PROCEEDINGS OF THE 16TH SYMPOSIUM ON RING THEORY

HELD AT THE TOKYO-TO KYÔIKU KAIKAN, TOKYO

SEPTEMBER 8—10, 1983

EDITED BY

TOYONORI KATO

University of Tsukuba

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OKAYAMA, JAPAN

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PREFACE

This volume consists of the articles presented at the 16th Symposium on Ring Theory held in Tokyo, Japan, on September 8 - 10, 1983.

The annual Symposium on Ring Theory, founded in 1968, has made a great contribution to the development of the ring theory in Japan with the main cooperation of Professors Shizuo Endo, Manabu Harada, Takasi Nagahara, Hiroyuki Tachikawa, Hisao Tominaga and Yukio Tsushima.

The Symposium and these Proceedings were financially supported by the Scientific Research Grant of the Educational Ministry of Japan through the arrangements by Professor Toshiro Tsuzuku at Hokkaido University.

Finally, I would like to take this oppotunity to thank Professors Kanzo Masaike, Hiroyuki Tachikawa, Hisao Tominaga, and the graduate students specialized in the ring theory at the University of Tsukuba for their close cooperation.

November 1983

Toyonori Kato

ON DERIVATIONS OF PRIME RINGS

Yasuyuki HIRANO and Hisao TOMINAGA

1. About a quarter of a century ago, Posner [14] proved the following longstanding theorems.

Theorem 1. Let R be a prime ring of characteristic not 2, and S, T derivations of R such that the iterate ST is also a derivation. Then one at least of S, T is zero.

Theorem 2. Let R be a prime ring with center C, and D a derivation of R such that $[x,x^D] = xx^D - x^D x \in C$ for all $x \in R$. Then, if D is not trivial, R is commutative.

Since 1959, many authors (see References) have been concerning these theorems. In this lecture, we shall exhibit several recent results they obtained, together with comments or proofs.

We begin with subjects related to Theorem 1. If R is a prime ring of characteristic 2, Theorem 1 fails. In fact, the square of a derivation of such R is always a derivation.

Now, let R be a prime ring of characteristic not 2. If D_1, D_2 and D_3 are derivations of R such that $D_1D_2D_3$ is also a derivation, can we conclude that at least one of the D_i's is trivial? The following example shows that the answer is no!

Example. Let us consider the ring $R = (\mathbb{C})_2$ and the element $a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ in R. Then we can easily see that

$Inn(a)^3 = Inn(a)$, where $Inn(a)$ denotes the inner derivation induced by a.

An additive subgroup U of a ring R is called a Lie ideal if $[r,u] \in U$ for all $r \in R$ and $u \in U$.

Let R be a prime ring of characteristic not 2, and S, T derivations of R. If ST is trivial, then, as a particular case of Theorem 1, either $S = 0$ or $T = 0$. The following theorem shows that if $ST = 0$ on a non-central Lie ideal of R, then the same is true.

Theorem 3 (Bergen, Herstein and Kerr [1, Theorem 4]). Let R be a prime ring with center C, char $R \neq 2$, and $U \not\subset C$ be a Lie ideal of R. Suppose that S and T are derivations of R such that $U^{ST} = 0$. Then either $S = 0$ or $T = 0$.

To prove Theorem 3, they required the following results.

Proposition 1 (Herstein [5, Theorem 1]). Let R be any ring, T a derivation of R such that $T^3 \neq 0$. Then the subring generated by all r^T, $r \in R$, contains a nonzero ideal of R.

Proposition 2 ([6, Theorem (1)]). Let R be any ring, and $T \neq 0$ a derivation of R. Suppose that $a \in R$ is such that $[a,x^T] = 0$ for all $x \in R$. If R is not of characteristic 2, a must be in the center of R.

Proposition 3 (Bergen, Herstein and Kerr [1, Theorems 1 and 2]). Let R be a prime ring with center C, char $R \neq 2$, and let U be a Lie ideal of R. Let $T \neq 0$ be a derivation of R.

(1) If $U^{T^2} = 0$ then $U \subset C$.

(2) If $U \not\subset C$ then $V_R(U^T) = C$.

Martindale and Miers [11] proved the following theorem concerning the iterates of derivations. Note that the second assertion follows also from Proposition 2.

Theorem 4 (Martindale and Miers [11]). Let R be a prime ring and let D and T be derivations of R such that $[x^{D^n}, y^{T^n}] = 0$ for all $x, y \in R$. Then either R is commutative or $D^{3n-1} = 0$ or $T^{3n-1} = 0$. Furthermore, if $n = 1$ and char $R \neq 2$, then either R is commutative, or $D = 0$, or $T = 0$.

Next, we state several results which generalize Theorem 2. Let R be a ring with center C, S a subset of R. An (additive group) endomorphism T of R is said to be centralizing (resp. skew-centralizing) on S if $[s^T, s] \in C$ (resp. $(s^T, s) = s^T s + s s^T \in C$) for every $s \in S$. Mere generally, T is defined to be semicentralizing on S if $[s^T, s] \in C$ or $(s^T, s) \in C$ for every $s \in S$. In case $S = R$, we say simply T is centralizing (resp. skew-centralizng) or semicentralizing according as so is T on R.

Theorem 5 (Hirano, Kaya and Tominaga [9], see also Mayne [13]). Let U be a nonzero ideal of a prime ring R, T a nontrivial derivation of R. If T is centralizing (resp. skew-centralizing) on U, then R is commutative.

Theorem 6. Let T be a semicentralizing derivation of a prime ring R. If T is not trivial, then R is commutative.

For the proof of Theorem 6, the following result is needed.

Proposition 4 (Kharchenko [10, Corollary 5]). Suppose the prime ring R satisfies a nontrivial identity involving derivations. Then the central closure RZ is a primitive ring with nonzero socle whose skew field is finite-dimensional over Z.

Finally, we exhibit a theorem of Herstein.

Theorem 7 (Herstein [8]). Let R be a prime ring with center C and suppose that $T \neq 0$ is a derivation of R such that $(x^T)^n \in C$ for all $x \in R$. Then either R is commutative or is an order in a 4-dimensional simple algebra.

For the proof of Theorem 7, the following results are required.

Proposition 5 (Giambruno and Herstein [3, Theorem 2]). Let R be a prime ring, I a nonzero ideal of R, and T a derivation of R. If $(x^T)^n = 0$ for all $x \in I$, where n is a fixed positive integer, then $T = 0$.

Proposition 6 (Herstein [7, Theorem]). Let R be a prime ring with center C, and suppose that $a \in R \backslash C$ is such that $[a,x]^n \in C$ for all $x \in R$ with some fixed positive integer n. Then R is an order in a 4-dimensional simple algebra.

Remark. For an automorphism (instead of a derivation), corresponding to Theorems 5 and 6, the following has been proved (Hirano, Kaya and Tominaga [9]): Let U be a nonzero ideal of a prime ring R with center C, and T a nontrivial automorphism of R. If T is semicentralizing on U, then R is commutative.

2. In this section, we shall give the proofs of Theorems 5 and 6. In preparation for proving Theorem 5, we state first three lemmas.

Lemma 1. Let R be a prime ring, and K a right ideal of R.

(1) If K is nonzero and commutative, then R is commutative.

(2) Let T be a derivation of R. If K is nonzero and T is trivial on K, then T is itself trivial.

(3) Let T be a nontrivial derivation of R. If I is a nonzero ideal of R and if $xI^T = 0$ (resp. $I^T x = 0$) for some $x \in R$, then $x = 0$.

(4) If there exists a positive integer n such that $x^n = 0$ for all $x \in K$, then $K = 0$.

Proof. (1) is [12, Lemma 4] and (2) and (3) are respectively [12, Lemma 2] and [14, Lemma 1] with routine proofs. (4) is immediate by [4, Lemma 1.1].

Lemma 2. Let $T: x \to x'$ be an endomorphism of R, and U an additive subgroup of R. Let $[U] = \{u \in U \mid [u',u] \in C\}$ and $(U) = \{u \in U \mid (u',u) \in C\}$.

(1) Let $u, v \in [U]$ (resp. (U)). Then $u+v \in [U]$ (resp. (U)) if and only if $u-v \in [U]$ (resp. (U)).

(2) If $v \in (U)$, then $[v',v^2] = [v,v'^2] = 0$.

Proof. (1) follows from $[u'-v',u-v] = -[u'+v',u+v] + 2([u',u]+[v',v])$ (resp. $(u'-v',u-v) = -(u'+v',u+v) + 2((u',u)+(v',v))$, and (2) is obvious by $[x,y^2] = [(x,y),y]$.

Lemma 3. Let $T: x \to x'$ be a derivation of a prime ring R with char $R \neq 2$ which is semicentralizing on a nonzero ideal U, and let $[U]$, (U) be as in Lemma 2.

(1) If $v \in U$ $[U]$, then $(v^2)' = 0$ and $v^2v' = v'v^2 = 0$. In particular, if $U = R$ and $v \notin [R]$, then $v'^4 = 0$.

(2) If $C \cap U = 0$ and $v \in U\backslash[U]$, then $v'^3 = 0$ and $v^2 \neq 0$.

(3) If $C \cap U$ is nonzero, then T is centralizing on U.

Proof. (1) Since $(v^2)' = (v',v) \in C$ and $[v',v^2] = 0$ by Lemma 2 (2), we have $[(v^2+v)',v^2+v] = [(v^2-v)',v^2-v]$ $= [v',v] \notin C$, which means that $v^2+v \notin [U]$ and $v^2-v \notin [U]$. Then, by Lemma 2 (1), $(v^2+v)-(v^2-v) = 2v \notin [U]$ shows that $2v^2 = (v^2+v)+(v^2-v) \in (U)$, and so $v^2 \in (U)$. Hence, $2(v^2)'v^2 = ((v^2)',v^2) \in C$, i.e., $(v^2)'v^2 \in C$. Furthermore, by Lemma 2 (2), $0 = (v^2)'[(v^2+v)',(V^2+v)^2] = 2(v^2)'[v',v^3] =$ $2(v^2)'v^2[v',v]$, i.e., $(v^2)'v^2[v',v] = 0$. Since $(v^2)'v^2 \in C$ and R is prime, $[v',v] \neq 0$ implies $(v^2)'v^2 = 0$. Noting here that $(v^2)' \in C$, we get

(i) $\quad (v^2)' = (v',v) = 0$ and $(v'',v)+(v',v') = (v',v')' = 0$.

Since $v^2+v \notin [U]$, we can apply (i) to see that $2v'v^2 =$ $(v',v^2+v) = ((v^2+v)',v^2+v) = 0$, and so

(ii) $\quad v'v^2 = v^2v' = 0$ and $v^2v'' = (v^2v')' = 0$.

Now, we assume that $U = R$. If $v' \notin [R]$, then $(v')^2v''$ $= 0$ by (ii). Since $[y,(y')^2] = 0$ (Lemma 2 (2)), by (i) we have $2(v')^4 = (v')^2((v'',v)+(v',v')) = 0$, i.e., $(v')^4 = 0$. Thus, we assume henceforth that $v' \in [R]$. Then, by Lemma 2 (1), either $v+v' \notin [R]$ or $v-v' \notin [R]$. We assume first that $v+v' \notin [R]$. Then, by (i) we have

(iii) $\quad (v',v'') = (v+v',(v+v')') = 0$.

Since $[v',v] \in C$, (iii) proves that $v'v'' \in C$. Hence, by (i) and (ii), we get

$v'(v''v'+(v')^2) = (v')^2v''+(v')^3 = (v^2+(v')^2+(v',v))(v'+v'')$

$= (v+v')^2(v+v')' = 0.$

Obviously, if $v''v' = 0$ then $(v')^3 = 0$. On the other hand, if $v''v' \neq 0$ then $v''v'(v'v'' + (v')^2) = 0$ gives $v'v'' + (v')^2 = 0$, whence it follows that $v'' + v' = 0$. This together with (iii) implies $(v')^2 = 0$. Also, in case $v - v' \notin [R]$, we can show that $(v')^3 = 0$.

(2) Observe that $vv' = -v'v$ and $uu' = \pm u'u$ for every $u \in U$. We prove first that $v^2 \neq 0$. In fact, if $v^2 = 0$ then for any $x \in R$ we have

$vxv'v + xvxv'v = \{(v+xv)(v+xv)' \pm (v+xv)'(v+xv)\}v = 0.$

Replace x by $-x$ in the above to get $-vxv'v + xvxv'v = 0$. Hence $vRv'v = 0$, and therefore $v'v = 0$. But this contradicts $v \in [U]$.

Next, we claim that $vv'^2 = 0$. Noting that $v^2v' = 0$ by (1), for any $x \in R$ we have

$$-v^2xvv'^2 - vxv^2xvv'^2$$
$$= \{(v+vxv)(v+vxv)' \pm (v+vxv)'(v+vxv)\}vv' = 0,$$

and similarly $v^2xvv'^2 - vxv^2xvv'^2 = 0$. Hence $v^2Rvv'^2 = 0$, and therefore $vv'^2 = 0$ by $v^2 \neq 0$.

Now, for any $x \in R$ we have

$vxv'^3 + xvxv'^3 = \{(v+xv)(v+xv)' \pm (v+xv)'(v+xv)\}v'^2 = 0,$

and similarly $-vxv'^3 + xvxv'^3 = 0$. Hence $vRv'^3 = 0$, and therefore $v'^3 = 0$.

(3) Suppose U contains an element v not contained in $[U]$. Choose an arbitrary nonzero $c \in C \cap U$. Because $c' \in C$, we have $[v'+c', v+c] = [v',v] \notin C$, and so $v+c \notin [U]$. Then by (1),

$$0 = [\{(v+c)^2\}', v] = [2cv' + 2c'v + (c^2)', v] = 2c[v',v],$$

i.e., $[v',v] = 0$. This contradiction proves that $[U] = U$.

Corollary 1. Let $T: x \to x'$ be a derivation of a prime ring R, and U a nonzero ideal of R.

(1) If T is skew-centralizing on U, then it is centralizing on U.

(2) If T is semicentralizing on U and U^T is a left (resp. right) ideal of R, then T is centralizing on U.

Proof. We may assume that T is nontrivial and char R $\neq 2$.

(1) According to Lemma 3 (3), it suffices to show that $C \cap U$ is nonzero. Suppose, to the contrary, that $C \cap U = 0$. Then, for any $u \in U$ and $x \in R$,

$$(u^2x + uxu)' = \{(u+ux)^2 - u^2 - (ux)^2\}' = 0$$

and

$$(xu^2 + uxu)' = (u+xu)^2 - u^2 - (xu)^2 \ ' = 0.$$

From those above, we readily obtain $[x,u^2]' = 0$. This means that $DT = 0$, where $D = \mathrm{Inn}(u^2)$. Then, Theorem 1 shows that $D = 0$, which tells us that $u^2 = 0$ for all $u \in U$. But, this is impossible by Lemma 1 (4).

(2) Suppose, to the contrary, that U contains an element v not contained in [U]. In view of Lemma 3 (3), it suffices to consider the case that $C \cap U = 0$. Let u be an arbitrary element of U. If $uv^2 \in [U]$ then it is easy to see that either $v + uv^2 \notin [U]$ or $v - uv^2 \notin [U]$ (Lemma 2 (1)). Hence, by Lemma 3,

$$\begin{aligned}(u'v^2)^4 &= (u'v^2)\{(u'v^2)^3 \pm v'(u'v^2)^2 + v'^2(u'v^2)\}\\ &= u'v^2(v \pm uv^2)'^3 = 0\end{aligned}$$

for all $u \in U$. Now, choose $r \in R$ such that $r' \neq 0$. Then, $r'u = (ru)' - ru' \in U'$ for all $u \in U$, i.e., $r'U \subset U'$, and hence U' contains a nonzero ideal Rr'U. Since $Rr'Uv^2$ is a nil left ideal of bounded index, we get $v^2 = 0$ by

Lemma 1 (4). But, this is impossible by Lemma 3 (2).

We are now ready to complete the proof of Theorem 5.

Proof of Theorem 5. In view of Corollary 1 (1), T is centralizing on U. We consider the ring $R_1 = \{\begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \mid x, y \in R\}$ with center $C_1 = \{\begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \mid x, y \in C\}$, where R is regarded as a subring of R_1 in an obvious way (see [15]). As is easily seen, T gives rise to a ring homomorphism $x \to x^* = \begin{pmatrix} x & x' \\ 0 & x \end{pmatrix}$ of R into R_1 and $[u^*,u] \in C_1$ for all $u \in U$. First, we claim that $[u',u] = 0$, or equivalently $[u^*,u] = 0$, for all $u \in U$. If char $R = 2$, then

$$0 = [[u + uu', (u + uu')'], u] = [[uu', u'] + [u, (uu')'], u]$$
$$= [u', u]^2 + [u[u, u''], u].$$

Since $[u,u''] = [u,u']' \in C$, the last shows that $[u',u]^2 = 0$, and hence $[u',u] = 0$. On the other hand, if char $R \neq 2$ then

$$4\begin{pmatrix} 0 & u^2[u',u] \\ 0 & 0 \end{pmatrix} = 2(u^*,u)[u^*,u] = [(u^2)^*, u^2] \in C_1,$$

i.e., $u^2[u',u] \in C$. Hence, $0 = [u', u^2[u',u]] = 2[u',u]^2 u$, and therefore $[u',u] = 0$.

Now, linearizing $[u^*,u] = 0$ gives $[u,v^*] = [u^*,v]$ for all $u, v \in U$, and then

$$(u - u^*)[u,v^*] = u[u,v^*] - [u,u^*v^*] = u[u^*,v] - [u^*,uv] = 0.$$

Hence, noting that $x^*[u,v^*] = [u,(xv)^*] - [u,x^*]v^*$ $(u, v \in U, x \in R)$, we get $(u^* - u)x^*[u,v^*] = 0$, which becomes $u'x[u,v] = 0$, i.e., $u'R[u,v] = 0$. Thus, we get $U = V_U(U) \cup K$, where $K = \{u \in U \mid u' = 0\}$. Since $U \neq K$ by Lemma 1 (2), U coincides with its center, and therefore R is commutative by Lemma 1 (1).

Next, in preparation for proving Theorem 6, we state two more lemmas.

Lemma 4 ([2, Lemma 1]). Let R be a prime ring with an idempotent $f \neq 0, 1$. If T is a derivation of R such that $(f + fx - fxf)^T = 0$ for all $x \in R$, then $T = 0$.

Lemma 5. Let R be a prime ring and let Q denote the Martindale quotient ring of R. Let p, q, r be elements of Q. If there exists a nonzero ideal U of R such that $puqur = 0$ for all $u \in U$, then one, at least, of p, q, r is zero.

Proof. If x, y are elements of Q such that $xUy = 0$, then x or y is zero. Using this fact, we can prove the lemma in the same way as in the proof of [14, Lemma 2].

Proof of Theorem 6. In view of Theorem 2, it suffices to show that T is centralizing. We may assume that char $R \neq 2$. According to Proposition 4, the central closure S of R is a primitive ring with nonzero socle. In view of [2, Lemma 4], we can extend T in a unique way to a derivation of S, which will be also denoted as T. For the convenience of notation, let us write $x^T = x'$ for every $x \in S$. Now, let e be an arbitrary idempotent in S. Then there exists a nonzero ideal U of R such that $fU \subset R$ and $Uf \subset R$. For any $u \in U$, we have $eu(eu)' = \pm(eu)'eu$, and therefore $e(eu)'eu = (eu)'eu$. Hence we see that $(ee' - e')ueu = 0$ for all $u \in U$, and so $ee' = e'$ by Lemma 5. Similarly, we can show that $e'e = e'$. We see therefore that $e' = (e^2)' = ee' + e'e = 2e'$, that is, $e' = 0$. Noting here that $f + fx - fxf$ is an idempotent for every idempotent $f \in S$ and every $x \in S$ and that T is nonzero, we see that S has no nontrivial idempotents (Lemma 4). Hence S has to be a division ring, and so R is a domain. Now, by Lemma 3 (1), we conclude that T is centralizing.

Addendum. Recently, in their paper "On the centralizer of ideals and nil derivations" [J. Algebra 83 (1983), 520-530], Felzenszwalb and Lanski have proved that if D is a nil derivation on an ideal I of a ring R containing no nonzero nil right ideals then $I^D = 0$, and have led to a generalization of Proposition 5.

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H-SEPARABLE EXTENSIONS OF SIMPLE RINGS

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Introduction. Throughout this report A will be a ring with the identity 1, C the center of A, B a subring of A containing 1 and $D = V_A(B)$, the centralizer of B in A. This report is a continuation of the author's previous paper [9]. When the author wrote it, he was very careless. Because, he did not know that G. Azumaya had proved in [3] that if A is a simple ring and S is a simple C-subalgebra of A with $[S:C] < \infty$ then we have $S = V_A(V_A(S))$. Thus (2) Proposition 2 [9] is contained in his works. In §1 we will study on the structure of a simple ring A where D is simple with $[D:C] < \infty$ and $B = V_A(D)$ in relation with Azumaya's works and [9]. All results in §1 will be obtained immediately by applying the already known results on H-separable extensions. In §2 we will study on H-separable extensions of closed irreducible rings in the sense of [1], and prove that in the case where B is a right closed irreducible ring A is an H-separable extension of B and right B-finitely generated projective, if and only if following three conditions are satisfied;

(1) A is also a right closed irreducible ring

(2) $V_A(V_A(B)) = B$

(3) D is a simple C-algebra with $[D:C] < \infty$.

In this case A is a free Frobenius extension of B having a free basis consisting of [D:C] elements (Theorems 2 and 3).

