PROCEEDINGS OF THE
18TH SYMPOSIUM ON RING THEORY

HELD AT THE SANSEN-SÔ, YAMAGUCHI

JULY 29—31, 1985

EDITED BY

KIYOICHI OSHIRO

Yamaguchi University

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OKAYAMA, JAPAN

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PREFACE

The 18th Symposium on Ring Theory was held at Sansensō, Yuda, Yamaguchi City, Japan, on July 29-31, 1985. Nearly one hundred participants attended the Symposium.

This volume consists of the articles presented at the Symposium.

The Symposium and the Proceedings were financially supported by the Scientific Reserch Grants of the Educational Ministry of Japan through the arrangements by Professor Katsumi Shiratani, Kyushu University.

This Symposium has continued to the present with cooperation of Professors Shizuo Endo, Manabu Harada, Hiroyuki Tachikawa, and Hisao Tominaga.

I wish to express my hearty thanks to Yasuyuki Hirano and Hiroaki Komatu of Okayama University for the publication of the Proceedings.

Finally I would like to thank Professors Hisao Tominaga and Hiroyuki Tachikawa, and staffs of the Department of Mathematics, Yamaguchi University, for their close cooperation.

November 1985

K. Oshiro

PSEUDO-RANK FUNCTIONS ON CROSSED PRODUCTS OF FINITE GROUPS OVER REGULAR RINGS

Jiro KADO

In this note, we shall announce some results in our recent papers [10, 11].

I.1. Extensions of pseudo-rank functions

Let R be a regular ring and we use $FP(R)$ to denote the set of all finitely generated projective left R-modules. For modules A, B, $A \lesssim B$ implies that A is isomorphic to a submodule of B and we use $n.A$ to denote the direct sum of n copies of A.

Definition [2, p.226]. A pseudo-rank function on R is a map $N:R \longrightarrow [0,1]$ such that

(1) $N(1) = 1$.

(2) $N(rs) \leq N(r)$ and $N(rs) \leq N(s)$ for all $r,s \in R$.

(3) $N(e+f) = N(e)+N(f)$ for all orthogonal idempotents $e, f \in R$.

If, in addition

(4) $N(r) > 0$ for all non-zero $r \in R$,

then N is called a rank function. We use $P(R)$ to denote the set of all pseudo-rank functions on R.

Definition [2,p.232]. A dimension function on FP(R) is a map $D:FP(R) \longrightarrow \mathbb{R}^+$ such that

(1) $D({}_RR) = 1$

(2) If A, $B \in FP(R)$ and $A \lesssim B$, then $D(A) \leq D(B)$.

(3) $D(A \oplus B) = D(A) + D(B)$ for all $A, B \in FP(R)$.

Let D(R) denote the set of all dimension functions on FP(R).

Pseudo-rank functions on R and dimension functions on FP(R) are equivalent functions as follows.

Lemma I.1 [2, Prop.16.8]. There is a bijection $\Gamma_R: P(R) \longrightarrow D(R)$ such that $\Gamma_R(P)(Rr) = P(r)$ for all $P \in P(R)$ and $r \in R$.

Our main objective is to study a crossed product R*G of a finite multiplicative group G over a regular ring R. A crossed product R*G of G over R is an asso-ciative ring which is a free left R-module containing an element $\bar{x} \in R*G$ for each $x \in G$ and the set generated by the symbols $\{\bar{x}: x \in G\}$ is a basis of R*G as a left R-module. Hence every element $\alpha \in R*G$ can be uniquely written as a sum $\alpha = \Sigma_{x \in G}\, r_x \bar{x}$ with $r_x \in R$. The addition in R*G is the obvious one and the multiplication is given by the formulas

$$\bar{x}\bar{y} = t(x,y)\overline{xy} \qquad r\bar{x} = \bar{x}r^{\tilde{x}}$$

for all $x, y \in G$ and $r \in R$. Here the twisting $t: G \times G \longrightarrow U(R)$ is a map from $G \times G$ to the group of units of R and for fixed $x \in G$, the map $\tilde{x}: r \longrightarrow r^{\tilde{x}}$ is an automorphism of R. We assume throughout this note that the order $|G|$ of G is invertible in R. The Lemma 1.1 of [17] implies that $R*G$ is also a regular ring. First we will study the question whether a pseudo-rank function P of R can be extended to one of $R*G$. We shall show that P is extensible to $R*G$ if and only if P is G-invariant, i.e., $P(r) = P(r^{\tilde{x}})$ for all $r \in R$ and $x \in G$. We always view R as a subring $R*G$ via the embedding $r \longrightarrow r1$. Then there exists a restriction-map $\theta: P(R*G) \longrightarrow P(R)$.
We consider the same connections between $D(R*G)$ and $D(R)$. For all $D \in D(R*G)$ and $A \in FP(R)$, define $(D|_R)(A) = D(R*G \otimes_R A)$. We can easily see that $D|_R$ is a dimension function on $FP(R)$ and $\Gamma_{R*G}(N)|_R = \Gamma_R(N|_R)$.

Lemma I.2 [10,Lemma 2]. Let N be in $P(R*G)$ and D be in $D(R*G)$. Then we have that $(N|_R)(r) = (N|_R)(r^x)$ and that $(D|_R)(Rr) = (D|_R)(Rr^r)$ for all $r \in R$ and all $x \in G$.

Now we shall define an extended dimension function on $R*G$ for a G-invariant $D \in D(R)$. Note that for $A \in FP(R*G)$, ${}_RA \in FP(R)$.

Proposition I.3 [10,Prop.3]. Let D be a G-invariant dimension function on FP(R). Put $D^G(A) = |G|^{-1}D({}_RA)$ for all $A \in FP(R*G)$. Then D^G is a dimension function on $FP(R*G)$ and $D^G|_R = D$.

Corollary I.4 [10,Cor.4]. Let P be a G-invariant pseudo-rank function on R. Define $P^G(\alpha) = (\Gamma_R(P))^G(R*G\alpha)$ for all $\alpha \in R*G$, then

(1) P^G is a pseudo-rank function on $R*G$ and $P^G|_R = P$

(2) We have $P^G(\alpha) = |G|^{-1}\sum_1^n P(r_i)$, if ${}_R(R*G\alpha) \cong \oplus_1^n Rr_i$, where $r_i \in R$.

Definition [2, Ch.19]. Let P be in $P(R)$. R admits a pseudo-metric δ by the rule:$\delta(r,s) = P(r-s)$. Note that δ is a metric if and only if P is a rank function. We call δ the P-metric. Let R be the completion of R with respect to δ and we call it the P-completion of R. R is a unit-regular, left and right self-injective ring by [2, Th.19.7] . There exists a natural ring map $\phi:R \longrightarrow \bar{R}$ and a continuous map $\bar{P}:\bar{R} \longrightarrow [0,1]$ such that $\bar{P}\phi = P$. By [2, Th. 19.6], $\bar{P}$ is a rank function on R. Put $\ker P = \{ r \in R: P(r) = 0\}$, which is a two-sided ideal. P induces the rank function $\bar{P}$ on $R/\ker P$. Then R is equal to the $\bar{P}$-completion of $R/\ker P$ and $\ker = \ker P$.

Now let $R*G$ be a given crossed product of a finite group G over a regular ring R and let P be a G-invariant pseudo-rank function. Since P is G-invariant, $\ker P$ is G-invariant ideal and therefore each automorphism $\tilde{x}$ induces an automorphism $\tilde{\tilde{x}}$ of $R/\ker P$ and $\tilde{\tilde{x}}$ is uniformly continuous with respect to the induced metric. Consequently we have an automorphism of $\bar{R}$, which is again denoted by $\bar{x}$ such that $\phi(r)^{\tilde{x}} = \phi(r^{\tilde{x}})$ for all $r \in R$.

Let a map $t'\colon G \times G \longrightarrow U(\bar{R})$ be $t'(x,y) = \phi(t(x,y))$ for all $x, y \in G$. Here of course $t\colon G \times G \longrightarrow U(R)$ is the given map for $R*G$. We define a crossed product $\bar{R}*G$ of G over $\bar{R}$ using multiplication formula $(a\bar{x})(b\bar{y}) = (ab^{\bar{x}^{-1}}t'(x,y))\overline{xy}$ for $a, b \in R$ and $x, y \in G$, and define a map $\bar{\phi}\colon R*G \longrightarrow \bar{R}*G$ by the rule: $\bar{\phi}(\sum_{x\in G} r_x\bar{x}) = \sum_{x\in G}\phi(r_x)\bar{x}$. Then $\bar{\phi}$ is a ring homomorphism and the following diagram is commutative ;

$$\begin{array}{ccc} R & \xrightarrow{\quad\phi\quad} & \bar{R} \\ \downarrow & & \downarrow \\ R*G & \xrightarrow{\quad\bar{\phi}\quad} & \bar{R}*G \end{array}$$

Proposition I.5 [10,Prop.6]. Let P be a G-invariant pseudo-rank function on R, let $\bar{R}$ be a P-completion, let $\bar{P}$ be a continuous extension of P and let $\phi\colon R \to \bar{R}$ the natural map. Then we have the relationship between P^G and $(\bar{P})^G$ such that the following diagram is commutative ;

$$\begin{array}{ccc} R*G & \xrightarrow{P^G} & [0,1] \\ \downarrow & & \\ \bar{R}*G & \xrightarrow{(\bar{P})^G} & [0,1]. \end{array}$$

Definition [2, Ch.16 and Appendix]. For a regular ring R, we view P(R) as a subset of the real vector space $\mathbb{R}^R$, which we equip with the product topology . Then P(R) is a compact convex subset of $\mathbb{R}^R$ by [2, Prop.16.17] . A extreme point of P(R) is a point P $\in$ P(R) which cannot be expressed as a positive convex combination of distinct two pionts of P(R). We use $\partial_e P(R)$ to denote the set of all extreme points of P(R). The important result is that P(R) is equal to the closure of the convex hull of $\partial_e P(R)$ by Krein-Milman Theorem.

Theorem I.6 [10,Th.7]. Let R*G be a crossed product of a finite group G over a regular ring R with $|G|^{-1} \in$ R. Let P be a G-invariant extreme point of P(R), let $\bar{R}$ be the P-completion of R, let $\phi: R \longrightarrow \bar{R}$ be the natural ring map and let $\bar{P}$ be the continuous extension of P over $\bar{R}$.

(1) The crossed product $\bar{R}*G$ of G over R defined above, is the completion of R*G with respect to P^G-metric.

(2) The extension P^G can be expressed as a

positive convex combination of finite distinct elements in $\partial_e(R*G)$, i.e., $P^G = \sum_1^n \alpha_i N_i$, where $N_i \in \partial_e P(R*G)$, $0 < \alpha_i < 1$ and $\sum_1^n \alpha_i = 1$.

For $N \in \partial_e P(R*G)$, we have the following relationship between N and $(N|_R)^G$.

Theorem I.7 [10,Th.,10]. Let $R*G$ be a crossed product of a finite group G over a regular ring R with $|G|^{-1} \in R$ and let N be extremal pseudo-rank function on $R*G$. Then we have $(N|_R)^G = \alpha N + (1-\alpha)N'$ for some $N' \in P(R*G)$ and some positive real number $\alpha \leq 1$.

Remark. For a G-invariant element $P \in \partial_e P(R)$, let $N_1, \ldots, N_t$ be elements in $\partial_e P(R*G)$ associative with P. We can easily prove that $\{ N_1, \ldots, N_t \}$ is equal to the set $\{ N \in \partial_e P(R*G) : \Theta(N) = N|_R = P \}$, where $\Theta : P(R*G) \longrightarrow P(R)$, by Theorem I.6 and Theorem I.7. Unfortunately we don't know whether $N|_R$ is always extremal for any extremal pseudo-rank function N on R*G or not.

Now we consider a pseudo-rank function P which is not necessarily G-invarint. For each $x \in G$, put $P^x(r) = P(r^{\tilde{x}^{-1}})$ for all $r \in R$. Then P^x is also a pseudo-rank function and $\ker P^x = (\ker P)^x$. Put t(P)

$= \sum_{x \in G} |G|^{-1} P^x$, then $t(P)$ is G-invariant pseudo-rank function with $P \leq |G| t(P)$. We call $t(P)$ to the trace of P.

Proposition I.8 [10,Prop.10]. Let $R*G$ be a crossed product of a finite group G over a regular ring R with $|G|^{-1} \in R$. Let P be in $\partial_e P(R)$ which is not necessarily G-invariant and let $t(P)$ be the trace of P. Then the extension $t(P)^G$ can be expressed as a positive convex combination of finite distinct elements in $\partial_e(R*G)$.

Corollary I.9 [10,Cor.11]. Let $R*G$ be a crossed product of a finite group G over a regular ring R with $|G|^{-1} \in R$. If $\partial_e P(R)$ is a finite set, then $\partial_e P(R*G)$ is also a finite set.

I.2. Isomorphism of θ.

Definition [1, p.202]. A partially ordered abelian group is an abelian group K equiped with a partial order $<$ which is translation invariant. The positive cone of K is the set $K^+ = \{ x \in K; x > 0\}$. If the partial order on K is directed (upward or downward), then K is called a directed abelian group. An ordered-unit in K is an element $u > 0$ such that for any $x \in K$, there exists a positive integer n for which $x \leq nu$. We denote by a pair

(G,u) a partially ordered abelian group with order-unit u.

Definition [1, §15]. For a unit-regular ring T, the Grothendieck group $K_0(T)$ is an abelian group with generators [A], where [A] is the isomorphism class for $A \in FP(T)$ and with relation $[A \oplus B] = [A] + [B]$. Every element of $K_0(T)$ has the form [A] - [B] for some $A, B \in FP(T)$. $K_0(T)$ is a partially ordered abelian group with order-unit [T] and positive cone $K_0(T)^+ = \{ [A]: A \in FP(T)\}$ by [1, Prop. 15.2] .

We shall study conditions under which Θ is a homeomorphsim.

Theorem I.10 [10,Th.15]. Let R be a left self-injective, regular ring of Type II_f and R*G be a crossed product of a finite group G over R with $|G|^{-1} \in R$. We assume any $M \in Max(R)$ is G-invariant. Let Θ: $\partial_e P(R*G) \longrightarrow \partial_e P(R)$ be a natural restriction map. Then the following conditions are equivalent:

(1) Θ is a homeomorphism.

(2) The natural map f: $K_0(R) \longrightarrow K_0(R*G)$,defined by $f([A]) = [R*G \otimes_R A]$ for $A \in FP(R)$, is an isomorphism as a partially ordered abelian group with order-unit.

(3) B(R) = B(R*G).

II.1. Relations between $P(R*G)$ and $P(R^G)$

Definition [15]. Let T be a ring with identity element 1 and let G be a finite group of automorphisms of T with $|G|^{-1} \in T$. The skew group ring, $T*G$, is defined to be a crossed product which has a trivial twisting map. Throughout this paper, put $e = |G|^{-1}\sum_{x \in G}\bar{x}$ and $\psi : e(T*G)e \longrightarrow T^G$ given by $\psi[e(\sum_{x \in G} r_x\bar{x})e] = \sum_{x \in G} t(r_x)$, where $t(r) = |G|^{-1}\sum_{x \in G} r^{\tilde{x}}$ for $r \in T$. Then e is an idempotent and ψ is an isomorphism by [15 , Lemma 0.1].

In this section, we shall study the relation between $P(R*G)$ and $P(R^G)$ (resp. $\partial_e P(R*G)$ and $\partial_e P(R^G)$). If $R*G$ and R^G are Morita equivalent, then K.R.Goodearl has shown by a general situation that there is a bijection between $P(R*G)$ and $P(R^G)$ in [1,Cor.16.9]. We shall define maps between $P(R*G)$ and $P(R^G)$, which are more concrete than Goodearl's Theorem, without the assumption of Morita Equivalence.

Let R be a unit-regular ring and let G be a finite subgroup of Aut(R) with $|G|^{-1} \in R$. The skew group ring $R*G$ is a regular ring by [18]. Unfortunately we don'n know whether $R*G$ is unit-regularor not. Then, from now on, we assume that $R*G$

is unit-regular in many cases. We regard $R*Ge$ as a (left $R*G$, right R^G)-bimodule.

There exists a natural functor $\mu : FP(R^G) \longrightarrow FP(R*G)$ by the rule $\mu(M) = R*Ge \otimes_{R^G} M$. Then we have a positive homomorphism $\bar{\mu} : K_0(R^G) \longrightarrow K_0(R*G)$, defined by $\bar{\mu}([M]) = [\mu(M)]$. Set $F = \{ N \in P(R*G): N(e) = 0\}$. Then μ also induces a map $\mu^*: P(R*G)-F \longrightarrow P(R^G)$ by the rule $\mu^*(N)(a) = N(e)^{-1} D_N(\mu(R^G a))$ for any $N \in P(R*G)-F$ and any $a \in R^G$, where D_N is the dimension function corresponding to N. In fact, since $\mu(R^G a) = R*Ge \otimes R^G a \cong R*Gea$, we have $D_N(\mu(R^G a)) = N(ea)$. Then $\mu^*(N)(a) = N(e)^{-1}.N(ea)$ for all $a \in R^G$. Thus $\mu^*(N)$ is a pseudo-rank function by the isomorphism υ: $eR*Ge \longrightarrow R^G$ and [1, Lemma 16.2].

Proposition II.1 [11,Prop.1]. Let $\mu^*: P(R*G)-F \longrightarrow P(R^G)$ be the map given above . If $N \in P(R*G)-F$ is extremal in $P(R*G)$, then $\mu^*(N)$ is also extremal.

In general, there may not exist any map from $P(R^G) \longrightarrow P(R*G)$. From now on, we assume that R is a finitely generated, projective, left R^G-module. For any $A \in FP(R*G)$, define $\lambda(A) = Hom_{R*G}(R*Ge, A)$. Since $Hom_{R*G}(R*Ge, R*G) \cong eR*G \cong R$ as left R^G-modules, then $\lambda(A)$ is a finitely generated, projective, left R^G-module. The functor λ induces a positive homomorphism $\bar{\lambda}: K_0(R*G) \longrightarrow K_0(R^G)$ by the rule; $\bar{\lambda}([A]) = [\lambda(A)]$. Since $Hom_{R*G}(R*Ge, R*G) \cong eR*G \cong R$ as left R^G-modules, we have $\bar{\lambda}([R*G]) = [_{R^G}R]$.

We define

$$\lambda^*(Q)(x) = D_Q(R)^{-1} D_Q(\lambda(R*Gx))$$

for any $Q \in P(R^G)$ and for all $x \in R*G$, where D_Q is the dimension function corresponding to Q. Then $\lambda^*(Q)$ is a pseudo-rank function on $R*G$.

Remark II.1. We note the following relation that $\lambda^*(Q)(e) = D_Q({}_{R^G} R)^{-1}$ for all $Q \in P(R^G)$, because $\lambda(R*Ge) \cong eR*Ge \cong R^G$.

Now we shall determine pseudo-rank functions on R^G from ones on $R*G$.

Theorem II.2 [11,Th.2]. Let R be a unit-regular ring, G a finite subgroup of $Aut(R)$ with $|G|^{-1} \in R$ and $R*G$ a skew group ring of G over R. Put $e = |G|^{-1} \sum_{x \in G} \bar{x}$ and set $F = \{ N \in P(R*G): N(e) = 0\}$. We assume that $R*G$ is a unit-regular ring and that R is a finitly generated, projective, left R^G-module. Then we have the following results;

(1) $\bar{\mu}: K_0(R^G) \longrightarrow K_0(R*G)$ is an order-embedding map and $\bar{\lambda}\,\bar{\mu}$ = identity.

