PROCEEDINGS OF THE 19TH SYMPOSIUM ON RING THEORY

HELD AT SHINSHU UNIVERSITY, MATSUMOTO

AUGUST 21—23, 1986

EDITED BY

YASUO IWANAGA

Shinshu University

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OKAYAMA, JAPAN

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TABLE OF LECTURES

The asterisks (*) above stand for the speakers at the symposium.

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PREFACE

This volume is the Proceedings of the 19th Symposium on Ring Theory, which was held at Shinshu University, Matsumoto, on August 21-23, 1986. The symposium consisted of thirteen talks including two special lectures from the knot theory by Dr. T. Kobayashi and the commutative ring by Prof. H. Matsumura, to whom I'd like to give my gratitude.

The proceedings contains eleven articles by the speakers. Some of them are expository and the complete or final versions will appear elsewhere.

We should like to acknowledge the financial assistance from the Grant-in-Aid for Scientific Research from the Ministry of Education through the arrangements by Professor K. Shiratani. We appreciate his arrangements.

We wish also to extend our thanks to all speakers of the symposium, to Professor K. Kishimoto for his well arrangement and kind hospitality at Shinshu University, and to Professor H. Tominaga for his compilation of the proceedings.

Shinshu University (Nagano), November 1986

Yasuo Iwanaga

PROCEEDINGS OF THE 19TH
SYMPOSIUM ON RING THEORY (1986)

ON H-SEPARABILITY OF GROUP RINGS

Kazuhiko HIRATA

Let $k[G]$ be the group ring of a finite group G with a coefficient field k, and C the center of $k[G]$. Assume that the characteristic of k does not divide the order of G. Denote by g_x and c_x the number and the sum of elements in the conjugate class of G containing the element x of G, respectively.

Lemma 1. $u = \sum_{c_x}(1/g_x)c_x c_{x^{-1}}$ is a unit in C.

Proof. We can prove that $\{(1/g_x)c_x\}$ and $\{c_{x^{-1}}\}$ is a pair of dual bases of C over k, that is, if $c_y(1/g_x)c_x = \sum_{c_z}(1/g_z)c_z a_{zx}$, $a_{zx} \in Z$, then $c_{z^{-1}}c_y = \sum_{c_{x^{-1}}} a_{zx} c_{x^{-1}}$. On the other hand, C is a separable algebra over k in the sense of that, for any field extension L of k, C_L is a semisimple L-algebra. Then u is a unit in C by Theorem 71.6 in [2] p.482.

Let v be the inverse of u in C, $uv = 1$.

Corollary 2. $\sum_{c_x}(1/g_x)c_x \otimes c_{x^{-1}}v$ is a separability idempotent in $C \otimes_k C$.

Proof. It is clear that $c(\sum(1/g_x)c_x \otimes c_{x^{-1}}v) = (\sum(1/g_x)c_x \otimes c_{x^{-1}}v)c$ for any $c \in C$ and $\sum(1/g_x)c_x c_{x^{-1}}v = 1$.

Let p be the projection map of $k[G]$ to C defined by $p(a) = (1/n)\sum_{x \in G} xax^{-1}$ for $a \in k[G]$, where n is the order of G.

Corollary 3. $\{x \cdot p;\ x^{-1}v\}$ $(x \in G)$ is a system of projective bases of $k[G]$ over C.

The final version of this paper has been submitted for publication elsewhere.

Proof. For the identity 1 of G, we have $\sum_{x\in G}(x\cdot p)(1)x^{-1}v = \sum_{x\in G} p(x)x^{-1}v = \sum_{x\in G}(1/g_x)c_x x^{-1}v = \sum_{c_x}(1/g_x)c_x c_{x^{-1}}v = 1$. Now, for any $y\in G$, we have $\sum_{x\in G}(x\cdot p)(y)x^{-1}v = \sum_{x\in G} p(yx)x^{-1}v = \sum_{x\in G} p(yx)(yx)^{-1}vy = y$.

Now consider the two-sided $k[G]$-module $k[G]\otimes_C k[G]$. Then, for each $x\in G$, the element $(1/n)\sum_{y\in G} y\otimes xy^{-1}$ is in $(k[G]\otimes_C k[G])^{k[G]} = \{\xi\in k[G]\otimes_C k[G] \mid a\xi = \xi a$, for all $a\in k[G]\}$. Therefore the map m_x, for $x\in G$, defined by $m_x(a) = ((1/n)\sum_{y\in G} y\otimes xy^{-1})a$, $a\in k[G]$, is a two-sided $k[G]$-homomorphism of $k[G]$ to $k[G]\otimes_C k[G]$. The map n_x, for $x\in G$, defined by $n_x(\sum a_i\otimes b_i) = \sum a_i x^{-1}vb_i$, $\sum a_i\otimes b_i\in k[G]\otimes_C k[G]$, is a two-sided $k[G]$-homomorphism of $k[G]\otimes_C k[G]$ to $k[G]$. Then it is easily verified that $\sum_{x\in G} m_x\circ n_x$ is the identity map of $k[G]\otimes_C k[G]$. Thus we have proved the following corollary.

Corollary 4. $k[G]\otimes_C k[G]$ is a two-sided $k[G]$-direct summand of the direct sum of n-copies of $k[G]$.

If this is the case, then it holds that $k[G]\otimes_C k[G] \cong \mathrm{Hom}_C(k[G], k[G])$ and $k[G]$ is finitely generated projective over C, see [5] p.112. Therefore $k[G]$ is a central separable C-algebra by Theorem 2.1 [1].

Let H be a subgroup of G and $G = \sum_{i=1}^{r} y_i H$ a coset decomposition of G by H. Denote by h_x and d_x the number and the sum of elements in the H-conjugate class of G containing the element x of G, respectively. Let Δ be the centralizer of $k[H]$ in $k[G]$. Then $\{d_x\}$ is a k-basis of Δ. By the same way as in Lemma 1, it can be verified that $\{(1/h_x)d_x\}$ and $\{d_{x^{-1}}\}$ form a pair of dual bases of Δ over k. Let q be the map of Δ to C defined by $q(a) = (1/r)\sum_i y_i a y_i^{-1}$, for $a\in\Delta$. It can be shown that q does not depend on the choice of y_i and q is the projection of Δ to C.

Proposition 5. $\{(1/h_x)d_x\cdot q;\ d_{x^{-1}}v\}$ is a system of projective bases of Δ over C, where x runs over all the representatives of H-conjugate classes of G.

Proof. If we notice that $q(d_x) = (h_x/g_x)c_x$ and c_x is a sum of some d_x's, the calculation is similar to the proof in Corollary 3 and we shall omit it.

Let D be the centralizer of Δ in k[G]. Then $D \supset k[H]$ and the centralizer of D in k[G] is equal to Δ.

Proposition 6. k[G] is an H-separable extension of D.

Proof. For a representative x of an H-conjugate class of G, define $s_x : k[G] \longrightarrow k[G] \otimes_D k[G]$ by $s_x(a) = ((1/r)\sum_i y_i \otimes (1/h_x) d_x y_i)a$ and $t_x : k[G] \otimes_D k[G] \longrightarrow k[G]$ by $t_x(\sum a_i \otimes b_i) = \sum a_i d_{x^{-1}} v b_i$, respectively. As $(1/r)\sum_i y_i \otimes (1/h_x) d_x y_i^{-1}$ is in $(k[G] \otimes_D k[G])^{k[G]}$ and $d_{x^{-1}} v$ is in Δ, s_x and t_x are two-sided k[G]-homomorphisms, respectively. If we notice that $\sum_{d_x} (1/h_x) d_x y_i^{-1} d_{x^{-1}} v$ is contained in D, it is easily verified that $\sum s_x \circ t_x$ is the identity map of $k[G] \otimes_D k[G]$, where the sum is taken over all the H-conjugate classes of G. Therefore $k[G] \otimes_D k[G]$ is a two-sided k[G]-direct summand of a direct sum of finite copies of k[G] and k[G] is an H-separable extension of D.

Even if the characteristic of k divides the order of G, if the index of H in G is a unit in k, k[G] is always a separable extension of k[H] by Proposition 3.1 [6]. In this case it happens that k[G] may or not be an H-separable extension of D. Let k be a field of characteristic two. Take $G = S_3$ the symmetric group of degree three and $H = \langle (12) \rangle$. Then $G = H + (13)H + (23)H$ is a coset decomposition of G by H. Put $x_1 = (12)$, $x_2 = (13) + (23)$ and $y = (123) + (132)$. Then we have $\Delta = k1 + kx_1 + kx_2 + ky$ and $D = k[G]^{\Delta} = \Delta$. The projection q of Δ to C is given by $q(a) = (1/3)(1 \cdot a \cdot 1 + (13)a(13) + (23)a(23))$ for $a \in \Delta$. Then $\{q,\ x_2 \cdot q,\ y \cdot q;\ 1+y,\ x_2,\ 1\}$ is a system of projective bases of Δ over C. Define maps $s_i : k[G] \longrightarrow k[G] \otimes_D k[G]$ $(i = 1, 2, 3)$ by $s_1(a) = (1/3)(1 \otimes 1 + (13) \otimes (13) + (23) \otimes (23))a$, $s_2(a) = (1/3)(1 \otimes x_2 + (13) \otimes x_2(13) + (23) \otimes x_2(23))a$ and $s_3(a) = (1/3)(1 \otimes y + (13) \otimes y(13) + (23) \otimes y(23))a$, respectively. Also define maps $t_i : k[G] \otimes_D k[G] \longrightarrow k[G]$ $(i = 1, 2, 3)$ by $t_1(\sum a_i \otimes b_i) = \sum a_i (1 + y) b_i$, $t_2(\sum a_i \otimes b_i) = \sum a_i x_2 b_i$ and $t_3(\sum a_i \otimes b_i) = \sum a_i b_i$, respectively. Then $\sum_{i=1}^{3} s_i \circ t_i$ is the identity map of $k[G] \otimes_D k[G]$ and k[G] is an H-separable extension of D. Next take $G = S_4$ and $H = \langle (13), (1234) \rangle$ a 2-Sylow subgroup of G. Put $x = \sum (12)$, $y = \sum$

(123), $z = \sum(12)(34)$ and $w = \sum(1234)$. Then we have $x^2 = 6 + 3y + 2z = y$, $y^2 = 8 + 4y + 8z = 0$, $z^2 = 3 + 2z = 1$ and $w^2 = 6 + 3y + 2z = y$. Therefore $kx + ky + k(z - 1) + kw$ is the radical of C. Since C is five dimensional over k, C is a local ring. On the other hand, as there are eight H-conjugate classes of G, Δ is eight dimensional over k. Therefore Δ is not C-projective and k[G] is not an H-separable extension of D.

Addendum. A. Hattori defined the rank element for finitely generated projective modules [3]. From Corollary 3 we know $\mathrm{rank}_C k[G] = n/u$. This is found in [4] in connection with the separability idempotent, Example 4 and Proposition 3.1.

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AUTOMORPHISMS OF FINITE ORDER OF KAC-MOODY LIE ALGEBRAS

Zenji KOBAYASHI

1. In this note, we will classify all automorphisms of prime order of the affine Lie algebra $A_{n-1}^{(1)}$ up to conjugacy in the group of all automorphisms of $A_{n-1}^{(1)}$. To do this, we will use non abelian group cohomology of some finite cyclic group acting on $PGL_n(\mathbb{C}[t,t^{-1}])$.

2. We first recall some facts about Kac-Moody Lie algebras and associated groups(see refs.[1] and [2] for details).

A symmetrizable generalized Cartan matrix $A=(a_{ij})$ is an $n\times n$ matrix of integers satisfying $a_{ii}=2$ for all i; $a_{ij}\leq 0$ if $i\neq j$; DA is symmetric for some non-degenerate diagonal matrix D. We fix such a matrix A, assumed for simplicity to be indecomposable. A is called finite(resp. affine) type when DA is positive definite (resp. positive semi-definite and not positive definite).

The Kac-Moody Lie algebra $\mathfrak{g}=\mathfrak{g}(A)$ is the Lie algebra over $\mathbb{C}$ generated by symbols h_i, e_i and $f_i (1\leq i\leq n)$ with defining relations: $[h_i,h_j]=0$; $[e_i, f_j]=\delta_{ij}h_i$; $[h_i,e_j]=a_{ij}e_j$, $[h_i,f_j]=-a_{ij}f_j$; $(\mathrm{ad}(e_i))^{1-a_{ij}}(e_j)=(\mathrm{ad}(f_i))^{1-a_{ij}}(f_j)=0$ $(i\neq j)$. $\mathfrak{g}$ is graded by $\mathbb{Z}^n$ with e_i of degree $\alpha_i: j\mapsto\delta_{ij}$, f_i of degree $-\alpha_i$ and h_i of degree 0. The set Δ of roots is the set of degree α for which the subspace $\mathfrak{g}_\alpha$ of elements of degree α in $\mathfrak{g}$ is non-trivial. Let $\mathfrak{n}_+$(resp. $\mathfrak{h}$, resp. $\mathfrak{n}_-$) be the subalgebra of $\mathfrak{g}$ generated by the e_i(resp. h_i,

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The detailed version of this paper has been submitted for a publication elsewhere.

resp. f_i)($1 \leq i \leq n$). Then $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+$ and $\mathfrak{h} = \mathbb{C}h_1 + \cdots + \mathbb{C}h_n$. Concordantly with this decomposition of $\mathfrak{g}$ we have $\Delta = \Delta_- \cup \{0\} \cup \Delta_+$.

Difine $r_i \in \mathrm{Aut}(\mathbb{Z}^n)$, by $r_i(\alpha_j) = \alpha_j - a_{ji}\alpha_i$, and put $S = \{r_1, \cdots, r_n\}$ then S generates the Weyl group W, and (W,S) is a Coxeter system. W preserves the set Δ of roots. A real(resp. imaginary) root is an element of $\Delta_R = W(\Pi)$ (resp. $\Delta_I = \Delta - \Delta_R$), where $\Pi = \{\alpha_1, \cdots, \alpha_n\}$.

3. We now construct the group G associated to the Lie algebra $\mathfrak{g}$. A $\mathfrak{g}$-module V, or (V, π), where $\pi: \mathfrak{g} \mapsto \mathrm{End}_{\mathbb{C}}(V)$, is called integrable if $\pi(e)$ is locally nilpotent whenever $e \in \mathfrak{g}_\alpha$, $\alpha \in \Delta_R$. ($\mathfrak{g}$, ad) is an integrable $\mathfrak{g}$-module.

Let G^* be the free product of the additive groups $\mathfrak{g}_\alpha$ ($\alpha \in \Delta_R$), with canonical inclusoins $i_\alpha: \mathfrak{g}_\alpha \mapsto G^*$. For any integrable $\mathfrak{g}$-module (V, π), define a homomorphism $\pi^*: G^* \mapsto \mathrm{Aut}_{\mathbb{C}}(V)$ by $\pi^*(i_\alpha(e)) = \exp \pi(e)$. Let N^* be the intersection of all $\mathrm{Ker}(\pi^*)$, put $G = G^*/N^*$, and let $q: G^* \mapsto G$ be the canonical homomorphism. For $e \in \mathfrak{g}_\alpha$ ($\alpha \in \Delta_R$), put $\exp(e) = q(i_\alpha(e))$, so that $U_\alpha = \exp(\mathfrak{g}_\alpha)$ is an additive one-prameter subgroup of G.

Example: Let A be the Cartan matrix of a simple finite-dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{C}$(i.e. A is of finite type). Then the group G associated to $\mathfrak{g} = \mathfrak{g}(A)$ is the connected simply-connected algebraic group associated to $\mathfrak{g}$. Now let $\widetilde{A}$ be the extended Cartan matrix of $\mathfrak{g}$ (in this case, $\widetilde{A}$ is of affine type). Then the group $\widetilde{G}$ associated to $\mathfrak{g}(\widetilde{A})$ is a central extension by $\mathbb{C}^*$ of $G(\mathbb{C}[t, t^{-1}])$.

To any integrable $\mathfrak{g}$-module (V, π) we associate the homomorphism(again denoted by) $\pi: G \mapsto \mathrm{Aut}_{\mathbb{C}}(V)$ satisfying $\pi(\exp(e)) = \exp \pi(e)$ for $e \in \mathfrak{g}_\alpha$ ($\alpha \in \Delta_R$). The homomorphism associated to ($\mathfrak{g}$, ad), denoted Ad, maps G into $\mathrm{Aut}_{\mathbb{C}}(V)$. The kernel of Ad is the center C of G, and Ad(G) acts faithfully on $\mathfrak{g}/c$, where c is the center of $\mathfrak{g}$.

By the study of such a group G, Kac and Peterson proved the invariance of generalized Cartan matrices and the description of $\mathrm{Aut}(\mathfrak{g})$.

Theorem 1 [2]. (a) If $\mathfrak{g}_1=\mathfrak{g}(A_1)$ is a Kac-Moody Lie algebra, with center c_1, such that $\mathfrak{g}_1/c_1$ is isomorphic to $\mathfrak{g}/c$, then $A=A_1$ up to a bijection of index sets.

(b) Any automorphism of the Kac-Moody Lie algebra $\mathfrak{g}=\mathfrak{g}(A)$ can be written in the form $\lambda\sigma$ or $\omega\lambda\sigma$ where $\sigma\in \mathrm{Ad}(G)$; $\lambda(e_i)=\lambda_{i_k}e_{i_k}$, $\lambda(f_i)=\lambda_{i_k}^{-1}f_{i_k}$, $i=1,\cdots,n$, for some $\lambda_i\in\mathbb{C}^*$ and a permutation $i\mapsto i_k$ preserving the matrix A; $\omega(e_i)=-f_i$, $\omega(f_i)=-e_i$, $i=1,\cdots,n$.

4. We now classify all automorphisms of prime order of the affine Lie algebra $A_{n-1}^{(1)}$ up to conjugacy(see ref. [3] for details).

Let $\mathfrak{g}$ be the affine Lie algebra over $\mathbb{C}$ of type $A_{n-1}^{(1)}$ $(n\geq 2)$, i.e. the Kac-Moody Lie algebra defined by the symmetrizable generalized Cartan matrix of affine type $A_{n-1}^{(1)}=(a_{ij})$ satisfying $a_{ij}=2$ if $i=j$; $=-1$ if $|i-j|=1$ or $n-1$; $=0$ otherwise. Then we have a universal central extension over $\mathbb{C}$; (1) $0\to\mathbb{C}z\to\mathfrak{g}\xrightarrow{\eta}\mathfrak{sl}_n(\mathbb{C}[t,t^{-1}])\to 0$, where $z=h_1+\cdots+h_n$, $\eta(e_i)=E_{i,i+1}$, $\eta(e_n)=tE_{n1}$, $\eta(h_i)=E_{ii}-E_{i+1,i+1}$, $\eta(h_n)=E_{11}-E_{nn}$, $\eta(f_i)=E_{i+1,i}$, $\eta(f_n)=t^{-1}E_{1n}$ $i=1,\cdots,n$; E_{ij} is the matrix unit with 1 in the i,j position and 0 elsewhere; $\mathfrak{sl}_n(\mathbb{C}[t,t^{-1}])=\{X\in M_n(\mathbb{C}[t,t^{-1}])\mid \mathrm{tr}\,X=0\}$.

By the universality of (1) and Theorem 1-(b),

$$\mathrm{Aut}(\mathfrak{g})\simeq \mathrm{Aut}_{\mathbb{C}}(\mathfrak{sl}_n(\mathbb{C}[t,t^{-1}]))$$
$$\simeq\begin{cases}\mathrm{Aut}_{\mathbb{C}\text{-alg}}(\mathbb{C}[t,t^{-1}])\ltimes PGL_2(\mathbb{C}[t,t^{-1}]) & \text{when } n=2\\ (\langle\tau\rangle\times \mathrm{Aut}_{\mathbb{C}\text{-alg}}(\mathbb{C}[t,t^{-1}]))\ltimes PGL_n(\mathbb{C}[t,t^{-1}]) & \text{when } n\geq 3,\end{cases}$$

where τ is the involutive automorphism induced by the Dynkin diagram automorphism of A_{n-1}. More precisely, τ is defined by $\tau(e_i)=-e_{n-i}$ and $\tau(f_i)=-f_{n-i}$ $i=1,\cdots,n$, $\tau(e_n)=-e_n$ and $\tau(f_n)=-f_n$. $PGL_n(\mathbb{C}[t,t^{-1}])$ acts on $\mathfrak{sl}_n(\mathbb{C}[t,t^{-1}])$ by conjugation.

From now on we will let $R=\mathbb{C}[t,t^{-1}]$. Let θP be an element of $\mathrm{Aut}_{\mathbb{C}}(\mathfrak{sl}_n(R))$, where $\theta\in\langle\tau\rangle\times\mathrm{Aut}_{\mathbb{C}\text{-alg}}(R)$ and $P\in PGL_n(R)$. θP is of order $k(k\geq 2)$ if and only if $\theta^k=1$ and $(\theta^{k-1}\cdot P)\cdots(\theta\cdot P)P=I_n$ in $PGL_n(R)$, where $\theta\cdot$ note the action of θ on $PGL_n(R)$. Let θP_1 and θP_2 be elements of order k. θP_1 is conjugate to θP_2 under $PGL_n(R)$ i.e. $\theta P_1=Q^{-1}(\theta P_2)Q$ for some $Q\in PGL_n(R)$, if and only if $P_1=(\theta\cdot Q^{-1})P_2Q$ for some $Q\in PGL_n(R)$. The condition $(\theta^{k-1}\cdot P)\cdots(\theta\cdot P)P=I_n$

(resp. $P_1=(\theta \cdot Q^{-1})P_2Q$) coincides with the cocycle condition (resp. the coboundary condition) of the group cohomology $H^1(\mathbb{Z}_k, PGL_n(R))$ under the action of θ (=a generator of $\mathbb{Z}_k$) on $PGL_n(R)$.

Let σ (resp. ε_2) be the automorphism of R induced by $t \mapsto t^{-1}$ (resp. $t \mapsto -t$), then the set $\{\sigma, \varepsilon_2, \tau, \tau\sigma, \tau\varepsilon_2\}$ is a set of representatives of the conjugacy classes of order 2 of $\langle\tau\rangle \times \mathrm{Aut}_{\mathbb{C}\text{-alg}}(R)$. Let ε_k be the automorphism of R induced by $t \mapsto \zeta_k t$ (ζ_k =k-th primitive root of unity), then the set $\{\varepsilon_k, (\varepsilon_k)^2, \cdots (\varepsilon_k)^{(k-1)/2}\}$ is a set of representatives of the conjugacy classes of order k(k is an odd prime) of $\langle\tau\rangle \times \mathrm{Aut}_{\mathbb{C}\text{-alg}}(R)$.

5. We determine some cohomologies $H^1(\mathbb{Z}_k, PGL_n(R))$ in the following situation:(1) k=2 (a) trivial action; (b) σ-action; (c) ε_2-action; (d) τ-action; (e) $\tau\sigma$-action; (f) $\tau\varepsilon_2$-action; (2) k= odd prime (a) trivial action; (b) ε_k-action. At first, we determine $H^1(\mathbb{Z}_k, GL_n(R))$ with the above actions, using Grothendick's "theory of descent" and the normalization of symmetric bilinear forms over R-modules.

