such that A_1 has cube-zero radical and A_2 is a serial ring.

Furthermore each of these conditions are equivalent to the corresponding left side version.

In the theorems above h(M) dinotes the height (= Loewy length) of M, namely $h(M) := \min\{ n \in \mathbb{N} \cup \{0\} \mid MJ^n = 0\}$.

In section 1, we introduce the basic tools used in the following sections. Section 2 is devoted to the structure of indecomposable projective left modules. In section 3 we examine the structure of indecomposable projective right modules on the whole using the method of Sumioka [7]. Finally in section 4 we give the proof of Theorem 2.

Proofs will be given only in outline. See [0] for details.

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1. Preliminaries

1.1. Throughout the note all modules are (unital) finitely generated. We write homomorphisms on the opposite side to scalar multiplications. For homomorphisms $p : K \longrightarrow L$ and $q : L \longrightarrow M$ of left A-modules and for a decomposition $D : L = \bigoplus_{i=1}^{n} L_i$ of L, $(p,D) = (p_i)_{i=1}^{T\,n}$ and $(D,q) = (q_i)_{i=1}^{n}$ are matrix expressions of p and q relative to D, respectively. In addition to the definition of right n^{th} local type for n any natural number, we define the dual notion: A is called to be of <u>left n^{th} colocal type</u> in case for every indecomposable left A-module M, the n^{th} socle $soc^n M$ of M is indecomposable. It should be noted that if A has a selfduality, then A is of right n^{th} local type iff A is of left n^{th} colocal type. Further as easily verified, when A is of right n^{th} local type, A is of finite representation type iff A/J^n is of finite representation type.

Since the property to be of n^{th} local (colocal) type is Morita invariant, we may assume that A is a basic ring. We put $pi(A) := \{e_1, \ldots, e_p\}$ to be a basic set of primitive idempotents of A.

1.2. Definition. Let $D : L = \bigoplus_{i=1}^{n} L_i$ be a decomposition of a right A-module L, $p : K \longrightarrow L$ be a homomorphism and $j \in \{1,\ldots,n\}$. Then the pair (p,D) (or $p : K \longrightarrow \bigoplus_{i=1}^{n} L_i$) is called j-fusible in case there is a homomorphism $q : \bigoplus_{i\neq j} L_i \longrightarrow L_j$ such that the diagram

$$\begin{array}{ccc} K & \xrightarrow{(p_i)^T_{i\neq j}} & \bigoplus_{i\neq j} L_i \\ \| & & \downarrow q \\ K & \xrightarrow{p_j} & L_j \end{array}$$

commutes where $(p,D) = (p_i)^T{}_{i=1}^{n}$. The pair (p,D) is called fusible in case (p,D) is j-fusible for some $j = 1,\ldots n$. Finally (p,D) is called infusible in case (p,D) is not fusible.

1.3. Let I be a two-sided ideal of A and $e,f \in pi(A)$. Then we have the canonical isomorphisms $Hom_A(fA, eA/eI) \xrightarrow{\sim} eAf/eIf \xrightarrow{\sim} Hom_A(Ae, Af/If)$. We denote by p^t the image of every $p \in Hom_A(fA,eA/eI)$ or inverse image of every $p \in Hom_A(Ae, Af/If)$ under the composition of these isomorphisms.

Proposition. Let $e,f_1,\ldots,f_n \in pi(A)$, $\ell > m$, $j \in \{1,\ldots,n\}$ and $p = (p_i)_{i=1}^{n} : \bigoplus_{i=1}^{n} f_iA \longrightarrow eJ^m/eJ^\ell$ be a homomorphism. Then the following statements are equivalent:

(1) $p(f_jA) \leqslant \sum_{i\neq j} p(f_iA)$.

(2) $p^t : Ae/J^{\ell-m}e \longrightarrow \bigoplus_{i=1}^{n} Af_i/J^\ell f_i$ is j-fusible

where p^t is the induced map by the homomorphism $(p_i^t)_{i=1}^{n}$.

Proof. Direct calculation.

In futur p^t shall always mean the above induced homomorphism when the domain of p is of the form as above.

Corollary. Under the same situation as above but $\ell = m + 1$, the following statements are equivalent:

(1) $\bar{p} : \bigoplus_{i=1}^{n} f_i A/f_i J \longrightarrow eJ^m/eJ^{m+1}$ (the induced map by p) is a monomorphism,

(2) $p^t : Ae/Je \longrightarrow \bigoplus_{i=1}^{n} Af_i/J^{m+1}f_i$ is infusible.

In particular if $p : \bigoplus_{i=1}^{n} f_i A \longrightarrow eJ^m$ is a projective cover of eJ^m, then $p^t : Ae/Je \longrightarrow \bigoplus_{i=1}^{n} Af_i/J^{m+1}f_i$ is infusible.

1.4. Proposition. Let $0 \longrightarrow K \xrightarrow{p} L \xrightarrow{q} M \longrightarrow 0$ be a non-split exact sequence of right A-modules and $D : L = \bigoplus_{i=1}^{n} L_i$ be a decomposition of L $(n \geqslant 2)$. Then we have

(1) If M is indecomposable, then (p,D) is infusible.

(2) If K is simple, each L_i is local and (p,D) is infusible, then M is indecomposable.

Proof. See [1] or [2].

Corollary. Let $p : \bigoplus_{i=1}^{n} f_i A \longrightarrow eJ^m$ be a homomorphism such that the induced map $\bar{p}$: top(dom(p)) $\longrightarrow$ top(cod(p)) is a monomorphism and the sequence

$$0 \longrightarrow Ae/Je \xrightarrow{p^t} \bigoplus_{i=1}^{n} Af_i/J^{m+1}f_i \longrightarrow M \longrightarrow 0$$

be exact. Then M is indecomposable.

Proof. Clear from Corollary 1.3 and Proposition 1.4.

2. Structure of indecomposable projective left modules

For an A-module M, we put $|M| :=$ the composition length of M.

2.1. Proposition. Let A be of n^{th} local type, n any natural number and $e \in pi(A)$. Then $J^n e$ is uniserial.

Proof. It suffices to prove that $|J^m e/J^{m+1}e| \leq 1$ for every $m \geqslant n$. Suppose $|J^m e/J^{m+1}e| \geqslant 2$ for some $m \geqslant n$. Then we have a homomorphism $p : Af_1 \oplus Af_2 \longrightarrow J^m e/J^{m+1}e$; f_1, $f_2 \in pi(A)$ such that the induced map $\bar{p} : \oplus_{i=1}^{2} Af_i/Jf_i \longrightarrow J^m e/J^{m+1}e$ is a monomorphism. Putting $L = \oplus_{i=1}^{2} f_i A/f_i J^{m+1}$, we have an exact sequence $0 \longrightarrow eA/eJ \xrightarrow{p^t} L \longrightarrow M \longrightarrow 0$ where M is indecomposable by Corollary 1.4. But $p^t(eA/eJ) \leqslant LJ^m \leqslant LJ^n$. Hence $top^n M = top^n L$ is decomposable, a contradiction.

2.2. Definition. ([3]) Let ${}_AL \lesssim {}_AM$. Then L is called to be a waist in M in case $0 \neq L \neq M$ and for each ${}_AN \lesssim {}_AM$ it holds that $L \lesssim N$ or $N \lesssim L$.

Proposition. Let A be with selfduality, of right 2^{nd} local type and $e \in pi(A)$. Then $J^2 e$ is a waist in Ae if $J^2 e \neq 0$.

Proof. Deduced from the following lemmas:

Lemma 1. Let ${}_AM$ be nonsimple indecomposable. Then $soc(JM) = soc\ M$.

Lemma 2. Let ${}_AM$ be local and soc^2M indecomposable. Then $\mathrm{soc}(J^2M) = \mathrm{soc}\,M$ if $J^2M \neq 0$.

Proof. Clear from Lemma 1.

Lemma 3. Let A be of left 2^{nd} local type, ${}_AM$ be local and J^2M be a nonzero uniserial module. Then J^2M is a waist in M.

Proof. Suppose J^2M is not a waist in M. Then for some $X \leqslant M$, $J^2M \nleqslant X$ and $X \nleqslant J^2M$. But $J^2M \cap X = J^tM$ for some $t \geqslant 3$. Hence $M/J^tM \geqslant (J^2M/J^tM) \oplus (X/J^tM)$ where $J^2M/J^tM \neq 0$ and $X/J^tM \neq 0$. On the other hand since $\mathrm{soc}^2(M/J^tM)$ is indecomposable and $J^2(M/J^tM) \neq 0$, we have that $\mathrm{soc}(M/J^tM) = \mathrm{soc}(J^2M/J^tM)$ is simple by Lemma 2. This is a contradiction.

We get Theorem 1 (1) from Propositions 2.1 and 2.2.

Corollary. Let A be with selfduality, of right 2^{nd} local type, $e \in \mathrm{pi}(A)$ and $h = h(Ae)$. Then we have $\mathrm{soc}^{h-t}(Ae) = J^te$ for every $t = 0,\dots,h$.

2.3. Lemma. Let ${}_AL_1$, ${}_AL_2$ be local with height $\geqslant 3$ such that for each $i = 1,2$, soc^3L_i is uniserial and J^2e_i is a uniserial waist in Ae_i where Ae_i is the projective cover of soc^3L_i. Suppose ${}_AK$ is simple and there exists an isomorphism $p_i\colon K \longrightarrow \mathrm{soc}L_i$ for each $i = 1,2$. Consider the exact sequence:

$$0 \longrightarrow K \xrightarrow{p=(p_1,p_2)} L_1 \oplus L_2 \xrightarrow{q=\begin{bmatrix}q_1\\ q_2\end{bmatrix}} M \longrightarrow 0 \quad .$$

Then soc^2M is decomposable if $p : K \longrightarrow \oplus_{i=1}^{2} socL_i$ is fusible.

Proof. See [0].

Proposition. Let A be with selfduality, of right 2^{nd} local type and ${}_AL_1$, ${}_AL_2$ be local with height ≥ 3 such that soc^3L_i are uniserial and $|L_1| \leq |L_2|$. Then for every isomorphism $r : socL_1 \longrightarrow socL_2$, r is extendable to a monomorphism $L_1 \longrightarrow L_2$ if r is extendable to a homomorphism $soc^2L_1 \longrightarrow soc^2L_2$.

Proof. Put $K = socL_1$, p_1= identity map of $socL_1$ and $p_2=r$. Let the sequence $0 \longrightarrow K \xrightarrow{p=(p_1,p_2)} L_1 \oplus L_2 \xrightarrow{q} M \longrightarrow 0$ be exact. If r is extendable to a homomorphism $soc^2L_1 \longrightarrow soc^2L_2$, then $p:K \longrightarrow soc^2L_1 \oplus soc^2L_2$ is fusible. Hence by Lemma 2.3, soc^2M is decomposable thus M is decomposable. Therefore $p: K \longrightarrow L_1 \oplus L_2$ is fusible thus r is extendable to a homomorphism $q:L_1 \longrightarrow L_2$ since $|L_1| \leq |L_2|$ where q is monic since $socL_1$ is simple.

2.4. Indecomposable projective modules with height ≥ 4

Throughout the rest of this section, A is an artinian ring with selfduality and of right 2^{nd} local type.

Proposition 1. Let $e,f \in pi(A)$ and $fJe/fJ^2e \neq 0$. Then Af is uniserial if $h(Ae) \geq 4$.

Proof. See [0].

<u>Proposition</u> 2. Assume that $e \in pi(A)$, $h(Ae) \geqslant 4$ and Ae is not uniserial. Then

(1) all simple submodules of Je/J^2e are mutually isomorphic, and

(2) $J^2e/J^3e \cong J^3e/J^4e$.

Proof. See [0].

<u>Proposition</u> 3. Assume that $e,f,g \in pi(A)$, $h(Ae) \geqslant 5$, Ae is not uniserial, $fJe/fJ^2e \neq 0$ and $J^2e/J^3e \cong Ag/Jg$. Then $fAf/fJf \cong gAg/gJg$ as rings.

Proof. See [0].

