PROCEEDINGS OF THE
20TH SYMPOSIUM ON RING THEORY

HELD AT OKAYAMA UNIVERSITY, OKAYAMA

AUGUST 27—29, 1987

EDITED BY

HIDETOSHI MARUBAYASHI
Naruto University of Education

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OKAYAMA, JAPAN

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DEDICATED TO PROFESSOR HISAO TOMINAGA FOR HIS 60TH BIRTHDAY

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PREFACE

The 20th Symposium on Ring Theory was held at Okayama University, Okayama, on August 27-29, 1987. More than one hundred participants attended the symposium. This volume is dedicated to Professor H. Tominaga for his 60th birthday, who is one of the founders and the greatest contributor to the development of the symposium.

The proceedings contain fifteen articles presented at the symposium including the ones given by two special guests, Professor J.S. Golan, Israel and Professor A. Haghany, Iran.

We should like to acknowledge the financial assistance from the Grant-in-Aid for Scientific Research from the Ministry of Education through the arrangements by Professor K. Shiratani. We appreciate his arrangements.

We are most grateful to the participants and, in particular, the speakers who contributed to success of the symposium. Our best thanks are due to Professor H. Tominaga for his help with the organizational work. Finally we should like to thank Professor T. Nagahara and Dr. H. Komatsu for their kind hospitality and compilation of the proceedings.

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Hidetoshi Marubayashi
Naruto University of Education
Naruto, 772 Japan

PROCEEDINGS OF THE 20TH
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THE SKEW FORMAL POWER SERIES RINGS OVER TAME ORDERS IN A SIMPLE ARTINIAN RING

Hidetoshi MARUBAYASHI and Akira UEDA

Introduction. It is well-known that if D is a commutative Krull domain, then so is D[[x]], the formal power series ring over D in an indeterminate x. The purpose of this paper is to investigate this problem in non-commutative situation. However, there are several obstructions in non-commutative case, for example, if R is a prime Goldie ring, then R[[x]] is not necessary to be a prime Goldie ring (see [11]). So, in this paper, we restrict ourself to the rings which are noetherian prime rings. Let R be a noetherian prime ring and let σ be an automorphism of R. We introduce, in Section 1, a notion of σ-maximal orders, and it is established that R[[x,σ]], the skew formal power series ring over R in an indeterminate x, is a maximal order (then it is a Krull order in the sense of [2]) if and only if R is a σ-maximal order. This is an affirmative answer to Ramras's conjecture ([21]) in the case R is a regular local ring and σ is of finite order. Let D be a noetherian integrally closed domain and let Λ be a tame D-order in the sense of [6] with an automorphism σ. In Section 2, we show that Λ[[x,σ]] is a v-HC order with enough v-invertible ideals. This is proved by using the result in Section 1 and the idealizers

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This paper is in final form and no version of it will be submitted for publication elsewhere.

defined in [8]. In the case of orders over a Krull domain, v-HC orders are nothing else than tame orders (see [14]). When σ is of infinite order, then $\Lambda[[x,\sigma]]$ is not a tame order over its center. Section 3 is concerned with v-invertible ideals of $\Lambda[[x,\sigma]]$. This paper is a continuation of [12] and [14] ∿ [18], and the authors assume that the reader is familiar with [14].

1. Throughout this paper, R will be an order in a simple artinian ring Q, otherwise stated, i.e., R is a prime Goldie ring and all rings are assumed to have an identity element. Let σ be an automorphism of R. Then it is extended to an automorphism σ of Q by $\sigma(ac^{-1}) = \sigma(a)(\sigma(c))^{-1}$, where $a \in R$ and c is a regular element in R. Let $R[[x,\sigma]]$ be a skew formal power series ring over R in an indeterminate x, i.e., $R[[x,\sigma]] = \{ f(x) = \sum_{i=0}^{\infty} r_i x^i \mid r_i \in R \}$ and the multiplication is defined by $xr = \sigma(r)x$ for every $r \in R$. It is well-known that $R[[x,\sigma]]$ is a prime ring (furthermore $R[[x,\sigma]]$ is semi-prime if R is semi-prime). But it is not necessary to be an order in a simple artinian ring as it has been shown in [11]. If R is a noetherian prime ring, then $R[[x,\sigma]]$ is a prime Goldie ring, because $R[[x,\sigma]]$ is also a noetherian ring which follows as in commutative algebras (see, Theorem 4 of [25, p. 138]). So, in this case, $R[[x,\sigma]]$ has a simple artinian quotient ring $Q(R[[X,\sigma]])$. We always denote by Q(T) the (classical two-sided) quotient ring of a ring T.

In the remainder of this section, R is a noetherian order in Q with an automorphism σ. To give a necessary and sufficient condition for $R[[x,\sigma]]$ to be a maximal order in $Q(R[[x,\sigma]])$, we shall introduce a notion of a σ-maximal order as follows; an order R in Q is said to be σ-maximal if, for any over-ring T of R, $R \subseteq T \subseteq Q$ such that $aTb \subseteq R$ for some regular elements a, b in Q and $\sigma(T) = T$, then $R = T$ follows. A one-sided R-ideal I is called σ-invariant if $\sigma(I) = I$. Note that R is a σ-maximal order if and only if $O_l(A) = R = O_r(A)$ for any σ-invariant ideal A of R (see, the proof of Proposition 3.1 of [19, Chap. I]), where $O_l(A) = \{q \in Q \mid qA \subseteq A\}$, the left order of A and $O_r(A) = \{q \in Q \mid Aq \subseteq A\}$, the right order of A. For any right R-ideal I, let $I[[x,\sigma]] =$

$\{a(x) = \sum_{i=0}^{\infty} a_i x^i \mid a_i \in I\}$. Then it is a right $R[[x,\sigma]]$-ideal. Similarly we define $[[x,\sigma]]J$ for any left R-ideal J. In the case of noetherian rings, we have

Lemma 1.1. Let I be any right R-ideal. Then $I[[x,\sigma]] = I\cdot R[[x,\sigma]]$.