Theorem 2. $H^1(\mathbb{Z}_k, GL_n(R))$ is: (1)-(a) $\left\{ I_{a,b} = \begin{bmatrix} I_a & \\ & -I_b \end{bmatrix} \;\middle|\; a+b=n \right\}$; (b) $\left\{ J_{a,b,c} = \begin{bmatrix} I_a & & \\ & tI_b & \\ & & -I_c \end{bmatrix},\ J^*_{a,b,c} = \begin{bmatrix} I_a & & \\ & -tI_b & \\ & & -I_c \end{bmatrix} \;\middle|\; a+b+c=n \right\}$; (c) $\{ I_n \}$; (d) $\left\{ K_1 = \begin{bmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{bmatrix},\ K_2 = \begin{bmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ t & 1 & \end{bmatrix} \right\}$; (e) $\{ K_1 \}$; (f) $\{ K_1 \}$; (2)-(a) $\left\{ I_{a_0,\cdots,a_{k-1}} = \begin{bmatrix} I_{a_0} & & & \\ & \zeta_k I_{a_1} & & \\ & & \ddots & \\ & & & \zeta_k I_{a_{k-1}} \end{bmatrix} \;\middle|\; \sum_{m=0}^{k-1} a_m = 0 \right\}$; (b) $\{ I_n \}$.

Since the sequence $1 \to R^* \to GL_n(R) \to PGL_n(R) \to 1$ is exact, the sequence $H^1(\mathbb{Z}_k, GL_n(R)) \to H^1(\mathbb{Z}_k, PGL_n(R)) \to H^2(\mathbb{Z}_k, R^*)$ is exact (ref.[4]).

Using this exact sequence, we can determine $H^1(\mathbb{Z}_k, PGL_n(R))$.

Theorem 3. $H^1(\mathbb{Z}_k, PGL_n(R))$ is : (1)-(a) $\{ I_{a,b} \mid a+b=n,\ a\geqslant b\}$ (n=odd), $\left\{ I_{a,b}, \begin{bmatrix} 0 & t^{-1} & & \\ 1 & 0 & & \\ & & \ddots & \\ & & 0 & t^{-1} \\ & & 1 & 0 \end{bmatrix} \middle|\ a\geqslant b \right\}$ (n=even); (b) $\{ J_{a,b,c}, J^*_{d,e,f} \mid a\geqslant b+c,\ d\geqslant e+f,\ d\neq 0 \}$; (c) $\{ I_n \}$; (d) $\{ K_1, K_2\}$ (n=odd), $\left\{ K_1, K_2, K_3= \begin{bmatrix} & & -1 & 0 \\ & \iddots & 0 & 1 \\ -1 & 0 & & \\ 0 & 1 & & \end{bmatrix} \right\}$(n=even); (e) $\{ K_1 \}$ (n=odd), $\left\{ K_1, K_3, K_4= \begin{bmatrix} & & 1 & 0 \\ & \iddots & 0 & t^{-1} \\ 1 & 0 & & \\ 0 & t^{-1} & & \end{bmatrix}, K_5= \begin{bmatrix} & & -1 & 0 \\ & \iddots & 0 & t^{-1} \\ -1 & 0 & & \\ 0 & t^{-1} & & \end{bmatrix} \right\}$(n=even); (f) $\{ K_1 \}$; (2)-(a) $\{ I_{a_0,\cdots,a_{k-1}} \mid (a_0,\cdots,a_{k-1})$ runs a set of representatives of the equivalence relation generated by $(a_0,\cdots,a_{k-1})\sim(a'_0,\cdots,a'_{k-1}) \Leftrightarrow a'_0=a_1,\cdots,a'_{k-2}=a_{k-1},\ a'_{k-1}=a_0 \}$ (When k is not a divisor of n);

$\left\{ I_{a_0,\cdots,a_{k-1}}, \begin{bmatrix} L & & \\ & \ddots & \\ & & L \end{bmatrix} \middle|\ L= \begin{bmatrix} & & t^{-j} \\ 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} \right.$ k×k matrix and $j=1,\cdots,(k-1)/2 \Big\}$ (When k is a divisor of n); (b) $\{ I_n \}$.

6. As an application of Theorem 3, we classify all automorphisms of prime order of $\mathfrak{g}$ up to conjugacy. In this note, we only state the case n=2, other cases are found in [3].

Theorem 4. Let $\mathfrak{g}=\mathfrak{g}(A_1^{(1)})$. A complete set of representatives of automorphisms of prime order k up to conjugacy in Aut($\mathfrak{g}$) is the following: (1) k=2 (a) $e_1\mapsto -e_1,\ f_1\mapsto -f_1,\ e_2\mapsto -e_2,\ f_2\mapsto -f_2$; (a') $e_1\mapsto e_2,\ f_1\mapsto f_2,\ e_2\mapsto e_1,\ f_2\mapsto f_1$; (b) $e_1\mapsto -e_1,\ f_1\mapsto -f_1,\ e_2\mapsto \frac{1}{2}[[f_2,f_1],f_1],\ f_2\mapsto \frac{1}{2}[[e_2,e_1],e_1]$; (b') $e_1\mapsto e_1,\ f_1\mapsto f_1,\ e_2\mapsto -\frac{1}{2}[[f_2,f_1],f_1],\ f_2\mapsto -\frac{1}{2}[[e_2,e_1],e_1]$; (b'') $e_1\mapsto f_2,\ f_1\mapsto e_2,\ e_2\mapsto f_1,\ f_2\mapsto e_1$; (c) $e_1\mapsto e_1,\ f_1\mapsto f_1,\ e_2\mapsto -e_2,\ f_2\mapsto -f_2$; (2) $k\geqslant 3$ Put $\zeta=\zeta_k$ (a) For $a=1,\cdots,(k-1)/2$, $e_1\mapsto \zeta^a e_1,\ f_1\mapsto \zeta^{-a} f_1,\ e_2\mapsto \zeta^a e_2,\ f_2\mapsto \zeta^{-a} f_2$

(b) For $b=1,\cdots,(k-1)/2$, $e_1 \mapsto e_1$, $f_1 \mapsto f_1$, $e_2 \mapsto \zeta^b e_2$, $f_2 \mapsto \zeta^{-b} f_2$.

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PROCEEDINGS OF THE 19TH
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LINK GRAPHS AND TILED ORDERS*

Hisaaki FUJITA

In [4], B. J. Müller introduced the concept of links between prime ideals in fully bounded noetherian rings to study localizability of semiprime ideals. Recently in [5], he initiated a detailed study of the link graphs. In this note we shall announce some results concerning link graphs of tiled orders over a local Dedekind domain. Detailed proofs of them can be found in [1], [2].

First, we shall recall links between prime ideals of a fully bounded noetherian ring of Krull dimension one. Let P and Q be nonzero prime (or maximal) ideals of a (fully) bounded noetherian ring of Krull dimension one. Then there exists a link from P to Q if $P \cap Q \supsetneq QP$ holds. (See [4].)

Let R be a local Dedekind domain with the maximal ideal πR and the quotient ring K. Let Λ be a tiled R-order in $(K)_n$ (i.e., an R-order in the full $n \times n$ matrix ring $(K)_n$ over K which contains n orthogonal idempotents of $(K)_n$). Since Λ is finitely generated as an R-module and R is local, Λ is a semiperfect bounded noetherian prime ring of Krull dimension one. By [3, Lemma 1.1], we may assume $\Lambda = (\pi^{\lambda_{ij}}R) \subset (R)_n$. Let Λ be basic and for each $1 \leq k \leq n$ put $M_k = (\pi^{m_{kij}}R)$ where $m_{kij} =$ 1 (if $i = j = k$) and λ_{ij} (otherwise). Then $M_1, \ldots, M_n$ are the maximal ideals of Λ.

* This note is a summary of [1] and [2].

Proposition 1. Let Λ be a basic tiled R-order in $(K)_n$ and let $Q(\Lambda)$, be the quiver of Λ defined by A. Wiedemann and K. W. Roggenkamp [7]. Then there is a link from M_i to M_j if and only if there is an arrow from i to j in $Q(\Lambda)$.

If global dimension of Λ is finite, $Q(\Lambda)$ contains no loops. Hence as an easy consequence of the proposition, we obtain

Corollary 2. If $\mathrm{gl.dim}(\Lambda) < \infty$, all maximal ideals of Λ is idempotent.

An ideal of a ring is said to be eventually idempotent if some power of it is idempotent. In connection with above fact, we note the following proposition.

Proposition 3. All maximal ideals of Λ are eventually idempotent.

In what follows we shall confine ourselves to tiled R-orders between $(R)_n$ and $(\pi R)_n$ (i.e., $(R)_n \supset \Lambda \supset (\pi R)_n$).

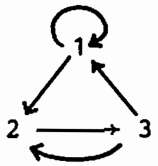
Theorem 4. Let Λ be a basic tiled R-order between $(R)_n$ and $(\pi R)_n$, $Q(A)$ the quiver of $R/\pi R$-algebra $A = \Lambda/(\pi R)_n$ and $M_1, \ldots, M_n$ the maximal ideals of Λ. Then there is a link from M_i to M_j if and only if there is an arrow from i to j in $Q(A)$, or else i is a non-domain and j is a non-range in $Q(A)$.

We shall give an example to illustrate the contents of the theorem.

Example 5. Let $\Lambda = \begin{pmatrix} R & \pi R & \pi R \\ \pi R & R & \pi R \\ \pi R & R & R \end{pmatrix}$ and $A = \Lambda/(\pi R)_3$. Then the quiver of A is given by

$$\begin{array}{ccc} & 1 & \\ & & \\ 2 & \longrightarrow & 3 \end{array} .$$

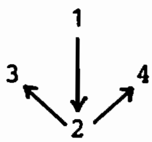
The non-domains of $\mathcal{Q}(A)$ are $\{1, 3\}$ and the non-rages of $\mathcal{Q}(A)$ are $\{1, 2\}$. So, the link graph of Λ is given by



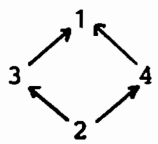
Let Λ be a tiled R-order between $(R)_n$ and $(\pi R)_n$ and $A = \Lambda/(\pi R)_n$. Then the quiver of A has the full information of Λ. However, as for the link graph, there exist tiled R-orders Λ and Γ between $(R)_n$ and $(\pi R)_n$ with the same link graph, but the quiver of $A = \Lambda/(\pi R)_n$ is different from that of $B = \Gamma/(\pi R)_n$.

Example 6. Let $\Lambda = \begin{pmatrix} R & \pi R & \pi R & \pi R \\ R & R & \pi R & \pi R \\ R & R & R & \pi R \\ R & R & \pi R & R \end{pmatrix}$ and $\Gamma = \begin{pmatrix} R & R & R & R \\ \pi R & R & \pi R & \pi R \\ \pi R & R & R & \pi R \\ \pi R & R & \pi R & R \end{pmatrix}$.

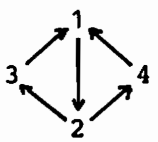
Then $\mathcal{Q}(A)$ and $\mathcal{Q}(B)$ are given by



and



Hence by Theorem 4, the link graphs of Λ and Γ are given by



The relationship between such Λ and Γ is clarified by the following theorem.

Theorem 7. Let Λ and Γ be basic tiled R-orders between $(R)_n$ and $(\pi R)_n$. Then the following statements are equivalent.

(1) Λ is isomorphic with Γ as rings.

(2) The link graphs of Λ and Γ are equal except for the numbering of the vertices.

(3) $\Gamma = u\Lambda u^{-1}$ for some regular element $u \in (R)_n$.

For a tiled R-order $\Lambda = (\Lambda_{ij})$ between $(R)_n$ and $(\pi R)_n$, let $d(\Lambda)$ denote the number of Λ_{ij}'s which are equal to πR.

Corollary 8. Under the same assumption of the theorem, if Λ is isomorphic with Γ, then $d(\Lambda) = d(\Gamma)$.

Proposition 9. Let Λ and Γ be basic tiled R-orders between $(R)_n$ and $(\pi R)_n$ with the same link graph, and put $A = \Lambda/(\pi R)_n$. Then if the quiver of A is disconnected, then $\Lambda = \Gamma$.

Next, we shall state some results on global dimension of tiled R-orders between $(R)_n$ and $(\pi R)_n$.

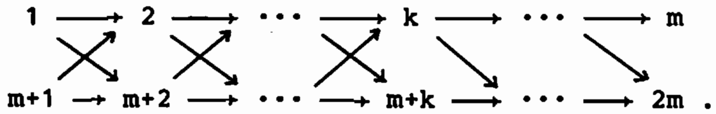
Proposition 10. Let Λ be a basic tiled R-order between $(R)_n$ and $(\pi R)_n$ and put $A = \Lambda/(\pi R)_n$. Then if the quiver of A is disconnected, $\mathrm{gl.dim}(\Lambda) = \infty$.

Proposition 11. Under the same assumption as Prop. 10, if the quiver of A is a tree, $\mathrm{gl.dim}(\Lambda) \leq 3$.

Remark. Let the quiver of A be a tree. Then it follows from the proof of Prop. 11 that $\mathrm{gl.dim}(\Lambda) \leq 2$ if and only if $Q(A)$ has a unique source or a unique sink. This fact is a special case of [6, Theorem].

Finally, we shall give an example of an ascending chain of tiled R-orders between $(R)_n$ and $(\pi R)_n$ whose global dimensions are increasing.

Example 12. Let $m \geq 2$, $n = 2m$ and $1 \leq k \leq m - 1$, and let Λ_k be the basic tiled R-order between $(R)_n$ and $(\pi R)_n$ such that the quiver of $\Lambda_k/(\pi R)_n$ is given by



Then $\Lambda_1 \subset \Lambda_2 \subset \cdots \subset \Lambda_{m-1}$, $gl.dim(\Lambda_k) = k + 3$ for $1 \leq k \leq m - 2$ and $gl.dim(\Lambda_{m-1}) = \infty$.

Concluding remark. Utilizing above results, I have written a program to compute a list of all non-isomorphic tiled R-orders between $(R)_n$ and $(\pi R)_n$. If $n = 5$ (resp. 6), there are 45 (resp. 244) non-isomorphic ones. Let $G(n) = Max\{gl.dim(\Lambda) \mid \Lambda$ is a tiled R-order such that $(R)_n \supset \Lambda \supset (\pi R)_n$ and $gl.dim(\Lambda)$ is finite$\}$. Then $G(n) = 1, 2, 3, 3, 4$ where $n = 2, 3, 4, 5, 6$, respectively.

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COMPRESSIBLE ALGEBRAS

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Dedicated to Professor Hisao Tominaga on his 60th birthday

1. Introduction.

For a ring R with identity, R is called *compressible* if $Z(eRe) = eZ(R)$ for every idempotent e of R, where $Z(R)$ denotes the center of the ring R. When every idempotent is central, then obviously R is compressible; thus, in some sense, compressibility is a measure of centrality of idempotents.

The concept of compressibility was introduced by S.K. Berberian in [4] and several classes of rings related to operator algebras were shown to be compressible. In response to a question of Berberian, A. Page [10] gave a partial affirmative answer for the von Neumann regular ring case, that is, for any n the $n \times n$ matrix ring over a regular compressible ring is compressible. But G.M. Bergman [5] constructed a (non-commutative) integral domain, satisfying a polynomial identity, for which the 2×2 matrix ring over the domain is not compressible, thereby showing that compressibility need not be a Morita invariant property.

However, E.P. Armendariz and J.K. Park [1], [2] proved that separable algebras, regular P.I. group algebras and prime P.I. group algebras are compressible. Also in [2] they observed the compressibility of some interesting classes of rings such as biregular rings and skew group rings, etc.

This paper is in final form and no version of it will be submitted for publication elsewhere.

Furthermore, in [3] E.P. Armendariz, H.K. Koo and J.K. Park showed that every semiprime group algebra is compressible. Indeed, they proved that every $n \times n$ matrix ring over a semiprime group algebra is compressible. By this fact idempotents of semiprime group algebras may be considered as being almost central.

For the non-semiprime group algebra case, K. Motose [8] constructed a noncompressible group algebra of a finite group. This example, coupled with Bergman's example shows the limitations of compressibility. But, however, K. Motose [8] found some classes of non semiprime group algebras which are compressible.

In this semi-expository paper, after recalling what we consider the interesting results and examples already done by several authors, we establish the compressibility of endomorphism rings of projective modules over commutative rings using the same line of investigation in [1], [10]. Also we consider the possibility of characterizing separable algebras via compressibility.

For compressible endomorphism rings, we provide some examples. As we see later, these examples show that the result of Page as well as those of Armendariz and Park may not be extended beyond the endomorphism ring of a projective module with finite rank.

We can prove, however, that the endomorphism ring of a projective module (not necessarily of finite rank) over a commutative Noetherian ring or over a commutative domain is always compressible.

For the possibility of characterizing separable algebras via compressibility, we give some compressible but not separable algebras which suggests other additional strong hypotheses will be necessary.

2. Semiprime Compressible Rings.

In this section we collect some interesting results and examples about semiprime compressible rings which are due to several people.

We start with the following

Proposition 1. [3] *Let R be a semiprime ring and let $Q(R)$ be the maximal right quotient ring of R. If $Z(R) = Z(Q(R))$, then R is compressible.*

Corollary 2. *(a) If R is semiprime, then $Q(R)$, the Martindale quotient ring $Q_0(R)$, and the extended central closure $RZ(Q)$ are compressible.*

(b) A semiprime rationally complete ring is compressible.

(c) Every regular right self-injective ring is compressible.

(d) If R is a semiprime P.I.*-ring and $Z(R)$ is self-injective, then R is compressible.*

A ring is called an *I-ring* if every non-nil right ideal contains a non-zero idempotent.

By Proposition 1, we are able to characterize prime compressible I-rings.

Proposition 3. [3] *If R is a prime I-ring, then R is compressible if and only $Z(R) = Z(Q(R))$.*

For the compressibility of regular Baer rings, D. Castella obtained the following via the relation between $Z(R)$ and $Z(Q(R))$.

Theorem 4. [6] *Let R be a regular Baer ring without a non-zero central abelian idempotent. Then R is compressible if and only if $Z(R) = Z(Q(R))$.*

Corollary 5. *Let R be a regular Baer ring. Then R is compressible if and only if $R = R_1 \oplus R_2$ with R_1 reduced and $Z(R_2) = Z(Q(R_2))$.*

In a reduced ring, every idempotent is central and so it is compressible. Also every abelian regular ring is compressible. Hence we have a naturally raised question: "Is every regular P.I.-ring compressible?". But the answer is negative, as Y. Hirano's example in [3] shows.

A ring R is called *biregular* if for each $a \in R$ the two-sided ideal RaR is generated by a central idempotent.

Theorem 6. [3] *(a) Every biregular ring is compressible; in particular, every simple ring is compressible.*

(b) Every finitely generated semiprime algebra over a commutative regular ring is compressible.

(c) Every semiprime finite centralizing extension of a commutative regular ring is compressible.

As a direct consequence of Theorem 6, a semiprime group ring $R[G]$ with G finite, as well as the $n \times n$ matrix ring $\mathrm{Mat}_n(R)$ over a commutative regular ring R, is compressible.

Moreover, for the matrix ring case, A. Page obtained the compressibility of the $n \times n$ matrix ring over a regular compressible ring.

Theorem 7. [10] *For any n, the $n \times n$ matrix ring over a regular compressible ring is compressible.*

Because of Page's result, combined with the compressibility of a semiprime group ring of a finite group over a commutative regular ring, we raise the question:

Question 8. Let R be a regular compressible ring and G be a finite group whose order is invertible in R. Then is $R[G]$ compressible?

Theorem 7 is a partial affirmative answer for a question raised in [4]. But in answering negatively to the question, Bergman [5] constructed a (non-commutative) P.I. domain such that the 2×2 matrix ring over the domain is not compressible. This domain given by Bergman eliminates a large class of rings, prime Goldie rings, for which compressibility is a Morita invariant property.

In contrast to P.I. domain given by Bergman, Armendariz and Park showed that every Azumaya algebra over a commutative ring is compressible. In particular, the matrix ring $\mathrm{Mat}_n(R)$ is compressible whenever R is commutative.

Theorem 9. [1] *Every Azumaya algebra is compressible.*

In the following example, due to Armendariz, there is a noncompressible prime regular ring which eliminates various potential conjecture about compressibility when combined with Bergman's example.

Example 10. [4] Let H be an infinite dimensional complex Hilbert space, B the $*$-algebra of all bounded linear operators on H, F the ideal of operators of finite rank. For a proper subfield $\mathbb{K}$ of the complex field $\mathbb{C}$ that is closed under complex conjugation, let $A = F + \mathbb{K}1$ be the set of all operators $x = \lambda 1 + a$ with $\lambda \in \mathbb{K}$ and $a \in F$, where 1 is the identity operator of H. Then A is prime, unit regular but not compressible.

However, for the group algebra case, every semiprime group algebra is always compressible as Armendariz, Koo and Park have shown.

Theorem 11. [3] *For a positive integer n, every $n \times n$ matrix ring over a semiprime group algebra $K[G]$ over a field K is compressible. In particular, every semiprime group algebra $K[G]$ over a field K is compressible.*

Without the condition of semiprimeness, a group algebra is not always compressible.

Example 12. [8] (1) The group algebra $K[S_4]$ is not compressible, where K is a finite field of 4 elements.

(2) Let K be a finite field of 4 elements. For the quarternion group Q of order 8, *i.e.*,

$$Q = \langle x, y \mid x^4 = 1 \ , \ x^2 = y^2 \ , \ y^{-1}xy = x^{-1} \rangle \ ,$$

let g be the automorphism of Q defined by $x^g = xy$ and $y^g = x$. Now let T be the semidirect product of Q by a cyclic group $\langle g \rangle$ of order 3 with respect to this action. Then T is isomorphic to $SL(2,3)$ and $e = 1 + g + g^2$ is an idempotent in the group algebra $K[T]$. In this case $Z(eK[T]e) \neq eZ(K[T])$ and hence $K[T]$ is not compressible.

In spite of Example 12, Motose proved that some non semiprime group algebras can be compressible.

Theorem 13. [8] *The group algebra $K[G]$ of a finite p-nilpotent group G over a field K is compressible.*

We close this section with the following example which shows that the group ring $R[G]$ may not be compressible even though the order of G is invertible and R is a P.I. domain.

Example 14. Let R be the P.I. domain given by Bergman such that $\mathrm{Mat}_2(R)$ is not compressible. Note that R may be an algebra over the complex number field $\mathbb{C}$. Take $G = S_3$. Then the order of G is invertible in R. Now $R[G] = R\otimes_{\mathbb{C}} \mathbb{C}[G] = R\otimes_{\mathbb{C}}(\mathbb{C}\oplus\mathbb{C}\oplus\mathrm{Mat}_2(\mathbb{C})) = R\oplus R\oplus\mathrm{Mat}_2(R)$. But since $\mathrm{Mat}_2(R)$ is not compressible, $R[G]$ is not compressible.

3. Compressible Endomorphism Rings.

Continuing the line of investigation in Theorems 7 and 9, in this section we concentrate on the compressibility of endomorphism rings of projective modules.

Recall that a *Morita context* consists of two rings R and S, two bimodules ${}_SM_R$ and ${}_RN_S$ and two bimodule homomorphisms (called pairings) $(\ ,\ ) : N \otimes_S M \to R$ and $[\ ,\ ] : M \otimes_R N \to S$ satisfying the associativity conditions $n[m, n'] = (n, m)n'$ and $m(n, m') = [m, n]m'$ for m, m' in M and n, n' in N. A Morita context $(R,\ {}_SM_R,\ {}_RN_S,\ S)$ is *nondegenerate* if the modules ${}_SM$, M_R, ${}_RN$, N_S and two pairing $(\ ,\ )$, $[\ ,\ ]$ are faithful. For details, see [9].