(2) For any $Q \in P(R^G)$, there exists some $N \in P(R*G)-F$ such that $Q(a) = N(e)^{-1} N(ae)$ for any $a \in R^G$.

Next we shall determine a condition that $R*G$ and R^G are Morita equivalent.

Proposition II.3 [11,Prop.3]. Let R be a unit-regular ring and let G be a finite subgroup of $Aut(R)$ with $|G|^{-1} \in R$. We assume that $R*G$ is also a unit-regular ring. The following conditions are equivalent.

(1) $R*Ge$ (resp. $eR*G$) is a generator as a $R*G$-module.

(2) $N(e) > 0$ for all $N \in \partial_e P(R*G)$.

II.2. X-outer automorphisms

In this section, let R be a directly finite, left self-injective, regular ring and G a finite group of automorphisms of R with $|G|^{-1} \in R$. It is known that both $R*G$ and R^G are directly finite, left self-injective, regular rings ([18]) and that such rings are unit-regular rings ([1, Th.9.17]). K.R.Goodearl has shown that there exists a bijection $\partial_e P(R) \longrightarrow Max(R)$ by the rule; $P \longrightarrow \ker P$ and that $R/\ker P$ is a simple self-injective regular ring with the unique rank function [6, II.14.5]. We use repeatedly that fact.

Definition [16]. An automorphism g of R is called an X-inner if there exists a non-zero element $x \in R$ such that $rx = xr^g$ for all $r \in R$. If g is not X-inner, we call g X-outer. For a subgroup G of Aut(R), we call G X-outer if all $g \neq 1 \in G$ are X-outer. Let Z(R) be the center of R.

First we shall determine the structure of Max(R*G) for an X-outer group G. The following Lemma has been essentially proved in [7], but we shall prove it in this note for the sake of completeness. We denote the set of all central idempotents of a ring T by B(T).

Lemma II.4 [11,Lemma 4]. Let R be a directly finite, left self-injective, regular ring and G a finite group of automorphisms of R with $|G|^{-1} \in R$. We assume that G is X-outer. Then we see that $\mathrm{Max}(R*G) = \{ (\bigcap_{g \in G} M^g)*G : M \in \mathrm{Max}(R) \}$.

Proposition II.5 [11,Prop.5]. Let R be a directly finite, left self-injective, regular ring and G a finite group of automorphisms of R with $|G|^{-1} \in R$. We assume that G is X-outer. Then we have $\partial_e P(R*G) = \{ t(Q)^G : Q \in \partial_e P(R) \}$

Lemma II.6 [11,Lemma 6]. Let R be a directly finite, left self-injective, regular ring and G a finite group of automorphisms of R with $|G|^{-1} \in R$. We assume that G is X-outer. Then we have the following results:

(1) $N(e) = n^{-1}$ for all $N \in \partial_e P(R*G)$, where $n = |G|$.

(2) $R*G \cong M_n(R^G)$.

Now , using Lemma II.6, we shall prove a interesting result with respect to "a normal basis" of R over R^G.

Proposition II.7 [11,Prop.7]. Let R be a directly finite, left self-injective, regular ring and G a finite group of automorphisms of R with $|G|^{-1} \in$ R. We assume that G is X-outer. Then $R \cong R^G[G]$ as R^G-modules.

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The detailed version of this paper has been submitted for a publication elsewhere.

ON FPF-RINGS

Shigeru KOBAYASHI

A ring R is called right finitely pseudo-Frobenius (FPF) if every finitely generated faithful right R-modules is a generator in the category of right R-modules.

FPF-rings include quasi-Frobenius rings, pseudo-Frobenius rings, commutative self-injective rings, Prüfer domains, and almost valuation rings.

Recently, C. Faith [1] has shown that a commutative ring R is FPF if and only if (1) The total quotient ring K of R is injective, (2) Every finitely generated faithful ideal is projective. In particular, as in the case that R is a commutative semiprime ring, he has also shown that R is FPF if and only if the total quotient ring K of R is injective and R is semihereditary.

On the other hand, S. Page [7] has shown that a (Von Neuman) regular ring R is (right) FPF if and only if R is isomorphic to a finite direct product of full matrix rings over abelian regular self-injective rings. Therefore we shall require a characterization of arbitrary FPF-rings, which includes the above results.

The results of this paper will be found in [4], [5] and [6], which will be appeared in Osaka J. Math.

theorem of C. Faith. If R is a regular ring, the condition (ii) says that R is a right self-injective. Furthermore, the conditions (i) and (iii) imply that R is isomorphic to a finite direct product of full matrix rings over abelian regular self-injective rings by [3, Corollary of Theorem 2]. Therefore the theorem of S. Page follows.

Next we consider semihereditary FPF-rings. If R is a commutative semiprime FPF-ring, then by Theorem 1.1, we can see that R is semihereditary. However, for arbitrary non-singular FPF-ring R, it is not known wheather R is semihereditary. Therefore in the next theorem, we shall give a characterization of semihereditary FPF-rings, and by this characterization, we shall give a necessary and sufficient condition for non-singular FPF-rings to be a semihereditary.

Theorem 1.2 [4, Theorem 2]. Let R be a ring. Then the following conditions are equivalent.

(1) R is a right semihereditary right FPF-ring.

(2) (i) R is right bounded and right non-singular.

(ii) For any positive integer n, $(nR)_R$ has the extending property of modules for $L_r(nR)$, where $L_r(nR)$ is the lattice of right R-submodules of $(nR)_R$.

(iii) For any finitely generated idempotent right ideal I of R, there exists a central idempotent e of R such that RI = eR.

In this paper, we shall concerned with non-singular rings. The results of this paper will appear in [4], [5], and [6]. Therefore we will ommit the proofs.

1. A characterization of non-singular FPF-rings.

The purpose of this section is to give a characterization of non-singular FPF-rings.

First of all we require some definitions and lemmas.

Definition 1.1. A ring R is right bounded if every essential right ideal contains a nonzero two-sided ideal of R which is essential as a right ideal.

Lemma 1.1 [4, Lemma 1]. For a right non-singular ring R, the following conditions are equivalent.

(1) R is right bounded.

(2) FOr any finitely generated right R-module M, $r_R(Z_r(M))$ is an essential right ideal of R, where $Z_r(M)$ the singular submodule of M and $r_R(-)$ is the right annihilator ideal.

Lemma 1.2 [4, Lemma 2]. Let R be a right non-singular right bounded ring. Then for any finitely generated right R-module M,

M is faithful if and only if $M/Z_r(M)$ is faithful.

By [3, Proposition 1], we know that right non-singular right FPF-rings are right bounded. Therefore, by virture of Lemma 1.2, if we show that right non-singular right bounded rings are right FPF, it suffices to show that any finitely generated non-singular faithful module is a generator.

Further, by [7], the maximal right quotient ring Q of a right non-singular right FPF-ring R, is a flat epimorphic extension of R. By using Pierce stalk, we know that Q is the classical left quotient ring of R.

Now we can give a characterization of non-singular FPF-rings.

Theorem 1.1 [4, Theorem 1]. Let R be a ring and Q be the maximal right quotient ring of R. Then the following conditions are equivalent.

(1) R is a right non-singular right FPF-ring.

(2) (i) R is right bounded.

(ii) Q is the classical left quotient ring of R.

(iii) For any finitely generated right ideal I of R, $Tr_R(I) \oplus r_R(I) = R$ (as ideal), where $Tr_R(-)$ is the trace ideal.

Remark. If R is a commutative semiprime ring, the condition (iii) of (2) of Theorem 1.1 shows that R is a semihereditary and the condition (ii) implies that the total quotient ring of R coincides the maximal quotient ring of R. Hence Theorem 1.1 follows the

Corollary 1.1 [4, Corollary 1]. Let R be a non-singular right FPF-ring. Then R is right semiheredirary if and only if for any positive integer n, nR has the extending property of modules for $L_r(nR)$.

Corollary 1.2 [4, Corollary 2]. Let R be a right semihereditary right FPF-ring. Then R is left FPF if and only if R is left bounded.

By using the language of stalk, we can give an another characterization of commutative semiprime FPF-rings.

Corollary 1.3 [4, Corollary 3]. Let R be a commutative ring. Then the following conditions are equivalent.

(1) R is a semiprime FPF-ring.

(2) $R\oplus R$ has the extending property of modules for $L_r(R\oplus R)$ and all stalks of R are Prüfer domain.

2. Applications.

In this section, we apply Theorem 1.1 and Theorem 1.2 to determine the structure of some type of FPF-rings.

First of all, we consider about the theorem of S. page. As we mentioned in the introduction, S. Page has determined the structure of regular FPF-rings. On the other hand, if R is a non-singular right FPF-ring, then R is regular if and only if R is right continuous, where a ring R is right continuous if (1) R has the extending property for right ideals, (2) for any element x of R such that xR is isomorphic to a direct summand of R, there exists an idempotent e of R such that $xR = eR$.

Therefore we are intrested in the structure of non-singular right quasi-continuous, right FPF-rings, where a ring R is right quasi-continuous if (1) R has the extending property for right ideals, (2) for any idempotents e, f in R such that $eR \cap fR = 0$, $eR \oplus fR$ is a direct summand of R.

We generalize the theorem of S. Page, as follows

Theorem 2.1 [5, Theorem 2]. Let R be a non-singular right FPF-ring and Q be the maximal right quotient ring of R. Then the following conditions are equivalent.

(1) R is right continuous.

(2) $id(R) = id(Q)$.

(3) $R \cong R_1 \times \prod_{i=1}^{t} M_{n(i)}(S_i)$, where R_1 is a non-singular right FPF-ring whose maximal right quotient ring is an abelian regular self-injective ring, and each S_i is an abelian regular self-injective ring and $n(i) \geqslant 2$.

Next we consider about noetherian non-singular FPF-rings and hereditary FPF-rings.

In [2], C. Faith and S. Page have proved that two-sided noetherian non-singular right FPF-rings are isomorphic to a finite direct product of bounded Dedekind prime rings (= hereditary noetherian prime rings whose nonzero ideals are invertible).

On the other hand. in the case that ring R is right noetherian non-singular two-sided FPF-rings, we have the followinf similar result.

Theorem 2.2 [6, Proposition 1]. Let R be a right noetherian non-singular two-sided FPF-ring. Then R is isomorphic to a finite direct product of Dedekid prime rings.

Further, for hereditary FPF-rings, we have the fI following.

Theorem 2.3 [6, Proposition 2]. Let R be a right hereditary ring. Then the following conditions are equivalent.

(1) R is two-sided FPF.

(2) R is isomorphic to a finite direct product of Dedekid prime rings.

Finally, we consider a matrix representation of right semihereditary right FPF-rings.

We can not give a precise matrix representation of semihereditary FPF-rings so far. However, under the Morita equivalence, we cangive a representation.

Theorem 2.4 [6, Theorem 1]. Let R be a right semihereditary right FPF-ring. Then R is Morita equivalent to the type of ring.

$$\begin{pmatrix} D & D & \cdots & D & I^* \\ \vdots & \ddots & & \vdots & \vdots \\ D & D & \cdots & D & I^* \\ I & I & \cdots & I & O_1(I) \end{pmatrix}$$

Where D is a right semihereditary right FPF-ring whose maximal right quotient ring is an abelian regular self-injective ring and I is a finitely generated faithful right ideal of D, and I^* is the dual module of I and $O_l(I)$ is the left order of I, i.e. $O_l(I) = \{q \in Q_{max}(D) \mid qI \subseteq I\}$.

Definition 2.1. A ring R is Prüfer prime ring if R is two-sided semihereditary and two-sided Goldie, and doesnot contain proper finitely generated idempotent ideals.

If R is a prime right semihereditary right FPF-ring, and has the restricted minimum condition (= for any essential right ideal I of R, R/I is artinian), then we have a matrix representation.

Theorem 2.5 [6, Corollary]. Let R be a prime right semihereditary right FPF-ring with the restricted right minimum condition. Then R is isomorphic to the type of ring of Theorem 2.4, and in this case the ring D of Theorem 2.4 is a bounded Prüfer prime ring.

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ON FPF TRIVIAL EXTENSION RINGS

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Let R be a ring with identity and M an (R, R)-bimodule. The cartesian product R x M with componentwise addition and multiplication given by (r, m)(r', m') = (rr', rm' +mr') becomes a ring. This ring is called the trivial extension of R by M and denoted by $R \ltimes M$. In [F] Faith gave a characterization of FPF trivial extension $R \ltimes M$ in case M is a faithful module over a commutative ring R. Here a ring is said to be right FPF provided that every finitely generated faithful right module is a generator in the category of all right modules.

This paper is concerned with a problem mentioned in [F]: when is $R \ltimes M$ right FPF for a faithful bimodule M over a noncommutative ring? We shall give a necessary condition for $R \ltimes M$ to be right FPF under a certain hypothesis.

Throughout this paper R will be a ring with identity, M an (R, R)-bimodule and all modules unital. We shall treat a right $R \ltimes M$-module as a couple (X, u_X), where X is a right R-module and u_X is an R-homomorphism of $X \otimes_R M$ to X with $u_X.(u_X \otimes 1_M) = 0$. The connection between a right $R \ltimes M$-module X and a couple (X, u_X) is given by a relation

$$x.(a, m) = xa + u_X(x, m) \quad \text{for a in R, m in M, x in X.}$$

The detailed version of this paper will be submitted for a publication elsewhere.

Proposition 1. If a right $R \ltimes M$-module (X, u_X) is a generator, then there exist a finite number of g_i in $\mathrm{Hom}_R(X, R)$ such that

(i) $\sum \mathrm{Im}\, g_i = R$,

(ii) $g_i . u_X = 0$ and

(iii) $\mathrm{Ker}\, u_X \subset \mathrm{Ker}\, (g_i \otimes 1_M)$.

Moreover the converse holds if M is injective as a right R-module.

Proposition 2. Let I be a right ideal of R and N a right R-submodule of M such that $IM \subset N$. Let

$$u_{(I,\ N)}: (I,\ N) \otimes_R M \to (I,\ N)$$

be a homomorphism given by

$$u_{(I,\ N)}((a,\ n) \otimes m) = (0,\ am) \quad \text{for } a \text{ in } I,\ n \text{ in } N,\ m \text{ in } M.$$

If $((I,\ N),\ u_{(I,\ N)})$ is a generator over $R \ltimes M$, then

$$t_R(I) + l_R(M) = R$$

where $t_R(I) = \mathrm{Hom}_R(I,\ R)(I)$ is the trace ideal of I and $l_R(M)$ is the left annihilator of M in R.

Corollary. Assume that M is faithful as a left R-module. If $((I,\ N), u_{(I,\ N)})$ is a generator, then I is a generator. Moreover the converse holds if, in addition, M is flat as a left R-module and injective as a right R-module.

Proposition 3. Assume that M is flat and faithful as a left R-module and injective as a right R-module. Then the following are equivalent.

(a) Every finitely generated faithful right ideal I of R such that $Im = 0$, m in M, implies $m = 0$ is a generator.

(b) Every finitely generated faithful right ideal of $R \ltimes M$ is a generator.

Now we consider the following condition on a ring R.

(#) Every finite subset of R generating R as a right ideal also generates R as a left ideal.

Remark. If R is commutative or a finite product of local rings, the condition (#) is satisfied.

Theorem. Assume that R satisfies the condition (#) and that M is faithful as a left R-module and nonsingular as a right R-module. If $R \ltimes M$ is right FPF, then M is injective as a right R-module and a maximal right quotient ring of $R \ltimes M$ takes the form of a trivial extension.

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ON PRIME RIGHT IDEALS OF INTERMEDIATE RINGS OF A FINITE NORMALIZING EXTENSION

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Introduction and definition.

Throughout this report, S will present a ring extension of a ring R with common identity 1. Let I be a right ideal of R, and $b_R(I) = \{r \in R \mid Rr \subseteq I\}$. As in [5], I is called a prime right ideal provided that if $XY \subseteq I$, X, Y are right ideals of R, then either $X \subseteq I$ or $Y \subseteq I$. It is clear that a maximal right ideal is a prime right ideal. If I is a prime right ideal, then $b_R(I)$ is a prime ideal. Let R' be a ring, M a R-R'-bimodule. M is said to be a torsionfree R-R'-bimodule if $r_M(X) = \ell_M(Y) = 0$ for every essential ideal X of R and every essential ideal Y of R', where $r_M(X)$ (resp. $\ell_M(Y)$) is the right (resp. left) annihilator of X (resp. Y) in M. Moreover, M is said to be a finite normalizing R-R-bimodule if there exist elements $a_1, a_2, \cdots, a_n$ of M such that $M = \sum_{i=1}^{n} Ra_i$ and $Ra_i = a_iR'$ for $i = 1, 2, \cdots, n$. Such a system $\{a_1, a_2, \cdots, a_n\}$ is called a normalizing generator of M. We say that S is a finite normalizing extension of R if S is a finite normalizing R-R-bimodule.

The purpose of this report is to give a "cutting

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down" theorem for a prime right ideal of finite normalizing extension. We previously studied a "cutting down" theorem for a prime ideal (cf. [1], [2], [3], [4] and [6]). In [3], Heinicke and Robson exhibited a "cutting down" theorem for a prime right ideal: If S is a finite normalizing extension of a ring R, T is a ring with $R \subseteqq T \subseteqq S$, and J is a prime right ideal of T, then there exist right ideals $H_1, H_2, \cdots, H_h$ of R such that $\cap_{i=1}^{h} H_i = J \cap R$ and, for each i, $H_i/(J \cap R)$ is a prime right R-module. In the author's paper [7], we obtained another representaion of a "cutting down" theorem for a prime right ideal of a prime torsionfree finite normalizing extension. Namely, there exist prime right ideals $K_1, K_2, \cdots, K_s$ of R such that $\cap_{i=1}^{s} K_i = J \cap R$. In this report, we shall prove a "cutting down" theorem without the condition "torsionfree".

1. Preliminaries.

Throughout this report, suppose that S is a finite normalizing extension of a ring R, and T is a ring with $R \subseteqq T \subseteqq S$. Let P be a prime ideal of T. P is said to be a standard setting if (1) S is a prime ring, and (2) $A \cap T \nsubseteq P$ for each non zero ideal A of S. In [1], [2], [3], [4], [6] and [8], the following results are well-known.

Proposition 1.1 ([3, Proposition 2.2]). Let P be a prime ideal of T. Then there exists a prime ideal Q of S such that $Q \cap T \subseteqq P$ and, for each ideal $A \supsetneqq Q$, A

$\cap T \nsubseteq P$.

By Proposition 1.1, S/Q is a prime finite normalizing extension of $R/(Q \cap R)$, $T/(Q \cap T)$ is a ring with $S/Q \supseteq T/(Q \cap T) \supseteq R/(Q \cap R)$, and $P/(Q \cap T)$ is a standard setting.

Theorem 1.2 (Cutting down, [3, Theorem 2.13 and 5, Theorem 2.2]). Let P be a prime ideal of T. If P is a standard setting, then (1) R is a semiprime ring, (2) there exists a set $\{P_1, P_2, \cdots, P_m\}$ of at most n (the number of normalizing generators of S over R) prime ideals of R such that $\cap_{i=1}^{m} P_i = 0$ and the prime rings R/P_i are all isomorphic, and (3) there exists a subset $\{P_k\}_k$ of $\{P_1, P_2, \cdots, P_m\}$ such that $P \cap R = \cap_k P_k$.

Theorem 1.3 ([1, Proposition 3.3, and Lemmas 5.2 and 5.3]). If S is a prime ring, then S embeds in the right Martindale quotient $Q(S)$, and there exist orthogonal idempotents $f_1, f_2, \cdots, f_m$ in $V_{Q(S)}(R)$ such that $f_1 + f_2 + \cdots + f_m = 1$ and $r_R(f_i) = P_i$ for all $i = 1, 2, \cdots, m$. In this case, we obtain that $f_iQ(S)f_j$ is a torsionfree f_iR-f_jR-bimodule and f_iSf_j is a torsionfree finite normalizing f_iR-f_jR-bimodule.