<u>Proof of Theorem 1 (4),(5)</u>. Suppose Ae is not uniserial and $h(Ae) \geqslant 4$. Let $p : \bigoplus_{i=1}^{n} P_i \longrightarrow Je/J^3e$ be a projective cover of Je/J^3e where each ${}_AP_i$ is indecomposable. Then $n \geqslant 2$. By Proposition 2, there is an $f \in pi(A)$ such that every $P_i \cong Af$. And $J^2e/J^3e \cong Ag/Jg$ for some $g \in pi(A)$. If we put $L_i := (P_i)p$ for $i = 1,2$, then $L_i \cong Af/J^2f$, $J^2e/J^3e \lneqq L_i \leqslant Je/J^3e$, $L_1 \cap L_2 = J^2e/J^3e$ and $\mathrm{top}L_i \cong Af/Jf$ for each $i = 1,2$. Since we have an exact sequence $0 \longrightarrow J^2e/J^3e \longrightarrow L_1 \oplus L_2 \longrightarrow L_1+L_2 \longrightarrow 0$ where $J^2e/J^3e \cong Ag/Jg$, $L_1 \oplus L_2 \cong (Af/Jf)^2$ and $L_1 + L_2$ is colocal, there exists an infusible homomorphism $Ag/Jg \longrightarrow (Af/Jf)^2$. Therefore $(fA/fJ)^2$ is isomorphic to a direct summand of gJ/gJ^2. Hence $\dim (gJf/gJ^2f)_{fAf/fJf} \geqslant 2$. If $h(Ae) \geqslant 5$ or k is algebraically closed, then by Propsition 3 $d := \dim {}_{gAg/gJg}(gJf/gJ^2f) = \dim (gJf/gJ^2f)_{fAf/fJf} \geqslant 2$. Hence $(Ag/Jg)^d$ is isomorphic to a direct summand of Jf/J^2f

and $d \geqslant 2$. Thus $|Jf/J^2f| \geqslant 2$. This contradicts the uniseriality of Af. Hence Ae must be uniserial.

3. Structure of indecomposable projective right modules

3.1. Lemma. Let $0 \longrightarrow K \xrightarrow{p} L \xrightarrow{q} M \longrightarrow 0$ be a nonsplit exact sequence of left A-modules such that K is simple, $D : L = \bigoplus_{i=1}^{n} L_i$ is a decomposition of L and for each $i = 1,\dots,n$, $h(L_i) = m + 1$ $(m \geqslant 1)$, $JL_i = soc^m L_i$ and $L_i \cong Ae_i/I_i$ for some $e_i \in pi(A)$ and $I_i \leqslant J^m e_i$. Then $JM = soc^m M$ if (p,D) is infusible.

Proof. See [0].

Proposition. Let A be with selfduality, of right 2^{nd} local type, $m \geqslant 2$, $e, f_1,\dots,f_n \in pi(A)$ and $p : \bigoplus_{i=1}^{n} f_i A \longrightarrow eJ^m/eJ^{m+1}$ be a projective cover of eJ^m/eJ^{m+1}. Then $p^t : Ae/Je \longrightarrow \bigoplus_{i=1}^{n} Jf_i/J^{m+1}f_i$ is infusible.

Proof. By Corollary 2.2 , Proposition 1.4 and Lemma 3.1. See [0] for details.

3.2. Proof of Theorem 1 (2). Let $p : \bigoplus_{i=1}^{n} f_i A \longrightarrow eJ^m$ be a projective cover of eJ^m and $f_i \in pi(A)$ for each $i = 1, \dots, n$. There is some $u_i \in eJ^m f_i \setminus eJ^{m+1} f_i$ for each $i = 1, \dots, n$ such that the i^{th} coordinate map of p is the right multiplication by u_i. Then $eJ^m = \sum_{i=1}^{n} u_i A$ where each $u_i A$ is local. If eJ^m is not a direct sum of local modules, then we have $\sum_{i=1}^{n} u_i a_i = 0$ for some $a_i \in A$ and $u_j a_j \neq 0$

for some $j = 1,\dots,n$. We may assume that there is some $g \in pi(A)$ such that $u_j a_j g \neq 0$ and $a_i = f_i a_i g$ for each $i = 1,\dots,n$. As easily seen, $a_i \in f_i Jg$ for each $i = 1, \dots,n$. From these facts we can show that $p^t : Ae/Je \longrightarrow \oplus_{i=1}^{n} Jf_i/J^{m+1}f_i$ is fusible. This contradicts Proposition 3.1. See [0] for details.

3.3. <u>Proof of Theorem 1 (3),(6)</u>. Suppose $|LJ^s/LJ^{s+1}| \geqslant 2$ for some $s \geqslant 1$. Then LJ^s is a direct sum of local modules and $L = vA$ for some $v \in eJ^2g \setminus eJ^3g$ and for some $g \in pi(A)$. Hence $vJ^s = u_1A \oplus u_2A \oplus \dots$ for some $u_i \in eJ^{2+s}f_i \setminus eJ^{3+s}f_i$. Then for each $i = 1,2$, there is some $a_i \in gJ^sf_i$ such that $u_i = va_i$. Define the map $p_i : Ag/J^3g \longrightarrow J^sf_i/J^{s+3}f_i \hookrightarrow Af_i/J^{s+3}f_i$, $x \longmapsto (xa_i + J^{s+3}f_i)$, for each $i = 1,2$. Then p_1 and p_2 are both monomorphisms since if we put $\tilde{v} := v + J^3g$ and $\tilde{u}_i := u_i + J^{s+3}f_i$, then $soc(Ag/J^3g) = A\tilde{v}$ and $soc(J^sf_i/J^{s+3}f_i) = A\tilde{u}_i$ are simple modules and $(A\tilde{v})p_i = A\tilde{u}_i$ for each $i = 1,2$. In particular Ag is uniserial. In case $s \geqslant 2$, by the above $A\tilde{v} \xrightarrow{(p_1,p_2)} \oplus_{i=1}^{2} J^sf_i/J^{s+3}f_i$ is fusible. By Proposition 2.3 $A\tilde{v} \xrightarrow{(p_1,p_2)} \oplus_{i=1}^{2} Af_i/J^{s+3}f_i$ is fusible, say 2-fusible. Then we can show $\bar{u}_2A \leqslant \bar{u}_1A$ where $\bar{u}_i := u_i + eJ^{s+3}$ for each $i = 1,2$. This contradicts the linear independency of $\bar{u}_1A$ and $\bar{u}_2A$. In case k is algebraically closed, we may assume $s = 1$. By the similar argument we can show the linear dependency of $\bar{u}_1A$ and $\bar{u}_2A$. See [0] for details.

4. QF rings of right 2^{nd} local type

4.1. Proof of Theorem 2. Let (x)' be the left side version of (x) for each $x = 1,3$. We show (1) $\Rightarrow$ (3)' $\Leftrightarrow$ (3) $\Rightarrow$ (6) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (1). Note (2) $\Leftrightarrow$ (1)' since A has a selfduality. Put $D := \mathrm{Hom}_A(?,A)$ to be the selfduality of A.

(1) $\Rightarrow$ (3)'. Let $e \in pi(A)$ and $h := h(Ae) \geqslant 4$. Then J^2e is a uniserial waist in Ae. Hence $soc^2 eA = D(Ae/J^2e)$ is a waist in $eA = D(Ae)$ and $soc^2 eA = eJ^{h-2}$ is a direct sum of local modules for $h-2 \geqslant 2$. But since $eJ^{h-2} \leqslant eA$ and eA is colocal, eJ^{h-2} is local. Hence $|Je/J^2e| = |soc^2(eA)/soc(eA)| = 1$ and Ae is uniserial.

(3)' $\Leftrightarrow$ (3). Clear from the fact that height and uniseriality is preserved by D.

(3) $\Rightarrow$ (6). It suffices to prove that if A is an indecomposable ring and $J^3 \neq 0$, then A is serial. (Note we assume that A is basic.) Put Q to be the right quiver of A, namely the oriented graph with vertex set $\{1,2,\ldots,p\}$ where $pi(A) = \{e_1,\ldots,e_p\}$ and with n_{ij} arrows $i \longrightarrow j$ iff $\dim(e_iJe_j/e_iJ^2e_j)_{e_jAe_j/e_jJe_j} = n_{ij}$. Note that A is an indecomposable ring iff Q is connected. $J^3 \neq 0$ implies that $h(e_iA) \geqslant 4$ for some $i = 1,\ldots,p$, therefore e_iA is uniserial by (3). There is a unique $j = 1,\ldots,p$ such that $i \longrightarrow j$ in Q for e_iA is nonsimple and uniserial. Then we obtain the following

(a) $i \neq j$, and

(b) e_jA is uniserial and $h(e_jA) \geqslant 4$.

Now put $E := \{i \in \{1,\ldots,p\} \mid e_iA$ is uniserial with height $\geqslant 4\}$. Since A is basic QF there exists a permutation p such that $soc(e_iA) \cong e_{p(i)}A/e_{p(i)}J$. By (a) and (b), we have $i \in E \Leftrightarrow p(i) \in E$. We show $E = \{1,\ldots,p\}$. Suppose there is some $\ell \in \{1,\dots,p\} \setminus E$. Then $\ell \longrightarrow j$ in Q $(\ell \neq j)$ for some $j \in E$ by (a), (b) and by the fact that Q is connected. If $e_\ell A$ is uniserial, then $\mathrm{soc}(e_\ell A) \cong e_i A/e_i J$ for some $i \in E$ by (a) and (b). Hence $\ell = p^{-1}(i) \in E$, a contradiction. If $e_\ell A$ is not uniserial, then $h(e_\ell A) = 3$ and $\mathrm{soc}(e_\ell A) = e_\ell J^2 = e_j J/e_j J^2$. But $e_j J/e_j J^2 = e_i A/e_i J$ for some $i \in E$ by (b). Hence $\ell = p^{-1}(i) \in E$, a contradiction. Thus eA is uniserial for each $e \in \mathrm{pi}(A)$ i.e. A is serial.

(6) $\Rightarrow$ (4). Clear from the fact that for a serial ring A, A is QF iff the admissible sequence of A is constant.

(4) $\Rightarrow$ (5). Let M_A be indecomposable and $h := h(M) \geqslant 3$. Then A/J^h is QF by (4). Let $0 \longrightarrow K \longrightarrow \oplus_{i=1}^m P_i \longrightarrow M \longrightarrow 0$ be a projective cover of M over A/J^h with each P_i indecomposable. Then $\mathrm{soc}(\oplus_{i=1}^m P_i) \nleq K$ implies that $\mathrm{soc} P_i \nleq K$ for some $i = 1,\dots,m$ and then $P_i \cap K = 0$. Hence $P_i \hookrightarrow M$. But since P_i is injective, P_i is isomorphic to a direct summand of M. Hence $P_i \cong M$ for M is indecomposable. Further $P_i = eA/eJ^h$ for some $e \in \mathrm{pi}(A)$.

(5) $\Rightarrow$ (1). Clear.

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TILTED ALGEBRAS OF FINITE DIMENSIONAL HEREDITARY ALGEBRAS OF TAME TYPE

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Introduction. Let A be a finite dimensional k-algebra where k is assumed to be an algebraically closed field for the simplification. Following [6], a right A-module T is called a tilting module provided it satisfies the following

1) p.dim $T_A \leq 1$,
2) $Ext^1_A(T,T) = 0$,
3) There is an exact sequence $0 \to A_A \to T'_A \to T''_A \to 0$ with $T', T'' \in \text{add } T_A = \{X_A \,;\, X_A < \oplus T_A^{(n)} \text{ for some } n\}$.

A finite dimensional algebra $B = End(T_A)$ is called a tilted algebra provided T_A is a tilting module over a finite dimensional hereditary algebra A[6]. In this note we construct certain tilted algebras and study the torsion theory $(\mathcal{T},\mathcal{F})$ over the category of all finitely generated right A-modules mod A.

§1. Preliminaries. In this section we give the exposition of the representation theory of algebras and Brenner-Butler Theorem.

(1.1) We give the definition of the Auslander-Reiten quiver of an algebra A([1], see [9] for short and expository report). Let M,M' $\in$ mod A. Then f : M $\to$ M' is called an irreducible map if f is not a splittable monomorphism and not a splittable epimorphism, moreover, the factorization f = f"f' implies that f' is a splittable monomorphism or f" is a splittable epimorphism.

An exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is called an Auslander-Reiten sequence if it doesn't split, A and C are indecomposable, and, for not a splittable epimorphism h : $X \to C$, there exists j : $X \to B$ such that $h = gj$. It holds that $A = \tau C$ and $C = \tau^{-1} A$ where $\tau = DTr(\tau^{-1} = TrD)$ is the Auslander-Reiten transformation.

The Auslander-Reiten quiver Q(A) of an algebra A consists of the set of vertices and the set of arrows. The vertices of Q(A) consist of the equivalence calasses of Ind A = {X ∈ mod A ; X is indecomposable} and there exists an arrow $[M] \to [M']$ for $M, M' \in$ Ind A if there exists an irreducible map $M \to M'$.

(1.2) Let A be a finite dimensional hereditary algebra of tame type whose representations are identified with the representations of an Euclidian graph. We assume A to be basic and connected. We summarize the properties of mod A (see [3,4,5] for detail).

Ind A is the disjoint union of Ind $\mathcal{P}$, Ind $\mathcal{R}$, and Ind $\mathcal{I}$ where $\mathcal{P},\mathcal{R},\mathcal{I}$ are the subcategories of mod A consisting of all preprojective, regular, preinjective modules, respectively. $\mathcal{P},\mathcal{R},\mathcal{I}$ are characterized as follows;

1) $X \in \mathcal{P}$ <=> There is a positive integer n such that $\tau^n X=0$,
2) $X \in \mathcal{I}$ <=> There is a positive integer n such that $\tau^{-n} X=0$,
3) $X \in \mathcal{R}$ <=> There is a positive integer n such that $\tau^n X \cong X$.