Proof. It is clear that $I\cdot R[[x,\sigma]] \subseteq I[[x,\sigma]]$. To prove the converse inclusion, let c be a regular element of R such that $cI \subseteq R$. Since R is noetherian, there are elements $a_1, \cdots, a_n$ of R such that $cI = a_1R + \cdots + a_nR$. Let $f(x) = \sum_{i=0}^{\infty} f_i x^i$ be any element of $I[[x,\sigma]]$. Then $cf_i = \sum_{j=1}^{n} a_j r_{ij}$ for some $r_{ij} \in R$. Hence $cf(x) = \sum_{i=0}^{\infty}(\sum_{j=1}^{n} a_j r_{ij})x^i = \sum_{j=1}^{n} a_j(\sum_{i=0}^{\infty} r_{ij}x^i) \in cI\cdot R[[x,\sigma]]$, and so $f(x) \in I\cdot R[[x,\sigma]]$. Thus $I[[x,\sigma]] = I\cdot R[[x,\sigma]]$.

We can extend σ to an automorphism of $R[[x,\sigma]]$ by $\sigma(f(x)) = \sum_{i=0}^{\infty} \sigma(a_i)x^i$ for every $f(x) = \sum_{i=0}^{\infty} a_i x^i \in R[[x,\sigma]]$. Then σ is an inner automorphism given by $\sigma(f(x)) = xf(x)x^{-1}$. We require some notation. If X, Y are subsets of Q, we write $(X:Y)_l = \{q \in Q \mid qY \subseteq X\}$ and $(X:Y)_r = \{q \in Q \mid Yq \subseteq X\}$. Let I be a right R-ideal. Then we write $I_v = (R:(R:I)_l)_r$, a right R-ideal and if $I = I_v$, then it is called a right v-R-ideal (a right v-ideal if there are no confusions). Similarly we define ${}_vJ = (R:(R:J)_r)_l$ for any left R-ideal J, and J is called a left v-ideal if $J = {}_vJ$. An R-ideal A is said to be a v-ideal if $A_v = A = {}_vA$.

Lemma 1.2. Let A be an ideal of $R[[x,\sigma]]$ such that $A = A_v$. Then it is σ-invariant.

Proof. This is proved in the same way as in Lemma 2.5 of [15].

The following lemma is very useful to compute the left and right order of an ideal.

Lemma 1.3. Let q be an element in $Q(R[[x,\sigma]])$. Then q =

$x^{-n}q(x)$ for some non-negative integer n and some $q(x) \in Q[[x,\sigma]]$.

Proof. By the same way as Lemma 2' of [24], we can find a regular element $c(x) = c_n x^n + c_{n+1}x^{n+1} + \cdots \in R[[x,\sigma]]$ such that $c(x)q = r(x) \in R[[x,\sigma]]$ and c_n is regular in R. Then we have $c(x)^{-1} = x^{-n}d(x)$ for some $d(x) \in Q[[x,\sigma]]$ (see [10, p. 7]). Hence $q = c(x)^{-1}r(x) = x^{-n}d(x)r(x)$ and $d(x)r(x) \in Q[[x,\sigma]]$, as desired.

Theorem 1.4. Let R be a noetherian order in Q and let σ be an automorphism of R. Then R is a σ-maximal order in Q if and only if $R[[x,\sigma]]$ is a maximal order in $Q(R[[x,\sigma]])$.

Proof. To prove the necessity, let A be any non-zero ideal of $R[[x,\sigma]]$ and let q be any element in $O_l(A)$. Then $q = x^{-n}q(x)$ by Lemma 1.3, where $q(x) = q_0 + q_1 x + \cdots \in Q[[x,\sigma]]$. Put $A_i = \{a_i \in R \mid a_i x^i + a_{i+1}x^{i+1} + \cdots \in A\} \cup \{0\}$ for all non-negative integers i and $A^* = \bigcup_{i=0}^{\infty} A_i$. First assume that $A = A_v$, then A_i is a σ-invariant ideal of R, because A is σ-invariant by Lemma 1.2, and so is A^*. Assume that $A_0 = A_1 = \cdots = A_{i-1} = 0$ and $A_i \neq 0$. Then we can take a regular element $a_i \in A_i$ by Goldie's theorem and $a(x) = a_i x^i + a_{i+1}x^{i+1} + \cdots \in A$. It follows that $qa(x) = x^{-n}q(x)a(x) \in A$ and $q(x)a(x) \in x^n A$. Hence $q_0 = q_1 = \cdots = q_{n-1} = 0$, because $(x^n A)_0 = \cdots = (x^n A)_{n+i-1} = 0$ and a_i is regular. Thus we have $q = q_n + q_{n+1}x + \cdots$. To prove that $q \in R[[x,\sigma]]$, let b_k be any element in A^* and let $b(x) = b_k x^k + b_{k+1}x^{k+1} + \cdots \in A$. Then $qb(x) \in A$ entails that $q_n b_k \in A^*$ and so $q_n \in O_l(A^*) = R$ as noted before Lemma 1.1. Assume that $q_n, \cdots, q_{n+j} \in R$ and set $q_j(x) = q - (q_n + q_{n+1}x + \cdots + q_{n+j}x^j)$. Then, since $q_j(x)A \subseteq qA - (q_n + q_{n+1}x + \cdots + q_{n+j}x^j)A \subseteq A$, it follows that $q_{n+j+1} \in R$ by the same way as the above (note that $\sigma(A^*) = A^*$) and that $q \in R[[x,\sigma]]$ by an induction. Hence $O_l(A) = R[[x,\sigma]]$. Now, let A be arbitrary ideal of $R[[x,\sigma]]$. Then we have $R[[x,\sigma]] \subseteq O_l(A) \subseteq O_l(A_v) = R[[x,\sigma]]$ and so $R[[x,\sigma]] = O_l(A)$. Similarly, $O_r(A) = R[[x,\sigma]]$ and therefore, $R[[x,\sigma]]$ is a maximal order in $Q(R[[x,\sigma]])$ by Proposition 3.1 of [19, Chap. I]. To prove the sufficiency, let

B be a σ-invariant ideal of R, then $R[[x,\sigma]] = O_l(B[[x,\sigma]]) = O_l(B)[[x,\sigma]]$ (the last equality follows from the σ-invariantness of $O_l(B)$). Thus $R = O_l(B)$ and $R = O_r(B)$ by the right version of it. Hence R is a σ-maximal order in Q.