Let f_i be as in Theorem 1.2. Let us set $S_{ij} = S \cap f_iQ(S)f_j = S \cap f_iSf_j$, $T_{ij} = T \cap f_iQ(S)f_j = T \cap f_iTf_j$, $S_i = S_{ii} + f_iR$ and $T_i = T_{ii} + f_iR$ for all $i, j = 1, 2, \cdots, m$. We immediately obtain $f_iR \subseteq T_i \subseteq S_i \subseteq f_iSf_i$

$\subseteq f_iQ(S)f_i$ and $T_{ii} \subseteq S_{ii} \subseteq f_iSf_i$. Let us set $R^* = \sum_{i=1}^m f_iR$, $S^* = \sum_{i=1}^m f_iSf_i$ and $T^\# = \sum_{i,j=1}^m T_{ij}$.

Theorem 1.4 ([2, Corollary 2.25 and Theorem 4.6]). $T^\#$ is an essential R-R-subbimodule of T. In this case, there exists a non zero ideal U of S such that $0 \neq U \cap T \subseteq T^\#$.

2. Prime right ideals of an intermediate ring of a finite normalizing extension.

Use the notation in the section 1. Let S be a prime finite normalizing extension of a ring R, and T a ring with $R \subseteq T \subseteq S$. Let J be a prime right ideal of T such that $b_T(J)$ is a standard setting. Let us set $h_i(J) = \{t_i \in T_i \mid t_if_iT^\#T \subseteq J\}$. Then we immediately obtain that $h_i(J)$ is a right ideal of T_i. In this situation, we have the following

Lemma 2.1. $h_i(J) = T_i$ if and only if $f_iT^\#T \subseteq J$.

Proof. If $h_i(J) = T_i$, then, by assumption, we have $T_if_iT^\#T \subseteq J$, and so $f_iT^\#T \subseteq J$. Conversely, if $f_iT^\#T \subseteq J$, then, for all $t_i + f_ir \in T_i$ ($t_i \in T_i$, $f_ir \in f_iR$), it is easy seen that $(t_if_i + f_ir)T^\#T \subseteq f_it_if_iT^\#T + f_irf_iT^\#T \subseteq f_iT^\#T + f_iT^\#T \subseteq J$. Hence we have $T_i \subseteq h_i(J)$, and so $h_i(J) = T_i$.

Lemma 2.2. There exists f_i such that $h_i(J) \neq T_i$.

Proof. If $h_i(J) = T_i$ for all $1 = 1, 2, \cdots, m$, then, by Lemma 2.1, we have $T^{\#}T \subseteq f_1T^{\#}T + f_2T^{\#}T + \cdots + f_mT^{\#}T \subseteq J$. By Theorem 1.4, there exists an non zero ideal U of S such that $0 \neq U \cap T \subseteq T^{\#}$, which contradicts that $b_T(J)$ is a standard setting.

By Lemma 2.2, we may assume that $f_iT^{\#}T \nsubseteq J$ for $i = 1, 2, \cdots, s$, and $f_iT^{\#}T \subseteq J$ for $i = s+1, \cdots, m$. In this situation, we shall prove the following

Lemma 2.3. $b_T(J) \cap R \subseteq \cap_{i=1}^{s} P_i$.

Proof. Since $T^{\#}$ is an essential R-R-subbimodule of T, there exists a non zero ideal U of S such that $0 \neq U \cap T \subseteq T^{\#}$. Therefore, since $b_T(J)$ is a standard setting, we have $(U \cap T)f_iT^{\#}T \nsubseteq b_T(J)$ for $i = 1, 2, \cdots, s$, and so $TT^{\#}f_if_iT^{\#}T \nsubseteq b_T(J)$. Let us set $Q'_{(i)} = \{t_i \in T_i \mid TT^{\#}f_it_if_iT^{\#}T \subseteq b_T(J)\}$ for each $i = 1, 2, \cdots s$. By the correspondence of prime ideals in a Morita context

$$C_i = \begin{pmatrix} T & TT^{\#}f_i \\ f_iT^{\#}T & T_i \end{pmatrix},$$

$Q'_{(i)}$ is a prime ideal of T_i which corresponds to $b_T(J)$. By [3, Proposition 2.11], we have $Q'_{(i)} \cap f_iR = 0$. Since $TT^{\#}f_i(b_T(J) \cap R)f_iT^{\#}T \subseteq Tb_T(J)T \subseteq b_T(J)$, we obtain $f_i(b_T(J) \cap R)f_i \subseteq f_iR \cap Q'_{(i)} = 0$, and hence $b_T(J) \cap R \subseteq r_R(f_i) = P_i$. This implies $b_T(J) \cap R \subseteq \cap_{i=1}^{s} P_i$.

Lemma 2.4. For each $i = 1, 2, \cdots, s$, $h_i(J)$ is a prime right ideal of T_i such that $A'_{(i)} \cap T_i \nsubseteq h_i(J)$ for each non zero ideal $A'_{(i)}$ of S_i.

Proof. To see that $h_i(J)$ is a prime right ideal of T_i, let $a, b \in T_i$ with $aT_ib \subseteq h_i(J)$ and $b \notin h_i(J)$. Let U be as in Theorem 1.4. Then, from $af_iT^{\#}T(U \cap T)f_ibf_iT^{\#}T \subseteq aT_ibf_iT^{\#}T \subseteq J$, we have either $af_iT^{\#}T \cap J$ or $(U \cap T)f_ibf_iT^{\#}T \subseteq J$. If $(U \cap T)f_ibf_iT^{\#}T \subseteq J$, then, since $b_T(J)$ is a standard setting and $f_ibf_iT^{\#}T$ is a right ideal of T, we obtain $bf_iT^{\#}T \subseteq J$, and so $b \in h_i(J)$, which is contradictory. Hence we have $af_iT^{\#}T \subseteq J$, and so $a \in h_i(J)$. This implies that $h_i(J)$ is a prime right ideal of T_i. Next we claim that $b_{T_i}(h_i(J)) \cap f_iR = 0$. Let $f_ir \in b_{T_i}(h_i(J)) \cap f_iR$ ($r \in R$). Then we have $f_iT^{\#}TT^{\#}f_irf_iT^{\#}T \subseteq T_if_irf_iT^{\#}T \subseteq J$, and so, by assumption for i, we have $TT^{\#}f_irf_iT^{\#}T \subseteq J$. Since $TT^{\#}f_irf_iT^{\#}T$ is an ideal of T, we have $f_ir \in Q'_{(i)} \cap f_iR = 0$, where $Q'_{(i)}$ is as in the proof of Lemma 2.3. Therefore we have $b_{T_i}(h_i(J)) \cap f_iR = 0$. Finally, if there exists a non zero ideal $A'_{(i)}$ of S_i such that $A'_{(i)} \cap T_i \subseteq b_{T_i}(h_i(J))$, then, by [2, Proposition 2.20], we have $0 \neq A'_{(i)} \cap f_iR \subseteq b_{T_i}(h_i(J)) \cap f_iR$, which contradict to $b_{T_i}(h_i(J)) \cap f_iR = 0$. This completes the proof.

Theorem 2.5. For each $i = 1, 2, \cdots, s$, $h_i(J) \cap f_iR$ is a prime right ideal of f_iR, and $b_{f_iR}(h_i(J) \cap f_iR) = 0$.

Proof. First, let X, Y be right ideals of f_iR such that $XY \subseteq h_i(J) \cap f_iR$ and $Y \not\subseteq h_i(J) \cap f_iR$. Then we immediately obtain that f_iRYT_i is an essential f_iR-f_iR-subbimodule of T_i. By the canonical epimorphism $R \longrightarrow f_iR$, we may regard that f_iRY is an essential R-R-

subbimodule of T_i. Hence $f_iRYT_i \cap T_{ii}$ is an essential R-R-subbimodule of T_{ii}. By [2, Corollary 2.25], there exists a non zero ideal A of S such that $A \cap T_{ii} \subseteq f_iRYT_i \cap T_{ii}$. Since $A \cap f_iAf_i$ is an ideal of S_i, and since $A \cap T_{ii} = (A \cap f_iAf_i \cap T_i) \cap T_{ii}$ is an ideal of T_i, we have $XT_i(A \cap T_{ii}) \subseteq X(f_iRYT_i \cap T_{ii}) \subseteq Xf_iRYT_i \subseteq h_i(J)$, and so, by Lemma 2.4, either $XT_i \subseteq h_i(J)$ or $A \cap T_{ii} \subseteq h_i(J)$. If $A \cap T_{ii} = (A \cap f_iAf_i \cap T_i) \cap T_{ii} \subseteq h_i(J)$, then we have either $A \cap f_iAf_i \cap T_i \subseteq h_i(J)$ or $T_{ii} \subseteq h_i(J)$. The first case is contradictory to Lemma 2.4, and the second case is contradictory to [3, Proposition 2.6]. Therefore we obtain $X \subseteq XT_i \cap f_iR \subseteq h_i(J) \cap f_iR$. Thus $h_i(J) \cap f_iR$ is a prime right ideal of f_iR. Next, if $b_{f_iR}(h_i(J) \cap f_iR) \neq 0$, then $b_{f_iR}(h_i(J) \cap f_iR)T_i$ is an essential f_iR-f_iR-subbimodule of T_i. It follows from [2, Corrolary 2.22] that there exists a non zero ideal B of S such that $B \cap T_i = B \cap f_iBf_i \cap T_i \subseteq b_{f_iR}(h_i(J) \cap f_iR)T_i \subseteq h_i(J)$. By [2, Proposition 2.22], $B \cap f_iBf_i$ is a non zero ideal of S_i. This contradicts to Lemma 2.4. Therefore $b_{f_iR}(h_i(J) \cap f_iR) = 0$.

The following theorem is a "cutting down" theorem for a prime right ideal.

Theorem 2.6. Let S be a prime finite normalizing extension of a ring R, and T a ring with $R \subseteq T \subseteq S$. If J is a prime right ideal of T such that $b_T(J)$ is a standard setting, then there exist prime right ideals $K_1, K_2, \cdots, K_s$ of R such that $\cap_{i=1}^{s} K_i = J \cap R$ and

$b_R(K_i) = P_i$. In this case, $b_R(J \cap R) = \cap_{i=1}^{s} P_i \supsetneqq b_T(J) \cap R$.

Proof. By Lemma 2.2, we assume that $f_i T^{\#} T \nsubseteq J$ for $i = 1, 2, \cdots, s$, and $f_i T^{\#} T \subseteqq J$ for $i = s + 1, \cdots, m$. For each $i = 1, 2, \cdots, s$, let us set $K_i = \{r \in R \mid f_i r \in h_i(J) \cap f_i R\}$. Then, by Theorem 2.5, we immediately obtain that, for each $i = 1, 2, \cdots, s$, K_i is a prime right ideal of R, and $b_R(K_i) = P_i$. If $r \in J \cap R$, then, for $i = 1, 2, \cdots, s$, we have $f_i r f_i T^{\#} T \subseteqq r T^{\#} T \subseteqq J$, and so $f_i r \in h_i(J) \cap R$. This implies $J \cap R \subseteqq \cap_{i=1}^{s} K_i$. Conversely, for all $r \in \cap_{i=1}^{s} K_i$, since $f_i r \in h_i(J) \cap f_i R$, we have $f_i r f_i T^{\#} T \subseteqq J$ for each $i = 1, 2, \cdots, s$. On the other hand, noting that $f_i T^{\#} T \subseteqq J$ for $i = s+1, \cdots, m$, it follows that $r T^{\#} T \subseteqq \sum_{i=1}^{s} f_i r f_i T^{\#} T + \sum_{i=s+1}^{m} f_i r f_i T^{\#} T \subseteqq J + \sum_{i=s+1}^{m} f_i r T^{\#} T \subseteqq J + \sum_{i=s+1}^{m} f_i T^{\#} T \subseteqq J$. Therefore $r(U \cap T) \subseteqq J$, where U is as in Theorem 1.4, and then, since $b_T(J)$ is a standard setting, we have $r \in J \cap R$. Consequently, we obtain $J \quad R = \quad {}_{i=1}^{s} K_i$. Finally, since $b_R(J \cap R)$ is an ideal contained in K_i for each $i = 1, 2, \cdots, s$, we obtain $b_R(J \cap R) \subseteqq \cap_{i=1}^{s} b_R(k_i) = \cap_{i=1}^{s} P_i \subseteqq \cap_{i=1}^{s} K_i = J \cap R$, and then we obtain $b_R(J \cap R) = \cap_{i=1}^{s} P_i \supsetneqq b_T(J) \cap R$.

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UNIVERSAL COVERS OF REPRESENTATION-FINITE SELF-INJECTIVE ALGEBRAS

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This article gives an outline of some results on relations between iterated tilted algebras of Dynkin type and representation-finite self-injective algebras without proofs. A detailed account will appear elsewhere.

Throughout this article, we assume that all algebras and modules are finitely generated over a commutative Artin ring K. The ordinary duality functor $\mathrm{Hom}_K(?,I)$ will be denoted by D, where I is the injective envelope of the K-module K/radK. Every homomorphisms operate from the opposite side of the scalar.

For an algebra A, the trivial extension algebra $T(A) = A \ltimes DA$ of A by its minimal injective cogenerator DA is defined over the underlying additive group $A \oplus DA$ by giving its multiplication as

$$(a_1,q_1)\cdot(a_2,q_2) = (a_1\cdot a_2, a_1\cdot q_2 + q_1\cdot a_2)$$

for any (a_1,q_1), $(a_2,q_2) \in A \oplus DA$.

It is easy to see that T(A) becomes a symmetric algebra and hence is self-injective.

A basic module T_A over an algebra A is called a tilting module [5] if it satisfies the following three properties:

(T_1) proj. dim $T_A \leq 1$,

(T_2) $\mathrm{Ext}^1_A(T,T) = 0$, and

(T_3) There is a short exact sequence

$$0 \longrightarrow A_A \longrightarrow T' \longrightarrow T'' \longrightarrow 0$$

such that both T' and T'' are in the class add T_A.

For a tilting module T_A, putting $B = \mathrm{End}(T_A)$, we call a triple $(B, {}_BT_A, A)$ a tilting triple[2].

In the paper [10], Tachikawa and the author proved

<u>Proposition 1</u>. Let (B,T,A) be a tilting triple. Then there is a stable equivalence $\mathfrak{L}$: $\underline{\mathrm{mod}}\text{-}T(A) \cong \underline{\mathrm{mod}}\text{-}T(B)$.

An algebra A is called an iterated tilted algebra of type X if there is a sequence of tilting triples $(A_n, T^{(n)}, A_{n-1})$, $(A_{n-1}, T^{(n-1)}, A_{n-2})$, ... , $(A_1, T^{(1)}, A_0)$ such that $A = A_n$ and A_0 is a hereditary algebra of type X.

Tachikawa[9] proved

<u>Proposition 2</u>. Let A_0 be a hereditary algebra of Dynkin type X. Then the trivial extension $T(A_0)$ is representation-finite of Cartan class X.

Hughes and Waschbüsch[7] proved

<u>Proposition 3</u>. Assume the trivial extension $T(A)$ be representation-finite of Cartan class X. Then there is a tilted algebra A' of Dynkin type X such that $T(A) \cong T(A')$.

Here, any endomorphism ring of a tilting module over a hereditary algebra is called a tilted algebra[5]. Of course, any tilted algebra is an iterated tilted algebra.

Assem, Happel and Roldán[1] and Tachikawa and the author[11] have proved

Proposition 4. The trivial extension $T(A)$ is representation-finite of Cartan class X if and only if A is an iterated tilted algebra of Dynkin type X.

For a simple projective module eA, putting $T_A = (1-e)A \oplus TrD(eA)$, we always have a tilting module T_A. This special kind of tilting modules is called an APR-tilting module and was first introduced by Auslander, Platzeck and Reiten as a generalization of reflection functors for hereditary algebras. In the above proposition, we can choose a sequence of tilting triples $(A_n, T^{(n)}, A_{n-1})$, ... , $(A_1, T^{(1)}, A_0)$ with $A = A_n$ and A_0 being a hereditary algebra of Dynkin type X in such a way that all tilting modules $T^{(i)}_{A_{i-1}}$ are APR-tilting modules. For proof, see [11]. In this case, we call A an APR-iterated tilted algebra of Dynkin type X. From the above proposition, we have

Corollary 5. Any iterated tilted algebra of Dynkin type is, in fact, an APR-iterated tilted algebra.

In the study of the trivial extension algebra $T(A)$, Hughes and Waschbüsch[7] introduced the following doubly infinite matrix algebra:

$$\hat{A} = \begin{bmatrix} \cdot & \cdot & & & & & \\ & \cdot & \cdot & & & & \\ & & A_{n-1} & M_{n-1} & & & \\ & & & A_n & M_n & & \\ & & & & A_{n+1} & M_{n+1} & \\ & & & & & \cdot & \cdot \\ & & & & & & \cdot \quad \cdot \end{bmatrix}$$

in which matrices are assumed to have only finitely many entries different from zero, $A_n = A$ and $M_n = DA$ for all integers n, all the remaining entries are zero, and multiplication is induced from the canonical maps $A \otimes_A DA \xrightarrow{\sim} DA$, $DA \otimes_A A \xrightarrow{\sim} DA$, and zero maps $DA \otimes_A DA \to 0$.

It is easy to see that the infinite matrix algebra $\hat{A}$ is one of examples of locally bounded categories defined by Bongartz and Gabriel.

A locally finitely generated module over $\hat{A}$ is defined by giving a set of finitely generated A-modules $\{X_n\}_{n \in \mathbb{Z}}$ and a set of A-homomorphisms $\{f_n : X_n \otimes_A DA \to X_{n+1}\}_{n \in \mathbb{Z}}$ such that $f_{n+1} \cdot f_n \otimes DA = 0$ for all integers n.

The category of all locally finitely generated $\hat{A}$-modules is denoted by Mod-$\hat{A}$. The category mod-$\hat{A}$ is defined as the full subcategory of Mod-$\hat{A}$ consisting of all $\hat{A}$-modules $\{X_n, f_n\}_{n \in \mathbb{Z}}$ such that $\bigoplus_{n \in \mathbb{Z}} X_n$ is finitely generated over K (this is equivalent to saying that $X_n = 0$ for only a finite number of n's). We call an object in the category mod-$\hat{A}$ a finitely generated $\hat{A}$-module.

Similarly, a finitely generated module over the trivial extension algebra $T(A)$ is defined by giving a finitely generated A-module X and a A-homomorphism f: $X \otimes_A DA \to X$ such that $f \cdot f \otimes DA = 0$.

Hughes and Waschbüsch defined a functor

$$F_A : \text{mod-}A \longrightarrow \text{mod-}T(A)$$

by $F_A(\{X_n, f_n\}) = (\bigoplus_n X_n, \bigoplus_n f_n)$.

They proved[7,13]

Proposition 6. The functor F_A is exact and preserves indecomposable modules, almost split sequences, irreducible maps, and the composition length of modules.

In the case where $T(A)$ is representation-finite, the functor F_A is, in fact, a covering functor in the sense of Bongartz and Gabriel and further ind-$\hat{A}$ has no oriented cycles. Therefore, we can consider $\hat{A}$ as the universal cover of $T(A)$.

