PROCEEDINGS OF THE 17TH SYMPOSIUM ON RING THEORY

HELD AT THE UNIVERSITY OF TSUKUBA

DECEMBER 4—6, 1984

EDITED BY

Yôichi MIYASHITA

University of Tsukuba

1984

OKAYAMA, JAPAN

PROCEEDINGS OF THE 17TH SYMPOSIUM ON RING THEORY

HELD AT THE UNIVERSITY OF TSUKUBA

DECEMBER 4—6, 1984

EDITED BY

YÔICHI MIYASHITA

University of Tsukuba

1984

OKAYAMA, JAPAN

CONTEXT

PREFACE

The 17th Symposium on Ring Theory was held at University of Tsukuba, Japan, on December 4 - 6, 1984. This volume consists of the articles presented at the Symposium. Besides these, there were a lecture dedicated to the memory of Professor Takeshi Onodera, given by Toyonori Kato, and an expository lecture given by Yasuo Iwanaga, on Artin's problem.

The Symposium and the Proceedings were financially supported by the Scientific Research Grants of the Educational Ministry of Japan through the arrangements by Professors Toshiro Tsuzuku at Hokkaido University, and Mieo Nishi at Hiroshima University.

This Symposium has continued to the present with cooperation of Professors Shizuo Endo, Manabu Harada, Hiroyuki Tachikawa, and Hisao Tominaga.

I wish to express my hearty thanks to Algebra staffs of Department of Mathematics, Okayama University for the publication of the Proceedings.

Finally I would like to thank Takayoshi Wakamatsu, Hisaaki Fujita, and the graduate students specialized in the ring theory at University of Tsukuba for their best help in making the meeting run smoothly.

December 1984

Y. Miyashita

ON REGULAR RINGS WITH INVOLUTION

Tadashi YANAI

This is a summary of my paper [3] written jointly with Y. Hirano. In this note, we first classify (von Neumann) regular rings having no nontrivial symmetric idempotents (Theorem 2). Next, we determine the structure of regular rings having finitely many symmetric idempotents (Theorem 6). Further, we extend these results to commutative p.p. rings (Corollary 8).

Throughout this note, R will represent an associative ring with involution (an anti-automorphism of order 2) * . An element a of R is said to be symmetric if $a^*=a$. A symmetric idempotent is called a projection. A projection is said to be minimal if it can not be represented as a sum of two orthogonal nontrivial projections.

Lemma 1. ([2,p.18]) Let R be a semiprime ring with involution *. If a minimal right ideal ρ of R contains a symmetric element s such that $s\rho\neq 0$, then ρ contains a nontrivial projection.

From this result and [2,Theorem 2.3.4], we shall completely classify regular rings having no nontrivial projections.

Theorem 2. R is a regular ring with involution * and has no nontrivial projections if and only if R is (1) a division ring, (2) a direct sum of a division ring and its opposite with (a,b)*=(b,a), or (3) the 2×2 matrix ring over a field with $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

This result and [2,Theorem 2.1.7] imply the following:

Corollary 3. Let R be a 2-torsion free ring with involution *. Then the following are equivalent:

(a) R is semiprime and every nonzero symmetric element is invertible,

(b) R is regular without nontrivial projections,

(c) R is (1) a division ring, (2) a direct sum of a division ring and its opposite with (a,b)*=(b,a), or (3) the 2×2 matrix ring over a field with $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Our second purpose is to determine the structure of regular rings with finitely many projections. To prove the theorem, we need the following useful results.

Proposition 4. Let R be a regular ring with involution * and has finitely many projections. Then, a minimal projection e of R is either

(I) a sum of two orthogonal primitive idempotents f,g with f*=g, or

(II) a primitive idempotent.

Proposition 5. ([1,Theorem 3.1.1]) Let D be a division ring with involution. If every symmetric element of D is algebraic over a finite field, then D is commutative.

Theorem 6. R is a regular ring with involution * having finitely many projections if and only if R is a finite direct sum of rings of the following types:

(1) a division ring,

(2) a direct sum of a division ring and its opposite with (a,b)*=(b,a),

(3) the 2×2 matrix ring over a field with $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$,

(4) a finite dimensional matrix ring over a finite field,

(5) a direct sum of a finite dimensional matrix ring over a finite field and its opposite with (a,b)*=(b,a).

Now, we extend these results. We say that R is a p.p. ring if every principal one-sided ideal of R is projective.

Proposition 7. ([1,Lemma 3.1]) R is a commutative p.p. ring if and only if its classical quotient ring Q is a regular ring and all idempotents of Q are in R.

Clearly, the involution of R can be uniquely extended to its classical quotient ring in this case, so from this result, Theorems 2 and 6, we have the following:

Corollary 8. Let R be a commutative p.p. ring with involution *.

(1) If R has no nontrivial projections, then R is (a) a domain, or (b) a direct sum of a domain and its opposite with $(a,b)^*=(b,a)$.

(2) If R contains no infinite number of orthogonal projections, then R is a finite direct sum of rings of types (a) and (b).

References

[1] G.M. Bergmann: Hereditary commutative rings and centers of hereditary rings, Proc. London Math. Soc. 23 (1971), 214-236.

[2] I.N. Herstein: Rings with involution, Chicago Lectures in Mathematics (1976).

[3] Y. Hirano and T. Yanai: Von Neumann regular rings with only finitely many symmetric idempotents (submitted).

Department of Mathematics,
Faculty of Science,
Okayama University

PROJECTIVE MODULES OVER DIRECTLY FINITE REGULAR RINGS SATIFYING THE COMPARABILITY AXIOM

Mamoru KUTAMI

This paper is an abstract of the author's paper [5], but some new results are added to them. In [2], J. Kado has studied simple directly finite regular rings satisfying the comparability axiom, and completely determined the directly finiteness of projective modules over these rings.

In the section 1 of this paper, without the assumption of simplicity in [2], we shall study directly finite projective modules over directly finite regular rings satisfying the comparability axiom. In Theorem 1.6, we shall give a criterion of the directly finiteness of projective modules over these rings. Using this criterion, in Theorem 1.7, we shall show the following result: Let R be a directly finite regular ring satisfying the comparability axiom. If P and Q are directly finite projective R-modules, then P $\oplus$ Q is directly finite.

In the section 2, $\aleph_0$-continuous projective modules over directly finite regular rings satisfying the comparability axiom are investigated. We show that every directly infinite $\aleph_0$-continuous projective modules over these rings is completely reducible.

Throghtout this paper, R is a ring with identity and R-modules are unitary right R-modules. If M and N are R-modules, then the notation N $\lesssim$ M (resp. N $\lesssim^{\oplus}$ M) means that N is isomorphic to a submodule of M (resp. N is iso-morhic to a direct summand of M). For a submodule N of an R-module M, N <⊕ M means that N is a direct summand of M. For a cardinal number α and an R-module M, αM denotes a direct sum of α-copies of M.

1. directly finite projective modules

First we recall some definitions and well-known results (cf. [1]).

Definition. A ring R is directly finite if xy = 1 implies yx = 1, for all x, y $\in$ R. An R-module M is directly finite if $End_R(M)$ is directly finite. A ring (a module M) is directly infinite if it is not directly finite. It is well-known that M is directly finite if and only if M is not isomorphic to a proper direct summand of M itself. A regular ring R is said to satisfy the comparability axiom provided that, for any x, y $\in$ R, either $xR \lesssim yR$ or $yR \lesssim xR$, or equivalently, for any finitely generated projective R-modules P and Q, either $P \lesssim Q$ or $Q \lesssim P$. A ring R is said to be unit-regular if, for each x $\in$ R, there is a unit (i.e. an invertible element) u of R such that xux = x.

Lemma 1.1. (a) Every directly finite regular ring satisfying the comparability axiom is unit-regular (cf. [1, Theorem 8.12]).

(b) Let R be a unit-regular ring. Then,

(1) Every finitely generated projective R-

module is directly finite ([1, Proposition 5.2]).

(2) Let B, A_1, A_2, ... be projective R-modules. If each A_n is finitely generated and $A_1 \oplus \ldots \oplus A_n \lesssim B$ for all n, then $\oplus A_n \lesssim B$ ([1, Proposition 4.8]).

(3) Let A be a finitely generated projective R-module. If B and C are any R-modules such that $A \oplus B \cong A \oplus C$, then $B \cong C$ ([1, Theorem 4.14]).

(4) Let A, B and C be projective R-modules such that $A = B \oplus C$. If C is finitely generated, then A is directly finite if and only if B is directly finite.

An R-module M is said to have the exchange property if, for any direct decomposition $G = M' \oplus C = \oplus_{i\varepsilon I} D_i$ with $M' \cong M$ and the index set I, there are submodules $D_i' \leq D_i$ ($i \varepsilon I$) such that $G = M' \oplus (\oplus_{i\varepsilon I} D_i')$.

Lemma 1.2. Every projective module over a regular ring has the exchange property.

Let R be a regular ring, and let P be a countably generated, but not finitely generated, projective R-module which has a cyclic decomposition $P = \oplus_{i=1}^{\infty} P_i$ satisfying the condition

(*) $P_i \gtrsim P_{i+1}$ for all i, and there exists no nonzero R-module X such that $X \lesssim P_i$ for all i.

Consider the following conditions on $\{P_i\}$:

(A) There exists a positive integer m such that

(1) for each $i \geq m$, $P_i \lesssim t_i P_{i+1}$ for some positive

integer t_i, and

(2) $\oplus_{i=m}^{\infty} P_i \lesssim tP_m$ for some positive integer t.

(B) There exists an increasing sequence $1 = i_1 < i_2 < \dots$, of positive integers such that $P_{i_n} \gtrsim \aleph_0 P_{i_{n+1}}$ for $n = 1, 2, \dots$.

(C) There exists a positive integer m for which the condition (1) of (A) holds.

Lemma 1.3. Let R be a directly finite regular ring satisfying the comparability axiom. Then, for a countably generated, but not finitely generated, projective R-module P with a cyclic decomposition $P = \oplus_{i=1}^{\infty} P_i$ satisfying (*), either (B) or (C) hold, but not both.

Proposition 1.4. Let R be a directly finite regular ring satisfying the comparability axiom. For a countably generated, but not finitely generated, projective R-modle P with a cyclic decomposition $P = \oplus_{i=1}^{\infty} P_i$ satisfying (*) and (C), the following are equivalent:

(a) P is directly finite.

(b) There exists a positive integer n > m such that $\oplus_{i=n}^{\infty} P_i \lesssim P_m$.

(c) There exists a positive integer t > 1 such that $\oplus_{i=m}^{\infty} P_i \lesssim tP_m$.

Proposition 1.5. Let R be a directly finite regular ring satisfying the comparability axiom. Then, a countably generated, but not finitely generated, projective R-module P with a cyclic decomposition $P = \oplus_{i=1}^{\infty} P_i$ satisfying (*) and (B) is directly finite.

We are now in a position to prove the main theorem.

Theorem 1.6. Let R be a directly finite regular ring satisfying the comparability axiom. Then for a projective R-module P, the following conditions are equivalent:

(a) P is directly finite.

(b) (1) P is finitely generated or

(2) P is a countably generated R-module with a cyclic decomposition $P = \oplus_{i=1}^{\infty} P_i$ satisfying (*) and (A), or (*) and (B).

(c) (1) P is finitely generated or

(2) P is a countably generated R-module with a cyclic decomposition $P = \oplus_{i=1}^{\infty} P_i$ satisfying (*) and the following condition:

(#) For each positive integer k, there exists a positive integer x(k) such that $\oplus_{i=x(k)}^{\infty} P_i \lesssim P_k$.

If, in addition, R has the nonzero socle, then a projective R-module P is directly finite if and only if P is finitely generated.

Proof. (a) $\leftrightarrow$ (b) has proved in [5], and (c) $\rightarrow$ (b) follows from Lemma 1.3 and Proposition 1.4. (a), (b) $\rightarrow$ (c). Assume that P is a countably generated, but not finitely generated, directly finite projective R-module with a cyclic decomposition $P = \oplus_{i=1}^{\infty} P_i$ satisfying (*), and that $P = \oplus_{i=1}^{\infty} P_i$ does not satisfy (#). Then there exists a positive integer k such that $P_k \not\lesssim \oplus_{i=n}^{\infty} P_i$ for all n, and so we can choose an increasing sequence $k+1 = k_1 < k_2 < \dots$ of positive integers such that $P_k \lesssim^{\oplus} \oplus_{i=k_n}^{k_{n+1}-1} P_i$ for $n = 1, 2, \dots$ by Lemma 1.1 (b). Therefore we have that $\aleph_0 P_k \lesssim^{\oplus} \oplus_{i=k_1}^{\infty} P_i <^{\oplus} P$, which contradicts the directly finiteness of P.

Remark 1. Let R be a nonzero simple directly finite regular ring satisfying the comparability axiom. Then, every non-finitely generated directly finite projective R-module P is a countably generated module with a cyclic decomposition $P = \oplus_{i=1}^{\infty} P_i$ satisfying (*) and (A), because R has a strictly positive dimension function ([1, Corollary 16.15]).

Remark 2. A directly finite regular ring R with the comparability axiom is classified into two cases:

(1) All non-finitely generated directly finite projective R-modules P have a cyclic decomposition $P = \oplus_{i=1}^{\infty} P_i$ satisfying (*) and (A).

(2) All non-finitely generated directly finite projective R-modules P have a cyclic decomposition $P = \oplus_{i=1}^{\infty} P_i$ satisfying (*) and (B).

Example. There exists a non-simple directly finite regular ring satisfying the comparability axiom which has the case 1 of above Remark 2. Choose a field F, let V be an uncountable-dimensional vector space over F, and set $Q = \mathrm{End}_F(V)$. Set $J = \{x \in Q \mid \dim_F(xV) < \infty\}$ and set $R = F + J$. Then R is a non-simple directly finite regular ring satisfying the comparability axiom with a nonzero socle J (see [1, Example 5.15, p.237 and p.238]). Set $R_n = M_{2^n}(R)$ for all $n = 1, 2, \ldots$. Map each $R_n \to R_{n+1}$ along the diagonal, i.e., map $x \to \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$ and set $S = \varinjlim R_n$. Let f be the natural map $R_n \to S$. Then S is a non-simple directly finite regular ring satisfying the comparability axiom with the zero socle, and $f\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}S \oplus f\begin{pmatrix} x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}S \oplus \ldots$ is a countably generated, but not finitely generated, directly finite projective S-module with (*) and (A), where x is contained in a simple right ideal of R. Therefore S is a desired one.

From Lemma 1.1, Theorem 1.6 and Remark 2, we have the following result.

Theorem 1.7. Let R be a directly finite regular ring satisfying the comparability axiom. If P and Q are directly finite projective R-modules, then so is $P \oplus Q$.

Lemma 1.8. Let R be a directly finite regular ring satisfying the comparability axiom, and let P and Q be countably generated, but not finitely generated, directly finite projective R-modules with cyclic decompositions P $= \oplus_{i=1}^{\infty} P_i$ and $Q = \oplus_{i=1}^{\infty} Q_i$ satisfying (*). If there exists a positive integer m such that $P_1 \oplus \dots \oplus P_m \gtrsim Q$, then $P \oplus\!\gtrsim Q$.

Proof. We can choose an increasing sequence $n_{m+1} < x(n_{m+1}) < n_{m+2} < x(n_{m+2}) < \dots$ of positive integers such that $P_i \gtrsim Q_{n_i} \gtrsim Q_{x(n_i)} \oplus \dots \oplus Q_{x(n_{i+1})-1}$ for each $i > m$ by (*) and Theorem 1.6. Noting that $P_1 \oplus \dots \oplus P_m \gtrsim Q_1 \oplus \dots \oplus Q_{x(n_{m+1})-1}$, we can conclude that $P \oplus\!\gtrsim Q$.

Theorem 1.9. Let R be a directly finite regular ring satisfying the comparability axiom. Then,

(a) For countably generated projective R-modules P and Q, either $P \lesssim Q$ or $Q \lesssim P$ hold.

(b) If P and Q are countably generated, but not finitely generated, directly finite projective R-modules such that $P \gtrsim Q$, then $P \oplus\!\gtrsim Q$.

Proof. Let $P = \oplus_{i=1}^{\infty} P_i$ and $Q = \oplus_{i=1}^{\infty} Q_i$ be cyclic decompositions of P and Q. (a) Assume that $P \not\lesssim Q$. Then there exists a positive integer n such that $P_1 \oplus \dots \oplus P_n \not\lesssim Q$ by Lemma 1.1 (b), and so $P_1 \oplus \dots \oplus P_n \gtrsim Q_1 \oplus \dots \oplus Q_m$ for all m. Hence $P \oplus\!> P_1 \oplus \dots \oplus P_n \gtrsim Q$.

(b) Assume that $P \gtrsim Q$. It is sufficient from Lemma 1.8 to show that $P_1 \oplus \dots \oplus P_n \not\lesssim Q$ for all n. There exists a positive integer n_1 such that $Q_1 \lesssim P_1 \oplus \dots \oplus P_{n_1}$ and

$Q_1 \not\lesssim P_1 \oplus \dots \oplus P_{n_1-1}$, and so there exists a positive integer m_1 such that $Q_1 \oplus \dots \oplus Q_{m_1} \lesssim P_1 \oplus \dots \oplus P_{n_1}$ and $Q_1 \oplus \dots \oplus Q_{m_1+1} \not\lesssim P_1 \oplus \dots \oplus P_{n_1}$. Using Lemma 1.1 (b), we have a direct sum decomposition $Q_{m_1+1} = X_{m_1+1} \oplus Y_{m_1+1}$ such that $Y_{m_1+1} \neq 0$ and $P_1 \oplus \dots \oplus P_{n_1} \cong Q_1 \oplus \dots \oplus Q_{m_1} \oplus X_{m_1+1}$. Noting that $Q_1 \oplus \dots \oplus Q_{m_1+1} \lesssim P$, again by Lemma 1.1 (b), there exists a positive integer $n_2 (> n_1)$ such that $Y_{m_1+1} \lesssim P_{n_1+1} \oplus \dots \oplus P_{n_2}$ and $Y_{m_1+1} \not\lesssim P_{n_1+1} \oplus \dots \oplus P_{n_2-1}$, and so there exists a positive integer $m_2 (> m_1)$ such that $Y_{m_1+1} \oplus Q_{m_1+1} \oplus \dots \oplus Q_{m_2} \lesssim P_{n_1+1} \oplus \dots \oplus P_{n_2}$ and $Y_{m_1+1} \oplus Q_{m_1+1} \oplus \dots \oplus Q_{m_2+1} \not\lesssim P_{n_1+1} \oplus \dots \oplus P_{n_2}$. Then we have a direct sum decomposition $Q_{m_2+1} = X_{m_2+1} \oplus Y_{m_2+1}$ such that $Y_{m_2+1} \neq 0$ and $P_{n_1+1} \oplus \dots \oplus P_{n_2} \cong Y_{m_1+1} \oplus Q_{m_1+1} \oplus \dots \oplus Q_{m_2} \oplus X_{m_2+1}$. Continuing this procedure, we have that $P \cong Q$ and so $P \oplus\gtrsim Q$.