Corollary 1.5. R is a maximal order in Q if and only if $R[[x,\sigma]]$ is a maximal order in $Q(R[[x,\sigma]])$ for all automorphism σ of R.

Remark. The theorem gives an affirmative answer to Ramras's conjecture in more general setting (see [21, p.255]), and the theorem will be used in section 2 to study the skew formal power series rings over tame orders.

2. Let D be a noetherian, integrally closed domain with field of quotients K, Σ be a central simple K-algebra with finite dimension over K and Λ be a (classical) D-order in Σ, i.e., Λ is a subring of Σ which is finitely generated as D-modules, $D \subsetneq \Lambda$ and $\Lambda K = \Sigma$. Recall the definition of D-tame orders ([6]), Λ is called a tame D-order in Σ if Λ is D-reflexive, namely, $\Lambda = \bigcap \Lambda_p$ and Λ_p is a hereditary noetherian ring, where p runs over all minimal prime ideals of D. In non-commutative situation, there is a nice generalization of tame D-orders as will be given below; let $\underline{C}$ be a right Gabriel topology corresponding to the torsion theory cogenerated by $E(Q/R)$, the right R-injective hull of Q/R. Then $\underline{C} = \{C:\text{right ideal of } R \mid (R:r^{-1}C)_l = R \text{ for all } r \in R\}$, where $r^{-1}C = \{x \in R \mid rx \in C\}$. Similarly we can define the left Gabriel topology $\underline{C}'$ on R. If I is a right ideal of R, then we write $\widetilde{I} = \{r \in R \mid rC \subseteq I \text{ for some } C \in \underline{C}\}$. Note that $I \subseteq \widetilde{I} \subseteq I_v$. If $I = \widetilde{I}$, then we say that I is $\underline{C}$-closed. Similarly, we can define the concept of $\underline{C}'$-closed left ideals of R. An order R (not necessary to be noetherian) is called a v-HC order if it satisfies the following two conditions:

(K) : ${}_v(A(R:A)_l) = O_l(A)$ for any ideal A of R with $A = {}_vA$, and $((R:B)_rB)_v = O_r(B)$ for any ideal B of R with $B = B_v$.

(C) : R satisfies the maximum condition on $\underline{C}$-closed right ideals as well as $\underline{C}'$-closed left ideals.

A v-ideal A of R is said to be v-invertible if $(A(R:A)_r)_v = R = {}_v((R:A)_l A)$. Note that if A is v-invertible, then $(R:A)_r = (R:A)_l$ and we denote it by A^{-1}. We say that R has enough v-invertible ideals if any v-ideal of R contains a v-invertible ideal of R. Note that "VHC orders" in the sense of [14] are equivalent to "v-HC orders" if the orders have enough v-invertible ideals (see Proposition 1.1 of [17]). In this section, we shall prove that $\Lambda[[x,\sigma]]$ is a v-HC order with enough v-invertible ideals if Λ is a tame D-order with an automorphism σ. To prove this, we use σ-invariant idealizers, localizations and some properties of formal power series rings. We begin with a property of a hereditary noetherian prime ring (an HNP ring for short) which is concerned with Jacobson radical. So let R be an HNP ring with non-zero Jacobson radical J(R) (we always denote by J(T) the Jacobson radical of a ring T). Then R is a bounded and $J = J(R) = M_1 \cap \cdots \cap M_n$ by Theorem 4.13 of [5], where $M_1, \cdots, M_n$ is the full set of all maximal ideals of R. Let $K = K_1 \cap \cdots \cap K_m$ be a semimaximal right ideal such that RK = R, where K_i is a maximal right ideal. Assume that $R/K_1 \cong \cdots \cong R/K_{i_1}, \cdots, R/K_{i_{l-1}+1} \cong \cdots \cong R/K_{i_l}$ $(1 \leq i_1 \leq \cdots \leq i_l = m)$ and we may assume that, without loss of generality, that $M_j = \mathrm{ann}_R(R/K_{i_j}) = \{r \in R \mid (R/K_{i_j})r = 0\}$ $(j = 1, \cdots, l)$. Set $C = I_R(K) = \{r \in R \mid rK \subseteq K\}$, the idealizer of K in R and $U_j = R/K_{i_j}$. Then C is an HNP ring by Theorem 4.3 of [22]. It follows from 1.3 of [22] that U_j is a right C-module of length 2 and that $0 \longrightarrow S_j \longrightarrow U_j \longrightarrow T_j \longrightarrow 0$ is a non-split exact sequence, where $S_j = (C + K_{i_j})/K_{i_j}$ and $T_j = R/(C + K_{i_j})$.

Lemma 2.1. Under the same notation as above, we have $J(R) \cap C \subseteq J(C)$ and $J(C)^2 \subseteq J(R) \cap C$.

Proof. Set $N_j = \mathrm{ann}_C(S_j)$ and $L_j = \mathrm{ann}_C(T_j)$. Then, by Proposition 2.2 of [9], $\{N_1, \cdots, N_l, L_1, \cdots, L_l$, and $M_{l+1} \cap C, \cdots, M_n \cap C\}$ is the full set of maximal ideals of C. Since C is bounded, by Proposition 4.2 of [19, Chap. I], we have $J(C) = \cap_{j=1}^{l} N_j \cap \cap_{j=1}^{l} L_j \cap \cap_{i=l+1}^{n} (M_i \cap C)$. Hence $J(R) \cap C \subsetneqq J(C)$ and $J(C)^2 \subseteqq J(R) \cap C$, because $L_j N_j \subsetneqq M_j \cap C$, $M_j \cap C \subseteqq L_j$ and $M_j \cap C \subseteqq N_j$.