By the way, by slightly modifying the proof of Proposition 1, we get

Proposition 7.[12] Let (B,T,A) be a tilting triple. Then the stable categories $\underline{\text{mod}}\text{-}\hat{A}$ and $\underline{\text{mod}}\text{-}\hat{B}$ are equivalent and we have the following commutative diagram of functors:

$$\begin{array}{ccc} \underline{\text{mod}}\text{-}\hat{A} & \xrightarrow{\ \sim\ } & \underline{\text{mod}}\text{-}\hat{B} \\ {\scriptstyle F_A}\downarrow & & \downarrow{\scriptstyle F_B} \\ \underline{\text{mod}}\text{-}T(A) & \underset{\text{£}}{\xrightarrow{\ \sim\ }} & \underline{\text{mod}}\text{-}T(B) \end{array} .$$

Therefore, in the case of $T(A)$ being representation-finite, the tilting process preserves the stable part of the universal cover of mod-$T(A)$. So it is interesting to know the change of the configurations in the sense of Riedtmann.

Proposition 8.[11] Let (B,T,A) be an tilting triple and $\mathfrak{L}$ the equivalence given by Proposition 1, then we have the following short exact sequence for any T(A)-module X:

$$0 \to \mathrm{Hom}_{T(A)}(T\otimes_A T(A),X) \to \mathfrak{L}(X) \to \Omega_{T(B)}(\mathrm{Ext}_A^1(T,X)) \to 0.$$

In the above, $\Omega_{T(B)}$ denotes the loop space functor of Heller.

By Proposition 4, in our study, we may assume that T_A is an APR-tilting module. Assume eA be simple and put $T_A = (1-e)A \oplus \mathrm{TrD}(eA)$ and $B = \mathrm{End}(T_A)$. Using the above proposition, we can give a description of the change of the configurations as follows:

Proposition 9.

(a) $\mathfrak{L}(eT(A)/soc) = \Omega_{T(B)}(\hat{e}T(B)/soc)$.

(b) For an idempotent $f \neq e$,

$$\mathfrak{L}(fT(A)/soc) = \Omega^{-1}_{T(B)}(\mathrm{Hom}_A(T,fA/\mathrm{rad}\, fA)).$$

(c) For an idempotent $f \neq e$, assume that rad fA has no direct summands isomorphic to eA, then

$$\mathfrak{L}(fT(A)/soc) = \hat{f}T(B)/soc.$$

In the above $\hat{e} \in B$ is the idempotent corresponding to the direct summand $\mathrm{TrD}(eA) \mathrel{\triangleleft\!\oplus} T_A$ and, similarly, $\hat{f} \in B$ is the idempotent corresponding to the direct summand $fA \mathrel{\triangleleft\!\oplus} T_A$.

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The detailed version of this paper will be submitted for publication elsewhere.

NOTE ON COVERINGS OF TRIVIAL EXTENSIONS*

Hiroshi OKUNO

Let A be an artin algebra over a commutative artin ring C. $T(A)$ denotes the trivial extension of A by an A-A-bimodule $DA = \mathrm{Hom}_C(A,I)$, where I is the injective envelope of $C/\mathrm{rad}\, C$ over C. Let $\hat{A}$ be the doubly infinite matrix algebra without identity:

$$\begin{bmatrix} \ddots & \ddots & & & & \\ & \ddots & \ddots & & & \\ & & A_{n-1} & N_{n-1} & & \\ & & & A_n & N_n & \\ & & & & A_{n+1} & N_{n+1} \\ & & & & & \ddots \quad \ddots \\ & & & & & \quad \ddots \quad \ddots \end{bmatrix}$$

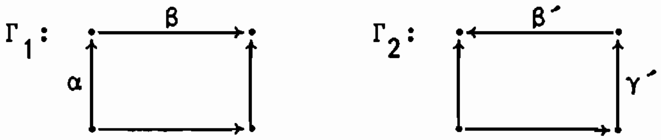
in which matrices are assumed to have only finitely many entries different from zero, $A_n = A$ and $N_n = DA$ for all integers n, and all remaining entries are zero. The identity maps $A_n \to A_{n+1}$ and $N_n \to N_{n+1}$ induce an automorphism ν_A of $\hat{A}$.

In [1] Hughes and Waschbüsch stated that if C is a field and the quiver of A has no oriented cycles then $\hat{A} \cong \hat{B}$ is equivalent to $T(A) \cong T(B)$. But unfortunately, as Tachikawa pointed out in the Informal Problem Session of the International Conference on Representations of Algebras

* This note is a summary of [2].

at Carleton in 1984, this is not true in general, because there is a counter example given by Wakamatsu.

Example 1. Let Γ_1 and Γ_2 be the following quivers:



Let A be the path algebra $k\Gamma_1$ with a relation $\alpha\beta = 0$ and B the path algebra $k\Gamma_2$ with a relation $\gamma'\beta' = 0$, where k is a field, then $T(A) \cong T(B)$ but $\hat{A} \not\cong \hat{B}$.

Also Tachikawa proposed the problem to give a necessary and sufficient condition for A and B to force $\hat{A} \cong \hat{B}$ there. This note gives a solution of the problem. At first we will give some definitions.

A pair of C-algebras (A,B) is said to be a D-pair if A and B have the following triangular matrix decompositions:

$$A \cong \begin{pmatrix} S_1 & M_1 & & & & 0 \\ & S_2 & M_2 & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & 0 & & & S_{n-1} & M_{n-1} \\ & & & & & S_n \end{pmatrix}$$

and

$$B \cong \begin{pmatrix} S_n & DM_{n-1} & & & & 0 \\ & S_{n-1} & DM_{n-2} & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & S_2 & DM_1 \\ 0 & & & & & S_1 \end{pmatrix} .$$

Let $1_A = e_1 + e_2 + \cdots + e_m$ be a decomposition into orthogonal primitive idempotents of A . And let $1_{A_j} = e_1^j + e_2^j + \cdots + e_m^j$ be the decomposition induced by the decomposition above. Let $Q_{\hat{A}}$ be the quiver of $\hat{A}$, and $\{ e_i^j \}$ is regarded as the set of vertices of $Q_{\hat{A}}$. ν_A induces a graph automorphism of $Q_{\hat{A}}$. A complete ν_A-slice Λ of $Q_{\hat{A}}$ is a full subquiver of $Q_{\hat{A}}$ such that the vertices of Λ is a set of representatives of the ν_A-orbits of vertices of $Q_{\hat{A}}$ and a vertex f_i iscontained in Λ if there exists a chain $f_t \to \cdots \to f_i \to \cdots \to f_s$ where f_t and f_s are contained in Λ (cf. [1]) . If Λ is a complete ν_A-slice and $\{ f_1, f_2, \cdots , f_m \}$ is the set of vertices of Λ , we put $\mathrm{Alg}(\Lambda) = (\Sigma_{j=1}^m f_j)\hat{A}(\Sigma_{j=1}^m f_j)$.

Theorem 1. Let A be a connected basic artin algebra over a commutative artin ring C , then the following are equivalent for a C-algebra B :

(1) $\hat{A} \cong \hat{B}$;

(2) (A,B) is a D-pair ;

(3) $B \cong \mathrm{Alg}(\Lambda)$ for some complete ν_A-slice Λ of $Q_{\hat{A}}$.

Note that if (A,B) is a D-pair then $T(A) \cong T(B)$. It is also true that $\hat{A} \cong \hat{B}$ implies $T(A) \cong T(B)$, however the proof is not obvious. In order to prove this, we have to show the existence of a C-algebra isomorphism from $\hat{A}$ to $\hat{B}$ commuting with ν_A and ν_B .

Next we consider the following condition (N-O) for A :

(N-O) The quivers of algebras whose trivial extensions are isomorphic to T(A) have no oriented cycles.

Theorem 2. Let A be a connected basic artin algebra satisfying (N-O) , then $\hat{A} \cong \hat{B}$ if and only if $T(A) \cong T(B)$.

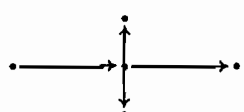
In [4] Yamagata proved that if T(A) is of finite representation type then the quiver of A has no oriented cycles. Then we have the following.

Corollary. Let A be a connected basic artin algebra, and assume that T(A) is of finite representation type. Then $\hat{A} \cong \hat{B}$ if and only if $T(A) \cong T(B)$.

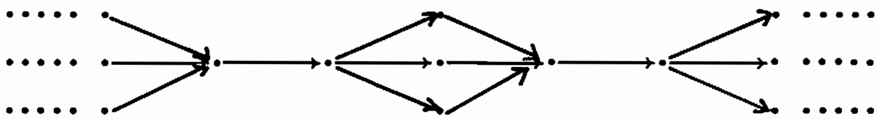
Let A be a path algebra $k\Gamma$, where k is a field and Γ is a tree graph. Then it is easy to check that A satisfies (N-O) , and we can calculate the algebras whose trivial extentions are isomorphic to T(A) by Theorem 1.

Example 2. Let Ω be the following quiver:

Ω:

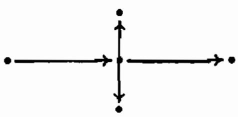


Let A be the path algebra $k\Omega$, where k is a field. Then $Q_{\hat{A}}$ is the following quiver:

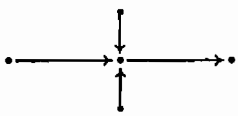


If $T(A) \cong T(B)$, then B is the path algebra $k\Omega'$ where Ω' is one of the following quivers with relations:

(i)

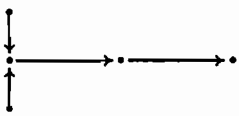


(ii)

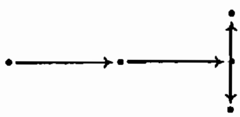


(i) and (ii) have no relations.

(iii)

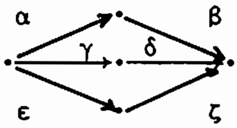


(iv)



The relations of (iii) and (iv) are all paths of length 3.

(v)



The relation of (v) is $\alpha\beta = \gamma\delta = \varepsilon\zeta$.

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REPRESENTATIONS OF ORDERS AND ONE-POINT EXTENSION ALGEBRAS

Kenji NISHIDA

1. Let R be a complete discrete valuation ring with prime element π and residue field k. Let K be a quotient field of R and Σ a semisimple K-algebra. An R-order Λ in Σ is a subring of Σ such that:

1) R is contained in the center of Λ,
2) Λ is a finitely generated R-module,
3) $K\Lambda = \Sigma$.

A right Λ-lattice M is a finitely generated right Λ-module which is torsionfree over R. Let $L(\Lambda)$ be the category of all right Λ-lattices. We study the Auslander-Reiten quiver of $L(\Lambda)$. Recently many results have appeared about this problem(see, for example, [6]). Under some conditions, we can give a mehtod of adapting the Auslander-Reiten quiver of the category of socle projective mosules over a right peak algebra to determine that of $L(\Lambda)$. We shall report here the outline of this result and the details will appear in [3].

2. Let Γ be a hereditary R-order in Σ and I a proper Γ-ideal in Λ such that $KI=\Sigma$. Put $A=\Lambda/I$ and $B=\Gamma/I$. Then A is a subring of B. Let $C=\begin{pmatrix} A & B \\ 0 & B \end{pmatrix}$ and $\mathcal{C}$ a full subcategory of mod C such that $(X_A, Y_B, \phi) \in \mathcal{C} \Longleftrightarrow X \neq 0$, $\overline{\phi}: X_A \to Y_A$ is injective, Im $\overline{\phi}$ B=Y and Y_B is projective. Here we identify a right C-module with a triple (X_A, Y_B, ϕ) such that $\phi: X \otimes_A B \to Y_B$ a B-homomorphism and $\overline{\phi} \in \mathrm{Hom}_A(X,Y)$ is the adjoint of ϕ.

Now define the functor $H: \mathcal{L}(\Lambda) \to \mathcal{C}$ by $H(M)=(M/MI, M\Gamma/MI, \phi)$ where ϕ is the adjoint of the canonical inclusion $M/MI \hookrightarrow M\Gamma/MI$. Then we have

THEOREM 1. [1,5]

H induces a representation equivalence $\mathcal{L}(\Lambda) \approx \mathcal{C}$.

Remark. A functor $F: \mathcal{A} \to \mathcal{B}$ for additive categories $\mathcal{A}$, $\mathcal{B}$ is called a representation equivalence if;

a) H preserves every isomorphism,

b) if $A \cong A_1 \oplus A_2$ in $\mathcal{A}$, then $F(A) \cong F(A_1) \oplus F(A_2)$,

c) for every $B \in \mathcal{B}$, there exists $A \in \mathcal{A}$ such that $F(A) \cong B$.

3. In what follows, we assume I=rad Γ. Then B is semi-simple. Let $S_1, \ldots, S_t$ be the representatives of the non-isomorphic simple left B-modules, $K_i = \mathrm{End}_B S_i$ $(i=1,\ldots,t)$, $G = S_1 \oplus \ldots \oplus S_t$ and $E = \mathrm{End}_B G = K_1 \times \ldots \times K_t$. Let $C' = \begin{pmatrix} A & G \\ 0 & K \end{pmatrix}$. Then C' is a one-point coextension k-algebra of A by G. Define the functor ρ : mod C $\to$ mod C' with $\rho(X_A, Y_B, \phi) = (X_A, Y \otimes_B G, \psi)$ where $\psi : X \otimes_A G \to Y \otimes_B G$ is $\psi(x \otimes g) = \phi(x \otimes 1) \otimes g$. Then ρ is a category equivalence. Let $\Phi = \rho H : \mathcal{L}(\Lambda) \to$ mod C'. Then Im Φ has very nice property. In order to state this, we need results due to Simson[7].

A basic artinian ring C is called a <u>right peak ring</u> if $\mathrm{soc}(C_C)$ is projective.

PROPOSITION 1. [7] C is a right peak ring $\Longleftrightarrow C = \begin{pmatrix} A & M \\ 0 & K \end{pmatrix}$ where K is a product of division rings, ${}_A M$ is faithful and M_K is finitely generated.

Let mod_{sp} C be a full subcategory of mod C consisting of modules having a projective socle.

PROPOSITION 2.[7] $(X_A, Y_K, \phi) \in \text{mod}_{sp}\ C \Leftrightarrow \overline{\phi}: X_A \to \text{Hom}_K(M,Y)$ is injective.

The category $\text{mod}_{sp}\ C$ for a right peak k-algebra C has enough injectives, almost split sequences, and is studied widely by Simson and others(cf.[7]).

By Proposition 2 we have $\text{Im}\ \Phi \subset \text{mod}_{sp}\ C'$. Let $\text{mod}_{sp}^{\circ}\ C'$ be the full subcategory of $\text{mod}_{sp}\ C'$ consisting of modules having no simple projective direct summands.

THEOREM 2. [2]

Φ induces a representation equivalence $L(\Lambda) \approx \text{mod}_{sp}^{\circ}\ C'$.

4. Decompose the hereditary order $\Gamma = \Gamma_1 \oplus \ldots \oplus \Gamma_k$ into the direct sum of indecomposable rings. Let G_{ij} $(j=1,\ldots a_i; i=1,\ldots,k)$ be the representatives of nonisomorphic indecomposable projective Γ-lattices, where for $i(i=1,\ldots,k)$ G_{ij} $(j=1,\ldots,a_i)$is a Γ-lattice. For $i(i=1,\ldots,k)$, we number so that;

$$\begin{cases} G_{ij} = G_{ij+1} \cdot \text{rad}\ \Gamma_i (j=1,\ldots,a_i-1) \\ G_{ia_i} = G_{i1} \cdot \text{rad}\ \Gamma_i \ \ (\text{see}[4]). \end{cases}$$

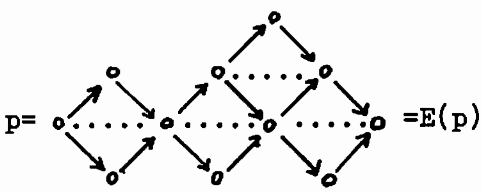
Then we can state the relations of Auslander-Reiten quivers Q and Q' of $L(\Lambda)$ and $\text{mod}_{sp}\ C'$, respectively, as follows.

THEOREM 3. Q' is obtained from Q by identifying every simple projective module $(0, \text{End}_B\ (G_{ij+1}/G_{ij}), 0)$ with an indecomposable injective module $E((0, \text{End}_B(G_{ij}/G_{ij-1}), 0))$ where if $j=a_i$ then a_i+1 is replaced by 1 and G_{ia_i} is identified with $G_{i1} \cdot \text{rad}\ \Gamma_i$ and if $j=1$ then 0 is replaced by a_i and the same identification is done as before.

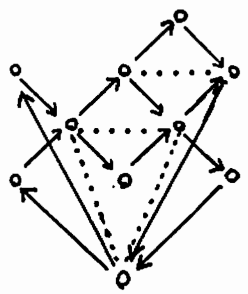
5. Example. Let $\Lambda = \begin{pmatrix} R & \pi & \pi & \pi & \pi & \pi \\ R & R & \pi & R & \pi & \pi \\ R & R & R & R & \pi & \pi \\ \pi & \pi & \pi & R & \pi & \pi \\ R & \pi & \pi & R & R & \pi \\ R & R & \pi & R & R & R \end{pmatrix}$ where we abbreviate πR to π and $\Gamma = (R)_6$. Then rad $\Gamma \subset \Lambda \subset \Gamma$ and

$$C' = \begin{pmatrix} k & 0 & 0 & 0 & 0 & 0 & k \\ k & k & 0 & k & 0 & 0 & k \\ k & k & k & k & 0 & 0 & k \\ 0 & 0 & 0 & k & 0 & 0 & k \\ k & 0 & 0 & k & k & 0 & k \\ k & k & 0 & k & k & k & k \\ 0 & 0 & 0 & 0 & 0 & 0 & k \end{pmatrix}$$

The Auslander-Reiten quiver of mod_{sp} C' is



where dotted lines indicate τ-orbits. Thus by Theorem 3 the Auslander-Reiten quiver of $L(\Lambda)$ is



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The detailed version of this paper will be submitted for publication elsewhere.

ON A CONSTRUCTION OF DTr-INVARIANT MODULES OVER ONE-POINT EXTENSION ALGEBRAS

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This note is a summary of the paper [5]* by the author and J. Miyachi.

In the study of finite dimensional algebras over a field k of infinite representation type (i.e. each of which has infinite number of indecomposable modules), V. Dlab and C. M. Ringel showed that in each of the Auslander-Reiten quivers of tame hereditary algebras, stable tubes (in particular homogeneous tubes) play an important role [3], [7]. A homogeneous tube is a basic component which contains $\mathbb{N}$-number of indecomposable modules which are constructed by extensions of the simplest one in it.

Here we characterize a part of DTr-invariant modules over one-point extension algebras, and construct homogeneous tubes by using it.

1. In this section, we recall fundamental notations and definitions.

Throughout this note, we deal only with finite dimensional algebras over a field k , and finite dimensional (usually left) modules. Let A be an algebra. We denote by P(X) , the projective cover of an A-module X , and by E(Y) , the injective hull of an A-module Y . The k-dual $\mathrm{Hom}_k(-,k)$ is denoted by D , and the A-dual $\mathrm{Hom}_A(-,A)$

* This paper was contributed to Tsukuba J. Math.

is denoted by $-^*$.

Let X , Y be A-modules. We call a homomorphism $f : X \longrightarrow Y$ irreducible if (1) f is neither a splitable monomorphism nor a splitable epimorphism, (2) given a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & {\scriptstyle g}\searrow \qquad \nearrow{\scriptstyle h} & \\ & Z & \end{array} ,$$

either g is a splitable monomorphism or h is a splitable epimorphism.

Theorem (Auslander, Reiten [1] [2])
The following statements are equivalent for a non-split exact sequence of A-modules

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

where L and N are indecomposable.

(1) f , g are irreducible.