Corollary 1.10. Let R be a directly finite regular ring satisfying the comparability axiom, and let P and Q be directly finite projective R-modules. If $P \lesssim Q$ and $Q \lesssim P$, then $P \cong Q$.

Let I be a set. Denote by $|I|$ the cardinal number of I.

Corollary 1.11. Let R be a directly finite regular ring satisfying the comparability axiom, and let P be a directly finite projective R-module such that $P = A \oplus B = C \oplus D$. If $A \cong C$ and $|B| = |D|$, then $B \cong D$.

2. $\aleph_0$-continuous projective modules

Let M be an R-module and let A(M) be the family of all submodules A of M such that A contains a countably generated essential submodule. We say that M is $\aleph_0$-continuous if M satisfies the following conditions (C_1) and (C_2):

(C_1) For any $A \in A(M)$ there exists a submodule A^* of M such that A is an essential submodule of A^* and $A^* <\oplus M$.

(C_2) For any $A \in A(M)$ with $A <\oplus M$, any exact sequence $0 \to A \to M$ splits.

Then note that, if M is nonsingular and $\aleph_0$-continuous then so is every direct summand of M (see [3] and [4]).

Lemma 2.1. Let R be a regular ring and let P be a non-finitely generated $\aleph_0$-continuous projective R-module with a cyclic decomposition $P = \oplus_{i \in I} P_i$. Then,

(a) We have a decending chain condition on `$\lesssim$' for

$\{P_i\}_{i\varepsilon I}$.

(b) If there exists $i_0 \varepsilon I$ such that $P_{i_0} \lesssim P_i$ for all $i \varepsilon I$, then we have a decending chain condition on '≦' for cyclic submodules of P_{i_0}, and hence we have a simple submodule of P_{i_0}.

Proof. (a) Let $P_{i_1} \gtrsim P_{i_2} \gtrsim \ldots$ with a non-isomorphic monomorphism $f_{i_n} : P_{i_{n+1}} \to P_{i_n}$ for $n = 1, 2, \ldots$. Then, for each n, there exists a nonzero submodule Q_{i_n} of P_{i_n} such that $P_{i_n} = f_{i_n}(P_{i_{n+1}}) \oplus Q_{i_n}$, and we see that $\oplus_{n=1}^{\infty} Q_{i_n} <\oplus P$ and $Q_{i_1} \oplus (\oplus_{n=1}^{\infty} f_{i_1} \ldots f_{i_n}(Q_{i_{n+1}}))$ $(\cong \oplus_{n=1}^{\infty} Q_{i_n})$ is a non-direct summand of P_{i_1}, which contradicts the $\aleph_0$-continuity of P.

(b) For each $i \varepsilon I$, let $f_i: P_{i_0} \to P_i$ be a monomorphism. Assume that there exists a family $\{A_i\}_{i=1}^{\infty}$ of cyclic submodules of P_{i_0} such that $A_1 > A_2 > \ldots$. Then, for each n, we can choose a nonzero submodule B_n of A_n such that $A_n = A_{n+1} \oplus B_n$, and we see that $\oplus f_{i_n}(B_n) <\oplus P$ and $\oplus B_n$ $(\cong \oplus f_{i_n}(B_n))$ is a non-direct summand of P_{i_0}, which contradicts the $\aleph_0$-continuity of P.

Theorem 2.2. Let R be a directly finite regular ring satisfying the comparability axiom and let P be a non-finitely generated projective R-module. Then, P is $\aleph_0$-continuous if and only if P is completely reducible.

Proof. Let $P = \oplus_{i \in I} P_i$ be a cyclic decomposition of P. " if part " is clear. " only if part " Assume that P is $\aleph_0$-continuous. From Lemma 2.1, there exists a simple projective R-module S such that $S \lesssim P_i$ for all $i \in I$. Note that each P_i is directly finite $\aleph_0$-continuous. Then, in view of Lemma 1.1 and [4, Theorem 2], we can choose a positive integer t_i such that $t_i S \gtrsim P_i$ for each i, and so $P = \oplus_{i \in I} P_i \lesssim^{\oplus} \oplus S$. Thus P is completely reducible.

In view of [3, Theorem 9], we have shown that $\aleph_0$-continuous directly finite projective modules over directly finite regular rings satisfying the comparability axiom are finitely generated. For $\aleph_0$-continuous directly infinite projective modules over these rings, we have the following.

Corollary 2.3. Let R be a directly finite regular ring satisfying the comparability axiom. Then every directly infinite $\aleph_0$-continuous projective R-module is completely reducible.

References

[1] K. R. Goodearl: Von Neumann regular rings, Pitman, London-San Francisco-Melbourne, 1979.

[2] J. Kado: Projective modules over simple regular rings Osaka J. Math. 16 (1979), 405-412.

[3] M. Kutami: $\aleph_0$-continuous directly finite projective modules over regular rings, Okayama J. Math. 25 (1983), 157-163.

[4] M. Kutami: $\aleph_0$-continuous modules, Osaka J. Math. 21 (1984), 31-38.

[5] M. Kutami: On projective modules over directly finite regular rings satisfying the comparability axiom, to appear in Osaka J. Math.

Department of Mathematics
Yamaguchi University
Yoshida, Yamaguchi 753
Japan

ON ARTINIAN RINGS WHOSE PROJECTIVE INDECOMPOSABLES ARE DISTRIBUTIVE

Yoshito YUKIMOTO

1. Introduction

A module $L \neq 0$ is called local (or hollow) if $L = L_1 + L_2$ implies $L = L_1$ or $L = L_2$. Especially a noetherian module is local if and only if it has a unique maximal submodule.

A module M is called distributive if $X \cap (Y + Z) = (X \cap Y) + (X \cap Z)$ for every submodules X, Y, Z in M. It is clear that any sub-(or factor) module of a distributive module is distributive.

We call a ring R right locally distributive, right LD in abbreviation, if it is right artinian and every projective indecomposable right R-module is distributive. It is evident that every local right module over a right LD-ring is distributive. The class of right LD-rings is a generalization of the class of right serial rings.

In this note right LD-rings are studied, mainly to construct a number of right LD-algebras.

2. Right LD-rings

The following lemma, shown by Fuller, is basic to study distributive modules over a semiperfect ring.

Lemma 1. Let R be a semiperfect ring. The following conditions on a right R-module M are equivalent:

(1) M is distributive.

(2) For every primitive idempotent e of R, the set

$\{xeR \mid x \in M\}$ of all homomorphic images of eR in M is linearly ordered.

(3) For every primitive idempotent e in R, the right eRe-module Me is uniserial.

Proof. See Fuller [1].

Theorem 2. The following conditions on a right artinian ring R are equivalent:

(1) Every projective indecomposable right R-module eR is distributive.

(2) For every primitive idempotent e and f of R, eRf is a uniserial right fRf-module.

(3) Every submodule in a projective indecomposable right R-module eR is characteristic, and the lattice of two-sided ideals in R is distributive.

Proof. (1)$\Leftrightarrow$(2) is a special case of Lemma 1.

(1)$\Rightarrow$(3). Every submodule in eR is a sum of local submodules and every local submodule is characteristic in eR by Lemma 1. Hence every submodule in eR is characteristic.

Let $\{e_i\}_{i=1}^{n}$ be a complete set of primitive idempotents, and let I,J,K be two-sided ideals in R. Then by the distributivity of e_iR,

$$e_i(I \cap (J+K)) = e_iI \cap (e_iJ + e_iK)$$
$$= (e_iI \cap e_iJ) + (e_iI \cap e_iK) = e_i(I \cap J) + e_i(I \cap K).$$

Summing up each sides of the equations (i=1,...,n), we have $I \cap (J+K) = (I \cap J) + (I \cap K)$.

(3)$\Rightarrow$(1). Let A be a submodule on a projective indecomposable module eR. Since A is characteristic in eR,

$eReA = A$. We notice that there exists a two-sided ideal A' ($= RA = ReA$) satisfying $eA' = A$.

If X,Y,Z are any submodules in eR, then

$$eX' \wedge (eY' + eZ') = e(X' \wedge (Y' + Z'))$$
$$= e((X' \wedge Y') + (X' \wedge Z')) = (eX' \wedge eY') + (eX' \wedge eZ').$$

Hence eR is distributive.

A right artinian ring is called right LD if it satisfies the equivalent conditions in Theorem 2.

3. Construction of right LD-algebras

We begin with a general remark on modules. For a module M we denote by $H(M)$ the inclusion-ordered set of all local submodules in M. A homomorphism $f : M \longrightarrow N$ of modules induces a correspondence : $H(M) \longrightarrow H(N)$, $X \longmapsto f(X)$. This correspondence is not a mapping in general (the image of some submodule M by f may be 0). If M is a module of finite length and f is an epimorphism, then there is a natural surjection

(*) $\{X \in H(M) \mid X \not\subseteq Ker(f)\} \longrightarrow H(N)$.

In fact, for every $Y \in H(N)$ there exist $X_1, \dots, X_n \in H(M)$ such that $f^{-1}(Y) = X_1 + \dots + X_n$, and $f(X_i) = Y$ for some $i \in \{1,\dots,n\}$. Moreover if M is distributive, i is unique by Lemma 1, and (*) is bijective.

In this section a method to construct some right LD-algebras is presented. We introduce some terminology.

Suppose C is a fixed set. A pair (P,t) of a set P and a mapping $t : P \longrightarrow C$ is called a C-set. When (P,t) and (P',t') are C-sets, a mapping $t : P \longrightarrow P'$ is called a C-set homomorphism if $t = t'f$. Moreover, in case that P and P' are posets, f is called a C-poset homomorphism if f is both a C-set homomorphism and a poset homomorphism.

A subposet U of a poset P is said to be an upper part of P if $x \in U$, $y \in P$ and $x \leq y$ imply $y \in U$. In particular, when P is finite, U is an upper part of P if and only if it is of the form $\{x \in P \mid x \geq p_1\} \cup \dots \cup \{x \in P \mid x \geq p_n\}$ $(p_1, \dots, p_n \in P)$.

Definition. Let C be a set. A family of finite C-posets $\{(P_1,t_1),\dots,(P_n,t_n)\}$ is called an admissible system (of C-posets) if it satisfies the following conditions($i=1,\dots,n$):

(1) Every poset P_i has a unique maximal element m_i.

(2) $C = \{t_1(m_1),\dots,t_n(m_n)\}$ $(t_i(m_i) \neq t_j(m_j)$ if $i \neq j)$.

(3) For every $c \in C$ the subposet $\{x \in P_i \mid t_i(x) = c\}$ is linearly ordered.

(4) For every $a \in P_i$, there exist $j \in \{1,\dots,n\}$ and a C-poset isomorphism from an upper part of P_j to $\{x \in P_i \mid x \leq a\}$.

Remark 1. Suppose that the conditions (1),(2),(3) of the above definition are satisfied and that f is a C-poset isomorphism from an upper part of P_j to $\{x \in P_i \mid x \leq a\}$. Then j is determined by $t_j(m_j) = t_i f(m_j) = t_i(a)$.

Let b_0 be any element in P_j and $b_0 \leq \dots \leq b_r = m_j$ be a chain with b_{k-1} being maximal in $\{x \in P_j \mid x \lneqq b_k\}$ $(k \in \{1,\dots,r\})$. Then $f(b_{k-1})$ is maximal in $\{x \in P_i \mid x \lneqq f(b_k)\}$ by (3). Therefore $f(b_0)$ is determined inductively, and the isomorphism in (4) of the above definition is unique.

Remark 2. By a similar argument we can replace (4) with

(4') If $a \in P_i$ is maximal in $P_i \setminus \{m_i\}$, there exist $j \in \{1,\dots,n\}$ and a C-poset isomorphism from an upper part of P_j to $\{x \in P_i \mid x \leq a\}$.

If R is a right LD-ring with the Jacobson radical J, and $\{e_i\}_{i=1}^{n}$ is a basic set of primitive idempotents for R, then by the first remark of this section, the posets $H(e_1R), \ldots, H(e_nR)$ form an admissible system with the mapping

$$\text{top}(\ ) \ (:= (\ )/(\ )J) : H(e_iR) \longrightarrow T(R) \qquad (i \in \{1, \ldots, n\}),$$

where $T(R)$ denotes the set of all isomorphism class of simple right R-modules.

Theorem 3. For any admissible system $\{(P_i, t_i)\}_{i=1}^{n}$ of C-posets, there exists a right LD-ring R such that $H(e_iR)$ is isomorphic to (P_i, t_i) ($T(R)$ is identified with C by a bijection β:
the isomorphism class of $\text{top}(e_iR) \longmapsto t_i(m_i)$), where $\{e_i\}_{i=1}^{n}$ is a basic set of primitive idempotents for R.

Proof. Since the C-poset isomorphism of (4) in the definition of admissible systems is uniquely determined by an element $a \in P_i$ (Remark 1), we denote the isomorphism by $\bar{a}$. Letting any element in P_j outside the domain of definition of $\bar{a}$ correspond to no element, the isomorphism $\bar{a}$ is extended to a correspondence $: P_j \longrightarrow P_i$, which operates P_j on the left. This extension is so trivial that it is also denoted by $\bar{a}$.

For two correspondences $\bar{a}_1 : P_i \longrightarrow P_j$ and $\bar{a}_2 : P_k \longrightarrow P_h$ ($a_1 \in P_j$, $a_2 \in P_h$), we define $\bar{a}_1\bar{a}_2 = 0$ if the composition $\bar{a}_1 \circ \bar{a}_2$ of the correspondences $= \emptyset$ or $h \neq i$, and otherwise $\bar{a}_1\bar{a}_2 = \bar{a}_1 \circ \bar{a}_2$ the composition of correspondences. Then the disjoint union of $\{\bar{a} \mid a \in P_i\}_i$ and $\{0\}$ forms a semigroup S with the multiplication defined above. If $a_1 \leq a_2$ in P_i, there exists $x \in S$ satisfying $\bar{a}_1 = \bar{a}_2 x$ by (4) in Definition.

Let $R := KS$ be the semigroup algebra of S over a field K. Then R is an artinian algebra over K with the Jacobson radical $\{\sum k_a \bar{a} \mid a \neq m_i$ for any i, and $k_a \in K\}$ and $\{\bar{m}_i\}_{i=1}^{n}$ is a basic set of primitive idempotents for R.

For any element $x \neq 0$ in $\bar{m}_j R \bar{m}_i$, $x = k_1 \bar{a}_1 + \dots + k_s \bar{a}_s$ with some $\bar{a}_1, \dots, \bar{a}_s : P_j \longrightarrow P_i$ (distinct) and $k_1, \dots, k_s \in K \setminus \{0\}$. Since $a_1, \dots, a_s \in P_i$ and $t_i(a_1) = \dots = t_i(a_s) = t_j(m_j)$, there exists uniquely the maximal element $a(x)$ of $\{a_1, \dots, a_s\}$ by (3) in the definition of admissible posets. If $a_u = a(x)$ $(u \in \{1, \dots, s\})$,

$x = \bar{a}_u (k_u +$ an element of the Jacobson radical$)$

and $xR = \overline{a(x)}R$. Therefore R is right LD by Theorem 2. It is easily verified that $\alpha_i : H(m_i R) \longrightarrow P_i$; $xR \longmapsto a(x)$ is an isomorphism of poset, and that the diagram

$$\begin{array}{ccc} H(\bar{m}_i R) & \xrightarrow{\alpha_i} & P_i \\ \text{top}(\ )\downarrow & & \downarrow t_i \\ T(R) & \xrightarrow{\beta} & C \end{array}$$

is commutative.

4. Right and left LD-rings

If R is a right LD-ring with a basic set $\{e_i\}_{i \in I}$ of primitive idempotents, we construct a semigroup S_R from the admissible system $\{(H(e_i R), (\ )/(\ )J)\}_{i \in I}$. Symmetrically, we have a semigroup ${}_R S$, the left version of S_R, from the admissible system $\{(H(Re_i), (\ )/J(\ ))\}_{i \in I}$, if R is a left LD-ring with a basic set $\{e_i\}_{i \in I}$ of primitive idempotents, where correspondences operate on the right.

The semigroup algebra KS_R (resp. $K_R S$) over a field K is considered a model of right (resp. left) LD-ring R with respect to the submodule-lattice structure of the projective indecomposable right (resp. left) R-modules.

However, if R is a right and left LD-ring, the "one-sided model" KS_R or K_RS is two-sided.

Lemma 4. Let e be an idempotent of a ring R, and suppose that every submodule in eR is chatacteristic. Then

(i) $Rx \leq Ry \Longrightarrow xR \leq yR$, for any $x, y \in eR$,

(ii) $zRxR = zxR$, for any $x \in eR$ and $z \in Re$.

Proof. (i). If $Rx \leq Ry$, there is $r \in eRe$ satisfying $x = ry$. Since yR is characteristic in eR, $xR = ryR \leq yR$.

(ii). From $eRxR = eRexR = xR$ the result follows.

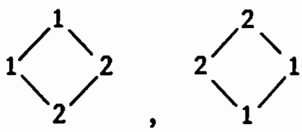
Proposition 5. Let R be a right and left LD-ring with a basic set $\{e_i\}_{i \in I}$ of primitive idempotents. Then $S_R \cong {}_RS$, $KS_R \cong K_RS$ is a right and left LD-ring, and one of the admissible systems $\{H(e_iR)\}_{i \in I}$, $\{H(Re_i)\}_{i \in I}$ is obtained by the other.

Proof. Adopting the notation in the proof of Theorem 3, via $xR \longmapsto Rx$ $(x \in e_iRe_j)$ a bijection : $S_R \longrightarrow {}_RS$ is defined by Lemma 4,(ii). The rest follows immediately.

5. Examples

The construction of right LD-rings in the section 3 is useful especially in case that the Loewy length is small.

(1) From the admissible system of C-posets $(C = \{1, 2\})$ with the Hasse diagram;



a QF-LD-ring is given, where the numbers on the vertices are their values in C.

(2) Let R be a right LD-ring with the admissible system of $\{1,2\}$-posets;

$$\begin{array}{c} 1 \\ | \\ 1 \end{array} \quad , \quad \begin{array}{cl} 2 & \\ | & \\ 1 & a \\ | & \\ 1 & b \end{array} .$$

Then R is not left LD, since there is no element x in S_R satisfying $\bar{b} = x\bar{a}$.

Reference

[1] K. R. Fuller: Rings of left invariant module type, Comm. in Algebra, 6(1978), 153-167.

Department of Mathematics
Osaka City University

ON P.P. RINGS AND GENERALIZED P.P. RINGS

Masayuki ÔHORI

Introduction. Recently, A. G. Naoum [4] and Y. Hirano [2] defined the concept of commutative generalized p.p. rings and obtained several results concerning these rings. On the other hand, G. M. Bergman [1] gave sheaf-theoretic characterizations of commutative p.p. rings. In the present note we deal with (non-commutative) generalized p.p. rings and extend some of their results to non-commutative rings.