Lemma 2.2. Let R be a noetherian order in Q with an automorphism σ and let I be a right R-ideal. Then $(R[[x,\sigma]]:I[[x,\sigma]])_l = [[x,\sigma]](R:I)_l$ and $(I[[x,\sigma]])_v = I_v[[x,\sigma]]$.

Proof. As in Lemma 3.1.5 of [3].

A v-ideal A of R is called v-idempotent if $_v(A^2) = A = (A^2)_v$. The following lemma shows that some important properties in R are inherited to $R[[x,\sigma]]$.

Lemma 2.3. Let R be a noetherian order in Q with an automorphism σ. Then, for any σ-invariant v-R-ideal A, we have
(1) If A is v-invertible, then so is $A[[x,\sigma]]$.
(2) If A is v-idempotent, then so is $A[[x,\sigma]]$.

Proof. (1) $R[[x,\sigma]] \supseteqq (A[[x,\sigma]](R[[x,\sigma]]:A[[x,\sigma]])_r)_v = (A[[x,\sigma]](R:A)_r[[x,\sigma]])_v \supseteqq (A(R:A)_r[[x,\sigma]])_v = (A(R:A)_r)_v[[x,\sigma]] = R[[x,\sigma]]$ by Lemma 2.2. Hence $R[[x,\sigma]] = (A[[x,\sigma]](R[[x,\sigma]]:A[[x,\sigma]])_r)_v$ and similarly, $A[[x,\sigma]]$ has a left inverse as v-ideals. Hence $A[[x,\sigma]]$ is v-invertible.
(2) By Lemmas 1.1 and 2.2, we have $((A[[x,\sigma]])^2)_v = ((A\cdot R[[x,\sigma]])(A\cdot R[[x,\sigma]]))_v = (A^2\cdot R[[x,\sigma]])_v = (A^2[[x,\sigma]])_v = (A^2)_v[[x,\sigma]] = A[[x,\sigma]]$. Hence $A[[x,\sigma]]$ is v-idempotent.

Let R and T be orders in Q. Then R is said to be right equivalent to T if $aT \subseteqq R$ and $bR \subseteqq T$ for some units a, b in Q.

Lemma 2.4. Let R be a v-HC order and let σ be an automorphism of R. Then there is a one-to-one correspondence between σ-invariant v-idempotent ideals A of R and σ-invariant over-rings T of R which are right equivalent to R and $T = T_v$, given by ;

$A \longrightarrow O_r(A) = (R:A)_r$, $T \longrightarrow (R:T)_l$.

Proof. If A (T) is σ-invariant, then so is $O_r(A)$ ($(R:T)_l$). Hence the lemma is proved by similar way as in Lemma 1.6 of [14] (see, Corollary 4.5 of [22]).

A subset C of R is called a reguler Ore set of R if any element in C is regular and R satisfies the Ore condition with respect to C. We denote by R_C the ring of quotients of R with respect to C. Let P be a semi-prime ideal of R and $C(P) = \{c \in R \mid c \text{ is regular mod } P\}$. If C(P) is a regular Ore set of R, then we denote by R_P the ring of quotients of R with respect to C(P). Similarly, let $\mathfrak{a}$ be a finite intersection of minimal prime ideals of D. Then $C(\mathfrak{a}) = \{d \in D \mid d \text{ is regular mod } \mathfrak{a}\}$ is a regular Ore set of a D-order Λ and we just write $\Lambda_{\mathfrak{a}}$ for the ring of quotients of Λ with respect to $C(\mathfrak{a})$. Following [8], a finite intersection of maximal right v-ideals of R is said to be a semi-maximal right v-ideal of R ("maximal right v-ideals" means maximal amongst right v-ideals of R).

In the remainder of this paper, let Λ be a tame D-order in a central simple K-algebra Σ with finite dimension over K and let σ be any automorphism of Λ, where D is a noetherian, integrally closed domain with its quotient field K. Then $\Lambda = \cap \Lambda_P$, where Λ_P is an HNP ring whose Jacobson radical is the unique maximal invertible ideal and P runs over all maximal v-invertible ideals of Λ. Furthermore there is a one-to-one correspondence between the set of all maximal v-invertible ideals P of Λ and the set of all minimal prime ideals p of D, given by $P \longrightarrow p = P \cap D$ and $\Lambda_P = \Lambda_p$ (see the proof of Proposition 3.1 of [14]). It is implicitly known that a right Λ-ideal is a right v-ideal if and only if it is D-reflexive. This follows from the fact that $I_v = \cap I\Lambda_p = \cap I_p$ (see the proof of Proposition 2.11 of [14] and Lemma 5.1 of [20]). Note that if Λ and Γ are D-orders, then they are right equivalent.

Lemma 2.5. Let Γ be a D-reflexive order in Σ containing Λ

such that $\sigma(\Gamma) = \Gamma$ and there is no D-orders between Γ and Λ which are σ-invariant and D-reflexive. Then $\Lambda = I_{\Gamma}(K) = \{\gamma \in \Gamma \mid \gamma K \subseteq K\}$, the idealizer of K in Γ, for some σ-invariant, semi-maximal right v-ideal K of Γ such that $_v(\Gamma K) = \Gamma$ and K is a semi-prime v-idempotent ideal of Λ.