1. To begin with we will introduce the following characterization of H-separable extensions which has been proved in [10] in Japanese;

Proposition 1. Let A be a ring with the center C and

B a subring of A. If $V_A(V_A(B)) = B$, the following conditions are equivalent;

(i) A is an H-separable extension of B and right B-finitely generated projective

(ii) A is a left $A\otimes_C D^\circ$-generator, and D is C-finitely generated projective, where $D = V_A(B)$.

Proof. Assume (i). Since A is right B-finitely generated projective, it is a generator of the category of left $Hom(A_B, A_B)$-modules. But $Hom(A_B, A_B) \cong A\otimes_C D^\circ$ (See Remark 3 [9]). Thus A is a left $A\otimes_C D^\circ$-generator. It is also well known that D is C-finitely generated projective. Thus we have (ii). Conversely, assume (ii). Set $\Lambda = A\otimes_C D^\circ$. Then since A is a left Λ-generator, A is a right $Hom({}_\Lambda A, {}_\Lambda A)$-finitely generated projective. But we have $Hom({}_\Lambda A, {}_\Lambda A) = Hom({}_A A_D, {}_A A_D) \cong V_A(D)$ $= B$. Hence A is right B-finitely generated projective. On the other hand, by Morita Theorem we have the following isomorphism

$$\eta : \Lambda \longrightarrow Hom(A_{Hom({}_\Lambda A, {}_\Lambda A)}, A_{Hom({}_\Lambda A, {}_\Lambda A)}) = Hom(A_B, A_B)$$

This isomorphism is exactly given by $\eta(a\otimes d^\circ)(x) = axd$, for $a, x \in A$ and $d \in D$. Then A is an H-separable extension of B (See Remark 3 [9]). Thus we have (i).

By a simple ring we mean simply a ring which has no non zero proper two sided ideal. Concerning with Azumaya's commutor theorem on simple ring we have

Theorem 1. Let A be a simple ring and S a simple C-subalgebra of A such that $[S:C] < \infty$, and set $B = V_A(S)$. Then we have

(1) A is a left $A\otimes_C S^\circ$-generator, and $V_A(B) = S$

(2) A is an H-separable and Frobenius extension of B

(3) Every automorphism which fixes all elements of B

is an inner automorphism

(4) B is also simple if and only if $B_B \lessdot\oplus A_B$.

Proof. (1) is due to Theorem 3.7 [3]. (2). A is an H-separable extension of B and right B-finitely generated projective by (1) and Proposition 1. Then by Theorem 4 [7] A is also a Frobenius extension of B. (3). For the same reason as Remark 5 [9] (2) yields (3). (4). Suppose that B is simple. Since $\mathrm{Hom}(A_B, B_B) \neq 0$, A is a right B-generator, and we have $B_B \lessdot\oplus A_B$. Conversely if $B_B \lessdot\oplus A_B$, A is left $A\otimes_C S°$-projective by (1) and Proposition 3.2 [4]. Then $B \cong \mathrm{Hom}({}_{A\otimes S°}A, {}_{A\otimes S°}A)$ is simple, because $A\otimes_C S°$ is simple.

A ring S is said to be right relatively separable extension of B in A in the case where $B \subset S \subset A$ and the map π_S of $A\otimes_B S$ to A such that $\pi_S(a\otimes s) = as$ for $a \in A$ and $s \in S$ splits as A-S-map. Now we consider the case where A and B satisfy the following condition;

(#) A, B and D are simple rings with $[D:C] < \infty$ and $B = V_A(D)$.

This condition is equivalent to the one that A is an H-separable extension of a simple ring B and right B-finitely generated projective (See Theorem 1 [9]). Denote by $\underline{T}$ the class of simple C-subalgebras of D and by $\underline{S}_r$ (resp. $\underline{S}_l$) the one of simple subrings which are right (resp. left) relatively separable extensions of B in A. Then there exist mutually inverse one to one correspondences between $\underline{T}$ and $\underline{S}_r$ (= $\underline{S}_l$) which are obtained by letting each subring belonging to $\underline{T}$ or $\underline{S}_r$ correspond to its centralizer in A (Theorem 2 [9]).

Proposition 2. Let A and B satisfy the condition (#), and define $\underline{S}_r$ as above. Then for any simple subring S of A which contains B, we have;

(1) If $S \in \underline{S}_r$, then A is A-$SV_A(S)$-irreducible, in particular, A is A-BD-irreducible

(2) If A is A-S-completely reducible, then $S \in \underline{S}_r$

(3) Suppose that A contains an irreducible A-B-submodule. Then, $S \in \underline{S}_r$ if and only if A is A-S-completely reducible.

Proof. (1). Let M be an A-BD-submodule of A. Then since B is simple, we see $M \cap B = 0$. Let $\{f_i, a_i\}$ be a dual basis for A_B, that is, $f_i \in Hom(A_B, B_B)$, $a_i \in A$ with $x = \Sigma a_i f_i(x)$ for any $x \in A$. But $Hom(A_B, A_B) \cong A \otimes_C D°$ (See Remark 3 [9]). Hence for each i, there exists $\Sigma y_{ij} \otimes d°_{ij}$ in $A \otimes_C D°$ such that $f_i(x) = \Sigma y_{ij} x d_{ij}$ for each $x \in A$. Then for each $a \in M$, we have $a = \Sigma a_i y_{ij} a d_{ij}$ with $\Sigma y_{ij} a d_{ij} \in M \cap B = 0$. Hence $M = 0$, which implies that A is A-BD-irreducible. For each S in $\underline{S}_r$, A and S satisfy the condition (#). Hence A is A-$SV_A(S)$-irreducible. (2). Since A is H-separable over B, we have A-A-isomorphisms $A \otimes_B A \cong Hom(D_C, A_C) \cong A \oplus A \oplus \cdots \oplus A$. Thus if A is A-S-completely reducible, $A \otimes_B A$ and $A \otimes_B S$ are so. Hence the map π_S of $A \otimes_B S$ to A splits as A-S-map. Thus $S \in \underline{S}_r$. (3). Let M be an irreducible A-B-submodule of A, and $S \in \underline{S}_r$. Since A is A-BD-irreducible by (1), we have $A = MD$. Thus we see that A is finitely generated completely reducible as A-B-module. Hence A satisfies the descending chain conditions for A-B-submodules and for A-S-submodules. Then A contains an irreducible A-S-submodule N. Then again by (1) we have $A = NV_A(S)$, which implies that A is completely reducible as A-S-module.

2. Following Nakayama-Azumaya [1] we say that a ring A is a right closed irreducible ring in the case where A is the endomorphism ring of a left vector space $\underline{m}$ over a divi-

sion ring K; A = Hom($_K\underline{m}$,$_K\underline{m}$). In this case $\underline{m}_A$ is faithful, irreducible and isomorphic to a right ideal of A, and we have K = Hom($\underline{m}_A$,$\underline{m}_A$). Thus A is a (right) ideal irreducible ring in the sense of [1]. In addition since A = Hom($_K\underline{m}$,$_K\underline{m}$) with $_K\underline{m}$ injective and $\underline{m}_A$ projective, we see that A is left A-injective. We see also that C, the center of A, is a field and equal to the center of K. In [1] it is shown that if A is a right closed irreducible ring and S is a simple C-subalgebra of A with $[S:C] < \infty$ then we have $S = V_A(V_A(S))$ and $V_A(S)$ is also a right closed irreducible ring. Furthermore Theorem 36.2 [2] shows that in this case A has a right $V_A(S)$-free bais consisting of [S:C] elements. In addition we have;

Theorem 2. Let A be a right closed irreducible ring and S a simple C-subalgebra of A with $[S:C] < \infty$. Then A is an H-separable extension of $V_A(S)$.

Proof. Set $B = V_A(S)$. By Theorem 10 [1] $A\otimes_C S^\circ$ is also a right closed irreducible ring whose smallest ideal is of the form $\underline{a}\otimes_C S^\circ$, where $\underline{a}$ is the smallest ideal of A. If the annihilator of A in $A\otimes_C S^\circ$ is not zero, it contains $\underline{a}\otimes_C S^\circ$. Then, $\underline{a} = \underline{a}AS = 0$, a contradiction. Hence $A\otimes_C S^\circ$ can be regarded as a subring of $\mathrm{Hom}(A_B,A_B)$. But $A\otimes_C S^\circ$ is left self injective, since it is right closed irreducible. Hence $A\otimes_C S^\circ$ is a left $A\otimes_C S^\circ$-direct summand of $\mathrm{Hom}(A_B,A_B)$. On the other hand A is right B-finitely generated free by Theorem 36.2 [2]. Hence A is a left $\mathrm{Hom}(A_B,A_B)$-generator. Then A is a left $A\otimes_C S^\circ$-generator. Then since $S = V_A(B)$ and $[S:C] < \infty$, A is an H-separable extension of B by Proposition 1.

In the case where A has a faithful minimal right ideal A has also a faithful minimal left ideal (See Theorem 2 [1]), and therefore is called strongly primitive ring in [2]. Now we will deal with H-separable extensions of strongly primitive rings, and show that the converse of Theorem 2 holds. The next remark about H-separable extensions is alredy well known. So, we will state it without proof.

Remark. If A is an H-separable extension of B such that ${}_B B \lessdot\!\oplus\, {}_B A$, then we have $B = V_A(V_A(B))$ and $\underline{a} = A(\underline{a} \cap B)$ for any ideal $\underline{a}$ of A (See Proposition 1.2 [6] and (2.2) [8]).

Proposition 3. Let B be a strongly primitive ring and A an H-separable extension of B such that ${}_B B \lessdot\!\oplus\, {}_B A$. Then A is a primitive ring.

Proof. Let J be the radical of A. Then $J = A(J \cap B)$. All elements of $J \cap B$ are quasi-regular in A, and their quasi-inverses belong to B, because $B = V_A(D)$. Then we have $J \cap B = 0$, because B has the zero radical. Next let $\underline{z}$ be the smallest ideal of B, whose existance is assured in Theorem 1 [1]. Then we have $\underline{A} = A(\underline{A} \cap B) \supset A\underline{z}A$ for any ideal $\underline{A}$ of A. Thus $A\underline{z}A$ is the smallest ideal of A. Then since $J = 0$, there exists a maximal right ideal I of A such that $A\underline{z}A \not\subset I$. If A/I is not faithful, its annihilator in A contains $A\underline{z}A$, and we have $A\underline{z}A \subset I$, a contradiction. Hence A/I is a faithful irreducible right A-module. Similarly, we see that there exists a maximal left ideal K of A such that A/K is faithful irreducible. Thus A is both left and right primitive.

Theorem 3. Let B be a right closed irreducible ring and A an H-separable extension of B. Then we have

(1) A is a right closed irreducible ring

(2) $B = V_A(V_A(B))$

(3) If furthermore A is right B-finitely generated projective, then D $(= V_A(B))$ is a simple C-algebra, and A is a free Frobenius extension of B having a left (or right) B-free

basis consisting of [D:C] elements.

Proof. At first we will show that A contains a faithful minimal left ideal. By Proposition 3 A has a faithful irreducible left module M. Since B is strongly primitive, B has a faithful minimal left ideal $\underline{l}$. But M is faithful. Hence there exists an x in M such that $0 \neq \underline{l}x \cong \underline{l}$. This isomorphism is extended to a left B-homomorphism of M to B, since B is left B-injective. Thus we have $0 \neq \mathrm{Hom}({}_BM, {}_BB) \subset \mathrm{Hom}({}_BM, {}_BA)$. Then since A is an H-separable extension of B, for an A-A-module $X = \mathrm{Hom}({}_CM, {}_CA)$ we have $\mathrm{Hom}({}_BM, {}_BA) = X^B = D \otimes_C X^A$ (See (1.3) [8]). Thus we have $X^A = \mathrm{Hom}({}_AM, {}_AA) \neq 0$. This means that A has a faithful minimal left ideal isomorphic to M. Then by Theorem 2 [1] A has also a faithful minimal right ideal $\underline{r}$. Set $L = \mathrm{Hom}(\underline{r}_A, \underline{r}_A)$ and $A^* = \mathrm{Hom}({}_L\underline{r}, {}_L\underline{r})$. Clearly A $\subset A^*$, and we can easily see that $\underline{r}$ is also a faithful minimal right ideal of A*. On the other hand A is an H-separable extension of a left self injective ring B. Hence A is also left self injective by (2.3) [8], and we have ${}_AA \lessdot\!\oplus\, {}_AA^*$. Let $A^* = A \oplus N$ as left A-module. Then $\underline{r} = \underline{r}A^* = \underline{r} \oplus \underline{r}N$, and we have $\underline{r}N = 0$. Hence $N = 0$, and $A = A^*$. Thus we have (1). Nextly since B is left B-injective, we have ${}_BB \lessdot\!\oplus\, {}_BA$. Then we have (2) by the remark above Proposition 3. Denote the center of B by Z. Z is a field, and we see $C \subset Z = V_D(D)$ by (2). Now assume that A is right B-finitely generated projective. Then we have $\mathrm{Hom}(A_B, A_B) = e(B)_n e$, where $(B)_n$ is the n x n-full matrix ring over B and $e = e^2 \in (B)_n$. $(B)_n$ is also a right closed irreducible ring. Therefore, $(B)_n = \mathrm{Hom}({}_K\underline{m}, {}_K\underline{m})$ for some division ring K and a left K-module $\underline{m}$. Then, $e(B)_n e = \mathrm{Hom}({}_K\underline{m}e, {}_K\underline{m}e)$, and we see that $A \otimes_C D^\circ$ (= $\mathrm{Hom}(A_B, A_B) = e(B)_n e$) is also a right closed irreducible ring. Then its radical is zero, and D has no nilpotent ideal except 0. But $[D:C] < \infty$.

Hence D is a semisimple ring whose center is Z. This implies that D is a simple ring. Then D is a Frobenius C-algebra. On the other hand (1), (2) and Theorem 36.2 [2] shows that A has a right B-free basis consisting of [D:C] elements. Furthermore, Theorem 4 [7] shows that A is a Frobenius extension of B, and consequently, A has also a left B-free basis which consists of [D:C] elements. Thus we have finished the proof.

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ON CONNECTEDNESS OF p-GALOIS EXTENSIONS OF RINGS

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The purpose of this note is to give necessary and sufficient conditions for a p^n-cyclic extension B over a connected ring A of prime characteristic p to be connected. The details of proof and related results will be seen in the joint work[1] of M. Ferrero and the author. When both A and B are commutative, these are considered in [3].

Let A be a ring with an identity. A is said to be <u>connected</u> if 0 and 1 are only idempotents of its center $C(A)$. A two sided simple ring with an identity is a typical example of a(non commutative) connected ring.

<u>Preliminaries</u>. In this note we assume that A is an algebra over a prime field $GF(p)$ of characteristic $p > 0$ and G is a cyclic group of order p^n with a generator σ. A G-Galois extension B of A such that $B_A \oplus > A_A$ (i.e., A_A is a direct summand of B_A) is said to be a p^n-cyclic extension. As is konwn in [2], if there are a derivation D of A and an element a of A such that $D^p - D = I_a$ (the inner derivation effected by a) and $D(a) = 0$ then $M = (X^p - X - a)A[X;D]$ becomes a two sided ideal of $A[X;D]$ where $A[X;D] = \{\sum X^i a_i;\ a_i \in A\}$ is a skew polynomial ring of derivation type whose multiplication is given by $cX = Xc + D(c)$ for $c \in A$. Moreover, $T = A[X;D]/M$ becomes a p-cyclic extension of A by $\sigma(xc) = (x + 1)c$ where $x = X + M$ and $c \in A$, and conversely,

if T^* is a p-cyclic extension of A then there are a derivation D^* of A and an element a^* of A such that ${D^*}^p - D^* = I_{a^*}$, $D^*(a^*) = 0$ and $T^* \approx A[X;D^*]/(X^p - X - a^*)A[X;D^*]$.

Let now B be a p^n-cyclic extension of a connected ring A and $T = B^H = \{b \in B;\ \tau(b) = b$ for any $\tau \in H\}$ where $H = (\sigma^p)$. Since T/A is a p-cyclic extension (with a Galois group $G|T$), we may assume that $T = A[X;D]/(X^p - X - a)A[X;D]$ for some derivation D of A and an element a of A such that $D^p - D = I_a$ and $D(a) = 0$. Hence, if A and B are assumed to be commutative, then $T = A[X]/(X^p - X - a)$ for some $a \in A$. The following theorem is proved in [3].

Theorem 1. Assume that B is commutative.

(1) B is connected if and only if T is connected.

(2) T is connected if and only if $X^p - X - a$ is irreducible in $A[X]$.

(3) $X^p - X - a$ is irreducible if and only if $a \notin A^p - A = \{c^p - c;\ c \in A\}$.

<u>Connected Cyclic Extensions</u>. In this section we shall generalize Theorem 1 to the non commutative case. For this, we shall introduce some notions about polynomials in $A[X;D]$.

A monic polynomial $f(X)$ in $A[X;D]$ is said to be a <u>generator</u> if $f(X)A[X;D] = A[X;D]f(X)$. A generator $f(X)$ is said to be <u>weakly-irreducible</u> (abbreviate w-irreducible) in $A[X;D]$ if $f(X)$ has no proper monic factor of degree ≥ 1 which is a generator.

Remark 1. If $f(X)$ is a generator in $A[X;D]$, then $f(X)$ is contained in $C(A_o)[X]$ where $A_o = \{c \in A; D(c) = 0\}$. Hence the notion of w-irreducibility of $f(X)$ in $A[X]$ coincides with the irreducibility of $f(X)$ in $C(A)[X]$ since each generator in $A[X]$ is contained in $C(A)[X]$. Hereafter, we denote $A[X;D]$ by R.

Lemma 2. Let $f(X) = X^p - X - a$ be a generator in R.

(1) If $R/f(X)R$ is connected then A is connected and $f(X)$ is w-irreducible in R.

(2) Let A be connected. Then $f(X)$ is either w-irreducible or a product of generators of degree = 1.

Let M be a group and N a normal subgroup of M. Then N is said to be a small subgroup of M if $NM' \neq M$ for any proper subgroup M' of M.

Lemma 3. Let A be connected and let S/A be an M-Galois extension for a finite group M. If S^N is connected for a small subgroup N of M, then S is connected.

Let B, $H = (\sigma^p)$ and $T = B^H = R/(X^p - X - a)R$ be same as in the preceeding section. Then, by making use of Lemmas 2 and 3, we have the following main theorem.

Theorem 4. (1) B is connected if and only if T

is connected.

(2) T is connected if and only if $X^p - X - a$ is w-irreducible in R.

(3) $X^p - X - a$ is w-irreducible in R if and only if $a \in \{c \in A_o; I_c = D^p - D\}$ but $a \notin A_o{}^p - A_o$.

Proof. (1) Let B be connected. Then, by Lemma 2, (1) and induction argument on n, we can see that the connectedness of T. The converse is a direct consequence of Lemma 3.

(2) By Lemma 2,(1), if T is connected then $X^p - X - a$ is w-irreducible. Conversely, suppose $X^p - X - a$ is w-irreducible and T is disconnected. Then we can see that D is an inner derivation of A, and hence, we may assume $R = A[X]$ and $C(T) \simeq C(A)[X]/(X^p - X - a)C(A)[X]$. Since $X^p - X - a$ is irreducible in $C(A)[X]$ by Remark 1, $C(T)$ is connected by Theorem 1, a contradiction.

(3) This is a direct consequence of Lemma 2,(2).

As an immediate consequence of Theorem 4, we have the following

Corollary 5. Let A be a two sided simple ring. Then the following conditions are equivalent.

(a) B is connected.

(b) B is a two sided simple ring.

(c) T is connected.

(d) T is a two sided simple ring.

(e) $X^p - X - a$ is w-irreducible in R.

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ON TWO RINGS OF M. HARADA

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Recently, in his papers [7]~[9], M. Harada has studied the following conditions:

(*) Every non-small R-module contains a non-zero injective submodule.

$(*)^*$ Every non-cosmall R-module contains a non-zero projective direct summand.

Concerning these conditions, he has then discovered two new rings; one is a perfect ring with (*) and the other is a semi-perfect ring with $(*)^*$. And, in his main theorems, very interesting ideal theoretic characterizations of these rings have been given. However, if we carefully read his papers, we see that several important problems remain on these rings. Actually, we should investigate the following fundamental problems:

1) Are these rings in fact new ?

2) Are these rings left-right symmetric ?

3) What kinds of relation are there between these two rings ?

4) Applications ?

Now, our purpose of this paper is to briefly announce some results which answer to these problems. Details will appear in [12]~[15].

1. Preliminaries.

Throughout this paper, we assume that all rings R considered are associative rings with identity, all R-modules are unitary and all homomorphisms between R-modules are written on the opposite side of scalars. The notation M_R is used to stress that M is a right R-module.

Let M be an R-module. We use E(M), J(M), S(M) and Z(M) to denote its injective hull, Jacobson radical, socle and singular submodule, respectively. Further, by $\{J_i(M)\}$ and $\{S_i(M)\}$, we denote its descending Loewy chain and ascending Loewy chain, respectively;

$J_0(M) = M$	$S_0(M) = 0$
$J_1(M) = J(M)$	$S_1(M) = S(M)$
$J_2(M) = J(J_1(M))$	$S_2(M)/S_1(M) = S(M/S_1(M))$
....	....

For submodules A and B of M with $A \subseteq B$, the notation $A \subseteq_e B$ means that A is an essential submodule of B; while $A \subseteq_c B$ (in M) means that A is a co-essential submodule of B, i.e., B/A is a small submodule of M/A.

For two R-modules M and N, we use $M \underset{\sim}{\subseteq} N$ to stand for 'M is isomorphic to a submodule of N'. The term ACC means the ascending chain condition.