Throughout this note the word "ring" will mean "non-zero associative ring with identity element". For any ring R, $J(R)$ denotes the Jacobson radical of R. The set of all idempotents of R is denoted by $E(R)$. A ring R is called normal if every idempotent of R is central. A right (resp. left) p.p. ring is a ring in which every principal right (resp. left) ideal is projective. A ring R is called a generalized right (resp. left) p.p. ring if for any element a of R, a^nR (resp. Ra^n) is projective for some positive integer n (depending on a). A ring which is both generalized right and left p.p. is said to be a generalized p.p. ring. Let a be an element of a ring R. We call a π-regular if there exists an integer n and an element b of R such that $a^nba^n = a^n$. Thus R is a π-regular ring if and only if every element of R is π-regular. Evidently, π-regular rings are generalized p.p. rings and it is easily verified that every reduced, generalized right p.p. ring is a (right and left) p.p. ring.

1. First, we generalize Hirano's theorems to non-commutative rings. The details of contents of this section will appear in [5]. We state the first of our theorems which contains [2, Theorem 2].

Theorem 1. Let R be a ring with normal, classical right quotient ring Q. Then the following are equivalent:

1) R is a generalized p.p. ring.

2) Every element of R is π-regular in Q and $E(Q) = E(R)$.

A ring R is called local if $R/J(R)$ is a division ring.

Let R be a commutative ring and let $\underline{q}$ be a proper ideal of R. Recall that $\underline{q}$ is a primary ideal of R provided every zero-divisor of $R/\underline{q}$ is nilpotent.

Proposition 1. Let R be a commutative ring and let Q be a classical quotient ring of R. Then the zero ideal (0) is a primary ideal of R if and only if Q is a local ring with nil Jacobson radical.

By Theorem 1 and Proposition 1, we see that the next contains [2, Corollary 3].

Theorem 2. Let R be a ring with normal, classical right quotient ring Q. Then the following are equivalent:

1) Q is a π-regular ring, $E(Q) = E(R)$ and R has no infinite sets of orthogonal idempotents.

2) R is a finite direct sum of rings whose classical right quotient rings are local rings with nil Jacobson

radicals.

In the proof of the above theorem, the following result of I. Kaplansky [3, Theorem 2.1] is essential.

Let R be a ring in which every non-nil right ideal contains a non-zero idempotent. Then either R contains an infinite number of orthogonal idempotents, or $R/J(R)$ is Artinian.

A. G. Naoum [4, Theorem 1.9] and Y. Hirano [2, Theorem 5] independently gave characterizations of commutative generalized p.p. rings by means of localization. Now we are going to generalize these results to non-commutative rings using the notion of central localization. For the definition of central localization, see [6, §1.7]. Let R be an arbitrary ring with center C and let S be a multiplicative subset of C, that is, S is a subset of C which contains 1 and is closed under multiplication in R. We let R_S be the localization of R by S.

Proposition 2. Let R be a ring with center C. Suppose that R has a normal, classical right quotient ring Q and that every element of R is π-regular in Q. Then for any multiplicative subset S of C, Q_S is a classical right quotient ring of R_S.

By the above proposition, the next contains [2, Theorem 5].

Theorem 3. Let R be a ring with center C and suppose that R has a normal, classical right quotient ring Q. Then the following are equivalent:

1) R is a generalized p.p. ring and for any maximal ideal $\underline{m}$ of C, the set of nilpotent elements of Q_S is invariant under right multiplication by elements of Q_S, where S is the complement of $\underline{m}$ in C.

2) Every element of R is π-regular in Q and for any maximal ideal $\underline{m}$ of C, Q_S is a local ring with nil Jacobson radical, where S is the complement of $\underline{m}$ in C.

Remark. Theorem 3 remains true if maximal ideals are replaced by prime ideals in 1) and 2).

2. In this section we give a theorem on normal, generalized p.p. rings of which the result of G. M. Bergman mentioned at the introduction is an immediate consequence.

Let R be a ring with center C and let $B(R)$ be the set of all central idempotents of R. For any $e, f \in B(R)$, we define $e \dotplus f = e + f - 2ef$, $e \cdot f = ef$ (product in R). Under these operations $B(R)$ becomes a Boolean ring. The set $X(R) = \text{Spec } B(R)$ of all maximal ideals of $B(R)$ endowed with the Zariski topology is a compact, totally disconnected Hausdorff space. For any $x \in X(R)$, we denote R/xR by R_x, which is called the stalk of R at x. Similarly for any C-algebra S and $x \in X(R)$, we set $S_x = S/xS$. Let $a \in R$ and $x \in X(R)$. Define a_x to be the coset $a + xR \in R_x$. The support of an element $a \in R$, written supp a, is the set $\{x \in X(R) \mid a_x \neq 0_x\}$, which is closed.

First, we need a lemma.

Lemma. Let R be a ring with classical right quotient ring Q. Suppose that for any $a \in R$ there exists a positive integer n such that for every integer $m \geq n$, supp a^m = supp a^n is open and closed, and that for any $x \in X(R)$ every zero-divisor of the stalk R_x is nilpotent. Then for any $x \in X(R)$, Q_x, the stalk of Q at x, is a classical right quotient ring of R_x.

Theorem 4. Let R be a normal ring with classical right quotient ring Q. Then the following are equivalent:

1) R is a generalized p.p. ring and for any $x \in X(R)$, the set of nilpotent elements of Q_x is invariant under right multiplication by elements of Q_x.

2) (i) For any $a \in R$, there exists a positive integer n such that for every integer $m \geq n$, supp a^m = supp a^n is open and closed.

(ii) For any $x \in X(R)$, every zero-divisor of R_x is nilpotent.

(iii) For any $x \in X(R)$, the set of nilpotent elements of Q_x is invariant under right multiplication by elements of Q_x.

3) For any $x \in X(R)$, Q_x is a local ring with nil Jacobson radical.

4) Q is a π-regular ring and $E(Q) = E(R)$.

Corollary 1. Let R be a normal ring with classical right quotient ring Q. Then the following are equivalent:

1) R is a p.p. ring.

2) (i) For any $a \in R$, supp a is open and closed.

(ii) For any $x \in X(R)$, R_x is an integral domain.

3) For any $x \in X(R)$, Q_x is a division ring.

4) Q is a von Neumann regular ring and $E(Q) = E(R)$.

Obviously, the above corollary contains [1, Lemma 3.1].

Corollary 2. Let R be a normal ring. Then the following are equivalent:

1) R is a generalized p.p. ring in which every non-zero-divisor is invertible and for any $x \in X(R)$, the set of nilpotent elements of R_x is invariant under right multiplication by elements of R_x.

2) For any $x \in X(R)$, R_x is a local ring with nil Jacobson radical.

3) R is a π-regular ring.

Corollary 3. Let R be a normal ring. Then the following are equivalent:

1) R is a p.p. ring in which every non-zero-divisor is invertible.

2) For any $x \in X(R)$, R_x is a division ring.

3) R is a von Neumann regular ring.

References

[1] G. M. Bergman: Hereditary commutative rings and centers of hereditary rings, Proc. London Math. Soc. (3) 23 (1971), 214-236.

[2] Y. Hirano: On generalized p.p. rings, Math. J. Okayama Univ. 25 (1983), 7-11.

[3] I. Kaplansky: Topological representation of algebras II, Trans. Amer. Math. Soc. 68 (1950), 62-75.

[4] A. G. Naoum: A generalization of Endo's theorem on p.p. rings, Islamabad J. Sci. 6 (1979), 67-69.

[5] M. Ôhori: On non-commutative generalized p.p. rings, to appear in Math. J. Okayama Univ.

[6] L. W. Rowen: Polynomial Identities in Ring Theory, Academic Press, New York-London-Toronto-Sydney-San Francisco, 1980.

Department of Mathematics
Shinshu University

LOCALIZATION IN ABELIAN CATEGORIES

Toyonori KATO and Tamotsu IKEYAMA

Let $\mathcal{A}$ be an abelian category and $(\mathcal{T}, \mathcal{F})$ a torsion theory for $\mathcal{A}$ in the sense of Dickson [2]. Recall that an object $L \in \mathcal{A}$ is $\mathcal{T}$-injective provided each diagram in $\mathcal{A}$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & T & \longrightarrow & 0 \quad \text{(exact)} \\ & & \downarrow & & & & & & \\ & & L & & & & & & \end{array}$$

with $T \in \mathcal{T}$ induces a morphism $M \longrightarrow L$ satisfying

$$K \longrightarrow M \longrightarrow L \;=\; K \longrightarrow L.$$

An object $L \in \mathcal{A}$ is called local at $(\mathcal{T}, \mathcal{F})$ if $L \in \mathcal{F}$ and L is $\mathcal{T}$-injective. We mainly concentrate in this note our attention on the full subcategory $\mathcal{L}$ of $\mathcal{A}$ consisting of all local objects at $(\mathcal{T}, \mathcal{F})$ in $\mathcal{A}$:

$$\mathcal{L} = \{L \in \mathcal{A} \mid L \in \mathcal{F} \text{ and } L \text{ is } \mathcal{T}\text{-injective}\}.$$

A morphism in $\mathcal{A}$

$$\lambda : M \longrightarrow L$$

is called the localization of M at $(\mathcal{T}, \mathcal{F})$ if

L1. $\operatorname{Ker} \lambda \in \mathcal{T}$ and $\operatorname{Cok} \lambda \in \mathcal{T}$,

L2. $L \in \mathcal{L}$.

Most of authors, including Lambek [6], have been studying localization in the fundamental situation when

A is the category of modules and the torsion theory (T, F) is hereditary except Ikeyama [4](cf. Tachikawa and Ohtake [8]).

The main purpose of this note is to discuss localization in the most general situation as mentioned above, extending some main results on localization in the fundamental situation to the corresponding ones in our general situation.

Our arguments in this note are categorical, so the dualistic version of our theorems also yields results on colocalization initiated by McMaster [7].

See the forthcoming paper [5] for the proofs and details in this note.

Let A be an abelian category. Recall that a pair (T, F) of classes of objects in A forms a torsion theory for A if

T1. $T \cap F = \{0\}$.

T2. If $T \longrightarrow X \longrightarrow 0$ is exact in A with $T \in T$, then $X \in T$.

T3. If $0 \longrightarrow X \longrightarrow F$ is exact in A with $F \in F$, then $X \in F$.

T4. For each $M \in A$ there exists an exact sequence in A

$$0 \longrightarrow tM \longrightarrow M \longrightarrow M/tM \longrightarrow 0$$

with $tM \in T$ and $M/tM \in F$(see Dickson [2]).

In the following, unless otherwise specified, let A be an abelian category, (T, F) a torsion theory for A and L the full subcategory of A consisting of all local objects at (T, F) in A.

Theorem 1. Let

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

be an exact sequence in A. Suppose that each M_i has its localization $M_i \longrightarrow L_i$ at (T, F) $(i = 1,2,3)$. Then the induced sequence

$$0 \longrightarrow L_1 \longrightarrow L_2 \longrightarrow L_3 \longrightarrow 0$$

is also exact in the category L.

Remark. In the above theorem, the category L may fail to be abelian (see Example 6 in the end of this note). Thus Theorem 1 states exactly in conclusion that, for the induced sequence

$$L_1 \xrightarrow{\alpha} L_2 \xrightarrow{\beta} L_3,$$

we have

(1) β is the cokernel of α in the category L,

(2) β has the kernel in the category L,

and

(3) α is a monomorphism in the category L.

The condition (3) is an essential part in this theorem which has been already obtained by Ikeyama [3, Lemma 3.2].

Let

$$C = \{C \in A \mid L \longrightarrow C \longrightarrow 0 \text{ is exact in } A \text{ with } L \in L\}.$$

Theorem 2. The category $\mathcal{L}$ is abelian if and only if, for each $C \in \mathcal{C}$, there exists an exact sequence in $\mathcal{A}$

$$0 \longrightarrow C/tC \longrightarrow L$$

with $L \in \mathcal{L}$.

Before stating our final theorem, we consult with

Lemma 3 ([1,p.19]). Let $L \in \mathcal{L}$. Suppose that L has its injective hull $E(L)$ in $\mathcal{A}$. Then $E(L)/L \in \mathcal{F}$.

Theorem 4. Suppose that each $F \in \mathcal{F}$ has its injective hull $E(F)$ in $\mathcal{A}$. Then the following conditions are equivalent:

(1) The category $\mathcal{L}$ is abelian.

(2) $E(E(L)/L) \in \mathcal{F}$ for each $L \in \mathcal{L}$.

(3) There exists a hereditary torsion theory for $\mathcal{A}$ at which $\mathcal{L}$ consists of all local objects.

This theorem combining with Lemma 3 yields

Corollary 5. Let R be a right hereditary ring with identity and $(\mathcal{T}, \mathcal{F})$ a torsion theory for Mod-R. Then the category $\mathcal{L}$ of all local right R-modules at $(\mathcal{T}, \mathcal{F})$ is always abelian.

We close this note with an example which shows that the category $\mathcal{L}$ is not necessarily abelian even if $\mathcal{A}$ is the category of modules.

Example 6 (Ikeyama [4, Example 5]). Let K be a field,

$$R = \left\{ \begin{pmatrix} a & K & K & K \\ 0 & K & K & K \\ 0 & 0 & a & K \\ 0 & 0 & 0 & K \end{pmatrix} \middle| \ a \in K \right\}$$

a subring of K_4, $E = (K\ K\ K\ K)$, $L = (0\ K\ K\ K)$, $X = (0\ 0\ K\ K)$ and $M = (0\ 0\ 0\ K)$ right R-modules by matrices operations. Then as is shown in Ikeyama [4, Example 5], we have

(1) $E = E(L)$,

(2) $\mathrm{Hom}_R(L/M, E(L)) = 0$,

(3) $\mathrm{Hom}_R(L/M, E(L)/L) = 0$,

(4) $\mathrm{Hom}_R(X/M, E(L)/L) \neq 0$.

Now, let $(\mathcal{T}, \mathcal{F})$ be a torsion theory for Mod-R generated by L/M, that is,

$$\mathcal{F} = \{F \in \text{Mod-}R \mid \mathrm{Hom}_R(L/M, F) = 0\}.$$

Then L is local at $(\mathcal{T}, \mathcal{F})$ by (2) and (3), nevertheless $E(E(L)/L) \notin \mathcal{F}$ by (4). Thus the category $\mathcal{L}$ is not abelian in view of Theorem 4.

References

[1] D. E. Bell, Torsion theories and relative homological algebra, Ph. D. Thesis, Vanderbilt University, Tennessee, 1971.

[2] S. E. Dickson, A torsion theory for abelian categories, Trans. Amer. Math. Soc., 121 (1966), 223-235.

[3] T. Ikeyama, Finitely hereditary torsion theories, Comm. Algebra, 10 (1982), 2233-2251.

[4] T. Ikeyama, Localizations with respect to non-hered-

itary torsion theories, Tsukuba J. Math., 7 (1983), 185-189.

[5] T. Kato and T. Ikeyama, Localization in abelian categories, to appear.

[6] J. Lambek, Torsion Theories, Additive Semantics and Rings of Quotients, Lecture Notes in Math. 177, Springer-Verlag, 1971.

[7] R. J. McMaster, Cotorsion theories and colocalization, Can. J. Math., 27 (1975), 618-628.

[8] H. Tachikawa and K. Ohtake, Colocalization and localization in abelian categories, J. Algebra, 56 (1979), 1-23.

Institute of Mathematics
University of Tsukuba

SIMPLE MODULES OF A DIRECT SUM OF UNIFORM MODULES

Syuhei TANIGUCHI

In a paper of M. Harada [2], a right Artinian serial (resp. coserial) ring is characterized as a right QF-2 (resp. QF-2*) ring satisfying that the class of all finite direct sums of hollow (resp. uniform) modules is closed under submodules (resp. factor modules). In his another paper [1], a new class of right Artinian rings satisfying the above condition and that any hollow module is quasi-projective is determined as a generalization of right serial rings. In this note, we shall give a class of rings which is a generalization of right coserial rings.

Throughout this note, R denotes a right Artinian ring with identity element and every module is a finitely generated unitary right R-module. For a module M, we denote its socle and injective hull as $Soc(M)$ and $E(M)$, a direct sum of k-copies of M as $M^{(k)}$, and put $S_0(M) = 0$ and $S_n(M)/S_{n-1}(M) = Soc(M/S_{n-1}(M))$, inductively.

Let U and V be uniform modules with $Soc(U) = Soc(V)$, and set $S = Soc(U)$ and $E = E(U)$, then we may assume that V is a submodule of E. We shall write Δ for $End_R(S)$. We can obtain the mapping ϕ from $End_R(E)$ to Δ by the restriction to S. Since E is injective, ϕ is an epimorphism. While we shall denote the image of the restriction mapping from $Hom_R(U,V)$ to Δ as $\Delta(U,V)$ and $\Delta(U)$ instead of $\Delta(U,U)$. It is known that $\Delta(U)$ is a subdivision ring of Δ, so we shall denote the left dimension of over $\Delta(U)$ as $\dim U$, if it is finite.

A right coserial ring satisfies the following conditions:

d-I : Every factor module of any finite direct sum of uniform modules is also a direct sum of uniform modules.

d-II : Every uniform module is quasi-injective.

Main Theorem. The following statements are equivalent:

(1) R satisfies the conditions d-I and d-II.

(2) R satisfies the condition d-I for direct sum of three uniform modules, and the condition d-II.

(3) For every indecomposable injective module E with $S = Soc(E)$, there are two uniserial modules A and B such that $E/S = A/S \oplus B/S$, and no factor of composition series of A/S is isomorphic to any one of B/S.

Regarding the condition d-I, we will consider the following conditions for a direct sum D of uniform modules, the former one is an equivalent to d-I.

d-(*) : Every factor module, with respect to any simple submodule, of D is also a direct sum of uniform modules.

d-(**) : Every simple submodule of D is contained in a non-trivial direct summand of D.

Lemma. Let $\{U_i\}_{i=1}^{n+1}$ be a set of uniform modules. If $D' = \sum_{i=1}^{n} \oplus U_i$ satisfies d-(**), then $D = \sum_{i=1}^{n+1} \oplus U_i$ satisfies d-(**).

Lemma. Let $\{U_i\}_{i=1}^{n+1}$ be a set of uniform modules with $|U_i| = n$. Then if $D = \sum_{i=1}^{n+1} \oplus U_i$ satisfies d-(*), then D does d-(**).