Now for a free R-module F, we may check easily that the derived

Morita context $(R, {}_AF_R, {}_RF^*_A, A)$ is nondegenerate, where ${}_RF^*_A$ = Hom (F_R, R_R) and $A = \mathrm{End}_R(F)$. In this case, note that two pairings are $(\ ,\ )$ and $[\ ,\ ]$ such that $(f, m) = f(m)$ and $[m, f](x) = mf(x)$ for m, x in F and f in F^*.

The following may already be known. But for completeness we give its proof.

Lemma 15. *The maximal right quotient ring of the column finite matrix ring over a commutative Noetherian ring R is the column finite matrix ring over the maximal quotient ring $Q(R)$ of R with the same rank.*

Proof. Let A be the column finite matrix ring over the commutative Noetherian ring R indexed by a set I, then A is the endomorphism ring of the free R-module F with rank equal to the cardinality of the set I. In this circumstance, we may observe that the derived Morita context $(R, {}_AF_R, {}_RF^*_A, A)$ of the module F_R is nondegenerate. Hence by B.J. Mueller [9, Theorem 19], the maximal quotient context $(Q(R), Q({}_AF_R), Q({}_RF^*_A), Q(A))$ is right normalized. Now since R is Noetherian, the maximal quotient module $Q(F_R)$ of F_R is a free $Q(R)$-module having rank the cardinality of the set I, by K. Louden [7, Corollary 1.10]. Hence the maximal right quotient ring $Q(A)$ of A is the endomorphism ring of the free $Q(R)$-module $Q(F_R)$. Therefore it is the column finite matrix ring over $Q(R)$ indexed by I.

Lemma 16. *Let R be a commutative ring and I be a finitely generated ideal of R satisfying $I^2 = I$. Then $I = eR$ for some idempotent e of R.*

With these preparations, we have one of our main results.

Theorem 17. *The endomorphism ring of free module over a commutative Noetherian ring is compressible.*

Proof. Let A be the endomorphism ring of a free module F over a commutative Noetherian ring R. Then A is the column finite matrix ring over R. To prove that A is compressible, let a matrix E be an idempotent in A. Say $E = (x_{ij})$ with x_{ij} in R and $B = \sum x_{ij}R$. Since $x_{ij} = \sum x_{ik}x_{kj}$, the ideal B satisfies $B = B^2$.

Now since R is commutative Noetherian, the ideal B is finitely generated and so by Lemma 16, $B = eR$ for some idempotent e in R.

By these facts we claim that the ideal AEA and the left annihilator ideal $\ell_A(AEA)$ of AEA satisfies $AEA \cap \ell_A(AEA) = 0$ and $AEA + \ell_A(AEA)$ is dense in A. Since $B = eR$, we have $x_{ij}e = x_{ij}$ for every i, j and so $Ee = E$. Now suppose X is in $AEA \cap \ell_A(AEA)$. Then $X = \sum a_i E b_i$ with a_i, b_i in A. Hence $Xe = \sum a_i E b_i e = \sum a_i E e b_i = \sum a_i E b_i = X$. For indices α, β let $E_{\alpha\beta}$ be the matrix with 1 in (α, β)-position and 0 elsewhere. First we show that $eE_{\alpha\alpha}$ is in AEA for every α. Say $e = x_{ij}a + x_{km}b$ where x_{ij} and x_{km} are entries of E and a, b are in R. Then we have $E_{\alpha i}EE_{j\alpha}a = x_{ij}aE_{\alpha\alpha}$ and $E_{\alpha k}EE_{m\alpha}b = x_{km}bE_{\alpha\alpha}$ and so $eE_{\alpha\alpha} = (x_{ij}a + x_{km}b)E_{\alpha\alpha} = x_{ij}aE_{\alpha\alpha} + x_{km}bE_{\alpha\alpha} = E_{\alpha i}EE_{j\alpha}a + E_{\alpha k}EE_{m\alpha}b$ is in AEA. Now since $X = Xe$ is in $AEA \cap \ell_A(AEA)$ and $eE_{\alpha\alpha}$ is in AEA for every α, we have $XE_{\alpha\alpha} = XeE_{\alpha\alpha} = 0$ for every α. Hence $X = 0$.

Now to prove that $AEA + \ell_A(AEA)$ is dense in A, let X be an element of A such that $X(AEA + \ell_A(AEA)) = 0$. Since $E(1-e) = 0$, we have $(1-e)I$ is in $\ell_A(AEA)$, where I denotes the identity matrix. Hence $X(1-e)I = X(1-e) = 0$ and so $X = Xe$. By our previous result, since $eE_{\alpha\alpha}$ is in AEA for every α, $XeE_{\alpha\alpha} = XE_{\alpha\alpha} = 0$. So $X = 0$.

Finally for the compressibility of A, let t be an element of $Z(EAE)$. Define a map f from $AEA + \ell_A(AEA)$ to A by $f(\sum x_i E y_i + z) = \sum x_i t y_i$. Since $AEA \cap \ell_A(AEA) = 0$, f is well defined by the technique of Armendariz and Park [1, Proposition 1]. In this case, f is an (A, A)-bimodule homomorphism. Now observing that $AEA + \ell_A(AEA)$ is dense in A, there exists q in $Z(Q(A))$ such that $f(x) = qx$ for every x in $AEA + \ell_A(AEA)$. By Lemma 15 $q = q_0 I$ for some q_0 in $Q(R)$. Now $qE = q_0 IE = q_0 E$ and so $qE = q_0(x_{ij}) = (q_0 x_{ij})$ is in A. Thus $q_0 x_{ij}$ is in R. Hence $q_0 e$ is in R. Let $s = q_0 e$. Then s is in $R = Z(A)$ and so $t = qE = q_0 eE = sE$ is in $EZ(A)$. So A is compressible.

Every corner of a compressible ring is always compressible. Hence we get the following immediately.

Corollary 18. *The endomorphism ring of a projective module over a commutative Noetherian ring is compressible.*

As we already mentioned in the previous section, Bergman [5] constructed a noncompressible 2×2 matrix ring $\mathrm{Mat}_2(R)$ over a P.I. domain R; for the commutative domain case, we have

Theorem 19. *The endomorphism ring of a projective module over a commutative domain is compressible.*

Proof. In view of Theorem 17 and Corollary 18, it is enough to consider compressibility for the free module case. Let A be the endomorphism ring of a free module F over a commutative domain R. Then A is the column finite matrix ring over R. Now let E be a nonzero idempotent matrix in A. Say $E = (x_{ij})$. We claim that the ideal $B = \sum x_{ij} R$ is equal to R. For each index k, let A_k be the ideal of R generated by the entries of the k^{th} column of E. Then A_k is finitely generated. Since $E^2 = E$, we have $BA_k = A_k$. Now assume to the contrary that $B \neq R$. Then there is a maximal ideal M of R containing B. Let S be the complementary set of M in R. Then $S^{-1}BA_k = S^{-1}A_k$. In this case $(S^{-1}B)(S^{-1}A_k) = S^{-1}A_k$. Note that $S^{-1}M$ is maximal in $S^{-1}R$, $S^{-1}B \neq 0$ and $S^{-1}A_k$ is finitely generated. Now let $J(S^{-1}R)$ be the Jacobson radical. Then since $S^{-1}R$ is a local ring, we have $S^{-1}B \subseteq S^{-1}M \subseteq J(S^{-1}R)$. So by the Nakayama lemma

$S^{-1}A_k = 0$. Hence $A_k = 0$ for all k. Hence $E = 0$ — a contradiction. Therefore $B = R$. Now by K. Louden [7, Corollary 1.10] the maximal quotient module $Q(F_R)$ of F_R is a free module over the maximal quotient ring $Q(R)$ of R. As in Lemma 15, $Q(A)$ is the endomorphism ring of $Q(F_R)$.

Finally for the compressibility of A, we have $AEA \cap \ell_A(AEA) = 0$ and $AEA + \ell_A(AEA)$ is dense in A by the same method as in Theorem 17. Hence if t is in $Z(EAE)$ then $t = Eq$ for some q in $Q(R)$. So $(x_{ij})q$ is in A and hence $x_{ij}q$ is in R for every i, j. Therefore $Bq \subseteq R$ and so q is in R. Thus A is compressible.

It would be interesting to know if Theorems 17 and 19 extend to commutative (semiprime) Goldie rings.

4. Examples.

As we already mentioned, every separable algebra is compressible. In particular the $n \times n$ matrix ring over a commutative ring is compressible. This result was partially extended in Theorems 17 and 19. But in general the endomorphism ring of a free module over a commutative ring will not be compressible.

Example 20. Consider the polynomial ring $\mathbb{Z}[x_1, x_2, \ldots]$ over the ring $\mathbb{Z}$ of integers with commuting indeterminates $x_1, x_2, \ldots$. Let I be the ideal generated by $x_1x_n^2 - x_n$, x_ix_j with $n \geq 2$, $i, j \geq 2$ and $i \neq j$. Then $R = \mathbb{Z}[x_1, x_2, \ldots]/I$ is a commutative ring. Now let A be the endomorphism ring of a free R-module of countably infinite rank. Then the diagonal matrix E in A with $\overline{x_1x_{i+1}}$ in (i, i)-position is an idempotent, where $^-$ denotes the canonical image in R. For convenience of our notation, $\overline{x_1}^{-1}E$ denotes the diagonal matrix with $\overline{x_{i+1}}$ in (i, i)-position. Then in this case EAE is a commutative ring and $Z(EAE) = EAE$. Note that $\overline{x_1}^{-1}E = E(\overline{x_1}^{-1}E)E$ is in EAE.

But we claim that $\overline{x_1}^{-1}E$ is not in $EZ(A)$. For, if $\overline{x_1}^{-1}E$ is in $EZ(A)$, then there exists a in R such that $\overline{x_1x_2}a = \overline{x_2}$, $\overline{x_1x_3}a = \overline{x_3}, \ldots, \overline{x_1x_n}a = \overline{x_n}, \ldots$. Say $a = f(\overline{x_1}, \overline{x_2}, \ldots, \overline{x_m})$ with $f(x_1, x_2, \ldots, x_m) \in \mathbb{Z}[x_1, x_2, \ldots, x_m]$. Then since $\overline{x_n} = \overline{x_1x_n}a$ for all $n = 1, 2, 3, \ldots$ and $\overline{x_i}$, $\overline{x_j}$ are orthogonal for $i, j \geq 2$ and $i \neq j$, $f(x_1, x_2, \ldots, x_m) = g(x_1) + u(x_2, x_3, \ldots, x_m)$ with nonzero polynomial $g(x_1)$ in $\mathbb{Z}[x_1]$ and $u(x_2, x_3, \ldots, x_m)$ in $\mathbb{Z}[x_2, x_3, \ldots, x_m]$. Now for $k > m$, we have $\overline{x_k} = \overline{x_1x_k}a = \overline{x_1x_k}g(\overline{x_1})$. Hence $x_1x_kg(x_1) - x_k$ is in I. Wherefore there are $b_1, b_2, \ldots, b_s$ in $\mathbb{Z}$ such that $b_1x_1x_k + b_2x_1^2x_k + \cdots + b_sx_1^sx_k - x_k = h(x_1, x_k)(x_1x_k^2 - x_k)$ for some polynomial $h(x_1, x_k)$ in $\mathbb{Z}[x_1, x_k]$. So if we substitute $x_1x_k^2$ for x_k in $b_1x_1x_k + b_2x_1^2x_k + \cdots + b_sx_1^sx_k - x_k$ then its value should be 0. But this is impossible. Hence $\overline{x_1}^{-1}E$ is not in $EZ(A)$ and so A is not compressible.

The next example shows that Theorem 7 can not be extended beyond the endomorphism ring of a free module with finite rank even for commutative regular rings.

Example 21. Let $\mathbb{C}$ be the field of complex numbers and let R be the set of all sequences from $\mathbb{C}$ which are eventually real. Then R is a commutative regular ring. Let A be the endomorphism ring of a countably infinite direct sum of copies of R. Now for each positive integer k let e_k be the element of R with 1 in the k^{th} position and 0 elsewhere. Then the diagonal matrix E in A with e_n in (n, n)-position is an indempotent. In this case every element of EAE is a diagonal matrix with $e_na_{nn}e_n$ in (n, n)-position for some a_{nn} in R. Hence EAE is commutative and so $Z(EAE) = EAE$. Now let a_{nn} be the sequence with i in the n^{th} position and 0 elsewhere, where $i^2 + 1 = 0$. Then the diagonal matrix in A with $a_{nn}e_n$ in (n, n)-position is in EAE but it is not in ER. Hence A is not compressible.

Finally, the following two examples show that in order for a compressible ring to be separable some additional quite strong conditions are probably necessary.

Example 22. Let F be the free algebra with variables x, y, z and w over the real number field $\mathbb{R}$. Let I be the ideal of F generated by x^2+1, y^2+1, z^2+1, $xy+yx$, $xy-z$, $xw+wx$, $yw+wy$, $zw+wz$ and w^2. Then the ring $A = F/I$ is a local ring and $Z(A) = \mathbb{R}$. So A is compressible because A has only idempotents 0 and 1. It can be checked that A is finitely generated projective over $\mathbb{R}$. But A is not an Azumaya algebra because the Jacobson radical $J(A)$ of A is the ideal generated by $w+I$.

Example 23. Let $\mathbb{Z}$ be the ring of integers and let

$$A = \begin{pmatrix} \mathbb{Z} & 2\mathbb{Z} \\ 2\mathbb{Z} & \mathbb{Z} \end{pmatrix} .$$

Then A is prime and $Z(A) = \mathbb{Z}$. Also A is finitely generated projective over $\mathbb{Z}$ and A is compressible. In this case the enveloping algebra $A \otimes_{\mathbb{Z}} A^{op}$ is not isomorphic to $\mathrm{Hom}_{\mathbb{Z}}(A, A)$. Take f in $\mathrm{Hom}_{\mathbb{Z}}(A, A)$ such that

$$f\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} .$$

But there do not exist $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ in A such that

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = a_1 \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} b_1 + a_2 \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} b_2 + \cdots + a_n \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} b_n .$$

So A is not an Azumaya algebra.

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SIGNIFICANT EXAMPLES IN MODULAR REPRESENTATIONS OF FINITE GROUPS *)

Shigeo KOSHITANI

Throughout this note let G be a finite group, let p be a fixed prime number > 0, and let (F,R,K) be a splitting p-modular system for G (see [1, p.15] or [11, p.47]), namely, R is a complete discrete valuation ring of rank one with a maximal ideal (π) generated by $\pi \in R$ such that $p \in (\pi)$, K is the quotient field of R with characteristic 0, $F = \bar{R} = R/(\pi)$, F is a field of characteristic p, and both F and K are splitting fields for all subgroups of G. We write FG, RG and KG for the group algebras of G over F, R and K, respectively. Here all groups are finite and all modules are finitely generated right modules. By an RG-lattice M we mean an R-free finitely generated RG-module. For such an M, let $\bar{M} = M/M\pi$, so that $\bar{M}$ is an FG-module, and let $M_K = M \otimes_R K$, so that M_K is a KG-module (hence semi-simple).

§1. Problems

Since it is considered that all finite simple groups G have been determined, as the next steps we want to get the following four things. Namely,

(1) the ordinary character table of G,

(2) the decomposition matrix D of G with respect to p,

*) This is a report of my result whose complete version will probably be published elsewhere.

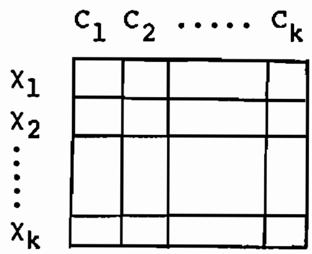
(3) the Loewy structure of all projective indecomposable modules (p.i.m.'s) over FG,

(4) all indecomposable FG-modules.

Concerning these problems we need more explanation. That is,

(1) Let $\{\chi_1, \ldots, \chi_k\}$ be the set of all irreducible K-characters of G, and let $\{C_1, \ldots, C_k\}$ be the set of all conjugacy classes of G (their numbers are always the same k). The ordinary character table

$$(a_{ij})_{i,j} =$$



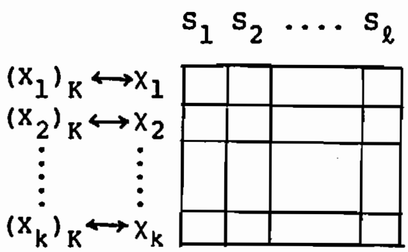
of G is a $k \times k$-matrix over K such that

$$a_{ij} = \chi_i(g_j) \qquad \text{for any } g_j \in C_j$$

and for all $i,j = 1, \ldots, k$ (cf. [3]).

(2) Let $\{S_1, \ldots, S_\ell\}$ be the set of all simple FG-modules. As is well known (see e.g. [4, Chap.I]), there are RG-lattices $X_1, \ldots, X_k$ such that $\{(X_i)_K \mid i=1, \ldots, k\}$ is the set of all simple (irreducible) KG-modules, so that we may assume $(X_i)_K$ affords χ_i for each $i = 1, \ldots, k$. Then the decomposition matrix

$$D = (d_{ij})_{i,j} =$$



of G with respect to p is defined as a $k \times \ell$-matrix over $\mathbb{Z}_{\geq 0}$ such that d_{ij} is the multiplicity of S_j in the composition factors of an FG-module $\overline{X}_i$ for each $i = 1,\dots,k$ and $j = 1,\dots,\ell$. Now, let $\{P_1,\dots,P_\ell\}$ be the set of all p.i.m.'s over FG such that P_j is the projective cover of S_j for each j. Then the Cartan matrix

$$C = (c_{jj'})_{j,j'} = \begin{array}{c|c|c|c|c|} & P_1 & P_2 & \cdots & P_\ell \\ \hline S_1 & & & & \\ \hline S_2 & & & & \\ \hline \vdots & & & & \\ \hline S_\ell & & & & \\ \hline \end{array}$$

of FG is defined as an $\ell \times \ell$-matrix over $\mathbb{Z}_{\geq 0}$ such that $c_{jj'}$ is the multiplicity of S_j in the composition factors of $P_{j'}$, for each $j,j' = 1,\dots,\ell$. Then, it is known ${}^tD \cdot D = C$ where ${}^tD$ is the transposed matrix of D (see e.g. [4, p.67]). This means once we get the matrix D, we automatically get all composition factors of all p.i.m.'s over FG.

(3) Let J be the Jacobson radical of FG. For an FG-module $X \neq 0$ and a positive integer n, let

$$L_n(X) = XJ^{n-1}/XJ^n$$

and this is called the n-th Loewy layer of X. The Loewy series of X is defined as

$$\begin{array}{c} S_{11} \cdots\cdots S_{1,r_1} \\ S_{21} \cdots\cdots\cdots\cdots S_{2,r_2} \\ \vdots \\ S_{m1} \cdots S_{m,r_m} \end{array}$$

where $L_n(X) = S_{n1} \oplus \dots \oplus S_{n,r_n}$ and all S_{nr} are simple FG-modules for each $n = 1,\dots,m$ and m is the Loewy length of X, namely the least positive integer m' with $XJ^{m'} = 0$ (see e.g.

[1, p.174] or [11, pp.25-26]). Therefore, the Loewy series of all p.i.m.'s P_j's are of course more explicit than the Cartan matrix C of FG.

Here, in the present note, we consider the problem (3) for a concrete finite simple group G.

§2. Tools and lemmas

In this section we give several tools and lemmas which are useful for calculating the Loewy series of modules. We use the same notation as in §1.

Lemma 1 (Frobenius Reciprocity, Nakayama Relations, Shapiro's Lemma). Let H be a subgroup of G, and let X and Y be respectively FG- and FH-modules. Then for any non-negative integer n,

$$\mathrm{Ext}_{FG}^{n}(X, Y^{\uparrow G}) \cong \mathrm{Ext}_{FH}^{n}(X_{\downarrow H}, Y),\ \mathrm{Ext}_{FG}^{n}(Y^{\uparrow G}, X) \cong \mathrm{Ext}_{FH}^{n}(Y, X_{\downarrow H})$$

as F-spaces, where $Y^{\uparrow G} = Y \otimes_{FH} FG$, so that $Y^{\uparrow G}$ is an FG-module and $X_{\downarrow H}$ is the restriction of X to FH (so an FH-module).

Proof. See e.g. [2, Proposition 1.12].

Lemma 2. Let X, Y and Z be FG-modules. Then for any non-negative integer n,

$$\mathrm{Ext}_{FG}^{n}(X \otimes_F Y, Z) \cong \mathrm{Ext}_{FG}^{n}(X, Y^* \otimes_F Z)$$

as F-spaces, where $X \otimes_F Y$ is an FG-module by the action $(x\otimes y)g = xg\otimes yg$ for all $x \in X$, $y \in Y$ and $g \in G$, and Y^* is the dual of Y, namely $Y^* = \mathrm{Hom}_F(Y,F)$ and by the action $(y)(\sigma g) = (yg^{-1})\sigma$ for all $y \in Y$, $g \in G$ and $\sigma \in Y^*$, Y^* is an FG-module, too.

Proof. See [2, Lemma 1.4].

Lemma 3 (Thompson). If $d_{ij} \neq 0$ for an irreducible K-character χ_i and a simple FG-module S_j, then there is an RG-lattice M such that $M_K \cong (\chi_i)_K$, $L_1(\overline{M}) = S_j$ and $\{d_{i1} \times S_1, \ldots, d_{i\ell} \times S_\ell\}$ is the set of all composition factors of $\overline{M}$.

Proof. See e.g. [11, I Corollaries 17.4 and 17.5].

For an FG-module X, we call X a trivial source module if X is an indecomposable direct summand of $I_H{}^{\uparrow G}$ for a subgroup H of G, where I_H is the trivial one-dimensional FH-module.

Lemma 4. (1) For any trivial source FG-module X, there is an RG-lattice M such that $\overline{M} \cong X$.

(2) (Scott) If X and Y are trivial source FG-modules (so that there are RG-lattices M and N such that $\overline{M} \cong X$ and $\overline{N} \cong Y$ from (1)), then

$$\mathrm{Hom}_{FG}(X,Y) \cong \mathrm{Hom}_{\overline{RG}}(\overline{M},\overline{N}) \cong \mathrm{Hom}_{RG}(M,N)/[\mathrm{Hom}_{RG}(M,N)]\pi$$

as F-spaces and this implies

$$\dim_F[\mathrm{Hom}_{FG}(X,Y)] = (\chi_M, \chi_N)_G$$

where χ_M and χ_N are respectively the K-characters of G afforded by M_K and N_K, and $(\chi_M, \chi_N)_G$ is the inner product.

Proof. See e.g. [11, II Theorem 12.4 and I Proposition 14.8].

The above Scott's theorem Lemma 4(2) is very powerful for calculating Loewy series of trivial source modules. It has been used for instance by P. Landrock, G. Michler and D. Benson (see the references of [6]).

Lemma 5 (Landrock). For simple FG-modules S and T and any positive integer n,

$$\dim_F[\mathrm{Hom}_{FG}(L_n(P_T),S)] = \dim_F[\mathrm{Hom}_{FG}(L_n(P_{S^*}), T^*)]$$

where P_T and P_{S*} are the projective covers of T and $S*$, respectively.