In 3) the smallest such n is called the τ-period of X. $\mathcal{R}$ is an exact abelian subcategory of mod A and has a simple regular module, i.e., a regular module which has no nonzero proper regular submodule. Using the simple regular modules we can define the regular composition series, the regular length, the regular socle, the regular top for a module in $\mathcal{R}$ same as in mod A. $\mathcal{R} = \mathcal{R}_{t_1} \times \cdots \times \mathcal{R}_{t_h} \times \mathcal{H}$ $(0 \le h \le 3,\ t_i \ge 2)$, where

$R_{t_i}(H)$ is the category of all regular modules of τ-period $t_i(1)$. If $X \in P$, $Y \in R$, $Z \in I$, then $\mathrm{Hom}_A(Z,X) = \mathrm{Hom}_A(Z,Y) = \mathrm{Hom}_A(Y,X) = 0$.

(1.3) We shall state Brenner-Butler Theorem and related results(see [2,6]). Let $B = \mathrm{End}(T_A)$ with T tilting and put $T = \{M \; ; \; \oplus T^{(n)} \twoheadrightarrow M\}$, $T = \{M \; ; \; \mathrm{Hom}_A(T,M) = 0\}$. Then (T,F) forms a torsion theory over mod A. Let $F = \mathrm{Hom}_A(T,-)$, $F' = \mathrm{Ext}^1_A(T,-)$: mod A → mod B, $G = - \otimes_B T$, $G' = \mathrm{Tor}^B_1(-,T)$: mod B → mod A and $Y = \mathrm{Im}\, F$, $X = \mathrm{Im}\, F'$. Then

Brenner-Butler Theorem.

a) ${}_BT$ is tilting and $A = \mathrm{End}({}_BT)$ canonically.

b) $G'F = GF' = 0$, $F'G = FG' = 0$, further,

F and G induce a category equivalence $T \cong Y$, F' and G' induce a category equivalence $F \cong X$.

(X,Y) forms a torsion theory over mod B by a), moreover, if A is hereditary, i.e., B is a tilted algebra, then (X,Y) is a splitting torsion theory, thus $N \in X$ or $N \in Y$ for each $N \in \mathrm{Ind}\, B$. If A is hereditary, then T is tilting <=> $\mathrm{Ext}^1_A(T,T) = 0$ and $|\mathrm{Ind}\, T|$, the number of the non-isomorphic representatives in Ind T, equals to the number of the isomorphism classes of the simple A-modules.

§2. Tilted algebras of tame hereditary algebras.

Let A be a finite dimensional hereditary algebra whose representations are identified with representations of an Euclidian graph, further, A basic and connected. T is a tilting module which is multiplicity-free, i.e., $T \cong \oplus T_i$ such that each T_i is indecomposable and $i \neq j \Rightarrow T_i \not\cong T_j$.

PROPOSITION 2.1. ([7, (3.2)]). The following are equivalent.

i) $\mathcal{T}$ is infinite.

ii) $\mathcal{I} \subset \mathcal{T}$.

iii) T has no nonzero preinjective direct summand.

PROPOSITION 2.2. ([7, (3.2*)]). The following are equivalent.

i) $\mathcal{F}$ is infinite.

ii) $\mathcal{P} \subset \mathcal{F}$.

iii) T has no nonzero preprojective direct summand.

PROPOSITION 2.3. ([7, (3.3)]). One of the following occurs.

(I) $T \in \mathcal{P}$, or $T \in \mathcal{I}$.

(II) $T = T' \oplus T''$, $0 \neq T' \in \mathcal{P}$ and $0 \neq T'' \in \mathcal{R}$.

(II*) $T = T' \oplus T''$, $0 \neq T' \in \mathcal{I}$ and $0 \neq T'' \in \mathcal{R}$.

(III) $T = T' \oplus T'' \oplus T'''$, $0 \neq T' \in \mathcal{P}$, $T'' \in \mathcal{R}$, and $0 \neq T''' \in \mathcal{I}$.

In the following we shall study the cases (II) and (II*), thus assume $T = T' \oplus T''$, $0 \neq T' \in \mathcal{P}$ or $\mathcal{I}$ and $0 \neq T'' \in \mathcal{R}$. The proofs of the following results are omitted so that the reader is referred to [8].

LEMMA 2.4. If S is a simple regular module with τ-period t, then not all the $\tau^i S (i=0,\dots,t-1)$ are contained in Ind T".

COROLLARY. If Ind $T'' \subset \mathcal{R}_t$, then $|\text{Ind } T''| < t$.

LEMMA 2.5. If $X \in \mathcal{R}_t \cap \operatorname{Ind} T''$, then $\operatorname{reg}.\ell(X) < t$. Therefore, if $X \neq 0$, then $t \geq 2$.

By the above lemma, for T'', we only study $\mathcal{R}_t (t \geq 2)$. It is noted that the number of categories $\mathcal{R}_t (t \geq 2)$ is less than 4(see (1.2)).

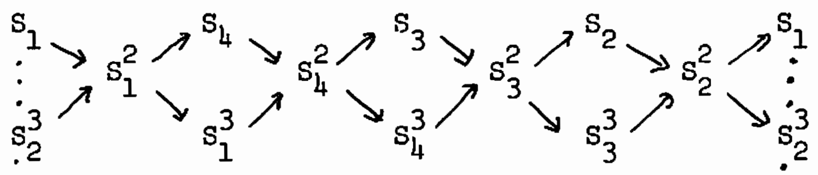
LEMMA 2.6. If $X \in \operatorname{Ind} \mathcal{T}$ and $\tau X \in \mathcal{F}$, then $X \in \operatorname{Ind} T$.

LEMMA 2.7. If $X \in \mathcal{R}$, then $t(X)$, $X/t(X) \in \mathcal{R}$, where $t(X)$ is the torsion submodule of X with respect to $(\mathcal{T},\mathcal{F})$.

COROLLARY. There exists a simple regular module $S \in \operatorname{Ind} T''$.

By the above results we firstly decide $T'' \in \mathcal{R}_t$ such that $\operatorname{Ext}_A^1(T'',T'') = 0$ and a simple regular module $S \in \operatorname{Ind} T''$. Then we seek for $T' \in \mathcal{P}$ or $T' \in \mathcal{I}$ with $\operatorname{Ext}_A^1(T' \oplus T'', T' \oplus T'') = 0$ such that $|\operatorname{Ind} (T' \oplus T'')|$ equals to the number of the isomorphism classes of the simple A-modules. We may assume that Ind T' contains a projective (injective) direct summand, if $T' \in \mathcal{P}(\mathcal{I})$(see the proof of [7, (3.2)]).

Example. Consider the component $\mathcal{R}_4$. Let $S = S_1$ be a simple regular module of τ-period 4 and $S_{i+1} = \tau^i S (i=1,2,3)$. The Auslander-Reiten quiver is the following (dotted lines are identified);



We only consider subquiver

$$\begin{array}{ccccccccc} S_2 & & & & S_1 & & & & S_4 \\ & \searrow & & \nearrow & & \searrow & & \nearrow & \\ & & S_2^2 & & & & S_1^2 & & \\ & & & \searrow & & \nearrow & & & \\ & & & & S_2^3. & & & & \end{array}$$

The other T" 's are obtained by applying τ^i for some i. The list of Ind T" with $\operatorname{Ext}_A^1(T'',T'') = 0$ such that Ind T" contains S_1 or S_2 or S_3 and consists of the above subquiver is the following:

$\{S_1\}$, $\{S_2\}$, $\{S_4\}$, $\{S_2,S_4\}$, $\{S_2^2,S_1\}$, $\{S_2^2,S_2\}$, $\{S_1^2,S_4\}$, $\{S_1^2,S_1\}$, $\{S_2^3,S_2,S_4\}$, $\{S_2^3,S_2\}$, $\{S_2^3,S_1\}$, $\{S_2^3,S_4\}$, $\{S_2^3,S_2^2,S_2\}$, $\{S_2^3,S_2^2,S_1\}$, $\{S_2^3,S_1^2,S_1\}$, $\{S_2^3,S_1^2,S_4\}$.

Remark. It is not necessarily the case that there exists some T' $\in \mathcal{P}$ or $\mathcal{I}$ with T'$\oplus$T" tilting corresponding to each T" above (for example, consider $\tilde{E}_7$:

$$\begin{array}{ccccccccccccc} & & & & & & 5 & & & & & & \\ & & & & & & \downarrow & & & & & & \\ 4 & \to & 3 & \to & 2 & \to & 1 & \leftarrow & 6 & \leftarrow & 7 & \leftarrow & 8 \end{array}$$

and $T'' = S_3^3 \oplus S_1$).

§3. Our standing assumption of this section is the same as §2. Let T = T' $\oplus$ T", 0$\neq$T' $\in \mathcal{P}$ or $\mathcal{I}$ and 0$\neq$T" $\in \mathcal{R}_t$ $(t\geq 2)$, a tilting module. If T' $\in \mathcal{P}$, then $\mathcal{H},\mathcal{I},\mathcal{R}_{t'} \subset \mathcal{T}$ for $t'\geq 2$ and $\mathcal{R}_{t'} \cap$ Ind T" = ϕ. Dually, if T' $\in \mathcal{I}$, then $\mathcal{H},\mathcal{P},\mathcal{R}_{t'} \subset \mathcal{F}$ for $t'\geq 2$ and $\mathcal{R}_{t'} \cap$ Ind T" = ϕ(cf. Propositions 2.1,2.2 also [7, (3.2),(3.2*)]). Thus we only study the remaining components in order to decide the torsion theory $(\mathcal{T},\mathcal{F})$. Put $\mathcal{S}(\mathcal{S}')$ = {S $\in \mathcal{R}_t$; S is simple regular and is not a regular composition factor of τT"(T")}.

THEOREM 3.1. Let X $\in$ Ind $\mathcal{R}_t$.

1) If T' is preprojective, then;

a) X $\in \mathcal{T}$ <=> the regular top of X is in $\mathcal{S}$ or there is an epimorphism $M \to X$ with $M \in \text{Ind } T''$.

b) $X \in \mathcal{F} \iff X \subset \tau M$ with $M \in \text{Ind } T''$.

2) If T' is preinjective, then;

a) $X \in \mathcal{T} \iff$ there is an epimorphism $M \to X$ with $M \in \text{Ind } T''$.

b) $X \in \mathcal{F} \iff$ the regular socle of X is in $\mathcal{S}'$ or $X \subset \tau M$ with $M \in \text{Ind } T''$.

Put;

$\mathcal{T}_0 = \{X \in \text{Ind } \mathcal{I}$; There is no chain of irreducible maps from X to any τM, $M \in \text{Ind } T\}$,

$\mathcal{F}_0 = \{X \in \text{Ind } \mathcal{P}$; There is no chain of irreducible maps from any M, $M \in \text{Ind } T$, to $X\}$,

$\mathcal{T}_1 = \{X \in \text{Ind } \mathcal{I}$; There is a chain of irreducible maps from M to X, or $X=M$ for $M \in \text{Ind } T\}$,

$\mathcal{F}_1 = \{X \in \text{Ind } \mathcal{P}$; There is a chain of irreducible maps from X to τM, or $X=\tau M$ for $M \in \text{Ind } T\}$.

THEOREM 3.2. 1) If $T' \in \mathcal{P}$, then $\mathcal{F}_0 \subset \mathcal{F} \cap \text{Ind } \mathcal{P} \subset \mathcal{F}_1$.
2) If $T' \in \mathcal{I}$, then $\mathcal{T}_0 \subset \mathcal{T} \cap \text{Ind } \mathcal{I} \subset \mathcal{T}_1$.

PROPOSITION 3.3. 1) Let $T' \in \mathcal{P}$ and $X \in \text{Ind } (\mathcal{P}-\mathcal{F}_1)$. Then $X \in \mathcal{T} \iff \text{Hom}_A(X,\tau T'') = 0$.