Theorem. Let $\{U_i\}_{i=1}^n$ be a set of uniform modules such that $Soc(U_i) \cong Soc(U_1)$. $D = \sum_{i=1}^n \oplus U_i$ satisfies d-(**) if and only if for any elements $\bar{\delta}_1, \bar{\delta}_2, \ldots, \bar{\delta}_n$ in Δ, there are an integer t and $\bar{y}_i \in \Delta(U_i, U_t)$ for any i such that $\sum_{i=1}^n \bar{y}_i \bar{\delta}_i = 0$ and $\bar{y}_t \neq 0$.

Proof. Put $S = Soc(U_1)$ and $E = E(U_1)$, then we can assume that all U_i are submodules of E. Let $p_i : D \to U_i$ and $j_i : U_i \to D$ be the projection and the injection. Assume that D satisfies d-(**). Let $\bar{\delta}_1, \bar{\delta}_2, \ldots, \bar{\delta}_n$ be elements in Δ, and $S^* = \{\sum_{i=1}^n \bar{\delta}_i(s) \mid s \in S\} \subset D$. Then there is a direct decomposition $D = D_1 \oplus D_2$ such that $D_1 \supset S^*$ and D_2 is uniform. Let $q : D \to U_2$ be the projection. Since $End_R(D_2)$ is a local ring, there is an integer t such that $qj_t p_t|_{D_2}$ is a unit. Let $z_i \in End_R(E)$ be an extension of $p_t q j_i \in Hom_R(U_i, U_t)$ for any i. Then z_t is an isomorphism. Put $y_i = z_t^{-1} z_i$, and $\bar{y}_i = \phi(y_i) \in \Delta(U_i, U_t)$. Then we get that $\sum_{i=1}^n \bar{y}_i \bar{\delta}_i = 0$. Conversely, let $S^* = \{\sum_{i=1}^n \bar{\delta}_i(s) \mid s \in S\}$ be a simple submodule of D, and δ_i elements in $End_R(E)$ such that $\bar{\delta}_i = \phi(\delta_i)$. By our assumption, there are an integer t and $\bar{y}_i \in \Delta(U_i, U_t)$ such that $\sum_{i=1}^n \bar{y}_i \bar{\delta}_i = 0$ and $\bar{y}_t \neq 0$. Let $y_i \in End_R(E)$ be an extension of $\bar{y}_i$ such that $y_i(U_i) \subset U_t$. Let $D' = \{u - f(u) \mid u \in \sum_{i \neq t} \oplus U_i\} \subset D$, where $f : \sum_{i \neq t} \oplus U_i \to U_t$ is a homomorphism given by setting $f(\sum_{i \neq t} x_i) = \sum_{i \neq t} y_t^{-1} y_i(x_i)$. Then $D = D' \oplus U_t$ and $D' \supset S^*$.

Corollary. Let U be a uniform module. Then $D = U^{(k+1)}$ satisfies d-(**) if and only if $\dim U \leq k$.

Corollary. Let $\{U_i\}_{i=1}^n$ be a set of uniform modules with $Soc(U_i) \cong Soc(U_1)$ and $k_i = \dim U_i$ for all i. Then $D = \sum_{i=1}^n \oplus U_i^{(k_i)}$ satisfies d-(**) if and only if there is a monomorphism from some U_i to another U_t.

Proof. We can take a set of linearly independent elements $\{\bar{\delta}_{ij}\}_{j=1}^{k_i}$ in Δ over (U_i). Applying Theorem for the set $\{\bar{\delta}_{ij}\}_{i,j}$, there is a non-zero element $\bar{y}_{ij} \in \Delta(U_i,U_t)$ for some pair $i \neq t$, which induces a monomorphism from U_i to U_t. Conversely if U_i is a submodule of U_t, then $\Delta(U_t) \subset \Delta(U_i,U_t)$. Therefore $U_i \oplus U_t(k_t)$ satisfies d-(**).

Proposition. When any indecomposable injective module E has $S_2(E) = E$, Main Theorem holds.

Proof. We shall show only the implication from (2) to (3). Let E be an indecomposable injective module with $S = \mathrm{Soc}(E)$. Since the condition d-(**) holds for any direct sum of three submodules of E, which are of length two. It is enough to show that $E/S = U/S \oplus V/S$ where U and V are only submodules of length two. If U/S is isomorphic to V/S via f, then a submodule W with $W/S = \{\bar{u} + f(\bar{u}) \mid \bar{u} \in U/S\}$ must be equal to either U or V. Therefore it must be $U/S \not\approx V/S$.

In following three lemmas, we shall assume that the statement (2) in Main Theorem holds.

Lemma. Let U be a uniform module with $S = \mathrm{Soc}(U)$. Then there are two submodules V_1 and V_2 of U such that $U/S = V_1/S \oplus V_2/S$, $S_2(V_1)$ and $S_2(V_2)$ are uniserial and $S_2(V_1)/S \not\approx S_2(V_2^{\,2})/S$, if $U \neq S$.

Proof. By the condition d-I, there are submodules

$\{V_i\}_{i=1}^n$ such that $U/S = \sum_{i=1}^n \oplus V_i/S$ and V_i/S are uniform. On the other hand put $E = E(U)$ and $E' = S_2(E)$, then $E' = \ell_E(J^2)$ and E' is R/J^2-injective, where J denotes the Jacobson radical of R and $\ell_E(J^2)$ the left annihilator of J^2 on E. Then from the above proposition, there are two submodules B' and C' such that $E'/S = B'/S \oplus C'/S$, $|B'|$, $|C'| \leq 2$ and $B'/S \not\cong C'/S$. Therefore we can get that $n \leq 2$ and $S_2(V_1)/S \not\cong S_2(V_2)/S$.

Lemma. Let U be a module. If $S_2(U)$ is uniserial, then U is also uniserial.

Proof. Assume that U is not uniserial. Then there are submodules A, B and C of a factor module of U such that A and B are uniserial modules of length three, $S_2(A) = C$ and $A/C \not\cong B/C$. And put $S = \mathrm{Soc}(C)$ and $D = A \oplus B$, and let S^* be a simple submodule of D. Then we can get that $D/S^* = V \oplus S$ for some uniform module $V = A + B$ and some simple module S, and that S is homomorphic image of neither A nor B, which is a contradiction.

Let E be an indecomposable injective module with $S = \mathrm{Soc}(E)$. Then since $S_k(E) = \ell_E(J^k)$ and $J^n = 0$ for any k and some n, E is of finite length. Hence there are two uniserial submodules A and B such that $E/S = A/S \oplus B/S$ and $S_2(A)/S \not\cong S_2(B)/S$, if $E \neq S$, from above lemmas.

Lemma. Let E, S, A and B be as above, and set $A_i = S_i(A)$ and $B_j = S_j(B)$. Then $A_{i+1}/A_i \not\cong B_{j+1}/B_j$ for any pair i, j.

Proof. We proceed by induction on $i+j$. The case of $i = j = 1$ is done. Assume that $i+j > 2$ and that A_{i+1}/A_i

is isomorphic to B_{j+1}/B_j. Then A_{i+1}/A_i is isomorphic to no factor of composition series of $A_i/S \oplus B_j/S$, by induction hypothesis. Put $K = A_i + B_j$, $C_0 = A_{i+1} + B_j$ and $C_2 = A_i + B_{j+1}$, then C_0/K is isomorphic to C_2/K via some isomorphis f, and let C_1 be a submodule of E such that $C_1/K = \{\bar{c} + f(\bar{c}) \mid \bar{c} \in C_0/K\}$. Then C_1 is a hollow module with a maximal submodule K. Put $D = C_1 + C_2$ and let S^* be a simple submodule of D. Then we can show that there is a uniform module $U = C_1 + C_2$ and a direct sum D' of uniform modules $\{U_t\}_{t=1}^n$ such that $D/S^* = U \oplus D'$, and that there is an integer t such that the composition mapping ν_t of the natural homomorphisms $C_1 \to D \to D/S^* \to D' \to U_t$ is not zero. And it holds that $\nu_t(C_1)/\nu_t(K) \cong C_1/K \cong A_{i+1}/A_i$, which means that A_{i+1}/A_i is isomorphic to some factor of composition series of U_t, and to some one of $A_i/S \oplus B_j/S$, by comparing the factors of composition series. Therefore we get a contradiction.

We had prove the implication from (2) to (3) in Main Theorem. It is enough to show the other implication that $D = \sum_{k=1}^n \oplus U_k$ satisfies d-(*), where $\{U_k\}_{k=1}^n$ is the set of uniform modules. Assume that every indecomposable injective module has the form in statement (3). If there is a monomorphism from some U_i to another U_j, then D satisfies d-(**), hence D satisfies d-(*) by induction on n. So showing following lemma, we can make an end of the proof.

Lemma. Let E be an indecomposable injective module with $S = \mathrm{Soc}(E)$. Assume that A and B are uniserial modules such that $E/S = A/S \oplus B/S$, and set $A_i = S_i(A)$ and $B_j = S_j(B)$. Let $U_k = A_{i_k} + B_{j_k}$; $1 \leq i_1 < i_2 < \cdots < i_n$, $j_1 > j_2 > \cdots > j_n \geq 1$, then $D = \sum_{k=1}^n + U_k$ satisfies d-(*).

Proof. Let S^* be a simple submodule of D. And set $V_k = A_{i_{k+1}} + B_{j_k}$ and $D' = (A_{i_1}/S) \oplus (\sum_{k=1}^{n-1} \oplus V_k) \oplus (B_{j_n}/S)$. Then we can show that $D/S^* = D'$.

References

[1] M. Harada: On maximal submodules of a finite direct sum of hollow modules III, (to appear).

[2] M. Harada: Serial rings and direct decompositions, J. Pure and Applied Algebra 31 (1984), 55-61.

[3] S. Taniguchi: Finite direct sum of uniform modules, Osaka J. Math. (to appear).

Department of Mathematics
Osaka City University

ON QF-1 ALGEBRAS

Ryohei MAKINO

In this note we shall announce the main results obtained in [9], [10] and [11].

Thrall [19] defined QF-1 algebras as follows: For a (finite dimensional) algebra A over a field, A is said to be QF-1 if every faithful representation of A coincides with its own second commutator. He proposed to give an internal characterization of QF-1 algebras, but this problem remains unsolved at the present time except in special cases. An algebra A is said to be of left colocal type (resp. local-colocal type) if every indecomposable (finitely generated) left A-module has a simple socle (resp. a simple socle or a simple top). It should be noted that all examples of QF-1 algebras known until now are either QF or of local-colocal type.

1. QF-1 algebras of local-colocal type and left serial QF-1 algebras

Let M be a left module over a ring A and K, L submodules of M. We denote $M = K \nabla L$ if the following conditions are satisfied:

(1) $M = K + L$.

(2) Both K and L are serial modules such that $|K|, |L| \geq 2$.

(3) $K \cap L$ coincides with the socle $S(M)$ of M.

We shall say a module M to be of type I in case $M = K \nabla L$ for suitable submodules K and L of M. By

Tachikawa's theorems in [15] and [16] it is known that if A is an algebra of local-colocal type over a field P with the radical N, then indecomposable left A-modules M are classified into the next four types;

(1) type I,

(2) the P-dual module $M^* = Hom_P(M, P)$ is of type I,

(3) the top $T(M) = M/NM$ of M is simple and the radical NM of M is of type I,

(4) serial.

We shall call indecomposable A-modules M satisfying (2) and (3) to be of type II and type III, respectively. Now we can state the next theorem, which is a solution of Thrall's problem to the case of algebras of local-colocal type.

Theorem A ([10, Theorem 5.5]). Let A be an algebra of local-colocal type, N the radical of A, e a primitive idempotent and K, L serial submodules of Ae. Then in order that A may be QF-1 it is necessary and sufficient that the following conditions are satisfied:

(1) If Ae is serial and Ne is projective, then every composition factor of Ae is isomorphic to a minimal left ideal. And the same holds for a primitive right ideal eA that is serial.

(2) If Ae is of type II, then every composition factor of Ae is isomorphic to a minimal left ideal. And the same holds for a primitive right ideal eA that is of type II.

(3) Let Ae be of type III and $Ne = K \nabla L$. If K is projective, then every composition factor of $K \nabla S^2(L)$ is isomorphic to a minimal left ideal. And the same holds

for a primitive right ideal eA that is of type III.

(4) Let Ae be of type II and $Ne = K \oplus L$. If $(Ae/K)^*$ is projective, then K^* is embedded into a primitive right ideal. And the same holds for a primitive right ideal eA that is of type II.

(5) Let Ae be of type II and $Ne = K \oplus L$. Let f be a primitive idempotent such that $Af/Nf \simeq K/NK$ and $\phi: eA \longrightarrow fN$ the projective cover of fN. If $(Ae/K)^*$ is projective, $(Ae/L)^*$ is nonprojective and eA is of type II, then the kernel $Ker(\phi)$ of ϕ is nonserial. And the same holds for a primitive right ideal eA that is of type II.

Tachikawa [18] proved that a left serial QF-1 algebra is of left colocal type. He established also the following structure theorem of algebras of left colocal type.

Theorem. (Tachikawa [16]) In order that an algebra A may be of left colocal type it is necessary and sufficient that the following conditions are satisfied:

(1) A is left serial.

(2) eN is either serial or a direct sum of two serial modules for any primitive idempotent e of A.

From Theorem A and the above results due to Tachikawa we have the next theorem, which is a solution of Thrall's problem to the case of left serial algebras.

Theorem B ([9, Theorem 3.1]). Let A be a left serial algebra and e, f primitive idempotents of A. Then in order that A may be QF-1 it is necessary and sufficient that the following conditions are satisfied:

(1) eN is either serial or a direct sum of two serial modules.

(2) If Ne is projective, then every composition factor of Ae is isomorphic to a minimal left ideal.

(3) If eN is not serial, then every composition factor of eA is isomorphic to a minimal right ideal.

(4) If $eN = K \oplus L$, where K and L are nonzero serial modules, and $(eA/K)^*$ is projective, then K^* is embedded into a primitive left ideal.

2. QF-1 algebras with infinitely many minimal faithful modules

H. Tachikawa gave the author a problem whether or not there exists a QF-1 algebra with infinitely many minimal faithful modules. We can answer this problem and show the existence of QF-1 algebras with infinitely many minimal faithful modules.

Let A be a diserial ring and M a standard left A-module such that $M = X_1 \vee \cdots \vee X_m$ with $NX_i = U_{i1} \oplus U_{i2}$ (see [7] for the definition of diserial rings and standard modules and for the meaning of $M = X_1 \vee \cdots \vee X_m$). Assume $S(U_{11}) \cong S(U_{m2}) \neq 0$ and $S^2(X_1/U_{12}) \not\cong S^2(X_m/U_{m1})$. Let $M^* = \oplus_{j=1}^{n} M_j$ with $M_j = M$ and $\phi\colon \oplus_{j=1}^{n} S_{j1} \longrightarrow \oplus_{j=1}^{n} S_{jm}$ an isomorphism where $S_{j1} = S(U_{11})$ ($\subset M_j$) and $S_{jm} = S(U_{m2})$ ($\subset M_j$), and let $\overline{M}^* = M^*/W$ with $W = \{x-\phi(x) \mid x \in \oplus_{j=1}^{n} S_{j1}\}$. If a left A-module K is indecomposable and isomorphic to a module $\overline{M}^*$ constructed as above, then K will be said to be of second kind. Donovan-Freislich [4] proved that if A is an algebra defined by a Brauer graph, then for left A-modules the following (#) holds:

(#) A left module is indecomposable if and only if it is either standard or of the second kind.

Now we shall assume the following situation. Let A be a basic diserial algebra and $1 = \Sigma_{i=1}^{r} e_i + \Sigma_{j=1}^{s} f_j + g$ $(r, s \geq 1)$ a decomposition into orthogonal primitive idempotents of A. Put $e = \Sigma_{i=1}^{r} e_i + g$ and $f = \Sigma_{j=1}^{s} f_j + g$. Assume that the next conditions hold:

(1) $(1-f)Af = 0$ and $(1-e)A(1-f) = 0$.

(2) eNg' is an ideal of eAe and $B = eAe/eNg$ is an indecomposable serial algebra.

(3) $C = fAf$ is an indecomposable weakly symmetric algebra.

Under this situation we have the following propositions.

Proposition A ([11, Proposition 2.3]). Let A be an algebra each of whose simple modules is one-dimensional. Assume that B is QF-1 and the above (#) holds for left C-modules. Further assume that there exists no standard left C-module into which $T(Cg) \oplus T(Cg)$ is embedded. Then

(1) If M is a minimal faithful left A-module having a direct summand isomorphic to Ag, then M is balanced;

(2) If the length of the first dominant chain end of B is 2, then A is QF-1.

Proposition B. If C is of infinite type, then A has infinitely many minimal faithful modules.

Let B be an indecomposable serial P-algebra each of whose simple modules is one-dimensional, and β a multiplicative Cartan basis of B. Assume that Be_0 is simple projective for a primitive idempotent e_0 ($\in \beta$). Let G be a graph, F an edge in G and N a node joined by F, and assume that the following conditions are satisfied:

(1) G is not a Brauer tree, i.e. G contains a cycle or G has at least two exceptional nodes.

(2) Except F there is no edge joined by N, and N is not exceptional.

(3) Any path on G starting from N and ending at N passes through an odd number of nodes (on counting N at the start and N at the end and all other repetitions).

Let C be a P-algebra defined from G and γ the multiplicative basis of C which is used in defining C from G (cf. [4]). Assume that C is diserial and Cf_0 is serial, where f_0 is the primitive idempotent corresponding to F. Further, concerning β and γ assume the following:

$$e_0 = f_0 \ (= g) \quad \text{and} \quad \beta \cap \gamma = \{g\}.$$

Now put $\alpha = \beta \cup \gamma$ and define a multiplication on α as follows

$$\begin{cases} xy = (xy \text{ in } B) & \text{if } x, y \in \beta \\ xy = (xy \text{ in } C) & \text{if } x, y \in \gamma \\ xy = 0 & \text{otherwise,} \end{cases}$$

where $x, y \in \alpha$. Let A be the P-algebra with the multiplicative basis α. Then this algebra A satisfies the above situation. By Donovan-Freislich's result stated above, (#) holds for left C-modules. Moreover, by the assumption on G there exists no standard left C-module into which $T(Cg) \oplus T(Cg)$ is embedded. Thus by Proposition A, if the length of the first dominant chain end is 2, then A is QF-1. Moreover, since G is not a Brauer tree, A has infinitely many minimal faithful modules by Proposition B.

References

[1] V. P. Camillo: Balanced rings and a problem of Thrall, Trans. Amer. Math. Soc. 149(1970), 143-153.

[2] V. P. Camillo and K. R. Fuller: Balanced and QF-1 algebras, Proc. Amer. Math. Soc. 34(1972), 373-378.

[3] V. Dlab and C. M. Ringel: Rings with the double centralizer property, J. Algebra 22(1972), 480-501.

[4] P. W. Donovan and M. R. Freislich: The indecomposable modular representations of certain groups with dihedral Sylow subgroup, Math. Ann. 238(1978), 207-216.