<u>Proof</u>. See [11, I Lemma 9.10].

§3. <u>Applications</u>

In this section, we shall try to show a little bit how to use the lemmas in §2 for calculating the Loewy series of p.i.m.'s with respect to the following special case

(**) G is the Tits simple group ${}^2F_4(2)'$ of order 17971200 $(= 2^{11}\cdot 3^3\cdot 5^2\cdot 13)$ and $p = 3$

(the complete version of this example can be seen in [10]). So we assume (**) throughout this section. The reason why we have been interested in this special group G is explained in our previous note [6] and the introduction of [9].

G. Hiss obtained the decomposition matrix D of G in his paper [5]. Namely,

<u>Proposition 1</u> (Hiss). The decomposition matrix D of the principal block of G in the situation (**) has the form

$D =$

	I	26	26*	77	124_1	124_2	572	1099_1	1099_2
I	1								
26		1							
26*			1						
78	1			1					
300		1	1		1	1			
325				1	1	1			
624_1		1	1				1		
624_2		1	1				1		
650	1			1			1		
1300_1				1		1		1	
1300_2				1	1				1
2048_1		1	1	1	1	1	1	1	
2048_2		1	1	1	1	1	1		1

and all the other blocks 27, 27^*, 351_0, 351, 351^*, 675, 702_1, 702_2 and 1728 are of defect 0 (so that semi-simple, nothing to do), where $I = I_G$ is the trivial FG- or KG-module and for other simple FG- and KG-modules we denote each of them by its dimension, together with a subscript if there is more that one simple module of the same dimension which is not the dual of the first one.

<u>Proof</u>. See [5].

By [12], G has a maximal subgroup V such that V is a semi-direct product of a group of order 2^{10} by the symmetric group on 3 letters, so that $|G:V| = 2925$. Since the Sylow 3-subgroup of V is cyclic of order 3, by the results of Dade, Janusz and Kupisch (see [4, Chap.VII]), all indecomposable FV-modules in the principal block are

$$I_V, \quad 1_V, \quad \begin{matrix} I_V \\ 1_V \end{matrix}, \quad \begin{matrix} 1_V \\ I_V \end{matrix}, \quad P_{I_V} = \begin{matrix} I_V \\ 1_V \\ I_V \end{matrix}, \quad P_{1_V} = \begin{matrix} 1_V \\ I_V \\ 1_V \end{matrix}$$

where I_V is the trivial FV-module and 1_V is the other simple FV-module of F-dimension one in the principal block. Then, by [5, Lemma 1],

$$(5) \qquad \chi_{I_V}{}^{\uparrow G} = \chi_I + \chi_{624_1} + \chi_{624_2} + \chi_{650} + \chi_{351_0} + \chi_{675}$$

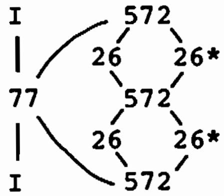
Hence, by Proposition 1,

$$(6) \qquad I_V{}^{\uparrow G} = X \oplus 351_0 \oplus 675$$

for an FG-module X such that all composition factors of X are

(7) I, I, 26, 26, 26^*, 26^*, 77, 572, 572 and 572.

By so complicated computations and by (5), (6) and (7), X has the form

(8) $X =$ 

,

so that X is a trivial source module. Hence, by (5),(6) and Lemma 4(2), $\dim_F[\mathrm{End}_{FG}(X)] = 4$. By (2) and Proposition 1, all composition factors of P_I are

(9) I, I, I, 77, 77 and 572.

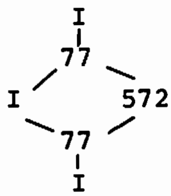
By Lemma 1, there is an exact sequence

(10) $$0 \longrightarrow Y \longrightarrow I_V{}^{\uparrow G} \longrightarrow I \longrightarrow 0$$

of FG-modules. Thus, for a simple FG-module S, (10) induces a long exact sequence

$$\begin{aligned} & 0 \longrightarrow \mathrm{Hom}_{FG}(I,S) \longrightarrow \mathrm{Hom}_{FG}(I_V{}^{\uparrow G},S) \longrightarrow \mathrm{Hom}_{FG}(Y,S) \\ (11) \quad & \longrightarrow \mathrm{Ext}^1_{FG}(I,S) \longrightarrow \mathrm{Ext}^1_{FG}(I_V{}^{\uparrow G},S) \longrightarrow \mathrm{Ext}^1_{FG}(Y,S) \\ & \longrightarrow \cdots\cdots \end{aligned}$$

of F-spaces. Hence, by (11), (9) and Lemma 1, we get

(12) $P_I =$ 

.

As a matter of fact, by Proposition 1 and Lemma 3, we have known $\mathrm{Ext}^1_{FG}(I,77) \neq 0$. Anyway, from (12) and Lemma 5, we get

$$\dim_F[\mathrm{Hom}_{FG}(L_n(P_{77}), I)] = \begin{cases} 1 & \text{if } n = 2 \text{ or } 4 \\ 0 & \text{otherwise} \end{cases}$$

$$\dim_F[\mathrm{Hom}_{FG}(L_n(P_{572}), I)] = \begin{cases} 1 & \text{if } n = 3 \\ 0 & \text{otherwise .} \end{cases}$$

By complicated calculations (which are more than 50 times of the above?), and by making use of [7], [8] and [9], we finally obtain

Theorem (Koshitani [10]). Let $p=3$. Then the Loewy and socle series of the p.i.m.'s over $F[{}^2F_4(2)']$ and $F[{}^2F_4(2)]$ are completely determined. In particular, their Loewy lengths are 9.

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FIFTEEN YEARS OF COMMUTATIVE ALGEBRA (1971-85)

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§1. INTRODUCTION.

In this historical survey we will consider some of the main developments of the theory of noetherian rings during the period 1971-85. Since this report is intended for non-specialists we start with some basic definitions.

A ring will always mean a commutative ring with unit element. A local ring (R,m) will mean a noetherian ring R with a unique maximal ideal m. Spec(A) denotes the set of prime ideals of the ring A. We introduce a topology on Spec(A) by taking the sets $V(I) := \{P \in \mathrm{Spec}(A) : P \supseteq I\}$, where I runs over the set of all ideals, as the closed sets; this is called Zariski topology.

The dimension of a ring A is defined by

$$\begin{aligned} \dim A &:= \text{the supremum of the length of prime ideal chains} \\ &= \sup \{s : P_0 \supset P_1 \supset \dots \supset P_s,\ P_i \in \mathrm{Spec}(A)\}. \end{aligned}$$

The height of a prime ideal P, denoted by ht P, is defined by

$$\begin{aligned} \mathrm{ht}\, P &:= \sup \{s : P = P_0 \supset P_1 \supset \dots \supset P_s\} \\ &= \dim A_P . \end{aligned}$$

For a noetherian ring, one can show that ht P can not exceed the number of generators of P, hence ht P is always finite. The height of an arbitrary ideal I is defined by

$$\mathrm{ht}\, I := \min \{\mathrm{ht}\, P : P \supseteq I\}.$$

The support of an A-module M is defined by

This paper is in final form and no version of it will be submitted for publication elsewhere.

$\text{Supp } M := \{P \in \text{Spec}(A): \ M_P \neq 0\}.$

If M is finitely generated and I = ann M, the annihilator of M, then Supp M = V(I) ≅ Spec(A/I). The dimension of M is defined by dim M := dim A/I.

When (A,m) is a local ring and dim M = d, a system of elements $a_1,\ldots,a_d$ of m such that $\ell(M/a_1M + \ldots + a_dM) < \infty$ (where ℓ denotes the length of module) is called a system of parameters (s.o.p. for short) of M. An ideal of A generated by an s.o.p. of A is called a parameter ideal.

Part I. Classification of local rings

There is the following hierarchy of local rings:

regular ⇒ complete intersection ⇒ Gorenstein ⇒ Cohen-Macaulay ⇒ Buchsbaum.

(These notions have global versions also.) Remarkable progress was made on the last three classes of rings.

§2. REGULAR RINGS.

The basic properties of regular local rings (homological characterization, unique factorization) had been proved by Serre, Auslander-Buchsbaum before 1970.

§3. COMPLETE INTERSECTION.

If a local ring A of dimension d is of the form $R/(a_1,\ldots,a_s)$ where R is a regular local ring of dimension d+s, then A is called a complete intersection (c.i. for short). More generally a local ring A is called c.i. if its completion $\hat{A}$ is c.i. in the above sense. When A is a homomorphic image of a regular local ring the two definitions are equivalent. A local ring (A,m) is c.i. iff $H_3(A,A/m,A/m) = 0$, where $H_*(\ )$ is the homology module of André-Quillen. (Michel André, Homologie des algèbres commutatives, Springer Verlag 1974). Using this homological characterization, Bulgarian mathematician L.L.Avramov (Soviet Math. Dokl. 16 (1975),1413-1417) proves that,if f: (B,n) → (A,m) is a flat local homomorphism of local rings, then A is a c.i. iff B and A/nA are c.i. As a corollary he proved that localization preserves complete intersection property. Global complete intersection presents more difficult but interesting problems which are closely connected with algebraic geometry. For these, see Greco et al., Complete intersections, Springer LN 1092 (1985).

§4. COHEN-MACAULAY RINGS.

Let A be a ring, I an ideal and M an A-module. A sequence $a_1,\ldots,a_r$ of elements of I is called an M-sequence in I if

$$(1)\quad \begin{cases} a_1 \text{ is a non-zerodivisor on } M, \\ a_2 \text{ is } \text{- - - - -} \quad \text{on } M/a_1M, \\ \vdots \\ a_r \text{ is } \text{- - - - - - - -} \quad \text{on } M/(a_1M+\ldots+a_{r-1}M). \end{cases}$$

Sometimes the following additional condition is required:

(2) $M \neq a_1M + \ldots + a_rM$.

The supremum of the lengths of M-sequences in I is denoted by depth(I,M). One can prove that

$$\begin{aligned} \operatorname{depth}(I,M) &= \inf \{ i : \operatorname{Ext}^i_A(A/I, M) \neq 0 \} \\ &= \inf \{ i : H^i_I(M) \neq 0 \}, \end{aligned}$$

where $H^*_I(\ )$ is the local cohomology functor. When (A,m) is a local ring we write depth M for depth(m,M). It is easy to see that depth $M \leq$ dim M. The module M is called a Cohen-Macaulay (C-M for short) module if the equality depth M = dim M holds. The local ring A is called a C-M ring if it is C-M module. If A is a C-M local ring then we have:

a) ht I + dim A/I = dim A for every ideal I ;

b) every s.o.p. of A is an A-sequence, and $\ell(A/q) = e(q)$ holds for every parameter ideal q, where e( ) denotes the multiplicity.

A noetherian ring A is called C-M if A_m is C-M for all maximal ideals m. Then:

c) A is C-M iff the unmixedness theorem holds in A, i.e. if an ideal I of height r is generated by r elements then every associated prime of I has height r. ($r \geq 0$).

d) If A is a regular local ring and B is an integral domain containing A such that B is a finitely generated A-module, then

B is C-M $\Leftrightarrow$ B is free as A-module.

e) If $A = A_0 + A_1 + \dots$ is a noetherian graded ring such that $A_0 = k$ is an infinite field and $A = k[A_1]$, then

A is C-M $\Leftrightarrow$ there exist $y_1,\dots,y_d \in A_1$, algebraically independent over k, such that A is a free module over $k[y_1,\dots,y_d]$.

These results were obtained before 1970.

Determinantal varieties.

Let k be a field and $X = (x_{ij}), 1 \leq i \leq r, 1 \leq j \leq s$, be a matrix with rs variables as components; let I_t be the ideal of the polynomial ring $k[X] := k[x_{ij}]$ generated by the $t \times t$ minor determinants of X ($t \geq \min(r,s)$). Then $k[X]/I_t$ is C-M. (Hochster-Eagon, Amer. J. Math. 93(1971)).

Invariant Theory.

Let k be a field and $G = (Gm)^r = \overbrace{k^* \times \dots \times k^*}^{r}$ be an algebraic torus over k. Let G operate on the polynomial ring $k[X] = k[X_1,\dots,X_n]$ by

$$aX_i = \left(\prod_{j=1}^{r} a_j^{t_{ij}}\right)X_i, \qquad a = (a_1,\dots,a_r).$$

Then the ring of invariants $k[X]^G$ is C-M. (Hochster, Annals of Math. 96(1972). This paper of Hochster was also epoch-making as the first work to combine commutative ring theory with topology and combinatorics. The starting point is the following easy

Lemma. Let A be a local ring and I,J ideals. Then there is the following exact sequence:

$$0 \to A/I\cap J \to (A/I)\oplus(A/J) \to A/I+J \to 0.$$

Therefore, if A/I and A/J are both k-dimensional C-M and A/I+J is (k-1)-dimensional C-M, then $A/I\cap J$ is k-dimensional C-M, as one can easily see from the long exact sequence of local cohomology or Ext.

With this in mind, one defines a 'polytope of ideals' of a noetherian ring R as a function which associates to the unions of faces of a d-dimensional polytope P some ideals of R, in such a way that inclusion relation is reversed and $\cup$, $\cap$ of unions of faces correspond to $\cap$, + of ideals. This machinery enables one to transform problems of ideals to problems of combinatorial structure of polytopes.

Advancing further, Hochster-Roberts (Adv. in math. 13(1974)) proved the following big theorem.

Theorem. Let k be a field, S a regular noetherian k-algebra and G a linearly reductive* algebraic group over k acting k-rationally on S. Then the ring of invariants S^G is C-M.
(*i.e. such that every linear representation is completely reducible. In characteristic p such groups are just tori.)

The ring $k[X]/I_t$ mentioned above is also the ring of invariants of some group, hence the result of Hochster-Eagon follows from this theorem in the case of characteristic zero.

Stanley-Reisner Rings.

Rings of the form $k[X_1,\dots,X_n]/I$, where k is a field and I is an ideal generated by square-free monomials (e.g. $I=(X_1X_2, X_2X_3)$ or $I=(X_1X_2X_3)$), are called Stanley-Reisner rings. Given such an ideal I, one can define an abstract complex Σ by

$$X_{i_1}\dots X_{i_p} \notin I \iff (i_1,\dots,i_p) \in \Sigma$$

$(i_1 < i_2 < \dots < i_p)$. Then:

Theorem (Reisner 1976). The ring $k[X_1,\dots,X_n]/I$ is C-M iff certain reduced homology groups with coefficients in k of Σ and certain subcomplexes of Σ vanish.

This important result was immediately used by Stanley to solve the so-called Upper Bound Conjecture of combinatorics affirmatively. Stanley, The Upper Bound Conjecture and Cohen-Macaulay rings, Stud. in Appl. Math. 54 (1975), cf. also Stanley, Combinatorics and Commutative Algebra, Birkhäuser 1984.

§ 5. GORENSTEIN RINGS.

Gorenstein rings are C-M rings which have particularly good homological properties, and can be characterized in many different ways.

1957. Grothendieck proved that, if a local ring A is a homomorphic image of a regular local ring R, then

$$K_A := \mathrm{Ext}_R^t(A,R) \qquad t = \dim R - \dim A$$

depends only on A and not on the choice of R, and he defined

A is a Gorenstein ring if A is C-M and $K_A \simeq A$.

1963. H. Bass proved that, for a local ring (A,m) of dimension n and of residue field k, the following conditions are equivalent, and when they are satisfied he called the ring A Gorenstein.

1) $\mathrm{inj.dim}_A A < \infty$;

2) The minimal injective resolution of A has the form

$$0 \to A \to I^0 \to \dots \to I^n \to 0, \qquad I^q = \bigoplus_{htP=q} E_A(A/P).$$

($E_A(N)$ denotes the injective hull of the A-module N.)

3) $\mathrm{Ext}_A^i(k,A) = \begin{cases} 0 & (i \neq n) \\ k & (i=n). \end{cases}$

4) A is C-M, and every parameter ideal of A is irreducible (in the sense that it is not an intersection of two larger ideals.)

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1971. Herzog-Kunz (Springer LN 238) systematically reconstructed the theory of canonical modules, and discussed Gorenstein rings on this basis. They define the canonical module K_R of an arbitrary local ring R. It does not exist in general, but it exists for a wide class of local rings, and

1) K_R is a finitely generated R-module,

2) if R is Gorenstein then K_R exists and $\simeq R$,

3) conversely, if R is C-M and $K_R \simeq R$ then R is Gorenstein,

4) if R is C-M, the minimal injective resolution of K_R is of the form $0 \to K_R \to I^0 \to I^1 \to \dots \to I^n \to 0$, $I^q = \bigoplus_{htP=q} E(R/P)$,

5) if R is C-M and M is an i-dimensional C-M R-module, then

$$\begin{cases} \mathrm{Ext}_R^{n-i}(M, K_R) \text{ is again i-dimensional and C-M,} \\ \mathrm{Ext}_R^j(M, K_R) = 0 \quad \text{for } j \neq n-i, \\ \text{if we set } M^{\square} = \mathrm{Ext}_R^{n-i}(M,K_R) \text{ then } M^{\square\square} \simeq M. \end{cases}$$

1973. Kei-ichi Watanabe (Osaka J.Math.11) studied the invariants of a finite subgroup G of $GL_n(k)$ acting linearly on $R = k[X_1,\dots, X_n]$ and proved the following theorem.

Theorem. i) If G is contained in $SL_n(k)$, then R^G is Gorenstein.

ii) The converse is also true provided that G does not contain pseudo-reflections.

In the proof of ii) he used canonical modules.

1978. S. Goto - K. Watanabe (J. Math. Soc. Japan 30) considered graded rings of the form $R = R_0 + R_1 + \dots$, where R_0 is a field k and R is finitely generated over k as a ring. They reconstructed the theories of injective hulls and canonical modules in the category of graded R-modules, and discovered an important invariant which they denoted a(R):

$$a(R) := -\min\{i : (K_R)_i \neq 0\}$$

$$= \max\{i : \underline{H}^n_m(R)_i \neq 0\} \qquad (n = \dim R),$$

where $\underline{H}^*_m$ denotes graded local cohomology and $(\ )_i$ denotes the degree i part.

§ 6. BUCHSBAUM RINGS.

For a parameter ideal q of a local ring (R,m), it holds in general that $\ell(R/q) \geqslant e(q)$. The equality holds iff R is C-M. Buchsbaum asked in 1965 whether the difference $\ell(R/q) - e(q)$ is independent of the choice of q, and W. Vogel in 1973 showed that it is not necessarily so. He then proceeded to study the class of local rings for which $\ell(R/q) - e(q)$ is independent of q, calling them Buchsbaum rings. Being a Buchsbaum ring is a weaker condition than C-M, but if R is Buchsbaum and P is a prime ideal other than m then R_P is C-M.

Let (R,m) be a local ring with R/m = k and let M be a finitely generated R-module. A sequence $a_1,\dots,a_r \in m$ is called a weak M-sequence if

$$(a_1,\dots,a_{i-1})M : a_i = (a_1,\dots,a_{i-1})M : m \qquad (1 \leqslant i \leqslant r).$$

Theorem (Stückrad-Vogel 1973) The following are equivalent:

(1) Every s.o.p. of M is a weak M-sequence,

(2) There is a constant c(M) such that, for every s.o.p. $\underline{x} = \{x_1,\dots,x_d\}$ of M, $\ell(M/\underline{x}M) - e(\underline{x},M) = c(M)$ holds.

When these conditions hold M is called a Buchsbaum (Bbm for short) module. The local ring R is Bbm iff it is a Bbm module.

Characterization by local cohomology. Let R,m,k,M be as above. For the local cohomology it holds in general that

$$H^i_m(M) = \lim_{\nu\to\infty} \mathrm{Ext}^i_R(R/m^\nu, M).$$

Therefore there exist canonical maps

$$\phi^i: \mathrm{Ext}^i_R(k,M) \to H^i_m(M).$$

Theorem (Stückrad-Vogel). If ϕ^i are surjective for all $i \neq \dim M$, then M is Bbm. When R is regular the converse is also true.

Buchsbaum rings and modules were intensively studied in East Germany and in Japan, by Schezel, Stückrad, Trung(Hanoi), Vogel, Goto, Shimoda, Ikeda, Yamaguishi among others. In particular, Goto produced many examples of Buchsbaum rings.

Part II. Pathologies of Noetherian rings.

§ 7. COUNTER-EXAMPLES.

The following are some of the difficult problems which arose in the theory of noetherian rings.

1° If A is a noetherian integral domain, is the integral closure $\tilde{A}$ of A in its quotient field a finitely generated A-module ?

2° If A is a local ring with completion $\hat{A}$, are the properties of A such as normal or reduced(= without nilpotents) inherited by $\hat{A}$?

3° Are the lengths of maximal prime ideal chains in a local domain constant ? (A noetherian ring A is said to be catenary if for each pair of prime ideals P,Q with $P \supset Q$, the length of any maximal prime ideal chain between P and Q depends only on P and Q. The above problem is equivalent to asking whether every noetherian ring is catenary.)

Y. Akizuki constructed a counterexample of 1° in 1935. 2° is closely connected with 1° (if $\hat{A}$ is reduced then $\tilde{A}$ is finite over A), and there are counterexamples due to Akizuki and Nagata.

In 1955 Emil Artin visited Japan for a conference on Algebraic Number Theory, and when he met Nagata (then 28 years old) he urged him to settle the problem 3°, saying that all great algebraists of this century had tried it without success. Taking up this challenge Nagata started to work on it, and within a month or so he found counterexamples.

Nagata was not only eager to find counter-examples, but also looked for a nice set of axioms to avoid pathologies. He defined a class of noetherian rings which he called pseudo-geometric rings (and which are now called Nagata rings) in his book Local Rings (Interscience, 1962). This class was designed to avoid the pathologies of type 1°. Not much later, Grothendieck defined a smaller class, called excellent rings, in Ch.4 of EGA (1964). This class avoids all three types 1°,2°, 3° of pathologies. For some time, however, the relation between these two classes was not clarified.

1977. Christel Rotthaus (Math. Z. 152) invented a very complicated, but very powerful, new technique to construct bad noetherian rings, by which she constructed Nagata rings which are not excellent.

1980. T. Ogoma (Jap. J. Math. 6) applied the method of Rotthaus successfully to find non-catenary normal Nagata rings. The non-catenary rings of Nagata were such that their normalizations were catenary, and whether normal noetherian rings were catenary had been a long-standing problem.

§8. EXCELLENT RINGS AND RELATED TOPICS.

Problem: Suppose a noetherian ring A has a property P. Does the polynomial ring A[X] have the same property ?

Answers:

a) P = Nagata. Yes (by Nagata himself).

b) P = excellent. Yes (by Grothendieck).

c) P = catenary. No.(Nagata's examples of 1955 answered also this problem negatively. If $A[X_1,...,X_n]$ are catenary for all n then A is called universally catenary. Ratliff showed that if $A[X_1]$ is catenary then A is universally catenary.

<u>Problem</u>: Suppose a noetherian ring A has a property P. Does the formal power series ring A[[X]] have the same property ? Usually this problem is much more difficult than the preceding one.