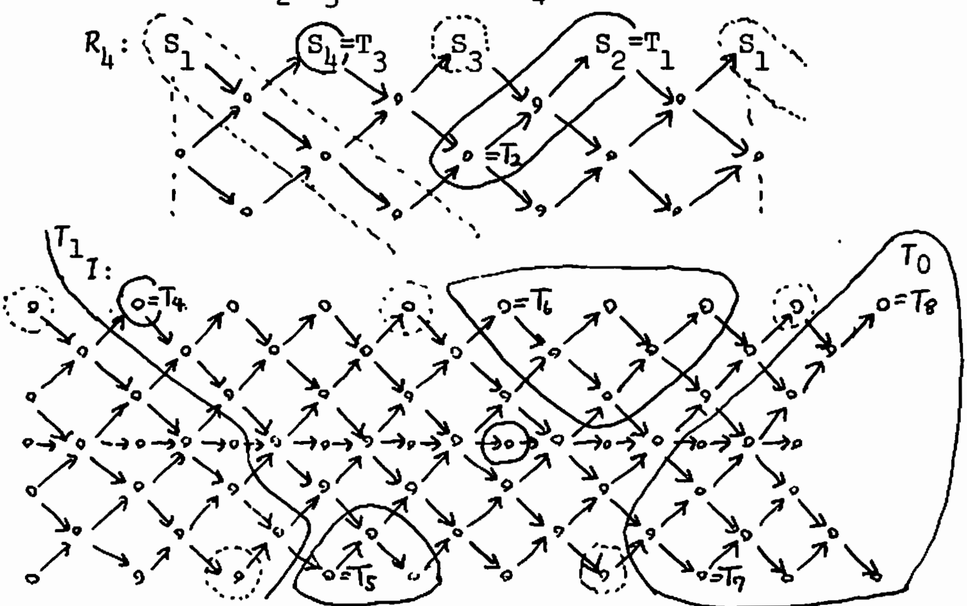
2) Let $T' \in \mathcal{I}$ and $X \in \text{Ind } (\mathcal{I}-\mathcal{T}_1)$. Then $X \in \mathcal{F} \iff \text{Hom}_A(T'',X) = 0$.

Example. Consider the path algebra A of the quiver

$$\begin{array}{c} \phantom{4 \to 3 \to 2 \to {}} 5 \phantom{{} \leftarrow 6 \leftarrow 7 \leftarrow 8} \\ \phantom{4 \to 3 \to 2 \to {}} \downarrow \phantom{{} \leftarrow 6 \leftarrow 7 \leftarrow 8} \\ 4 \to 3 \to 2 \to 1 \leftarrow 6 \leftarrow 7 \leftarrow 8 \end{array}$$

of type $\tilde{E}_7$. The simple regular representations $S_1=000\overset{1}{1}110$, $S_2=111\overset{0}{1}100$, $S_3=011\overset{1}{1}000$, $S_4=001\overset{0}{1}111$ form the τ-orbit. Let $T_1=S_2$, $T_2=112\overset{1}{3}321=S_2^3$, $T_3=S_4$.

Then $T'' = T_1 \oplus T_2 \oplus T_3 \in \mathcal{R}_4$. Let I be indecomposable injective and n be a nonnegative integer. Then; $\mathrm{Ext}_A^1(\tau^{4n+k} I, T'') \cong \mathrm{DHom}_A(\tau^{-(k+1)} T'', I) = 0 \ (0 \le k \le 3) \iff k=0: I=I_4,\ k=2: I=I_8$, where I_i is the indecomposable injective module corresponding to each $i \in \tilde{E}_7$. Let $T_4 = 112\overset{1}{2}210 = \tau^8 I_4$, $T_5 = 111\overset{1}{2}211 = \tau^6 I_8$, $T_6 = 111\overset{0}{1}110 = \tau^4 I_4$, $T_7 = 000\overset{0}{0}100 = \tau^2 I_8$, $T_8 = 100\overset{0}{0}000 = I_4$ and $T' = \oplus_{i=4}^{8} T_i$. Then $T = T' \oplus T''$ is tilting and $B = \mathrm{End}(T_A)$ is a tilted algebra. The Auslander-Reiten quiver of A is $\mathcal{P} \times \mathcal{R}_2 \times \mathcal{R}_3 \times \mathcal{R}_4 \times \mathcal{H} \times \mathcal{I}$ by [5]. Thus $\mathcal{P}, \mathcal{R}_2, \mathcal{R}_3, \mathcal{H} \subset \mathcal{F}$ and $\mathcal{R}_4$, $\mathcal{I}$ is the following;



,where $\mathcal{T}(\mathcal{F})$ is encircled by the solid (dotted) line.

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TILTING MODULES AND TORSION THEORIES

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Introduction

Brenner-Butler[5] first introduced the notion of a tilting module in a rather restrictive form, and Happel-Ringel[6] generalized their work and extensively developed the theory of tilting modules, which has a close connection with the work of Auslander-Smalø[2] and [3]. Let A be an artin algebra. Recall that a finitely generated A-module T is said to be a tilting module if it satisfies the following three properties;

(1) $\operatorname{projdim} T \leq 1$.

(2) $\operatorname{Ext}_A^1(T,T) = 0$.

(3) There is an exact sequence $0 \to A \to T' \to T'' \to 0$ with T', T'' direct sums of direct summands of T.

Let T_A be a tilting module, then T_A induces a torsion theory $(\mathcal{T}, \mathcal{F})$ on $\mathcal{M}_A$, the category of finitely generated right A-modules, where $\mathcal{T} = \{X \mid \operatorname{Ext}_A^1(T,X) = 0\}$ and $\mathcal{F} = \{X \mid \operatorname{Hom}_A(T,X) = 0\}$. The torsion class $\mathcal{T}$ consists of the modules generated by T, and the torsionfree class $\mathcal{F}$ consists of the modules cogenerated by $\operatorname{DTr} T$. Let $B = \operatorname{End}(T_A)$, then by the Theorem of Brenner-Butler ${}_BT$ is also a tilting module, thus as above induces a torsion theory on ${}_B\mathcal{M}$, the category of finitely generated left B-modules. Hence by the duality D, we get a torsion theory $(\mathcal{X}, \mathcal{Y})$ on $\mathcal{M}_B$, the category of finitely generated right B-modules, where $\mathcal{X} = \{X \mid X \otimes T = 0\}$ and $\mathcal{Y} = \{X \mid \operatorname{Tor}_B^1(X,T) = 0\}$. The torsion class $\mathcal{X}$ consists of the modules generated by

TrD U, and the torsionfree class $\mathcal{Y}$ consists of the modules cogenerated by U, where $U_B = D(_BT)$. Let $F = \mathrm{Hom}_A(_BT_A, ?)$, $F' = \mathrm{Ext}^1_A(_BT_A, ?)$ be the functors from $\mathcal{M}_A$ to $\mathcal{M}_B$, and $G = ? \otimes_B T_A$, $G' = \mathrm{Tor}^B_1(?, _BT_A)$ those from $\mathcal{M}_B$ to $\mathcal{M}_A$. The following theorem is the main result on tilting modules.

Theorem of Brenner-Butler(see [5], [6]). Let T_A be a tilting module with $\mathrm{End}(T_A) = B$. Then $_BT$ is also a tilting module with $\mathrm{End}(_BT) = A$. With the above notations, $\mathcal{T}$ and $\mathcal{Y}$ are equivalent under the restrictions of F and G which are mutually inverse to each other, and similarly $\mathcal{F}$ and $\mathcal{X}$ are equivalent under the restrictions of F' and G' which are mutually inverse to each other.

Happel-Ringel[6] showed that in case A is hereditary $(\mathcal{X}, \mathcal{Y})$ is always splitting, thus by the Theorem of Brenner-Butler B is of finite representation type whenever A is. From this point of view, we ask when $(\mathcal{X}, \mathcal{Y})$ is splitting. The main aim of this note is to answer this problem.

In §1, we study, in general situation, torsion theories on $\mathcal{M}_A$, the category of finitely generated right modules over an artin algebra A.

In §2, we study torsion theories on $\mathcal{M}_A$ such that every injective module is a torsion module (note that for a tilting module T_A the associated torsion class $\mathcal{T}$ contains every injective module). We show that the work of Auslander-Smalø[2] and [3] has a close connection with the theory of tilting modules, and that certain torsion theories conversely determine tilting modules.

In §3, we give an answer to the above problem and some other necessary conditions for $(\mathcal{X}, \mathcal{Y})$ to be splitting.

Throughout this note, we deal only with artin algebras over a fixed commutative artinian ring C. We denote by D duality $\mathrm{Hom}_C(?,I)$, where I is the injective envelope of C/rad C over C. For an artin algebra A, we denote by $\mathcal{M}_A$ the category of finitely generated right A-modules and by τ (resp. τ^{-1}) DTr (resp. TrD). All modules are finitely generated and most modules are right modules. We refer to [1] for DTr and Auslander-Reiten sequences, and to [2] for Auslander-Reiten sequences in full subcategories. We freely use the results of [1].

1. Preliminaries

We first recall some definitions Let A be an artin algebra. The pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of $\mathcal{M}_A$ is said to be a torsion theory on $\mathcal{M}_A$ provided X belongs to $\mathcal{T}$ if and only if $\mathrm{Hom}_A(X,Y) = 0$ for all Y in $\mathcal{F}$, and Y belongs to $\mathcal{F}$ if and only if $\mathrm{Hom}_A(X,Y) = 0$ for all X in $\mathcal{T}$. The modules in $\mathcal{T}$ are said to be torsion modules, and those in $\mathcal{F}$ torsionfree. Clearly the torsion class $\mathcal{T}$ is closed under extensions and factor modules, and the torsionfree class $\mathcal{F}$ is closed under extensions and submodules. For any module X, there is a torsion submodule t(X) such that X/t(X) is torsionfree, where t is a subfunctor of the identity functor called the idempotent radical. If t(X) is always a summand of X, $(\mathcal{T}, \mathcal{F})$ is said to be splitting, which is clearly equivalent to that $\mathrm{Ext}_A^1(Y,X) = 0$ for all $X \in \mathcal{T}$ and all $Y \in \mathcal{F}$. Throughout this section, we fix the above notations. Let $\mathcal{C}$ be a full subcategory of $\mathcal{M}_A$ closed under extensions. A module $X \in \mathcal{C}$ is said to be Ext-projective (resp. Ext-injective) if $\mathrm{Ext}_A^1(X,Y) = 0$ (resp. $\mathrm{Ext}_A^1(Y,X) = 0$) for all $Y \in \mathcal{C}$ (see [2]).

The following two lemmas are well known.

Lemma 1. Let $X \in \mathcal{T}$ be indecomposable. Then X is Ext-injective in $\mathcal{T}$ iff $X \simeq t(J)$ for some indecomposable injective module J.

Lemma 2. Let $X \in \mathcal{F}$ be indecomposable. Then X is Ext-projective in $\mathcal{F}$ iff $X \simeq P/t(P)$ for some indecomposable projective module P.

Lemma 3([7], cf.[2]). Let $X \in \mathcal{T}$ be indecomposable. Then

(1) X is Ext-projective in $\mathcal{T}$ iff $\tau X \in \mathcal{F}$.

(2) Assume that X is not Ext-projective in $\mathcal{T}$. Then $t(\tau X)$ is indecomposable, and for the Auslander-Reiten sequence $0 \to \tau X \to E \to X \to 0$ the induced sequence $0 \to t(\tau X) \to t(E) \to X \to 0$ is the Auslander-Reiten sequence in $\mathcal{T}$.

The dual statement of Lemma 3 is the following.

Lemma 4. Let $X \in \mathcal{F}$ be indecomposable. Then

(1) X is Ext-injective in $\mathcal{F}$ iff $\tau^{-1}X \in \mathcal{T}$.

(2) Assume that X is not Ext-injective in $\mathcal{F}$. Then $\tau^{-1}X/t(\tau^{-1}X)$ is indecomposable, and for the Auslander-Reiten sequence $0 \to X \to E \to \tau^{-1}X \to 0$ the induced sequence $0 \to X \to E/t(E) \to \tau^{-1}X/t(\tau^{-1}X) \to 0$ is the Auslander-Reiten sequence in $\mathcal{F}$.

Proposition 5([7]). The following are equivalent;

(1) $(\mathcal{T}, \mathcal{F})$ is splitting.

(2) $\mathcal{T}$ is closed under τ^{-1}.

(3) $\mathcal{F}$ is closed under τ.

Proposition 6. Let J be the anihilator ideal of $t(D(A))$, then $t(D(A)) \simeq D(A/J)$.

Proof. Clearly, $(D\mathcal{F}, D\mathcal{T})$ is a torsion theory on ${}_A\mathcal{M}$. Denote by $\bar{t}$ the idempotent radical for $(D\mathcal{F}, D\mathcal{T})$. Then we have $t(D(A)) \simeq D(A/\bar{t}(A))$. Setting $J = \bar{t}(A)$, we are done.

2. Tilting modules and torsion theories

Throughout this section, A is an artin algebra and $(\mathcal{T}, \mathcal{F})$ is a torsion theory on $\mathcal{M}_A$.

Lemma 7. Assume that $\mathcal{T}$ contains every injective module. Let $X \in \mathcal{T}$ be indecomposable. If X is Ext-projective in $\mathcal{T}$, then $\operatorname{projdim} X \leq 1$.