(2) given any homomorphism $s : L \longrightarrow X$ which is not a splitable monomorphism, there is a $t : M \longrightarrow X$ such that $t\ f = s$.

(3) given any homomorphism $u : Y \longrightarrow N$ which is not a splitable epimorphism, there is a $v : Y \longrightarrow M$ such that $g\ v = u$.

For an arbitrary non-projective indecomposable A-module N (non-injective indecomposable A-module L), there uniquely exists the extension with above properties up to isomorphism. We call it an Auslander-Reiten sequence. Here L is given by $DTr\ N$, the Auslander-Reiten translation of N . It is the composition of D and Tr (transpose) which is defined as follows. Let

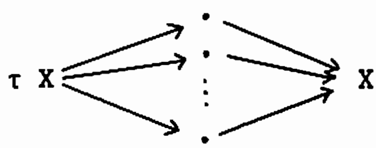
$$P_1 \xrightarrow{p} P_0 \longrightarrow N \longrightarrow 0$$

be the minimal projective presentation of N . Then we set

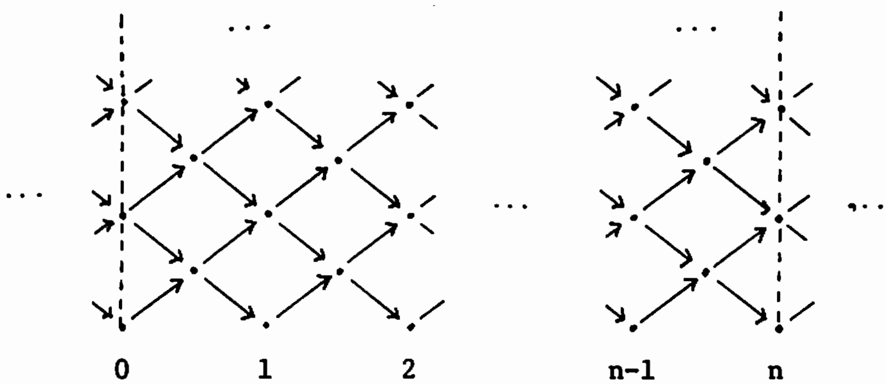
$$\mathrm{Tr}\, N = \mathrm{Cok}\ (P_0^* \xrightarrow{p^*} P_1^*) .$$

After this DTr is denoted by τ .

Given an algebra A , the Auslander-Reiten quiver of A is a directed graph which has as vertices the isomorphism classes of indecomposable A-modules, and if there is an irreducible homomorphism $f : X \longrightarrow Y$, we write an arrow $[X] \longrightarrow [Y]$, where $[X]$ denotes the isomorphism class of the module X . By the previous theorem, an Auslander-Reiten quiver locally has the following form.



A connected component $\mathcal{C}$ of an Auslander-Reiten quiver is said to be a stable tube of rank n if $\mathcal{C}$ is the form of $\mathbb{Z}A_\infty/n$, namely $\mathcal{C}$ has the following form



where vertical dotted lines on both side are identified. In particular if $n = 1$, we say C a homogeneous tube. We know the Auslander-Reiten quiver of a tame hereditary algebra has infinite number of stable tubes. And almost all are homogeneous. In the sequel, we construct homogeneous tubes in more general case.

2. For an algebra A, and an A-module M, we denote by $R = R(A,M)$ the one-point extension of A by M, namely

$$R = \begin{pmatrix} A & M \\ 0 & k \end{pmatrix}.$$

It is well known that the category of R-modules is equivalent to the category of representations of the bimodule ${}_AM_k$. It has as objects the triples $({}_kU, {}_AX, \phi)$ with an A-homomorphism $\phi : {}_AM \otimes_k U \longrightarrow {}_AX$, and a morphism from $({}_kU, {}_AX, \phi)$ to $({}_kU', {}_AX', \phi')$ is given by a pair (α, β) of a k-linear map $\alpha : {}_kU \longrightarrow {}_kU'$, and an A-homomorphism $\beta : {}_AX \longrightarrow {}_AX'$ satisfying $\beta\,\phi = \phi'(1\otimes\alpha)$. After this, we write $(\dim_k U, X, \phi)$ for (U, X, ϕ) and we will call $V = (\dim_k U, X, \phi)$ just an R-module.

Given an R-module $V = (n, X, \phi)$, we consider the following commutative diagram in mod A.

$$(A)\qquad \begin{array}{ccccccccccc}
0 & \longrightarrow & \operatorname{Ker}\nu & \xrightarrow{\iota} & Y & \xrightarrow{\nu} & P(\operatorname{Cok}\phi) & \xrightarrow{\varepsilon} & \operatorname{Cok}\phi & \longrightarrow & 0 \\
 & & \chi\downarrow\wr & & \mu\downarrow & \text{exact} & \downarrow\rho & & \| & & \\
0 & \longrightarrow & \operatorname{Ker}\phi & \xrightarrow[\lambda]{} & M^n & \xrightarrow[\phi]{} & X & \xrightarrow[\pi]{} & \operatorname{Cok} & \longrightarrow & 0
\end{array}$$

This construction is as follows. In the bottom row, morphisms are canonical. Since $P(\mathrm{Cok}\ \phi) \xrightarrow{\varepsilon} \mathrm{Cok}\ \phi \longrightarrow 0$ is the projective cover, we can take $\rho \in \mathrm{Hom}_A(P(\mathrm{Cok}\ \phi), X)$ such that $\varepsilon = \pi\rho$. For the pair (ϕ, ρ), we take the pull-back $(Y; \mu, \nu)$. Then this square is exact, and $\mathrm{Ker}\ \nu$ is isomorphic to $\mathrm{Ker}\ \phi$.

Using this diagram, we get the first result which characterizes a part of DTr-invariant R-modules.

Theorem 1. Let $V = (1, X, \phi)$ be a non-projective indecomposable R-module.

(I) If ϕ is an epimorphism, the following two statements are equivalent.

(1) $\tau_R V \simeq V$.

(2) (a) ${}_AX \simeq E(\mathrm{top}(\mathrm{Ker}\ \phi))$.

(b) $\dim_k \mathrm{Hom}_A(M,X) = 2$.

(II) If ϕ is not an epimorphism, the following two statements are equivalent.

(1) $\tau_R V \simeq V$.

(2) (a) ${}_AX \simeq \tau_A(\mathrm{Cok}\ \phi)$.

(b) $\dim_k \mathrm{Hom}_A(M,X) = 2$.

(c) In the commutative diagram (A), $\mathrm{Im}\ \iota \subset \mathrm{rad}\ Y$.

Corollary 2. Let $V = (1, X, \phi)$ be a non-projective indecomposable R-module.

(I) If ϕ is a monomorphism, the following two statements are equivalent.

(1) $\tau_R V \simeq V$.

(2) (a) ${}_AX \simeq \tau_A(\mathrm{Cok}\ \phi)$.

(b) $\dim_k \mathrm{Hom}_A(M,X) = 2$.

(II) If ϕ is not an epimorphism and $\text{proj.dim}_A \text{Cok}\ \phi = 1$, the following two statements are equivalent.

(1) $\tau_R V \simeq V$.

(2) (a) ϕ is a monomorphism.

(b) ${}_A X \simeq \tau_A(\text{Cok}\ \phi)$.

(c) $\dim_k \text{Hom}_A(M,X) = 2$.

Remark. In this theorem, easy calculations show that if $\tau_R V \simeq V$ then ${}_A X$ is indecomposable. This fact is very useful to applications.

This theorem is essentially obtained by the following proposition.

Proposition 3. Let $V = (n, X, \phi)$ be a non-projective indecomposable R-module. Then $\tau_R V$ is isomorphic to the R-module $(\dim_k \text{Hom}_A(M,\tau_A(\text{Cok}\ \phi)\oplus I_V) - n, \tau_A(\text{Cok}\ \phi)\oplus I_V, \Phi)$ with some Φ . Here I_V is the injective A-module $D(Q^*)$ where Q is the direct summand of $P(Y)$ such that $P(Y) = Q\oplus P(\text{Ker}\ \varepsilon)$.

Corollary 4. Let $V = (n, X, \phi)$ be a non-projective indecomposable R-module. Then

(1) If ϕ is an epimorphism, $\tau_R V$ is isomorphic to $(\dim_k \text{Hom}_A(M,E(\text{top}(\text{Ker}\ \phi))) - n, E(\text{top}(\text{Ker}\ \phi)), \Phi)$ with some Φ .

(2) If ϕ is a monomorphism, $\tau_R V$ is isomorphic to $(\dim_k \text{Hom}_A(M,\tau_A(\text{Cok}\ \phi)) - n, \tau_A(\text{Cok}\ \phi), \Phi)$ with some Φ .

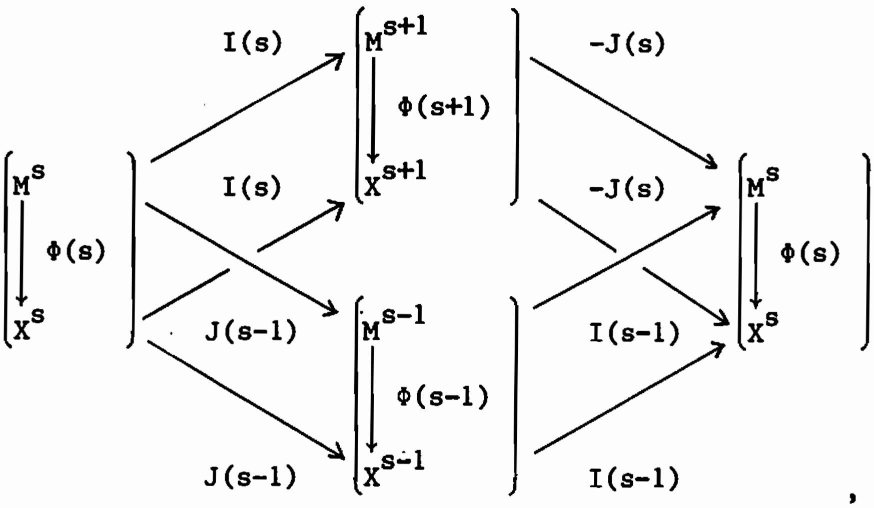
(3) If $\text{proj.dim}_A \text{ Cok } \phi = 1$, $\tau_R V$ is isomorphic to $(\dim_k \text{Hom}_A(M, \tau_A(\text{Cok } \phi) \oplus E(\text{top}(\text{Ker } \phi))) - n,$ $\tau_A(\text{Cok } \phi) \oplus E(\text{top}(\text{Ker } \phi)), \bar{\phi})$ with some $\bar{\phi}$.

Now we know that the τ-invariant module constructed in the theorem belongs to a homogeneous tube $\mathcal{C}$ [4] . We next consider a construction of it. Before this, we recall some definitions. A module V contained in $\mathcal{C}$ is called quasi-simple if there does not exist an irreducible monomorphism $W \longrightarrow V$ with $W \neq 0$. And a module V in $\mathcal{C}$ has quasi-length s if there is a chain $V_1 \longrightarrow V_2 \longrightarrow \cdots \longrightarrow V_s = V$ of irreducible monomorphism with V_1 quasi-simple [6] . In the following theorem, we denote $V(s)$ the module in $\mathcal{C}$ which has the quasi-length s .

Theorem 5. Let $V = (1, X, \phi)$ be a non-projective indecomposable R-module. And assume $\tau_R V \simeq V$. Then V is quasi-simple, and $V(s) = (s, X^s, \Phi(s))$, where

$$\Phi(s) = \begin{pmatrix} \phi & \psi & & 0 \\ & \phi & \ddots & \\ & & \ddots & \psi \\ 0 & & & \phi \end{pmatrix}$$

with ψ being an arbitrary A-homomorphism which is linearly independent of ϕ . Further the Auslander-Reiten sequence which has the end-term $V(s)$ has the following form



,

where $I(s) = \begin{pmatrix} E(s) \\ \hline 0 \end{pmatrix}$, $J(s) = (0 \mid E(s))$ with $E(s)$ the unit matrix of degree s .

Recently Ringel considered the stable separating tubular families, and he made $\mathbb{P}_1 k$ - family of stable tubes [7] . In connection with it, we get the following.

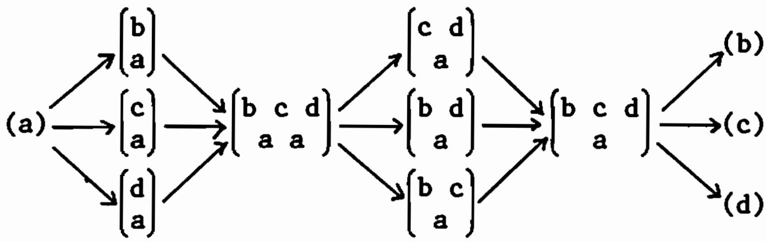
Proposition 6. Let $V = (1, X, \phi)$ be a non-projective indecomposable R-module. Assume $\tau_R V \simeq V$, ϕ a monomorphism, $\mathrm{End}_A(X) = k$, and k an infinite field. Then we can make $| k |$ - family of homogeneous tubes. ($| \ |$ means the cardinal number.)

Example. We observe these statements by a famous example. Let

$$A = \left\{ \begin{pmatrix} a & x & y & z \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix} \in M_4(k) \right\}$$

with k an algebraically closed field.

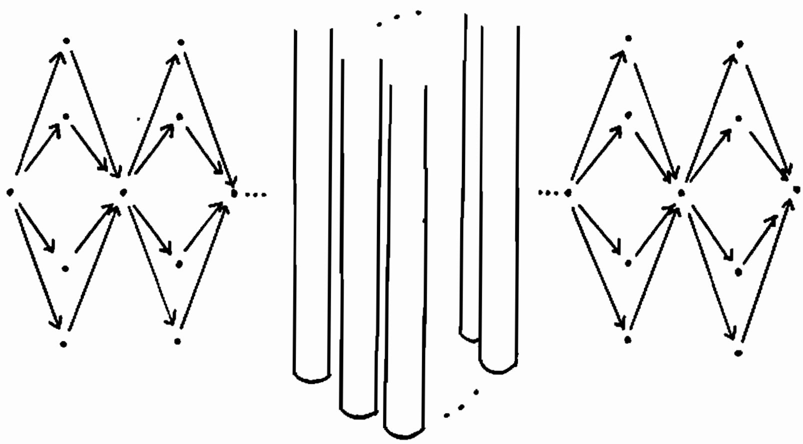
The Auslander-Reiten quiver of A is as follows.



Here, for example, $\begin{pmatrix} b\ c \\ a \end{pmatrix}$ means the indecomposable A-module N such that top N $\simeq S_b \oplus S_c$ and soc N $\simeq S_a$, where $S_\#$ means the simple A-module corresponding to the idempotent # . We take a simple A-module (a) as M in the statements. Then

$$R(A,M) = \left\{ \begin{pmatrix} a & x & y & z & u \\ 0 & b & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & d & 0 \\ 0 & 0 & 0 & 0 & e \end{pmatrix} \in M_5(k) \right\} .$$

The Auslander-Reiten quiver of R(A,M) is as follows.



In the middle part of the above quiver, there are $\mathbb{P}_1 k$ - family of stable tubes. Three of them are rank 2 and the others are homogeneous. Now let's observe the situation of the theorem. For $(\alpha,\beta) \in \mathbb{P}_1 k$, let

$$V(\alpha,\beta) = \left(1, \begin{pmatrix} b\ c\ d \\ a\ a \end{pmatrix}, \phi(\alpha,\beta) \right),$$

where $\phi(\alpha,\beta)$ is an inclusion

$$\phi(\alpha,\beta) : (a) \longrightarrow \begin{pmatrix} b\ c\ d \\ a\ a \end{pmatrix} .$$

Then $\operatorname{Cok} \phi(\alpha,\beta) \simeq \begin{pmatrix} b\ c\ d \\ a \end{pmatrix}$ for almost all $(\alpha,\beta) \in \mathbb{P}_1 k$.

(Except three cases. In fact these three cases correspond to three stable tubes of rank 2 in the above Auslander-Reiten quiver.) And in these cases the conditions (I), (2) of Corollary 2 are satisfied. So we can construct τ-invariant modules which generate all homogeneous tubes.

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ON CONNECTED GALOIS EXTENSIONS AND DISCONNECTED GALOIS EXTENSIONS OF A CONNECTED RING

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Let A be a ring with an identity 1. By C(A) and $\mathcal{B}(C(A))$ we denote the center of A and the set of all idempotents of C(A). Then A is said to be <u>connected</u> (resp. <u>disconnected</u>) if the cardinality $|\mathcal{B}(C(A))| = 2$ (resp. $|\mathcal{B}(C(A))| > 2$). The purpose of this note is to study about connected Galois extensions and disconnected Galois extensions over a connected ring. Thus, throughout in this study, we assume that A is a connected ring and B is a G-Galois extension over A with a finite group G. In [1], M. Ferrero and the author studied on the connectedness of p-Galois extensions. As a sequel, we study about the connectedness of G-cyclic extensions of Kummer type in §1. While, in §2, we study about disconnected Galois extensions B/A and $|\mathcal{B}(C(A))|$. The detail of §1 will be seen in [2] and that of §2 will be seen in the forthcoming paper [3].

An element e of $\mathcal{B}(C(A))$ is said to be a G- <u>idempotent</u> of B if there holds either $\tau(e) = e$ or $\tau(e)e = 0$ for any $\tau \in G$. The following theorem [1, Lemma 1.8] plays important role in this study.

Theorem. If $|\mathcal{B}(C(A))| > 2$ then there exists a non-trivial G-idempotent.

A G-idempotent e of B is said to be of length m if $(G:G_e) = m$ where $G_e = \{\tau \in G; \tau(e) = e\}$.

Let ρ be an automorphism of A. Then a monic polynomial $f(X)$ of a skew polynomial ring of automorphism type $A[X;\rho]$ is said to be a generator if $f(X)A[X;\rho] = A[X;\rho]f(X)$. A generator $f(X)$ of $A[X;\rho]$ is said to be w-irreducible if the degree of $f(X)$ is minimal in the set of generators of degree $\geqq 1$.

1. CONNECTED STRONGLY CYCLIC EXTENSIONS. In this section, we assume that an integer $n(>1) \in U(A)$, the set of all invertible elements of A, $C(A)$ contains a primitive n-th root ζ of 1 such that $1 - \zeta^i \in U(A)$ for $i = 1, 2, \ldots, n - 1$, and G is a cyclic group of order n with a generator σ. A G-Galois extension B of A is said to be a G-strongly cyclic extension if there exists $x \in U(B)$ such that $\sigma(x) = x\zeta$ and $B_A \oplus> A_A$.

It is known that A has a G-strongly cyclic extension B if and only if there exist an automorphism ρ of A and a generator $X^n - \alpha$ of $A[X;\rho]$ such that $\rho(\zeta) = \zeta$ and $\alpha \in U(A)$. Moreover, if this is the case, $B \cong A[X;\rho]/(X^n - \alpha)A[X;\rho]$. The main theorem of this section is the following.

Theorem 1.1. Let $B = A[X;\rho]/(X^n - \alpha)A[X;\rho]$ be a G-strongly cyclic extension over A. Then B is connected if and only if $X^n - \alpha$ is w-irreducible.

Proof. Assume $f(X) = X^n - \alpha$ is not w-irreducible. Then $f(X) = g(X)h(X)$ for some proper generators $g(X)$

and $h(X)$. Hence $nX^{n-1} = f'(X) = g'(X)h(X) + g(X)h'(X)$. Let x be the coset of X in $B = A[X;\rho]/(X^n - \alpha)A[X;\rho]$. Since $x \in U(B)$, we can see that $nx^{n-1} = f'(x) = g'(x)h(x) + g(x)h'(x)$ is an invertible element of B. Therefore, $(g(x))$ and $(h(x))$ are co-maximal ideal of B and $B \cong B/(g(x)) \oplus B/(h(x))$. Thus $f(X)$ is w-irreducible if B is connected.