Proof. First of all, note that Γ is a tame D-order by Proposition 3.2 of [8] and Proposition 3.1 of [14]. By Lemma 2.4 there is a σ-invariant v-idempotent ideal K of Λ such that $\Gamma = O_r(K)$. Put $\Gamma' = I_{\Gamma}(K)$. Then Γ' is a σ-invariant D-order such that $\Lambda \subseteq \Gamma' \subseteq \Gamma$. Since $\Gamma' K \subseteq K$, we have $\Gamma'_p K_p = (\Gamma' K)_p \subseteq K_p$ for any minimal prime ideal p of D and so $(\cap_p \Gamma'_p)K \subseteq \cap K_p = K$, showing the reflexiveness of Γ'. Thus $\Gamma' = \Gamma$ or $\Gamma' = \Lambda$. If $\Gamma' = \Gamma$, then K is an ideal of Γ and so $K = \Gamma K = O_r(K)K = (\Lambda:K)_r K$. It follows that $K = K_v = ((\Lambda:K)_r K)_v = O_r(K) = \Gamma$, a contradiction. Hence we have $\Gamma' = \Lambda$. Since Λ_p is an HNP ring for every minimal prime ideal p of D, we have $\Gamma_p = O_r(K_p) = (\Lambda_p:K_p)_r K_p = O_r(K_p)K_p = \Gamma_p K_p$, because K_p is an idempotent ideal of Λ_p. This entails that $_v(\Gamma K) = \Gamma$. To prove that K is a semi-maximal right v-ideal of Γ, let $\{p_1, \cdots, p_n\}$ be the full set of minimal prime ideals of D such that $K_{p_i} \subsetneq \Lambda_{p_i}$ and put $\ell = p_1 \cap \cdots \cap p_n$, a semi-prime ideal. Then K_ℓ is an idempotent ideal of an HNP ring $\Lambda_\ell = \Lambda_B$, where $B = Q_1 \cap \cdots \cap Q_n$ and Q_i is the maximal v-invertible ideal of Λ such that $p_i = Q_i \cap D$ (see Lemma 2.1 of [15]). Furthermore, $\Gamma_\ell = (O_r(K))_\ell = O_r(K_\ell)$ and $\Lambda_\ell = I_{\Gamma_\ell}(K_\ell)$. Hence, by Theorem 5.3 of [22], K_ℓ is a semi-maximal right ideal of Γ_ℓ, and so K is a semi-maximal right v-ideal of Γ by Lemma 2.4 of [8] (note that K is D-reflexive). It follows from Theorem 5.2 of [22] that Λ_ℓ / K_ℓ is a semi-simple artinian ring. Furthermore Λ/K is embedded in Λ_ℓ/K_ℓ and each element in $C(\ell)$ is regular in Λ/K, because $K = \cap K_p = K_\ell \cap \cap K_p$ $(p \neq p_i) = K_\ell \cap \Lambda$. Hence Λ_ℓ/K_ℓ is the semi-simple artinian quotient ring of Λ/K and so K is a semi-prime ideal of Λ by Goldie's theorem. This completes the proof.

With the same notation and assumptions as in Lemma 2.5, let

p be any minimal prime ideal of D other than p_i $(1 \le i \le n)$, then $K_p = \Lambda_p$ entails that $K_p = \Gamma_p$, since K is a right ideal of Γ. Hence $\{p_1, \cdots, p_n\}$ is also the full set of minimal prime ideals of D such that $K_{p_i} \subsetneqq \Gamma_{p_i}$. Since Λ and Γ are both tame D-orders, there are maximal v-invertible ideals $P_1, \cdots, P_n$ of Γ and $Q_1, \cdots, Q_n$ of Λ, respectively, satisfying $\Gamma_{P_i} = \Gamma_{p_i}$, and $\Lambda_{Q_i} = \Lambda_{p_i}$. For any maximal v-invertible ideal P of Γ, we have $\sigma(C(P)) = C(\sigma(P))$ and so $\sigma(\Gamma_P) = \Gamma_{\sigma(P)}$. Thus we have $A = P_1 \cap \cdots \cap P_n$ is σ-invariant, semi-prime and v-invertible ideal of Γ. Similarly, $B = Q_1 \cap \cdots \cap Q_n$ is also a σ-invariant, semi-prime and v-invertible ideal of Λ(see Theorem 1.13 of [14]). Hence $\Gamma_A = \Gamma_{P_1} \cap \cdots \cap \Gamma_{P_n} = \Gamma_{p_1} \cap \cdots \cap \Gamma_{p_n} = \Gamma_\ell$, $I_{\Gamma_\ell}(K_\ell) = \Lambda_\ell = \Lambda_{p_1} \cap \cdots \cap \Lambda_{p_n} = \Lambda_{Q_1} \cap \cdots \cap \Lambda_{Q_n} = \Lambda_B$. Furthermore, $A\Gamma_A = J(\Gamma_A)$ and $B\Lambda_B = J(\Lambda_B)$ by Theorem 3 of [1]. By Lemma 2.1, $A\Gamma_\ell \cap \Lambda_\ell \subsetneqq B\Lambda_\ell$ and $(B\Lambda_\ell)^2 \subsetneqq A\Gamma_\ell \cap \Lambda_\ell$. So it follows that $A \cap \Lambda \subsetneqq B$ and $B^2 \subsetneqq A \cap \Lambda$, which are used in Lemma 2.8.

Lemma 2.6. Under the same notation and assumptions as before, we have

(1) $\Lambda[[x,\sigma]] = I_{\Gamma[[x,\sigma]]}(K[[x,\sigma]])$ and $O_r(K[[x,\sigma]]) = \Gamma[[x,\sigma]]$.

(2) $K[[x,\sigma]]$ is a semi-prime and v-idempotent ideal of $\Lambda[[x,\sigma]]$.

(3) $K[[x,\sigma]]$ is a semi-maximal right v-ideal of $\Gamma[[x,\sigma]]$ and $K[[x,\sigma]] \supsetneqq A[[x,\sigma]]$.

(4) $_v(\Gamma[[x,\sigma]]\cdot K[[x,\sigma]]) = \Gamma[[x,\sigma]]$.

Proof. (1) follows from Lemma 2.5.

(2) follows from Lemmas 2.3 and 2.5.

(3) Write $K = K_1 \cap \cdots \cap K_k$, where K_i is a maximal right v-ideal of Γ $(1 \le i \le k)$. Since K_ℓ is a semi-maximal right ideal of Γ_ℓ, it follows that $K_\ell \supseteqq J(\Gamma_\ell) = A\Gamma_\ell$. Hence $K_i \supsetneqq K \supsetneqq A\Gamma_\ell \cap \Gamma = A$ and so $K_i[[x,\sigma]] \supsetneqq A[[x,\sigma]]$, a v-invertible ideal of $\Gamma[[x,\sigma]]$ by Lemma 2.3. So, by the same way as in Lemma 2.2 of [18], we have $K_i[[x,\sigma]]$ is a maximal right v-ideal of $\Gamma[[x,\sigma]]$. Therefore $K[[x,\sigma]] = K_1[[x,\sigma]] \cap \cdots \cap K_k[[x,\sigma]]$ is a semi-maximal right v-ideal of $\Gamma[[x,\sigma]]$ and $K[[x,\sigma]] \supsetneqq A[[x,\sigma]]$.