Proof. We may assume that X is not projective. Let $0 \to \tau X \to I_0 \to I_1$ be the minimal injective resolution. By the definition of τ^{-1}, we get the minimal projective resolution

$$\operatorname{Hom}_A(D(I_0), A) \xrightarrow{\rho} \operatorname{Hom}_A(D(I_1), A) \to X \to 0.$$

Since $\tau X \in \mathcal{F}$ and $D(A) \in \mathcal{T}$, we get

$$\begin{aligned} \operatorname{Ker} \rho &\simeq \operatorname{Hom}_A(D(\tau X), A) \\ &\simeq \operatorname{Hom}_A(D(A), \tau X) \\ &= 0. \end{aligned}$$

Proposition 8(cf.[3]). Assume that $\mathcal{T}$ contains every injective module. Let T be the direct sum of the non-isomorphic indecomposable Ext-projective modules in $\mathcal{T}$. Then T generates every module in $\mathcal{T}$ iff T is a tilting module.

Proof. "if" part: See [6].

"only if" part: We have only to show that there is an exact sequence $0 \to A \to T' \to T'' \to 0$ with T', T'' Ext-projective modules in $\mathcal{T}$. Note that T is faithful, since T generates D(A). Let $t_1, \ldots, t_r \in T$ be generators over C and define an A-homomorphism $f : A \to T^{(r)}$ by $f(a) = (t_1 a, \ldots, t_r a)$ for any $a \in A$. From the exact sequence $0 \to A \xrightarrow{f} T^{(r)} \to \mathrm{Cok}\, f \to 0$, we get the exact sequence of functors

$$\mathrm{Hom}_A(T^{(r)},?) \xrightarrow{\psi} \mathrm{Hom}_A(A,?) \to \mathrm{Ext}_A^1(\mathrm{Cok}\, f,?) \to \mathrm{Ext}_A^1(T^{(r)},?),$$

where $\psi = \mathrm{Hom}_A(f,?)$. Since T generates every module in $\mathcal{T}$, ψ is monic on $\mathcal{T}$. Therefore the functor $\mathrm{Ext}_A^1(\mathrm{Cok}\, f,?)$ vanishes on $\mathcal{T}$, thus Cok f is Ext-projective in $\mathcal{T}$.

Proposition 9([4], cf.[6]). Let n be the number of non-isomorphic simple modules. Let $T = T_1 \oplus \ldots \oplus T_m$ be a direct sum of non-isomorphic indecomposable modules. Suppose $\mathrm{projdim}\, T \leq 1$ and $\mathrm{Ext}_A^1(T,T) = 0$. Then $m \leq n$, and $m = n$ iff T is a tilting module.

Proposition 10([2]). Assume that $(\mathcal{T}, \mathcal{F})$ is induced by a tilting module. Then $\mathcal{T}$, $\mathcal{F}$ have Auslander-Reiten sequences.

Proof. This proposition directly follows from Lemma 3 (2), Lemma 4 (2) and the Theorem of Brenner-Butler.

Theorem 11(cf.[2], [3]). Let T be the direct sum of the non-isomorphic indecomposable Ext-projective modules in $\mathcal{T}$. Then, $\mathcal{T}$ has the same number of non-isomorphic indecomposable Ext-projective modules and Ext-injective modules iff T generates every module in $\mathcal{T}$. In that case, $\mathcal{T}$ has Auslander-Reiten sequences.

Proof. By proposition 6, we may assume that $\mathcal{T}$ contains every injective module. Then, this theorem is an immediate consequence of propositions 8, 9 and 10.

At the end of this section, we give, in an applicable form, a condition for $(\mathcal{T}, \mathcal{F})$ to determine a tilting module as the direct sum of the non-isomorphic indecomposable Ext-projective modules in $\mathcal{T}$.

Theorem 12([8]). Assume that $\mathcal{T}$ contains every injective module. Suppose that either $\mathcal{T}$ or $\mathcal{F}$ contains only a finite number of non-isomorphic indecomposable modules. Let T be the direct sum of the non-isomorphic indecomposable Ext-projective modules in $\mathcal{T}$. Then T is a tilting module.

If both $\mathcal{T}$ and $\mathcal{F}$ contain infinitely many non-isomorphic indecomposable modules, $\mathcal{T}$ does not necessarily contain an Ext-projective module. Consider, for example, the case in which A is connected, hereditary and of infinite representation type. Let $\mathcal{T}$ be the full subcategory of the pre-injective modules, and $\mathcal{F}$ that of modules without a pre-injective summand. Then $(\mathcal{T}, \mathcal{F})$ is a torsion theory on $\mathcal{M}_A$ and $\mathcal{T}$ contains every injective modules, whereas $\mathcal{T}$ does not have any Ext-projective module.

3. Splitting torsion theories induced by tilting modules

Throughout this section, we use the same notations as in the introduction.

Theorem 13([7]). The following are equivalent;

(1) $(\mathcal{T}, \mathcal{F})$ is splitting.

(2) $F(\tau X) \simeq \tau(FX)$ for all $X \in \mathcal{T}$.

(3) $F'(\tau^{-1}X) \simeq \tau^{-1}(F'X)$ for all $X \in \mathcal{F}$.

(4) $\operatorname{injdim} X \leq 1$ for all $X \in \mathcal{F}$.

The next lemma is included in the Theorem of Brenner-Butler.

Lemma 14. Denote by t the idempotent radical for both the torsion theories $(\mathcal{T}, \mathcal{F})$ and $(\mathcal{X}, \mathcal{Y})$. Then

(1) For $(\mathcal{T}, \mathcal{F})$, $t(X) \simeq GFX$ and $X/t(X) \simeq G'F'X$.

(2) For $(\mathcal{X}, \mathcal{Y})$, $t(X) \simeq F'G'X$ and $X/t(X) \simeq FGX$.

Proof of Theorem 13.

(1)$\Longrightarrow$(2): By Lemma 3, Proposition 5, Lemma 14 and the Theorem of Brenner-Butler.

(2)$\Longrightarrow$(1): By Proposition 5.

(1)$\Longrightarrow$(3): By Lemma 4, Proposition 5, Lemma 14 and the Theorem of Brenner-Butler.

(3)$\Longrightarrow$(1): By Proposition 5.

(1)$\Longrightarrow$(4): Let $0 \to X \xrightarrow{\mu} J$ be the injective envelope of $X \in \mathcal{F}$ and set $L = \operatorname{Cok} \mu$. On the exact sequence $0 \to X \to J \to L \to 0$, by applying the functor F we get the exact sequence $0 \to FJ \to FL \to F'X \to 0$, and thus the exact sequence of functors

$$\operatorname{Ext}^1_B(?, FJ) \to \operatorname{Ext}^1_B(?, FL) \to \operatorname{Ext}^1_B(?, F'X).$$

By Lemma 1 and the Theorem of Brenner-Butler, FJ is Ext-injective in $\mathcal{Y}$, thus $\mathrm{Ext}_B^1(?,FJ)$ vanishes on $\mathcal{Y}$, and by the assumption on $(\mathcal{X},\mathcal{Y})$ $\mathrm{Ext}_B^1(?,F'X)$ also vanishes on $\mathcal{Y}$. Therefore $\mathrm{Ext}_B^1(?,FL)$ vanishes on $\mathcal{Y}$, thus FL is Ext-injective in $\mathcal{Y}$. Hence again by Lemma 1 and the Theorem of Brenner-Butler, $L \simeq GFL$ is injective.

(4)$\Longrightarrow$(1): Let $0 \to X \to I_0 \to I_1 \to 0$ be the minimal injective resolution of $X \in \mathcal{F}$, on which by applying the functor F we get the exact sequence $0 \to FI_0 \to FI_1 \to F'X \to 0$, and thus the exact sequence of functors

$$\mathrm{Ext}_B^1(?,FI_1) \to \mathrm{Ext}_B^1(?,F'X) \to \mathrm{Ext}_B^2(?,FI_0).$$

Since $\mathcal{Y}$ contains every projective module and is closed under submodules, $\mathrm{Ext}_B^2(?,FI_0)$ vanishes on $\mathcal{Y}$ because by Lemma 1 and the Theorem of Brenner-Butler FI_0 is Ext-injective in $\mathcal{Y}$. Similarly FI_1 is Ext-injective in $\mathcal{Y}$, thus $\mathrm{Ext}_B^1(?,FI_1)$ vanishes on $\mathcal{Y}$. Therefore $\mathrm{Ext}_B^1(?,F'X)$ vanishes on $\mathcal{Y}$, which together with the Theorem of Brenner-Butler completes the proof.

In what follows, we assume that $(\mathcal{X},\mathcal{Y})$ is splitting. Let $\{S_1, \ldots, S_n\}$, $\{P_1, \ldots, P_n\}$ and $\{I_1, \ldots, I_n\}$ be the complete sets of simple, indecomposable projective and indecomposable injective modules respectively such that $\mathrm{top}\, P_i \simeq S_i \simeq \mathrm{soc}\, I_i$ for all i.

Connecting lemma([6]). $\tau^{-1}FI_i \simeq F'P_i$ for all i.

Proposition 15([7]). For any i, the minimal left almost split homomorphism starting at FI_i is of the form

$$FI_i \to F(I_i/S_i) \oplus F'(\mathrm{rad}\, P_i).$$

Remark. In case A is hereditary, this proposition is due to Happel-Ringel [6]. Their situation was, however, contrary to ours. They used this proposition to prove that $(\mathcal{X}, \mathcal{Y})$ is splitting.

Proposition 16 ([7]). If injdim $S_i > 1$, then $P_i \in \mathcal{T}$.

Proof. Let X be a non-zero indecomposable summand of I_i/S_i and suppose that X is not injective. Then FX is not Ext-injective in $\mathcal{Y}$, thus by Lemma 4 and the assumption on $(\mathcal{X}, \mathcal{Y})$ we conclude that $\tau^{-1}FX \in \mathcal{Y}$. Therefore by Proposition 15 we conclude that $F'P_i \simeq \tau^{-1}FI_i = 0$.

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FINITELY HEREDITARY TORSION THEORIES

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This note is an abstract of the author's papers [4, 5 and 6] with some new proofs. We introduce new notions on torsion classes called cyclic-hereditary and finitely hereditary. Let $\mathcal{T}$ be a torsion class and $\mathfrak{F}(\mathcal{T}) = \{X_R \leq R \mid R/X \in \mathcal{T}\}$. Then it is well-known that $\mathcal{T} \to \mathfrak{F}(\mathcal{T})$ gives a bijective correspondence between the hereditary torsion classes and the right Gabriel topologies over a ring R. There exists, however, a non-hereditary torsion class $\mathcal{T}$ such that $\mathfrak{F}(\mathcal{T})$ forms a right Gabriel topology. We characterize such torsion classes as cyclic-hereditary torsion ones in Theorem 2. In Theorem 7 we show that every torsion class over a ring Morita-equivalent to a commutative ring is finitely hereditary. We close this note with some applications of Theorem 7.

Throughout this note we consider in the category mod-R of unital right R-modules over a ring R with unit and retain the notations used in the above introduction. For the definition and basic properties of torsion classes and torsion theories see [2].

We begin this note with the following definition.

A non-empty set $\mathfrak{F}$ of right ideals of R is called a right Gabriel topology provided it satisfies the following conditions:

(1) If $X \in \mathfrak{F}$ and $X \leq Y_R \leq R$, then $Y \in \mathfrak{F}$.

(2) If $X, Y \in \mathfrak{F}$, then $X \cap Y \in \mathfrak{F}$.

(3) If $X \in \mathcal{J}$ and $a \in R$, then $(X; a) \in \mathcal{J}$.

(4) If $X \in \mathcal{J}$ and $Y_R \leq R$ such that $(Y; x) \in \mathcal{J}$ for every $x \in X$, then $Y \in \mathcal{J}$.

(In the above conditions $(X; a) = \{r \in R \mid ar \in X\}$ for every right ideal X and $a \in R$.)

Then, as noted in the introduction, $T \to \mathcal{J}(T)$ gives a bijective correspondence between the hereditary torsion classes; i.e., the torsion classes closed under taking submodules and the right Gabriel topologies on R (see [3] and [8]). There exists, however, a non-hereditary torsion class T such that $\mathcal{J}(T)$ forms a right Gabriel topology as follows.

Example 1([4, Example 4]). The class T of injective modules over the ring $\mathbb{Z}$ of integers forms a torsion class over $\mathbb{Z}$, since the ring is hereditary and Noetherian. The torsion class is obviously non-hereditary and $\mathcal{J}(T) = \{\mathbb{Z}\}$ forms a right Gabriel topology.

Definition. We call a torsion class to be cyclic-hereditary (resp. finitely hereditary) provided it is closed under taking submodules of cyclic (resp. finitely generated) torsion modules.

Theorem 2([4, Theorem 5]). Let T be a torsion class. Then $\mathcal{J}(T)$ forms a right Gabriel topology if and only if T is cyclic-hereditary.