Answers:

a) P = universally catenary. No (by H. Seydi, Bull. Soc. Math. France, <u>98</u>(1970)).

b) P = Nagata. Yes (by J. Marot, C.R.Paris, <u>277</u>(1973)). In his proof Marot used a difficult and half-forgotten theorem of Yoshiro Mori, and this latter was later given a new and elegant proof by J. Nishimura (J. Math. Kyoto Univ. <u>15</u> (1975)).

c) P = excellent. The answer is yes in almost all cases.

When A is finitely generated (as ring) over a field k,

1969 Kunz proved it for the case char(k) = p with $[k:k^p] < \infty$.

1973 Matsumura, for the case char(k) = 0.

1975 Valabrega, for general k.

When A is semi-local, Rotthaus proved it in 1979 (Nagoya Math. J. <u>74</u>.)

When A contains the rational numbers, Rotthaus proved it in 1980 (Math. Ann. <u>253</u>) by using Hironaka's resolution of singularities of excellent rings containing the rational numbers and by introducing some other brilliant new ideas. Her solution was such that if resolution of singularity over a field of characteristic p is settled, then her proof would also work for such excellent rings. Thus she has almost settled the problem, although the case when A contains no fields remains untouched.

Let $A \to B$ be a homomorphism of noetherian rings, and let $f: \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ be the induced map. If $P \in \mathrm{Spec}(A)$, then $f^{-1}(P)$ is homeomorphic to $\mathrm{Spec}(B \otimes_A k(P))$, where k(P) = the quotient field of A/P = the residue field of A_P. The algebra $B \otimes_A k(P)$ over the field k(P) is called the fibre over P. In particular, when A is a local ring and B is the completion $\hat{A}$, the fibres are called <u>formal fibres</u> of A. Now the definition of an excellent ring is the following: a ring A is excellent if

0) A is noetherian,

1) all formal fibres of all local rings of A are smooth,(an algebra over a field k is said to be smooth if it is a regular ring, and remains so after any coefficient extension),

2) A is universally catenary,

3) for every A-algebra B of finite type, the singular locus Sing(B):= {P: B_P is not regular} is closed in Spec(B).

When A is a local ring the last condition follows from 1) and can be omitted. The condition 1) is the condition which guarantees that good properties of A_P are inherited by $(A_P)^\wedge$, and it was Grothendieck who first recognized its importance. Noetherian rings which satisfy 1) are called G-rings.

J. Nishimura constructed an example of a G-ring A such that A[[X]] is not a G-ring.

R. Y. Sharp (J. of Algebra 44(1977)) modified the definition of excellent rings as follows and called the resulting rings acceptable rings: in axiom 1), replace 'smooth' by 'Gorenstein', and in 3), replace 'singular locus' by 'non-Gorenstein locus'. He showed that a theory parallel to that of excellent rings can be built for acceptable rings, and that homomorphic images of Gorenstein rings, and more generally rings which possess dualizing complexes, are acceptable.

Part III. Homological Tools.

§9. MINIMAL INJECTIVE RESOLUTIONS.

Let A be a noetherian ring and M an A-module. Let

$$0 \to M \to I^0 \to I^1 \to \dots \to I^j \to \dots$$

be the minimal injective resolution of M. Since every injective module over A is a direct sum of indecomposable ones and since the indecomposable injectives are of the form $E_A(A/P)$ with some prime ideal P (theory of Matlis, 1958), one defines the Bass number $\mu^j(P,M)$ as the number of times $E_A(A/P)$ occurs in I^j; we have

$$\mu^j(P,M) = \dim_{k(P)} \mathrm{Ext}^j_{A_P}(k(P), M_P).$$

1963. Bass proved that

$$\text{Min } \{j : \mu^j(P,M) \neq 0\} = \text{depth}_{A_P}(M_P),$$

$$\text{Sup } \{j : \mu^j(P,M) \neq 0\} = \text{inj.dim}_{A_P}(M_P).$$

$$\text{inj.dim}_{A_P}(M_P) = \infty \Rightarrow \mu^j(P,M) \neq 0 \text{ for } j \geqslant \dim A_P.$$

<u>1971</u>. Foxby proved a part of the following theorem.

<u>1976</u>. Fossum-Foxby-Griffith-Reiten (Publ. IHES <u>45</u>, 1976) proved

$$\text{depth}_{A_P}(M_P) \leqslant j \leqslant \text{inj.dim}_{A_P}(M_P) \Rightarrow \mu^j(P,M) \neq 0.$$

§ 10. <u>LOCAL COHOMOLOGY</u>.

Let A be a noetherian ring and I be an ideal. For each A-module M , set $\Gamma_I(M) := \{x \in M : I^\nu x = 0 \text{ for some } \nu > 0\}$. Then Γ_I is a left exact functor. We denote its derived functors by $H^i_I(M)$ and call them the local cohomology functors. We have

$$H^i_I(M) = \varinjlim_{\nu} \text{Ext}^i_A(A/I^\nu, M).$$

When (R,m) is a local ring and M is an R-module, we write $H^i(M)$ for $H^i_m(M)$. If dim M = d and depth M = r, then

$$d = \sup \{i : H^i(M) \neq 0\},$$

$$r = \inf \{i : H^i(M) \neq 0\}.$$

In particular, M is C-M iff $H^i(M)$ is different from 0 for only one value of i.

<u>Local duality</u>. Let (R,m) be a complete d-dimensional Gorenstein local ring. Let $E = E_R(R/m)$ be the injective hull of R/m. Then

$$H^s(\cdot) = \text{Hom}_R(\text{Ext}^{d-s}_R(\cdot,R), E).$$

When R is not Gorenstein but C-M, then the formula remains valid if one replaces, in Ext(·,R) of the right hand side, R by the canonical module K_R.

Grothendieck created the theory of local cohomology and made it public in a seminar at Harvard in 1961. Later the lecture was published as Springer LN <u>41</u>. Local cohomology was extensively used in algebraic geometry by Grothendieck and his school in the Sixties, but its usefulness in commutative algebra was recognized only in the Seventies. R.Y.Sharp's expository work of 1970 (Local cohomology theory in commutative algebra, Quart. J. Math. Oxford <u>21</u>, 425-434) had some influence, but the decisive event was the appearance of the big paper of Peskine and Szpiro: Dimension projective finie et cohomologie locale, Publ. IHES 42, (1973),47-119, in which they solved many problems, including some of the homological conjectures (see §12) in important special cases, by skillful use of local cohomology and by many other new ideas. For instance, in proving the intersection conjecture in characteristic p, they used local cohomology to estimate the dimension of a module, and to calculate local cohomology they used Frobenius maps.

For the graded version of local cohomology, see the 1978 paper of Goto-Watanabe cited above.

§11. DUALIZING COMPLEXES.

For some purposes the category of modules is not adequate, and one has to work in the larger 'category of complexes'. The need of such a machinery was first recognized by Grothendieck and was hinted in his talk at the Edinburgh Congress,1958. But he did not systematically develop the theory. Early in the Sixties Verdier invented the notion of derived category in his thesis, and using this theory Grothendieck outlined his theory of duality to Hartshorne, who filled in the details and wrote the thick book Residue and Duality, Springer LN 20, 1966. Like local cohomology, it took some years till the theory began to be used in commutative algebra. R.Y.Sharp, Dualizing complexes for commutative Noetherian rings, Math. Proc. Camb. Phil. Soc. 78 (1975), 369-386, simplified the theory by avoiding the use of derived category. Paul Roberts, Two applications of dualizing complexes over local rings, Ann. Scient. Ec. Norm. Sup. 9 (1976), 103-106, applied dualizing complexes to give a new proof of the intersection theorem of Peskine and Szpiro, and to an independent proof of the theorem of non-vanishing of $\mu^i(P,M)$ obtained also by Fossum-Foxby-Griffith-Reiten (cf. §9).

Fix a noetherian ring A. Consider complexes

$$M^{\bullet} : \quad \dots \to M^i \to M^{i+1} \to \dots$$

of A-modules as objects of a category. A morphism $\phi : M^{\bullet} \to N^{\bullet}$ is a fimily (ϕ^i) of A-module homomorphisms $\phi^i : M^i \to N^i$ such that

$$\begin{array}{ccccccc} \to & M^{i-1} & \to & M^{i} & \to & M^{i+1} & \to \\ & \downarrow \phi^{i-1} & & \downarrow \phi^{i} & & \downarrow \phi^{i+1} & \\ \to & N^{i-1} & \to & N^{i} & \to & N^{i+1} & \to \end{array}$$

is commutative. If ϕ induces isomorphisms $H^i(M^\cdot) \to H^i(N^\cdot)$ for all i, then ϕ is called a quasi-isomorphism (quism for short). The derived category is, roughly speaking, the category in which the complexes of A-modules are the objects and the quisms are the isomorphisms.

Example: Let $\to P^{-i} \to P^{-i+1} \to \dots \to P^{-1} \to P^0 \to M \to 0$ be a projective resolution of a module M. We can identify M with the complex which has M in degree zero and 0 elsewhere. The morphism

$$\begin{array}{cccccccc} \to & P^{-i} & \to \dots \to & P^{-1} & \to & P^0 & \to & 0 \\ & \downarrow & & \downarrow & & \downarrow & & \\ \to & 0 & \to \dots \to & 0 & \to & M & \to & 0 \end{array}$$

is a quism, hence we can 'identify' M with $P^\cdot$. Similarly with injective resolutions. What is essential is the notion of quism, and one can dispense with derived categories.

For two complexes $M^\cdot, N^\cdot$ we set

$$\underline{\mathrm{Hom}}_A(M^\cdot,N^\cdot) = \bigoplus_{p,q} \mathrm{Hom}_A(M^p,N^q), \qquad M^\cdot \underline{\otimes} N^\cdot = \bigoplus_{p,q} M^p \otimes N^q$$

where $\mathrm{Hom}(M^p,N^q)$ lies in degree q-p, while $M^p \otimes N^q$ in degree p+q, and the differential operators are defined suitably. When (A,m) is a local ring, a complex $D^\cdot$ is called a <u>dualizing complex</u> of A if

(1) it is a bounded complex of injective modules

$$0 \to D^{-n} \to \dots \to D^0 \to 0$$

with finitely generated cohomology modules $H^i(D^\cdot)$; and

(2) for each complex $M^\cdot$ which is bounded above or below and has finitely generated cohomology modules, it holds that

$$\underline{\mathrm{Hom}}(\,\underline{\mathrm{Hom}}(M^\cdot,D^\cdot),\ D^\cdot\,) \approx M^\cdot$$

where $\approx$ denotes quism.

Or equivalently, a dualizing complex of A is a complex $D^\cdot$ with finitely generated cohomology such that, for each integer i,

$$D^{-i} = \bigoplus_{P\in \mathrm{Spec}(A),\ \dim(A/P)=i} E(A/P).$$

It is unique (if it exists). If A is a homomorphic image of a Gorenstein ring then $D^\bullet$ exists. If A is C-M then $D^\bullet$ is quasi-isomorphic to the canonical module of A, so that we don't need the theory of dualizing complexes as long as we consider C-M rings.

Let A be an n-dimensional local ring with a dualizing complex $0 \to D^{-n} \to \dots \to D^0 \to 0$. Then A is a Buchsbaum ring iff the complex $0 \to D^{-n+1} \to \dots \to D^0 \to 0$ is quasi-isomorphic to a complex of vector spaces over the residue field.

For details of the theory of dualizing complexes, see

Paul Roberts, Homological invariants of modules over commutative rings, Les Press de l'université de Montréal, 1980.

H.-B. Foxby, A homological theory of complexes of modules, Preprint Series No.19a,b, Copenhagen Univ. Math. Inst. 1981.

P. Schenzel, Dualisierende Komplexe in der lokalen Algebra und Buchsbaum-Ringe, Springer LN 907, 1981.

Part IV. Problems, Solved and Unsolved.

§12. HOMOLOGICAL CONJECTURES.

There are a number of conjectures which are grouped (by Hochster) under the vague name of homological conjectures.

(1) Auslander Conjecture (Zero-divisor Conjecture).

Let R be a local ring and M be a finitely generated R-module with proj.dim $M < \infty$. Then every M-sequence is also an R-sequence.

(2) Bass Conjecture.

Let R be a local ring and suppose there exists a finitely generated R-module M of finite injective dimension. Then R is C-M. (The converse is true and easy to prove.)

(3) Intersection Conjecture. (Peskine-Szpiro)

If R is local and M,N are finitely generated non-zero R-modules such that $M \otimes N$ has finite length, then

$$\dim N \leq \text{proj.dim } M.$$

(3) implies (1) and (2), and Peskine-Szpiro (loc.cit.) proved these conjectures for the cases

(a) R contains a field of characteristic p,
and(b) R is essentially of finite type over a field (i.e. R is of the form A_P, where P is a prime ideal of A and A is finitely generated over a field as ring).

(4) Existence of Big C-M module. (Hochster)
Let R be a local ring and $\underline{x} = \{x_1,\ldots,x_d\}$ be an s.o.p. of R. Then there exists a (not necessarily finitely generated) R-module M such that $\underline{x}$ is an M-sequence (with $\underline{x}M \neq M$).

(5) Existence of Small C-M module.(Hochster)
Let R be a complete local ring and $\underline{x}$ be an s.o.p. of R. Then there exists a finitely generated R-module M such that $\underline{x}$ is an M-sequence.

Since a finitely generated C-M module M over $\hat{R}$ is a big C-M module over R, (5) implies (4). Hochster proved that (4) implies (3) and many other important consequences. And around 1974 he proved (4) for the case when R contains a field. He proved it first for the case of characteristic p by using Frobenius maps, and he reduced the characteristic zero case to the p case by applying Artin Approximation theorem, a technique which had been already used by Peskine-Szpiro (loc.cit.). The unequal characteristic case (i.e. when R contains no fields) remains still open, in spite of tenacious efforts of Hochster.

Applications of Hochster's theorem.
1°) The proof of Fossum-Foxby-Griffith-Reiten (loc.cit.) of the non-vanishing of $\mu^i(P,M)$ used big C-M modules of Hochster. Their theorem itself is valid in the unequal characteristic case also.
2°) Solution of the Syzygy Problem by Evans-Griffith (1981).

Theorem. Let (R,m) be a local catenary domain containing a field, and M be a non-free R-module of rank r and of finite projective dimension satisfying S_k. Then $r \geqslant k$.
(A module M said to satisfy S_k if

$$\operatorname{depth}_{R_P}(M_P) \geqslant \min(k, \operatorname{ht} P)$$

holds for every prime ideal P. A module M is called a k-th

<u>syzygy</u> if there exists an exact sequence of the form

$$0 \to M \to F_{k-1} \to \cdots \to F_0 \to N \to 0$$

with F_i free. Auslander-Bridger (1969) showed that, when R is C-M and proj.dim $M < \infty$, M is a k-th syzygy iff it satisfies S_k.)

In their proof they used big C-M modules. The theorem has many geometric applications. For example,

<u>Theorem</u> (Evans-Griffith). Let R be a regular local ring containing a field of characteristic zero with dim $R \geqslant 7$. Let P be a prime ideal of height 2 such that R/P is normal isolated singularity. Then R/P is a complete intersection (i.e. P is generated by two elements).

(6) <u>Conjectures on Intersection Multiplicities</u>. (Serre)

Let (R,m) be a local ring and M,N be finitely generated R-modules satisfying proj.dim $M < \infty$, proj.dim $N < \infty$ and $\ell(M \otimes N) < \infty$ (this last condition is equivalent to $\operatorname{Supp} M \cap \operatorname{Supp} N = \{m\}$). Then each $\operatorname{Tor}_i(M,N)$ has finite length, and one can define

$$\chi(M,N) := \sum_{i \geqslant 0} (-1)^i \ell(\operatorname{Tor}_i(M,N)).$$

This number is called the intersection multiplicity of M and N. (Let $X = \operatorname{Spec} R$, $x = m$, $Y = V(I)$ and $W = V(J)$. Then $\chi(R/I, R/J)$ is the intersection multiplicity of Y and W on X at the point x, at least when R is the local ring of the point x on a smooth algebraic variety.)

Serre proved that, when R is regular, it holds that

$$\dim M + \dim N \leqslant \dim R.$$

Furthermore, he proved that, under the additional assumption that R is unramified,

(M1) $\dim M + \dim N < \dim R \Rightarrow \chi(M,N) = 0$,

(M2) $\dim M + \dim N = \dim R \Rightarrow \chi(M,N) > 0$.

(A local ring is said to be <u>unramified</u> if either it contains a field or the residue field characteristic p is not in m^2.) Serre conjectured that these would hold for all regular local rings. Some poeple conjectured that even the hypothesis of regularity may be superfluous.

1974. Peskine and Szpiro proved that (Ml), (M2) hold in the graded case under the assumption proj.dim $M < \infty$ only.
1983. Hochster-Dutta-McLaughlin found a counterexample to (Ml) in the case R is a hypersurface isolated singularity, proj.dim M is finite but proj.dim N is infinite.
1985. P. Roberts (Bull. AMS) proved (M1) under the assumption that both M and N have finite projectvie dimension but R may be either a c.i. or an isolated singularity. The proof depends on the intersection theory of Fulton-Baum-Macpherson.

References.

M. Hochster, Topics in the homological theory of modules over commutative rings, Regional Conference Series 24, AMS, 1974.
E.G.Evans - P.Griffith, The Syzygy Problem, Ann. of Math. 114 (1981),323-353.
-----------,Syzygies, London Math. Soc. Lect. Note Series 106, Camb. Univ. Press, 1985.

§13. SERRE CONJECTURE ON PROJECTIVE MODULES.

Serre conjectured that all finitely generated projective modules over a polynomial ring $k[X_1,...,X_n]$ are free. (Sur les modules projectifs, Sem.Dubreil-Pisot 1960/61), where k is a field. Geometrically, this is equivalent to saying that an algebraic vector bundle over an affine space k^n is necessarily trivial. The case n=1 is trivial, and the case n=2 was proved by Seshadri shortly afterwards. But the general case remained open until 1976, when Quillen and Suslin independently proved the conjecture for the more general case k= a principal ideal domain. Then it was asked whether the theorem holds true even when k is a regular local ring of arbitrary dimension. Lindel has obtained some positive results in this line, but I don't know whether the general problem is still open at present. The theorem of Quillen-Suslin is useful in the problem of complete intersection. Namely, when a certain ideal I of $k[X_1,...,X_n]$ is given, one takes a minimal projective resolution $\dots \to L_0 \to I \to 0$, and proves that L_0 has rank r. Then L_0 is free of rank r by the theorem, so that one can conclude that I is generated by r elements.

References.

Lam, Serre's Conjecture, Springer LN 635, 1978.

Kunz, Einführung in die komm. Alg. u. alg. Geometrie, Kapitel IV. Vieweg & Sohn 1979, English ed. 1986.

Part V. Some Other Topics.

Here we will collect some more topics with the names of main researchers, with or without comments, just to show the variety and richness of commutative algebra of today.

§14. LINKAGE (liaison).

Two proper ideals I,J of a local Gorenstein ring R said to be linked if there is a R-regular sequence $\underline{a} = a_1,\ldots,a_s$ in $I \cap J$ such that $J = (\underline{a}):I$ and $I = (\underline{a}):J$. The set of all ideals of R which can be obtained from I by a finite sequence of links is called the linkage class of I. One looks for n.a.s. conditions for two ideals to be in a same class, and in particular, characterizations of ideals in the linkage class of a complete intersection ideal.

After sporadic works by Dubreil (1935), Apéry (1945) and Gaeta (1952), Peskine-Szpiro, Liaison des variétés algébriques I, Invent. Math. 26 (1974), rediscovered the notion and systematically developed the theory. Note that J.Watanabe's work of 1973 (Nagoya Math. J. 50, 227-232) was also in this direction. After them, Hartshorne, P.Rao, Buchweitz, Kustin, Miller, and among others, Huneke and Ulrich.

§15. ALGEBRAS WITH STRAIGHTENING LAWS (ASL)

An ASL is a k-algebra generated by a finite poset (=partially ordered set) over k satisfying certain axioms. The homogeneous coordinate ring of a Grassmann variety is a typical example. The foundation of the theory was laid by DeConcini-Procesi-Eisenbud, and was first published by Eisenbud in

Introduction to algebras with straightening laws, in Ring Theory and Algebra III, Dekker, 1980.

Later they gave a fuller account in DC.-P.-E., Hodge Algebras, Astérisque 91 (1982). Hodge Algebra is a generalization of ASL, but Hibi showed that every graded algebra has a structure of Hodge algebra. K.Watanabe and Hibi have been investigating many examples of ASL, sometimes in connection with lattice theory and combinatorics.

§16. MAXIMAL COHEN-MACAULAY MODULES AND BUCHSBAUM MODULES.

If A is a d-dimensional local ring, a C-M (resp. Bbm) module over A of dimension d is called a maximal C-M module (resp. maximal Bbm module). Following the example of representation theory of non-commutative rings à la Gabriel, Auslander and Reiten, and under the strong influence of M. Auslander, some commutative ring theorists started,rather recently,to work on the classification of maximal C-M modules (mainly over isolated singularity). Beside Auslander and Reiten, Herzog, Buchweitz, Knörrer, Greuel, Yoshino. On the other hand, S.Goto succeeded in classifying the maximal Bbm modules over a regular local ring completely (1985).

§17. ARTINIAN RINGS.

In 1984 J. Watanabe observed that there is a remarkable parallelism between combinatorics and the theory of Artin rings. Let A be an Artin local ring let $\mu(I)$ denote the minimum number of generators of an ideal I. Set

$$d(A) = \text{Max}\{\mu(I) : \text{all ideals}\},$$

$$r(A) = \text{Min}\{\ell(A/yA) : \text{all non-unit element } y \text{ of } A\},$$

and call d(A) the Dilworth number of A and r(A) the Rees number of A. He proved that $d(A) \leqslant r(A)$. According to him the number d(A) corresponds to the Dilworth number of a finite poset P, which is defined as the minimum number of disjoint chains into which P is decomposed.

Other interesting results have been obtained by Iarrobino and Emsalem.

§18. COMBINATORICS AND COMMUTATIVE ALGEBRA.

After the pioneering works of Hochster and Reisner, Richart Stanley of MIT led the movement to combine combinatorics and commutative algebra. He also used advanced results of algebraic geometry such as Hard Lefschetz Theorem and intersection homology to solve combinatorial problems. Now there is an active interplay between combinatorics and commutative ring theory. Baclawski, Björner, Hibi.

§19. BLOWING UP.

Starting from a local ring A and an ideal I, one can form the Rees ring $R(I,A) = A + I + I^2 + \ldots$ and the associated graded ring $G(I,A) = A/I + I/I^2 + I^2/I^3 + \ldots$. The scheme Proj(R(I,A)) is the blowing-up of Spec(A) along V(I). These rings were extensively studied. Herrmann, Orbanz, Goto, Shimoda, Ikeda, Valla.

§20. FREE RESOLUTIONS AND FINITE FREE COMPLEXES.

Buchsbaum and Eisenbud, and more recently Akin and Lascoux, made many important discoveries about complexes of finite free modules and free resolutions of (say) determinantal ideals. Also Northcott, Eagon, Bruns.

§21. ARTIN APPROXIMATION THEORY.

Mike Artin (Publ. IHES 36, 1969) started the approximation theory, which runs, roughly, as follows: given a set of algebraic equations with coefficients in a Henselian ring A, any solution in the completion $\hat{A}$ can be approximated by solutions in A as closely as one wants. There are some variants of the theme, and one of the problems is to characterize the class of local rings A for which the above statement is true. M.Artin, Kurke, Pfister, Roczen, Popescu, Rotthaus. Note that some logicians contributed to this theory, via ultraproduts.