(4) easily follows, because $\Gamma[[x,\sigma]] \supseteqq {}_v(\Gamma[[x,\sigma]]\cdot K[[x,\sigma]]) \supsetneqq$

${}_v([[x,\sigma]]\Gamma\cdot K) = {}_v([[x,\sigma]](\Gamma K)) = [[x,\sigma]]\Gamma$ by Lemmas 1.1, 2.2 and 2.5.

Since $\Lambda[[x,\sigma]] = I_{\Gamma[[x,\sigma]]}(K[[x,\sigma]])$, and $K[[x,\sigma]]$ is a bounded, semi-maximal right v-ideal, we are in a position to apply Theorem 2.9 of [8] to prove that $\Lambda[[x,\sigma]]$ is a v-HC order with enough v-invertible ideals if $\Gamma[[x,\sigma]]$ is a v-HC order with enough v-invertible ideals. But it seems to us that there is a logical gap in his proof (see [8, p. 1381]). So, we shall prove this by using some properties of algebras and skew formal power series rings. The following lemma is maybe known. But we give the proof of it, because we can not find any paper which mentioned it.

Lemma 2.7. Let R be an order in Q satisfying the condition (C) and let A be a semi-prime v-invertible ideal of R. If K is a right v-ideal of R containing a power of A, then $K = KR_A \cap R$.

Proof. By Lemma 2.1 of [15], R_A exists and it is an HNP ring. Assume that $K_1 = KR_A \cap R \supsetneq K$. Then we can choose a regular element c in K_1 but not in K by Robson's theorem (see Theorem 1.21 of [4]). Then $c(c^{-1}K) \subseteq K$ implies that $c^{-1}K$ is a right v-ideal and $c^{-1}K \subsetneq R$. Hence $(c^{-1}K)R_A \subsetneq R_A$ by Lemma 2.3 of [8]. On the other hand, $R_A = c^{-1}(K_1R_A) = c^{-1}(KR_A) = (c^{-1}K)R_A$, because $K_1R_A = KR_A$. This is a contradiction and so $K_1 = K$, as desired.

Lemma 2.8. Under the same notation and assumptions as before, we have

(1) $B[[x,\sigma]]$ and $A[[x,\sigma]]$ are v-invertible ideals of $\Lambda[[x,\sigma]]$ and $\Gamma[[x,\sigma]]$, respectively, and $C(B[[x,\sigma]]) \subseteq C(A[[x,\sigma]])$.

(2) $A[[x,\sigma]] \subsetneq B[[x,\sigma]]$ and $(B[[x,\sigma]])^2 \subsetneq A[[x,\sigma]]$.

Proof. (1) By Lemma 2.3, $B[[x,\sigma]]$ and $A[[x,\sigma]]$ are v-invertible ideals of $\Lambda[[x,\sigma]]$ and $\Gamma[[x,\sigma]]$ respectively. To prove $C(B[[x,\sigma]]) \subsetneq C(A[[x,\sigma]])$, first of all, we shall prove that

$\Lambda[[x,\sigma]]/A[[x,\sigma]]$ has an artinian quotient ring. To prove this, let $\bar{\Lambda} = \Lambda/A$ and $\bar{D} = D/\mathfrak{a}$. Then $\bar{D}$ is a semi-prime noetherian ring and $\bar{\Lambda}$ is a $\bar{D}$-algebra. It is easily seen that $\bar{\Lambda}$ is $\bar{D}$-torsion-free, i.e., $\bar{\lambda}\bar{d} = \bar{0} \Rightarrow \bar{\lambda} = \bar{0}$, where $\bar{\lambda} \in \bar{\Lambda}$ and $\bar{d}$ is regular in $\bar{D}$. So $\bar{\Lambda}$ is embedded in $\bar{\Lambda} \otimes_{\bar{D}} Q(\bar{D}) = \{\bar{\lambda}\bar{c}^{-1} \mid \bar{\lambda} \in \bar{\Lambda}$ and $\bar{c}$: regular in $\bar{D}\}$ and $\bar{\Lambda} \otimes_{\bar{D}} Q(\bar{D})$ is artinian, because $Q(\bar{D})$ is a finite direct sum of fields. Hence $\bar{\Lambda} \otimes_{\bar{D}} Q(\bar{D})$ is the artinian quotient ring of $\bar{\Lambda}$. Then it is proved by the same way as in Theorem 2 of [14] $\bar{\Lambda}[[x,\sigma]]$ ($\simeq \Lambda[[x,\sigma]]/A[[x,\sigma]]$) has an artinian quotient ring. Since $B^2 \subseteq A$ and $\bar{\Lambda}/\bar{B} \simeq \Lambda/B$ is semi-simple, it follows that $B[[x,\sigma]]/A[[x,\sigma]]$ is nilpotent by Lemma 1.1 and so it is the nilpotent radical of $\Lambda[[x,\sigma]]/A[[x,\sigma]]$, because $\Lambda[[x,\sigma]]/B[[x,\sigma]]$ ($\simeq \Lambda/B[[x,\sigma]]$) is a semi-prime ring. Hence, by the regularity condition (see Theorem 2.3 of [4]), for an element c of $\Lambda[[x,\sigma]]$, c is regular mod $A[[x,\sigma]]$ if and only if it is regular mod $B[[x,\sigma]]$. Now let c be any element of $C(B[[x,\sigma]])$ and assume that $c\gamma \in A[[x,\sigma]]$, where $\gamma \in \Gamma[[x,\sigma]]$. By the left version of Lemma 2.5, there is a σ-invariant, semi-maximal left v-ideal L of Γ such that $\Lambda = I_\Gamma(L)$ $= \{\gamma \in \Gamma \mid L\gamma \subseteq L\}$ and $\Gamma = (L\Gamma)_v$. The last property implies that $\Gamma_A = (L\Gamma)\Gamma_A = L\Gamma_A$. Thus we have $\gamma L \subseteq [[x,\sigma]]L \subseteq [[x,\sigma]]\Lambda = \Lambda[[x,\sigma]]$ and $c\gamma L \subseteq A[[x,\sigma]]$. It follows that $\gamma L \subseteq A[[x,\sigma]]$ and so $\gamma \in \gamma L\Gamma_A \subseteq$ $A[[x,\sigma]]\cdot\Gamma_A \subseteq A[[x,\sigma]]\cdot\Gamma[[x,\sigma]]_{A[[x,\sigma]]}$, because $C(A) \subseteq C(A[[x,\sigma]])$. Hence $\gamma \in A[[x,\sigma]]\cdot\Gamma[[x,\sigma]]_{A[[x,\sigma]]} \cap \Gamma[[x,\sigma]] = A[[x,\sigma]]$ and so $c \in C(A[[x,\sigma]])$.
(2) follows from $A = A \cap \Lambda \subseteq B$ and $B^2 \subseteq A$ (see the proof of Lemma 2.6).