Definition. We say that an R-module M is an extending (resp. lifting) module if, for any submodule A of M, there exists a direct summand A^* of M with $A \subseteq_e A^*$ (resp. $A^* \subseteq_c A$).

Definition([7]~[9], cf [16]). An R-module M is said to be a small module if it is small in its injective hull, and M is said to be a non-small module if it is not a small module. Dually, M is said to be a cosmall module if there exists an exact sequence $T \xrightarrow{f} M \to 0$ with $\ker f \subseteq_e T$, and M is said to be non-cosmall if it is not cosmall.

Definition ([2],[3],[17]). A ring R is said to be right QF-3 if it has a minimal faithful right ideal, and R is said to be right QF-2 if every indecomposable projective right R-module has simple socle.

2. H-ring.

Harada studied the following condition ([7],[9]):

(*) Every non-small right R-module contains a non-zero injective submodule.

The following result is one of his main results:

Theorem 2.1 ([9 , Theorem 2.3]). A right artinian ring R satisfies (*) if and only if, for any primitive idempotent e in R with eR_R non-small, there exists an integer t ≥ 0 for which

a) $eR_R/S_k(eR_R)$ is injective for all $0 \leq k \leq t$, and

b) $eR_R/S_{t+1}(eR_R)$ is a small module.

Remark. 1) Rings with (*) are Morita invariant. 2) A left and right perfect ring with (*) is right artinian ([7 , Theorem 5]).

Now, we can easily show that (*) is equivalent to the condition: For any injective R-module E_R and any submodule A of E with A not small in E, A contains a non-zero direct summand of E. From this fact, we see that the condition (*) is weaker condition than the following:

(#) Every injective right R-module is a lifting module.

Here, a natural question arises:

How far is (*) from (#) ?

For this question, we have

Theorem 2.2([12]). The following conditions are equivalent for a given ring R:

1) R satisfies (#).

2) R is a right artinian ring with (*).

3) R is a right perfect ring with the condition: The family of all injective right R-modules is closed under taking small covers, i.e., for any exact sequence $P \xrightarrow{f} E \to 0$ where E is injective and ker f is small in P, P is also injective.

4) Every right R-module is expressed as a direct sum of an injective R-module and a small R-module.

Definition. We call that a ring R is a right H-ring if it satisfies the equivalent conditions in the above theorem. A left H-ring is symmetrically defined and a right and left H-ring is simply called an H-ring.

Combining Theorem 2.2 with Colby-Rutter's theorem [2 , Theorem 1.3], we have

Theorem 2.3. A right H-ring is both right and left QF-3.

The following is due to Harada:

Theorem 2.4([7]). Every indecomposable injective right R-module over a right H-ring is a cyclic hollow module.

3. Co-H-ring.

The following condition is dual to (*):

$(*)^*$ Every non-cosmall right R-module contains a non-zero projective direct summand.

Harada gave the following ideal theoretic characterization of semi-perfect rings with $(*)^*$:

Theorem 3.1([8], [9]). A semi-perfect ring R satisfies $(*)^*$ if and only if, for a complete set $\{e_i\} \cup \{f_j\}$ of orthogonal primitive idempotents of R with each e_iR_R small and each f_jR_R non-small, the following hold:

1) Each e_iR_R is injective.

2) For each e_iR, there exists integer $t_i \geq 0$ such that $J_t(e_iR_R)$ is projective for $0 \leq t \leq t_i$ and $J_{t_i+1}(e_iR_R)$ is a singular module.

3) For each f_jR, there exists e_iR such that $f_jR_R \subsetneqq e_iR_R$.

Therefore, if this is so, R is right QF-2.

Now, we note that $(*)^*$ is equivalent to the condition: For any projective module P_R and any submodule A of P, if A is not essential in P then there exists a proper direct summand $B \oplus P$ with $A \subseteq B$ (cf. [12]). Hence, $(*)^*$ is weaker than the following condition:

$(\#)^\#$ Every projective right R-module is an extending module.

By comparing $(*)^*$ with $(\#)^\#$, we obtain the following theorem which is dual to Theorem 2.2:

Theorem 3.2 ([12]). The following conditions are equivalent:

1) R satisfies $(\#)^{\#}$.

2) R satisfies $(*)^{*}$ and the ACC for right annihilator ideals of R.

3) The family of all projective right R-modules is closed under taking essential extensions.

4) Every right R-module is expressed as a direct sum of a projective module and a singular module.

Further, when this is so, R is a semi-primary right and left QF-3 ring and satisfies the ACC for left annihilator ideals (cf. [2]).

4. Connections with classical artinian rings.

Clearly, quasi-injective right R-modules are extending modules. Dually, if R is a right perfect ring, then quasi-projective right R-modules are lifting modules ([10],[11]).

Let us consider the following conditions:

a) Every injective right R-module is projective.

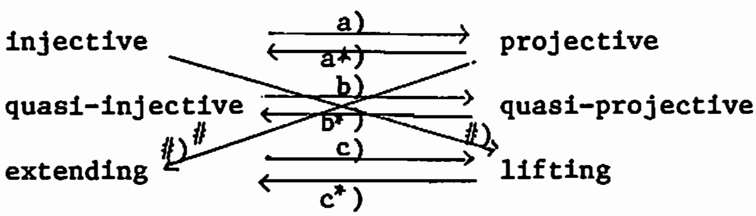
a^*) Every projective right R-module is injective.

b) Every quasi-injective right R-module is quasi-projective.

b^*) Every quasi-projective right R-module is quasi-injective.

c) Every extending right R-module is lifting.

c^*) Every lifting right R-module is extending.



As is well known, the following conditions are equivalent:

1) R is QF.

2) R satisfies a).

3) R satisfies a^*).

So, a) and a^*) are right-left symmetric.

On the other hand, the following conditions are equivalent ([1], [4]):

1) R is a unisetial ring.

2) R satisfies b).

3) R satisfies b^*).

Therefore, b) and b^*) are also right-left symmetric.

In view Figure above, a natural question arises: What is a ring R with c) or c^*) ? By using Theorems 2.2 and 3.2, we can study this question. Our conclusion is the following:

Theorem 4.1 ([13]). The following conditions are equivalent:

1) R is a generalized uniserial ring.

2) R satisfies c).

3) R is a right perfect ring with c^*).

So, 2) and 3) are left-right symmetric.

By above theorems we have immediately

Theorem 4.2. QF-rings and generalized uniserial rings are both H and co-H-rings.

QF-rings can be characterized in terms of H or co-H-rings as follows:

Theorem 4.3 ([12]). The following conditions are equivalent for a given ring R:

1) R is QF.

2) R is a right H-ring with $Z(R_R) = J(R)$.

3) R is a right co-H-ring with $Z(R_R) = J(R)$.

So, 2) and 3) are right-left symmetric.

Next, we shall study right non-singular right H- or co-H-rings. If R is a right co-H-ring, we see from Theorem 3.2 that every non-singular right R-module is projective. In case R is right non-singular, the converse also holds, because, in this case, a submodule A of a projective module P_R is a closed submodule if and only if P/A is non-singular. Therefore, by the Goodearl's work [6, Chapter 5], a right non-singular right co-H-ring is completely determined as it is Morita equivalent to a finite direct sum of upper triangular matrices over division rings.

A right non-singular right H-ring have also the same structure as the following shows:

Theorem 4.4 ([12]). The following conditions are equivalent for a given ring R:

1) R is right non-singular right H.

2) R is right non-singular right co-H.

3) R is Morita equivalent to a finite direct sum of upper triangular matrices over division rings.

Therefore, 1) and 2) are right-left symmetric.

5. Typical examples.

In view of Figure in § 4, the following questions arise:

1) Are right H-rings left H-rings ?

2) Are right co-H-rings left co-H-rings ?

3) Are right H-rings right co-H-rings ?

4) Are right co-H-rings right H-rings ?

However, in the case when R is an algebra over a field of finite dimension, these problems are all equivalent, as the following shows:

Theorem 5.1([12]). Let R be an algebra over a field of finite dimension. Then R is a left H-ring if and only R is a right co-H-ring.

In this section, we give two typical examples of left H- and right co-H-rings. From one of these, we see that the answers of the questions above are no.

Now, in order to make these examples, let us consider a special type of a co-H-ring. Let R be a co-H-ring with a complete set {e, f} of orthogonal primitive idempotents such that

1) eR_R is injective,

2) $eJ(R)_R \simeq fR_R$.

We put Q = eRe and X = $S(eR_R)$. Then, X_R is simple and isomorphic to $S(fR_R)$. Note that the projective cover of X is either eR or fR.

Under these situations, we have the following two theorems.

Theorem 5.2([12]). Q is a local QF-ring. So, $S(Q_Q)$ is simple and coincides with $S({}_QQ)$ and therefore

$$\begin{pmatrix} Q & Q/S \\ J & Q/S \end{pmatrix}$$

becomes a ring canonically, where $S = S(Q_Q)$ and $J(Q)$.

Theorem 5.3([12]). 1) If eR_R is a projective cover of X_R, then R is isomorphic to the ring

$$\begin{pmatrix} Q & Q/S \\ J & Q/S \end{pmatrix}$$

2) If fR_R is a projective cover of X_R, then R is isomorphic to the ring

$$\begin{pmatrix} Q & Q \\ J & Q \end{pmatrix}$$

Now, conversely for a given local QF-ring Q, we put

$$V(Q) = \begin{pmatrix} Q & Q \\ J & Q \end{pmatrix}$$

$$W(Q) = \begin{pmatrix} Q & Q/S \\ J & Q/S \end{pmatrix}$$

where $S = S(Q_Q) = S({}_QQ)$ and $J = J(Q)$. Then we have

Theorem 5.4([12]). 1) V(Q) is a H- and co-H-ring. 2) W(Q) is a left H- and right co-H-ring.

We can extend Theorems 5.3 and 5.4 as follows:

Theorem 5.5([15]). A ring R is a basic right co-H-ring with homogeneous socle if and only if it is represented as a matrix ring:

$$\begin{pmatrix} \overbrace{Q & \cdots & Q}^{k} & \overline{Q} & \cdots & \overline{Q} \\ J & Q & & & & \\ \vdots & \ddots & \ddots & \vdots & & \vdots \\ J & \cdots & J\ Q & \overline{Q} & \cdots & \overline{Q} \\ J & \cdots & J & \overline{Q} & \cdots & \overline{Q} \\ \vdots & & \vdots & \overline{J} & \ddots & \vdots \\ & & \vdots & \vdots & \ddots\ \overline{Q} & \overline{Q} \\ J & \cdots & J & \overline{J} & \cdots\ \overline{J} & \overline{Q} \end{pmatrix}$$

$k \geq 1$

for a suitable local QF-ring Q, where $\overline{Q} = Q/S(Q)$ and $J = J(Q)$. Furthermore, such a ring is also left H.

Here, we shall show that W = W(Q) above is not a left co-H-ring in general. Suppose that W is a left co-H-ring. We put

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 0 & \overline{1} \end{pmatrix}$$

Since We is injective, ${}_W Wf$ must be isomorphic to $J({}_W We)$; whence it follows $J = {}_Q J(Q) \simeq {}_Q(Q/S)$. As a result, J must be a cyclic left ideal of Q. However, this is not true in general.

For example, consider the ring

$$\overline{K[x,y]} = K[x,y]/(x^2,y^2)$$

where K is a field. This is a local QF-ring with Jacobson radical $\overline{(x,y)}$. As is easily computed, $\overline{(x,y)}$ is not a cyclic ideal. Thus if we take this ring as Q, W(Q) is a left H- and right co-H-ring but not right H- nor left co-H. As an interesting other property of this ring, we note that this W(Q) is left and right QF-3 and right QF-2 but not left QF-2. Therefore this ring gives a counter example of a ring which solve a problem raised in Fuller [3].

Remark. Althrough we did not give proofs of Theorems 5.2 $\sim$5.5, the following results are used for their proofs:

1) Let R be a one sided artinian ring and let e and f be primitive idempotents in R. Following Fuller [5], we say that the pair $(eR_R, {}_RRf)$ is an injective pair if

$$S(eR_R) \simeq fR_R/fJ \text{ and } S({}_RRf) \simeq {}_RRe/Je$$

where J = J(R). It is shown in [5] that if $(eR_R, {}_RRf)$ is an injective pair then eR_R and ${}_RRf$ is injective.

2) Let R be a right artinian ring and let M be a right R-module. The following criterion is due to Harada ([9]): M is a small module if and only if $MS({}_RR) = 0$.

6. Coincidence of left H-rings and right co-H-rings.

Now, from our results in §§ 2 $\sim$5, we can enoughly expect that Theorem 5.1 is valid for all rings. Indeed, this is true:

Theorem 6.1([14]). A ring R is a left H-ring if and only if R is a right co-H-ring.

In this paper, we only give a sketch of a proof. For a complete proof of this theorem, the reader is referred to [14].

6.A Proof of 'right co-H $\Rightarrow$ left H'

In this sub-section we assume that R is a basic right co-H-ring with E, a complete set of primitive orthogonal idempotents of R. Then we have a partition

$$E = \{e_{11}, \ldots, e_{1n(1)}\} \cup \ldots \cup \{e_{m1}, \ldots, e_{mn(m)}\}$$

such that

1) $e_{i1}R_R$ is injective for all i,

2) $e_{i1}R \supsetneq e_{i2}R \supsetneq \cdots \supsetneq e_{in(i)}R$, more precisely $J(e_{ik}R_R)_R \simeq e_{i,k+1}R_R$ for i and $k = 1, \ldots, n(i)-1$,

3) each $S(e_{ij}R_R)$ is simple; so $S(e_{i1}R)_R \simeq \cdots \simeq S(e_{in(i)}R)_R$ for all i.

Remark. Henceforth, we observe R by representing it as the matrix ring:

$$R = \begin{pmatrix} [e_{11},e_{11}] & \cdots\cdots & [e_{mn(m)},e_{11}] \\ \vdots & \ddots & \vdots \\ [e_{11},e_{mn(m)}] & \cdots\cdots & [e_{mn(m)},e_{mn(m)}] \end{pmatrix}$$

or

$$R = \begin{pmatrix} e_{11}Re_{11} & \cdots\cdots & e_{11}Re_{mn(m)} \\ \vdots & \ddots & \vdots \\ e_{mn(m)}Re_{11} & \cdots\cdots & e_{mn(m)}Re_{mn(m)} \end{pmatrix}$$

where $[e_{ij}, e_{kt}] = \mathrm{Hom}_R[e_{ij}R, e_{kt}R]$. So, we identify $e_{ij}R$ with ij-row and Re_{ij} with ij-column.

Lemma 6.A.1. R is left artinian.

Proof (sketch). We may show that ${}_{fRf}fRe$ is artinian for any f, e in E.

Step 1. As R is semi-primary QF-3, we see from [2] that $e_{i1}Re_{j1}$ is left artinian as a left $e_{i1}Re_{i1}$-module for for any i, j.

Step 2. Assume that $e_{pq}R$ is a projective cover of $S(e_{i1}R_R)_R$. If $p \neq i$, then

$$e_{i1}Re_{i1} \simeq \cdots \simeq e_{in(i)}Re_{in(i)} \quad \text{(as rings)}.$$

If p = i, then there exists s in $\{1, \ldots, n(i)\}$ such that

1) $e_{i1}Re_{i1} \simeq \cdots \simeq e_{is}Re_{is}$,

2) $e_{i,s+1}Re_{i,S+1} \simeq \cdots \simeq e_{i,n(i)}Re_{in(i)}$,

3) there exists a ring epimorphism:

$$e_{is}Re_{is} \longrightarrow e_{i,s+1}Re_{i,s+1}.$$

Consequently, $e_{ij}Re_{ij}$ is artinian as a left $e_{ij}Re_{ij}$-module for all e_{ij}.

Step 3. We observe $e_{ij}Re_{kt}$ for $i \neq k$. Put $f_j = e_{ij}$, $f_1 = e_{i1}$; $g_t = e_{kt}$, $g_1 = g_{k1}$. Then, note that f_jRg_t becomes a left f_1Rf_1-module. Now, we see that

$${}_{f_1Rf_1}f_jRg_t \simeq {}_{f_1Rf_1}f_1Rg_t$$

and there exists an epimorphism:

$${}_{f_1Rf_1}f_1Rg_1 \longrightarrow {}_{f_1Rf_1}f_1Rg_t$$

Hence it follows that ${}_{f_jRf_j} f_jRg_t$ is artinian.

Step 4. Put $e_1 = e_{i1}, \ldots, e_t = e_{in(i)}$ and observe e_jRe_i for $i \neq j$. If $i < j$, then e_jRe_i becomes a left e_iRe_i-module and

$$e_iRe_i{}^{e_jRe_i} \subsetneqq e_iRe_i{}^{e_iRe_i}$$

So, ${}_{e_jRe_j} e_jRe_i$ is artinian. If $i > j$, then e_iRe_i becomes a left e_jRe_j-module and

$$e_jRe_j{}^{e_jRe_i} \simeq e_jRe_j{}^{e_iRe_i}.$$

Hence ${}_{e_jRe_j} e_jRe_i$ is artinian.

By Step 1 $\sim$ Step 4, ${}_{fRf} fRg$ is artinian for all f,g in E and hence R is left artinian.

Lemma 6.A.2. Let e, f in E and assume that fR_R is the projective cover of $S(eR_R)$. Put $X = Hom_R(fR, S(eR_R))$. Then,

1) ${}_{eRe}X$ and X_{fRf} are simple.

2) ${}_{eRe}X \simeq {}_{eRe}S({}_{eRe}eRf)$ and $X_{fRf} \simeq S(eRf_{fRf})_{fRf}$

3) $S({}_{eRe}eRf) = S(eRf_{fRf})$.

Lemma 6.A.3. Let f in E and assume that fR_R is the projective cover of $S(e_{i1}R_R)$. Then,

1) $S_k({}_RRf) = S(e_{i1}R_R) + \ldots + S(e_{ik}R_R)$ for $k = 1, \ldots, n(i)$.

2) $S(e_{in(i)}R_R)(Rf/S_k(Rf)) = S(e_{in(i)}R_R)$ for $k < n(i)$ and

$$S(R_R)(Rf/S_{n(i)}(Rf)) = 0$$

whence $Rf/S_k(Rf)$ is a non-small left R-module for $k < n(i)$ and $Rf/S_{n(i)}(Rf)$ is a small left R-module.

Lemma 6.A.4. The following conditions are equivalent for a given f in E:

1) ${}_R Rf$ is a non-small module.
2) ${}_R Rf$ is injective.
3) fR_R is a projective cover of some $S(e_{i1}R_R)$.

Now, we are in a position to show that R is a left H-ring.

By Lemma 6.A.1, R is left artinian. Let Rf and $e_{i1}R$ be as in Lemma 6.A.3, i.e., fR_R is the projective cover of $S(e_{i1}R_R)$. We may show that $Rf/S_k({}_R Rf)$ is injective as a left R-module for k = 1, ..., n(i)-1 (cf. Lemma 6.A.4). In view of Lemma 6.A.3, $S_k({}_R Rf)$ is a two-sided ideal of R for $k < n(i)$; so, the factor ring $\overline{R} = R/S_k({}_R Rf)$ is considered. Then, we can see that $(\bar{e}_{i,k+1}\overline{R}_{\overline{R}}, {}_{\overline{R}}\overline{R}\bar{f})$ is an injective pair; whence ${}_{\overline{R}}\overline{R}\bar{f}$ is injective. Here, noting $S({}_{\overline{R}}\overline{R}\bar{f}) = \overline{S(e_{i,k+1}R_R)}$, we can show that ${}_R\overline{R}\bar{f}$ $(=Rf/S_k({}_R Rf))$ is injective.

6.B. Proof of 'left H $\Rightarrow$ right co-H'.

In this sub-section, we assume that R is a basic left H-ring and E is a complete set of orthogonal primitive idempotents of R. Since R is semi-primary QF-3, we have a partition: $E = \{e_{11}, \ldots, e_{1n(1)}\} \cup \ldots \cup \{e_{m1}, \ldots, e_{mn(m)}\} \cup G$ such that

1) each $e_{i1}R_R$ is injective,
2) $S(e_{i1}R_R) \simeq \ldots \simeq S(e_{in(i)}R_R)$ for all i,
3) $S(gR_R)$ is not simple for all g in G.

Remark. We observe R by identifying it with the matrix ring:

$$\begin{pmatrix} [e_{11},e_{11}] & \cdots & [e_{mn(m)},e_{11}] & [g_1 e_{11}] & \cdots & [g_t,g_1] \\ \vdots & & \vdots & \vdots & & \vdots \\ [e_{11},e_{mn(m)}] & \cdots & [e_{mn(m)},e_{mn(m)}] & [g_1,e_{mn(m)}] & \cdots & [g_t,e_{mn(m)}] \\ [e_{11},g_1] & \cdots & [e_{mn(m)},g_1] & [g_1,g_1] & \cdots & [g_t,g_1] \\ \vdots & & \vdots & \vdots & & \vdots \\ [e_{11},g_t] & \cdots & [e_{mn(m)},g_t] & [g_1,g_t] & \cdots & [g_t,g_t] \end{pmatrix}$$

where $G = \{g_1, \ldots, g_t\}$ and $[p,q] = \mathrm{Hom}_R(pR,qR)$ for p, q in E.

Lemma 6.B.1. Assume that $e_{i2}R_R \not\subseteq e_{ij}R_R$ for j = 3, .., n(i). Then

1) $e_{i2}R_R \not\subseteq gR_R$ for all g in G,
2) $e_{i2}R_R \simeq J(e_{i1}R_R)$.

Proof. We can take h in E such that hR_R is the projective cover of $S(e_{i1}R_R)$.