§22 NUMBER OF GENERATORS OF MODULES AND IDEALS.

Eisenbud-Evans, and independently Storch, proved in 1973 that every algebraic set in (affine or projective) n-space can be defined by n equations. In the affine case this means that for every ideal I of the polynomial ring $k[X_1,...,X_n]$, k a field, one can choose n elements $f_1,...,f_n$ of I so that I and $(f_1,..., f_n)$ have the same radical. Judith Sally proved many interesting results about number of generators of an ideal of a local ring. Also Bruns, Mohan-Kumar. See the book of Kunz cited in §13, and Sally's monograph Numbers of Generators of ideals in local rings, Dekker, 1978.

§23. DERIVATIONS AND DIFFERENTIALS.

Kunz, Lipman, Scheja, Storch, Radu, S.Suzuki, Nakai, Matsumura, Kimura, Niitsuma. Kimura-Niitsuma (J.Math.Soc.Japan 34, 1982) proved a conjecture of Kunz which says that, if S is a regular local ring of characteristic p and R is a subring such that $S \supseteq R \supseteq S^p$ and S is a finitely generated R-module, then R is regular iff S has a p-basis over R. See the new book of Kunz, Kähler Differentials, Vieweg & Sohn 1986.

§24. INTEGRAL DOMAINS.

Rationality (Asanuma), Finite generation (K.Yoshida, Onoda, Heinzer, Sally) , Semi-normality (Traverso, Swan, Greco).

§25. ASYMPTOTIC THEORY.

McAdam, Ratliff, Eakin. See McAdam, Asymptotic Prime Divisors, Springer LN 1023,(1983).

§26. RATIONALITY OF POINCARÉ-BETTI SERIES.

Gulliksen, Löfwall, Roos, Anick, Avramov.

See J.-E. Roos (ed.), Algebra, Algebraic Topology and Their Inter-Actions, Springer LN 1183,(1983).

and the articles by Anick-Halperin and others in Volume 38 (1985) of J. Pure and Appl. Algebra, which was dedicated to J.-E. Roos.

CHRONICLE.

1971 Herzog-Kunz introduced canonical modules.

Hochster-Eagon proved Cohen-Macaulayness of $k[X]/I_t(X)$.

Foxby's work on minimal injective resolution.

Ratliff characterized catenary rings.

1972 Hochster used polytopes in the study of C-Mness of rings generated by monomials, combining commutative ring theory with topology and combinatorics.

Ratliff: Catenary rings and the altitude formula.

1973 Peskine-Szpiro used local cohomology and Frobenius maps effectively.

Buchsbaum-Eisenbud gave an exactness criterion of complex of free modules.

Hochster proved the existence of big C-M module in characteristic zero.

Vogel started the theory of Buchsbaum rings.

Storch, and Eisenbud-Evans, independently proved that an algebraic set in n-space can be defined by n equations.

1974 Reisner gave a topological criterion of Cohen-Macaulayness of Stanley-Reisner rings.

Peskine-Szpiro, Liaison I.

Hochster-J.Roberts, Rings of invariants of reductive groups acting on regular rings are C-M.

K. Watanabe, Certain invariant subrings are Gorenstein.

M. André, Localisation de la lissité formelle.

----- , Homologie des algèbres commutatives.

1975 Stanley applied the theory of C-M rings to combinatorics (Upper Bound Conjecture).

Fossum-Foxby-Griffith-Reiten's work on minimal injective resolution.

Strong approximation theorem of Pfister-Popescu.

Hochster wrote the Regional Conference Monograph, proposing many homological conjectures.

1976 Quillen and Suslin proved that projective modules over polynomial rings are free.

P. Roberts, Two applications of dualizing complexes over local rings.

1977 Eisenbud-Buchsbaum, Algebra structure for finite free resolutions, and some structure theorems for ideals of codimension 3. (American J. Math. 99)

Rotthaus invented a powerful method to construct bad noetherian rings.

1978 Stanley, Hilbert functions of graded rings.(Adv. in Math. 28)

Goto-K.Watanabe, On graded rings, I, II.

J. Sally, Numbers of generators of ideals in local rings.

Lascoux, Syzygies des variétés déterminantales.(Adv. in Math. 30)

1979 Rotthaus proved that formal power series rings over an excellent semi-local ring are excellent.

1980 Rotthaus proved that formal power series rings over an excellent ring containing a field of char. zero are excellent.

Ogoma constructed an example of non-catenary normal ring.

DeConcini-Procesi-Eisenbud initiated the theory of ASL.

1981 Evans-Griffith, The Syzygy Problem.

Foxby, A homological theory of complexes of modules.

1982 Huneke introduced the notion of d-sequence.

Schenzel, Dualisierende Komplexe in der lokalen Algebra und Buchsbaum Ringe.

1983 Hochster-Dutta-Maclaughlin found a counterexample to a conjecture about multiplicity.

Stanley, Combinatorics and Commutative Algebra.

1984 P. Roberts solved the vanishing part of Serre conjecture on multiplicity, using the intersection theory of Fulton.

J. Watanabe found parallelism between Artin rings and finite posets.

1985 Maximal Cohen-Macaulay modules were studied by Auslander, Herzog, Buchweitz, Greuel, Knörrer, Yoshino.

Goto determined maximal Buchsbaum modules over regular local rings.

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ISOMORPHISM OF GROUP RINGS OF INFINITE NILPOTENT GROUPS

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This note is an abstract of the author's paper [4], and we shall state only some of our results. See [4] for details.

Let G be a group and R an integral domain of characteristic 0 in which no element $g \neq 1$ of G has order invertible. We denote by RG the group ring of G over R. In the case where G is finite, a remarkable consequence of an algebra isomorphism $RG \cong RH$ between two group rings is the lattice isomorphism induced between the set of normal subgroups of G and that of H. In fact it is an isomorphism which preserves many natural operations and properties on these sets. By virtue of this, the isomorphism $RG \cong RH$ preserves nilpotency, solvability, class of nilpotency and the derived length of G, and so on. In the case where G is infinite, however, such a lattice isomorphism is not yet established. In the integral case Sehgal [11, p.229] proposed the following problem : Does $\mathbb{Z}G \cong \mathbb{Z}H$, G solvable (nilpotent) $\Rightarrow$ H solvable (nilpotent) ? Röhl [9] has introduced a new notion of the torsion-length for infinite nilpotent groups and has shown that the answer to the nilpotent case is affirmative. In such a situation, however, the question whether the class of H coincides with that of G still remains with us.

The main purpose of [4] is to consider the question whether an isomorphism $RG \cong RH$ preserves the class of nilpotency of G, and it is proved that, under some strong assumptions on G, the answer is "yes" (Theorem 3 and Proposition 2). In the process of establishing this, it is shown that in the nilpotent case the isomorphism $RG \cong RH$ entails a lattice isomorphism between the periodic normal subgroups of G and H (Theorem 2).

In what follows, R always denotes a commutative ring with identity and, unless otherwise stated, G denotes an arbitrary group.

1. Central units of finite order in RG.

We denote by $\zeta(G)$ the center of G and by TG the set of elements of finite order in G. Also, denote by $U(RG)$ the unit group of the group ring RG. A unit of RG is said to be *trivial* if it is of the form rg, $r \in U(R)$, $g \in G$.

Following [7] we say that R is G-*adapted* if R is an integral domain of characteristic 0 in which no element $g \neq 1$ of G has order invertible. The following result, which is well known for the case $R = \mathbb{Z}$, plays a central role in the proof of Lemma 1.

Proposition 1. If R is G-adapted then any central unit of finite order in RG is trivial ; that is

$$T\zeta(U(RG)) = TU(R) \times T\zeta(G).$$

Remark. If $\alpha = \sum \alpha(g)g$ is a central element in RG, we see that the supporting subgroup $\langle g \in G \mid \alpha(g) \neq 0 \rangle$ is always polycyclic-by-finite. The above proposition therefore follows from a certain result of Sehgal on group rings of polycyclic-by-finite groups.

The isomorphism problem for group rings asks whether the groups G and H are necessarily isomorphic if their group rings RG and RH are isomorphic as R-algebras. When this is so for a group G, G is said to be *characterized* by RG. In [8], Röhl has proved that the circle group $G := (A,\circ)$ of every nilpotent ring A is characterized by its integral group ring

$\mathbb{Z}G$. $((A,o)$ is the group induced by the circle operation $a \circ b = a + b + ab$, $a,b \in A$.) This result is mainly based on the triviality of central units of finite order in any integral group ring and we see, by virtue of Prop.1, that the proof remains valid over any integral domain R of characteristic 0. Thus we have the following

Theorem 1. If R is an integral domain of characteristic 0 then the circle group $G := (A,o)$ of every nilpotent R-algebra A is characterized by RG.

2. Normal subgroup correspondence.

Throughout this section, R will denote a G-adapted ring.

The *torsion-length* of a nilpotent group, which is introduced by Röhl [9], is defined as follows. Let $\{1\} = \zeta_0(G) \subseteq \zeta_1(G) \subseteq \cdots \subseteq \zeta_i(G) \subseteq \cdots$ be the upper central series of G and we write $T_i(G) = TG \cap \zeta_i(G)$ for $i \geq 0$. If TG forms a subgroup of G, then $\{T_i(G)\}_{i \geq 0}$ is an ascending series of normal subgroups of G with the property that

$$T_i(G/T_j(G)) = T_{i+j}(G)/T_j(G)$$

for all $i,j \geq 0$. This formula can be seen by induction on i, keeping j fixed. In case G is nilpotent (of class n), we have a finite series $\{1\} = T_0(G) \subseteq T_1(G) \subseteq \cdots \subseteq T_n(G) = TG$ and the torsion-length $t(G)$ of G is defined to be the number of different terms $T_i(G)$ $(\neq \{1\})$. Note that since G is nilpotent TG is a normal subgroup of G (see e.g. [6, p.470]). Clearly, $t(G) = 0$ if and only if G is torsion-free. Moreover, since every nontrivial normal subgroup of G has a nontrivial intersection with the center $\zeta(G) = \zeta_1(G)$, we observe from the above formula that if $t(G) > 0$ then $t(G) = t(G/T_1(G)) + 1$.

The induction argument on $t(G)$ together with Prop.1 gives us the following result whose proof is almost identical to that of [9, Proposition on p.138].

Lemma 1. Let G be a nilpotent group and suppose that $RG \cong RH$ as R-algebras for some group H. Then H is nilpotent with $t(G) = t(H)$, and R is H-adapted.

However this does not any impact on our question of showing that the classes of G and H coincide, and we need to investigate the normal subgroup correspondence.

Before stating the next lemma, we introduce the following notation. For any normal subgroup N of G, let $\Delta_R(G,N)$ be the kernel of the natural epimorphism $RG \to R(G/N)$. It is equal to $\Delta_R(N)RG$, where $\Delta_R(N)$ denotes the augmentation ideal of RN. For X a subgroup of $U(RG)$ and I a (two-sided) ideal in RG, $X \cap (1+I) = \{ u \in X \mid u - 1 \in I \}$ forms a normal subgroup of X. In particular we write $U(1+I)$ for $U(RG) \cap (1+I)$. It is easy to verify that $\Delta_R(G, G \cap (1+I)) \subseteq I$, and hence $U(1 + \Delta_R(G, G \cap (1+I)))$ is normal in $U(1+I)$.

Lemma 2. Let G and H be nilpotent groups, let I be an ideal of RH contained in $\Delta_R(H)$, and assume that $RH/I \cong RG$ as R-algebras. Then the factor group $U(1+I)/U(1 + \Delta_R(H, H \cap (1+I)))$ is torsion-free.

Remark. In case H is finite, Lemma 2 is trivial because I is necessarily equal to $\Delta_R(H, H \cap (1+I))$. But in the infinite case, $I \neq \Delta_R(H, H \cap (1+I))$ in general.

Let us now suppose that we have an isomorphism $\theta : RG \to RH$ of R-algebras. We may assume that θ is augmented, that is, $\varepsilon_H \theta = \varepsilon_G$ where ε_G and ε_H are the augmentation maps of RG and RH respectively (see [11, p.64]). Following [1], for any normal subgroup N of G we define the normal subgroup ϕN of H as $\phi N = H \cap (1+\theta(\Delta_R(G,N)))$. It is well known [11, pp.94-95] that if R is an integral domain of characteristic 0 in which no rational prime is invertible, then ϕ yields an isomorphism of the lattices of the finite normal subgroups of G and H. However it seems to be hard to extend this result to arbitrary normal subgroups. For the case of nilpotent groups, we obtain the following result which is applied in our arguments. For convenience, we denote by $I(RG)$ the lattice of ideals of RG and by $L_{PN}(G)$ the lattice of periodic normal subgroups of G. (A group is called periodic if all of its elements are of finite order.) Also Δ_G denotes the map $L_{PN}(G) \to I(RG)$ defined by $\Delta_G(N) = \Delta_R(G,N)$.

Theorem 2. If G is nilpotent then the following hold :

(a) ϕ induces a lattice isomorphism between $L_{PN}(G)$ and $L_{PN}(H)$, and the diagram

$$\begin{array}{ccc} L_{PN}(G) & \xrightarrow{\phi} & L_{PN}(H) \\ \downarrow \Delta_G & & \downarrow \Delta_H \\ I(RG) & \xrightarrow{\theta} & I(RH) \end{array}$$

is commutative.

(b) Let $I \in I(RG)$, and set $N = G \cap (1+I)$, $K = H \cap (1+\theta(I))$. If $K \in L_{PN}(H)$, then $\phi N = K$.

Remarks:

(1) In the situation of (b) we have not been able to show that $K \in L_{PN}(H)$ if $N \in L_{PN}(G)$.

(2) It would be nice to know if it is true that, given a torsion-free normal subgroup N of G, there always exists a normal subgroup K of H such that $\theta(\Delta_R(G,N)) = \Delta_R(H,K)$; because in that case, by going mod TN Theorem 2 tells us that $\theta(\Delta_R(G,N)) = \Delta_R(H,\phi N)$ for any normal subgroup N of G and hence that ϕ is an isomorphism between the lattice of all normal subgroups of G and that of H.

3. Lie dimension subgroups.

The Lie powers $\Delta_R^{(n)}(G)$ $(n = 1,2,\cdots)$ of the augmentation ideal $\Delta_R(G)$ are defined inductively as follows :

$$\Delta_R^{(1)}(G) = \Delta_R(G), \quad \Delta_R^{(n+1)}(G) = [\Delta_R^{(n)}(G), \Delta_R(G)]RG,$$

where $[X,Y]$ denotes the R-submodule of RG generated by $[x,y] = xy - yx$, $x \in X$, $y \in Y$. Then by the n-th Lie dimension subgroup of G over R, written $D_{(n),R}(G)$, we understand $D_{(n),R}(G) = G \cap (1 + \Delta_R^{(n)}(G))$. Let $\gamma_n(G)$ denote the n-th term of the lower central series of G starting with $\gamma_1(G) = G$. We notice that $\gamma_n(G) \subseteq D_{(n),R}(G)$ for all n, because $\{D_{(n),R}(G)\}_{n \geq 1}$ is a central series of G (see [5]).

For the results of the next section we need the following two lemmas on Lie dimension subgroups. The first is an application of Theorem 2 and is analogous to a result of [2] on dimension subgroups (see also [3]). The second one is due to Sandling [10], and our results heavily depend on it.

Lemma 3. Suppose $RG \cong RH$ as R-algebras, where R is G-adapted. Then $D_{(n),R}(G) = \{1\}$ implies $D_{(n),R}(H) = \{1\}$.

Lemma 4. (1) For all $n \leq 6$, $D_{(n),\mathbb{Z}}(G) = \gamma_n(G)$.

(2) If G is a metabelian group, then $D_{(n),\mathbb{Z}}(G) = \gamma_n(G)$ for all $n \geq 1$.

4. Main theorem.

The ring R is assumed to be G-adapted in this section also.

For a nilpotent group G, cl(G) denotes its nilpotence class ; that is, cl(G) is the smallest integer c such that $\gamma_{c+1}(G) = \{1\}$. The application of Lemma 3 and Lemma 4 together with Theorem 2 gives us the following results.

Theorem 3. Let G be a nilpotent group and suppose that $RG \cong RH$ as R-algebras. If $cl(G/TG) \leq 5$, then $cl(G) = cl(H)$.

Proposition 2. Let G be a metabelian and nilpotent group, suppose that $RG \cong RH$ as R-algebras. Then H is metabelian and nilpotent, and $cl(G) = cl(H)$.

Remark. Taking the proof of Theorem 3 into consideration, it seems reasonable to conjecture that the condition $cl(G/TG) \leq 5$ can be dropped.

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ON ANTI-HOPFIAN MODULES AND RESTRICTED ANTI-HOPFIAN MODULES

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1. Introduction. One of the interesting properties of the p-Prüfer group $Z(p^\infty)$ is that it is isomorphic to every non-zero factor group of itself. In fact, Szélpál [8] proved that the p-Prüfer groups are characterized as non-simple Abelian groups with this property. In this paper, we shall study the structure of anti-Hopfian modules (non-simple modules all of whose non-zero factor modules are isomorphic) and restricted anti-Hopfian modules (non-simple modules all of whose non-zero proper factor modules are isomorphic). A restricted anti-Hopfian module has the striking property that every non-zero proper factor module is subdirectly irreducible. Non-simple modules with such property will be called restricted subdirectly irreducible, and will be studied in Section 2. Section 3 is devoted to the study of structure of anti-Hopfian modules and restricted anti-Hopfian modules and their endomorphism rings.

Throughout this paper, R will represent an associative ring with identity and all modules will be unitary right R-modules.

2. Restricted subdirectly irreducible modules.

Definitions. (a) A module M is said to be <u>uniserial</u> if the set of submodules of M is linearly ordered by inclusion.

This is a summary of my papers [4], [5] written jointly with Y. Hirano.

(b) A non-zero module M is said to be <u>subdirectly irreducible</u> if the intersection H of all its non-zero submodules is not 0. In this case, the submodule H is called the <u>heart</u> of M.

(c) A module M is called <u>completely subdirectly irreducible</u> if every non-zero factor module of M is subdirectly irreducible.

(d) A non-simple module M is called <u>restricted subdirectly irreducible</u> (resp. <u>restricted Artinian</u>) if each proper non-zero factor module of M is subdirectly irreducible (resp. Artinian).

In this section, we shall study the structure of the restricted subdirectly irreducible modules.

First, we state the following

Proposition 1 ([4, Proposition 1], [5, Lemma 1]). An R-module M is completely subdirectly irreducible if and only if M is Artinian and uniserial.

Example 2 ([5, Example 2]). $Z(p^{\infty})$ is completely subdirectly irreducible. In fact, every non-zero factor group of $Z(p^{\infty})$ is isomorphic to $Z(p^{\infty})$. But $Z(p^{\infty})$ is not Noetherian.

For any module M, we denote the <u>Jacobson radical</u> and the <u>socle</u> of M by Rad(M) and Soc(M), respectively.

We shall now give the following theorem which plays an important role in our study.

Theorem 3 ([5, Theorem 3]). Let M be an R-module. Then, M is restricted subdirectly irreducible if and only if one of the following holds:

(1) M is a direct sum of two simple modules;

(2) M is restricted Artinian and uniserial;

(3) M is Artinian, M/Soc(M) is non-zero uniserial, Soc(M)

is a direct sum of two simple modules and Soc(M) is a waist of M (that is, every submodule is comparable with Soc(M)).

Moreover, if $M \neq \mathrm{Rad}(M)$ and M satisfies (2) or (3), then M is local.

In case R is commutative, we have the following

Theorem 4 ([5, Theorem 4]). Let R be a commutative ring, and M an R-module such that $M \neq \mathrm{Rad}(M)$. Then, M is restricted subdirectly irreducible if and only if one of the following holds:

(1) M is a direct sum of two simple modules;

(2) M is local, Noetherian and uniserial;

(3) Soc(M) is a unique maximal submodule of M, and is a direct sum of two simple modules.

Let R be a Dedekind domain, K the field of fractions of R, and P a prime ideal of R. We denote by $R(P^{\infty})$ the P-primary part of K/R, which is called the <u>module of type</u> P^{∞} (see Kaplansky [6, p.335]). It is easily seen that $R(P^{\infty})$ is isomorphic to K/R_P, where R_P is the localization of R at P.

When R is a Dedekind domain, we can completely classify the restricted subdirectly irreducible R-modules as follows:

Theorem 5 ([5, Theorem 6]). Let R be a Dedekind domain, and M an R-module. Then, M is restricted subdirectly irreducible if and only if one of the following holds:

(1) $M \cong R/P \oplus R/Q$ for some prime ideals P and Q;

(2) $M \cong R/P^n$ for some prime ideal P and some positive integer n;

(3) M is isomorphic to $R(P^{\infty})$ for some prime ideal P;

(4) R is a discrete valuation ring and M is isomorphic to the field of fractions K of R.

As a particular case of Theorem 5, we have

Corollary 6 ([5, Corollary 7]). An Abelian group M is restricted subdirectly irreducible if and only if one of the following holds:

(1) $M \cong Z(p) \oplus Z(q)$ for some primes p and q;

(2) $M \cong Z(p^n)$ for some prime p and some positive integer n;

(3) $M \cong Z(p^\infty)$ for some prime p.

3. Anti-Hopfian modules and restricted anti-Hopfian modules.

Definitions. (e) A module M is said to be Hopfian if every surjective endomorphism of M is an automorphism.

(f) A submodule N of M is said to be a non-Hopf kernel (for M) if there exists an isomorphism of M/N to M.

(g) A non-simple module M is said to be anti-Hopfian if every proper submodule of M is a non-Hopf kernel.

(h) A non-simple module M is said to be restricted anti-Hopfian if any two non-zero proper factor modules of M are isomorphic. Clearly, every anti-Hopfian module is restricted anti-Hopfian.

(i) A ring R is said to be a (right) CH-ring if every cyclic right R-module is Hopfian. Clearly, every right Noetherian ring is a CH-ring. As is well known, every finitely generated module over a commutative ring R is Hopfian (see, e.g.,[2]). Hence, every commutative ring is a CH-ring.

The lattice of the R-submodules of M is denoted by $L_R(M) = L(M)$. We set $\omega^+ = \{\xi \mid \xi \leq \omega\}$, where ω denotes the first limit ordinal.

We shall now characterize the anti-Hopfian module M over a CH-ring R by the structure of $L(M)$.

Theorem 7 ([4, Theorem 2]). Let R be a CH-ring, and M an R-module. Then the following conditions are equivalent:

1) M is anti-Hopfian.

2) $L(M)$ is lattice isomorphic to ω^+ and the heart of M is a non-Hopf kernel.

When R is a right Noetherian ring, we have the following simple criterion for a module to be anti-Hopfian.

Corollary 8 ([4, Corollary 3]). Let R be a right Noetherian ring, and M an R-module. Then the following conditions are equivalent:

1) M is anti-Hopfian.

2) M is subdirectly irreducible and the heart of M is a non-Hopf kernel.

Remark 9 ([4, Example 4]). Let R be a left and right hereditary Noetherian prime ring which is not primitive and has no proper idempotent ideal. Let M be an indecomposable, injective right R-module, which is not torsion free. Then, M is anti-Hopfian by Corollary 8 and [7, Corollary 1].