Proof. The proof given here is different from [4, Theorem 5]. Suppose that $\mathcal{J}(T)$ forms a right Gabriel topology and Y/X a submodule of R/X in T. Then the epimorphism

$$\bigoplus_{y \in Y} R/(X;\, y) \longrightarrow Y/X \quad ((r_y) \longmapsto \sum y\, r_y)$$

induces $Y/X \in \mathcal{T}$. The converse implication follows from the same proof of the well-known assertion that $\mathfrak{F}(\mathcal{T})$ forms a right Gabriel topology for every hereditary torsion class $\mathcal{T}$.

A ring is called a right Duo ring provided every right ideal is two-sided.

Proposition 3. Every torsion class over a right Duo ring is cyclic-hereditary.

Proof. It follows from the fact that every cyclic submodule of a cyclic module M over a right Duo ring is isomorphic to a factor module of M.

We give two characterizations of cyclic-hereditary (resp. finitely hereditary) torsion classes.

Theorem 4(see [5, Proposition 1.3 with Remark in §1]). Let $\mathcal{T}$ be a torsion class. Then the following conditions are equivalent:

(1) $\mathcal{T}$ is cyclic-hereditary (resp. finitely hereditary).

(2) For every cyclic (resp. finitely generated) module M, M is torsion if and only if each of the elements in M is annihilated by some right ideal in $\mathfrak{F}(\mathcal{T})$.

Proof. (1) ⇒ (2). By [5, Proposition 1.3 with Remark in §1].

(2) ⇒ (1). Obvious.

Proposition 5 ([5, Proposition 1.1 with Remark in §1]). Let (T, F) be a torsion theory. Then T is cyclic-hereditary (resp. finitely hereditary) if and only if each of the injective hulls of the torsionfree modules has no non-zero torsion, cyclic (resp, finitely generated) submodule.

Now, we show that every torsion class over a ring Morita-equivalent to a commutative ring is finitely hereditary. To show this we need the following lemma.

Lemma 6([5, Lemma 2.1]). Let N be an essential submodule of a non-zero module M and $\{m_i \mid i = 1,\cdots, n\}$ a finite subset of the non-zero elements in M. Then there exists $r \in R$ such that $m_i r \in N$ for every i and $m_i r \neq 0$ for some i.

Theorem 7([5, Theorem 2.2]). Every torsion class over a ring Morita-equivalent to a commutative ring is finitely hereditary.

Now, we have the following trivial implications

hereditary torsion classes
$\Rightarrow$ finitely hereditary torsion classes
$\Rightarrow$ cyclic-hereditary torsion classes
$\Rightarrow$ torsion classes.

Moreover, Example 1 with the preceding theorem shows that the converse of the first implication is not true in general. The following two examples show that none of the rest of the converse is true.

Example 8([6, Example 2]). Let R be the subring of the form

$$\left\{ \begin{pmatrix} a & 0 & b \\ 0 & a & c \\ 0 & 0 & d \end{pmatrix} \,\middle|\, a, b, c, d \in K \right\}$$

of the ring of 3 × 3 full matrices over a field K and $\mathcal{T}$ the class of injective right R-modules. Then $\mathcal{T}$ forms a torsion class for mod-R, since R is right hereditary and Noetherian. Moreover, $\mathcal{T}$ is cyclic-hereditary but not finitely hereditary.

Example 9([6, Example 3]). Let R be the ring of 2 × 2 upper triangular matrices over a field and $I = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} R$ a right ideal of R. Then $\mathcal{T} = \{M \in \text{mod-}R \mid MI = M\}$ forms a torsion class which is not cyclic-hereditary.

Thus a cyclic-hereditary torsion class need not be finitely hereditary. The two notion, however, coincide when the corresponding torsionfree class is a TTF-class introduced in [7].

Theorem 10([6, Theorem]). Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory with a TTF-class $\mathcal{F}$. Then $\mathcal{T}$ is cyclic-hereditary if and only if it is finitely hereditary.

We close this note with applications of Theorem 7. The following theorem is well-known when the ring is commutative (see e.g. [9, Lemma 8.6, p.154]). We give a new proof making use of Theorem 7.

Theorem 11([5, Theorem 2.7]). If a ring R is Morita-equivalent to a commutative ring, then every idempotent ideal of R finitely generated as a one-sided ideal is generated by a single central idempotent element.

In [1] a ring R is called a right R-ring provided R_R is torsionfree for every non-trivial torsion theory for mod-R. Combining [1, Corollary 1.14] with the preceding theorem, commutative hereditary domains are right (and left) R-rings. Furthermore, we have the following.

Theorem 12([5, Theorem 2.4]). Every ring Morita-equivalent to a commutative Noetherian domain is a right (and left) R-ring.

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A CERTAIN TYPE OF COMMUTATIVE HOPF GALOIS EXTENSIONS*)

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Let R be a commutative algebra over the prime field GF(p) ($p \neq 0$), u an element in R, and m a positive integer. We denote by $H(u,p^m)$, the free Hopf algebra over R with basis $\{1, \delta, \ldots, \delta^{p^m-1}\}$ whose Hopf algebra structure is defined by

$$\delta^{p^m} = 0,$$

$$\Delta(\delta) = \delta\otimes 1 + 1\otimes\delta + u(\delta\otimes\delta), \quad \varepsilon(\delta) = 0 \quad \text{and}$$

$$\lambda(\delta) = \sum_{i=1}^{p^m-1}(-1)^i u^{i-1}\delta^i,$$

where Δ, ε and λ are the comultiplication, counit and antipode of $H(u,p^m)$, respectively. In this note we characterize commutative $H(u,p^m)$-Hopf Galois extensions of R and using this characterization, we show that a commutative $H(u,p^m)$-Hopf Galois extension is a cyclic p^m-extension [2], a purely inseparable extension [6], or a strongly radicial extension [7] according as u is invertible, or $u = 0$, or u is nilpotent. Moreover, for $H(u,p^2)$-Hopf Galois extensions A and B of R, we determine $H(u,p^2)$-module algebra isomorphisms from A to B and give a system of generators of the $H(u,p^2)$-Hopf Galois extension $A\cdot B$ of R. Finally, using the above results, we determine the isomorphism class group of $H(u,p)$-Hopf Galois extensions.

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*) This is drived from author's article [4] which includes all the proofs omitted here.

Throughout the following, R is a commutative algebra over $GF(p)$ $(p \neq 0)$, each $\otimes$, Hom, etc. is taken over R and each map is R-linear unless otherwise stated. By an R-algebra A we always assume that A is a ring extension of R with the same identity. All R-algebra homomorphisms are unitary. We freely use the notations, terminologies and the results of Hopf algebras and Galois objects in Sweedler [5] and Chase-Sweedler [1].

First we give some definitions. Let H be a finite cocommutative Hopf algebra over R. An R-algebra A is called an H-module algebra if A is an H-module such that the following conditions hold:

(1) $h(ab) = \sum_{(h)}(h_{(1)}a)(h_{(2)}b)$ and $h(1) = \varepsilon(h)1$,

where $\Delta(h) = \sum_{(h)}h_{(1)} \otimes h_{(2)}$. If A and B are H-module algebras and $f \in \mathrm{Hom}(A, B)$, then f is called an H-module algebra homomorphism if it is an H-module homomorphism and an R-algebra homomorphism. For an H-module algebra A, the smash product $A \# B$ is eqaul to $A \otimes H$ as an R-module with mutiplication

$$(a \# h)(b \# k) = \sum_{(h)}a(h_{(1)}b) \# h_{(2)}k \quad (a,b \in A, \ h,k \in H).$$

A commutative H-module algebra A is called an H-Hopf Galois extension of R if A is a finitely generated projective R-module and the map $\phi: A \# H \longrightarrow \mathrm{Hom}(A, A)$ defined by $\phi(a \# h)(x) = ah(x)$ is an isomorphism. Note that if A is a commutative $H(u,p^m)$-module algebra, then by (1), δ operates on A as follows:

$$\delta(ab) = \delta(a)b + a\delta(b) + u\delta(a)\delta(b) \quad \text{and} \quad \delta(1) = 0.$$

Now, we have the structure theorem of commutative $H(u,p^m)$-Hopf Galois extensions.

Theorem 1 ([4, Th.1.3]). Let A be an $H(u,p^m)$-Hopf Galois extension of R. Let $\delta_i = \delta^{p^{i-1}}$, $R_i = \{a \in A| \delta_i(a) = 0\}$, and H_i an R-Hopf subalgebra generated by δ_i $(1 \leq i \leq m)$. Then there exist $x_1, \ldots, x_m$ in A which satisfy the following conditions:

(1) $\delta_i(x_i) = 1$, $\delta_i(x_j) = 0$ and $(\delta_k)^{p-1}(x_{k+1}) = x_k$ $(1 \leq j < i \leq m, 1 \leq k \leq m-1)$.

(2) $\{x_1^{j_1} \ldots x_m^{j_m}\}_{0 \leq j_i \leq p-1}$ is a free basis of A.

(3) R_i is generated by $x_1, \ldots, x_{i-1}$ as an R-algebra and A is an H_i-Hopf Galois extension of R_i.

(4) $x_1^p - u^{p-1}x_1 \in R$ and $x_i^p = (u^{p^{i-1}})^{p-1}x_i + f_{i-1}(x_{i-1})$ $f_{i-1}(X) \in R_{i-1}[X]$ with $\deg f_{i-1}(X) \leq p-1$ $(2 \leq i \leq m)$.

If $u = 0$, then δ is an R-derivation on A and so $\delta(x_i^p) = 0$. Thus we have the following

Corollary 2 ([4, Cor.1.4]). Let A be an $H(0,p^m)$-Hopf Galois extension of R. Then there exist $x_1, \ldots, x_m$ in A such that $\delta_i(x_i) = 1$. Further we have $x_i^p \in R$ and $R[x_1, \ldots, x_m] = A$; and there exists an R-algebra isomorphism

$$A \cong R[X_1]/(X_1^p - x_1^p) \otimes \ldots \otimes R[X_m]/(X_m^p - x_m^p).$$

Let A be a commutative R-algebra, and $\mu: A \otimes A \longrightarrow A$ a map defined by $\mu(a \otimes b) = ab$. A is called a purely inseparable algebra over R if $\mathrm{Ker}(\mu)$ is contained in the Jacobson radical $J(A \otimes A)$ of $A \otimes A$ (cf. [6, Def.1 and Lemma 1 (a)]). A is called a strongly radicial over R

if A is a finitely generated projective R-module and $Ker(\mu)$ is a nil ideal (cf.[7]).

By Th.1, we have the following

Theorem 3 ([4,Th.1.9]). Let A be an $H(u,p^m)$-Hopf Galois extension of R.

(1) If u is contained in the Jacobson radical $J(R)$ of R, then A is a purely inseparable algebra.

(2) A is strongly radicial if and only if u is nilpotent.

Remark. Let A be an $H(u,p^m)$-Hopf Galois extension of R. If u is invertible, then $\sigma = u\delta + 1$ is an R-algebra automorphism of A of order p^m and $H(u,p^m) = R\langle\sigma\rangle$, where $\langle\sigma\rangle$ is the cyclic group generated by σ. Thus A is a cyclic p^m-extension of R in the sense of [2]. If u is idempotent, then Au is a cyclic p^m-extension of Ru and $A(1-u)$ is an $H(0,p^m)$-Hopf Galois extension of $R(1-u)$.

Now, let A_i be $H(u,p^2)$-Hopf Galois extensions of R $(i = 1, 2)$. Then by Th.1, there exist x_i, $y_i \in A$ such that the following conditions hold:

(2) $\delta(x_i) = 1$ and $\delta^{p-1}(y_i) = x_i$.

(3) $\{x_i^j y_i^k\}_{0 \leqq j,\, k \leqq p-1}$ is a free basis of A_i

(4) $x_i^p = u^{p-1}x_i + r_i$ and $y_i^p = (u^p)^{p-1}y_i + f_i(x_i)$, where $f_i(x_i) = \sum_{j=0}^{p-1} s_{ij}x_i^j$ $(r_i,\ s_{ij} \in R)$.

Under the above notations, we have the following

Theorem 4 ([4, Th.2.4]). There exists an $H(u,p^2)$-module algebra homomorphism $\psi: A_1 \longrightarrow A_2$ if and only if there exist $r \in R$ and $g(X) \in R[X]$ with $\deg g(X) \leqq p-1$ such that the following conditions hold:

(1) $r^p = u^{p-1}r + (r_1 - r_2)$.

(2) $g(x_2)^p = (u^p)^{p-1}g(x_2) + f_1(x_2 + r) - f_2(x_2)$.

(3) $\delta(g(x_2)) = g_{p-1}(x_2 + r) - g_{p-1}(x_2)$, where $g_{p-1}(x_i) = \delta(y_i)$.