As it has been pointed out in Lemma 2.8, there is a σ-invariant, semi-maximal left v-ideal L of Γ such that $\Lambda = I_\Gamma(L)$ and $O_1(L) = \Gamma$. Furthermore, L is a semi-prime, v-idempotent ideal of Λ and $(L\Gamma)_v = \Gamma$. Let p be any minimal prime ideal of D different from p_i ($1 \leq i \leq n$). Then $K_p = \Lambda_p$ implies that $\Gamma_p = O_1(K_p) = \Lambda_p$. So L_p is an ideal of Γ_p and $L_p = \Gamma_p$ follows, because $(L\Gamma)_v = \Gamma$. Hence we have $L \supseteq A$ by the same way as in Lemma 2.6 and so $L[[x,\sigma]] \supseteq A[[x,\sigma]]$. Furthermore, since A is a

σ-invariant, semi-prime v-invertible ideal of Γ, we have $A = A_1 \cap \cdots \cap A_m$, where A_i is a maximal σ-invariant v-invertible ideal (maximal amongst σ-invariant v-invertible ideals of Γ). In fact, $D_\sigma(\Gamma)$, the set of all σ-invariant, v-invertible ideals, is a free abelian group generated by maximal σ-invariant v-invertible ideals of Γ (see Theorem 1.13 of [14] and Lemmas 2.6, 2.7 of [15]). Furthermore, if $\Gamma[[x,\sigma]]$ is a v-HC order with enough v-invertible ideals, then $A_i[[x,\sigma]]$ is a maximal v-invertible ideal of $\Gamma[[x,\sigma]]$ by the same way as in Lemma 2.10 of [15] and by Lemma 1.12 of [14]. Hence $K[[x,\sigma]]\cdot\Gamma[[x,\sigma]]_{A_j} = \Gamma[[x,\sigma]]_{A_j}$ and $\Gamma[[x,\sigma]]_{A_j}\cdot L[[x,\sigma]] = \Gamma[[x,\sigma]]_{A_j}$ for any maximal v-invertible ideals A_j of $\Gamma[[x,\sigma]]$ different from $A_i[[x,\sigma]]$ $(1 \leq i \leq m)$.

Now we can summarize the properties of $\Lambda[[x,\sigma]]$ and $\Gamma[[x,\sigma]]$ what we have obtained in more general situation as follows:

Let R be a v-HC order with enough v-invertible ideals and let C be a subring of R satisfying the following;

(1) C satisfies the condition (C).

(2) There are v-idempotent, semi-prime ideals K and L of C such that $I_R(K) = C = I_R(L)$, $O_r(K) = R = O_l(L)$ and $_v(RK) = R = (LR)_v$.

(3) There are finite numbers $P_1, \cdots, P_m$ of maximal v-invertible ideals of R such that $KR_P = R_P$ and $R_PL = R_P$ for all maximal v-invertible ideals P of R different from P_i $(1 \leq i \leq m)$.

(4) Put $A = P_1 \cap \cdots \cap P_m$. Then $K \supsetneqq A$, $L \supsetneqq A$ and there is a semi-prime v-invertible ideal B of C such that $C(B) \subseteq C(A)$, $A \cap C \subsetneqq B$ and $B^2 \subsetneqq A \cap C$.

Following [14], we denote by S(R) the Asano overring of R, i.e., $S(R) = \bigcup X^{-1}$, where X ranges over all v-invertible ideals of R. We can now prove the theorem due to Fujita under the conditions above (Theorem 2.9 of [8]).

Lemma 2.9. Under the same notation and assumptions as in the above, C is a v-HC order with enough v-invertible ideals and S(C) = S(R).