Remark 10 ([4, Remark 6]). Right Artinian rings have no anti-Hopfian modules.

In case R is commutative, we can remove the condition that the heart of M is a non-Hopf kernel, in 2) of Theorem 7:

Theorem 11 ([4, Theorem 8]). Let R be a commutative ring, and M an R-module. Then, M is anti-Hopfian if and only if $L(M)$ is lattice isomorphic to ω^+.

Next, we shall consider a restricted anti-Hopfian module M with $M = Rad(M)$. When this is the case, for any non-zero proper submodule N of M, M/N is a non-simple R-module all of whose factor modules are isomorphic. Hence, M is a restricted anti-Hopfian module with $M = Rad(M)$ if and only if M/N is anti-Hopfian for every non-zero proper submodule N of M.

As is well known, every non-zero module has a subdirectly irreducible factor module (see, e.g., [1, p.95]). Hence every restricted anti-Hopfian module is restricted subdirectly irreducible.

In case R is a CH-ring, Theorem 3 and Theorem 7 enable us to characterize restricted anti-Hopfian R-modules M with $M = \mathrm{Rad}(M)$.

Theorem 12 ([5, Theorem 10]). Let R be a CH-ring, and M an R-module such that $M = \mathrm{Rad}(M)$. Then, M is restricted anti-Hopfian if and only if one of the following holds:

(1) 1a) The set of proper submodules of M forms a chain

$$0 \subsetneqq M_1 \subsetneqq M_2 \subsetneqq M_3 \subsetneqq \cdots$$

such that

$$\bigcup_{i \in N} M_i = M, \text{ and}$$

1b) M_2/M_1 is a non-Hopf kernel for M/M_1.

(2) 2a) The set of proper submodules of M forms a chain

$$\cdots \subsetneqq M_{-2} \subsetneqq M_{-1} \subsetneqq M_0 \subsetneqq M_1 \subsetneqq M_2 \subsetneqq \cdots$$

such that

$$\bigcap_{i \in Z} M_i = 0, \quad \bigcup_{i \in Z} M_i = M, \text{ and}$$

2b) for each i, M_{i+1}/M_i is a non-Hopf kernel for M/M_i.

(3) 3a) Soc(M) is a waist of M, and is a direct sum of two isomorphic simple modules and the set of proper submodules of M containing Soc(M) forms a chain

$$M_1 = \mathrm{Soc}(M) \subsetneqq M_2 \subsetneqq M_3 \subsetneqq \cdots$$

such that

$$\bigcup_{i \in N} M_i = M, \text{ and}$$

3b) for every simple submodule S of M, M_1/S is a non-Hopf kernel for M/S.

Corollary 13 ([5, Corollary 11]). Let R be a commutative ring, and M an R-module such that $M = \mathrm{Rad}(M)$. Then, M is restricted anti-Hopfian if and only if one of the following holds:

(1) The set of proper submodules of M forms a chain

$$0 \subsetneqq M_1 \subsetneqq M_2 \subsetneqq M_3 \subsetneqq \cdots$$

such that

$$\bigcup_{i \in \mathbf{N}} M_i = M,$$

that is, M is anti-Hopfian.

(2) The set of proper submodules of M forms a chain

$$\cdots \subsetneqq M_{-2} \subsetneqq M_{-1} \subsetneqq M_0 \subsetneqq M_1 \subsetneqq M_2 \subsetneqq \cdots$$

such that

$$\bigcap_{i \in \mathbf{Z}} M_i = 0, \quad \bigcup_{i \in \mathbf{Z}} M_i = M.$$

Definitions. (j) Following P.M. Cohn [3], an integral domain R (not necessarily commutative) is said to be a discrete valuation ring if there exists a prime element p such that every element of R is of the form

$$up^n \quad (n \geq 0,\ u \text{ a unit}).$$

When R is commutative, this reduces to the usual definition.

(k) A ring R is called left duo if every left ideal of R is two-sided.

Given a non-empty subset N of an R-module M, we put $\mathrm{Ann}_R(N) = \{r \in R \mid xr = 0 \text{ for all } x \in N\}$ (the annihilator of N).

We shall exhibit here some properties of anti-Hopfian modules and restricted anti-Hopfian modules, and the structure of their endomorphism rings.

Proposition 14 ([4, Theorem 5], [5, Proposition 12]). Let R be a CH-ring, and M an R-module.

(I) If M is anti-Hopfian, then

(1) every proper submodule of M is cyclic;

(2) $S = \mathrm{End}_R(M)$ is a (left duo) complete discrete valuation ring;

(3) M is injective as a left S-module;

(4) for every non-zero right ideal I of $\bar{R} = R/\mathrm{Ann}_R(M)$, $MI = M$, and so $\bar{R}$ is a prime ring.

(II) If M is not anti-Hopfian but restricted anti-Hopfian, and $M = \mathrm{Rad}(M)$, then

(1) every proper submodule of M is finitely generated;

(2) $S = End_R(M)$ is a division ring.

Proposition 15 ([4, Theorem 10], [5, Lemma 13]). Let R be a commutative ring, and M an R-module.

(I) If M is anti-Hopfian, then

(1) M is a quasi-injective R-module;

(2) $S = End_R(M)$ is a commutative, complete discrete valuation ring;

(3) M is isomorphic to $Q(S)/S$, where $Q(S)$ is the field of fractions of S;

(4) S is a homomorphic image of the completion of $\bar{R} = R/Ann_R(M)$ with respect to the $\mathfrak{p}$-adic topology, where $\mathfrak{p}$ is the annihilator of the heart of M.

(II) If M is not anti-Hopfian but restricted anti-Hopfian, and $M = Rad(M)$, then

(1) every proper submodule of M is cyclic;

(2) any two non-zero proper submodules are isomorphic;

(3) $\bar{R} = R/Ann_R(M)$ is a discrete valuation ring;

(4) M is an injective $\bar{R}$-module (so that M is a quasi-injective R-module).

Furthermore, in case R is commutative, we can describe explicitly the classes of anti-Hopfian R-modules and restricted anti-Hopfian R-modules.

Theorem 16 ([5, Theorem 14]). Let R be a commutative ring, and M an R-module. Then,

(I) M is restricted anti-Hopfian if and only if one of the following holds:

(1) M has exactly one non-zero proper submodule;

(2) M is a direct sum of two isomorphic simple modules;

(3) $S = End_R(M)$ is a discrete valuation ring, $M \cong Q(S)/S$ and $L_S(M) = L_R(M)$;

(4) $\bar{R} = R/Ann_R(M)$ is a discrete valuation ring and M is isomorphic to $Q(\bar{R})$.

(II) M is anti-Hopfian if and only if

(3) $S = End_R(M)$ is a discrete valuation ring, $M \cong Q(S)/S$ and $L_S(M) = L_R(M)$.

Finally, we shall extend Szélpál's theorem [8] to modules over a Dedekind domain.

Corollary 17 ([4, Theorem 9], [5, Corollary 15]). Let R be a Dedekind domain, and M an R-module. Then,

(I) M is restricted anti-Hopfian if and only if one of the following holds:

(1) $M \cong R/P^2$;

(2) $M \cong R/P \oplus R/P$, where P is a non-zero prime ideal of R;

(3) M is isomorphic to $R(P^\infty)$ for some prime ideal P;

(4) R is a discrete valuation ring and M is isomorphic to $Q(R)$.

In particular, if $M = Rad(M)$, the following statements are equivalent:

1) M is a restricted anti-Hopfian module.

2) M is a restricted subdirectly irreducible module.

(II) M is anti-Hopfian if and only if

(3) M is isomorphic to $R(P^\infty)$ for some prime ideal P.

Remark 18 ([4, Remark 11]). Let R be a commutative ring, and M an anti-Hopfian R-module. Then, by Proposition 15, $End_R(M)$ is a complete discrete valuation ring and $Q(End_R(M))/End_R(M) \cong M$ as R-modules.

Conversely, let T be a complete discrete valuation ring. Then, as was seen before, the T-module $Q(T)/T$ is anti-Hopfian. Since T is complete, it is easy to see that $End_T(Q(T)/T) \cong T$. In conclusion, every complete discrete valuation ring may be regarded as the endomorphism ring of a certain anti-Hopfian module.

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LEFT SERIAL RINGS OVER WHICH EVERY RIGHT MODULE WITH HOMOGENEOUS TOP IS A DIRECT SUM OF HOLLOW MODULES

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Throughout this note, R is a left and right artinian ring with identity 1 and J is the Jacobson radical of R. In [4], M. Harada has considered a left serial ring R satisfying the condition (*,2) that every maximal submodule of a direct sum of any two hollow modules is also a direct sum of hollow modules, and characterized such a ring by the structure of eR for each primitive idempotent e. And it is shown that the condition (*,2) is equivalent to saying that every factor module of $eJ \oplus eR$ is a direct sum of hollow modules. Here we consider the following condition on a projective indecomposable right module eR over a ring R.

(A): Every factor module of $eR \oplus eR$ is a direct sum of hollow modules.

Clearly if R is a ring of right local type, then all projective indecomposable right R-modules satisfy the condition (A), and as well known ([6]), R is left serial. The purpose of this note is to characterize a left serial ring R over which

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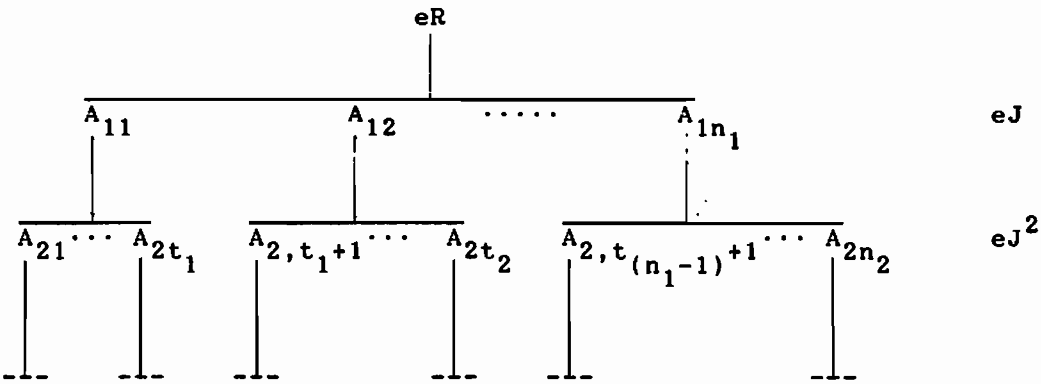
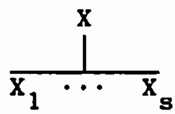
The detailed version of this paper has been submitted for publication elsewhere.

every projective indecomposable right module eR satisfies the condition (A) (i.e. a ring R as described in the title (see Theorem 1)) in terms of the structure of eR. Thus our result gives a generalization of rings of right local type.

Lemma. ([6, Corollary 4.2]) Let R be a left serial ring. For each primitive idempotent e and each natural number j, eJ^j is a direct sum of hollow modules, and eR has a structure expressed by the following diagram:



where each A_{ik} is a hollow module, $eJ^i = \bigoplus_{k=1}^{n_i} A_{ik}$ and



means $XJ = \bigoplus_{i=1}^{s} X_i$.

Theorem 1. For a primitive idempotent e, the following four statements are equivalent.

(1) eR satisfies the condition (A).

(2) Every factor module of $eR^{(n)}$ is a direct sum of hollow modules for each natural number n.

(3) If M is an R-module such that $M/MJ \simeq (eR/eJ)^{(n)}$ for some n, then $M \simeq \bigoplus_{i=1}^{n} eR/X_i$, where each X_i is a submodule of eR.

(4) Let C_i and D_i $(i = 1, 2)$ be submodules of eR such

that $eR \geqq C_i > D_i$. If $f : C_1/D_1 \longrightarrow C_2/D_2$ is an isomorphism, and C_1/D_1 is simple, then f or f^{-1} is extended to some homomorphism from eR/D_1 to eR/D_2 or one from eR/D_2 to eR/D_1, respectively.

The equivalence of (1) and (3) in the theorem tells us that the property of a ring R that the condition (A) holds for all projective indecomposable right R-modules is Morita invariant. So we will assume that R is always a basic ring. The condition (4) is used to check whether eR satisfies the condition (A) or not.

From now on, R is a left serial ring. If eR satisfies the condition (A), then the following two propositions hold.

Proposition 1. If eR/eJ^i is uniserial and $eJ^i = A_{i1} \oplus A_{i2} \oplus \cdots \oplus A_{ip}$ for some i, where $p \geqq 3$ and each $A_{ik} (1 \leqq k \leqq p)$ is a hollow module, then we have $A_{i1} \simeq A_{i2} \simeq \cdots \simeq A_{ip}$, and each A_{ik} is simple.

Proposition 2. If eR/eJ^i is uniserial and $eJ^i = A_{i1} \oplus A_{i2}$ for some i, where A_{i1} and A_{i2} are hollow modules, then both A_{i1} and A_{i2} are uniserial.

From these two propositions, eR with the condition (A) has one of the following structures.

(a) eR is a uniserial module.

(b_1) For some natural number i, eR/eJ^i is uniserial and $eJ^i = A_{i1} \oplus A_{i2}$, where A_{i1} and A_{i2} are uniserial modules which are not isomorphic to each other.

(b_2) For some natural number i, eR/eJ^i is uniserial and $eJ^i = A_{i1} \oplus A_{i2}$, where $A_{i1} \simeq A_{i2}$ and A_{i1} is uniserial.

(c) For some natural number i, eR/eJ^i is uniserial and $eJ^i = A_{i1} \oplus \cdots \oplus A_{ip}$ ($p \geqq 3$), where $A_{i1} \simeq A_{i2} \simeq \cdots \simeq A_{ip}$ are simple.

We put $\Delta := eRe/eJe$ and $\Delta(A) := \{ \bar{x} \in \Delta \mid x'A \leqq A, \bar{x}' = \bar{x}$ for some $x' \in eRe \}$, where A is a hollow module and $\bar{x}$ is the coset of x in Δ. Then Δ is a division ring and $\Delta(A)$ is a division subring of Δ. In the case that $eJ^i = A_{i1} \oplus A_{i2} \oplus \cdots \oplus A_{ip}$ $(p \geqq 2)$, we put $\Delta(A_{i1}) = \Delta_i$. If eR with the condition (A) has the structure (b_2) or (c), then the dimension $[\Delta : \Delta_i]_r$ of Δ as a right Δ_i-vector space is equal to the length of the top of eJ^i. Further in this case, the following condition (#) on Δ as a right Δ_i-vector space holds.

(#) Let V_1 and V_2 be subspaces of Δ_{Δ_i} and v_1 and v_2 be elements of Δ satisfying $|V_1| \leqq |V_2|$ and $v_1\Delta_i \cap V_1 = 0 = v_2\Delta_i \cap V_2$. Then there exists $\bar{x}$ in Δ such that $xV_1 \leqq V_2$ and $xv_1 \equiv v_2 \pmod{V_2}$.

By [4, Lemma 5], the following holds.

<u>Proposition 3</u>. Let $\Delta \geqq \Delta_i$ be division rings. If Δ and Δ_i satisfy the condition (#), then the left dimension $[\Delta : \Delta_i]_\ell$ is $\leqq 2$. In particular, if eR satisfies the condition (A), then $[\Delta : \Delta_i]_\ell \leqq 2$.

By this proposition, if eR with (A) has the structure (b_2) or (c), then we have $[\Delta : \Delta_i]_\ell = 2$.

Conversely, if eR has the structure (b_2) with $[\Delta : \Delta_i]_\ell = 2$, then eR satisfies the condition (A), and if eR has the structure (c) with the condition (#) for Δ and Δ_i, then eR satisfies the condition (A). As a consequence, we obtain the following main theorem.

<u>Theorem 2</u>. Let R be a left serial ring. The following are equivalent for each primitive idempotent e.

(1) eR satisfies the condition (A).

(2) eR has one of the structures (a); (b_1); (b_2) with $[\Delta : \Delta_i]_\ell = 2$; and (c) with the condition (#) for Δ and Δ_i.

Remark. If R is a finite dimensional algebra over a field, then $[\Delta : \Delta_i]_r = [\Delta : \Delta_i]_\ell$ holds. So for a primitive idempotent e, if eR satisfies the condition (A), it follows from $[\Delta : \Delta_i]_\ell \leqq 2$ (by Proposition 3) that $[\Delta : \Delta_i]_r \leqq 2$. Hence eR never has the structure (c). Further suppose that R is a finite dimensional algebra over an algebraically closed field. Then we have $[\Delta : \Delta_i]_r = [\Delta : \Delta_i]_\ell = 1$. Hence eR has the structure (a) or (b_1).

Examples

Here we give some examples of left serial rings having projective indecomposable modules with structures (b_1), (b_2), and (c) which satisfy the condition (A).

Example 1. Let k be a field and put

$$R := \begin{pmatrix} k & k & k & k \\ 0 & k & k & k \\ 0 & 0 & k & 0 \\ 0 & 0 & 0 & k \end{pmatrix}, \qquad e := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then every projective indecomposable R-module satisfies the condition (A) and eR has the structure (b_1). Note that R is not of right local type.

Example 2. Let $K \leqq L$ be fields with $[L : K] = 2$. Put

$$R := \begin{pmatrix} L & L & L \\ 0 & L & L \\ 0 & 0 & K \end{pmatrix} \quad \text{and} \quad e := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then eR has the structure (b_2) and satisfies the condition on the left dimension in Theorem 2. Also in this case every projective indecomposable module satisfies the condition (A) but R is not of right local type.

<u>Example 3</u>. (Asashiba [1]) Let F and G be division rings and M an (F, G)-bimodule having the dimension sequence (3,1,2,2,1) (see Dowbor, Ringel and Simson [2]). The existence of such an M follows from Schofield [5, section 13] and [2, Proposition 1]. Then $R := \begin{pmatrix} F & M \\ 0 & G \end{pmatrix}$ has exactly 5 non-isomorphic indecomposable modules and $[M : G]_r = 3$, say $M_G = A_1 \oplus A_2 \oplus A_3$ with each $A_i \simeq G_G$. Put $e_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $e_2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then we can identify $e_1J_R = M_G$. Since the set $S := \{e_2R, e_1R, e_1R/A_1, e_1R/(A_1 \oplus A_2), e_1R/e_1J\}$ consists of 5 non-isomorphic local modules, S is a complete set of representatives of isomorphism classes of indecomposable R-modules. Thus R is of right local type. Hence every projective indecomposable R-module satisfies the condition (A). In particular so does e_1R. Further since e_1J is isomorphic to a direct sum of three copies of a simple module, e_1R has the structure (c) and satisfies the condition (#).

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THE REPRESENTATION TYPES OF CERTAIN TRIANGULAR MATRIX ALGEBRAS

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In this talk, we summarize our results [7] and [8]. Throughout, we work over a fixed algebraically closed field k .

1. Main Results.

Let Λ be a finite dimensional self-injective algebra and assume that Λ is basic, connected and non-simple. For an integer $p \geq 2$, denote by $T_p(\Lambda)$ the algebra of the $p \times p$ upper triangular matrices over Λ . We ask when $T_p(\Lambda)$ is tame. So we may assume that Λ is representation-finite, otherwise $T_p(\Lambda)$ is wild [18]. Then, as well known, the universal cover of the stable Auslander-Reiten quiver of Λ is isomorphic to a Dynkin-translation-quiver $Z\Delta$ [11], where $\Delta = A_q$ $(q \geq 1)$, D_q $(q \geq 4)$ or E_q $(q = 6,7,8)$, and Δ is called the Dynkin class of Λ .

Theorem 1. Let Λ be as above. Then, $T_2(\Lambda)$ is tame if and only if Λ has Dynkin class A_3 .

Remark. The case $p > 2$ is rather easy. Denote by $J_p(\Lambda)$ the ideal of $T_p(\Lambda)$ consisting of the strictly upper triangular matrices. If $p > 2$ and $p \geq r \geq 2$, $T_p(\Lambda)/J_p(\Lambda)^r$ is tame if and only if Λ is a Nakayama algebra of Dynkin class A_q and $(p,q,r) = (3,2,2)$, $(4,1,3)$ or $(4,1,4)$ (cf. [16]).

Our second objective is the algebras of the form $\begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$ with A, B local. There has been the complete list of the representation-finite algebras of this type [2].

<u>Theorm 2.</u> Let $\Lambda = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$ be a finite dimensional algebra with A, B local. Then, Λ is tame if and only if Λ is isomorphic or anti-isomorphic to one of the following:

0) $\cdot \rightrightarrows \cdot$

1) $\cdot \xrightarrow{\mu} \cdot \circlearrowleft \alpha$ with $\alpha^2\mu = \alpha^6 = 0$

2) $\beta \circlearrowright \cdot \xrightarrow{\mu} \cdot \circlearrowleft \alpha$ with $\alpha^2 = \beta^2 = 0$

2') with $\alpha\mu\beta = \alpha^2 = \beta^2 = 0$

3_q) with $\alpha\mu - \mu\beta = \alpha^q\mu = \alpha^6 = \beta^3 = 0$, $q = 2,3$

4_q) with $\alpha\mu - \mu\beta = \alpha^q\mu = \alpha^4 = \beta^4 = 0$, $q = 2,3,4$

Recall now that an algebra Λ is said to be representation-finite if there are only a finite number of pairwise non-isomorphic indecomposable objects in mod Λ, the category of finite dimensional left Λ-modules, to be wild if there is an exact embedding mod $k\Omega \longrightarrow$ mod Λ, where $k\Omega$ is the path algebra of the quiver Ω: $\circlearrowright \cdot \circlearrowleft$, which is a representation equivalence with the corresponding full subcategory of mod Λ, and to be tame if Λ is neither representation-finite nor wild.

2. Basic Tools.

Covering techniques ([2], [3], [5] and [6]) will play an indispensable role in deciding the representation type of a given algebra. For a certain class of algebras, by taking appropriate Galois coverings, the problem can be reduced to the calculation of vector space categories, which have been classified in [14] (see also [9]). On the other hand, we will come across an algebra which can be obtained as a quotient of a suitable Galois covering of the tame local algebra $\tau \circlearrowright \cdot \circlearrowleft \sigma$ with $\sigma^2 = \tau^2 = 0$ [13], thus is tame. The similar argument will also apply to the situation that there is a Galois covering of a given algebra which has a wild algebra as a quotient.

2.1. <u>Locally Bounded Categories</u>.

A locally bounded category Λ is a k-category such that: a) distinct objects are not isomorphic; b) for each $x \in \Lambda$, the algebra $\Lambda(x,x)$ is local; c) for each $x \in \Lambda$, $\sum_{y \in \Lambda}[\Lambda(x,y):k]$ and $\sum_{y \in \Lambda}[\Lambda(y,x):k]$ are finite [2]. The support supp M of a Λ-module M is the full subcategory of Λ consisting of the objects $x \in \Lambda$ such that $M(x) \neq 0$. The dimension vector of a Λ-module M is the family <u>dim</u> M = $[M(x):k]_{x \in \Lambda}$. Let Γ_i $(i \in I)$ be a family of full subcategories of Λ. Denote by $\bigcup_{i \in I}\Gamma_i$ the full subcategory of Λ consisting of the objects of the Γ_i. For a family of objects $x_i \in \Lambda$ $(i \in I)$, we denote by $\{x_i\}_{i \in I}$ the full subcategory consisting of the objects x_i. Λ is said to be locally support-finite if for each $x \in \Lambda$, $\bigcup_{\substack{M(x)\neq 0 \\ M \in \text{ind}\,\Lambda}}$ supp M is finite [5].