Conversely, assume that B is disconnected. Then there exists a nontrivial G-idempotent e of B. For this e, we put $G_e = (\sigma^m)$. Then the length of e is m and $T_\sigma(e;m) = \sum_{i=0}^{m-1} \sigma^i(e) = 1$. Let $T = B^{\sigma^m}$. Then $T = \sum_{i=0}^{m-1} \oplus (x^{m'})^i A$ where $m' = n/m$. Further, we can see that T/A is a $(\sigma|T)$-strongly cyclic extension and $T = \sum_{i=0}^{m-1} \oplus \sigma^i(e)A$. If we put $y = x^{m'}$ and $y = \sum_{i=0}^{m-1} \sigma^i(e)a_i$ $(a_i \in A)$ then we have

(1) $a_i \in A^\rho = \{a \in A;\ \rho(a) = a\}$,

(2) $aa_i = a_i\rho^{m'}(a)$ for any $a \in A$ and $0 \leq i \leq m-1$,

(3) $\alpha = a_i^{\,m}$ for any $a_i \neq 0$.

In virtue of (1) and (2), we have the following decomposition of $f(X)$:

$$\begin{aligned} f(X) &= (X^{m'})^m - (a_i)^m \\ &= (X^{m'} - a_i)(((X^{m'})^{m-1} + (X^{m'})^{m-2}a_i + \cdots \\ &\qquad + (a_i)^{m-1}) \end{aligned}$$

for any $a_i \neq 0$.

Then by (1) and (3), wecan seethat the two factors of $f(X)$ are w-irreducible.

For an automorphism ρ of A, the index of the subgroup of inner automorphisms in (ρ) is said to be the <u>index</u> of ρ. If $X^n - \alpha \in A[X;\rho]$ is a generator and n is a prime, then the index of ρ is either n or 1 since $\rho^n = \widetilde{\alpha}^{-1}$, an inner automorphism generated by α^{-1}. From this we can see the following

Lemma 1.2. If n is prime, then a generator $X^n - \alpha$ in $A[X;\rho]$ is either w-irreducible or a product of gnerators of degree 1.

Combining Theorem 1.1 with Lemma 1.2, we have the following

Theorem 1.3. Let n be a prime. Then A has a connected G-strongly cyclic extension if and only if one of the following conditions (a) and (b) is satisfied.

(a) $(U(A):U(A)^n) \geq n$.

(b) A has an automorphism ρ of the index n such that $\rho(\zeta) = \zeta$ and $\{a \in A^{\rho};\ \rho^n = \widetilde{a}\} \neq \emptyset$.

2.DISCONNECTED GALOIS EXTENSIONS. In this section, we shall study the case that B is a disconnected ring. The detail of proofs will be seen in the forthcoming paper [3].

Theorem 2.1. Let B/A be a disconnected G-Galois extension, e a G-idempotent of B of the maximal length m and $G = \tau_1 G_e \cup \tau_2 G_e \cup \dots \cup \tau_m G_e$ ($\tau_1 = 1$) the left coset decomposition of G by G_e. Then we have the followings:

(1) $B = \sum_{i=1}^{m} \oplus \tau_i(e)B$ and $\{\tau_i(e);\ i = 1,2,\dots,m\}$ is linearly independent over A.

(2) If $|G_e|$ is an invertible element of A, then

$\tau_i(e)B$ is connected for $i = 1,2,\ldots,m$ and $|\mathcal{B}(C(B))| = 2^m$.

(3) If G_e is a normal subgroup, then $B^{G_e} = \sum_{i=1}^m \oplus \tau_i(e)A$, $\tau_i(e)B$ is a connected $(G_e|\tau_i(e)B)$-Galois extension over $\tau_i(e)A$ for $i = 1,2,\ldots,m$ and $|\mathcal{B}(C(B))| = 2^m$.

As direct consequences of Theorem 2.1, we have the following

Corollary 2.2. Under the same assumptions and notations as in Theorem 2.1,

(1) If A is an algebra over a field of characteristic 0, then $\tau_i(e)B$ is a connected $(G_{\tau_i(e)}|\tau_i(e)B)$-Galois extension over $\tau_i(e)A$ for $i = 1,2,\ldots,m$ and $|\mathcal{B}(C(B))| = 2^m$.

(2) If G is abelian, then $B^{G_e} = \sum_{i=1}^m \oplus \tau_i(e)A$, $\tau_i(e)A$ is a connected $(G_e|\tau_i(e)B)$-Galois extension over $\tau_i(e)A$ for $i = 1,2,\ldots,m$ and $|\mathcal{B}(C(B))| = 2^m$.

(3) If n is prime, then B is disconnected if and only if B is ring isomorphic to $A^{(n)}$, a direct sum of n-copies of A.

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The detailed version of §2 has been submitted for publication elsewhere.

AZUMAYA ALGEBRAS AND SKEW POLYNOMIAL RINGS

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This is a summary of the auther's paper [6] in preparation. In [4, 5], we have studied some Azumaya algebras induced by skew polynomial rings over commutative rings, and some skew polynomial rings of automorphism type whose coefficient rings are Azumaya algebras. In this note, we study certain skew polynomial rings of derivation type whose coefficient rings are Azumaya algebras. The main result is Theorem 4.

Throughout this note, B will mean a ring, D a derivation of B. We denote by $B[X;D]$ the skew polynomial ring defined by $aX = Xa + D(a)$ $(a \in B)$. By $B[X;D]_{(0)}$, we denote the set of all monic polynomials g in $B[X;D]$ with $gB[X;D] = B[X;D]g$. A ring extension B/A is called H-separable if $B \otimes_A B$ is B-B-isomorphic to a direct summand of a finite direct sum of copies of B. A polynomial g in $B[X;D]_{(0)}$ is called H-separable if $B[X;D]/gB[X;D]$ is an H-separable extension of B. We shall use the followng conventions. $V_B(A)$ = the centralizer of A in B. $B^D = \{b \in B \mid D(b) = 0\}$.

First, we shall state the following lemma which is useful in the proof of Theorem 4.

Lemma 1 ([3, Theorem 1]). Let B be an Azumaya C-algebra, and H a C-subalgebra of B. If B_H is projective then B/H is an H-separable extension.

The detailed version of this paper will be submitted for publication elsewhere.

Let $D^*: B[X;D] \to B[X;D]$ be the derivation defined by $D^*(\Sigma_i X^i c_i) = \Sigma_i X^i D(c_i)$. Concerning H-separable polynomials, we know the following which is complicated.

Lemma 2 ([4, Lemma 1.5], [9, Theorem 1.9]). Let f be in $B[X;D]_{(0)}$, and $\deg f = m$. If f is H-separable in $B[X;D]$, then there exist $y_i, z_i \in B[X;D]$ with $\deg y_i < m$ and $\deg z_i < m$ such that $ay_i = y_i a$, $az_i = z_i a$ $(a \in B)$ and $\Sigma_i D^{*m-1}(y_i)z_i \equiv 1 \pmod{fB[X;D]}$, $\Sigma_i D^{*k}(y_i)z_i \equiv 0 \pmod{fB[X;D]}$ $(0 \leqq k \leqq m-2)$, and convercely.

Definition. An H-separable polynomial f is called strongly H-separable if the elements $\{y_i, z_i\}$ in Lemma 2 are obtained from the center of B.

Lemma 3. Let f be in $B[X;D]_{(0)}$ and $\deg f = m$. Let C be the center of B, $d = D|C$, and $A = C^d$. Then, f is strongly H-separable in $B[X;D]$ if and only if ${}_AC$ is a finitely generated projective module of rank m and $\mathrm{Hom}({}_AC, {}_AC) = C[d]$.

Remark. If ${}_AC$ is a finitely generated projective module and $\mathrm{Hom}({}_AC, {}_AC) = C[d]$, then C/A is called a purely inseparable extension of exponent one (e.g. [10]). Hence the existence of a strongly H-separable polynomial characterizes a purely inseparable extension of exponent one.

Now, we shall state the main results of this note which is a generalization of [3, Theorem 3.3].

Theorem 4. Let B be an Azumaya C-algebra, D a derivation of B, $d = D|C$ and $A = C^d$. Assume that $B[X;D]_{(0)}$ contains a polynomial f of degree $m \geqq 2$. Then the following are equivalent:

(1) ${}_AC$ is a finitely generated projective module of rank m and $\mathrm{Hom}({}_AC, {}_AC) = C[d]$.

(2) f is strongly H-separable in $B[X;D]$.

(3) $S = B[X;D]/fB[X;D]$ is an Azumaya A-algebra with $V_S(C) = B$.

(4) $B[X;D]$ is an Azumaya $A[f]$-algebra with $V_{B[X;D]}(C) = B[f]$.

When this is the case, there holds the following:

(i) B is of prime characteristic p, and f is a p-polynomial of the form $\sum_{j=0}^{e} X^{p^j} b_{j+1} + b_0$ $(p^e = m)$, $b_{j+1} \in A$ $(1 \leqq j \leqq e)$ and $b_0 \in B^D$.

(ii) For any $\gamma \in A$, $S_\gamma = B[X;D]/(f + \gamma)B[X;D]$ is an Azumaya A-algebra with $V_{S_\gamma}(C) = B$.

(iii) $\{g \in B[X;D] \mid g$ is H-separable in $B[X;D]\}$ $= \{g \in B[X;D] \mid g$ is strongly H-separable $\} = \{f + \gamma \mid \gamma \in A\}$.

As a special case of Theorem 4, we have the following Corollary. Note that the centralizer conditions are superfluous in this case.

Corollary 5. Let B be a commutative ring, d a derivation of B, and $A = B^d$. Then B/A is a purely inseparable d-extension of exponent one and ${}_AB$ is projective rank m if and only if $B[X;d]$ is an Azumaya $A[f]$-algebra for some $f \in B[X;d]_{(0)}$ of degree m.

By means of [8, Theorem 6.1] and Theorem 4, we have the following proposition which is a generalization of [8, Theorem 6.3] and [1, Theorem 2].

Proposition 6. Let C be a commutative ring, d a derivation of C, and $A = C^d$. Let C/A is a purely inseparable d-extension of exponent one and ${}_AC$ is projective rank $m = p^e$. Assume that d satisfies a minimal polynomial $x^{p^e} + x^{p^{e-1}} a_e + \ldots + xa_1$ $(a_i \in A)$. Let E be an Azumaya A-algebra with C as an A-subalgebra, and E_C be projective. Then, if $B = V_E(C)$, there is a derivation D of B which is an extension of d, and an element u in B^D such that E is of the form $B[X;D]/(X^{p^e} + X^{p^{e-1}} a_e + \ldots + Xa_1 - u)B[X;D]$.

Remark. In [2], K. Hoechsmann studied skew polynomial rings of derivation type whose coefficient rings are simple algebras. Theorem 4 and Proposition 6 contains the main results [2, Theorem 3.1] as a special case. As an another application of Theorem 4, we have a generalization of R. Irving's theorem [7, Theorem 5.4]. The details will be appeared in [6].

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ON FLAT RING EXTENSIONS AND GABRIEL TOPOLOGY

Kozo SUGANO

1. Throughout this report every ring will have the identity 1, and every subring of it contain 1. All modules over a ring will be unital. All terminologies and notations are the same as in [5], [6] and [7].

Let G be a Gabriel topology on a ring R consisting of right ideals. As in [5] we will denote $\varinjlim_{\underline{a} \in G} \mathrm{Hom}(\underline{a}_R, R_R)$ by $R_{(G)}$, and $(R_{(G)})_{(G)}$ by R_G. A left R-module M is said to be G-divisible if $\underline{a}M = M$ holds for each $\underline{a}$ in G. If M is R-flat and G-divisible, we can make M a left $R_{(G)}$-module in the following way. Let $m \in M$ and $x \in R_{(G)}$ represented by $\xi \in \mathrm{Hom}(\underline{a}_R, R_R)$ with $\underline{a} \in G$. Then we have $m = \Sigma a_i m_i$ for some $a_i \in \underline{a}$ and $m_i \in M$. Define $xm = \Sigma \xi(a_i) m_i$. Then it is easily seen that this is well defined and gives M a left $R_{(G)}$-module structure. Furtheremore, we see $R_{(G)} \otimes_R M = M$ by the map $x \otimes m \longrightarrow xm$ ($x \in R_{(G)}$, $m \in M$) and $\mathrm{Hom}_R(M, N) = \mathrm{Hom}_{R_{(G)}}(M, N)$ for any left $R_{(G)}$-module N. The same goes with R_G and M, and we can obtain a simpler proof of Theorem 1.4 [4].

Theorem 1 (Theorem 1.4 [4]). Let M be a flat left R-module and G the class of right ideals $\underline{a}$ of R such that $\underline{a}M = M$. Then G is a Gabriel topology on R, and M is a faithful left R_G-module. There exists a ring isomorphism ι_M of R_G to a subring of $\mathrm{Bicom}(_R M)$ defined by $\iota_M(x)(m) = xm$ for each $x \in R_G$, $m \in M$.

2. Now consider the case where A is a ring and B is a

This report is the abstract of the author's forthcoming paper [9].

subring of A. Denote $D = V_A(B)$, the centralizer of B in A, and $C = V_A(A)$, the center of A. Suppose that A is flat as a left B-module, and let G be the Gabriel topology on B consisting of right ideals $\underline{a}$ of B such that $\underline{a}A = A$. It is easily seen that $\mathrm{Bicom}({}_BA)$ is a subring of $\mathrm{Hom}({}_DA_A, {}_DA_A)$, and that there exists a natural ring isomorphism ν of $\mathrm{End}({}_DA_A)$ to $V_A(D)$. Thus we have ring monomorphisms

$$B_G \xrightarrow{\iota_A} \mathrm{Bicom}({}_BA) \xrightarrow{i} \mathrm{Hom}({}_DA_A, {}_DA_A) \xrightarrow{\nu} V_A(V_A(B))$$

where ι_A is the map defined in §1, and i is the inclusion map. Denote the composition of the above maps by ϕ_A. Then we see that $\mathrm{Im}\phi_A = \{\Sigma\xi(a_i)m_i \mid \xi \in \mathrm{Hom}(\underline{a}_B, A_B),\ \underline{a} \in G,\ \Sigma a_i m_i = 1 \text{ with } a_i \in \underline{a},\ m_i \in A\}$. Now the question under what conditions $\mathrm{Im}\phi_A = V_A(V_A(B))$ holds comes out. Concerning with this problem we have

Theorem 2. If A is an H-separable extension of B and A is left B-flat, then we have $B_G \cong \mathrm{Im}\phi_A = V_A(V_A(B))$.

3. In this section we will deal with an H-separable extension of a regular ring. Note that A is regular if and only if every left A-module is A-flat. By Prop. 5.4 [2] we can see that a separable extension of a regular ring is always regular. Here we will give the other proof of it. This is an immediate consequence of the next lemma. A left A-module M is said to be (A,B)-projective if and only if M is isomorphic to a direct summand of $A\otimes_B M$ as left A-module.

Lemma 1. If a left A-module M is (A,B)-projective and B-flat, then M is also A-flat.

The proof of the above lemma is an easy exercise. This lemma shows that, if A is a left semisimple extension of B, every left A-module which is B-flat is A-flat. Now we have

Theorem 3. Let B be a regular ring and A an H-separable extension of B. Then A is also a regular ring, and we have $V_A(V_A(B)) = B$.

It is already known that, if M is a finitely generated flat left B-module such that $A \otimes_B M$ is A-projective, then M is B-projective (See Prop. I 11.6 [5]). If A is an H-separable extension of B, $A \otimes_B A$ is left (as well as right) A-finitely generated projective. Therefore we have

Proposition 1. Let A be an H-separable extension of B. If A is finitely generated flat as left B-module, then A is left B-projective. Consequently, A is left B-projective if one of the following conditions are satisfied;

(1) B is a right B-direct summand of A, and A is left B-flat.

(2) B is regular, and A is left B-finitely generated.

Finally we will deal with an H-separable extension of a left full linear ring, and give the complete improvement of Theorem 3 [8]. A left full linear ring is the endomorphism ring of a left vector space over a division ring. Such a ring is always regular, indecomposable as ring and a left injective module over itself. Thus by Theorems 2, 3 [8] and Prop. 1, Theorem 3, we have

Theorem 4. Let B be a left full linear ring. Then A is an H-separable extension of B, if and only if the following three conditions are satisfied;

(1) A is also a left full linear ring

(2) D is a simple C-algebra with $[D: C] < \infty$.

(3) $V_A(V_A(B)) = B$.

If these conditions are satisfied, A is a free Frobenius extension of B having a left (or right) B-free basis consisting of [D: C]-elements of A.

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THE SELF-DUALITY OF H-RINGS AND NAKAYAMA AUTOMORPHISMS OF QF-RINGS

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In 1978, M. Harada ([4]) has found a new class of artinian rings which includes QF-rings and Nakayama (artinian serial) rings. In [7]$\sim$[9], the first author has studied this ring and called it a H-ring. In this abstract, we shall discuss on the problem whether this ring has self-duality or not like QF-rings or Nakayama rings. Although we can not solve this problem, we shall show that the following three problems are mutually equivalent ones:

Problem A: Do basic left H-rings have Nakayama isomorphisms ?

Problem B: Do basic QF-rings have Nakayama automorphisms ?

Problem C: Do left H-rings have self-duality ?

Morita Duality. We start with Morita duality. Consider two rings R and S, and let ${}_R\mathcal{M}$ and $\mathcal{M}_S$ be the categories of all finitely generated left R-modules and right S-modules, respectively. If there exist contravariant functors C: ${}_R\mathcal{M} \longrightarrow \mathcal{M}_S$ and D: $\mathcal{M}_S \longrightarrow {}_R\mathcal{M}$ such that DC and CD are isomorphic to the identity functors of ${}_R\mathcal{M}$ and $\mathcal{M}_S$, respectively, then (C,D) is said to be a (Morita) duality between ${}_R\mathcal{M}$ and $\mathcal{M}_S$. We use ${}_R\mathcal{M} \sim \mathcal{M}_S$ to mean that there exists a duality between ${}_R\mathcal{M}$ and $\mathcal{M}_S$, and in this case, $\mathcal{M}_S$(resp. ${}_R\mathcal{M}$) is said to be dual to ${}_R\mathcal{M}$ (resp. $\mathcal{M}_S$). In particular, when ${}_R\mathcal{M} \sim \mathcal{M}_R$, R is said to be self-dual or to have self-duality.

From theorems of Morita [5], Azumaya [1] and Mueller [6] we see the following: Let R be a left artinian ring, and let E be the injective hull of R/J(R) as a left R-module where J(R) is the Jacobson radical of R. Then ${}_R\mathcal{M} \sim \mathcal{M}_{End(E)}$ if and only if E is finitely generated. And in this case End(E) is a right artinian ring. In particular, if E is finitely generated and End(E) is isomorphic to R, then R is self-dual.

This is a principal result for the study of self-duality. However, in spite of this result, it is not easy to find those artinian rings which have self-duality; even if we find an artinian ring with duality, it seems to be difficult to verify whether it is self-dual or not.

Finite dimensional algebra over a field, QF-rings and Nakayama rings are typical artinian rings which have self-duality. The reader is reffered to Waschbüsch [10] from which we can know an interesting history on the study of self-duality of Nakayama rings.