When this is the case, ψ is given by

$$\psi(x_1) = x_2 + r \quad \text{and} \quad \psi(y_1) = y_2 + g(x_2).$$

Moreover the coefficients of $g(X)$ is determined explicitly.

For $H(u,p^2)$-Hopf Galois extensions A_1 and A_2, we define the product of $H(u,p^2)$-Hopf Galois extension of R as follows:

(5) $A_1 \cdot A_2 = \{\sum a_{1i} \otimes a_{2i} \in A_1 \otimes A_2 \mid \sum \delta(a_{1i}) \otimes a_{2i} = \sum a_{1i} \otimes \delta(a_{2i})\}$,

where δ acts on $A_1 \cdot A_2$ by $\delta(a \otimes b) = \delta(a) \otimes b$ $(= a \otimes \delta(b))$. Then it is known that $A_1 \cdot A_2$ is an $H(u,p^2)$-Hopf Galois extension of R. We set

$$x = x_1 \otimes 1 + 1 \otimes x_2$$

and

$$y = y_1 \otimes \delta^p(y_2) + \delta(y_1) \otimes \delta^{p-1}(y_2) + \cdots + \delta^{p-1}(y_1) \otimes \delta(y_2) + \delta^p(y_1) \otimes y_2.$$

Then we have the following

Theorem 5 ([4, Th.2.5]). Under the above notations, we have

(1) $\delta(x) = 1$ and $\delta^{p-1}(y) = x$.

(2) $\{x^j y^k\}_{0 \leqq j,k \leqq p-1}$ is a free basis of $A_1 \cdot A_2$.

Let $Gal(H(u,p))$ be the group of $H(u,p)$-isomorphism classes of commutative $H(u,p)$-Hopf Galois extension of R with product defined by (5). If A is an $H(u,p)$-Hopf Galois extension of R, then by Th.1 there exists $x \in A$ such that $\delta(x) = 1$, $\{1, x, \ldots, x^{p-1}\}$ is a free basis of A and $x^p = u^{p-1}x + r$ for some $r \in R$. Thus we may write $A = R[x;r]$. Let $B = R[y;s]$ be another $H(u,p)$-Hopf Galois extension of R, and $z = x \otimes 1 + 1 \otimes y$. By Th.5, $A \cdot B$ has a free basis $\{1, z, \ldots, z^{p-1}\}$, $\delta(z) = 1$ and $z^p = u^{p-1}z + (r+s)$. Therefore $A \cdot B = R[z;r+s]$. Thus we have the following theorem which is a generalization of [1, Cor. 17.14 or 3, Th.2.4].

Theorem 6 ([4, Th.3.1.2]). Let R^+ be the additive group of R. Then there exists a group isomorphism

$$\psi: R^+/\{t^p - u^{p-1}t \mid t \in R\} \longrightarrow Gal(H(u,p))$$

defined by $\psi(\bar{r}) = (R[x;r])$.

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ON DUAL HOPF GALOIS EXTENSION

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Introduction. It was shown by A.A. Albert [1] that a cyclic p-algebra contains a purely inseparable extension as well as a cyclic extension of its center, which are related by inner actions. On the other hand, the quaternion algebra $\mathbb{R}(i,j)$ contains two cyclic extensions $\mathbb{R}(i)$ and $\mathbb{R}(j)$ of $\mathbb{R}$ related by inner actions. Further, N. Jacobson [7] showed that a certain type of p-algebra contains a pair of purely inseparable extensions of its center which are also related by inner actions.

In this paper, by making use of Hopf algebras, we shall generalize the above results; under suitable conditions, we shall show that if a central simple algebra A over a field K contains an H-Hopf Galois extension of K (in the sense of [17], see also [3], [18]) as a maximal commutative subalgebra, then A contains an H*-Hopf Galois extension of K. This will be done in §1. In §2, we shall treat with some type of Hopf algebras and show that the former two classical results cited above are typical examples of our theorems. In 3, we shall treat with the case in which a pair of purely inseparable extensions appears. The details of §1 and §2 will be found in [19], and so we shall state the results in these sections only with outline of proof or without proof.

Throughout this paper, K will denote a field, and H a finite commutative co-commutative Hopf algebra over K; ε (resp. Δ, λ) will denote the augmentation (resp. diagonalization, antipode) of H. Unadorned $\otimes$ and Hom will mean $\otimes_K$ and Hom_K. We shall denote by $-*$ the functor $\mathrm{Hom}_K(-,K)$. For Hopf algebras and Hopf Galois extensions, we refer to [3], [12], [17] and [18].

1. Dual Hopf Galois extension. Let A be a central simple K-algebra which contains an H-Hopf Galois extension L of K as a maximal commutative subalgebra. Suppose that A is a projective left L-module. Then the action of H on L can be extended innerly to the action on A and A becomes a smash product algebra $L \#_\sigma H$, where σ is a 2-cocycle associated to an A-innerization of the action of H on L (cf. [16] Corollaries 3.7, 3.8). But, in general, A would not be an H-module. The following proposition is fundamental in our study.

Proposition 1. The following conditions are equivalent:

(i) A is an H-module algebra.

(ii) The associated 2-cocylce σ is K-valued, i.e., $\sigma(g \otimes h) \in K$ for any $g, h \in H$.

Proof. The assertion follows from the paragraph preceding Lemma 1.11 of F.W. Long [9].

Corollary 2. Suppose the equivalent conditions in Proposition 1. If a homomorphism v gives an A-inner action and makes A an H-module algebra, then the K-module $v(H)$ forms a K-subalgebra of A.

From now on, we always assume the equivalent conditions in Proposition 1.

We define an H*-action on $v(H)$ by

$$x \cdot v(h) = \sum_{(h)} x(h_{(1)}) v(h_{(2)}), \quad x \in H^*.$$

As to this H^*-action we have the following

Proposition 3. Through the natural isomorphism $v(H) \simeq H$, the H^*-action on $v(H)$ is given by the canonical left H^*-module structure of H and $v(H)$ becomes an H^*-module algebra.

Theorem 4. Under the equivalent conditions in Proposition 1, $v(H)$ is an H^*-Hopf Galois extension of K.

Proof. Let σ be an associated normal 2-cocycle. Since σ is K-valued, the cocycle conditions ensure that σ is a unit-valued Harrison 2-cocyle. The multiplication in $v(H)$ is given by the formula

$$v(g)v(h) = \sum_{(g),(h)} \sigma(g_{(1)} \otimes h_{(1)}) v(g_{(2)} \otimes h_{(2)}).$$

Thus, $v(H) = H(\sigma)$ in the sense of [18] §2, and we get Theorem 4 by [18] Theorem 2.3.

Next we consider an A-innerization of the H^*-Hopf Galois extension $v(H) = H(\sigma)$. Since L/K is an H-Hopf Galois extension, we may write $L = H^*(u)$, where $u \in H \otimes H$ is a normal 2-cocycle in unit-valued Harrison cohomology (cf. [18] §2). We define $V : H^* \to H^*(u) \subset A$ by

$$(V(f))(h) = f(\lambda(h)), \quad f \in H^*.$$

By the laborious computations, we can prove that V gives an A-innerization of H^*-action on $v(H) = H(\sigma)$. Thus we get

Theorem 5. Let A be a K-central simple algebra which contains an H-Hopf Galois extension $H^*(u)$ of K as a maximal commutative subalgebra. Suppose that A is left $H^*(u)$-projective and the associated 2-cocycle σ is K-valued. Then A contains an H^*-Hopf Galois extension $v(H) = H(\sigma)$ of K such that $H^*(u)$ and $v(H)$ are related as follows: There exist homomorphisms $v : H \to v(H) \subset A$ and $V : H^* \to H^*(u) \subset A$ such that v gives an A-inner action extending the H-action on $H^*(u)$ and V gives an A-inner action extending the H^*-action on $v(H)$.

2. Dual Hopf algebra. In this section, we investigate the dual structure of group rings as Hopf algebra.

Proposition 6. Let p be a prime number different from the characteristic of K, and G a cyclic group of order p^n. If K contains a primitive p^n-th root of 1, then $(KG)^* \simeq KG$ as Hopf algebra.

Proof. Noting that the character group G^* is isomorphic to G, we can easily see the assertion.

Remark. Theorem 5 and Proposition 6 explain the phenomenon that a pair of cyclic extensions appears.

Next we review a Hopf algebra H_n introduced by A. Hattori [5] and K. Kosaki [8]. Let K be a field of characteristic $p \neq 0$. Then H_n is defined as a K-algebra

$$K[X_0, X_1, \cdots, X_{n-1}]/(X_0^p - X_0, X_1^p - X_1, \cdots, X_{n-1}^p - X_{n-1})$$

whose Hopf algebra structure is defined by

$$\Delta(x_i) = S_i(x_0 \otimes 1, \cdots, x_i \otimes 1; 1 \otimes x_0, \cdots, 1 \otimes x_i),$$

$$\varepsilon(x_i) = 0 \quad \text{and} \quad \lambda(x_i) = -x_i$$

where x_i denotes the residue class of X_i, and S_i the polynomials which define the addition of Witt vectors.

Proposition 7. Let K be a field of characteristic $p \neq 0$, and $G = \{1, \theta, \cdots, \theta^{p^n-1}\}$ a cyclic group of order p^n. Then $(KG)^*$ is isomorphic to H_n as Hopf algebra.

Proof. Let $\{e_i\}$ be a K-basis of $(KG)^*$ defined by $e_i(\theta^j) = \delta_{i,j}$ (Kronecker's delta). We define a homomorphism $\phi : H_n \to (KG)^*$ by $\phi(x_j) = \sum_{i=1}^{p^n-1} a_{ji}e_i$, where the coefficients $a_{ji} \in \mathbf{Z}/p\mathbf{Z}$ are determined by

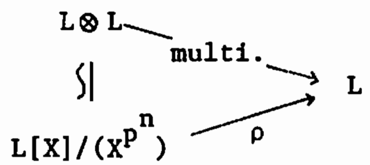
$$a_{j0} = 0 \quad \text{for all } j, \ (a_{01}, a_{11}, \cdots, a_{n-11}) = (1,0,\cdots,0),$$

$$(a_{0i}, a_{1i}, \cdots, a_{n-1i}) = \underbrace{(1,0,\cdots,0) + \cdots + (1,0,\cdots,0)}_{i \text{ terms}}$$

(in Witt vectors). Then, we can prove that ϕ is a Hopf algebra isomorphism.

Proposition 8. Let K be a field of characteristic $p \neq 0$, and L a commutative K-algebra. If L/K is an H_n-Hopf Galois extension then L is a purely inseparable K-algebra in the sense of M.E.Sweedler [13].

Proof. Since L/K is an H_n-Hopf Galois extension, $L \otimes L \simeq \mathrm{Hom}(H,L) \simeq H^* \otimes L \simeq KG \otimes L \simeq L[X]/(X^{p^n})$, where G is a cyclic group of order p^n. Using the above isomorphisms, we can prove that the following diagram is commutative:



where multi. is the multiplication, ρ is defined by $\rho(x) = 0$ (x the residue class of X). Thus the kernel of multi. is nilpotent.

Remark. Theorem 5, Proposition 7 and Proposition 8 explain the phenomenon that a pair of a cyclic extension and a purely inseparable extension appears.

3. Pair of purely inseparable extensions. In this section, we shall review the paper of N. Jacobson [7] from Hopf Galois theoretic view-point. Now, let K be a field of characteristic $p \neq 0$, and let $H(p^n)$ be a Hopf algebra defined as follows: $H(p^n)$ is a K-algebra freely generated by d with relation $d^{p^n} = 0$ and its Hopf algebra structure is given by $\Delta(d) = d \otimes 1 + 1 \otimes d$, $\varepsilon(d) = 0$ and $\lambda(d) = -d$. A. Nakajima and the author [10] showed that if $L = K(x_1) \otimes K(x_2) \otimes \cdots \otimes K(x_n)$, $x_i^p \in K$, is a purely inseparable extension of K of exponent one, then L/K is an $H(p^n)$-Hopf Galois extension and the partial converse holds, namely we have

Proposition 9. Let L be a commutative K-algebra. Then L/K is an $H(p^n)$-Hopf Galois extension if and only if L is isomorphic to $K(x_1, \cdots, x_n)$, $x_i^p \in K$, as K-algebra.

Proof. See [10] Proposition 2, Corollary 4 and Theorem 7.

Next, let $H[p^n]$ be a Hopf algebra defined as follows: $H[p^n]$ is a p^n-dimensional K-vector space with basis $\{h_i\}_{i=0}^{p^n-1}$, and its Hopf algebra structure is given by $h_0 = 1$, $h_i h_j = \binom{i+j}{i} h_{i+j}$, $\Delta(h_r) = \sum_{i=0}^{r} h_i \otimes h_{r-i}$, $\varepsilon(h_i) = \delta_{i,0}$, $\lambda(h_i) = (-1)^i h_i$. The Hopf algebra $H[p^n]$ is related to iterative higher derivations (cf. [14]).