[5] K. R. Fuller: Generalized uniserial rings and their Kupisch series, Math. Z. 106(1968), 248-260.

[6] K. R. Fuller: Double centralizers of injectives and projectives over artinian rings, Illinois J. Math. 14(1970), 658-664.

[7] K. R. Fuller: On a generalization of serial rings II, Comm. Algebra 8(1980), 635-661.

[8] R. Makino: Balanced conditions for direct sums of serial modules, Sci. Rep. Tokyo Kyoiku Daigaku Sec. A 12(1974), 181-201.

[9] R. Makino: Double centralizers of minimal faithful modules over left serial algebras, to appear in J. Algebra.

[10] R. Makino: QF-1 algebras of local-colocal type, preprint.

[11] R. Makino: QF-1 algebras which have infinitely many minimal faithful modules, preprint.

[12] K. Morita: On algebras for which every faithful representation is its own second commutator, Math. Z. 69(1958), 429-434.

[13] C. M. Ringel: Socle conditions for QF-1 rings, Pacific J. Math. 41(1973), 309-336.

[14] T. Sumioka: Tachikawa's theorem on algebras of left colocal type, preprint.

[15] H. Tachikawa: On rings for which every indecomposable right module has a unique maximal submodule, Math. Z. 71(1959), 200-222.

[16] H. Tachikawa: On algebras of which every indecomposable representation has an irreducible one as the top or the bottom Loewy constituent, Math. Z. 75(1961), 215-227.

[17] H. Tachikawa: Double centralizers and dominant dimensions, Math. Z. 116(1970), 79-88.

[18] H. Tachikawa: Balancedness and left serial algebras of finite type, Proc. ICRA 1974, Springer Lecture Notes No. 488(1975), 351-378.

[19] R. M. Thrall: Some generalizations of quasi-Frobenius algebras, Trans. Amer. Math. Soc. 64(1948), 173-183.

Department of Mathematics
Fukushima University

ON BRUMMUND'S METHOD FOR REPRESENTATION-FINITE ALGEBRAS

Yutaka Kawada

During last ten years representation theory has developed by introducing.the new combinatorial and homological tools: quivers, partially ordered sets, vectorspace categories, Auslander-Reiten sequences and covering spaces. Simultaneously it seems that the classical amalgamation, i.e.Brummund's method has been forgotton. We,however, belive that representation theory on Artin rings is nothing else an analysis of its Jacobson radicals, and also that Brummund's method is still useful with this viewpoint.

In the present paper, we shall first prepare a fundamental lemma (in terms of Brummund's method) on left Artin rings, and by applying it we shall get a refinement of J. Waschbüsch [8, Satz 3]. Let further A be a representation-finite and a finite dimensional algebra over a commutative field K (K arbitrary), and e and f primitive idempotents of A. Then, by only applying the fundamental lemma to several cases, we shall be able to determine easily the structure of the bimodule ${}_{fAf}[fAe]_{eAe}$ (Propositions 8, 12 and 14). The proofs are self-contained and so we do not assume even the results of J.P. Jans [4].

However, since this is an abstract of the above paper, the proofs in §§2-4 are all omitted. The details of this paper will appear elsewhere.

1. Fundamental lemma

Throughout this section, A is assumed to be a left Artin ring and N its Jacobson radical. Two primitive idempotents e and f in A are said to be isomorphic to each other ($e \approx f$ in notation) if and only if $Ae \approx Af$ (or equivalently $eA \approx fA$). For a left A-module M, we denote by $|M|$ the composition length of M. Then the following lemma will play a fundamental rôle in this paper.

Lemma 1. Let $\{e_j\}$ be primitive idempotents of A and let A-epimorphisms ε_j: $Ae_j \to Ae_j\varepsilon_j$ $(1 \leq j \leq n,\ n \geq 3)$ satisfy the following conditions:

(I) $Ne_j\varepsilon_j \supset \mathrm{soc}\ Ae_j\varepsilon_j \supset Af_{j-1}u_j\varepsilon_j \oplus Af_jv_j\varepsilon_j$ $(1 \leq j \leq n,\ u_1 = v_n = 0)$ where $\{f_j\}$ are primitive idempotents of A.

(II) $v_j \notin \sum_{k>j}(Au_k + Av_k + \mathrm{Ker}\ \varepsilon_k)e_kNe_j + Au_j + \mathrm{Ker}\ \varepsilon_j$ $(1 \leq j \leq n-1)$.

(III) $|Ae_j\varepsilon_j| = |Ae_k\varepsilon_k|$ whenever $e_j \approx e_k$.

Setting now $M = \bigoplus_{j=1}^{n} Ae_j\varepsilon_j/L$, $L = \sum_{j=1}^{n-1} Af_j(v_j\varepsilon_j - u_{j+1}\varepsilon_{j+1})$ and

$$\begin{pmatrix}\delta_1\\ \vdots\\ \delta_n\end{pmatrix}: \bigoplus_{j=1}^{n} Ae_j \xrightarrow{\oplus\varepsilon_j} \bigoplus_{j=1}^{n} Ae_j\varepsilon_j \xrightarrow{\pi} M$$

where π denotes the canonical projection, we have $M = \sum_{j=1}^{n} Ae_j\delta_j$.

If the next condition (*) is valid, then M is indecomposable.

(*) For any A-homomorphism γ_j: $Ae_j \to M$ $(1 \leq j \leq n)$ such that $e_j\gamma_j = \sum_{k<j} e_jx_ke_k\delta_k + e_j\delta_j + \sum_{k>j} e_jx_ke_k\delta_k$ with $x_k \in N$ for every

$k < j$, it holds that $|\mathrm{Ker}\ \gamma_j| \leqq |\mathrm{Ker}\ \varepsilon_j|$ (although $\mathrm{Ker}\ \gamma_j \subset Av_j + \mathrm{Ker}\ \varepsilon_j$ by (II)).

Proof. First of all it should be noted that from the construction of M the properties below follow:

(1) $v_j\delta_j = u_{j+1}\delta_{j+1}$ $(1 \leqq j \leqq n-1)$ and $\mathrm{Ker}\ \delta_j = \mathrm{Ker}\ \varepsilon_j$ $(1 \leqq j \leqq n)$.

(2) If $\sum_{j=1}^{n} z_j\delta_j = 0$ with $z_j \in Ae_j$ $(1 \leqq j \leqq n)$, then there exist $y_j \in Af_j$ $(1 \leqq j \leqq n-1)$ such that $z_j \equiv -y_{j-1}u_j + y_jv_j$ mod $\mathrm{Ker}\ \varepsilon_j$ $(1 \leqq j \leqq n)$ where we set $y_o = y_n = 0$.

(3) If $\sum_{k=1}^{n} z_k\delta_k = 0$ and if $z_j \equiv xu_j + yv_j$ mod $\mathrm{Ker}\ \varepsilon_j$ for some j $(2 \leqq j \leqq n-1)$, then we have $\sum_{k<j} z_k\delta_k + xu_j\delta_j = 0$ and at the same time $yv_j\delta_j + \sum_{k>j} z_k\delta_k = 0$.

We can next assume without loss of generality that there are pairwise non-isomorphic primitive idempotents $\{g_r \mid 1 \leqq r \leqq t\}$ such that

$$\{e_j \mid 1 \leqq j \leqq n\} = \{g_r (n_r \text{ times}) \mid 1 \leqq r \leqq t\} \text{ and } n = \sum_{r=1}^{t} n_r.$$

Under this situation assume now that M is decomposable; that is,

$$M = M_1 \oplus M_2\ , \quad M_1 \neq 0 \quad \text{and} \quad M_2 \neq 0.$$

Since $\bigoplus_{j=1}^{n} Ae_j = \bigoplus_{r=1}^{t} Ag_r^{(n_r)} \longrightarrow M$ is the projective cover of M, we can take the projective cover of M_1 as follows:

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix} : \bigoplus_{i=1}^{m} Ah_i = \bigoplus_{r=1}^{t} Ag_r^{(m_r)} \longrightarrow M_1$$

where, for brevity, we set

$$\{h_i \mid 1 \leqq i \leqq m\} = \{g_r (m_r \text{ times}) \mid 1 \leqq r \leqq t\}$$

with $0 \leqq m_r \leqq n_r$ $(1 \leqq r \leqq t)$ and $m = \sum_{r=1}^{t} m_r$. Then $\{h_i \alpha_i\}$ generate M_1 and are expressed in the form:

$$(4) \quad h_i \alpha_i = \sum_{j=1}^{n} x_{ij} \delta_j \text{ with } x_{ij} \in h_i A e_j \quad (1 \leqq i \leqq m).$$

Here, repeatedly applying the elementary substitutions to the generators $\{h_i \alpha_i\}$ of M_1, we can assume from the first that there exist a sequence $\{\nu(i)\}$ satisfying

$$(5) \quad 1 \leqq \nu(1) < \nu(2) < \cdots < \nu(m) \leqq n, \quad h_i = e_{\nu(i)} \quad (1 \leqq i \leqq m)$$

and , for each i,

$$(6) \quad x_{ij} \in N \text{ for every } j < \nu(i), \quad x_{i,\nu(i)} = e_{\nu(i)} \text{ and}$$

$x_{\ell,\nu(i)} = 0$ for every $\ell \neq i$.

Then we remark that if $\sum_{i=1}^{m} z_i \alpha_i = 0$ for $z_i \in Ae_{\nu(i)}$ $(1 \leqq i \leqq m)$ then $z_1 \in Av_{\nu(1)} + \operatorname{Ker} \varepsilon_{\nu(1)}$. To prove this, by (4) we have

$$\sum_{i=1}^{m} z_i \alpha_i = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} z_i x_{ij} \right) \delta_j = 0,$$

and by (6) we see that the coefficient of $\delta_{\nu(i)}$ in the above is just z_i, and so by (2) z_i is expressed in the form:

$$z_i = x_i u_{\nu(i)} + y_i v_{\nu(i)} + w_{\nu(i)} \text{ with } w_{\nu(i)} \in \operatorname{Ker} \varepsilon_{\nu(i)}$$

$(1 \leqq i \leqq m)$ and in particular, for the case where $\nu(1) = 1$, $z_1 = y_1 v_1 + w_1$. Hence we have only to consider the case where $\nu(1) > 1$, and by (3) together with (1) we have

$$\sum_{j<\nu(1)} \left(\sum_{i=1}^{m} z_i x_{ij}\right)\delta_j + x_1 u_{\nu(1)}\delta_{\nu(1)} = \sum_{j<\nu(1)-1} \left(\sum_{i=1}^{m} z_i x_{ij}\right)\delta_j +$$

$$\left\{\sum_{i=1}^{m}(x_i u_{\nu(i)} + y_i v_{\nu(i)} + w_{\nu(i)})x_{i,\nu(1)-1}\right\}\delta_{\nu(1)-1} + x_1 u_{\nu(1)}\delta_{\nu(1)}$$

$= 0,$

whence it follows by (2) again that

$$\sum_{i=1}^{m}(x_i u_{\nu(i)} + y_i v_{\nu(i)} + w_{\nu(i)})x_{i,\nu(1)-1} \equiv - x_1 v_{\nu(1)-1} + y u_{\nu(1)-1}$$

mod Ker $\varepsilon_{\nu(1)-1}$ for some $y \in A$, where $x_{i,\nu(1)-1} \in N$ by (6).

Therefore $x_1 \in N$; otherwise, we would get

$$v_{\nu(1)-1} \in \sum_{i=1}^{m}(Au_{\nu(i)} + Av_{\nu(i)} + \text{Ker } \varepsilon_{\nu(i)})e_{\nu(i)}Ne_{\nu(1)-1}$$

$$+ Au_{\nu(1)-1} + \text{Ker } \varepsilon_{\nu(1)-1},$$

which contradicts our assumption (II). Thus we obtain the above remark.

By using this remark we want to show that

$$(7)\quad |M_1| \geqq \sum_{i=1}^{m} |Ae_{\nu(i)}\varepsilon_{\nu(i)}| - m + 1.$$

In case $m = 1$, this is valid; because then $M_1 = Ae_{\nu(1)}\alpha_1$ and $|\text{Ker } \alpha_1| \leqq |\text{Ker } \varepsilon_{\nu(1)}|$ by our assumption (*), and hence

$$|M_1| = |Ae_{\nu(1)}| - |\text{Ker } \alpha_1| \geqq |Ae_{\nu(1)}| - |\text{Ker } \varepsilon_{\nu(1)}|$$

$$= |Ae_{\nu(1)}\varepsilon_{\nu(1)}|.$$

In case $m > 1$, it readily follows from the above remark that

$$\text{Ker } \alpha_1 \subset Av_{\nu(1)} + \text{Ker } \varepsilon_{\nu(1)},$$

and that

$$Ae_{\nu(1)}\alpha_1 \cap \sum_{i=2}^{m} Ae_{\nu(i)}\alpha_i \subset (Av_{\nu(1)} + \text{Ker } \varepsilon_{\nu(1)})\alpha_1.$$

Therefore, by the modularity on composition lengths we have

$$\left|\sum_{i=1}^{m} Ae_{\nu(i)}\alpha_i\right|$$

$$= \left|Ae_{\nu(1)}\alpha_1\right| - \left|Ae_{\nu(1)}\alpha_1 \cap \sum_{i=2}^{m} Ae_{\nu(i)}\alpha_i\right| + \left|\sum_{i=2}^{m} Ae_{\nu(i)}\alpha_i\right|$$

$$\geq \left|Ae_{\nu(1)}\alpha_1\right| - \left|(Av_{\nu(1)} + \mathrm{Ker}\ \varepsilon_{\nu(1)})\alpha_1\right| + \left|\sum_{i=2}^{m} Ae_{\nu(i)}\alpha_i\right|$$

$$= \left|Ae_{\nu(1)}\right| - \left|Av_{\nu(1)} + \mathrm{Ker}\ \varepsilon_{\nu(1)}\right| + \left|\sum_{i=2}^{m} Ae_{\nu(i)}\alpha_i\right|$$

$$= \left|Ae_{\nu(1)}\varepsilon_{\nu(1)}\right| - 1 + \left|\sum_{i=2}^{m} Ae_{\nu(i)}\alpha_i\right|,$$

because $\left|Av_{\nu(1)} + \mathrm{Ker}\ \varepsilon_{\nu(1)}/\mathrm{Ker}\ \varepsilon_{\nu(1)}\right| = 1$. In virtue of induction on m, we have further

$$\left|\sum_{i=2}^{m} Ae_{\nu(i)}\alpha_i\right| \geq \sum_{i=2}^{m}\left|Ae_{\nu(i)}\varepsilon_{\nu(i)}\right| - (m-1) + 1$$

and consequently we obtain (7).

Moreover, by our assumption (III) we can always set

$$|Ae_j\varepsilon_j| = \ell_r \quad \text{for } e_j = g_r \ (1 \leq r \leq t),$$

and, since $\{e_{\nu(i)} \mid 1 \leq i \leq m\} = \{g_r (m_r \text{ times}) \mid 1 \leq r \leq t\}$, (7) is reformed as:

$$|M_1| \geq \sum_{r=1}^{t} m_r\ell_r - m + 1.$$

In the same way as M_1, we have also

$$|M_2| \geq \sum_{r=1}^{t} (n_r - m_r)\ell_r - (n - m) + 1,$$

because, $\bigoplus_{r=1}^{t} Ag_r^{(n_r-m_r)} \longrightarrow M_2$ is a projective cover of M_2.

Consequently we obtain

$$|M| = |M_1| + |M_2| \geq \sum_{r=1}^{t} n_r \ell_r - n + 2 = \sum_{j=1}^{n} |Ae_j\mathcal{E}_j| - n + 2,$$

which contradicts the fact that

$$|M| = \left|\bigoplus_{j=1}^{n} Ae_j\mathcal{E}_j\right| - |L| = \sum_{j=1}^{n} |Ae_j\mathcal{E}_j| - n + 1.$$

Thus M must be indecomposable.

Remark 1. In almost all cases, the assumption (II) may be replaced by:

(II') $(Au_j + Av_j + \mathrm{Ker}\ \mathcal{E}_j)e_jNe_k \subset \mathrm{Ker}\ \mathcal{E}_k$ for $j \neq k$.

Of course (II') is a stronger assumption than (II) (in the situation that (I) holds).

Remark 2. Under the circumstances that (I),(II) and (III) hold, a sufficient condition for (*) is given by:

(IV) $u_{j+1} \notin (v_j + \mathrm{Ker}\ \mathcal{E}_j)e_jAe_{j+1} + Av_{j+1} + \mathrm{Ker}\ \mathcal{E}_{j+1}$ $(1 \leq j \leq n-1)$.

In fact we have then $\mathrm{Ker}\ \gamma_j \subset \mathrm{Ker}\ \mathcal{E}_j$; otherwise, there would exist $v_j + w_j$ $(w_j \in \mathrm{Ker}\ \mathcal{E}_j)$ such that $(v_j + w_j)\gamma_j = 0$ and hence by (3)

$$v_j\delta_j + (v_j + w_j)x_{j+1}\delta_{j+1} + \sum_{k>j+1}(v_j + w_j)x_k\delta_k = 0,$$

and by (2) further

$$(v_j + w_j)x_{j+1} \equiv -u_{j+1} + yv_{j+1} \bmod \mathrm{Ker}\ \mathcal{E}_{j+1} \text{ for some } y \in A,$$

which contradicts (IV).

The following is useful, but is not applied in the present paper:

Corollary 2. Under the same notations as in Lemma 1, assume (I),(II') and (III), and further assume $e_j \not\simeq e_{j+1}$ for each j ($1 \leqq j \leqq n-1$). Then M is indecomposable.

Proof. Since (II') implies (IV), we have Ker $\gamma_j \subset$ Ker ε_j; that is, the condition (*) is automatically satisfied. So the corollary is a direct consequence of Lemma 1.

2. Depths and Waschbüsch's result

In this section also, let A be a left Artin ring and N its Jacobson radical with ρ as its nilpotency index, i.e. $N^{\rho-1} \neq 0$ and $N^{\rho} = 0$. We set $N^0 = A$ for convenience.

Definition. (Cf. Gabriel [3]) Given an element u in A, $\max\left\{m \mid u \in N^m,\ 0 \leqq m \leqq \rho\right\}$ is called the depth of u, denoted by d(u). We shall express as $u \sim v$ in case d(u) = d(v).

The following is a refinement of Waschbüsch [8, Satz 3].

Lemma 3. Let A be a representation-finite left Artin ring, and e and f primitive idempotints of A. For given elements u and v in fNe, the statements below are valid:

(i) If $d(u) \geqq d(v)$, then there exist elements $a \in fAf$ and $b \in eAe$ such that $u \equiv avb \mod N^{d(u)+1}$.

(ii) If d(u) = d(v), then there further exist elements c $\in fAf$ and $d \in eAe$ such that $u \equiv cv + vd \mod N^{d(u)+1}$.

Remark. (ii) in Lemma 3 was already obtained by Waschbüsch [8, Satz 4]. (Also cf. (IV)).