Assume that there exists g in G such that $e_{i2}R_R \subsetsim gR_R$. We can assume that $gR_R \not\subseteq zR_R$ for any z in G − {g}. We put

$$A_1 = \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \cdot & & \cdot & \cdot & \cdot & & \cdot \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & [hR,S(e_{i1}R)] & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \cdot & & \cdot & \cdot & \cdot & & \cdot \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & [hR, S(e_{i2})] & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$A_g = \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & [hR, S(gR)] & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

where $Hom_R = (hR, S(e_{ij}R_R)) = [hR, S(e_{ij}R)]$, $i = 1, 2$ and $Hom_R(hR, S(gR_R)) = [hR, S(gR_R)]$.

We see that $A_1 = S(e_{i1}R_R) = S({}_R Rh)$; so A_1 is a two-sided ideal of R. We denote the factor ring R/A_1 by $\bar{R}$. Note that $J(e_{i1}R_R)$ and $e_{i2}R$ become right $\bar{R}$-modules, there exists an epimorphism: $J(e_{i1}R_R)_{\bar{R}} \rightarrow \overline{J(e_{i1}R_R)}_{\bar{R}}$ and an isomorphism : $e_{i2}R_{\bar{R}} \simeq \bar{e}_{i2}\bar{R}_{\bar{R}}$.

Further, we see that $A_1 + A_g = S_2({}_R Rh)$ and $A_2(Rh/A_1) \neq 0$. Hence $\bar{R}\bar{h}$ must be injective as a left R-module and hence so is as a left $\bar{R}$-module. Since the socle of ${}_{\bar{R}}\bar{R}\bar{h}$ is $\bar{A}_g$, we see that $(\bar{g}\bar{R}_{\bar{R}}, {}_{\bar{R}}\bar{R}\bar{h})$ is an injective pair. As a result, $\bar{g}\bar{R}_{\bar{R}}$ has simple socle. However this implies that gR_R has simple socle, a contradiction. Thus such g does not exist.

From this fact and the assumption, we see that $(\bar{e}_{i2}\bar{R}_{\bar{R}}, {}_{\bar{R}}\bar{R}\bar{h})$ is an injective pair; so $\bar{e}_{i2}\bar{R}_{\bar{R}}$ is injective. Since $\bar{e}_{i2}\bar{R}_{\bar{R}} \simeq e_{i2}R_{\bar{R}} \subsetneqq J(e_{i1}R_R)_{\bar{R}}$, we see $e_{i2}R_{\bar{R}} \simeq J(e_{i1}R_R)_{\bar{R}}$ and hence $e_{i2}R_R \simeq J(e_{i1}R_R)_R$ as desired.

By a similar argument, we can show the following

Lemma 6.B.2. 1) There is a permutation $\{e_{i2'}, \ldots e_{in(i)'}\}$ of $\{e_{i2}, \ldots, e_{in(i)}\}$ such that

$$J(e_{i1}R_R)_R^{k-1} \simeq e_{ik'}R_R$$

for $k = 2', \ldots, n(i)'$.

2) $\quad \neg(e_{ij}R_R \subsetneqq gR_R)$

for all e_{ij} and g in E.

3) $\quad \neg(S(e_{ij}R_R)_R \subsetneqq S(gR_R)_R)$

for all e_{ij}, and g in E; so G is empty.

Now, Lemma 6.B.2 shows that R is a right co-H-ring.

7. Application.

As an application of our study, we can show the following

Theorem 7.1([14]). If R is a right QF-3 and right generalized uniserial ring then R is a generalized uniserial ring.

For a proof of this theorem, the following two lemmas are needed.

Lemma 7.1. If R is a right QF-3 and right generalized uniserial ring, then R is a right co-H- and hence left H-ring.

Proof. This is easily shown by using Theorem 3.1.

Lemma 7.2. If R is a right QF-3 and right generalized uniserial ring, then so is $R/S(R_R)$.

Proof. We can assume that R is a basic ring. As in § 6, we observe R by identifying it with the matrix ring:

$$\begin{pmatrix} [e_{11},e_{11}] & \cdots & [e_{mn(m)},e_{11}] \\ \vdots & & \vdots \\ [e_{11},e_{mn(m)}] & \cdots & [e_{mn(m)},e_{mn(m)}] \end{pmatrix}$$

where $E = \{e_{ij}\}$ is a complete set of orthogonal primitive idempotents of R such that

1) each $e_{i1}R_R$ is injective,

2) $e_{in(i)}R_R \subsetneqq e_{i,n(i)-1}R_R \subsetneqq \cdots \subsetneqq e_{i2}R_R \subsetneqq e_{i1}R_R$.

(Note that R is a right co-H-ring by Lemma 7.1). Put $\bar{R} = R/S(R_R)$. Then, clearly,

3) $\bar{e}_{in(i)}\bar{R}_{\bar{R}} \subsetneqq \bar{e}_{i,n(i)+1}\bar{R}_{\bar{R}} \subsetneqq \cdots \subsetneqq \bar{e}_{i2}\bar{R}_{\bar{R}} \subsetneqq \bar{e}_{i1}\bar{R}_{\bar{R}}$

for all i. When $\bar{e}_{i1}\bar{R} \neq 0$, we take h in E such that hR_R is the projective cover of $S_2(e_{i1}R_R)_R$. Then we see that $(\bar{e}_{i1}\bar{R}_{\bar{R}}, {}_{\bar{R}}\bar{R}\bar{h})$ is an injective pair. As a result,

4) each $\bar{e}_{i1}\bar{R}_{\bar{R}}$ is injective if it is non-zero.

By 3) and 4), we see that $\bar{R}$ is a right co-H-ring. So, it is QF-3 by Theorem 3.2.

Proof of Theorem 7.1: We can assume that R is a basic ring. Let E = $\{e_{ij}\}$ be as in Lemma 7.3. We take h_i in E such that h_iR_R is a projective cover of $S(e_{i1}R_R)_R$ for i = 1, ..., m. Then, by the argument in § 6,

α) $S(e_{i1}R_R) + \dots + S(e_{ik}R_R) = S_k({}_RRh_i)$

for k = 1, ..., n(i) and moreover

β) $S_k({}_RRh_i)/S_{k-1}({}_RRh_i)$ is simple

(as a left R-module) for k = 1, ..., n(i).

Now, by induction of the sum of composition lengths of all $e_{ij}R_R$ together with Lemma 7.3, we see $R/S(R_R)$ is a generalized uniseral ring. In view of α) and β), this implies that R is a left generalized uniserial ring.

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THE INVARIANT SYSTEMS OF SERIAL RINGS AND THEIR APPLICATIONS TO THE THEORY OF SELF-DUALITY

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This note is a summary of the author's paper [12]. Proofs and details will be published in [12].

The structure of serial rings has been studied by many authors (cf. [1], [2], [3], [6], [9], [15], [16], [17] and [18]). In this note, we shall define the invariant systems of serial rings and prove that there is a larger class of serial rings which have self-duality.

1. The invariant systems of serial rings. Throughout this section, R denotes an indecomposable self-basic serial ring with the radical J. Let

$$1 = e_1 + e_2 + \cdots + e_n$$

be a decomposition of 1_R into a sum of mutually orthogonal primitive idempotents such that $Re_1, Re_2, \cdots, Re_n$ is a Kupisch series of R, i.e., $Re_1, Re_2, \cdots, Re_n$ satisfy the folllowing conditions:

$$c({}_R Re_i) \geq 2 \quad \text{for} \quad i = 2, 3, \cdots, n,$$

$$c({}_R Re_{[i+1]}) \leq c({}_R Re_i) + 1 \quad \text{for} \quad i = 1, 2, \cdots, n,$$

$$Re_i/Je_i \cong Je_{i+1}/J^2e_{i+1} \quad \text{for} \quad i = 1, 2, \cdots, n-1,$$

and

$$Re_n/Je_n \cong Je_1/J^2e_1 \quad \text{if} \quad Je_1 \neq 0,$$

where $c({}_R M)$ denotes the composition length of a module ${}_R M$ and $[k]$ denotes the least positive remainder of an integer k modulo n. Let us put $b_i = c({}_R Re_i)$, $i = 1, 2, \cdots, n$. The sequence $b_1, b_2, \cdots, b_n$ is called an admissible sequence of R. Let us put

$$R_i = e_i R e_i, \; J_i = e_i J e_i,$$
$$c_{ij} = c({}_{R_i} e_i R e_j) = c(e_i R e_{j R_j}).$$

Then $R_1, R_2, \cdots, R_n$ are the local uniserial rings with the radicals $J_1, J_2, \cdots, J_n$ respectively, and we have

$$c({}_{R_i} R_i) = c_{ii},$$

$$R_i/(J_i)^{c_{ij}} \simeq R_j/(J_j)^{c_{ij}}.$$

Moreover, we have the following lemma

Lemma 1.1. With the above notations, we have

$$c_{ii} = \{b_i - [b_i]\}/n + 1$$
$$c_{ij} = \{b_j - [j-i] - [b_j - [j-i]]\}/n + 1 \quad \text{if} \quad i \neq j,$$
$$c_{ij} \leq c_{[j-1]j} \leq c_{jj} \leq c_{ij} + 1 \quad \text{if} \quad i \neq j,$$
$$c_{ij} \leq c_{i[i+1]} \leq c_{ii} \leq c_{ii} + 1 \quad \text{if} \quad i \neq j,$$
$$|c_{ii} - c_{jj}| \leq 1,$$
$$c_{i[i+1]} \geq c_{jj} - 1.$$

In the rest of this section, <u>we shall assume that</u> $n \neq 1$. (Notice that R is uniserial if and only if $n = 1$.)

Let $y_i \in e_i R e_{[i+1]}$ be an element such that

$$e_i R e_{[i+1]} = R_i y_i = y_i R_{[i+1]}.$$

Then there exists a mapping $\phi_i : R_i \to R_{[i+1]}$ such that

$$r_i y_i = y_i \circ \phi_i(r_i) \quad \text{for all} \quad r_i \in R_i.$$

Let $\pi_{[i+1]} : R_{[i+1]} \to R_{[i+1]}/(J_{[i+1]})^{c_{i[i+1]}}$ be the natural homomorphism.

Lemma 1.2. $\pi_{[i+1]} \circ \phi_i : R_i \to R_{[i+1]}/(J_{[i+1]})^{c_{i[i+1]}}$ is an onto ring homomorphism.

For each i and each j, we shall define

$$x_{ii} = 1_{R_i},$$

$$x_{ij} = y_i y_{[i+1]} \cdots y_{[j-1]} \quad (i \neq j),$$
$$w_i = y_i y_{[i+1]} \cdots y_{[i+2]} y_{[i-1]} .$$

Then it is easy to prove that

$$e_i R e_j = R_i x_{ij} = x_{ij} R_j ,$$
$$J_i = R_i w_i = w_i R_i ,$$
$$\phi_i(w_i) \equiv w_{[i+1]} \pmod{(J_{[i+1]})^{c_{i[i+1]}}} .$$

Definition. With the above notations, the system $\mathcal{S} = \{n;\ b_i,\ R_i,\ w_i,\ \phi_i\}$ will be called an <u>invariant system</u> of R.

Since a Kupisch series of R is not uniquely determined by R, an invariant system of R is not uniquely determined by R. However, two invariant systems of R are "equivalent". (As for the definition of an equivalence, see §3 below.)

2. Serial systems. In this section, we shall define the notion of serial systems.

Let $b_1, b_2, \cdots, b_n$ be a sequence of positive integers satisfying the following conditions:

$$b_i \geq 2 \quad \text{for} \quad i = 2, 3, \cdots, n,$$
$$b_{[i+1]} \leq b_i + 1 \quad \text{for} \quad i = 1, 2, \cdots, n,$$

where $[k]$ denotes the least positive remainder of k modulo n. For convenience sake, we shall assume $n \geq 2$. Let us put

$$c_{ii} = \{b_i - [b_i]\}/n + 1,$$
$$c_{ij} = \{b_j - [j-i] - [b_j - [j-i]]\}/n + 1 \quad (i \neq j),$$

for $1 \leq i, j \leq n$. Then the c_{ij}'s satisfy Lemma 1.1. Let $R_1, R_2, \cdots, R_n$ be local uniserial rings with the radicals $J_1, J_2, \cdots, J_n$ respectively, such that

$$c({}_{R_i}R_i) = c_{ii} ,$$

$$R_i/(J_i)^{c_{i[i+1]}} \simeq R_{[i+1]}/(J_{[i+1]})^{c_{i[i+1]}} \quad \text{for all } i.$$

Let $\phi_i : R_i \to R_{[i+1]}$ be a mapping and $w_i \in R_i$, $i = 1, 2, \cdots, n$. Then the system $\mathcal{S} = \{n; b_i, R_i, w_i, \phi_i\}$ is called a serial system if the following four conditions are satisfied: For each i,

(i) $J_i = R_i w_i = w_i R_i$,

(ii) $\pi_{[i+1]} \circ \phi_i$ is an onto ring homomorphism where $\pi_{[i+1]} : R_{[i+1]} \to R_{[i+1]}/(J_{[i+1]})^{c_{i[i+1]}}$ denotes the natural homomorphism,

(iii) $\phi_i(w_i) \equiv w_{[i+1]} \pmod{(J_{[i+1]})^{c_{i[i+1]}}}$,

(iv) $r_i w_i = w_i \phi_{[i-1]} \circ \phi_{[i-2]} \circ \cdots \circ \phi_i(r_i)$ for all $r_i \in R_i$.

It is easy to prove that an invariant system of an indecomposable self-basic serial ring is a serial system.

The first main theorem is stated as follows.

Theorem 2.1. Let $\mathcal{S}$ be a serial system. Then there uniquely exists an indecomposable self-basic serial ring R such that $\mathcal{S}$ is an invariant system of R.

3. Isomorphisms between two serial rings. Let R be an indecomposable self-basic serial ring with the radical J. The notations are as in §1.

Let $Re'_1, Re'_2, \cdots, Re'_n$ be another Kupisch series of R, and $\mathcal{S}' = \{n; b'_i, R'_i, w'_i, \phi'_i\}$ be an invariant system of R which is constructed from $Re'_1, Re'_2, \cdots, Re'_n$ as in §1. Then there exist a unit $u \in R$ and an integer m such that

$$e_{[i-m]} = ue'_i u^{-1} \quad \text{and} \quad e'_i = u^{-1} e_{[i-m]} u \quad \text{for all } i.$$

For each i, let us put

$$\theta_i : R_{[i-m]} \ni r_{[i-m]} \to u^{-1} r_{[i-m]} u \in R'_i.$$

Then θ_i is a ring isomorphism. Next, there exists a unit $u'_{[i+1]} \in R'_{[i+1]}$ such that $y'_i u'_{[i+1]} = u^{-1} y_{[i-m]} u$, $i = 1, 2, \cdots, n$. Then we have

$$(u'_{[i+1]})^{-1} \phi'_i(\theta_i(r_{[i-m]}))u'_{[i+1]} \equiv \theta_{[i+1]}(\phi_{[i-m]}(r_{[i-m]}))$$
$$(\text{mod } (J'_{[i+1]})^{c'_{i[i+1]}}) \quad \text{for all} \quad r_{[i-m]} \in R_{[i-m]},$$
$$\theta_i(w_{[i-m]}) = w'_i \phi'_{[i+1]i}(u'_{[i+1]}) \cdots \phi'_{[i-1]i}(u'_{[i-1]})u'_i,$$

where $\phi'_{ji} = \phi'_{[i-1]} \circ \phi'_{[i-2]} \circ \cdots \circ \phi'_j$ $(i \neq j)$.

From the above discussion, we get the following definition. For convenience sake, we shall assume $n \geq 2$.

Definition. Let $\mathcal{S} = \{n;\ b_i,\ R_i,\ w_i,\ \phi_i\}$ and $\mathcal{S}' = \{n';\ b'_i,\ R'_i,\ w'_i,\ \phi'_i\}$ be serial systems. We shall say that $\mathcal{S}$ is equivalent to $\mathcal{S}'$ if the following three conditions are satisfied:

(i) $n = n'$,

(ii) there exists an integer m such that

$$b_{[i-m]} = b'_i \quad \text{for all} \quad i,$$

(iii) for each i, there exist a unit $u'_{[i+1]} \in R'_{[i+1]}$ and a ring isomorphism $\theta_i : R_{[i-m]} \to R'_i$ such that

$$(u'_{[i+1]})^{-1} \phi'_i(\theta_i(r_{[i-m]}))u'_{[i+1]} \equiv \theta_{[i+1]}(\phi_{[i-m]}(r_{[i-m]}))$$
$$(\text{mod } (J'_{[i+1]})^{c'_{i[i+1]}}) \quad \text{for all} \quad r_{[i-m]} \in R_{[i-m]},$$
$$\theta_i(w_{[i-m]}) = w'_i \phi'_{[i+1]i}(u'_{[i+1]}) \cdots \phi'_{[i-1]i}(u'_{[i-1]})u'_i,$$

where $\phi'_{ji} = \phi'_{[i-1]} \circ \phi'_{[i-2]} \circ \cdots \circ \phi'_j$ $(i \neq j)$.

The above relation is an equivalence relation of serial systems, and two invariant systems of the same indecomposable self-basic serial ring are equivalent.

The second main theorem is stated as follows.

Theorem 3.1. A necessary and sufficient condition for given two indecomposable self-basic serial rings to be isomorphic to each other is that their invariant systems be equivalent to each other.

Corollary 3.2. There is a bijective correspondence between all Morita equivalence classes of indecomposable serial rings and all equivalence classes of serial systems by

$$[R] \mapsto [\mathcal{S}_{R^\circ}],$$

where $[R]$ denotes the Morita equivalence class of an indecomposable serial ring R, R° denotes the basic ring of R, $\mathcal{S}_{R^\circ}$ denotes an invariant system of R°, and $[\mathcal{S}_{R^\circ}]$ denotes the equivalence class of $\mathcal{S}_{R^\circ}$.

4. Self-duality. The notations are as in 1. Let us put

$$E_i = E({}_R Re_i/J_i), \quad i = 1, 2, \cdots, n,$$
$$E = E_1 \oplus E_2 \oplus \cdots \oplus E_n,$$
$$S = \mathrm{End}({}_R E),$$

and $f_i : E \to E_i$ be the projection, $i = 1, 2, \cdots, n$. Then S is an indecomposable self-basic serial ring such that $Sf_1, Sf_2, \cdots, Sf_n$ is a Kupisch series for S and

$$c({}_S Sf_i) = b_i = c({}_R Re_i) \quad \text{for all } i,$$

and ${}_R E_S$ defines a Morita duality between the category of all finitely generated left R-modules and the category of all finitely generated right S-modules.

J. K. Haack [5] has proved the following theorem.

Theorem 4.1. Every factor ring of an indecomposable serial ring with either a constant or a strictly increasing admissible sequence has a self-dual. (The admissible sequence of R is said to be strictly increasing if $b_{k+1} < b_{k+2} < \cdots < b_n < b_1 < \cdots < b_k$ for some k. Notice that the admissible sequence of R is constant if and only if R is quasi-Frobenius.)

As one of the applications of Theorem 3.1, we get the following theorem which is a generalization of Theorem 4.1.

Theorem 4.1. Assume that

(*) $$\#\{i \mid b_i \equiv 1 \pmod{n}\} \leq 1.$$

Then there exists a ring isomorphism $\Gamma : R \to S$ such that

$$\Gamma(e_i) = f_i \quad \text{for all } i.$$

In particular, if R satisfies the condition (*), then every factor ring of R has a self-duality.

Theorem 4.2 is proved by the induction on n.

For a suitable e_i, we can prove that eRe satisfies the condition (*), where $e = 1 - e_i = e_1 + \cdots + e_{i-1} + e_{i+1} + \cdots + e_n$. Let us put $f = f_1 + \cdots + f_{i-1} + f_{i+1} + \cdots + f_n$. Then $fSf = \mathrm{End}({}_{eRe}E(eRe/eJe))$ and there exists a ring isomorphism $\Gamma' : eRe \to fSf$ such that

$$\Gamma'(e_j) = f_j \quad \text{for } j \neq i,$$

by the induction hypothesis. Using Theorem 3.1 and the ring isomorphism Γ', we can construct a ring isomorphism $\Gamma : R \to S$ such that

$$\Gamma(e_j) = f_j \quad \text{for all } j.$$

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ON THE HEREDITARITY OF TORSION CLASSES*

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Let R be a ring with identity. A torsion class of R-modules is said to be cyclic-hereditary, following Ikeyama [3], if it is closed under submodules of cyclic R-modules. Not all cyclic-hereditary torsion classes are hereditary.

In this paper, we attempt to find necessary and sufficient conditions under which cyclic-hereditary torsion classes are hereditary. It is shown that torsion class T of R-modules is hereditary if and only if (i) T is cyclic-hereditary and (ii) for each M ($\neq$ 0) ε T there exists x ($\neq$ 0) ε M such that Rx ε T. We can provide examples to show that these conditions (i) and (ii) are independent with each other and each of them does not always imply the hereditarity of T.

§ 1. Throughout this paper, R is a ring with identity and R-modules are unitary left R-modules. R-mod denotes the category of all R-modules. For R-modules M and M', we use $M' \leqq M$ to denote that M' is a submodule of M. Especially $\underline{m} \leqq {}_R R$ means that $\underline{m}$ is a left ideal of R.

A _pretorsion class_ T of R-modules is a subclass of R-mod closed under factor modules and direct sums. In addition, if T is closed under extensions, then it is called a _torsion class_.

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* Dedicated to Professor K. Murata for the celebration his sixtieth birthday.

A torsion class which is closed under submodules is said to be hereditary.