2.2. <u>Galois Coverings</u>.

Let Λ be a connected locally bounded category and G a group of k-linear automorphisms of Λ. Then G acts naturally on mod Λ by the left. We assume that the action of G on Λ is free, that is, $gx \neq x$ for any $g \in G \setminus \{1\}$ and any $x \in \Lambda$. Following [6], we can consider the quotient Λ/G and the Galois covering $F:\Lambda \longrightarrow \Lambda/G$. Then we have the push down functor F_λ:mod $\Lambda \longrightarrow$ mod Λ/G which is left adjoint to the induced functor $F.$:mod $\Lambda/G \longrightarrow$ mod Λ. If G acts freely on ind Λ, that is , ${}^{g}M \neq M$ for any $g \in G\setminus\{1\}$ and any $M \in \mathrm{ind}\,\Lambda$, then F_λ preserves the Auslander-Reiten sequences. We will freely use the following results.

Proposition 3 (see [6]). Let S be a quotient category of Λ with the natural embedding $\mathrm{mod}\,S \longrightarrow \mathrm{mod}\,\Lambda$, and $L = \{ X \in \mathrm{ind}\,S \mid {}^{g}M \notin \mathrm{ind}\,S$ for any $g \in G\setminus\{1\} \}$. Then there exists a set-theoretic injection $L \longrightarrow \mathrm{ind}\,\Lambda/G$. In particular, in case L is cofinite in $\mathrm{ind}\,S$, the following hold.
(1) If Λ/G is tame, so is S unless it is representaion-finite.
(2) If S is wild, so is Λ/G .

Proposition 4 ([5]). If Λ is locally support-finite and if G acts freely on $\mathrm{ind}\,\Lambda$, then the push down functor $F_\lambda:\mathrm{mod}\,\Lambda \longrightarrow \mathrm{mod}\,\Lambda/G$ is dense. In particular, if Λ is tame, so is Λ/G .

In what follows, we will deal only with a full subcategory A of a Galois covering U which is in fact a quotient category, thus we may consider $\mathrm{mod}\,A$ as a full subcategory of $\mathrm{mod}\,U$ by the natural embedding.

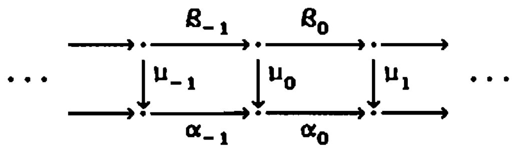
2.3. Vector Space Categories.

A vector space category K is an additive k-category together with a faithful functor $|\ |:K \longrightarrow \mathrm{mod}\,k$ such that every idempotent in K splits. Given a vector space category K , its subspace category $U(K)$ is defined as follows: its objects are triples of the form (U,X,Φ) , where U is a k-space, X is an object in K , and $\Phi:U \longrightarrow |X|$ is a k-linear map. A homomorphism from (U,X,Φ) to (U',X',Φ') is given by a pair (α,β) where $\alpha:U \longrightarrow U'$ is k-linear, $\beta:X \longrightarrow X'$ is a morphism in K such that $|\beta|\Phi = \Phi'\alpha$. Given a poset S , considered as a category, add kS is a vector space category. Conversely, assume that K is a vector space category with only 1-dimensional indecomposable objects, then K is of the form add kS for some poset S .

Let Λ be a one-point extension algebra of R by M , then any Λ-module is given by a triple $({}_kU, {}_RX, \Phi: {}_RM \otimes {}_kU \longrightarrow {}_RX)$. It is well known that $U(\mathrm{Hom}(M, \mathrm{mod}\, R))$ is representation equivalent to the full subcategory of mod Λ consisting of the Λ-modules without non-zero direct summands of the form $(k,0,0)$ or $(0,Y,0)$ with $\mathrm{Hom}(M,Y) = 0$. In case R is tame, if the vector space category $\mathrm{Hom}(M, \mathrm{mod}\, R)$ is tame, so is Λ .

3. About Theorem 1.

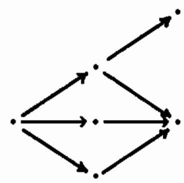
Let us consider first the case where Λ is a Nakayama algebra. Suppose that Λ is a Nakayama algebra of Dynkin class A_q . Then, $T_2(\Lambda)$ has the following universal Galois covering U:



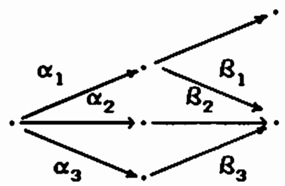
with $\alpha_i\mu_i - \mu_{i+1}\beta_i = \alpha_{i+q}\cdots\alpha_{i+1}\alpha_i = \beta_{i+q}\cdots\beta_{i+1}\beta_i = 0$ for all $i \in \mathbb{Z}$. If $q \leq 2$ then U is locally representation-finite [4], if $q = 3$ then U is locally support-finite and tame [17], and if $q \geq 4$ then U has a finite quotient which is wild [17]. Thus, in this case, $T_2(\Lambda)$ is tame if and only if $q = 3$.

In what follows, we assume that Λ is not a Nakayama algebra. Notice that Λ is a Nakayama algebra if Λ has Dynkin class A_q $(q \leq 2)$.

Consider next the case where Λ has Dynkin class A_q ($q \geq 4$), D_q ($q \geq 4$) or E_q ($6 \leq q \leq 8$). Then, as easily seen, the Auslander-Reiten quiver of Λ has the following full subquiver:



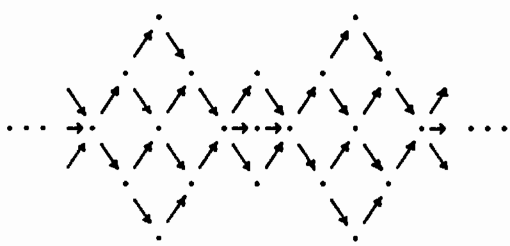
and, as a quotient, the Auslander algebra over Λ has the following algebra:



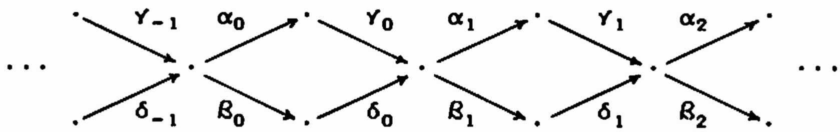
with $\beta_1\alpha_1 + \beta_2\alpha_2 + \beta_3\alpha_3 = 0$.

This is a concealed hereditary algebra of type $\tilde{\tilde{D}}_4$. Notice that $T_2(\Lambda)$ is representation equivalent to the Auslander algebra over Λ [1], because Λ is assumed to be representation-finite. Thus $T_2(\Lambda)$ is wild.

It only remains the case of Λ having Dynkin class A_3 . Then, the universal cover of the Auslander-Reiten quiver of Λ is the following:

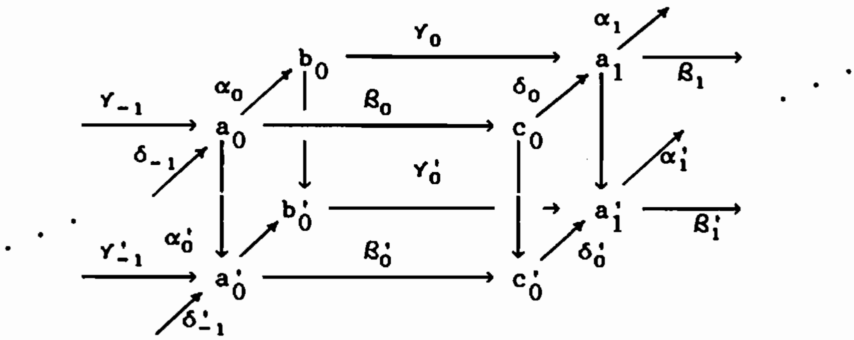


Thus, since Λ is standard [12], Λ has the following universal Galois covering:



with $\gamma_i\alpha_i - \delta_i\beta_i = \alpha_{i+1}\delta_i = \beta_{i+1}\delta_i = 0$ for all $i \in Z$ [6].

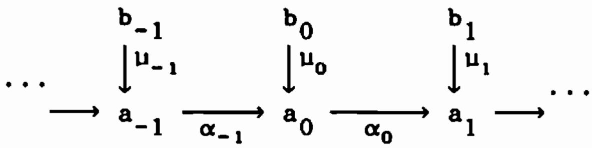
Hence, by [5], it suffices to prove that the following locally bounded category U is locally support-finite and tame:



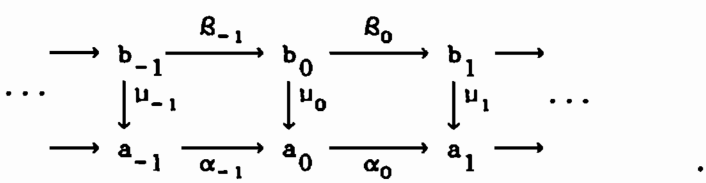
with $\alpha_{i+1}\delta_i = \beta_{i+1}\gamma_i = \alpha'_{i+1}\delta'_i = \beta'_{i+1}\gamma'_i = 0$ for all $i \in Z$ and all the squares commutative.

4. About Theorem 2.

We have only to consider the algebras of ordinary quiver $\cdot \longrightarrow \cdot \circlearrowleft$ or $\circlearrowright \cdot \longrightarrow \cdot \circlearrowleft$, which have the following coverings:



or



For only (2), the Galois covering is not locally support-finite, but, this is a quotient of the following tame algebra:

$\beta \circlearrowleft b \underset{\nu}{\overset{\mu}{\rightleftarrows}} a \circlearrowright \alpha$ with $\alpha^2 = \beta^2 = \nu\alpha\mu = \mu\beta\nu = \nu\mu = \mu\nu = 0$ (see [4] for details). Thus, (2) is tame.

Remark. Given a representation $\psi \circlearrowleft V \circlearrowright \phi$ of the quiver $\tau \circlearrowleft \cdot \circlearrowright \sigma$ with relations $\sigma^2 = \tau^2 = 0$, by defining the representation $\psi \circlearrowleft V \xrightarrow{1} V \circlearrowright \phi$, we obtain a full exact embedding. Since the above algebra is a Galois covering of the algebra: $\tau \circlearrowleft \cdot \circlearrowright \sigma$ with $\sigma^2 = \tau^2 = \tau\sigma\tau = 0$, with Galois group $\approx Z/2Z$, by Proposition 3, the category of the finite dimensional representations of the quiver
$\tau \circlearrowleft \cdot \circlearrowright \sigma$ with relations $\sigma^2 = \tau^2 = 0$ is similar to that of the quiver $\tau \circlearrowleft \cdot \circlearrowright \sigma$ with relations $\sigma^2 = \tau^2 = \tau\sigma\tau = 0$. Note that the latter is a finite dimensional algebra.

5. An Example of Tameness.

As an example for proofs of tameness of algebras in our theorems, we show that the following algebra is tame:

(3_3) $\beta \circlearrowleft b \xrightarrow{\mu} a \circlearrowright \alpha$ with $\alpha\mu - \mu\beta = \alpha^6 = \beta^3 = 0$.

Take the universal Galois covering U with Galois group $\approx Z$:

$$\begin{array}{ccccccccc} & \longrightarrow & b_{-1} & \xrightarrow{\beta_{-1}} & b_0 & \xrightarrow{\beta_0} & b_1 & \longrightarrow & \\ \cdots & & \downarrow{\scriptstyle \mu_{-1}} & & \downarrow{\scriptstyle \mu_0} & & \downarrow{\scriptstyle \mu_1} & & \cdots \\ & \longrightarrow & a_{-1} & \xrightarrow[\alpha_{-1}]{} & a_0 & \xrightarrow[\alpha_0]{} & a_1 & \longrightarrow & \end{array}$$

with $\alpha_i\mu_i - \mu_{i+1}\beta_i = \alpha_{i+5}\cdots\alpha_{i+1}\alpha_i = \beta_{i+2}\beta_{i+1}\beta_i = 0$ for all $i \in Z$. For each $n \in Z$, let A_{2n} be the following full subcategory of U:

$$\begin{array}{ccccccccccc} & & & & & & & b_{n+4} & \longrightarrow & b_{n+5} & \longrightarrow & b_{n+6} \\ & & & & & & & \downarrow & & \downarrow & & \\ a_n & \longrightarrow & a_{n+1} & \longrightarrow & a_{n+2} & \longrightarrow & a_{n+3} & \longrightarrow & a_{n+4} & \longrightarrow & a_{n+5} & \end{array}$$

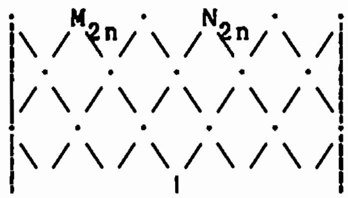
and let A_{2n-1} be the following full subcategory of U:

$$\begin{array}{ccccccccccc} & & & & & & b_{n+3} & \longrightarrow & b_{n+4} & \longrightarrow & b_{n+5} \\ & & & & & & \downarrow & & \downarrow & & \downarrow \\ a_n & \longrightarrow & a_{n+1} & \longrightarrow & a_{n+2} & \longrightarrow & a_{n+3} & \longrightarrow & a_{n+4} & \longrightarrow & a_{n+5} \end{array}$$

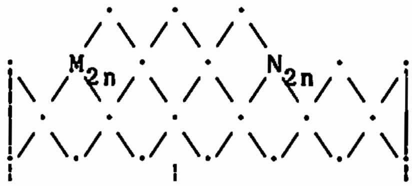
these are concealed hereditary algebras of type $\tilde{E}_8$, and for $l, m \in Z$ with $l \leq m$, as before, let $A_{l,m}$ be the full subcategory of U consisting of the objects of the A_n , $l \leq n \leq m$. Then, as an algebra, $A_{2n-1,2n+1}$ is isomorphic to

$$\begin{bmatrix} k & DN_{2n} & 0 \\ 0 & A_{2n} & M_{2n} \\ 0 & 0 & k \end{bmatrix}$$

where $M_{2n} = 0001\,{}^{1}_{1}{}^{1}_{1}{}^{0}$ and $N_{2n} = 0111\,{}^{1}_{1}{}^{1}_{1}{}^{1}$ are regular modules belonging to the same tube:



The vector space categories $\mathrm{Hom}(M_{2n}, \mathrm{mod}\ A_{2n})$ and $\mathrm{Hom}(\mathrm{mod}\ A_{2n}, N_{2n})$ belong to the pattern $(\tilde{E}_8, 5)$, and $\mathrm{ind}\ A_{2n-1,2n+1} = P_{2n} \cup R_{2n} \cup Q_{2n}$, where P_{2n} consists of the objects of $\mathrm{ind}\ A_{2n,2n+1}$ with restriction to A_{2n} being preprojective, Q_{2n} consists of the objects $\mathrm{ind}\ A_{2n-1,2n}$ with restriction to A_{2n} being preinjective and R_{2n} consists of the regular objects of $\mathrm{ind}\ A_{2n}$ except that the above tube changes to the following:

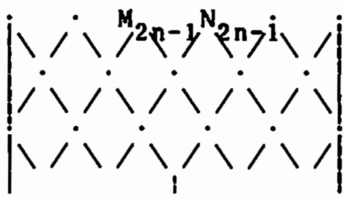


Thus, $A_{2n-1,2n+1}$ is tame.

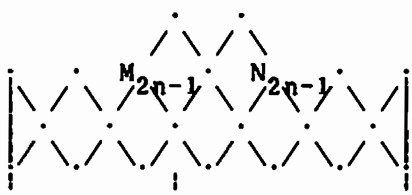
Similarly, $A_{2n-2,2n}$ is isomorphic to

$$\begin{bmatrix} k & DN_{2n-1} & 0 \\ 0 & A_{2n-1} & M_{2n-1} \\ 0 & 0 & k \end{bmatrix},$$

where $M_{2n-1} = 111\substack{0\\1}\substack{0\\1}\substack{0\\0}$ and $N_{2n-1} = 000\substack{0\\0}\substack{1\\0}\substack{1\\0}$ are regular modules:



The vector space categories $\mathrm{Hom}(M_{2n-1}, \mathrm{mod}\ A_{2n-1})$ and $\mathrm{Hom}(\mathrm{mod}\ A_{2n-1}, N_{2n-1})$ belong to the pattern $(\tilde{E}_8, 5)$, and $\mathrm{ind}\ A_{2n-2,2n} = P_{2n-1} \uplus R_{2n-1} \uplus Q_{2n-1}$, where P_{2n-1} consists of the objects of $\mathrm{ind}\ A_{2n-1,2n}$ with restriction to A_{2n-1} being preprojective, Q_{2n-1} consists of the objects of $\mathrm{ind}\ A_{2n-2,2n-1}$ with restriction to A_{2n-1} being preinjective and R_{2n-1} consists of the regular objects of $\mathrm{ind}\ A_{2n-1}$ except that the above tube changes to the following:



Thus, $A_{2n-2,2n}$ is tame.

For $l,m \in Z$ with $l \leq m$, $A_{l-1,m+1}$ is the one-point extension of $A_{l,m+1}$ by the module with support in A_l and with restriction to it being M_l . The vector space category $Hom(M_l, \text{mod } A_{l,m+1})$ is isomorphic to $Hom(M_l, \text{mod } A_l)$, and $\text{ind } A_{l-1,m+1} = \text{ind } A_{l-1,l+1} \cup \text{ind } A_{l,m+1}$. Therefore $\text{ind } U = \bigcup_{n \in Z} \text{ind } A_{n-1,n+1}$, in particular, U is locally support-finite and tame. Thus, $(T\text{-}3_3)$ is tame.

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HEREDITARY ORDERS OVER $\mathbb{P}^1$ AND EXTENDED DYNKIN DIAGRAMS

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§0. This is an exposition of Lenzing's papers [5] and [6]. Let Δ be a Dynkin quiver and $\tilde{\Delta} = \Delta \cup \{*\}$ be an extended Dynkin quiver. The representations of $\tilde{\Delta}$ were classified in [2]. They divide into three classes of modules, namely preprojective, preinjective and regular modules. The category of regular modules turns out to be equivalent to a direct sum of categories $U_{e(x)}$ indexed by points x of the projective line $\mathbb{P}^1$, where U_m denotes a uniserial category with m simple objects and we put $e(x) = 1$ for almost all x and the integers $e(x) \neq 1$ are the same as the numbers of points in branches of the graph Δ (for example if Δ is of type E_6, they are 2, 3, 3) ([3]). This phenomenon had not been explained intrinsically until [5] and [6] appeared. In characteristic zero, one can reduce to Mckay correspondence, considering vector bundles over $\mathbb{P}^1$ with a polyhedral group action. In general, Lenzing constructed an abelian category $\mathbb{F}/\mathbb{F}_0$ as a localization of some functor category on $k\tilde{\Delta}$-modules. The category $\mathbb{F}/\mathbb{F}_0$ is very similar to the category of coherent modules over a curve and the category of objects of finite length in $\mathbb{F}/\mathbb{F}_0$ is equivalent to the category of regular

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$k\tilde{\Delta}$-modules. In this report we directly construct an equivalence between $k\tilde{\Delta}$-modules and Λ-modules, where Λ is an order over $\mathbb{P}^1$ made from the Dynkin graph Δ.

§1. Let $X = \mathbb{P}^1$ be the projective line over an algebraically closed field k. Suppose that we are given a finite subset B of X and positive integers e_x for $x \in B$. From these data we shall define two objects, a quiver Δ and an $\mathcal{O}_X$-algebra Λ. The points of Δ are 0 and (x, i) where $x \in B$ and $0 < i < e_x$. We put 0 = (x, 0) for $x \in B$. The arrows of Δ are $(x, i) \to (x, i-1)$ where $0 < i < e_x$. For $x \in X$ let $\mathfrak{m}_x$ denote the $\mathcal{O}_X$-ideal defining the point x. Let $M_\Delta(\mathcal{O}_X)$ be the matrix algebra over $\mathcal{O}_X$ with size $\#\Delta$ and let e_{pq} ($p, q \in \Delta$) be the matrix units. Then Λ is the subalgebra of $M_\Delta(\mathcal{O}_X)$ whose (p, q) entry is $\mathcal{O}_X$ if there is a path from q to p, otherwise it is $\mathfrak{m}_x$ where p = (x, i).

Now assume that the underlying graph of Δ is Dynkin (i.e. A_n, D_n, E_6, E_7, E_8). Let $\tilde{\Delta} = \Delta \cup \{*\}$ be a quiver such that it contains the quiver Δ and its underlying graph is an extended Dynkin (i.e. $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$) and there is no arrow from *. We denote by $k\Delta$ and $k\tilde{\Delta}$ the path algebras of Δ and $\tilde{\Delta}$ respectively. Let r be an integer 0, 1, 2, 3 or 5 according as Δ is of type A, D, E_6, E_7 or E_8. Let TrD be Auslander-Reiten's functor for $k\Delta$-modules. Since $k\Delta$ is contained in Λ as the constants, any $k\Delta$-module M extends to a Λ-module $\Lambda \otimes_{k\Delta} M$. Define a Λ-module Z by $Z = \Lambda \otimes_{k\Delta} (\mathrm{TrD})^r(k\Delta) \oplus \Lambda e_{00} \otimes \mathcal{O}_X(1)$. It can be shown that $(\mathrm{End}_\Lambda Z)^{op} \cong k\tilde{\Delta}$, so we have a functor $Z \otimes_{k\tilde{\Delta}}$ from $k\tilde{\Delta}$-modules to Λ-modules.

Theorem. The derived functor $Z \overset{\mathbb{L}}{\otimes}_{k\tilde{\Delta}} : D(k\tilde{\Delta}) \longrightarrow D(\Lambda)$ is an equivalence from the derived category of $k\tilde{\Delta}$-modules to the derived category of quasi-coherent Λ-modules.

The proof does not use the classification of regular modules. There exist almost split sequences in locally projective Λ-modules, so we can speak of the Auslander-Reiten quiver of them. Roughly speaking, the theorem is proved by comparing the Auslander-Reiten quivers of $k\tilde{\Delta}$-modules and of locally projective Λ-modules.

One can deduce easily from the theorem that the above functor induces an equivalence from the category of regular modules to the category of Λ-modules of finite length. By the definition of Λ, the latter is equivalent to the direct sum of the categories $U_{e(x)}$ where x runs through the points of $\mathbb{P}^1$. Thus we obtain a constructive proof of the structure theorem for regular modules as stated in section 0. The following is also proved by using the equivalence of the theorem.

Corollary. For any finitely generated $k\tilde{\Delta}$-module M, the ring $\oplus_{n\geq 0} \mathrm{Hom}_{k\tilde{\Delta}}(M, (\mathrm{TrD})^n M)$ is a finitely generated module over its center which is of Krull dimension ≤ 2.

The fact $D(k\tilde{\Delta}) \cong D(\Lambda)$ may be also proved as follows. There is an equivalence $D(\Lambda) \cong D(T)$ where $T = \mathrm{End}_{\Lambda}(\Lambda \oplus \Lambda e_{00} \oplus \mathcal{O}_X(1))^{op}$ ([4] Proposition 4.1). This finite dimensional algebra T was called the canonical algebra and studied by Ringel. Then it suffices to find an equivalence $D(T) \cong D(k\tilde{\Delta})$, which is probably known.

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