Proposition 10. The Hopf algebras $(H[p^n])^*$ and $H(p^n)$ are isomorphic.

Proof. If $\{D_i\}_{i=0}^{p^n-1}$ is a dual basis of $\{h_i\}_{i=0}^{p^n-1}$, then we get easily $D_iD_j = D_{i+j}$ $(i+j < p^n)$, especially $(D_1)^{p^n} = 0$. We define a homomorphism $\phi : (H[p^n])^* \to H(p^n)$ by $\phi(D_1^i) = d^i$. Then, ϕ is an algebra isomorphism and indeed a Hopf algebra isomorphism.

Proposition 11. Let L be a commutative K-algebra. Then L/K is an $H[p^n]$-Hopf Galois extension if and only if L is isomorphic to $K(x)$, $x^{p^n} \in K$, $x^{p^{n-1}} \notin K$, as K-algebra.

Proof. If L is isomorphic to $K(x)$, $x^{p^n} \in K$, then we can define the action of D_i to x^j by $D_i(x^j) = \binom{j}{i}x^{j-i}$ $(j \geq i)$ and $D_i(x^j) = 0$ $(j < i)$. Then, as is easily seen, $k(x)$, and hence L, is an $H[p^n]$-module algebra and L/K is an $H[p^n]$-Hopf Galois extension. Conversely, if L/K is an $H[p^n]$-Hopf Galois extension, then M. Weisfeld [14] Theorem 2 asserts that L is of the desired form (the assumption that L is a field is unnecessary and the assumption that L/K is an $H[p^n]$-Hopf Galois extension works well).

Theorem 12 (N. Jacobson [7]). Let A be a p-algebra of degree p^n (order p^{2n}) over K. Suppose that A contains an $H(p^n)$-Hopf Galois extension $L = K(x_1,\cdots,x_n)$, $x_i^p \in K$, of K and that A is left L-projective. Then, A contains an $H[p^n]$-Hopf Galois extension $L' = K(x)$, $x^{p^n} \in K$, $x^{p^{n-1}} \notin K$, of K.

Proof. It is enough to check the condition in Proposition 1. Since the action of $H(p^n)$, especially of d, can be extended A-innerly to that on A, we take such an element a of A that gives the A-inner action of d, i.e., $d(\ell) = a\ell - \ell a$, $\ell \in L$. We define $v : H(p^n) \to A$ by $v(d^i) = a^i$. Then, v gives an A-inner action of $H(p^n)$ and A becomes an $H(p^n)$-module. Now, by Theorem 4, Propositions 10 and 11, we get the assertion.

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ON RADICALS OF SKEW POLYNOMIAL RINGS

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Let K be a ring, not necessarily with 1. The following result is due to Amitsur [1]:

The Jacobson radical $J(K[X])$ of the polynomial ring $K[X]$ is the polynomial ring $N[X]$ where $N = J(K[X]) \cap K$ is a nil ideal of K.

It is natural to study on radicals (the Jacobson radical, the lower nil radical, etc.) of (i) the skew polynomial ring of automorphism type, (ii) that of derivation type and (iii) the skew group ring.

The case (i) (and (iii)) is considered by Bedi and Ram in [2]. The case (ii) is considered by Jordan when K is a right Noetherian ring [6]. Recently, we determined the structure of the Jacobson radical and the lower nil radical of the case (ii) [4].

This note is an abstract on some results of [2] and [4]. In §1, we shall consider the automorphism type and the details can be seen in [2]. §2 is devoted to the derivation type, and the details can be seen in [4].

1. Automorphism type. Throughout this section, we assume that σ is an automorphism of K and we put

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$R = K[X;\sigma]$ is a skew polynomial ring of automorphism type (i.e., the multiplication is given by $bX = X\sigma(b)$ for all $b \in K$).

An ideal will mean a two-sided ideal. An ideal I of K is called a σ-ideal if $\sigma(I) = I$. If I is a σ-ideal of K then $\{ \sum_i X^i a_i \ ; \ a_i \in I \}$ is an ideal of R which we will denote by $I[X;\sigma]$. The Jacobson radical of R will be denoted by $J(R)$.

Let $I = \{a \in K \ ; \ Xa \in J(R)\}$. Then I is a σ-ideal of K and $XI[X;\sigma] = \{\sum_{i\geq 1} X^i a_i \ ; \ a_i \in I\} \subseteq J(R)$. Further, it can be seen that $I \cap J(K) \subseteq J(R)$. Thus $(I \cap J(K)) + XI[X;\sigma] \subseteq J(R)$.

We now have the following key result.

Lemma 1.1. If $J(R) \neq 0$ then $I \neq 0$.

Using this, the following theorem can be obtained by the same way as in [1].

Theorem 1.2. $J(R) = (I \cap J(K)) + XI[X;\sigma]$ where $I = \{a \in K \ ; \ Xa \in J(R)\}$.

It is now natural to ask about the following questions:

(i) Is $J(R) = I[X;\sigma]$?

It is true if and only if $I \subseteq J(K)$. Equivalently, this is so if and only if $Xa \in J(R)$ implies $a \in J(R)$.

(ii) Is $J(R) \cap K$ a nil ideal of K ?

(iii) Does $J(K) = 0$ imply $J(R) = 0$?

To answer these questions, let $K = \sum_{i\in \mathbf{Z}} \oplus S_i$, $S_i = S$

an arbitrary ring and $\sigma : K \to K$ such that $\sigma(\sum_{i\in\mathbf{Z}} a_i) = \sum_{i\in\mathbf{Z}} b_i$ where $b_i = a_{i-1}$. Then, it is easy to see that $I = K$. Thus we have a negative answer for above questions, choosing a suitable S.

Related with question (iii), we have the following

Theorem 1.3. Let $K = A_1 \oplus \cdots \oplus A_t$ where $A_1, \cdots, A_t$ are simple rings with 1 and let σ be any automorphism of K. Then $J(R) = 0$.

For an automorphism σ of finite order we can give an affirmative answer for the above questions. More generally, an automorphism σ of K is said to be of locally finite order if for every $a \in K$ there exists an integer $n(a) \geq 1$ such that $\sigma^{n(a)}(a) = a$. In this case we have

Corollary 1.4. If σ is of locally finite order then I is a nil ideal and $J(R) = I[X;\sigma]$.

Corollary 1.5. If σ is of locally finite order and $J(K)$ is a locally nilpotent ideal then $J(R) = J(K)[X;\sigma]$ and $J(R)$ is a locally nilpotent ideal.

2. Derivation type. Throughout this section, we assume that D is a derivation of K and we put $R = K[X;D]$ is a skew polynomial ring of derivation type (i.e., the multiplication is given by $bX = Xb + D(b)$ for all $b \in K$).

As in §1, an ideal will mean a two-sided ideal. An ideal I of K is called a D-ideal if $D(I) \subseteq I$. If I

is a D-ideal of K then $\{\sum_i X^i a_i \; ; \; a_i \in I\}$ is an ideal of R which we will denote by $I[X;D]$. It is clear that if M is an ideal of R then $M \cap K$ is a D-ideal of K.

Following [5, p.194], for every ordinal α, we define an ideal $N_K(\alpha)$ and a D-ideal $\mathcal{D}(\alpha)$ of K as follows:

(i) $N_K(0) = 0$ and $\mathcal{D}(0) = 0$.

(ii) Suppose $N_K(\alpha)$ (resp. $\mathcal{D}(\alpha)$) has been defined for every ordinal α less than the ordinal β. Then $N_K(\beta)$ (resp. $\mathcal{D}(\beta)$) is defined as follows:

Case I: $\beta = \gamma + 1$ is not a limit ordinal. $N_K(\beta)$ (resp. $\mathcal{D}(\beta)$) is the sum (= the union) of all ideals I of K (resp. D-ideals E of K) such that $I^s \subseteq N_K(\gamma)$ (resp. $E^s \subseteq \mathcal{D}(\gamma)$) for some integer s.

Case II: β is a limit ordinal. Then

$$N_K(\beta) = \sum_{\gamma<\beta} N_K(\gamma) \quad (\text{resp. } \mathcal{D}(\beta) = \sum_{\gamma<\beta} \mathcal{D}(\gamma)).$$

There exists an ordinal τ (resp. ρ) such that $N_K(\tau) = N_K(\tau+1)$ (resp. $\mathcal{D}(\rho) = \mathcal{D}(\rho+1)$). By $\mathcal{D}(K)$ we denote $\mathcal{D}(\rho) = \mathcal{D}(\rho+1)$. As it is known, $N_K(\tau) = N_K(\tau+1)$ is called the lower nil radical of K. Hereafter, by $L(A)$ we denote the lower nil radical of a ring A.

The following theorem can be proved using transfinite induction.

Theorem 2.1. For any ordinal α, $N_R(\alpha)$ $K = \mathcal{D}(\alpha)$ and $N_R(\alpha) = \mathcal{D}(\alpha)[X;D]$.

As a direct consequence of this, we have the following

Corollary 2.2. (i) For any ordinal α, $N_R(\alpha) = 0$ if and only if $\mathcal{D}(\alpha) = 0$.

(ii) $\mathcal{D}(K) = L(R) \cap K$ and $L(R) = \mathcal{D}(K)[X;D]$.

Next we shall consider the Jacobson radical. We now put $S = J(R) \cap K$ and we have the following key lemma, as in §1.

Lemma 2.3. If $J(R) \neq 0$ then $S \neq 0$.

Using this, we can also obtain the following theorem (cf. [1]).

Theorem 2.4. $J(R) = S[X;D]$ where $S = J(R) \cap K$.

It is a natural question to ask that whether $S = J(R) \cap K$ is nil. It seems this still remain open in general. Concerning with this problem we can prove that S is nil if K is commutative.

For an ideal I of K we put $M(I) = \{a \in I \,;\, D^i(a) \in I$ for $i \geq 1\}$. Then $M(I)$ is the maximal D-subideal of I. Then we have

Theorem 2.5. If K is a commutative ring then $S = M(L(K))$ and is the maximum nil D-ideal of K.

It is clear that $M(I) = I$ if and only if I is a D-ideal. Hence we can obtain

Corollary 2.6. If K is a commutative ring then $J(R) = \sum_{i=0}^{\infty} \oplus X^i L(K)$ if and only if $D(L(K)) \subseteq L(K)$. Moreover, if this is the case, $J(R)$ coincides with the set of all nilpotent elements of R.

The condition $D(L(K)) \subseteq L(K)$ is satisfied if the abelian group (K,+) is torsion free (see [3], Proposition 1).

We can easily see that $L(K) \supseteq M(L(K)) \supseteq \mathcal{D}(K)$. It is natural to ask whether $M(L(K)) = \mathcal{D}(K)$. It is true under some finiteness condition of D on L(K).

For every $a \in K$, we denote by $T_m(a)$ the ideal of K generated by $\{a, D(a), \cdots, D^{m-1}(a)\}$. Let V be a subset of K. We say that D satisfies the condition (F) on V if the following condition holds:

(F) For every $a \in V$, there exists a positive integer $m(a)$ such that $D^{m(a)}(a) \in T_{m(a)}(a)$.

Then we can obtain

Theorem 2.7. If (F) is satisfied on $N_k(\alpha)$ then $M(N_K(\alpha)) = \mathcal{D}(\alpha)$.

Hence we obtain that if (F) is satisfied on L(K), then $M(L(K)) = \mathcal{D}(K)$ and $L(R) = M(L(K))[X;D]$. Moreover, if K is commutative then $J(R) = L(R)$.

Assuming (F) is satisfied on $N_K(\alpha)$, we can also prove that $N_R(\alpha) = \sum_{i=0}^{\infty} \oplus X^i N_K(\alpha)$ if and only if $N_K(\alpha)$ is a D-ideal. Hence, if (K,+) is torsion free, $N_R(\gamma) = N_K(\gamma)[X;D]$ for all $\gamma \leq \alpha$.

As an application of the above results we can prove

Theorem 2.8. Let K be a commutative ring with 1, and D a derivation of K such that $D(L(K)) \subseteq L(K)$. Then $\sum_{i=0}^{n} X^i a_i \in R$ is invertible if and only if a_0 is invertible in K and a_i is nilpotent for $i \geq 1$.

Hence, if K is a commutative ring with 1 such that (K,+) is torsion free, then the unit group U(R) of R can be obtained by U(R) = U(K) + L(K)[X;D] where U(K) is the unit group of K.

Remark 2.9. After submitting our work [4], the author obtained an affirmative answer for the following question proposed by Prof. Kaplansky: Is $\mathcal{D}$(K) the intersection of all D-prime ideals of K? Here, a D-ideal I of K is called a D-prime ideal if AB $\subseteq$ I for any two D-ideals A and B of K implies that either A $\subseteq$ I or B $\subseteq$ I (cf. [6]).

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