For a class T of R-modules, let us put

$$L(T) = \{\underline{m} \leq {}_R R \mid R/\underline{m} \in T\}.$$

If T is a hereditary torsion class, then L(T) satisfies:

(1°) If $\underline{m} \in L(T)$ and $\underline{n}$ is a left ideal of R such that $\underline{m} \leq \underline{n}$, then $\underline{n} \in L(T)$.

(2°) If $\underline{m}$ and $\underline{n}$ belong to L(T), then $\underline{m} \cap \underline{n} \in L(T)$.

(3°) If $\underline{m} \in L(T)$ and $a \in R$, then $(\underline{m} : a) \in L(T)$.

(4°) If $\underline{m}$ is a left ideal of R and there exists $\underline{n} \in L(T)$ such that $(\underline{m} : a) \in L(T)$ for all $a \in \underline{n}$, then $\underline{m} \in L(T)$.

A family of left ideals of R satisfying the conditions (1°) to (4°) is called a left Gabriel topology on R. It is well-known that $T \to L(T)$ gives a bijective correspondence between hereditary torsion classes of R-modules and left Gabriel topologies on R. There exists, however, a non-hereditary torsion class T such that L(T) is left Gabriel. Such a torsion class can be characterized as one closed under submodules of cyclic R-modules and is called cyclic-hereditary [3, Theorem 5].

We refer to Stenström [8] for more information about torsion theories for R-mod.

§ 2. For a torsion class T of R-modules, as was shown by [3, Lemma 4], L(T) satisfies only (1°) and (4°) of the preceding section. In case R is commutative, it is easily seen that (1°) implies (3°) and hence L(T) satisfies (3°) (cf. [3, Corollary 6]). This, however, is not the case in general.

Example 2.1. For a field K, let $R = \begin{pmatrix} K & K \\ 0 & K \end{pmatrix}$. Then $I = \begin{pmatrix} 0 & K \\ 0 & K \end{pmatrix}$ is an idempotent ideal of R and $T = \{_R M \mid IM = M\}$ is a torsion class of R-modules. In this situation, L(T) coincides with $\{\underline{m} \leqq {}_R R \mid R = I + \underline{m}\}$ and does not satisfy (2°) as well as (3°). For, both $\begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix}$ and $\{\begin{pmatrix} a & a \\ 0 & 0 \end{pmatrix} \mid a \in K\}$ belong to L(T), but their intersection does not belong to L(T).

It is, therefore, natural to ask when L(T) satisfies (3°). Concerning this we have

Lemma 2.2. Let T be a class of R-modules closed under isomorphic images. Then L(T) satisfies (3°) if and only if T is closed under cyclic submodules of cyclic R-modules.

Proof. For a left ideal $\underline{m}$ of R and $a \in R$, $0 \to R/(\underline{m}:a) \to R/\underline{m}$ is exact, from which the lemma follows. //

Especially we have

Corollary 2.3 ([3, Theorem 5]). Let T be a torsion class of R-modules. Then L(T) is left Gabriel if and only if T is cyclic-hereditary.

As was shown in [3, Example 4], not all cyclic-hereditary torsion classes are hereditary. We now consider conditions under which cyclic-hereditary torsion classes are hereditary.

For a class T of R-modules, let us put

$$T_o = \{_R M \mid Rx \in T \text{ for all } x \in M\}.$$

If T is a pretorsion class, then T_o coincides with

$$\{_RM \mid M' \in T \text{ for all } M' \leq M\}.$$

This is the largest class of R-modules which is contained in T and is closed under submodules. If T is a torsion class, T_o becomes the largest hereditary torsion class contained in T.

<u>Proposition</u> 2.4. For a pretorsion class T of R-modules, the following conditions are equivalent:

(1) T is closed under submodules.

(2) (i) T is closed under cyclic submodules of cyclic R-modules.

(ii) For each $M \in T$ and each $x \in M$ there exists a cyclic submodule of M in T containing x.

(3) (i) T is closed under cyclic submodules of cyclic R-modules.

(ii) For each $M \in T$ and each $x \in M$ there exist $y \in M$ and $a \in R$ such that $Ry \in T$ and $\ell_R(x) = \ell_R(ay)$.

<u>Proof</u>. The implications (1) → (2) → (3) are obvious. (3) → (1). Let $x \in M \in T$. Then, by assumption, $Ry \in T$ and $\ell_R(x) = \ell_R(ay)$ for some $y \in M$ and $a \in R$. Since $\ell_R(y) \in L(T)$ and L(T) satisfies (3°) by Lemma 2.2, $\ell_R(x) = (\ell_R(y) : a) \in L(T)$. Hence $Rx \in T$ and thus $M \in T_o$. //

§ 3. Let L be a family of left ideals of R and let

$$L_o = \{\underline{m} \leq {}_RR \mid (\underline{m} : a) \in L \text{ for all } a \in R\}.$$

This is the largest family of left ideals of R which satisfies (3°) and is contained in L. Furthermore, if L satisfies (1°), then so does L_o. If L satisfies (4°), then so does L_o and L_o is the largest left Gabriel topology contained in L.

For each R-module M, define t(M) and $t_o(M)$ to be

$$t(M) = \{x \in M \mid \ell_R(x) \in L\}$$

and

$$t_o(M) = \{x \in M \mid \ell_R(x) \in L_o\},$$

respectively. Both t(M) and $t_o(M)$ are merely subsets of M and $t_o(M) \subset t(M)$. If L satisfies (4°), then $t_o(M)$ is the largest submodule of M contained in t(M), and t_o may be regarded as a left exact radical of R-mod corresponding to the left Gabriel topology L_o.

Even if L satisfies (4°), t(M) is not always a submodule of M unlike $t_o(M)$.

Example 3.1. For a field K, let $R = \begin{pmatrix} K & K \\ 0 & K \end{pmatrix}$. Put $I = \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix}$ and $L = \{\underline{m} \leq {}_R R \mid I \leq \underline{m}\}$. Then I is an idempotent left ideal, but not a right ideal. Hence L satisfies (1°) to (4°) except for (3°). However, $t(R) = \begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix}$ and is not a left ideal of R.

The following proposition gives conditions for t(M) to be a submodule of M.

Proposition 3.2. Let L be a family of left ideals of R and let t(M) and $t_o(M)$ be as above. If L satisfies (4°), then the following conditions are equivalent:

(1) t(M) is a submodule of M for all R-modules M.

(2) $t(M) = t_o(M)$ for all R-modules M.

(3) $L = L_o$.

(4) L satisfies (3°).

Moreover, if L satisfies (4°) and if one of the conditions (1) to (4) holds, then t may be regarded as a left exact radical of R-mod and the corresponding left Gabriel topology on R coincides with L.

Proof. The equivalences (1) $\rightleftarrows$ (2) and (3) $\rightleftarrows$ (4) are obvious. (2) → (3). Assume (2). Then t may be regarded as a left exact radical of R-mod and the corresponding left Gabriel topology on R coincides with L. Hence L must be left Gabriel. (3) → (2) is also trivial. //

Proposition 3.3. For a torsion class T of R-modules, let $L(T) = \{\underline{m} \leqq {}_R R \mid R/\underline{m} \in T\}$ and $t(M) = \{x \in M \mid \ell_R(x) \in L(T)\}$ for each R-module M. Then the following conditions are equivalent:

(1) t(M) is a submodule of M for all R-modules M.

(2) L(T) is left Gabriel.

(3) T is cyclic-hereditary.

Moreover, if one of the conditions (1) to (3) holds, then t may be regarded as a left exact radical of R-mod, the corresponding left Gabriel topology on R coincides with L(T) and further $T(t) = T_o$, where $T(t) = \{{}_R M \mid t(M) = M\}$.

Proof. We may only show the last statement. $M \in T(t)$ means that $\ell_R(x) \in L(T)$ for all $x \in M$, or equivalently, $Rx \in T$ for all $x \in M$, which shows that $M \in T_o$. //

Now we come to the main theorem of this paper. We need the following lemma which is a slight generalization of [9, Lemma 2.1].

Lemma 3.4. Let r and s be an idempotent preradical and a radical of R-mod, respectively. Then $r \leq s$ if and only if, for each R-module M ($\neq 0$) with $r(M) = M$, we have $s(M) \neq 0$.

Proof. The "only if" part is obvious. To prove the "if" part, assume that $r \nleq s$. Then there exists an R-module M such that $r(M) \nleq s(M)$. Hence $s(r(M)) \neq r(M)$. It follows that $r(M)/s(r(M)) \neq 0$ and $r(r(M)/s(r(M))) = r(M)/s(r(M))$. By assumption, $s(r(M)/s(r(M))) \neq 0$, a contradiction. //

Theorem 3.5. For a torsion class T of R-modules, the following conditions are equivalent:

(1) T is hereditary.

(2) (i) T is cyclic-hereditary.

(ii) For each M ($\neq 0$) ε T there exists x ($\neq 0$) ε M such that $Rx \in T$.

Proof. (1) $\rightarrow$ (2) follows from Proposition 2.4.
(2) $\rightarrow$ (1). Assume (2). We show that $T \subset T_o$. To do this, for each R-module M, put $t(M) = \{x \in M \mid \ell_R(x) \in L(T)\}$. Then, by Proposition 3.3, t may be regarded as a left exact radical of R-mod and $T(t) = T_o$. By assumption, for each M ($\neq 0$) ε T, there exists x ($\neq 0$) ε M such that $Rx \in T$. Hence $\ell_R(x) \in L(T)$ and $x \in t(M)$, which means that $t(M) \neq 0$. Thus, by Lemma 3.4, $T \subset T_o$. //

It is to be noted, in the preceding theorem, that the conditions (2)(i) and (2)(ii) are independent with each other. [3, Example 4] shows that (2)(i) does not imply (2)(ii) in general. The following example shows that (2)(ii) does not always imply (1) as well as (2)(i).

Example 3.6. Let $e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ in Example 2.1. Then, for any M ($\neq 0$) $\in T$, $M = IM = ReM$. Hence $eM \neq 0$ and so there exists $x \in M$ such that $ex \neq 0$. Since $ex = e(ex)$, $Rex \in T$. However, $L(T)$ does not satisfy (3°), as was pointed out in Example 2.1, and thus T is not (cyclic-)hereditary.

Corollary 3.7. Let R be commutative. Then a torsion class T of R-modules is hereditary if and only if, for each M ($\neq 0$) $\in T$, there exists x ($\neq 0$) $\in M$ such that $Rx \in T$.

Corollary 3.8 (cf. [5, Theorem 4]). For an ideal I ($\neq 0$) of R and the torsion class $T = \{{}_RM \mid IM = M\}$, the following conditions are equivalent:

(1) T is hereditary.

(2) (i) $\{\underline{m} \leqq {}_RR \mid R = I + \underline{m}\}$ is left Gabriel.

(ii) For each sequence $a_1, a_2, \ldots$ of elements of I, there exists $n > 0$ such that $R = I + \ell_R(a_1 a_2 \ldots a_n)$.

(3) (i) T is cyclic-hereditary.

(ii) For each R-module M ($\neq 0$) $\in T$ there exists x ($\neq 0$) $\in M$ such that $x \in Ix$.

Proof. (1) → (2). Let F be the free R-module with basis $x_1, x_2, \ldots$. For each sequence $a_1, a_2, \ldots$ of elements of I, let

$$y_n = x_n - a_n x_{n+1} \text{ for each } n > 0$$

and G the submodule of F generated by $y_1, y_2, \ldots$.

Then the factor module F/G is in T. Hence $R(x_1 + G) \in T$ and so $(G : x_1) \in L(T)$. As is easily seen, $L(T) = \{\underline{m} \leq {}_R R \mid R = I + \underline{m}\}$ and $(G : x_1) = \ell_R(a_1 a_2 \ldots a_n)$ for some $n > 0$.
(2) → (3). Let M ($\neq 0$) be an R-module and suppose that $IM = M$. As was shown in the proof of [5, Theorem 4], there exist sequences $a_1, a_2, \ldots$ of elements of I and $x_1, x_2, \ldots$ of elements of M such that $a_1 a_2 \ldots a_i x_i \neq 0$ for all $i > 0$. By assumption, $\ell_R(a_1 a_2 \ldots a_n) \in L(T)$ for some $n > 0$. Hence $Ra_1 a_2 \ldots a_n x_n$ belongs to T. (3) → (1) follows from Theorem 3.5. //

§ 4. A left Gabriel topology L on R is called a 1-topology if it has a basis B consisting of principal left ideals of R.

For a subset S ($\neq \phi$) of R, define a family L of left ideals of R to be

$$L = \{\underline{m} \leq {}_R R \mid \underline{m} \cap S \neq \phi\}.$$

If L is a left Gabriel topology on R, then L becomes a 1-topology with basis $\{Ru \mid u \in S\}$. Moreover, any 1-topology L' on R with basis B can be obtained in this way, i.e., $L' = \{\underline{m} \leq {}_R R \mid \underline{m} \cap S \neq \phi\}$, where $S = \{u \in R \mid Ru \in B\}$.

We now ask for conditions on S for L to be left Gabriel.

Lemma 4.1. If S satisfies:

(S0) For any $u, v \in S$ there exists $a \in R$ such that $auv \in S$, then L satisfies (4°).

Proof. Let $\underline{n} \in L$ and $\underline{m}$ a left ideal of R such that $(\underline{m} : a) \in L$ for all $a \in \underline{n}$. Then there exist $v \in \underline{n} \cap S$ and $u \in (\underline{m} : v) \cap S$.

By assumption, $c(uv) \in S$ for some $c \in R$. Thus $c(uv) \in \underline{m} \cap S$ and $\underline{m} \in L$. //

We note that the converse of this lemma is not true in general.

Example 4.2. Cozzens [2] has constructed an example of a left V-ring R which is a principal left ideal domain, but not a field. Take a non-trivial left ideal I of R and let $I = Ra$. Put $S = \{a\}$ and $L = \{\underline{m} \leqq {}_R R \mid \underline{m} \cap S \neq \phi\}$. Then L coincides with $\{\underline{m} \leqq {}_R R \mid I \leqq \underline{m}\}$ and satisfies (4°). However, S cannot satisfy (S0).

Concerning (3°) we have

Lemma 4.3. The following conditions are equivalent:

(1) L satisfies (3°).

(2) $(Ru : a) \in L$ for all $u \in S$ and $a \in R$.

(3) S satisfies:

(S1) For any $a \in R$ and $u \in S$ there exist $b \in R$ and $v \in S$ such that $bu = va$.

Proof. (1) → (2) is obvious. (2) → (3). Let $a \in R$ and $u \in S$. Then $(Ru : a) \cap S \neq \phi$. Thus $va = bu$ for some $b \in R$ and $v \in S$. (3) → (1). Let $\underline{m} \in L$ and $a \in R$. Then there exists $u \in \underline{m} \cap S$ and by assumption $bu = va$ for some $b \in R$ and $v \in S$. Hence $v \in (\underline{m} : a) \cap S$ and thus $(\underline{m} : a) \in L$. //

Combining Lemma 4.1 with Lemma 4.3, we have

Proposition 4.4. For a subset S $(\neq \phi)$ of R, let $L = \{\underline{m} \leqq {}_R R \mid \underline{m} \cap S \neq \phi\}$ and $t(M) = \{x \in M \mid \ell_R(x) \in L\}$ for each R-module M. Then the following conditions are equivalent:

(1) S satisfies (S0) and (S1).

(2) L is a 1-topology with basis $\{Ru \mid u \in S\}$.

(3) t(M) is a submodule of M for each R-module M and t may be regarded as a radical of R-mod.

Moreover, if one of the conditions (1) to (3) holds, then t is left exact and the corresponding left Gabriel topology is L.

Proof. (1) → (2) follows from Lemmas 4.1 and 4.3. (2) → (3) is well-known. (3) → (1). Let $u, v \in S$. Then $u(1 + Ru) = 0$ and so $R/Ru \in T(t)$. Likewise $R/Rv \in T(t)$. Moreover, Rv/Ruv is a homomorphic image of R/Ru and hence $Rv/Ruv \in T(t)$. Since t is a radical, T(t) is closed under extensions. The exactness of the sequence $0 \to Rv/Ruv \to R/Ruv \to R/Rv \to 0$ implies that $R/Ruv \in T(t)$. There exists $w \in S$ such that $w(1 + Ruv) = 0$ and $auv = w \in S$ for some $a \in R$. Thus S satisfies (S0) and, by Lemma 4.1, L satisfies (4°). The proposition then follows from Proposition 3.2. //

We now specialize the above discussion. For an ideal I ($\neq 0$) of R, we let

$$S = \{1 - b \mid b \in I\}.$$

Then S is multiplicatively closed and hence satisfies (S0) trivially. The class $T = \{{}_RM \mid IM = M\}$ of R-modules is a torsion class and, in this situation, L(T) coincides with $\{\underline{m} \leq {}_RR \mid \underline{m} \cap S \neq \phi\}$.

Proposition 4.5. For an ideal I ($\neq 0$) of R and the torsion class $T = \{{}_RM \mid IM = M\}$, let $S = \{1 - b \mid b \in I\}$

and $t(M) = \{x \in M \mid \ell_R(x) \in L(T)\}$ for each R-module M. Then the following conditions are equivalent:

(1) S satisfies (S1).

(2) L(T) is a 1-topology with basis $\{Ru \mid u \in S\}$.

(3) T is cyclic-hereditary.

(4) t(M) is a submodule of M for all R-modules M.

(5) t(M) is a submodule of M for all R-modules $M \in T$.

(6) $t(R/Ru) = R/Ru$ for all $u \in S$.

Moreover, if one of the conditions (1) to (6) holds, then t may be regarded as a left exact radical of R-mod,the corresponding left Gabriel topology on R coincides with L(T) and further $T(t) = \{_RM \mid x \in Ix \text{ for all } x \in M\}$.

Proof. The equivalence of (1) to (4) and (6) follows from Proposition 3.3 and Lemma 4.3. (4) → (5) is trivial. So we may prove (5) → (6).

Assume that t(M) is a submodule of M for $M \in T$. Let $u \in S$. If $R/Ru = 0$, then clearly $t(R/Ru) = R/Ru$. Suppose now that $R/Ru \neq 0$. Put $u = 1 - b$ for some $b \in I$, then $R = I + Ru$ and so $I(R/Ru) = R/Ru$. By assumption, t(R/Ru) is a submodule of R/Ru. Since $u(1 + Ru) = 0$, $1 + Ru \in t(R/Ru)$ and $a + Ru = a(1 + Ru) \in t(R/Ru)$ for all $a \in R$. Hence $t(R/Ru) = R/Ru$. //

Theorem 4.6. For an ideal I ($\neq 0$) of R and the torsion class $T = \{_RM \mid IM = M\}$, let $t(M) = \{x \in M \mid \ell_R(x) \in L(T)\}$ for each R-module M. Then the following conditions are equivalent:

(1) T is hereditary.

(2) t(M) is a nonzero submodule of M for all R-modules

M ($\neq$ 0) ε T.

(3) t(M) = M for all R-modules M ε T.

Proof. (1) $\rightarrow$ (3) follows from Proposition 3.3 since $T = T_o$ and (3) $\rightarrow$ (2) is trivial. (2) $\rightarrow$ (1). By Proposition 4.5, T is cyclic-hereditary and hence, by Theorem 3.5, T is hereditary. //

We note that t(M) $\leq$ IM for all R-modules M. Therefore, if t(I) = I holds, then by [4, Lemma 3.1] we have t(M) = IM and hence t(M) = M for all R-modules M ε T. By Theorem 4.6, T is hereditary. Especially we have

Corollary 4.7 (cf. [1, Theorem 6] and [7, Lemma 1.2]). For an idempotent ideal I ($\neq$ 0) of R, T is hereditary if and only if t(I) = I.

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ON REGULAR SELF-INJECTIVE RINGS

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Introduction. Let R be a ring with identity. R is said to be right bounded if every essential right ideal contains a non-zero two-sided ideal, which is essential as a right ideal.

In [2], we have determined the regular right bounded and right self-injective ring. More precisely we have proved the following theorem.

Theorem 1. Let R be a regular, right self-injective ring. Then R is right bounded if and only if

$R = \prod M_{n(i)}(T_i) \times \prod \mathrm{End}_{D_j}(V_j)$ where each T_i is an abelian regular self-injective ring and each V_j is a right vector space over a division ring D_j.

In this report, as an application of theorem 1, we shall give a necessary and sufficient condition for the maximal right quotient ring Q(R) of R to be type I_f.

Preliminaries. Let R be a regular ring. Then R is abelian provided all idempotent in R are central, and R is said to be directly finite if $xy = 1$ implies $yx = 1$, for all $x, y \in R$. Furthermore R is called a regular ring of bounded index of nilpotence if for any nilpotent element x of R, there exists a positive integer N such that $x^N = 0$.

A regular right self-injective ring R is called type I if R contains an idempotent e such that eR is faithful right R-module and eRe is an abelian regular ring. We call R type I_f if R is type I and directly finite.

We shall state basic results used in the following section.

Proposition 1 ([1, Theorem 3.2]). For a regular ring R, the following conditions are equivalent.

(1) R is abelian.

(2) R has no, non-zero nilpotent elements.

(3) All right (left) ideals of R are two-sided.

(4) Every non-zero right (left) ideal of R contains a non-zero central idempotent.

Proposition 2 ([1, Theorem 10.24]). A regular right self-injective ring R is type I_f if and only if R is isomorphic to a direct product of full matrix rings over abelian regular self-injective rings.

Proposition 3 ([2, Proposition 4]). Let R be a regular ring of bounded index of nilpotence. Then R is right bounded and every non-zero two-sided ideal of R contains a non-zero central idempotent of R.

<u>Application of Theorem 1.</u> Let R be a regular ring. We denote the maximal right quotient ring of R by Q(R).

In this section, we shall prove the following theorem.