Corollary 4. (Cf. [8, Satz 4]) Let A be a representation-finite left Artin ring, and e and f primitive idempotents of A.

Let further M be a subbimodule of ${}_{fAf}[fAe]_{eAe}$, and set $m = \min \{d(v) \mid v \in M\}$. Then, for any element $u \in M$ with $d(u) = m$, we have $M = (fAf)u(eAe) = fN^{m}e$.

Definition. In Corollary 4, u is called a bigenerator of M, expressed as $M = \langle u \rangle$ in notation. Also m is called the depth of M, denoted by d(M).

Remark. Let A be a left (resp. right) Artin ring. Then A is called a left (resp. right) Köthe ring if every idecomposable left (resp. right) A-module has a square-free top as well as a square-free socle. About twenty-three years ago we established a characterization of left Köthe rings and a classification of all possible left indecomposables of those. (Cf. [6]) For left (resp. right) Köthe rings we enjoy the following nice property: Every subbimodule M stated in Corollary 4 is always expressed as $M = (fAf)u = u(eAe)$. (Cf. [6, I, Corollaries 5·2 and 5·9])

3. Types of bimodules and Preliminary results

Hereafter A is always assumed to be a representation-finite and a finite dimensional algebra over a commutative field K (K arbitrary). For given (but fixed) primitive idempotents e and f of A, we set

$$V_m = fN^{m}e/fN^{m+1}e \quad (0 \leq m \leq \rho),$$

and futher set respectively

$$E = eAe/eNe \quad \text{and} \quad F = fAf/fNf.$$

Then V_m is regarded as an F-E-bimodule. Keeping these notations, we obtain the next.

Lemma 5. If $V_m \neq 0$ then we have either $\dim_E V_m = 1$ or $\dim_F V_m = 1$. Especially, in case $e \approx f$, $\dim_E V_m = \dim_F V_m = 1$.

Lemma 6. If $\dim_E V_m = 1$ then $\dim_F V_m \leqq 3$. Similarly if $\dim_F V_m = 1$ then $\dim_E V_m \leqq 3$.

Definition. (Cf. Dlab and Ringel [2]) Let M be a non-zero subbimodule of fAe with $m = d(M)$. Then M is called a bimodule of type (p,q), in case $\dim_F V_m = p$ and $\dim_E V_m = q$.

Remark. In the above, $1 \leqq pq \leqq 3$ by Lemmas 5 and 6. If fAe itself is of type (p,q), so is every non-zero subbimodule; because, we see that

$$p : q = \dim_K E : \dim_K F,$$

which does not depend on m.

The next lemma was first obtained by Jans [4] for the case where K is algebraically closed, and later his result was essentially generalized by Dlab and Ringel [1] to the case where K is an infinite field. (Also cf. Waschbüsch [8, Satz 2])

Lemma 7. $|N^m e/N^{m+1}e| \leqq 3$ for every $m \geqq 1$.

Previous to prove the lemma, assume that $|N^m e/N^{m+1}e| \geqq 4$. Then $N^m e/N^{m+1}e$ becomes a direct sum of at least four simples, whose possible forms are classified as follows:

(a) $S_1 \oplus S_1 \oplus S_1 \oplus S_1 \oplus \cdots\cdots$,

(b) $S_1 \oplus S_1 \oplus S_2 \oplus S_2 \oplus \cdots\cdots$,

(c) $S_1 \oplus S_1 \oplus S_1 \oplus S_2 \oplus \cdots\cdots$,

(d) $S_1 \oplus S_1 \oplus S_2 \oplus S_3 \oplus \cdots\cdots$,

(e) $S_1 \oplus S_2 \oplus S_3 \oplus S_4 \oplus \cdots\cdots$, where $\{S_i = Af_i/Nf_i \mid 1 \leq i \leq 4\}$ denote mutually distinct simples. However (a) is impossible by Lemma 6. The other cases also will be shown to be impossible.

Lemma 7-1. (b) is impossible.

Lemma 7-2. (c) is impossible.

Lemma 7-3. (d) and (e) are both impossible.

Remark. As is easily seen from its proofs, Lemmas 7-1 and 7-3 are valid for representation-finite left Artin rings.

4. Structures of bimodules

In this section also A is assumed to be a representation-finite and a finite dimensional algebra over a commutative field K (K arbitrary). We always denote by e (resp. f) a primitive idempotent of A.

The next is somewhat or well known. (Cf. Waschbüsch [8, Satz 1])

Proposition 8. eAe is a uniserial algebra; that is,

${}_{eAe}[eAe]$ as well as $[eAe]_{eAe}$ is serial. If $eNe = \langle w \rangle$ then $(eNe)^i = eAew^i = w^i eAe$ for every $i \geq 1$.

Lemma 9. Let $fNe \neq 0$ and $e \not\approx f$. If $d((fNe)(eNe)) \leq d((fNf)(fNe))$ (resp. $d((fNe)(eNe)) \geq d((fNf)(fNe))$, then there is an integer $h \geq 1$ such that $(fNf)(fNe) = (fNe)(eNe)^h$ (resp. $(fNf)^h(fNe) = (fNe)(eNe))$, and that each subbimodule of fNe is expressed as $(fNe)(eNe)^k$ (resp. as $(fNf)^k(fNe))$ for some $k \geq 0$.

Remark. Lemma 9 was obtained by Waschbüsch [8, Satz 4]. Also cf. Kupisch [7].

Though the two cases stated in Lemma 9 are not completely separated, corresponding to those fNe is called a bimodule of degree (1,h) (resp. of degree (h,1)); that is,

Degree (1,h): $(fNf)(fNe) = (fNe)(eNe)^h$ with $h \geq 1$, which contains the case where $(fNf)(fNe) = 0$, $(fNe)(eNe)^{h-1} \neq 0$ and where $(fNe)(eNe)^h = 0$.

Degree (h,1): $(fNf)^h(fNe) = (fNe)(eNe)$ with $h \geq 1$, which contains the case where $(fNe)(eNe) = 0$, $(fNf)^{h-1}(fNe) \neq 0$ and where $(fNf)^h(fNe) = 0$.

Observe that these terminologies are consistint for bimodules of degree (1,1).

Lemma 10. Let fNe be a bimodule (with $e \not\cong f$) of type (p,1) with $p \geq 2$. Then fNe is never of degree (h,1) with $h \geq 2$.

Lemma 11. Let fNe be a bimodule (with $e \not\cong f$) of type (p,1) with $p \geq 2$. Then fNe is never of degree (1,h) with $h \geq 2$.

Combining Lemmas 10 and 11 we obtain the following.

Proposition 12. Let fNe (with $e \not\cong f$) be a bimodule of type (p,q) with $pq \neq 1$. Then fNe is always of degree (1,1), that is, $(fNf)(fNe) = (fNe)(eNe)$ ($=0$ or $\neq 0$). More precisely, denoting by $fNe = \langle u \rangle$, it is expressed as follows:

(i) Case of type (2,1) [resp. of type (3,1)]:

$$fNe = ueAe = fAfu \oplus fAfua\ [\oplus\ fAfub] \text{ for some } a \text{ [and } b]$$

in $eAe \setminus eNe$.

(ii) Case of type (1,2) [resp. of type (1,3)]:

$$fNe = fAfu = ueAe \oplus aueAe\ [\oplus\ bueAe] \text{ for some } a \text{ [and } b]$$

in $fAf \setminus fNf$.

Finally we shall consider the case of type (1,1).

Lemma 13. Let fNe (with $e \not\approx f$) be a bimodule of type (1,1) and of degree (1,h). If $(eNf)(fNe) \subset (eNe)^4$, then we have either $h \leqq 2$ or $(eNe)^3 = 0$ (with h = 3).

By using Lemma 13 we obtain the following.

Proposition 14. Let fNe (with $e \not\approx f$) be a bimodule of type (1,1) and of degree (1,h) (resp. of degree(h,1)). Then necessarily $h \leqq 3$. More precisely, setting respectively $fNe = \langle u \rangle$, $fNf = \langle v \rangle$ and $eNe = \langle w \rangle$, fNe is expressed in the form:

Case of degree (1,h): fNe = ueAe = fAfu or $fAfu \oplus fAfuw$ or $fAfu \oplus fAfuw \oplus fAfuw^2$ (according to h = 1, 2 and 3).

Case of degree (h,1): fNe = fAfu = ueAe or $ueAe \oplus vueAe$ or $ueAe \oplus vueAe \oplus v^2ueAe$ (according to h = 1, 2 and 3).

As for another amalgamations we shall study those in a subsequent paper.

References

[1] V. Dlab and C.M. Ringel: On algebras of finite representation type, J. of Algebra 33 (1975), 306-394.

[2] V. Dlab and C.M. Ringel: Indecomposable representation of graphs and algebras, Mem. of A.M.S. 173 (1976)

[3] P. Gabriel: The universal cover of a representation-finite algebra, Math. Lect. Notes 903 (1981), 68-105.

[4] J.P. Jans: On the indecomposable representations of algebras, Ann. of Math. 66 (1957), 418-429.

[5] Y. Kawada: On Köthe's problem concerning algebras for which every indecomposable module is cyclic I-II, Proc. Japan Acad. 37 (1961), 282-293.

[6] Y. Kawada: ——————, I-III, Sci.Rep. of Tokyo Kyoiku Daigaku, Sect. A; I, 7 (1962), 154-230; II, 8 (1963), 1-62; III, 8 (1964), 165-250.

[7] H. Kupisch: Basisalgebren symmetrischer Algebren und eine Vermutung von Gabriel, J. of Algebra 55 (1968), 58-73.

[8] J. Waschbüsch: Über Bimoduln in Artinringen vom endlichen Modultyp, Comm. in Algebra 8 (1980), 105-151.

Yutaka Kawada
Department of Mathematics
Kyoto Institute of Technology,
Matsugasaki, Sakyo-ku, Kyoto, 606

EXT ALGEBRAS

Daisuke TAMBARA

In this report we show that in a special case we can recover a ring A to some extent from the algebra $\mathrm{Ext}_A^*(\bar{A}, \bar{A})$ when $\bar{A}$ is a suitable factor ring of A. In §1 we state the main theorem and a related duality of Ext algebras. §2 consists of examples.

§1 Let A be a ring and X be an A-module. Then we have a graded algebra $\mathrm{Ext}_A^*(X, X) = \bigoplus_{i \geq 0} \mathrm{Ext}_A^i(X, X)$ whose multiplication is Yoneda product. Our main result is the following.

Theorem 1. Let R be a commutative noetherian ring and $\mathfrak{m}$ be a maximal ideal. Let A be a finite R-algebra and put $A_0 = (A/\mathfrak{m}A)/\mathrm{rad}(A/\mathfrak{m}A)$, $B_i = \mathrm{Ext}_A^i(A_0, A_0)$, $B = \bigoplus_{i \geq 0} B_i$. We denote by $\mathrm{Ext}_{B\text{-gr}}^i$ the Ext-groups in the abelian category of graded B-modules. For a graded B-module X and $q \in \mathbb{Z}$, the graded B-module $X[q]$ is defined by $X[q]_n = X_{q+n}$. With these notations we assume the following condition about the graded algebra B.

(LR) $\mathrm{Ext}_{B\text{-gr}}^p(B_0, B_0[q]) = 0$ $(p + q \neq 0)$ and $\mathrm{Ext}_{B\text{-gr}}^p(B_0, B_0[-p])$ is finitely generated over $k = R/\mathfrak{m}$.

Put $J = \mathrm{rad}(A_{\mathfrak{m}})$. Then we have an isomorphism of graded algebras

$$\mathrm{gr}_J A_{\mathfrak{m}} = \bigoplus J^i/J^{i+1} \cong \mathrm{Ext}_B^*(B_0, B_0).$$

There is a duality for the graded algebras satisfying the condition (LR).

Theorem 2. Let A be a graded algebra over a field k such that $A_n = 0$ $(n < 0)$ and A_0 is semi-simple and finite dimensional over k. If A satisfies (LR), then $B = \mathrm{Ext}^*_A(A_0, A_0)$ satisfies (LR) and we have an isomorphism of graded algebras

$$A \cong \mathrm{Ext}^*_B(B_0, B_0).$$

To explain categorical background of these dualities, we introduce the derived category for differential graded algebras.

Let k be a commutative ring. C(k) denotes the category of (unbounded) cochain complexes of k-modules. DG-algebra over k is by definition the monoid object of C(k) with respect to $\otimes_k$ (tensor product of complexes). For a DG-algebra $\Lambda^\cdot$ we have natural notion of a DG-$\Lambda^\cdot$-module. It is a pair $(X^\cdot, u)$ of $X^\cdot \in C(k)$ and $u: \Lambda^\cdot \otimes_k X^\cdot \longrightarrow X^\cdot$. Two maps of DG-$\Lambda^\cdot$-modules $f, g: X^\cdot \longrightarrow Y^\cdot$ are said to be homotopic if there is a map $h: X^\cdot \longrightarrow Y^\cdot[-1]$ of graded $\Lambda^\cdot$-modules (ignoring differentials) such that $f - g = dh + hd$. Then we have a category $K(\Lambda^\cdot)$ whose objects are the DG-$\Lambda^\cdot$-modules and morphisms are the equivalence classes of maps of DG-$\Lambda^\cdot$-modules by homotopy relation. $K(\Lambda^\cdot)$ is triangulated category in the sense of [1] and so the class of morphisms (called quasi-isomorphisms) which induce isomorphisms on cohomology admits the calculus of fractions in both sides. Now the category $D(\Lambda^\cdot)$ is defined as the localization of $K(\Lambda^\cdot)$ with respect to the class of the quasi-isomorphisms. The derived functors

$\overset{\mathbb{L}}{\underset{\Lambda^\cdot}{\otimes}}$ of $\underset{\Lambda^\cdot}{\otimes}$ and $\mathbb{R}\mathrm{Hom}^\cdot_{\Lambda^\cdot}$ of $\mathrm{Hom}^\cdot_{\Lambda^\cdot}$ are also defined. The fundamental properties of these functors are the following spectral sequences.

$$(1.\ 1) \qquad \mathrm{Tor}_{-p}^{H(\Lambda^\cdot)\text{-gr}}(H(X^\cdot),\ H(Y^\cdot)[q]) \Rightarrow H^{p+q}(X^\cdot \overset{\mathbb{L}}{\underset{\Lambda^\cdot}{\otimes}} Y^\cdot).$$

(Eilenberg-Moore)

$$(1.\ 2) \qquad \mathrm{Ext}^p_{H(\Lambda^\cdot)\text{-gr}}(H(X^\cdot),\ H(Y^\cdot)[q]) \Rightarrow H^{p+q}\mathbb{R}\mathrm{Hom}^\cdot_{\Lambda^\cdot}(X^\cdot,\ Y^\cdot) = \mathrm{Hom}_{D(\Lambda^\cdot)}(X^\cdot,\ Y^\cdot(p+q))$$

(multiplicative but not necessarily convergent)

($Y^\cdot(n)$ is the complex with $Y^\cdot(n)^p = Y^{n+p}$.)

Let A be as in Th. 1. Take an injective resolution $A_0 \longrightarrow I^\cdot$ in A-modules. $\Lambda^\cdot = \mathrm{Hom}^\cdot_A(I^\cdot, I^\cdot)$ is a DG-algebra over R. We introduce some notations. $D^+_f(\Lambda^{\cdot op})$ is the full subcategory of $D(\Lambda^{\cdot op})$ consisting of the DG-$\Lambda^{\cdot op}$-modules $X^\cdot$ such that $H^i(X^\cdot) = 0$ $(i \ll 0)$ and $H^i(X^\cdot)$ are finitely generated over $k = R/\mathfrak{m}$. $\mathcal{C}_\mathfrak{m}$ is the class of R-modules which are supported in $\{\mathfrak{m}\}$ and artinian. $D^+_{\mathcal{C}_\mathfrak{m}}(A)$ is the full subcategory of D(A) consisting of the complexes $X^\cdot$ such that $H^i(X^\cdot) = 0$ $(i \ll 0)$ and $H^i(X^\cdot) \in \mathcal{C}_\mathfrak{m}$. Now we can state

Theorem 3. Let A and B be as in Th. 1 (assume B satisfy (LR)) and $\Lambda^\cdot$ be as above. Further assume $\mathrm{gr}_J A_\mathfrak{m}$ is noetherian. Then we have an equivalence of derived categories

$$D^+_{\mathcal{C}_\mathfrak{m}}(A) \simeq D^+_f(\Lambda^{\cdot op}).$$

Remark. (i) If $\hat{A}_\mathfrak{m}$ denotes the $\mathfrak{m}$-adic completion of A, then $D^+_{\mathcal{C}_m}(A)$ is equivalent to $D^-_f(\hat{A}_m{}^{op})^{op}$ that is the full subcategory of $D(\hat{A}_m{}^{op})^{op}$ formed by the bounded above complexes with finitely generated cohomology (Matlis duality).

(ii) A relation between $D(\Lambda^{\cdot op})$ and the derived category of the abelian category of the graded $H(\Lambda^{\cdot op}) = B^{op}$-modules is given by (1. 2).

§2 Examples.

(2. 1) Let A be a finite dimensional algebra over a field k. Consider the following condition

(DR) there is a function h: {simple A-modules} $\longrightarrow$ $\mathbb{Z}$ such that $Ext^p_A(S, S') = 0$ for $p \neq h(S') - h(S)$.

If A satisfies (DR), then the algebra B = $Ext^*_A(A/rad(A), A/rad(A))$ also satisfies (DR) and we have $A \cong Ext^*_B(B/rad(B), B/rad(B))$. Moreover there is an equivalence

$$D_f(A)^{op} \simeq D_f(B).$$

This is not the case where Th. 2 can apply, but principle is same.

Examples of such algebras come from Cohen-Macaulay partially ordered sets (CM-posets for short) ([2]). Let $A = k[\Gamma]$ be the poset algebra of a CM-poset Γ. Simple A-modules are indexed by points of Γ and A satisfies (DR) with the function h defined by $h(S_x) = height(x)$ $(x \in \Gamma)$ ([2]).

The following examples (2. 2) $\sim$ (2. 4) lie in the situation of Th. 2.

(2. 2) Trivial extensions and tensor algebras.

Let K be a finite dimensional semi-simple algebra over a field k and V be a K-bimodule finitely generated over k. Put $V^{\vee} = \mathrm{Hom}_K(V, K)$ (dual of the left K-module V). We have two algebras $A = K \oplus V$ (trivial extension) and $B = T_{K^{op}}(V^{\vee}) = \bigoplus_{i \geq 0} (V^{\vee})^{\otimes i}$ (tensor algebra). Then

$$B \cong \mathrm{Ext}_A^*(K, K), \; A \cong \mathrm{Ext}_B^*(K^{op}, K^{op}).$$

(2. 3) Symmetric algebras and exterior algebras.