Notation. Throughout this paper all rings considered are associative with identity, all R-modules are unitary and all homomorphisms between R-modules are written on the opposite side of scalars. The notation M_R (resp. ${}_RM$) is used to stress that M is a right (resp. left) R-module. For R-modules M and N, we use '$M \underset{\sim}{\subset} N$' to mean that M is isomorphic to a submodule of N. For an R-module M, by E(M), J(M) and S(M) we denote its injective hull, Jacobson radical and socle, respectively. And $M = J_0(M) \supseteq J_1(M) \supseteq J_2(M) \supseteq \dots$ and $0 = S_0(M) \subseteq S_1(M) \subseteq S_2(M) \subseteq \dots$ mean the descending Loewy chain and ascending Loewy chain of M, respectively, i.e., $J_i(M) = J(J_{i-1}(M)$ and $S_i(M)/S_{i-1}(M) =$

$S(M/S_{i-1}(M))$.

Let R be a left artinian ring and let E be a complete set (i.e. sum = 1) of orthogonal primitive idempotents. For convenience'sake, we put

$$(e,f) = (eR,fR) = \mathrm{Hom}_R(eR,fR)$$
$$[e,f] = [Re,Rf] = \mathrm{Hom}_R(Re,Rf)$$

for e, f in E. When E is arranged as $\{e_1, \ldots, e_n\}$, we can identify R with the following matrix rings:

$$\begin{pmatrix} e_1Re_1 & \cdots & e_1Re_n \\ \vdots & & \vdots \\ e_nEe_1 & \cdots & e_nRe_n \end{pmatrix},$$

$$\begin{pmatrix} (e_1,e_1) & \cdots & (e_n,e_1) \\ \vdots & & \vdots \\ (e_1,e_n) & \cdots & (e_n,e_n) \end{pmatrix},$$

$$\begin{pmatrix} [e_1,e_1] & \cdots & [e_1,e_n] \\ \vdots & & \vdots \\ [e_n,e_1] & \cdots & [e_n,e_n] \end{pmatrix}$$

We use the terms 'e_i-row' and 'e_i-column' instead of the terms i-row and i-column, respectively. So, we identify e_iR and Re_i with e_i-row and $e_i$column, respectively.

For f in E, the following basic result is due to Fuller ([3]):

Lemma 1. ${}_R Rf$ is injective if and only if there exists e in E such that (eR;Rf) is an injective pair, i.e.,

$${}_R Re/J({}_R Re) \simeq {}_R S({}_R Rf) \text{ and } fR_R/J(fR_R) \simeq S(eR_R)_R$$

When this is so, eR_R is also injective.

We note that if R is basic and e_i, e_j are in $E = \{e_1, \ldots, e_n\}$ such that $(e_iR;Re_j)$ is an injective pair, then

$$S(e_iR_R) = \begin{pmatrix} & & 0 & & \\ & 0 & \vdots & 0 & \\ & & 0 & & \\ 0 \cdots 0 & & S(e_iRe_{j\, e_jRe_j}) & & 0 \cdots 0 \\ & & 0 & & \\ & 0 & \vdots & 0 & \\ & & 0 & & \end{pmatrix}$$

$$S({}_R Re_j) = \begin{pmatrix} & & 0 & & \\ & 0 & \vdots & 0 & \\ & & 0 & & \\ 0 \cdots 0 & & S({}_{e_iRe_i} e_iRe_j) & & 0 \cdots 0 \\ & & 0 & & \\ & 0 & \vdots & 0 & \\ & & 0 & & \end{pmatrix}$$

<u>Nakayama automorphism.</u> Let R be a basic QF-ring, and let $\{e_1, \ldots, e_n\}$ be a complete set of orthogonal primitive idempotents of R. Then, there is a permutation $\{f_1, \ldots, f_n\}$ of $\{e_1, \ldots, e_n\}$ such that $(f_iR;Re_i)$ is an injective pair for i = 1, ..., n. This permutation is called the Nakayama permutation of $\{e_1, \ldots, e_n\}$. And if there exists an automorphism ϕ of R satisfying $\phi(e_i) = f_i$ for i = 1, ..., n, then ϕ is called a Nakayama automorphism of

R. Although we do not know whether such an automorphism exists or not, we will see later that the existence of this is essentially related to our study of self-duality.

For later use, we shall generalize 'Nakayama automorphism' to 'Nakayama isomorphism' for arbitrary basic artinian rings. Let R be a basic left artinian ring, and let $E = \{e_1, \ldots, e_n\}$ be a complete set of orthogonal primitive idempotents of R. Put $E_i = E({}_R(Re_i/J({}_R Re_i)))$ and $E = E_1 \oplus \ldots \oplus E_n$. Then the endomorphism ring $T = \mathrm{End}_R(E)$ is identified with the matrix ring:

$$\begin{pmatrix} [E_1,E_1] & \cdots & [E_1,E_n] \\ \vdots & & \vdots \\ [E_n,E_1] & \cdots & [E_n,E_n] \end{pmatrix}$$

where $[E_i,E_j] = \mathrm{Hom}_R(E_i,E_j)$. Let f_i be the matrix such that its (i,i) position is the unity of $[E_i,E_i]$ and all other entries are zero. Then $\{f_1, \ldots, f_n\}$ is a complete set of orthogonal primitive idempotents of T; $T = f_1T \oplus \ldots \oplus f_nT$. Here if there exists an isomorphism ϕ of R to T such that $\phi(e_i) = f_i$ for all i, then we call it a Nakayama isomorphism of R. Of course, when R is a basic QF-ring, it is just a Nakayama automorphism of R.

<u>H-ring.</u> Now our purpose of this paper is to investigate the self-duality of a new artinian ring which was found by M. Harada and studied by the first author. Among several characterizations of this ring ([9]), we adopt

here the following as its definition:

Definition. A ring R is a left H-ring if it is left artinian and its complete set E of orthogonal primititive idempotents is arranged as

$$E = \{e_{11}, \ldots, e_{1n(1)}, \ldots, e_{m1}, \ldots, e_{mn(m)}\}$$

for which

1) each $e_{i1}R_R$ is injective,

2) $e_{i1}R_R \gtrsim e_{i2}R_R \gtrsim \cdots \gtrsim e_{in(i)}$ for each i, and

more precisely

$$e_{i,k-1}R_R \simeq e_{ik}R_R \quad \text{or} \quad J(e_{i,k-1}R)_R \simeq e_{ik}R_R$$

for each k, and

3) $e_{ik}R_R \not\simeq e_{jt}R_R$ if $i \neq j$.

We look at the following two conditions both which are necessary and sufficient conditions of a left artinian ring R to be a left H-ring:

a) The family of all (finitely generated) injective left R-modules is closed under taking small covers.

b) The family of all (finitely generated) projective right R-modules is closed under taking essential extensions.

As these conditions are mutually dual, we obtain the following

Proposition 1. Let R be a left H-ring. If S is a ring such that ${}_R\mathcal{M} \sim \mathcal{M}_S$, then S is a left and right artinian left H-ring.

Example. QF-rings are clearly left and right H-rings. Nakayama rings are also left and right H-rings; whence so is the ring of all upper triangular matrices over a division ring D:

$$\begin{pmatrix} D & \cdots & D \\ & \ddots & \vdots \\ 0 & & D \end{pmatrix}$$

As an typical example, for a local QF-ring Q, the ring

$$\begin{pmatrix} Q & \cdots & Q & \overline{Q} & \cdots & \overline{Q} \\ J & \ddots & \vdots & \vdots & & \vdots \\ \vdots & \ddots & Q & & & \vdots \\ \vdots & & J & \overline{Q} & & \vdots \\ & & \vdots & \overline{J} & \ddots & \\ J & \cdots & J & \overline{J} & \cdots \overline{J} & \overline{Q} \end{pmatrix}$$

is a left H-ring, where $\overline{Q} = Q/S(Q)$ and $J = J(Q)$.

Problems A, B and C. As left H-rings and self-duality are Morita invaliants, in order to investigate the self-duality of left H-rings, we may restrict our attention to basic left H-rings. Therefore, hereafter, we assume that R is a basic left H-ring and E is a complete set of orthogonal primitive idempotents of R. E is therefore arranged as

$$E = \{e_{11}, \ldots, e_{1n(1)}, \ldots, e_{m1}, \ldots, e_{mn(m)}\}$$

for which

1) $e_{i1}R_R$ is injective for $1 \leq i \leq m$,

2) $e_{i1}R_R \gtrsim \cdots \gtrsim e_{in(i)}R_R$; more precisely there exists an isomorphism $\theta^{i}_{k,k-1}$ from $e_{ik}R_R$ to $J(e_{i,k-1}R_R)_R$

for $1 \leq i \leq m$, $1 \leq k \leq n(i)$.

We use later the notations:

$$\theta^i_{1,1} = \text{ the identity map of } e_{i1}R_R$$
$$\theta^i_{k,1} = \theta^i_{2,1} \cdots \theta^i_{k-1,k-2}\theta^i_{k,k-1}.$$

Now, we represent R as

$$R = \begin{pmatrix} (e_{11},e_{11}) & \cdots\cdots & (e_{mn(m)},e_{11}) \\ \vdots & & \vdots \\ (e_{11},e_{mn(m)}) & \cdots & (e_{mn(m)},e_{mn(m)}) \end{pmatrix}$$

$$= \begin{pmatrix} e_{11}Re_{11} & \cdots\cdots & e_{11}Re_{mn(m)} \\ \vdots & & \vdots \\ e_{mn(m)}Re_{11} & \cdots\cdots & e_{mn(m)}Re_{mn(m)} \end{pmatrix}$$

The following properties hold on R:

a) Each $S(e_{ij}R_R)_R$ is simple,

$S(e_{i1}R_R)_R \simeq \cdots \simeq S(e_{in(i)}R_R)_R$,

$S(e_{ij}R)_R \not\simeq S(e_{kt})_R$ if $i \neq k$.

b) For each $e_{i1}R$, there exists a unique g_i in E such that $(e_{i1}R;Rg_i)$ is an injective pair; whence ${}_RRg_i$ is injective.

c) $S_k({}_R Rg_i) = S(e_{i1}R_R) \oplus \dots \oplus S(e_{ik}R_R)$

for $1 \leq i \leq m$, $1 \leq k \leq n(i)$. So, $S_k({}_R Rg_i)$ is a two-sided ideal. In particular, $S({}_R Rg_i) = S(e_{i1}R_R)$ is a simple ideal. In the matrix representation,

$$S_k({}_R Rg_i) = \begin{pmatrix} & 0 & \\ & \vdots & \\ & 0 & \\ & X_1 & \\ 0 & \vdots & 0 \\ & X_k & \\ & 0 & \\ & \vdots & \\ & 0 & \end{pmatrix}$$

where $X_j = S({}_{e_{ij}Re_{ij}}e_{ij}Rg_i) = S(e_{ij}Rg_{ig_iRg_i})$ for $1 \leq j \leq k$.

d) $Rg_i/S_{k-1}({}_R Rg_i)$ is injective as a left R-module and isomorphic to $E({}_R Re_{ik}/J({}_R Re_{ik}))$ for $1 \leq i \leq m$, $1 \leq k \leq n(i)$. By g_{ik}, we denote the generator $g_i + S_{k-1}({}_R Rg_i)$ of $Rg_i/S_{k-1}({}_R Rg_i)$ for $1 \leq i \leq m$, $1 \leq k \leq n(i)$. Put

$G = Rg_{11} \oplus \dots \oplus Rg_{in(i)} \oplus \dots \oplus Rg_{m1} \oplus \dots \oplus Rg_{mn(m)}$,

$T = \mathrm{End}_R({}_R G)$.

Since ${}_R G$ is finitely generated and isomorphic to $E({}_R R/J(R))$, by Proposition 1, we have

Proposition 2. ${}_R\mathcal{M} \sim \mathcal{M}_T$; whence T is a left and right artinian left H-ring.

In order to investigate the structure of T, we express

$$T = \begin{pmatrix} [g_{11},g_{11}] & \cdots & [g_{11},g_{mn(m)}] \\ \vdots & & \vdots \\ [g_{mn(m)},g_{11}] & \cdots & [g_{mn(m)},g_{mn(m)}] \end{pmatrix}$$

where $[g_{ij},g_{kl}] = \mathrm{Hom}_R(Rg_{ij},Rg_{kl})$. Let h_{ij} be the matrix sxch that (ij,ij) position is the unity of $[g_{ij},g_{ij}]$ and all other entries are zero. Then $K = \{h_{11}, \ldots, h_{1n(1)}, \ldots, h_{m1}, \ldots, h_{mn(m)}\}$ is a complete set of orthogonal primitive idempotents of T; so

$$T = h_{11}T \oplus \cdots \oplus h_{1n(1)} \oplus \cdots \oplus h_{m1}T \oplus \cdots \oplus h_{mn(m)}T.$$

Further we have the following two propositions on T.

Proposition 3. T is a basic left H-ring such that

1) $h_{i1}T_T$ is injective for $1 \leq i \leq m$, and

2) $h_{i1}T_T \supsetneqq h_{i2}T_T \supsetneqq \cdots \supsetneqq h_{im(i)}T_T$ for $1 \leq i \leq m$.

Proposition 4. For e_{kt} in E and h_{kt} in K, the following are equivalent:

1) $(e_{i1}R;Re_{kt})$ is an injective pair.

2) $(h_{i1}R;Rh_{kt})$ is an injective pair.

In view of these propositions, we see that the structure of T is too similar to that of R, and want to raise the following problem: Does there exist an isomorphism ϕ from R to T satisfying $\phi(e_{ij}) = h_{ij}$ for all ij ? Namely, in other words, does R have a Nakayama isomorphism ?

Here we raise more explicitly the following three problems:

Problem A: Do basic left H-rings have Nakayama isomorphisms ?

Problem B: Do basic QF-rings have Nakayama automorphisms ?

Problem C: Do left H-rings have self-duality ?

Of course problems B and C are sub-problems of problem A. However, as stated in the introduction, we can prove that these are equivalent problems. Though, in this abstract, we only give a skech of its proof, detail will apper elsewhere.

Problem B & Problem A. We recall that $g_i = g_{i1}$ is the element of E such that $(e_{i1}R;Rg_{i1})$ is an injective pair for $1 \leq i \leq m$. Here we define two mappings

$$\sigma : \{1, \ldots, m\} \longrightarrow \{1, \ldots, m\}$$

$$\rho : \{1, \ldots, m\} \longrightarrow \{1, \ldots, n(1)\} \cup \ldots \cup \{1, \ldots, n(m)\}$$

by the rule $\sigma(i) = k$ and $\rho(i) = t$ if $g_{i1} = e_{kt}$; namely $(e_{i1}R;Re_{\sigma(i)\rho(i)})$ is an injective pair.

We note that $\{\sigma(1), \ldots, \sigma(m)\} \subseteq \{1, \ldots, m\}$ and $1 \leq \rho(i) \leq n(\sigma(i))$. Here we introduce a left H-ring of Type (*) as follows:

Definition. R is Type (*) if $\{\sigma(1), \ldots, \sigma(m)\}$ is a permutation of $\{1, \ldots, m\}$, and $\rho(i) = n(\sigma(i))$ for $1 \leq i \leq m$.

For example

$$\begin{pmatrix} D & \cdots & D \\ & \ddots & \vdots \\ 0 & & D \end{pmatrix} \qquad \text{(D: division ring)}$$

is Type (*), since m = 1 and $\rho(1) = n(i)$.

For a local QF-ring Q with $S(Q) \neq 0$, consider the ring:

$$R = \begin{pmatrix} \overbrace{Q & \cdots & Q}^{t} & \overbrace{\bar{Q} & \cdots & \bar{Q}}^{n-t} \\ J & \ddots & \vdots & \vdots & & \vdots \\ \vdots & \ddots & Q & \vdots & & \vdots \\ \vdots & & J & \bar{Q} & & \vdots \\ \vdots & & \vdots & \bar{J} & \ddots & \vdots \\ J & \cdots & J & \bar{J} & \cdots \bar{J} & \bar{Q} \end{pmatrix}$$

where $J = J(Q)$ and $\bar{Q} = Q/S(Q)$. Since m = 1 and $\rho(1) = t$ $n(1) = n$, this ring is Type (*) iff $t = n$.

Now, we must observe the structure of R and introduce two matrix rings P and R^*.

We put

$$R_{ij} = \begin{pmatrix} (e_{j1}, e_{i1}) & \cdots\cdots & (e_{jn(j)}, e_{i1}) \\ \vdots & & \vdots \\ (e_{j1}, e_{in(i)}) & \cdots\cdots & (e_{jn(j)}, e_{in(i)}) \end{pmatrix}$$

$$= \begin{pmatrix} e_{i1}Re_{j1} & \cdots & e_{i1}Re_{jn(j)} \\ \vdots & & \vdots \\ e_{in(i)}Re_{j1} & \cdots & e_{in(i)}Re_{jn(j)} \end{pmatrix}$$

So,

$$R = \begin{pmatrix} R_{11} & \cdots & R_{1m} \\ \vdots & & \vdots \\ R_{m1} & \cdots & R_{mm} \end{pmatrix}$$

Corresponding to $e_{ik}Re_{jt}$, we define

$$P_{ik,jt} = \begin{cases} e_{i1}Re_{j1} = (e_{j1}, e_{i1}) & \text{if } i \neq j \\ e_{i1}Re_{j1} = (e_{j1}, e_{i1}) & \text{if } i = j,\ k \leq t \\ J(e_{i1}Re_{j1}) = J((e_{j1}, e_{i1})) & \text{if } i = j,\ k > t \end{cases}$$

for ik, jt in $\{11, \ldots, 1n(1), \ldots, m1, \ldots, mn(m)\}$, and put

$$P_{ij} = \begin{pmatrix} P_{i1,j1} & \cdots & P_{i1,jn(j)} \\ \vdots & & \vdots \\ P_{in(i),j1} & \cdots & P_{in(i),jn(j)} \end{pmatrix}$$

Namely, when $i \neq j$,

$$P_{ij} = \begin{pmatrix} e_{i1}Re_{j1} & \cdots & e_{i1}Re_{j1} \\ \vdots & & \vdots \\ e_{i1}Re_{j1} & \cdots & e_{i1}Re_{j1} \end{pmatrix}$$

and when i = j,

$$P_{ij} = \begin{pmatrix} e_{i1}Re_{i1} & \cdots & e_{i1}Re_{i1} \\ & \ddots & \\ J(e_{i1}Re_{i1}) & & e_{i1}Re_{i1} \end{pmatrix}$$

We put

$$P = \begin{pmatrix} P_{11} & \cdots & P_{im} \\ \vdots & & \vdots \\ P_{m1} & \cdots & P_{mm} \end{pmatrix}$$

Then P becomes a ring by usual matrix operations. Let p_{ij} be the element of P such that its (ij,ij) position is the unity of $P_{ij,ij}$ and all other positions are zero. Then $\{ p_{11}, \ldots, p_{1n(1)}, \ldots, p_{m1}, \ldots, p_{mn(m)}\}$ is a complete set of orthogonal primitive idempotents of P; $P = p_{11}P \oplus \ldots \oplus p_{1n(1)}P \oplus \ldots \oplus p_{m1}P \oplus \ldots \oplus p_{mn(m)}P$.

We define a mapping

$$\begin{array}{cccc} \tau_{ik,jt}: & P_{ik,jt} & \longrightarrow & e_{ik}Re_{jt} \\ & \cap\!| & & \| \\ & (e_{jl}, e_{il}) & & (e_{jt}, e_{ik}) \end{array}$$

by the rule: $\alpha \rightarrow (\theta^{i}_{k,l})^{-1}\alpha\theta^{j}_{t,l}$. Then we see that $\tau_{ik,jt}$ is an epimorphism of abelian groups and

$$\tau_{ik,jt}\tau_{jt,pq} = \tau_{ik,pq}$$

for any ik, jt, pq in $\{11, \ldots, mn(m)\}$; whence

$$\tau = \begin{pmatrix} \tau_{11,11} & \cdots & \tau_{11,mn(m)} \\ \vdots & & \vdots \\ \tau_{mn(m),11} & \cdots & \tau_{mn(m),mn(m)} \end{pmatrix}$$

gives a ring epimorphism from P to R and then $\tau(p_{ik}) = e_{ik}$ for all ik.