Theorem 2. For a regular ring R, $Q(R)$ is type I_f if and only if

(1) R is right bounded.

(2) Every non-zero two-sided ideal of R contains a non-zero central idempotent of R.

In order to prove this theorem, we prepare the next lemma.

Lemma 1 ([2, Lemma 2]). Let R be a right bounded and regular ring such that every non-zero ideal contains a non-zero central idempotent of R. Then $Q(R)$ is also right bounded.

Proof. Let I be an essential right ideal of $Q(R)$. Then clearly $I \cap R \subseteq_e R_R$. Since R is right bounded, there exists a non-zero ideal J such that $J \subseteq_e I \cap R$. And from the assumption, J contains a non-zero central idempotent e of R. Note that e is also central in $Q(R)$. Thus I contains a non-zero central idempotent in $Q(R)$. Let H be the ideal generated by all the central idempotents in I. We claim that $H \subseteq_e Q_Q$. Assume not, then $l_Q(H)$, the left annihilator ideal of H in Q, is not zero and $H \cap l_Q(H) = 0$ since R is semi-prime. Hence there exists a non-zero central idempotent f in $J \cap l_Q(H)$ since $J \cap l_Q(H)$ is a non-zero two-sided ideal in R. On the other hand, f is in H because $f \in J \subseteq I$. But this contradicts that

$H \cap l_Q(H) = 0$. Consequently, H is an essential right ideal of Q(R), as claimed. Therefore Q(R) is right bounded.

Proof of Theorem 2. If Q(R) is type I_f, then by Proposition 2, Q(R) is isomorphic to a direct product of full matrix rings over abelian regular self-injective rings and by proposition 1, each full matrix ring over an abelian regular ring has bounded index of nilpotence. Thus there exist orthogonal central idempotents $e_1, e_2, \ldots$ in Q(R) such that $Q(R) = \prod e_n Q(R)$ and each $e_n Q(R)$ has bounded index of nilpotence. First we show that (2) holds. Let I be a non-zero two-sided ideal of R. Since $\oplus e_n Q(R)$ is an essential right ideal of Q(R), we have that $I \cap e_n Q(R) \neq 0$ for some positive integer n. On the other hand, $e_n R$ has bounded index of nilpotence because that $e_n Q(R)$ has bounded index. Thus $e_n R \cap I$ contains a non-zero central idempotent f of $e_n R$ by proposition 3. Now it is easy to see that f is a central idempotent of R. Hence (2) holds.
Next we shall show that (1) holds. Let J be an essential right ideal of R. Then for each n, we have that $J \cap e_n R \neq 0$. Thus $J \cap e_n R$ is an essential right ideal of $e_n R$.
On the other hand, $e_n R$ has bounded index of nilpotence, so $e_n R$ is right bounded. Therefore $J \cap e_n R$ contains a non-zero central idempotent in $e_n R$. Consequently, J contains a non—zero central idempotent in R. Let H be the ideal generated by all the central idempotents in J. We claim that H is an essential right ideal of R. For a non-zero element x in R, there exists a positive integer n such that $e_n x \neq 0$. Thus there exists a central idempotent

f in J such that $fe_n x \neq 0$ and in J since $e_n R$ is right bounded and the ideal generated by all central idempotents in $J \wedge e_n R$ is essential. Hence $fe_n x$ is in H. This implies that H is an essential right ideal of R, so (1) holds. Conversely, we assume that (1), (2) hold. Then by Lemma 1, Q(R) is right bounded. Hence Theorem 1 shows that $Q(R) = Q_1 \times Q_2$ where Q_1 is type I_f and Q_2 is a direct product of right full linear rings. Furthermore (2) states that every non-zero ideal of Q(R) contains a non-zero central idempotent. But it is easily seen that Q_2 does not satiesfy this property. Therefore we can conclude that Q(R) is type I_f.

Corollary. Let R be a regular, right self-injective ring. Then R is isomprphic to a finite direct product of full matrix ring over abelian regular self-injective rings if and only if

(1) R is right bounded.

(2) All prime ideals of R are maximal.

Proof. It is clear by Theorem 1 and 2.

Remark. Without (1) or (2), Theorem 2 can fail, as the following examples show.

A regular right bounded ring does not necessarily have the maximal right quotient ring such that type I_f. For example, choose a field F, let V be a countable-infinite dimensional vector space over F and set $R = End_F(V)$ and $M = \{x \in R \mid dim_F(xV) < \infty\}$. Then clearly

R is regular, right self-injective ring. Given any $x \in R - M$, we have $\dim_F(xV) = \dim_F(V)$ and so $xV \cong V$, whence $xR \cong R_R$. This shows that R is not directly finite. On the other hand, R is right bounded by Theorem 1. Furthermore, a regular ring R, which every non-zero ideal contains a non-zero central idempotent of R, also does not necessarily have the maximal right quotient ring such that type I_f. For example, choose fielfs $F_1, F_2, \ldots$, set $R_n = M_n(F_n)$ for all $n = 1,2, \ldots$, and set $R = \prod R_n$. Let M be a maximal ideal of R which contains $\oplus R_n$. Then R/M be a simple right and left self-injective regular ring, but not type I_f by [1, Example 10.7].

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Reflection Functors and Auslander-Reiten Translations for Self-Injective Algebras

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This report is an introductory version of reflection functors stated in the title and a generalization of the relation between them and Auslander-Reiten translations in hereditary algebras. Further we shall state the limit of announced results and a view for generalizations at end. The detailed proofs will appear elsewhere [7].

Let $\Sigma = (F_k, {}_kM_i)$, $k,i \in \Gamma_o$, be a K-species, i.e. Γ_o is a finite set of indices, F_k division rings with center K and ${}_kM_i$ finite K-dimensional F_k-F_i-bimodules on which K operates centrally. Then Σ associates with a valued graph Γ having an orientation Λ and a hereditary algebra A. Cf. Dlab-Ringel [4], pp.1-5.

Assume that (Γ,Λ) is connected and without cycle. Then A is a two-sided indecomposable, hereditary and basic finite dimensional K-algebra. For the orientation Λ and a vertex $j \in \Gamma_o$, we can define a new orientation $\sigma_j \Lambda$ by reversing the direction of arrows along all edges containing j. Since Γ has no cycle, there is an admissible ordering of vertices $k_1,k_2,\ldots,k_n$ with respect to Λ, i.e. each vertex k_t is a sink with respect to the orientation $\sigma_{k_{t-1}} \sigma_{k_{t-2}} \cdots \sigma_{k_1} \Lambda$.

In the case $j \in \Gamma_o$ is a sink (resp. a source) with respect to Λ, let us denote by Γ_o^j the set of starting (end) vertices of all arrows containing j. Then we can define a new species $\sigma_j\Sigma = (F_k, {}_kN_i)$, $k,i \in \Gamma_o$, such that ${}_kN_i = {}_kM_i$ for $k,i \notin \Gamma_o^j$ and ${}_jN_i = \mathrm{Hom}_K({}_iM_j, K)$ (resp. ${}_iN_j = \mathrm{Hom}_K({}_jM_i, K)$) for $i \in \Gamma_o^j$. And we shall denote the corresponding valued graph and the algebra by $(\Gamma, \sigma_j \Lambda)$ and $\sigma_j A$ respectively.

Denote $L(\Sigma)$ and mod-A the category of all representations of Σ in K and the category of all finitely generated right A-modules respectively. Given a sink (resp. a source) k of Σ, the reflection functors s_k^+ (resp. s_k^-) is defined as a functor: $L(\Sigma) \longrightarrow L(\sigma_k\Sigma)$, equivalently mod-A$\longrightarrow$mod-$\sigma_k$A. It is well known that for an object X of $L(\Sigma)$ (an A-module X) $s_k^+(X)$ (resp. $s_k^-(X)$) is non-zero indecomposable iff so is X, provided X is not isomorphic to a simple representation (a simple A-module) L_k which corresponds to $k \in \Gamma_o$, i.e. $L_k = (X_j; {}_i\phi_j : X_j \otimes_{K_j} {}_jM_i \longrightarrow X_i)$, $X_k = F_k$, $X_j = 0$ for $j \neq k$ and all ${}_i\phi_j = o$. And $s_k^+(L_k) = 0$ (resp. $s_k^-(L_k) = 0$).

Denote by $\underline{\mathrm{mod}}$-A (resp. $\overline{\mathrm{mod}}$-A) the stable category of mod-A, i.e. the objects of $\underline{\mathrm{mod}}$-A (resp. $\overline{\mathrm{mod}}$-A) are same with ones of mod-A, but the groups of morphisms are residue class groups of morphisms in mod-A by the subgroups generated by morphisms which factor through projective (resp. injective) A-modules. It is well known the Auslander-Reiten translation DT_r is a functor from $\underline{\mathrm{mod}}$-A to $\overline{\mathrm{mod}}$-A.

In case of A being a tensor algebra associated with a species Σ with no cycle Brenner and Butler [3] proved that $DT_r \cong s_{k_n}^+ s_{k_{n-1}}^+ \dots s_{k_1}^+$, where the composition goes through an admissible sequence $k_1, \dots, k_n \in \Gamma_o(\Sigma)$. This is the theorem which in this note I want to generalize for the case of self-injective algebras.

Put $\mathrm{Hom}_K(A,K) = D(A)$. Then $D(A)$ is an injective cogenerator right and left A-module. Further the full subcategories of all projective A-modules and all injective A-modules are equivalent by $- \otimes_A D(A)$ and $\mathrm{Hom}_A(D(A),-)$ with natural equivalences $\delta : \mathrm{Hom}_A(D(A),-) \otimes_A D(A) \cong 1_{\text{mod-}A}$ and $\gamma: 1_{\text{mod-}A} \cong \mathrm{Hom}_A(D(A), - \otimes_A D(A))$.

The trivial extension R of A by D(A), denoted by $A \ltimes D(A)$, is an algebra defined as follows: $R = A \oplus D(A)$ as an additive group and the multiplication is defined by $(a,q)(a',q') = (aa',aq'+qa')$ for $(a,q),(a',q') \in R$. It is well known that an R-module X has expressions $(X_A \otimes_A D(A)_A \xrightarrow{\phi} X_A)$ and $(X_A \xrightarrow{\psi} [{}_A D(A)_A, X_A]_A)$ with $\phi \cdot \phi \otimes D(A) = 0$ and $[D(A), \psi] \cdot \psi = 0$ which are considered as objects of mod-$A \ltimes (- \otimes D(A))$ and $[D(A),-] \rtimes$ mod-A respectively. Cf [8]. Here ϕ corresponds to ψ in an adjoint relation $\mathrm{Hom}_A(X \otimes D(A),X) \cong \mathrm{Hom}_A(X,\mathrm{Hom}_A(D(A),X))$, and $x(a,q) = xa + \phi(x \otimes q) = xa + \psi(x)(q)$ for $x \in X$, $a \in A$ and $q \in D(A)$. Denote $\mathrm{Im}\,\phi$ by V $(=XD(A))$. Since A is hereditary, V is injective and we have a decomposition $X \cong U \oplus V$ as A-modules, where $U \cong X/XD(A)$. Then $\phi(XD(A) \otimes D(A)) = 0$ (resp. $\psi(XD(A)) = 0$) and hence ϕ (resp. ψ) may be identified with $\phi | U \otimes D(A)$ (resp. $\psi | U$). According to $\phi = 0$ or $\phi \neq 0$ X is said to be of 1st kind or 2nd kind. Hereafter we shall use $(U \otimes D(A) \xrightarrow{\phi} V)$ (resp. $(U \xrightarrow{\psi} [D(A),V]))$ as a canonical expression of X in place of $(X \otimes D(A) \xrightarrow{\phi} X)$ (resp. $(X \xrightarrow{\psi} [D(A),X])$). X is of 1st kind iff X is an A-module and then X is indecomposable iff it is indecomposable as an A-module. For indecomposable R-modules X of 2nd kind the author proved in [6] the following theorem.

Theorem 1. For the case ϕ (resp. ψ) $\neq 0$, an R-module is indecomposable iff either one of the following conditions (i) and (ii) is satisfied:

(i) ϕ (resp. ψ) is an isomorphism and U_A is indecomposable projective.

(ii) $\phi: U \otimes D(A) \longrightarrow V$ (resp. $\psi : U \longrightarrow [D(A),V])$ is an epimorphism (resp. a monomorphism) but not an isomorphism, Ker ϕ (resp. Cok ψ) is indecomposable and Ker ϕ_A (resp. Im ψ_A) is large submodule of $U \otimes D(A)$ (resp. small submodule of $[D(A),V]$).

In the case (i) X is a projective and injective R-module.

Proposition 1. Let $(U \otimes D(A) \xrightarrow{\phi} V)$ and $(U \xrightarrow{\psi} [D(A),V])$ be the canonical expressions of the same indecomposable R-module X of 2nd kind. Then

Cok $\psi \cong T_r D$ (Ker ϕ) and Ker $\phi \cong DT_r$(Cok ψ).

Proof. By Theorem 1 $U \longrightarrow [D(A),V] \longrightarrow C \longrightarrow 0$ is a minimal projective resolutuion of C_A. Hence we have an exact sequence

$$0 \longrightarrow DT_r(C) \longrightarrow U \otimes D(A) \longrightarrow [D(A),V] \otimes D(A) \longrightarrow 0.$$

But $[D(A),V] \otimes D(A) \cong V$ and $\phi \cong \delta \cdot \psi \otimes D(A)$.

Here it is to be noted that R is a symmetric (i,e. self-injective) algebra. Therefore $\underline{\text{mod}}$-R = $\overline{\text{mod}}$-R .

From now on we shall consider Auslander-Reiten translations which are the endofunctors on $\underline{\text{mod}}$-R, and in order to emphasize the difference from $DT_r : \underline{\text{mod}}\text{-A} \longrightarrow \overline{\text{mod}}\text{-A}$ and $T_rD : \overline{\text{mod}}\text{-A} \longrightarrow \underline{\text{mod}}\text{-A}$ we shall denote them DT_r^R and T_rD^R respectively. And the relation between DT_r and DT_r^R is characterized by the following theorems:

Theorem 2. Let X be a non-projective indecomposable R-module.

(i) If X is a non-projective indecomposable A-module, then $DT_r^R(X) \cong DT_r(X)$.

(ii) Let X be a projective indecomposable A-module . If $X \otimes D(A)_A$ is non-projective, then $\operatorname{Ker} \lambda \cong DT_r(X \otimes D(A))$ provided $DT_r^R(X) = (U \otimes D(A) \xrightarrow{\lambda} V)$. If $X \otimes D(A)_A$ is projective, then $DT_r^R(X) \cong X \otimes D(A) \otimes D(A)$.

(iii) Let $X = (U \otimes D(A) \xrightarrow{\phi} V)$ with $\phi \neq 0$ and $DT_r^R(X) = (U' \otimes D(A) \xrightarrow{\lambda} V')$. If $\operatorname{Ker} \phi_A$ is not projective, then $\operatorname{Ker} \lambda \cong DT_r(\operatorname{Ker} \phi)$.

(iv) Let $X = (U \otimes D(A) \xrightarrow{\phi} V)$ with $\phi \neq 0$. If $\operatorname{Ker} \phi_A$ is projective, then $DT_r^R(X) \cong \operatorname{Ker} \phi \otimes D(A)$.

Theorem 3. Let X be an indecomposable R-module.

(i) If X be a non-injective indecomposable A-module, then $T_rD^R(X) \cong T_rD(X)$.

(ii) Let X be an injective indecomposable A-module. If $[DA),X]$ is non-injective, then $\operatorname{Cok} \zeta \cong T_rD([D(A),X])$ provided $T_rD^R(X) = (U \xrightarrow{\zeta} [D(A),V])$. If $[D(A),X]$ is injective, then $T_rD^R(X) \cong [D(A),[D(A),X]]$.

(iii) Let $X = (U \xrightarrow{\psi} [D(A),V])$ with $\psi \neq 0$ and $T_rD(X) \cong (U' \xrightarrow{\zeta} [D(A),V'])$. If $\operatorname{Cok} \psi_A$ is not injective, then $\operatorname{Cok} \zeta \cong T_rD(\operatorname{Cok} \psi)$.

(iv) Let $X = (U \xrightarrow{\psi} [D(A),V])$ with $\psi \neq 0$. If $\operatorname{Cok} \psi_A$ is injective, then $T_rD^R(X) \cong [D(A),\operatorname{Cok} \psi]$.

As was stated before, for a valued graph (Γ, Λ) without cycles associated with a species Σ we have hereditary algebras $A, \sigma_{k_1} A, \sigma_{k_2}\sigma_{k_1} A, \ldots$, and reflection functors $s_{k_j}^{+}$:

mod-$\sigma_{k_{j-1}} \ldots \sigma_{k_1} A \longrightarrow$ mod-$\sigma_{k_j}\sigma_{k_{j-1}} \ldots \sigma_{k_1} A$ for an admissible sequence $k_1, \ldots, k_n \in \Gamma(\Sigma)$.

In this case we can construct trivial extensions $A \ltimes D(A), \sigma_{k_1} A \ltimes D(\sigma_{k_1} A), \sigma_{k_2}\sigma_{k_1} A \ltimes D(\sigma_{k_2}\sigma_{k_1} A), \ldots$, which in the following we shall denote by $R, \sigma_{k_1} R, \sigma_{k_2}\sigma_{k_1} R, \ldots$.

Then it arises naturally a question whether there exist functors $S^+_{k_j}$: mod-$\sigma_{k_{j-1}} \ldots \sigma_{k_1} R \longrightarrow$ mod-$\sigma_{k_j}\sigma_{k_{j-1}} \ldots \sigma_{k_1} R$ such that $DT^R_r \cong S^+_{k_n} S^+_{k_{n-1}} \ldots S^+_{k_1}$. We shall show in the remaining part of this note how to define such functors and show the outline of the proof of the isomorphism.

In Theorem 3 for an indecomposable R-module X of 1st kind $DT^R_r(X) \cong DT_r(X)$ except X_A is projective, when $DT_r(X)$ vanishes and $DT^R_r(X) \cong (U \otimes D(A) \xrightarrow{\phi} V)$ is of 2nd kind and Ker $\phi \cong DT_r(X \otimes D(A))$ or $X \otimes D(A) \otimes D(A)$ according to $X \otimes D(A)$ is non-projective or projective.

This fact suggests us the following definition of S^+_k.

Definition 1. Let k be a sink of (Γ, Λ) and X a non-projective indecomposable R-module. An indecomposable $\sigma_k R$-module $S^+_k(X)$ is defined as follows:

(i) If X is of 1st kind and not isomorphic to L_k which is the simple R-module corresponding to the sink k, then $S^+_k(X) \cong s^+_k(X)$.

(ii) If X is isomorphic to L_k, then Ker $\phi' \cong s^+_k(L_k \otimes D(A))$ provided $S^+_k(X) = (U' \otimes D(\sigma_k A) \xrightarrow{\phi'} V')$.

(iii) If X is of 2nd kind and $X = (U \otimes D(A) \xrightarrow{\phi} V)$ and if Ker $\phi \not\cong L_k$, then Ker $\phi' \cong s^+_k(\text{Ker } \phi)$ provided

$S_k^+(X) = (U' \otimes D(\sigma_k A) \xrightarrow{\phi'} V')$.

(iv) If X is of 2nd kind and having $(U \otimes D(A) \xrightarrow{\phi} V)$ as the canonical expression and Ker $\phi \cong L_k$, then $S_k^+(X) = L'_k$ which is the simple σ_kR-module corresponding to the source k of (Γ, $\sigma_k\Lambda$).

Further, for the the case of a vertex k being a source with respect to (Γ, Λ) we make dually

Definition 2 (i) If X is of 1st kind and not isomorphic to L_k, then $S_k^-(X) \cong s_k^-(X)$.

(ii) If X is isomorphic to L_k which is simple R-module corresponding to the source k, then Cok ψ $\cong$ $s_k^-([D(A),L_k])$ provided $S_k^-(X) = (U \xrightarrow{\psi} [D(A),V])$.

(iii) If X is of 2nd kind having $(U \xrightarrow{\psi} [D(A),V])$ as the canonical expression and Cok $\psi \not\cong L_k$, then Cok $\psi' \cong s_k^-(\text{Cok } \psi)$ provided $S_k^-(X) = (U' \xrightarrow{\psi'} [(D(A),V'])$.

(iv) If X is of 2nd kind and having $(U \xrightarrow{\psi} [D(A),V])$ as the canonical expression and Cok $\psi \cong L_k$, then $S_k^-(X) \cong L'_k$ which is the simple σ_kR-module corresponding to the sink k of (Γ, $\sigma_k\Lambda$).

In Definition 1 it is clear that the correspondence: $X \longrightarrow S_k^+(X)$ is injective on the set of R-modules X of the cases (i) and (iv). Further in the case (iii), by Theorem 1 Ker ϕ_A is not injective. Hence Ker $\phi \not\cong L_k \otimes D(A)$. Therefore the correspondence is injective and consequently bijective on the set of all non-projective indecomposable R-modules, because by the following Proposition 2 $L_k \otimes D(A)$ is only one injective A-module of which the image by s_k^+ is not an injective $\sigma_k A$-module. Similarly the correspondence: $X \longrightarrow S_k^-(X)$ is bijective for all non-projective R-modules.

Proposition 2 (Dlab-Ringel [4]).