Let k be a field and V be a finite dimensional vector space over k. Put $A = S(V)$ (symmetric algebra) and $B = \Lambda(V^{\vee})$ (exterior algebra) where $V^{\vee}$ denotes the linear dual of V. Then

$$B \cong \mathrm{Ext}_A^*(k, k), \; A \cong \mathrm{Ext}_B^*(k, k).$$

Since both A and B are finite over their centers, Th. 3 can apply. In this case we have equivalences of simpler forms

$$D_f^-(B^{op})^{op} \simeq D_f^+(A^{\cdot op})$$

$$D_f^-(\hat{A})^{op} \simeq D_f^+(B^{\cdot op}) \qquad (\hat{A} = \prod_{i \geq 0} S^i(V))$$

where in the right sides $A^{\cdot}$ and $B^{\cdot}$ are viewed as DG-algebras with zero differentials.

As a deformed version we have the following pair (A, B) of algebras. Let $\lambda_{ij} \in k - \{0\}$ $(1 \leq i < j \leq n)$. Put $A = k[X_1, \ldots, X_n]$ with relations $X_j X_i = \lambda_{ij} X_i X_j$ $(i < j)$ and $B = k[Y_1, \ldots, Y_n]$ with relations $Y_i^2 = 0$, $Y_j Y_i = -\lambda_{ij}^{-1} Y_i Y_j$ $(i < j)$. Then we have

$$B \cong \mathrm{Ext}_A^*(k, k), \; A \cong \mathrm{Ext}_B^*(k, k).$$

(2. 4) Tree-like algebras.

Let k be a field and $n \in \mathbb{N}$. Let V_{ij} $(1 \leq i, j \leq n)$ be a family of finite dimensional vector spaces over k.

Put $K = k \times \ldots \times k$ (n factors), $V = \oplus V_{ij}$. V can be looked as a K-bimodule in the way $e_i V e_j = V_{ij}$ where $e_i = (0,\ldots, \overset{i}{1},\ldots, 0)$. We assume $V_{ii} = 0$ for all i. Take a base $\{x_{ij}^{(\alpha)}\}_\alpha$ of V_{ij}. The algebra A is the factor of the tensor algebra $T_K(V)$ by the ideal generated by $x_{ij}^{(\alpha)} \otimes x_{jk}^{(\beta)}$ ($\alpha > \beta$ if $j < k$, $\alpha \geq \beta$ if $j > k$). Then A is finite dimensional. Put $U_{ij} = \check{V}_{ij}$ (linear dual) and $U = \oplus U_{ij}$. U is a K-bimodule in the similar way. Let $\{y_{ij}^{(\alpha)}\}_\alpha$ be the dual base of U_{ij} for $\{x_{ij}^{(\alpha)}\}_\alpha$. The algebra B is the factor of the tensor algebra $T_K(U)$ by the ideal generated by $y_{ij}^{(\alpha)} \otimes y_{jk}^{(\beta)}$ ($\alpha \leq \beta$ if $j < k$, $\alpha < \beta$ if $j > k$). Then we have

$$B \cong \mathrm{Ext}_A^*(K, K), \quad A \cong \mathrm{Ext}_B^*(K, K).$$

This is only a special case of Th. 2, but provides an interesting example of finite global dimension. The construction above shows that for any K-bimodule V such that $e_i V e_i = 0$ for all i, there is an algebra A of finite global dimension (in fact $\leq \dim_k V$) such that $A/\mathrm{rad}(A) \cong K$ as algebras and $\mathrm{rad}(A)/\mathrm{rad}^2(A) \cong V$ as K-bimodules. Thus it occurs the following two problems.

(i) Does there exist a finite dimensional algebra A such that $\dim_k \mathrm{rad}(A)/\mathrm{rad}^2(A) < \mathrm{gl.dim}(A) < \infty$?

(ii) If A is a finite dimensional algebra of finite global dimension, is $\mathrm{Ext}_A^1(S, S) = 0$ for any simple A-module S?

For the latter we have

Proposition. (ii) is yes for the following two cases.

(a) $A = A_0 \oplus A_1 \oplus \ldots$ is a graded algebra such that A_0 is semi-simple and A is generated by A_1 over A_0.

(b) gl.dim(A) $\leqslant$ 2.

Remark. The author found in [3] that Roos mentioned a result (biduality of Löfwall) which seems relate to our Th. 2.

References

[1] R. Hartshone, Residues and Duality, L. N. in Math. Vol. 20, Springer-Verlag (1966).

[2] D. Quillen, Homotopy properties of the poset of non-trivial p-subgroups of a group, Adv. in Math. 28, 101-128 (1978).

[3] J. E. Roos, Relations between the Poincaré-Betti series of loop spaces and of local rings, in L. N. in Math. Vol. 740, Springer-Verlag (1979).

Department of Mathematics
Faculty of Science
Hokkaido University
Sapporo 060, Japan

ON SCHUR ALGEBRA OVER $\mathbb{Q}$

Michitaka HIKARI

Let A be a central simple algebra over $\mathbb{Q}$. If there exists a finite multiplicative subgroup G of A such that A is spanned by G with coefficients in $\mathbb{Q}$, then A is called a Schur algebra over $\mathbb{Q}$. The Schur subgroup $S(\mathbb{Q})$ of the Brauer group $Br(\mathbb{Q})$ consists of those algebra classes that contain a Schur algebra over $\mathbb{Q}$. By Benard [2] and Fields [3] an algebra class $[A]$ of $Br(\mathbb{Q})$ belongs to $S(\mathbb{Q})$ if and only if the index of A is either 1 or 2. This means that

$$S(\mathbb{Q}) = \{[(\frac{a,b}{\mathbb{Q}})] \mid a,\ b \in \mathbb{Q}\} ,$$

where $(\frac{a,b}{\mathbb{Q}})$ is a quaternion algebra over $\mathbb{Q}$, i.e. $(\frac{a,b}{\mathbb{Q}}) = \mathbb{Q} + \mathbb{Q}i + \mathbb{Q}j + \mathbb{Q}k,\ i^2 = a,\ j^2 = b,\ ij = -ji = k.$

Let Δ be a division algebra central over $\mathbb{Q}$. If $[\Delta] \in S(\mathbb{Q})$, then there exists a positive integer n such that $M_n(\Delta)$ is a Schur algebra.

For a positive integer n we set

$$S^{(n)}(\mathbb{Q}) = \{[\Delta] \in S(\mathbb{Q}) \mid M_n(\Delta) \text{ is a Schur algebra}\} .$$

By Amitsur [1], it is easily seen that

$$S^{(1)}(\mathbb{Q}) = \{[\mathbb{Q}],\ [(\frac{-1,-1}{\mathbb{Q}})],\ [(\frac{-3,-1}{\mathbb{Q}})]\}.$$

More generally, using the Brauer-Witt theorem we can prove the following

Theorem. Let Δ be a division algebra central over $\mathbb{Q}$ such that $[\Delta] \in S(\mathbb{Q})$. Let Π be the set of rational primes p with $\operatorname{inv}_p \Delta = \frac{1}{2}$. We set

$$n(\Pi) = \begin{cases} \prod_{p\in\Pi} p & \text{if } 2 \notin \Pi, \\ 2\prod_{p\in\Pi} p & \text{if } 2 \in \Pi \end{cases}$$

Then $[\Delta] \in S^{(m)}(Q)$ if and only if $\varphi(n(\Pi)) \mid 2m$.

The list of all set Π, which satisfies $\varphi(n(\Pi)) \mid 2m$ for $m \leq 5$, is given as follows.

m	Π
1	∅, {2}, {3}
2	∅; {2}, {3}, {5}, {2, 3}
3	∅, {2}, {3}, {7}
4	∅, {2}, {3}, {5}, {2, 3}, {2, 5}, {3, 5}
5	∅, {2}, {3}, {11}

Then we have

$$S^{(1)}(Q) = \{[Q],\ [(\tfrac{-1,-1}{Q})],\ [(\tfrac{-3,-1}{Q})]\},$$

$$S^{(2)}(Q) = \{[Q],\ [(\tfrac{-1,-1}{Q})],\ [(\tfrac{-3,-1}{Q})],\ [(\tfrac{-5,-2}{Q})],\ [(\tfrac{-3,-1}{Q})]\},$$

$$S^{(3)}(Q) = \{[Q],\ [(\tfrac{-1,-1}{Q})],\ [(\tfrac{-3,-1}{Q})],\ [(\tfrac{-7,-1}{Q})]\},$$

$$S^{(4)}(Q) = \{[Q],\ [(\tfrac{-1,-1}{Q})],\ [(\tfrac{-3,-1}{Q})],\ [(\tfrac{-5,-2}{Q})],$$
$$[(\tfrac{3,-1}{Q})],\ [(\tfrac{5,-2}{Q})],\ [(\tfrac{5,-3}{Q})]\},$$

$$S^{(5)}(Q) = \{[Q],\ [(\tfrac{-1,-1}{Q})],\ [(\tfrac{-3,-1}{Q})],\ [(\tfrac{-11,-1}{Q})]\}.$$

References

[1] S.A. Amitsur: Finite subgroups of division rings, Trans. Amer. Math. Soc. 80 (1955), 361-386.

[2] M. Benard: Quaternion constituents of group algebras, Proc. Amer. Math. Soc. 30 (1971), 217-219.

[3] K.L. Fields: On the Brauer-Speiser theorem, Bull. Amer. Math. Soc. 77 (1971), 223.

[4] T. Yamada: The Schur Subgroup of the Brauer Group, Lecture Notes in Math. 397, Springer, Berlin, 1974.

Department of Mathematics
Keio University

Introduction to $\sqrt{\text{Morita}}$ Theory

Mitsuhiro TAKEUCHI

This report is derived from the author's article [1] in preparation. Let us begin with a finite extension of fields A/k. It is known (see Rosenberg and Zelinsky [2] for example) that there is an isomorphism of abelian groups

(1) $$H^2(A/k) \cong Br(A/k)$$

where the left-hand side $H^2(A/k)$ denotes the second Amitsur cohomology group of the multiplicative group G_m with respect to the extension A/k and the right-hand side Br(A/k) denotes the Brauer group of Azumaya k-algebras split by A. Sweedler [3] shows that this isomorphism is realized as follows.

By an <u>A/k-algebra</u> we mean a pair (E,i) where E is a k-algebra and $i: A \to E$ a k-algebra map. Let

(2) $$\sigma = \sum_i a_i \otimes b_i \otimes c_i$$

be an element in $A \otimes_k A \otimes_k A$. For an A/k-algebra E we define a new multiplication $*_\sigma$ on E as follows. For x, y in E we put

(3) $$x *_\sigma y = \sum_i a_i x b_i y c_i .$$

If σ is an Amitsur 2-cocycle (this means σ is a unit in particular), then the product $*_\sigma$ is associative with unit

(4) $$e_\sigma = (\sum_i a_i b_i c_i)^{-1}.$$

We denote by E^σ the k-algebra E with product $*_\sigma$. In particular we have a k-algebra A^σ, and E^σ becomes an A^σ/k-algebra. On the other hand, we have an isomorphism of k-algebras

(5) $$A \cong A^{\sigma},\ a \leftrightarrow ae_{\sigma}.$$

Thus, for an Amitsur 2-cocycle σ for A/k we have an endofunctor

(6) $$E \mapsto E^{\sigma}$$

of the category $\underline{\underline{A}}_{A/k}$ of all A/k-algebras.

The endomorphism algebra $\mathrm{End}_k(A)$ has a natural structure of A/k-algebra. Sweedler [3] shows that the isomorphism (1) is realized by

(7) $$\sigma \mapsto \mathrm{End}_k(A)^{\sigma}.$$

This report is concerned with generalizations of the functor (6).

First note that A/k need not be a field extension to have the formation E^{σ}. Let A/k be an extension of commutative rings. For an Amitsur 2-cocycle σ for A/k and an A/k-algebra E, we can define the A^{σ}/k-algebra E^{σ} as above and we have the isomorphism (5). Hence the endofunctor (6) is well-defined in this case. Furthermore, the functor (6) is an equivalence with quasi-inverse $E \mapsto E^{\sigma^{-1}}$ corresponding to the inverse σ^{-1}.

In the following, let k be a commutative ring, and let A be a (non-commutative) k-algebra. Sweedler [3] generalizes the notion of Amitsur 2-cocycles as follows. The element in (2) is called a <u>(Sweedler) 2-cocycle</u> if the following two conditions are satisfied:

(8) $\sum_{i,j} a_i a_j \otimes b_j \otimes c_j b_i \otimes c_i = \sum_{i,j} a_i \otimes b_i a_j \otimes b_j \otimes c_j c_i,$

(9) There is an element e_{σ} in A such that

$$\sum_i a_i e_{\sigma} b_i \otimes c_i = 1 \otimes 1 = \sum_i a_i \otimes b_i e_{\sigma} c_i.$$

The element e_σ is uniquely determined. If A is commutative, condition (8) is equivalent to the Amitsur condition. If σ is a unit in addition, condition (9) follows from (8) with (4). Hence, in the commutative case, Amitsur 2-cocycles are just invertible 2-cocycles.

For an element σ in (2) put

$$\sigma(X,Y) = \sum_i a_i X b_i Y c_i \tag{10}$$

with indeterminates X, Y. The condition (8) (resp. (9)) is equivalent to condition (11) (resp. (12)).

$$\sigma(\sigma(X,Y),Z) = \sigma(X,\sigma(Y,Z)), \tag{11}$$

$$\sigma(e_\sigma,X) = X = \sigma(X,e_\sigma). \tag{12}$$

Thus σ is a 2-cocycle if and only if $\sigma(X,Y)$ is a <u>formal ring law</u>. For an A/k-algebra E, we have

$$x *_\sigma y = \sigma(x,y) \tag{13}$$

for $x, y \in E$. This observation shows immediately that E^σ is an associative k-algebra with unit e_σ. We have a k-algebra A^σ, and E^σ becomes an A^σ/k-algebra. One should note that the isomorphism (5) does not hold in the non-commutative case. Hence the functor (6) is from $\underline{\underline{A}}_{A/k}$ to $\underline{\underline{A}}_{A^\sigma/k}$.

Let σ be a 2-cocycle for A/k. If A is commutative, is invertible, i.e., an Amitsur 2-cocycle if and only if the functor (6) is an equivalence. Hence in general, we define the 2-cocycle σ to be <u>invertible</u> if the functor (6) is an equivalence from $\underline{\underline{A}}_{A/k}$ onto $\underline{\underline{A}}_{A^\sigma/k}$.

To proceed with the theory, I have to talk about <u>monoidal categories</u>. The reader is referred to Eilenberg and Kelly [4] for a generality of monoidal categories and monoidal functors.

A <u>monoidal category</u> is a combination of a category $\underline{\underline{M}}$, a bifunctor $\otimes$: $\underline{\underline{M}} \times \underline{\underline{M}} \to \underline{\underline{M}}$, an object $I_{\underline{\underline{M}}}$ in $\underline{\underline{M}}$, and two kinds of natural transformations

$$(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z), \tag{14}$$

$$X \otimes I_{\underline{\underline{M}}} \cong X \cong I_{\underline{\underline{M}}} \otimes X \tag{15}$$

for X, Y, Z in $\underline{\underline{M}}$. The natural transformations (14) and (15) are assumed to satisfy the <u>coherence</u> condition.

For a monoidal category $\underline{\underline{M}}$, a <u>monoid object</u> in $\underline{\underline{M}}$ means a triple (X,m,u), where X is an object in $\underline{\underline{M}}$, m: $X \otimes X \to X$ and u: $I_{\underline{\underline{M}}} \to X$ are maps in $\underline{\underline{M}}$ such that the associativity and the unit condition for usual monoids (or semigroups) are satisfied. The category of all monoid objects in $\underline{\underline{M}}$ is denoted by $\underline{\underline{M}}^{mon}$.

For two monoidal categories $\underline{\underline{M}}$, $\underline{\underline{N}}$, a <u>monoidal functor</u> $\underline{\underline{M}} \to \underline{\underline{N}}$ consists of a functor Γ: $\underline{\underline{M}} \to \underline{\underline{N}}$, a natural transformation

$$\Gamma(X) \otimes \Gamma(Y) \to \Gamma(X \otimes Y) \tag{16}$$

for X, Y in $\underline{\underline{M}}$, and a map in $\underline{\underline{N}}$

$$I_{\underline{\underline{N}}} \to \Gamma(I_{\underline{\underline{M}}}). \tag{17}$$

The structure maps (16) and (17) are assumed to commute with natural transformations (14) and (15) for $\underline{\underline{M}}$ and $\underline{\underline{N}}$. Such a monoidal functor Γ: $\underline{\underline{M}} \to \underline{\underline{N}}$ induces a functor $\tilde{\Gamma}$: $\underline{\underline{M}}^{mon} \to \underline{\underline{N}}^{mon}$ in a natural way. We call a monoidal functor Γ a <u>monoidal equivalence</u> if Γ is an equivalence and the structure maps (16) and (17) are isomorphisms. If Γ is a monoidal equivalence, then $\tilde{\Gamma}$ is obviously an equivalence.

The monoidal categories of <u>bimodules</u> play an essentail role in the $\sqrt{\text{Morita}}$ theory. For simplicity we fix a base ring k. All algebras are assumed to be over k. Let A be an algebra. We write $\underline{\underline{A}}_A$ instead of $\underline{\underline{A}}_{A/k}$. Let ${}_A\underline{\underline{M}}_A$ be the category of all A-bimodules over k. This means are the left and right actions of every element in k are the same on those A-bimodules. For A-bimodules M, N over k, the tensor product $M \otimes_A N$ of M_A and ${}_AN$ over A has a natural structure of an A-bimodule over k. The category ${}_A\underline{\underline{M}}_A$ becomes a monoidal category with tensor product $\otimes_A$ and unit object A. The monoid objects in ${}_A\underline{\underline{M}}_A$ are identified with the A/k-algebras. Hence we have

$$(18) \qquad \underline{\underline{A}}_A = ({}_A\underline{\underline{M}}_A)^{mon}.$$

More precisely, ${}_A\underline{\underline{M}}_A$ is a k-linear monoidal category in the sense that there is a canonical ring map $k \to \mathrm{cent}({}_A\underline{\underline{M}}_A)$, where the center $\mathrm{cent}({}_A\underline{\underline{M}}_A)$ mearns the endomorphism ring of the unit object A.

A monoidal functor Γ of general monoidal categories would not be a monoidal equivalence even if $\tilde{\Gamma}$ is an equivalence. But for monoidal categories of bimodules this does hold.

(19) Theorem [5, Lemma 5.12]. Let A and B be algebras. Let Γ: ${}_A\underline{\underline{M}}_A \to {}_B\underline{\underline{M}}_B$ be a monoidal functor. Γ is a monoidal equivalence if and only if $\tilde{\Gamma}$ is an equivalence.

In fact, the algebras A and B can be over different base rings.