We need the following proposition.

Proposition 5. For $\tau_{ik,jt}$,

1) if $j \neq \sigma(i)$, then $\tau_{ik,jt}$ is an isomorphism,

2) if $j = \sigma(i)$ and $t \leq \rho(i)$, then $\tau_{ik,jt}$ is also an isomorphism,

3) if $j = \sigma(i)$ and $t > \rho(i)$, then $\tau_{ik,jt}$ is not an isomorphism. Indeed, in this case,

$$\begin{aligned}\operatorname{Ker} \tau_{ik,jt} &= \{\alpha \in (e_{jl}, e_{il}) \mid \operatorname{Ker} \alpha = J_{\rho(i)}(e_{jl}R_R)\} \cup 0 \\ &= S({}_{e_{il}Re_{il}}e_{il}Re_{jl}) \\ &= S({}_{P_{ik,ik}}P_{ik,jt}) \\ &= S(e_{il}Re_{jl}{}_{e_{jl}Re_{jl}}) \\ &= S(P_{ik,jt}{}_{P_{jt,jt}})\end{aligned}$$

and Ker $\tau_{ik,jt}$ is simple as a left $P_{ik,ik}$-module and right $P_{jt,jt}$-module.

We replace $P_{ik,\sigma(i)t}$ in

$$P_{i\sigma(i)} = \begin{pmatrix} P_{i1,\sigma(i)1} & \cdots & P_{i1,\sigma(i)n(\sigma(i))} \\ \vdots & & \vdots \\ P_{in(i),\sigma(i)1} & \cdots & P_{in(i),\sigma(i)n(\sigma(i))} \end{pmatrix}$$

by $P_{ik,\sigma(i)t}/S(P_{ik,\sigma(i)t})$ for $k = 1, \ldots, n(i)$ and $j = \rho + 1, \ldots, n(\sigma(i))$, and denote it by $P^*_{i\sigma(i)}$. And we put

$$R^* = \begin{pmatrix} P_{11} & \cdots & P_{1,\sigma(i)-1} & P^*_{1\sigma(1)} & P_{1,\sigma(1)+1} & \cdots & P_{1m} \\ \vdots & & & & & & \vdots \\ P_{m1} & \cdots & P_{m,\sigma(m)-1} & P^*_{m\sigma(m)} & P_{m,\sigma(m)+1} & \cdots & P_{mm} \end{pmatrix}$$

and by e^*_{ik} we denote the element of R^* which corresponding to P_{ik} in P for each ik in {11, ..., mn(m)}. Then, by Proposition 5, we obtain

Theorem 1. R^* becomes a ring and τ induces a ring isomorphism τ^* from R^* to R satisfying $\tau^*(e^*_{ik}) = e_{ik}$ for all e^*_{ik}. Therefore R^* is a representative matrix ring of R.

We shall illustrate this theorem for two cases:

1) The case m = 1. Then

$$R^* = \left(\begin{array}{ccccccc} \overbrace{Q & \cdots & Q}^{f(1)} & \bar{Q} & \cdots & & \bar{Q} \\ J & \ddots & \vdots & \vdots & & & \vdots \\ \vdots & & Q & \vdots & & & \vdots \\ \vdots & & J & \bar{Q} & & & \vdots \\ \vdots & & \vdots & \bar{J} & \ddots & & \vdots \\ \vdots & & \vdots & \vdots & & \ddots & \vdots \\ J & \cdots & J & \bar{J} & \cdots & \bar{J} & \bar{Q} \end{array} \right)$$

where $Q = e_{11}Re_{11}$, $J = J(e_{11}Re_{11})$ and $\bar{Q} = Q/S(Q)$. (Then Q is a local QF-ring).

2) The case m = 2, $\sigma(1) = 2$ and $\sigma(2) = 1$. Then

$$R^* = \left(\begin{array}{cccccccccc} Q & \cdots & & Q & \overbrace{A & \cdots & A}^{f(1)} & \bar{A} & \cdots & \bar{A} \\ & \ddots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ & J(Q) & \ddots & & & & & & & \\ & & & Q & A & \cdots & A & \bar{A} & \cdots & \bar{A} \\ B & \cdots & B & \bar{B} \cdots \bar{B} & T & \cdots & & & \cdots & T \\ \vdots & & \vdots & \vdots & & \ddots & & & & \vdots \\ \vdots & & \vdots & \vdots & & J(T) & \ddots & & & \vdots \\ \underbrace{B & \cdots & B}_{f(2)} & \bar{B} \cdots \bar{B} & & & & & & T \end{array} \right)$$

where $Q = e_{11}Re_{11}$, $T = e_{21}Re_{21}$, $A = e_{11}Re_{21}$, $\bar{A} = A/S(A)$, $B = e_{21}Re_{11}$ and $\bar{B} = B/S(B)$.

Next, we put

$$Q(R) = \begin{pmatrix} e_{11}Re_{11} & e_{11}Re_{21} & \cdots & e_{11}Re_{m1} \\ e_{21}Re_{11} & e_{21}Re_{21} & \cdots & e_{21}Re_{m1} \\ \vdots & \vdots & & \vdots \\ e_{m1}Re_{11} & e_{m1}Re_{21} & \cdots & e_{m1}Re_{m1} \end{pmatrix}$$

Then $Q(R)$ becomes a ring; $Q(R) \simeq eRe \simeq pPp \simeq e^{*}R^{*}e^{*}$, where $e = e_{11} + e_{21} + \dots + e_{m1}$, $p = p_{11} + p_{21} + \dots + p_{m1}$ and $e^{*} = e^{*}_{11} + e^{*}_{21} + \dots + e^{*}_{m1}$.

The following hold on $Q(R)$.

Proposition 6. If R is Type(*), then $Q(R)$ is a basic QF-ring and R is (left and) right artinian.

Proposition 7. Assume that R is Type(*). Then R has a Nakayama isomorphism if and only if $Q(R)$ has a Nakayama automorphism.

The following is proved by using the representation R^{*} and induction on m.

Proposition 8. There exist basic left H-ring T_1, T_2, ..., T_n and ring epimorphisms $\phi_1: T_1 \to T_2$, $\phi_2: T_2 \to T_3$, ..., $\phi_{n-1}: T_{n-1} \to T_n$, $\phi_n: T_n \to R$ such that T_1 is Type(*) and $\mathrm{Ker}\ \phi_i$ is a simple two sided ideal of T_i, $i = 1, 2, .., n$.

Proposition 9. If R has a Nakayama isomorphism, then so does R/S for every simple ideal S of R.

We are now in a position to state the following

Theorem 2. If Problem B is affirmative, then so is Problem A.

In fact, if Problem B is affirmative, then T_1 in Proposition 8 has a Nakayama isomorphism, since $Q(T_1)$ has a Nakayama automorphism. Hence, by proposition 9, we see that R has a Nakayama isomorphism.

As a by-product, we see that R is right artinian, since T_1 in Proposition 8 is right artinian. Thus we have

Theorem 3. Every left H-rings is (left and) right artinian.

<u>Problem C & Problem A.</u> We put

$$R^*_{ij} = \begin{cases} P^*_{ij} & \text{if } j = \sigma(i) \\ P_{ij} & \text{if } j \neq \sigma(i) \end{cases}$$

and

$$R^*_{ik,jt} = \begin{cases} P_{ik,jt} & \text{if } j \neq \sigma(i) \\ P_{ik,jt} & \text{if } j = \sigma(i),\ t \leq \sigma(i) \\ P_{ik,jt}/S(P_{ik,jt}) & \text{if } j = \sigma(i),\ t > \sigma(i) \end{cases}$$

So,

$$R^* = \begin{pmatrix} R^*_{11} & \cdots & R^*_{1m} \\ \vdots & & \vdots \\ R^*_{m1} & \cdots & R^*_{mm} \end{pmatrix}$$

and

$$R^*_{ij} = \begin{pmatrix} R^*_{i1,j1} & \cdots & R^*_{i1,jn(j)} \\ \vdots & & \vdots \\ R^*_{in(i),j1} & \cdots & R^*_{in(i),jn(i)} \end{pmatrix}$$

Here, adding one row and one column to R^*, we make an extension ring $W_i(R)$ of R as follows: Put

$$W_i(R) = \begin{pmatrix} R^*_{11} & \cdots & & R^*_{1i} & Y_1 & R^*_{1,i+1} & \cdots & R^*_{1m} \\ \vdots & & & \vdots & \vdots & \vdots & & \vdots \\ R^*_{i1} & \cdots & & R^*_{ii} & Y_i & R^*_{i+1} & \cdots & R^*_{im} \\ X_1 & \cdots & X_{i-1} & X_i & Q & X_{i+1} & \cdots & X_m \\ R^*_{i+1,1} & \cdots & & R^*_{i+1,i} & Y_{i+1} & R^*_{i+1,i+1} & \cdots & R^*_{i+1,m} \\ \vdots & & & \vdots & \vdots & \vdots & & \vdots \\ R^*_{m1} & \cdots & & R^*_{mi} & Y_m & R^*_{m,i+1} & \cdots & R^*_{mm} \end{pmatrix}$$

where X_k is the last row of R^*_{ik} for $k = 1, \ldots, i-1, i+1, \ldots, m$, Y_k is the last column of R^*_{ki} for $k = 1, \ldots, m$,

$X_i = (R^*_{in(i),i1} \cdots R^*_{in(i),in(i)-1}\ J(R^*_{in(i),in(i)})$, and $Q = R^*_{in(i),in(i)}$. Then $W_i(R)$ becomes a ring from operations of R^*, and moreover it is a basic left H-ring.

For example, consider the case: $m = 2$, $n(1) = 1$, $n(2) = 2$, $\sigma(1) = 2$, $\sigma(2) = 1$, $\rho(1) = 2$. Then

$$\begin{pmatrix} Q & B & B \\ A & T & T \\ A & K & T \end{pmatrix}$$

where $Q = e_{11}Re_{11}$, $T = e_{21}Re_{21}$, $A = e_{21}Re_{11}$, $B = e_{11}Re_{21}$, $K = J(T)$,

$$W_1(R) = \begin{pmatrix} Q & Q & B & B \\ J & Q & B & B \\ A & A & T & T \\ A & A & K & T \end{pmatrix}$$

where $J = J(Q)$

$$W_2(R) = \begin{pmatrix} Q & B & B & B \\ A & T & T & T \\ A & K & T & T \\ A & K & K & T \end{pmatrix}$$

Proposition 10. Let W be in $\{W_1(R), \ldots, W_m(R)\}$. Then W has a Nakayama isomorphism if and only if R has a Nakayama isomorphism.

This proposition enable us to prove the following

Theorem 4. If Problem C is affirmative, so is Problem A.

Indeed, assume that Problem C is affirmative. Then, there exists an isomorphism ϕ_1 from R to T, where T is the ring described in Proposition 2; Recall that $T = h_{11}T \oplus \cdots \oplus h_{1n(1)}T \oplus \cdots \oplus h_{m1}T \oplus \cdots \oplus h_{mn(m)}T$ and

a) $h_{i1}T_T$ is injective,

b) $h_{i1}T_T \gtrsim h_{i2}T_T \gtrsim \cdots \gtrsim h_{in(i)}T_T$

for i = 1, ..., m.

We want to prove that there exists an isomorphism ϕ from R to T satisfying $\phi(e_{ij}) = h_{ij}$ for all e_{ij}. To prove this, we can assume by Proposition 10 that

$$n(1) < n(2) < \cdots < n(m) \quad \cdots \quad (*)$$

Putting $w_{ij} = \phi_1(e_{ij})$, $\{w_{ij}\}$ is a complete set of orthogonal primitive idempotents of T and

a′) $w_{i1}T_T$ is injective,

b′) $w_{i1}T_T \gtrsim w_{i2}T_T \gtrsim \cdots \gtrsim w_{in(i)}T_T$

for i = 1, ..., m. For $\{h_{ij}\}$ and $\{w_{ij}\}$, there exists an automorphism ϕ_2 of T such that $\{\phi_2(w_{ij})\} = \{h_{ij}\}$ (cf.[2,p42]). Then, comparing a), b) to a′), b′), together with (*), we see that $\phi_2(w_{ij})$ must be just h_{ij} for all w_{ij}. Hence putting $\phi = \phi_2\phi_1$, we get $\phi(e_{ij}) = h_{ij}$ for all e_{ij}.

From Theorems 2 and 4, we obtain the following

Theorem 5. Problems A, B and C are mutually equivalent ones.

Finally we note that all arguments above on left H-rings work on Nakayama rings; so the following problems are mutually equivalent ones:

Problem A′: Do basic Nakayama rings have Nakayama isomorphisms ?

Problem B′: Do basic Nakayama QF-rings have Nakayama automorphisms ?

Problem C′: Do Nakayama rings have self-duality ?

As Nakayama rings have self-duality ([10]), it follows that Problems A′ and B′ are affirmative. By a similar reason, we see that finite dimensional algebras over a field have Nakayama isomorphisms.

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ON THE PROJECTIVE INDECOMPOSABLE MODULES OVER THE GROUP ALGEBRAS OF GROUPS WHOSE SYLOW 3-SUBGROUPS ARE EXTRA-SPECIAL OF ORDER 27 OF EXPONENT 3 IN CHARACTERISTIC p = 3 *)

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All groups considered here are finite and all modules considered here are finitely generated right modules. Let FG be the group algebra of a finite group G over an algebraically closed field F of characteristic $p > 0$. Let J be the Jacobson radical of FG. For an FG-module $M \neq 0$, we write $j(M)$ for the Lowey length of M, that is, $j(M)$ is the least positive integer j such that $MJ^j = 0$. We are interested in the Loewy structure of the projective indecomposable modules (p.i.m.'s) over FG (see [4] and [12]).

For a while, let's assume that G is p-solvable. Even in this particular case, few on the Loewy structure of the p.i.m.'s over FG has been known if G has p-length > 1.

The next is the simplest example in this case. It is noted that K. Motose firstly remarked that $j(FG) = 4$ in the following situation (see [17, Proposition]).

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*) This is a report of my results, some of which have already been announced elsewhere and the rest of which would be published elsewhere.

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Example 1 ([7, VII 15.10 Example]). Let G be the symmetric group on 4 letters, and let $p = 2$. Then the Loewy and socle series of the p.i.m.'s over FG are

P_I $\qquad\qquad\qquad$ P_2

where I is the trivial FG-module and 2 is a simple FG-module of F-dimension two, and P_I and P_2 are respectively the projective covers of I and 2.

The thing we had to calculate was the following:

Example 2 (Koshitani [9]). Let $p = 3$ and let G be the semi-direct product of the elementary abelian group of order 9 by the special linear group $SL_2(\mathbb{F}_3) = SL(2,3)$ in a natural way. Then the Loewy and socle series of the p.i.m.'s over FG are completely determined (see [9, Theorem]). In particular, $j(FG) = 9$ (cf. [16, Theorem] and [15, Example 2.5]).

An advantage of Example 2 is the following.

Corollary to Example 2 ([10]). Let G be p-solvable and $p > 2$, and let B be a block ideal of FG with defect group

$$D = M(p) = \langle a, b, c \mid a^p = b^p = c^p = 1, a^{-1}ba = bc, a^{-1}ca = c, b^{-1}cb = c\rangle,$$

namely, the extra-special group of order p^3 of exponent p (see [6, p. 203]). Then the Loewy and socle series of the p.i.m.'s in B are completely determined. In particular, $j(B) = 4p - 3$ (cf. [8, Lemma 1.4]).

Now, let's consider non-p-solvable groups G whose Sylow 3-subgroups are isomorphic to M(3) in the above notation and let $p = 3$.

By [18], the Tits simple group ${}^2F_4(2)'$ has a maximal subgroup Aut(SL(3,3)), which is isomorphic to the semi-direct product $SL(3,3):Z_2$ of $SL_3(\mathbb{F}_3) = SL(3,3)$ by the cyclic group Z_2 of order 2 such that the action of Z_2 on SL(3,3) is the transpose-inverse. So that ${}^2F_4(2)$ has the following subgroups;

$$
{}^2F_4(2) = \mathrm{Aut}({}^2F_4(2)') \underset{2}{\gneq} {}^2F_4(2)' \underset{1600}{\gneq} SL(3,3):Z_2 = \mathrm{Aut}(SL(3,3))
$$

$$
\underset{2}{\gneq} SL(3,3) \underset{13}{\gneq} (Z_3 \times Z_3):GL(2,3) \underset{2}{\gneq} (Z_3 \times Z_3):SL(2,3)
$$

where $(Z_3 \times Z_3):GL(2,3)$ and $(Z_3 \times Z_3):SL(2,3)$ are respectively the semi-direct products of the elementary abelian group $Z_3 \times Z_3$ of order 9 by the general linear group GL(2,3) and the special linear group SL(2,3) in a natural way, and the numbers between two groups are the indices.

Hence it appears worth-while to get the Loewy and socle series of the p.i.m.'s over F[SL(3,3)] and F[Aut(SL(3,3))]. In fact, we get the following by making use of Example 2.

<u>Example 3</u> (Koshitani [11]). Let $p = 3$. Then the Loewy and socle series of the p.i.m.'s over F[SL(3,3)]

and F[Aut(SL(3,3))] are completely determined. In particular, the Loewy lengths of both of the group algebras are 9.

Concerning other examples, in which the Loewy and socle series of p.i.m.'s over group algebras are calculated, see [1], [2], [3], [4], [5], [12], [13] and [14].

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ON A CONJECTURE OF P.LANDROCK

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This article outlines the joint work[2] with T.Okuyama.

Let G be a finite group, k an algebraically closed field of prime characteristic p and let J be the Jacobson radical of the group ring kG. P.Landrock conjectured in his book[1] that

(L): J^i / J^{i+1} is self-dual as a (right) kG-module for all i.

Unfortunately this is not true in general, as the Mathieu group M_{11} shows for $p = 11$. However we acknowledge the significance of this conjecture for various reasons. So it seems to be reasonable to investigate when or for which groups it is true. Here we shall mention some results concerning it without proofs.

Lemma 1. (L) is true if and only if $\dim J^i e = \dim eJ^i$ for all i and every primitive idempotent e of kG.

Proposition 1. If every irreducible k-character is algebraically conjugate to its dual, then (L) is true.

The following corollary to the above Proposition shows that (L) is true for a large class of groups.

The detailed version will appear elsewhere.

Corollary 1. Let $|G| = p^a m$ with $(p,m) = 1$. Then (L) is true if there is an integer n such that $p^n \equiv -1 \mod m$.

Lemma 2. Let H be a normal subgroup of G and assume that $(p, [G:H]) = 1$. Then (L) is true for G if and only if it is true for H.

Since (L) is trivially true for S_n (the symmetric group on n letters), we can show the following by making use of Lemma 2.

Proposition 2. (L) is true for A_n.

One may notice from Lemma 1 that (L) is true if there is an anti-automorphism f of kG such that $f(e)kG \simeq ekG$ for any primitive idempotent e of kG. This is just the case for $G = GL_n(q)$ or $U_n(q)$.

In fact if we define f by $f(x) = {}^t x$ for $x \in G$, where ${}^t x$ denotes the transpose of x, then this enjoys the above condition. Thus we have

Proposition 3. (L) is true for $GL_n(q)$ and $U_n(q)$.

Also we have

Proposition 4. (L) is true for $SL_n(q)$.

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