Let $k_1, k_2, \ldots, k_n$ be an admissible sequence of (Γ, Λ), and L'_{t-1} and L'_t simple representations (modules) in $L(\sigma_{k_{t-1}} \sigma_{k_{t-2}} \cdots \sigma_{k_1} \Gamma)(\text{mod-}\sigma_{k_{t-1}} \sigma_{k_{t-2}} \cdots \sigma_{k_1} A)$ corresponding to a sink k_t and source k_{t-1} with respect to $\sigma_{k_{t-1}} \sigma_{k_{t-2}} \cdots \sigma_{k_1} \Lambda$. Then $P_t = s_{k_1}^- s_{k_2}^- \cdots s_{k_{t-1}}^- (L'_t)$ and $Q_{t-1} = s_{k_n}^+ s_{k_{n-1}}^+ \cdots s_{k_t}^+ (L'_{t-1}), 1 \leq t \leq n$, are an indecomposable projective representation (A-module) whose top constituent is isomorphic to L_t and indecomposable injective representation (A-module) whose bottom constituent is isomorphic to L_{t-1} respectively. Consequently, if $k_i \in \Gamma_o$ is a sink (resp. a source) with respect to Λ, and $t \neq i$, then $s_{k_i}^+(P_t)$ (resp. $s_{k_i}^-(P_t)$) and $s_{k_i}^+(Q_t)$ (resp. $s_{k_i}^-(Q_t)$) are indecomposable projective and injective $\sigma_{k_i} A$-modules respectively such that the top and the bottom constituents are respectively isomorphic to the simple module corresponding to k_t of $(\Gamma, \sigma_{k_i} \Lambda)$.

Now we shall give an outline of a proof of the following main theorem. For the details, however, c.f. [7].

Theorem 3. For any non-projective R-module X it holds that $S_{k_n}^+ S_{k_{n-1}}^+ \cdots S_{k_1}^+ (X) \cong DT_r^R(X)$ and $S_{k_1}^- S_{k_2}^- \cdots S_{k_n}^- (X) \cong T_r D^R(X)$.

Proof. In our proof we shall make the following abbreviation of notations; $\sigma_{k_i}\sigma_{k_{i-1}}\cdots\sigma_{k_1}\Lambda$ to $\sigma_i\sigma_{i-1}\cdots\sigma_1\Lambda$, $s^+_{k_i}$ to s_i, $s^-_{k_i}$ to s_i^{-1}, S_{k_i} to S_i and $S^-_{k_i}$ to S_i^{-1}.

The proof consists of the four parts (i)——(iv) which correspond to the classifications of indecomposable R-modules in Definition 1.

(i) X is of 1st kind and $X \not\cong L_1$. Let us denote by L_{i+1}, $1 \leq i \leq n-1$, the simple $\sigma_i\sigma_{i-1}\cdots\sigma_1$R-module corresponding to a sink k_{i+1} of $(\Gamma, \sigma_i\sigma_{i-1}\cdots\sigma_1\Lambda)$. If $s_is_{i-1}\cdots s_1(X) \not\cong L_{i+1}$ for each i, $1 \leq i \leq n-1$, then by the definition and Theorem of Brenner and Butler it holds that $S_nS_{n-1}\cdots S_1(X) \cong s_ns_{n-1}\cdots s_1(X) \cong DT^A_r(X) \cong DT^R_r(X)$. Hence hereafter we shall assume that $s_\ell s_{\ell-1}\cdots s_1(X) \cong L_{\ell+1}$ for some ℓ ($\ell \leq n$) but $s_is_{i-1}\cdots s_1(X) \not\cong L_{i+1}$ for $i = 1,2,\ldots,\ell-1$. Then it should be noted that $X \cong s_1^{-1}s_2^{-1}\cdots s_\ell^{-1}(L_{\ell+1}) \cong P_{\ell+1}$ and $P_{\ell+1}$ is a projective R-module such that Top $P_{\ell+1} \cong L_{\ell+1}$.

At the begining we shall consider the

Case 1); $X \otimes D(A) \cong P_{\ell+1} \otimes D(A) \cong Q_{\ell+1}$ is not projective. Let $(U \otimes D(A) \xrightarrow{\phi} V)$ be the canonical expression of $DT^R_r(X)$. Since $X \otimes D(A)$ ($\cong Q_{\ell+1}$) is not projective, by Theorem 1 Ker $\phi \cong DT_r(Q_{\ell+1}) \cong s_ns_{n-1}\cdots s_1(Q_{\ell+1})$.

On the other hand, if we put $S_nS_{n-1}\cdots S_{\ell+1}S_\ell\cdots S_1(X) = (U_1 \otimes D(A) \xrightarrow{\phi_1} V_1)$ then Ker $\phi_1 \cong s_ns_{n-1}\cdots s_{\ell+1}(L_{\ell+1} \otimes D(\sigma_\ell\sigma_{\ell-1}\cdots\sigma_1 A))$ because $S_\ell S_{\ell-1}\cdots S_1(X) \cong s_\ell s_{\ell-1}\cdots s_1(X) \cong L_{\ell+1}$, which is a simple $\sigma_\ell\sigma_{\ell-1}\cdots\sigma_1$A-module corresponding to a sink $k_{\ell+1}$ of $(\Gamma, \sigma_\ell\,\sigma_{\ell-1}\cdots\sigma_1\Lambda)$.

Now $L_{\ell+1} \otimes D(\sigma_\ell \sigma_{\ell-1} \cdots \sigma A)$ is isomorphic to an injective $\sigma_\ell \sigma_{\ell-1} \cdots \sigma_1 A$-module Q'_{+1} whose simple socle corresponds to a sink $k_{\ell+1}$ of $(\Gamma, \sigma_\ell \sigma_{\ell-1} \cdots \sigma_1 \Lambda)$. Since $k_{\ell+1}, k_{\ell+2}, \ldots, k_n, k_1, \ldots, k_\ell$ is a admissible sequence with respect to $(\Gamma, \sigma_\ell\ \sigma_{\ell-1} \cdots \sigma_1 \Lambda)$, by Proposition 2 it follows that $Q'_{\ell+1} \cong s_\ell s_{\ell-1} \cdots s_1 s_n s_{n-1} \cdots s_{\ell+2}(L'_{\ell+2})$ and $Q_{\ell+1} \cong s_n s_{n+1} \cdots s_{\ell+2}(L'_{\ell+1})$, where $L'_{\ell+1}$ is a simple $\sigma_\ell \sigma_{\ell+1} \cdots \sigma_1 A$-module which corresponds to a vertex $k_{\ell+1}$ of $(\Gamma, \sigma_{\ell+1} \sigma_\ell \cdots \sigma_1 \Lambda)$. Thus we have $Q'_{\ell+1} \cong s_\ell s_{\ell-1} \cdots s_1(Q_{\ell+1})$ and $\operatorname{Ker} \phi_1 \cong s_n s_{n-1} \cdots s_{\ell+1} s_\ell s_{\ell-1} \cdots s_1(Q_{\ell+1})$. It follows that $DT_r^R(X) \cong S_n S_{n-1} \cdots S_1(X)$.

Case 2): $X \otimes D(A) = Q_{\ell+1}$ is a projective A-module. Suppose $Q_{\ell+1} \cong P_j$. Then (Γ, Λ) must be the following linear diagram:

$$k_j \bullet \longrightarrow \bullet \longrightarrow \cdot \quad \cdots \quad \longrightarrow \bullet \longrightarrow \cdot\ k_{\ell+1}$$

and it follows $\ell+1 = 1$. But this is impossible and this case does not happen.

(ii) X is of 1st kind and $X \cong L_1$, where L_1 is a simple R-module corresponding to the sink k of (Γ, Λ), i.e. it is also a simple A-module.

At the begining we shall consider the Case 1): $L_1 \otimes D(A) = Q_1$ is not a projective A-module, where Q_1 is an injective A-module with $\operatorname{Soc} Q_1 \cong L_1$.

By Definition 1 $\operatorname{Ker} \phi_1 \cong s_1(L_1 \otimes D(A))$ where $S_1(X) = (U_1 \otimes D(\sigma_1 A) \xrightarrow{\phi_1} V_1)$ and s_1 is the reflection functor of mod-A to mod-$\sigma_1 A$ which is corresponding to the sink k_1 of (Γ, Λ). From the assumption of Case 1) we know $0 \neq DT_r(L_1 \otimes D(A))$. It follows that $s_i s_{i-1} \cdots s_1(L_1 \otimes D(A)) \neq 0$ for $1 \leqq i \leqq n$, since

$DT_r(L_1 \otimes D(A) \cong s_n s_{n-1} \cdots s_1(L_1 \otimes D(A))$. So if we put $S_n S_{n-1} \cdots S_1(X) = (U' \otimes Q \xrightarrow{\phi'} V')$, then by the definition $\operatorname{Ker} \phi' \cong s_n s_{n-1} \cdots s_1(L_1 \otimes D(A))$.

On the other hand, for the canonical expression $(U \otimes D(A) \xrightarrow{\phi} V)$ of $DT_r^R(X)$ we have $\operatorname{Ker} \phi \cong DT_r(L_1 \otimes D(A))$ by Theorem 3. Hence $DT_r^R(X) \cong S_n S_{n-1} \cdots S_1(X)$.

Case 2): $L_1 \otimes D(A)_A = Q_1$ is projective.
If $Q_1 \cong P_j$, then the graph (Γ, Λ) should be the following linear diagram:

$$\overset{k_j}{\cdot} \longrightarrow \cdot \longrightarrow \cdot \cdots \longrightarrow \overset{k_3}{\cdot} \longrightarrow \overset{k_2}{\cdot} \longrightarrow \overset{k_1}{\cdot}$$

and consequently $j=n$. Then it follows by Theorem 2 that $DT_r(X) \cong L_1 \cong L_1 \otimes D(A) \otimes D(A) \cong Q_1 \otimes D(A) \cong P_n \otimes D(A)$ $\cong Q_n \cong L_n$, where L_n, P_n, Q_n are simple, projective and injective R-modules respectively which are corresponding to vertex k_n of (Γ, Λ) and $\operatorname{Top}(P_n) \cong L_n \cong \operatorname{Soc}(Q_n)$.

On the other hand, we put $S_1(X) = (U_1 \otimes D(\sigma_1 A) \xrightarrow{\phi_1} V_1)$, $\operatorname{Ker} \phi_1 \cong s_1(L_1 \otimes D(A)) \cong s_1(P_1 \otimes D(A)) \cong s_1(Q_1) = s_1(P_1)$.

Now, in $(\Gamma, \sigma_{n-1}\sigma_{n-2} \cdots \sigma_1 \Lambda)$ the vertex k_n is a sink and the corresponding simple $\sigma_{n-1}\sigma_{n-2} \cdots \sigma_1 A$-module L_n' is projective. Hence by Proposition 2 $s_{n-1}s_{n-2} \cdots s_1(P_n)$ $\cong L_n'$. Hence if $(U_2 \otimes D(\sigma_{n-1}\sigma_{n-2} \cdots \sigma_1 A) \xrightarrow{\phi_2} V_2)$ is the canonical expression of $S_{n-1}S_{n-2} \cdots S_1(X)$, then $\operatorname{Ker} \phi_2 \cong$ L_n'. Therefore by the definition of S_n, $S_n S_{n-1} \cdots S_1(X) \cong$ L_n' and this follows $DT_r^R(X) \cong S_n S_{n-1} \cdots S_1(X)$.

For indecomposable R-modules X of 2nd kind corresponding to the classifications (iii) and (iv) of Definition 1 we can proceed the proofs similarly, but for the sake of the restriction of spaces we shall omit them.

Complementing the definition of morphisms we can obtain stable functors S_k^+ and S_k^- : $\underline{\text{mod}}$-R$\longrightarrow\underline{\text{mod}}$-$\sigma_k$R from Definitions 1 and 2. And we have

Theorem 4. Let $k_1, k_2, \ldots, k_r$ be an admissible sequence of (Γ , Λ). Then self-injective algebras R and $\sigma_{k_r} \ldots \sigma_{k_2} \sigma_{k_1}$ R are stably equivalent to each other. Especially by taking a suitable admissible sequence $\sigma_{k_r} \ldots \sigma_{k_2} \sigma_{k_1}$ R is isomorphic to a self-injective algebra with cube-zero radical.

Similarly if Γ is a Dynkin diagram A_n, $\sigma_{k_r} \ldots \sigma_{k_2} \sigma_{k_1}$ R can be taken a Nakayama algebra (= a serial algebra).

In the remaining part we would like to mension the relation between (Γ , Λ)(= the quiver of A) and the quiver of R (= A $\ltimes$ D(A)), where A in not necessary hereditary.

Proposition 3. For K-algebras A and R = A $\ltimes$ D(A) denote by $\Gamma = (\Gamma_o, \Gamma_1)$ and $\Gamma' = (\Gamma'_o, \Gamma'_1)$ the quivers (= (set of vertices, set of arrows)) of A and R respectively. Then $\Gamma_o = \Gamma'_o$ and

$$\Gamma'_1 = \Gamma_1 \cup \{(i \cdot\!\!\longrightarrow\!\cdot j) \mid e_i(\mathrm{Soc}_A A \cap \mathrm{Soc}\, A_A)e_j \neq 0\},$$

where e_i, e_j are primitive idempotents of A corresponding to $i, j \in \Gamma_o$.

Proof. Denote the Jacobson radicals of A and R by J and N respectively. Clearly N = J $\oplus$ D(A) $\subset$ R (=A $\oplus$ D(A)) and $N^2 = J^2 \oplus (JD(A) + D(A)J)$. It follows that $(i \cdot\!\!\longrightarrow\!\cdot j) \in \Gamma'_1$ iff $e_j(J/J^2)e_i \oplus e_j(D(A)/JD(A){+}D(A)J)e_i \neq 0$. But $(i \cdot\!\!\longrightarrow\!\cdot j) \in \Gamma_1$ iff $e_j(J/J^2)e_i \neq 0$. Thus $\Gamma'_1 \supset \Gamma_1$.

Now let (,) be an inner product : D(A) $\ltimes$ A$\longrightarrow$ K defined by (f , a) = f(a) for $f \in D(A)$, $a \in A$. Then $Soc_A A$ and $Soc\ A_A$ are the annihilators of JD(A) and D(A)J in A with respect to (,). Hence $Soc\ A_A \cap Soc\ _A A$ is the annihilator of JD(A) + D(A)J and

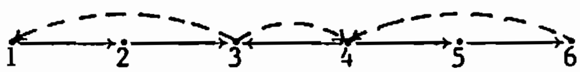
$$_A D(D(A)/JD(A) + D(A)J)_A \cong {}_A(Soc\ A_A \cap Soc\ _A A)_A \ .$$

Therefore $e_j(D(A)/JD(A) + D(A)J)e_i \neq 0$ iff $e_i(Soc\ _A A \cap Soc\ A_A\ )e_j \neq 0$. This concludes our proof.

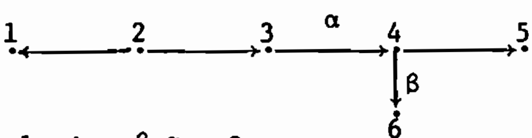
The next corollary characterizes the Brauer quiver of $A \ltimes D(A)$ where the quiver of A in a Dynkin diagram A_n.

Corollary 1. Let R be a DJK-algebra with a Brauer quiver Q. Then R is isomorphic to a trivial extension of a hereditary algebra A whose quiver is a Dynkin diagram A_n iff there is a linear subgraph of Q such that each non-looped cycle of Q has one and only one edge which is an edge of the linear subgraph too.

Consider the following quiver with no relation as the quiver of an algebra A. Then by Proposition 3 we know the dotted arrows in the diagram are arrows which should be added as arrows of the quiver $A \ltimes D(A)$.

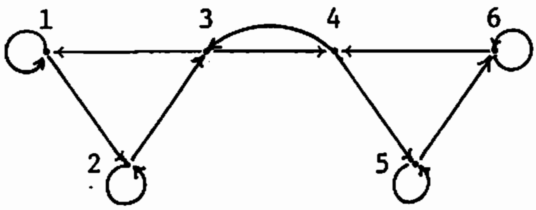


So the Brauer quiver of $A \ltimes D(A)$ is Example 1. Proposition 3 teaches us a DJK-algebra having the Brauer quiver of Example 2 never been constructed as a trivial extension of hereditary algebra, but constructed as a trivial extension $B \ltimes D(B)$ of tilted algebra B whose quiver is

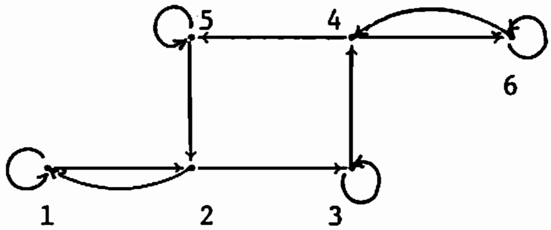


with zero relation $\beta\ \alpha = 0$.

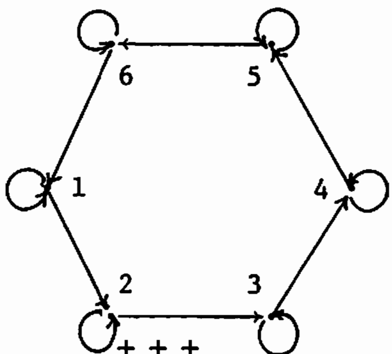
Example 1.



Example 2.

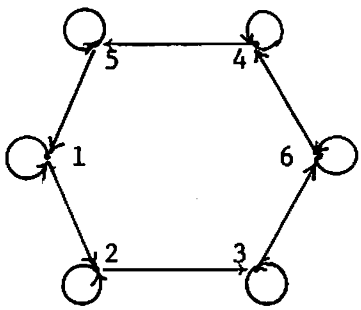


From our Theorem 4 it follows, however, that the DJK-algebra associated with the quiver of Example 1 is stably equivalent to a wreath-like algebra associated with the Brauer quiver



by our stable functor $S_1^+ S_2^+ S_3^+$.

The existence of DJK-algebras such as Example 2 seems to imply that our context in this note is more special than one in Gabriel-Riedtmann [5] . But our context is vaild to self-injective algebras of not only finite type but also of infinite type. Further we can extend our theory in this note, by which we know that the DJK-algebra having the quiver of Example 2 is also stably equivalent to a wreath-like algebra associated with the Brauer quiver



by a functor $S_1^+F_6$, where a stable functor F_6 corresponds to an APR-tilt. Cf.[1] and [2].

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SELF-INJECTIVE DIMENSION OF SERIAL RINGS

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This note is an abstract of [9], in which we gave the complete proofs. We shall state only some of our results and ideas. See [9] for details.

Following Eisenbud and Griffith [2,3], we say that a ring R is serial if both ${}_RR$ and R_R are direct sums of uniserial modules of finite length. Nakayama [8] established the well known theorem that each module over such a ring is a direct sum of factor modules of indecomposable projective modules and conversely. With each indecomposable serial ring R, Kupisch [6] associated a series $P_1,\ldots,P_n$ of non-isomorphic indecomposable projective left R-modules so that there exist epimorphisms $P_i \to NP_{[i+1]}$ for $1 \leqq i \leqq n$ where [i] is the least strictly positive remainder of i modulo n and N is the radical of R. Such a series $P_1,\ldots,P_n$ is called a left Kupisch series for R. Then let $c(i) = |P_i|$, the composition length of P_i. The series $(c(1),\ldots,c(n))$ is called the left admissible sequence of R corresponding to a left Kupisch series $P_1,\ldots,P_n$.

Fuller [4] showed that the global dimension of a serial ring is determined by its left admissible sequence. On the other hand we are interested in the self-injective dimension of serial rings. In the last section we shall give examples which answers negatively to Sumioka's problem whether the maximal quotient ring of a QF-3 artinian ring with self-injective dimension one on both sides is QF or not. (Cf. [10])

1. Preliminaries

Throughout this note, let R be an indecomposable serial ring with a left Kupisch series $P_1,\ldots,P_n$ and let $(c(1),\ldots, c(n))$ be the corresponding left admissible sequence. Furthermore let N be the radical of R, $n = n(R)$ and $c(R) = \min\{c(1),\ldots,c(n)\}$. Withour loss of generality, we can assume $c(R) = c(1)$.

For a left R-module M, we use the following notations.

$pd(M)$ = the projective dimension of M

$id(M)$ = the injective dimension of M

$|M|$ = the composition length of M

$P(M)$ = the projective cover of M

$E(M)$ = the injective hull of M

$(E^k(M))$ = the minimal injective resolution of M

$(P^k(M))$ = the minimal projective resolution of M

$\Omega^k(M)$ = the cokernel of $\Omega^{k-1}(M) \to E^{k-1}(M)$ where $\Omega^0(M) = M$

$\Lambda^k(M)$ = the kernel of $P^{k-1}(M) \to \Lambda^{k-1}(M)$ where $\Lambda^0(M) = M$.

The following is essentially due to Fuller [4].

Theorem 1. For an indecomposable left R-module M, both $id(M)$ and $pd(M)$ are determined only by the left admissible sequence of R, $|M|$ and M/NM. In particular both $id({}_RR)$ and $id(R_R)$ are determined only by the left admissible sequence of R

For our purpose to calculate $id({}_RR)$ and $id(R_R)$, we can abbreviate our serial ring as ${}_RR = (c(1),\ldots,c(n))$ by the above theorem. Recall that Kupisch [6] constructed a serial ring with the given left admissible sequence.

2. Periodicity of self-injective dimension.

The following is the key lemma for our theory.

Lemma 2. Let M be an indecomposable left R-module with $|M| = n$.

(1) If M is not injective, then $E^0(M) \cong E^1(M)$ and $|\Omega^2(M)| = n$.

(2) If M is not projective, then $P^0(M) \cong P^1(M)$ and $|\Lambda^2(M)| = n$.

So we have immediately,

Theorem 3. If a serial ring R is of finite global dimension, then $c(R) \leqq n(R)$.

Let ${}_R R = (c(1),\ldots,c(n))$ and ${}_{\tilde{R}}\tilde{R} = (c(1)+n,\ldots,c(n)+n)$. Let $\tilde{P}_1,\ldots,\tilde{P}_n$ be the corresponding left Kupisch series for $\tilde{R}$. Then we have the following.