For a 2-cocycle σ for A/k, we have a functor (6) from $\underline{\underline{A}}_A$ to $\underline{\underline{A}}_{A^\sigma}$. We show this comes from a k-linear monoidal functor ${}_A\underline{\underline{M}}_A \to {}_{A^\sigma}\underline{\underline{M}}_{A^\sigma}$. Let M be an A-bimodule over k. For $a \in A$ and $m \in M$ we define a new operation $*_\sigma$ by

$$(20) \qquad a *_\sigma m = \sigma(a,m), \quad m *_\sigma a = \sigma(m,a).$$

With this operation M becomes an A^σ-bimodule over k. Let M^σ denote this bimodule. We have a k-linear functor

(21) $$M \mapsto M^{\sigma},\ {}_{A}\underline{\underline{M}}_{A} \to {}_{A^{\sigma}}\underline{\underline{M}}_{A^{\sigma}}.$$

We make this into a monoidal functor. We take the identity $A^{\sigma} \to A^{\sigma}$ as the map (17). For M, N in ${}_{A}\underline{\underline{M}}_{A}$, the map (16)

(22) $$M^{\sigma} \otimes_{A^{\sigma}} N^{\sigma} \to (M \otimes_{A} N)^{\sigma}$$

is defined by $m \otimes n \longmapsto \sum_i a_i m b_i \otimes n c_i$. The map (22) has the following meaning. For an A-bimodule M over k, let

(23) $$T_A(M) = A \oplus M \oplus (M \otimes_A M) \oplus (M \otimes_A M \otimes_A M) \oplus \cdots$$

be the tensor A/k-algebra of M. This is the free A/k-algebra generated by M. Since both A^{σ}/k-algebras $T_A(M)^{\sigma}$ and $T_{A^{\sigma}}(M^{\sigma})$ contain M^{σ}, there is a natural A^{σ}/k-algebra map

(24) $$T_{A^{\sigma}}(M^{\sigma}) \to T_A(M)^{\sigma}$$

which is the identity on M^{σ}. This map preserves graduation and induces an A^{σ}-bimodule map

(25) $$M^{\sigma} \otimes_{A^{\sigma}} M^{\sigma} \to (M \otimes_A M)^{\sigma}$$

in the second degree. If we replace M with $M \oplus N$ and restrict the amp on $M^{\sigma} \otimes_{A^{\sigma}} N^{\sigma}$, we get map (22).

From this observation it follows that the functor (21) becomes a monoidal functor with structure map (22). One can check easily that the functor $({}_{A}\underline{\underline{M}}_{A})^{\text{mon}} \to ({}_{A^{\sigma}}\underline{\underline{M}}_{A^{\sigma}})^{\text{mon}}$ induced from this monoidal functor is precisely the functor (6).

As a corollary, the following statements are equivalent with each other.

(i) The 2-cocycle σ is invertible.

(ii) The monoidal functor (21) is a monoidal equivalence.

(iii) The functor (6) is an equivalence.

For algebras A, B, we denote by $A \underset{M}{\sim} B$ if algebras A and B are k-linearly Morita equivalent. We denote by $A \underset{\sqrt{M}}{\sim} B$ if there is a k-linear monoidal equivalence between monoidal categories ${}_A\underline{\underline{M}}_A$ and ${}_B\underline{\underline{M}}_B$, and in this case we say algebras A and B are <u>k-linearly $\sqrt{\text{Morita}}$ equivalent</u>.

For an invertible 2-cocycle σ for A/k, we have $A \underset{\sqrt{M}}{\sim} A^{\sigma}$.

If $A \underset{\sqrt{M}}{\sim} B$, the categories $\underline{\underline{A}}_A$ and $\underline{\underline{A}}_B$ are equivalent. Let A^{op} denote the opposite algebra to A. If $A^{op} \underset{\sqrt{M}}{\sim} B$, then the categories $\underline{\underline{A}}_A$ and $\underline{\underline{A}}_B$ are still equivalent. This follows since there is an isomorphism

(26) $$E \longleftrightarrow E^{op},\ \underline{\underline{A}}_A \cong \underline{\underline{A}}_{A^{op}}.$$

We have the following basic properties of $\sqrt{\text{Morita}}$ eqiva-lences. See [1] for the proof.

(27) <u>Let A and B be algebras (over k)</u>.

1. $A \underset{M}{\sim} B \Rightarrow A \underset{\sqrt{M}}{\sim} B \Rightarrow A \otimes_k A^{op} \underset{M}{\sim} B \otimes_k B^{op}$.

2. $A \underset{\sqrt{M}}{\sim} B \Rightarrow \text{center}(A) \cong \text{center}(B)$.

3. <u>The classes of the following k-algebras are closed with respect to</u> $\underset{\sqrt{M}}{\sim}$: <u>central, separable, simple.</u>

4. <u>A is Azumaya</u> $\Leftrightarrow A \underset{\sqrt{M}}{\sim} k$.

The theory is called $\sqrt{\text{Morita}}$ (suggested by Moss Sweedler) by property 1.

We showed the categories $\underline{\underline{A}}_A$ and $\underline{\underline{A}}_B$ are equivalent if $A \underset{\sqrt{M}}{\sim} B$ or if $A^{op} \underset{\sqrt{M}}{\sim} B$. We are concerned with the convese problem: when $\underline{\underline{A}}_A$ and $\underline{\underline{A}}_B$ are equivalent.

Before presenting our answer, let us explain the "k-linearlity" of $\underline{\underline{A}}_A$. Let $(\underline{\underline{A}}_A, A)$ denote the category of all A/k-algebra maps $E \to A$. If $\Lambda\colon \underline{\underline{A}}_A \to \underline{\underline{A}}_B$ is an equivalence, we have $\Lambda(A) \cong B$. Hence Λ induces an equivalence $(\Lambda, A)\colon (\underline{\underline{A}}_A, A) \to (\underline{\underline{A}}_B, B)$. On the other hand, there is a natural equiv-alence between categories $(\underline{\underline{A}}_A, A)$ and $\underline{\underline{A}}^{o}{}_{A}$, the category of all trivial non-unitary A/k-algebras [5, Proposition 1.1]. The category ${}_{A}\underline{\underline{M}}_{A}$ is identified with the subcategory of all trivial non-unitary A/k-algebras. Hence there is an embedding

(28) $$M \mapsto A \oplus M,\ {}_{A}\underline{\underline{M}}_{A} \hookrightarrow (\underline{\underline{A}}_{A}, A).$$

It is known [5,§2] that for every equivalence Λ the equivalence (Λ, A) induces an equivalence of subcategories Λ': ${}_{A}\underline{\underline{M}}_{A} \to {}_{B}\underline{\underline{M}}_{B}$. We say the equivalence Λ is k-linear if the induced equivalence Λ' is k-linear.

Let A_1 and A_2 be algebras. Assume the tensor product $A_1 \otimes_k A_2$ is zero. (If k is an extension of a field by a nil ideal, this means one of A_1 and A_2 is trivial). Let $A = A_1 \times A_2$. In this case, the functor

(29) $$(M_1, M_2) \mapsto M_1 \oplus M_2,\ {}_{A_1}\underline{\underline{M}}_{A_1} \times {}_{A_2}\underline{\underline{M}}_{A_2} \to {}_{A}\underline{\underline{M}}_{A}$$

becomes a monoidal equivalence. In particular, it induces an equivalence

(30) $$\underline{\underline{A}}_{A_1} \times \underline{\underline{A}}_{A_2} \to \underline{\underline{A}}_{A}.$$

Let A_1, A_2, B_1, B_2 be algebras such that $A_1 \otimes_k A_2 = 0 = B_1 \otimes_k B_2$. If $A_1 \underset{\sqrt{M}}{\tilde{}} B_1$ and $A_2{}^{op} \underset{\sqrt{M}}{\tilde{}} B_2$, then we have the following chain of equivalences

(31) $$\underline{\underline{A}}_{A} \sim \underline{\underline{A}}_{A_1} \times \underline{\underline{A}}_{A_2} \sim \underline{\underline{A}}_{B_1} \times \underline{\underline{A}}_{B_2} \sim \underline{\underline{A}}_{B}$$

where $A = A_1 \times A_2$ and $B = B_1 \times B_2$. The main result of this report is to claim the converse is true.

(32) Theorem [5, Theorem 5.14]. Let A and B be k-algebras. Every k-linear equivalence $\underline{\underline{A}}_{A} \sim \underline{\underline{A}}_{B}$ comes from a decomposition $A = A_1 \times A_2$, $B = B_1 \times B_2$ with $A_1 \otimes_k A_2 = 0 = B_1 \otimes_k B_2$ and $\sqrt{}$Morita equivalences $A_1 \underset{\sqrt{M}}{\tilde{}} B_1$, $A_2{}^{op} \underset{\sqrt{M}}{\tilde{}} B_2$.

This presentation seems determined uniquely up to isomorphism by the equivalence $\underline{\underline{A}}_{A} \sim \underline{\underline{A}}_{B}$.

Refernces

[1] M. Takeuchi, Morita Theory, in preparation.

[2] A. Rosenberg and D. Zelinsky, On Amitsur's complex, Trans. Amer. Math. Soc., vol. 97 (1960), pp. 327-356.

[3] M. Sweedler, Multiplication Alteration by Two-Cocycles, Ill. J. Math., vol. 15 (1971), pp. 302-323.

[4] S. Eilenberg and G. Kelly, Closed Categories, in: Proc. Conf. Categorical Algebra, La Jolla, 1965 (Springer-Verlag 1966), pp. 421-562.

[5] M. Takeuchi, Equivalences of Categories of Algebras, to appear Comm. in Algebra.

Institute of Mathematics
University of Tsukuba

ON A CLASS OF REPRESENTATION-FINITE QF-3 ALGEBRAS

Yasuo IWANAGA

This is a summary of my joint work with I. Assem, which has been announced at the 4th International Conference on Representations of Algebras (ICRA IV) at Carleton University in 1984. The complete form of the results will appear in the Proceedings of ICRA IV and thus, in this note, I don't give any proof.

1. Motivation

Let me start to mention about the motivation of this work. For a given finite dimensional algebra A over a field K, we consider the following three types of algebras which are all extensions of A and which are related to a notion of so-called QF-3 algebras:

(1) Trivial extension $T(A) = A \ltimes DA$, where $DA = \mathrm{Hom}_K(A,K)$;

(2) $\begin{pmatrix} A & 0 \\ DA & A \end{pmatrix}$;

(3) $\begin{pmatrix} A & 0 \\ A & A \end{pmatrix}$.

These are interesting algebras to investigate their representations because of the following reasons:

(1) T(A) is a symmetric algebra, i.e., $\mathrm{Hom}_K(T(A),K) \cong T(A)$ as T(A)-bimodules (thus, self-injective algebra and A is a homomorphic image of T(A);

(2) $\begin{pmatrix} A & 0 \\ DA & A \end{pmatrix}$ is a QF-3 algebra and closely related

to T(A) in the following sense. In Hughes-Waschbüsch's work (H-W), they succeeded to use an infinite matrix algebra

$$\hat{A} = \begin{bmatrix} \ddots & & & & 0 \\ & \ddots & A & & \\ & & DA & A & \\ & & & DA & A \\ 0 & & & & \ddots \end{bmatrix}$$

as a universal covering of T(A) in characterizing when T(A) is representation-finite. The 'finite' dimensional algebra $\begin{pmatrix} A & 0 \\ DA & A \end{pmatrix}$ is a homomorphic image of $\hat{A}$;

(3) For a representation-finite algebra A, $\begin{pmatrix} A & A \\ A & 0 \end{pmatrix}$ and the Auslander algebra have finite type simaltaneously. Here, Auslander algebra of A is an endomorphism algebra of a direct sum of all non-isomorphic indecomposable A-modules.

For a symmetric algebra A, it is certainly obvious that $\begin{pmatrix} A & 0 \\ DA & A \end{pmatrix}$ is isomorphic to $\begin{pmatrix} A & 0 \\ A & A \end{pmatrix}$.

Now, we'd like to raise the following problems according to each algebra of (1), (2) and (3):

(I) When is T(A) representation-finite?

(II) When is $\begin{pmatrix} A & 0 \\ DA & A \end{pmatrix}$ representation-finite?

(III) When is $\begin{pmatrix} A & 0 \\ A & A \end{pmatrix}$ representation-finite?

and

(IV) How is the relationship between representation types of algebras (1), (2) and (3)?

(I) has been solved completely. See (A-H-R), (H-W), (Y1) and (Y2). (II) and (III) are unsolved yet, and (III) was given by Auslander in (Au) and there is some partial answer in (M).

About (IV)

(i) (Assem) If $T(A)$ is representation-finite, then $\begin{pmatrix} A & 0 \\ DA & A \end{pmatrix}$ is simply connected, especially representation-finite, and moreover, if we assume A is a tilted algebra of Dynkin type, then the Auslander-Reiten quiver of $T(A)$ is obtained by identifying some rayers in the Auslander-Reiten quiver of $\begin{pmatrix} A & 0 \\ DA & A \end{pmatrix}$.

(ii) $\begin{pmatrix} A & 0 \\ DA & A \end{pmatrix}$ has the (left and right) maximal quotient ring

$$Q = \begin{pmatrix} A & \mathrm{Hom}_A(DA,A) \\ DA & A \end{pmatrix}$$

which is also QF-3, and if $\begin{pmatrix} A & 0 \\ A & A \end{pmatrix}$ is representation-finite, then so is Q. However, it is unknown whether $\begin{pmatrix} A & 0 \\ DA & A \end{pmatrix}$ is representation-finite or not.

2. Algebra $\begin{pmatrix} A & 0 \\ DA & A \end{pmatrix}$

As we mentioned earlier, $\begin{pmatrix} A & 0 \\ DA & A \end{pmatrix}$ is a homomorphic image of $\hat{A}$. Now, we can consider a more general situation: for any $t \geq 1$, let

$$A^{(t)} = \begin{bmatrix} A_0 & & & & 0 \\ Q_1 & A_1 & & & \\ & Q_2 & A_2 & & \\ & & \ddots & \ddots & \\ 0 & & & Q_t & A_t \end{bmatrix}$$

where $A_i = A$ and $Q_i = DA$ for all i and multiplication of $A^{(t)}$ is given by the map $DA \otimes_A DA \to 0$. Then $A^{(t)}$ is still a QF-3 algebra, and for a connected hereditary algebra A, TFAE:

(i) $A^{(t)}$ is representation-finite for any $t \geq 1$,

(ii) $A^{(1)}$ is representation-finite,

(iii) $T(A)$ is representation-finite,

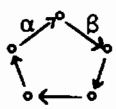
(iv) A is of Dynkin type, i.e., the ordinary quiver of A has a Dynkin diagram $\mathbb{A}_n$, $\mathbb{B}_n$, $\mathbb{C}_n$, $\mathbb{D}_n$, $\mathbb{E}_n$ $(n = 6, 7, 8)$, $\mathbb{F}_4$ and $\mathbb{G}_2$ as its underlying graph. However, these equivalences no longer hold even for an iterated tilted algebra A. (See (A-H-R) for the definition of an iterated tilted algebra.)

Next, we want to stress the difference of (I) and (II). In the case (I), that is, if $T(A)$ is representation-finite, then the ordinary quiver of A doesn't contain an oriented cycle (see (Y1)), thus A has to be a homomorphic image of some hereditary algebra. However, an algebra A with an oriented cycle or a loop might have a representation-finite $A^{(t)}$, and then, of course, $T(A)$ is representation-infinite.

Now, we'd like to see some examples concerning the problem (II).

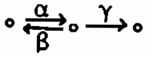
Examples

(1) Let A be the algebra given by the quiver



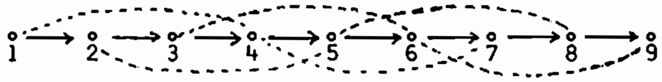
with the relation $\beta\alpha = 0$, then $A^{(1)}$ is representation-finite but $T(A)$ is not.

(2) Let A be the algebra with square-zero radical

given by the quiver  with relations $\beta\alpha = \alpha\beta = \gamma\alpha = 0$, then $A^{(1)}$ is representation-finite but $A^{(t)}$ is not for any $t \geq 2$.

On the other hand, if A is the algebra with square-zero radical given by the quiver  with $\beta\alpha = 0$, then A is a homomorphic image of a hereditary algebra, and $A^{(t)}$ is representation-finite for all $t \geq 1$, but T(A) is not.

(3) Let A be the Nakayama algebra given by the quiver



bound by the ideal generated by the set of constant length 3 (i.e. the dotted lines mean the zero-relations), then A is an iterated tilted algebra of Euclidean type and $A^{(1)}$ is still representation-finite.

(4) For a self-injective Nakayama algebra A with Löwy length n, $A^{(1)}$ is representation-finite iff $n \leq 3$.

Finally, we'd like to end this summary by stating the results on the problem (II) which we obtained. We got the criterions for two classes of algebras A when an algebra $A^{(t)}$ is representation-finite, and they are the following:

(i) $A^{(t)}$ for any $t \geq 1$ when A has the square-zero radical;

(ii) $A^{(1)}$ for any Nakayama algebra A.

(i) is given by constructing the separated diagram of $A^{(t)}/(\text{Rad } A^{(t)})^2$ and by applying Gabriel's theorem (G). In this case, we use the fact that $A^{(t)}$ is representation-finite iff so is $A^{(t)}/(\text{Rad } A^{(t)})^2$, which follows from that $A^{(t)}$ is QF-3. (ii) is given by constructing the ordinary quiver of $A^{(1)}$ from a given A and by looking at Bongartz's list (B1) of full convex subquivers and further, we consider the Galois covering $\tilde{A} \to A^{(1)}$ if necessary, namely, if A contains an oriented cycle. The result we need appears in (B2).

References

(A) I. Assem: On representations of the bimodule DA, to appear.

(A-H-R) I. Assem, D. Happel and O. Roldan: Representation-finite trivial extension algebras, to appear in J. of Pure Appl. Algebra.

(Au) M. Auslander: Representation dimension of artin algebras, Lecture Notes at Queen Marry College, London, 1971.

(B1) K. Bongartz: Critical simply connected algebras, Manuscripta Math. 46 (1984), 117-136.

(B2) K. Bongartz: A criterion for finite representation type, Math. Ann. 269 (1984), 1-12.

(G) P. Gabriel: Unzerlegbare Darstellungen I, Manuscripta Math. 6 (1972), 71-103.

(H-W) D. Hughes and J. Waschbüsch: Trivial extensions of tilted algebras, Proc. London Math. Soc. 46 (1983), 347-364.

(M) N. Marmaridis: On the representation type of certain triangular matrix algebras, Comm. Algebra 11 (1983), 1945-1964.

(Y1) K. Yamagata: On algebras whose trivial extensions are of finite representation type, Proc. of ICRA III, Lecture Notes in Math. 903 (1981), 364-371.

(Y2) K. Yamagata: On algebras whose trivial extensions are of finite representation type II, to appear.

Department of Mathematics
Faculty of Education
Shinshu University
Nagano, 380 Japan