Theorem 4. (Periodicity Theorem) The following statements hold for each k, $1 \leqq k \leqq n$.

(1) If $id(P_k) = \infty$, then $id(\tilde{P}_k) = \infty$.

(2) Assume $id(P_k) = m$.

  (i) If m is even, then $id(\tilde{P}_k) = m$.

  (ii) If m is odd, then $id(\tilde{P}_k) = \infty$.

Therefore our problem to calculate self-injective dimension of a serial ring R is reduced to the case $c(R) \leqq n(R)$. Let R be an indecomposable serial ring with $c(R) = 1$, that is, of the first category in the terminology of Murase [7]. Then Eisenbud and Griffith [3] applied Chase's result [1] and Murase's result [7] to show that R is of finite global dimension. We can show directly the existence of the upper bound depending only on n(R).

Theorem 5. Let R be an indecomposable serial ring with c(R) = 1. Then gl.dim R $\leqq$ n(R)-1.

3. Rivers and corivers

Let R be an indecomposable serial ring with c(R) $\geqq$ n(R) = n and $P_1,\dots,P_n$ its left Kupisch series. Let $X_k = P_k/N^nP_k$. By Lemma 2, we have directly,

Lemma 6. (1) X_k is injective if and only if $\Omega^2(M) = 0$. (2) If X_k is not injective, then there exists $\tau(k)$, $1 \leqq \tau(k) \leqq n$, such that $\Omega^2(X_k) \cong X_{\tau(k)}$.

With each indecomposable serial ring R with c(R) $\geqq$ n(R), we associate the following oriented graph G(R), which we call the river of R.

The set of verteces of G(R) = {1,...,n}, n = n(R).

The arrow i → j exists if and only if $\Omega^2(X_i) \cong X_j$.

A vertex i in G(R) is said to be an initial point if there exist a vertex j and an arrow j → i. Dually a final point is defined. Remark that a river has not necessarily any final point. A final point is said to be of height 0, and a non-final point k is said to be of height p if there exist verteces $k = k_0,k_1,\dots,k_p$ such that k_p is a final point and there exist arrows $k_{i-1} \to k_i$ for $1 \leqq i \leqq p$. Then we let $h(k) = p$. If each vertecex has height, we let $h(R) = \sup\{h(k)\}$, which we call the height of $G(R)$. Otherwise we let $h(R) = \infty$.

Lemma 7. The following conditions are equivalent for X_k.

(1) There exists X_h such that $\Omega^2(X_h) \cong X_k$.

(2) The projective cover of X_k is injective.

Lemma 8. A vertex k in G(R) is a final point if and only if X_k is injective. Hence G(R) has no final point if $c(R) > n(R)$.

We can show the dual statements of Lemma 7 and Lemma 8. Therefore, for an indecomposable serial ring R with $c(R) \geqq n(R) = n$, we can obtain the dual notion of the river, which we call the coriver of R and denote by G*(R). More precisely,

The set of verteces of $G^*(R) = \{1,\dots,n\}$.

The arrow $j \to i$ exists if and only if $\Lambda^2(X_i) \cong X_j$.

Also we define initial points and final points in G*(R) similarly, and we define coheights $h^*(k)$ and $h^*(R)$ dually.

4. Rivers and corivers for the case $c(R) = n(R)$

Let R be an indecomposable serial ring with $c(R) = n(R) = n$. Then we can obtain precise informations about the river G(R) and the coriver G*(R). Recall that we assume $|P_1| = n = n(R)$. We begin with the river.

Lemma 9. If X_k is not injective, then we have

$$k < \tau(k) = k + d([k+1]) - n$$

where $d(i) = |E(P_k/NP_k)|$.

Lemma 10. If neither X_k nor $X_{k'}$ is injective, then $k < k'$ implies $\tau(k) \leqq \tau(k')$.

From the above lemmas, we have immediately,

Proposition 11.

(1) G(R) contains no oriented cycle.

(2) Each connected component contains exactly one final point.

(3) Each connected component contains one point k such that X_k is projective, and then k is necessarily an initial point.

(4) The maximal height in each component is given by an initial point k such that X_k is projective.

(5) If k and p such that X_k is injective and X_p is projective belong to the same connected component, then $p \leqq k$ and each j, $p \leqq j \leqq k$, belongs to that component.

(6) The number of connected components is equal to that of the final points.

Corollary 12. The following statements hold for an indecomposable serial ring R with $c(R) = n(R)$.

(1) $id(X_k) = 2 \cdot h(k)$ for each k.

(2) $h(R) \leqq n(R)-1$.

(3) $\sup\{id(X_k)\} = 2 \cdot h(R)$.

We have the dual statements of the aboves for the coriver $G^*(R)$. We write down only the duals of Lemmas 9 and 10.

Lemma 9*. If X_k is not projective, then we have

$$k > \sigma(k) = k - c(k) + n$$

where $\Lambda^2(X_k) \cong X_{\sigma(k)}$.

Lemma 10*. If neither X_k nor $X_{k'}$ is projective, then $k < k'$ implies $\sigma(k) \leqq \sigma(k')$.

Now we can directly obtain the coriver $G^*(R)$ from the river $G(R)$ and conversely.

Theorem 13. (Transformation Theorem)

(1) Assume that k is not an initial point in $G(R)$. Let $p_1 < \cdots < p_t$ be all the points such that $\tau(p_1) = \cdots = \tau(p_t) = k$. Then $\sigma(k) = p_1$.

(2) Assume that k is an initial point in $G(R)$ such that X_k is not projective. Then there exists at least one point k' $(>k)$ which is not initial point in $G(R)$. Let p be the least number among them. Then $\sigma(k) = \sigma(p)$.

By Transformation Theorem, we can show the following.

Theorem 14. Let R be an indecomposable serial ring. Then we have

(1) $h(R) = h^*(R)$.

(2) $\sup\{id(X_k)\} = \sup\{pd(X_k)\}$.

5. Self-injective dimension for the case $c(R) = n(R)$

Throughout this section, R is assumed to be an indecomposable serial ring. We must compare the minimal injective resolution of P_k with that of X_k.

Lemma 15. Let X and Y be nonzero indecomposable R-modules, both of which are non-injective. Consider the following commutative diagram with exact rows.

$$(Y,X:\eta,\varepsilon,\eta') : \quad \begin{array}{ccccccccc} 0 & \to & Y & \to & E(Y) & \to & \Omega(Y) & \to & 0 \\ & & \downarrow \eta & & \downarrow \varepsilon & & \downarrow \eta' & & \\ 0 & \to & X & \to & E(X) & \to & \Omega(X) & \to & 0 \end{array}$$

(1) If η is a monomorphism, then η' is an epimorphism.
(2) If η is an epimorphism, then η' is a monomorphism.

Applying this lemma and its dual, we have

Theorem 16. Let R be an indecomposable serial ring with $n(R) = c(R) = n$, and Y an indecomposable R-module with $|Y| \geqq n$. Let $X_k = Y/N^nY$. Then there exists an integer s such that $id(Y) = 2\cdot s$ and $0 \leqq s \leqq h(k)$.

Theorem 16*. Let R be an indecomposable serial ring with $c(R) = n(R) = n$, and Y an indecomposable R-module with $|Y| \geqq n$. Let X_k be the submodule of Y with length n. Then there exists an integer s* such that $pd(Y) = 2\cdot s^*$ and $0 \leqq s^* \leqq h^*(k)$.

Iwanaga showed $id(R_R) = \sup\{pd(E) \mid E$ is indecomposable injective}. Hence we apply Transformation Theorem to have the main theorem.

Theorem 17. Let R be an indecomposable serial ring with $c(R) = n(R) = n$. Then $id({}_RR) = id(R_R) = 2\cdot h(R)$.

We are interested in the last term of the minimal injective resolution of a serial ring.

Theorem 18. Let R be an indecomposable serial ring with $c(R) = n(R) = n$. Let $id({}_RR) = 2h$. Then an injective indecomposable R-module Y is a direct summand of $E^{2h}({}_RR)$ if and only if $|Y| = n$ and $pd(Y) = 2h$.

6. Examples and Remark

For an indecomposable serial ring R with typical left admissible sequence, we can calculate $id({}_RR)$ explicitly because we can easily calculate its right admissible sequence.

Example 19. (Iwanaga [5]) Let R be an indecomposable serial ring with left admissible sequence $(c,c+1,\dots,c+(n-1))$. Then $id({}_RR) < \infty$ if and only if $c < n$ or $c \equiv 0$ (mod. n). If the conditions above are satisfied, then $id({}_RR) = id(R_R) \leqq 2$.

Example 20. Let $H = H(c,n)$ be an indecomposable serial ring with left admissible sequence $(c,c+1,\dots,c+1)$ and $n(H) = n$ ($c+1$ occurs $(n-1)$ times in the admissible sequence). Then the following statements hold.

I. If $c = 1$, then gl.dim $H = n-1$.

II. Assume $c > 1$. Then the following conditions are equivalent.

(1) $id({}_H H) < \infty$. (2) $id(H_H) < \infty$. (3) $(c+1, n) = 1$.

As such is the case, we have $id({}_H H) = id(H_H) = 2h$ where h is the least integer such that $0 \leqq h \leqq n-1$ and $1+h(c+1) \equiv 0$ (mod. n).

Let R be aQF-3 artinian ring with self-injective dimension one on both sides. Then Sumioka proved in [10] that R is Morita equivalent to a matrix ring over a QF ring if its maximal quotient ring Q is QF. He asked whether Q is QF or not. The following example answers negatively to his problem.

Example 21. Consider the bounden quiver

$$Q : \quad 2 \underset{\gamma}{\overset{\alpha}{\leftrightarrows}} 1 \underset{\delta}{\overset{\beta}{\rightleftarrows}} 3$$

with relation $\gamma\alpha = \delta\beta$, $\alpha\delta = \beta\gamma = 0$. Let $A = K(Q)$ be the bounden quiver algebra over a field K. Then A is a QF-2 algebra with minimal faithful left ideal Ae_1. Furthermore it is easily shown that $Ae_1/A\alpha \cong Hom_K(e_3A, K)$ and $Ae_1/A\beta \cong Hom_K(e_2A, K)$. Therefore Ae_1, $Ae_1/A\alpha$ and $Ae_1/A\beta$ form a complete set of non-isomorphic injective indecomposable left R-modules. Also it is easy to show $A\alpha \cong Ae_2$ and $A\beta \cong Ae_3$. Hence $id({}_A A) = 1$. Similarly we can show $id(A_A) = 1$. On the other hand we have

$$Q \cong \text{Bi-End}({}_A Ae_1) \cong \left\{ \begin{bmatrix} a & 0 & 0 & 0 \\ b & c & d & 0 \\ f & g & h & 0 \\ x & y & z & a \end{bmatrix} \middle| \begin{matrix} a,b,c,d,f \\ g,h,x,y,z \end{matrix} \in K \right\}.$$

Hence Q is a serial algebra with left admissible sequence (2,3). Therefore $id({}_Q Q) = id(Q_Q) = gl.dim\ Q = 2$ by Example 20.

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ON FINITE GROUP ALGEBRAS WITH RADICAL CUBE ZERO

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Let G be a finite group and k be an algebraically closed field of characteristic p, a prime number. Let B be a block algebra of the group algebra kG with defect group D and let $J(B)$ denote the Jacobson radical of B. It is well known that $J(B) = 0$ if and only if $D = 1$ if and only if there exists a projective simple kG-module in B. Furthermore it is true that $J(B)^2 = 0$ $(J(B) \neq 0)$ if and only if $p = 2$ and $|D| = 2$ if and only if there exists a projective indecomposable kG-module with Loewy length 2 in B.

In my talk I considered blocks B of finite groups with $J(B)^3 = 0$. Our main result is the following.

Theorem 1. $J(B)^3 = 0$ (and $J(B)^2 \neq 0$) if and only if one of the following conditions holds ;

(1) $p = 2$, D is a four group and B is isomorphic to the matrix ring over kD or is Morita equivalent to kA_4 where A_4 is the alternating group of degree 4,

(2) p is odd , $|D| = p$, the number of simple kG-modules in B is $p-1$ or $p-1/2$ and the Brauer tree of B is a straight line segment such that the exceptional vertex is in an end point(if it exists).

For the prime 2 we have the following.

Theorem 2. Assume $p = 2$. Let U be the projective indecomposable kG-module with $U/\mathrm{Rad}(U) = k_G$, the trivial kG-module. If Loewy length of U is 3 then a 2-Sylow subgroup of G is dihedral.

I had reported these reults also in Conference at Obelwolfach (Representation Theory of Finite Groups, 24-30,July 1983). A complete proof will be given elsewhere.

1. Examples

First we shall give some examples of blocks B with $J(B)^3 = 0$. The principal blocks of the following groups satisfy the condition.

(a) $p = 2$ and $G = \mathbb{Z}_2 \times \mathbb{Z}_2$. B has only one simple kG-module i.e. the trivial kG-module $k = k_G$ and the corresponding projective indecomposable kG-module has the following Loewy series,

$$\begin{array}{ccc} & k & \\ k & & k \\ & k & \end{array} .$$

(b) $p = 2$ and $G = A_4$. B has three simple kG-modules S_1, S_2 and S_3. Let U_i be the projective indecomposable kG-module corresponding to S_i. Then U_i has the following Loewy series, $U_i = \begin{array}{ccc} & S_i & \\ S_j & & S_k \\ & S_i & \end{array}$ where $\{i,j,k\} = \{1,2,3\}$.

(c) p is odd and $G = S_p$ (the symmetric group of degree p). B has a defect group of order p and its Brauer tree is the following;

Let S_i (U_i, resp.) be the simple (projective indecompos-able , Resp.) kG-module corresponding to the edge i. Then U_i has the following Loewy series; $U_1 = \begin{array}{c} S_1 \\ S_2 \\ S_1 \end{array}$,

$$U_i = \begin{array}{ccc} & S_i & \\ S_{i-1} & & S_{i+1} \\ & S_i & \end{array} \quad (2 \leqq i \leqq p-2) \quad \text{and} \quad U_{p-1} = \begin{array}{c} S_{p-1} \\ S_{p-2} \\ S_{p-1} \end{array}.$$

(d) p is odd and $G = A_p$ (the alternating group of degree p). B has a defect group of order p and its Brauer tree is the following;

Let S_i and U_i be as in (c). Then U_i has the following Loewy series ; $U_1 = \begin{array}{c} S_1 \\ S_2 \\ S_1 \end{array}$, $U_i = \begin{array}{ccc} & S_i & \\ S_{i-1} & & S_{i+1} \\ & S_i & \end{array}$

$$(2 \leqq i \leqq p-1/2-1) \quad \text{and} \quad U_{p-1/2} = \begin{array}{ccc} & S_{p-1/2} & \\ S_{p-1/2-1} & & S_{p-1/2} \\ & S_{p-1/2} & \end{array}.$$

Erdmann [6] shows that for each prime power q with q = 3 (mod.4) the group PSL(2,q) satisfies the assump-tion in Theorem 2. U has the following Loewy series;

$U = \begin{array}{ccc} & k & \\ S & & T \\ & k & \end{array}$ where S and T are some simple kG-modules.

2. Some Lemmas

In this section we shall give some lemmas which will be used to prove Theorem 1. Throughout this section, B is an arbitrary block algebra of a finite group G. Let

D be a defect group of B. For a positive integer n let n_p denote the p-part of n.

Lemma 1. There exists a simple kG-module S in B such that a vertex of S is D and a source of S is p'-dimensional.

This follows from the fact that there exists a simple kG-module S in B with $(\dim_k S)_p = |G:D|_p$.

Let Ω denote the Heller's syzygy functor. Then the following lemma follows from the fact that kG is a smmetric algebra.

Lemma 2. Let X be a kG-module with no projective direct summand. Then $\mathrm{Soc}(\Omega^1(X)) \cong X/\mathrm{Rad}(X)$.

Lemma 3. Let P be a nontrivial cyclic subgroup of D. Then there exists a kG-module X in B such that

(1) a vertex of each indecomposable direct summand of X is P and $(\dim_k X)_p = |G:P|_p$,

(2) $\Omega^1(X) \cong \Omega^{-1}(X)$.

The lemma can be proved by using the theory of vertices of Green and some properties of the Green correspondence [9],[11],[12].

3. Outline of Proof of Theorem 1.

If a block B satisfies one of the conditions (1) and (2) in Theorem 1 , then it is easy to show that $J(B)^3$

$= 0$ and $J(B)^2 \neq 0$. In the rest of this section we assume that $J(B)^3 = 0$ and $J(B)^2 \neq 0$ and we shall show that B satisfies one of the conditions (1) and (2).

Step 1. If X is a nonsimple nonprojective indecomposable kG-module in B, then $Soc(X) = Rad(X)$.

See [14].

Step 2. If p is odd, then $|D| = p$.

Our proof of Step 2 uses Lemma 2, Lemma 3 and the result of Erdmann [5].

Using the result of Brauer [2] Prop. (6G) and [15] a similar argument as in Step 2 shows the following.

Step 3. If $p = 2$, then D is elementary abelian.

Step 4. If $p = 2$, then D is a four group.

For a proof of Step 4, we use Lemma 1 and the results of Knörr [13] and Conlon [3]. We also require some results from the theory of Auslander-Reiten sequences [1], [17].

Step 5. Conclusion.

If $p = 2$, then by Step 4 the result follows from the result of Erdmann [7],[8]. If p is odd, then by Step 2 the result follows from the theory of Brauer-Dade [4] on blocks with cyclic defect groups and the result of Peacock [16].

4. Outline of Proof of Theorem 2.

Our proof of Theorem 2 uses the result of Webb [18] which says that if Rad(U)/Soc(U) is decomposable, then a 2-Sylow subgroup of G is dihedral. We also use the result of Fong [10] on self-dual simple kG-modules.

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EXT FOR BLOCKS WITH CYCLIC DEFECT GROUPS

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According to Reiten [4] we consider a k-algebra A given by a Brauer tree as a generalization of a block with a cyclic defect group. The aim of this note is to determine the indecomposable A-module M with $\mathrm{Ext}_A^n(S,M) \neq 0$ for certain values of n, where S is a fixed simple A-module whose projective cover is uniserial. Janusz's classification of indecomposable A-modules [3] and Heller's loop-space operation [2] enable us to compute $\mathrm{Ext}_A^n(S,M)$. The canonical walk by Alperin and Janusz [1] describes our result. The proofs will appear in Journal of Algebra.

Let A be a k-algebra given by a Brauer tree [4]. Then the edges correspond to the simple modules, there is a unique vertex called exceptional and the edges incident with each vertex give a counter-clockwise ordering.

Let S be a fixed simple module whose projective cover is uniserial. Then S corresponds to an edge E = PQ, where P is nonexceptional and E is a unique edge incident with P [3, Collary 7.3]. Put $Q_0 = P$, $F_0 = E$ and $Q_1 = Q$. For $i \geqq 0$ let F_{i+1} be the edge immediately following F_i around Q_{i+1} and let Q_{i+2} be the vertex such that $F_{i+1} = Q_{i+1}Q_{i+2}$. Thus we get the <u>canonical walk</u> Q_0, F_0, Q_1, F_1, Q_2, ... , Q_i, F_i, Q_{i+1}, ... [1].

Every nonprojective indecomposable module M is determined by a sequence $M_1, \ldots, M_t$ of submodules of M, where M_i has a unique maximal submodule $M_{i,1} \oplus M_{i,2}$ and $M_{i,j}$ is uniserial. Then we write $M = (M_1, \ldots, M_t)$ [4].

Theorem. Let M be a nonprojective indecomposable A-module, where A is a k-algebra given by a Brauer tree with e edges. Let S be a fixed simple module whose projective cover is uniserial. i and j denote the integers such that $0 < i < j < i+2e$ and $i-j$ is odd.

We have $\mathrm{Ext}_A^n(S,M) \neq 0$ only for $n \equiv i \pmod{2e}$ and $n \equiv j \pmod{2e}$ if and only if M is isomorphic to the A-module $(M_1, \ldots, M_t)$ with the property (P): F_i corresponds to $\mathrm{soc}\, M_{1,1}$ if the edge denoted by F_i does not appear between Q_{i+1} and Q_j in the canonical walk. Otherwise F_i corresponds to M_1/JM_1 $(M_{1,1} = 0)$; F_j corresponds to $\mathrm{soc}\, M_{t,2}$ or M_t/JM_t $(M_{t,2} = 0)$.

To compute $\mathrm{Ext}_A^n(S,M)$ we use Heller's loop-space operation Ω [2]: $\mathrm{Ext}_A^n(S,M) \simeq \mathrm{Ext}_A^n(S,\Omega^{-(n-1)}M)$.

Proposition 1. If a nonprojective indecomposable module $M = (M_1, \ldots, M_t)$ has the property (P) for fixed integers i and j such that $0 < i < j < i+2e$ and $i-j$ is odd, then $\Omega^{-1}M$ is isomorphic to a module $L = (L_1, \ldots, L_r)$ with the property (P) for $i-1$ and $j-1$. (If $i = 1$, we consider the property (P) for 0 and $j-1$.)

We know that $\mathrm{Ext}_A^1(S,M) \neq 0$ if and only if there is the exact sequence $0 \to M \to W \to S \to 0$ with W indecomposable:

Proposition 2. For the indecomposable A-module M with the property (P) for integers i and j such that $0 < i < j < i+2e$ and $i-j$ is odd, we have $\mathrm{Ext}_A^1(S,M) \neq 0$ if and only if $i \equiv 1 \pmod{2e}$ or $j \equiv 1 \pmod{2e}$.

These propositions immediately show our theorem.

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