Proof. By Theorem 2.23 of [14], we have $R = R_A \cap \cap R_P \cap S(R)$ and $R_A = \cap_{i=1}^{m} R_{P_i}$, where P runs over all maximal v-invertible ideals of R different from P_i $(1 \leq i \leq m)$. Furthermore, $(IS(R))_v = S(R) = {}_v(S(R)I)$ for any R-ideal I by Proposition 2.10 of [14]. The conditions $KR_P = R_P$ and $K \supsetneqq A$ imply that $K = KR_A \cap \cap R_P \cap S(R)$. So it follows that $C = I_R(K) = I_{R_A}(KR_A) \cap \cap R_P \cap S(R)$. We shall prove that $R_P = C_H$ for some semi-prime ideal H of C. To do this, write $P' = PR_P = M_1' \cap \cdots \cap M_k'$, a finite intersection of maximal ideals of R_P, and put $H = P' \cap C$ and $M_i = M_i' \cap C$. By Lemma 2.7 of [8], $\mathcal{C} = C(P') \cap C$ is a regular Ore set of C and $C_{\mathcal{C}} = R_P$. Then each M_i is a prime ideal of C and $C_H = R_P$ by the same way as in Proposition 1.1 of [13]. By Lemma 2.1 of [15], C_B exists and is an HNP ring. To prove $I_{R_A}(KR_A) \supsetneqq C_B$, first note that $KC_B \cap C = K$ by Lemma 2.7 and the condition (4). So KC_B is also a semi-prime ideal of C_B and is idempotent by the conditions (the fact that KC_B is an ideal follows from the proof of Theorem 1.31 of [4]). Thus, by Theorem 5.2 of [22], we have $I_{R*}(KC_B) = C_B$, where $R* = O_r(KC_B)$. Since $R* = O_r(KC_B) = O_r(K)C_B = RC_B \subseteq R_A$, we have $C_B = I_{R*}(KC_B) \subseteq I_{R_A}(KR_A)$. Hence we have $C = C_B \cap \cap C_H \cap S(R)$. This means that C is a ring of type II' in the sense of [8]. To prove that C has enough v-invertible ideals, let X be any ideal of C, then LXK is an ideal of R contained in X and so $S(R) \supseteq (XS(R))_v \supseteq (LXKS(R))_v = S(R)$. Hence $(XS(R))_v = S(R)$ and similarly, $S(R) = {}_v(S(R)X)$. This implies that C has enough v-invertible ideals by Lemma 1.3 of [8]. To prove that $S(R) = S(C)$, let Y be any v-invertible ideal of C, then we have $Y^{-1} \subseteq Y^{-1}S(R) = Y^{-1}(LYKS(R))_v \subseteq (Y^{-1}LYKS(R))_v \subseteq (Y^{-1}YS(R))_v = S(R)$ by Lemma 2.3 of [12]. Hence $S(C) \subseteq S(R)$. To prove the converse inclusion, let Z be any v-invertible ideal of R, then KZZ^{-1} and KZ are both ideals of C. Hence $Z^{-1} \subseteq S(C)Z^{-1} = {}_v(S(C)KZ)Z^{-1} \subseteq {}_v(S(C)KZZ^{-1}) = S(C)$, because C has enough v-invertible ideals and so $S(R) \subseteq S(C)$. Therefore $S(R) = S(C)$. Thus C is a v-HC order with enough v-invertible ideals by Lemma 1.1 of [8].

From Lemmas 2.6, 2.8, 2.9 and the notes before Lemma 2.9, we have

Lemma 2.10. Under the same notation and assumptions as in Lemma 2.5, if $\Gamma[[x,\sigma]]$ is a v-HC order with enough v-invertible ideals, then so is $\Lambda[[x,\sigma]]$.

We are now in a position to show the main theorem of this paper.

Theorem 2.11. Let D be a noetherian integrally closed domain with quotient field K and let Λ be a tame D-order in a central simple K-algebra Σ with finite dimension over K. If σ is an automorphism of Λ, then $\Lambda[[x,\sigma]]$ is a v-HC order with enough v-invertible ideals.

Proof. There is a maximal D-order which contains Λ (see Theorem 1.4 of [6]). So, we can easily see that Λ has only finite many v-idempotent ideals by the similar way as in Theorem 6.3 of [22] (see, also Lemma 2.4). Thus we have a chain of σ-invariant, v-idempotent ideals of Λ ; $\Lambda = I_0 \supsetneqq I_1 \supsetneqq \cdots \supsetneqq I_n$ such that there is no σ-invariant, v-idempotent ideals between I_i and I_{i+1} $(0 \leq i \leq n-1)$ and I_n is a minimal σ-invariant, v-idempotent ideal of Λ. Let $\Lambda_i = O_r(I_i)$. Then Λ_i is a tame D-order by Proposition 3.1 of [14] and Proposition 3.2 of [8]. Furthermore, there are no D-reflexive, σ-invariant D-orders between Λ_i and Λ_{i+1} by Lemma 2.4. In particular, Λ_n is a σ-maximal D-order. Hence it follows, by Theorem 1.4, that $\Lambda_n[[x,\sigma]]$ is a maximal order and so it is a Krull order in the sense of [2]. Hence $\Lambda[[x,\sigma]]$ is a v-HC order with enough v-invertible ideals by using Lemma 2.10, successively. This completes the proof.

Remark. (1) $\Lambda[[x]]$ is, of course, a tame D[[x]]-order. However, if σ is of an infinite order, then $\Lambda[[x,\sigma]]$ is not a tame $C(\Lambda[[x,\sigma]])$-order, where $C(\Lambda[[x,\sigma]])$ is the center of $\Lambda[[x,\sigma]]$, because $C(\Lambda[[x,\sigma]]) = \{d \in D \mid \sigma(d) = d\}$.

(2) Concerning polynomial rings over v-HC orders (tame D-orders), see Theorem 2.16 of [15] (Theorem 1.11 of [6]).

3. In this section, we shall investigate v-invertible ideals of $\Lambda[[x,\sigma]]$. By Theorem 1.13 of [14] and Theorem 2.11, $D(\Lambda[[x,\sigma]])$, the set of all v-invertible ideals, is a free abelian group generated by the maximal v-invertible ideals of $\Lambda[[x,\sigma]]$. So it suffices to study the maximal v-invertible ideals of $\Lambda[[x,\sigma]]$. Let us begin with the lemma in more general case.

Lemma 3.1. Let R be a v-HC order in a simple artinian ring Q. Let S be a set consisting of v-invertible ideals of R which is closed under v-multiplication ; $X \circ Y = (XY)_v$ for $X, Y \in S$. Put $T = \cup_{X \in S} X^{-1}$. If I is an ideal of R, then $(IT)_v$ is an ideal of T.

Proof. First we shall prove that $(IT)_v = (X^{-1}IT)_v$ for any element X in S. Since $IT \subsetneq X^{-1}IT \subsetneq X^{-2}IT \subsetneq \cdots \subsetneq T$, there is a natural number n such that $(X^{-n}IT)_v = (X^{-(n+1)}IT)_v$ by Lemma 2.4 of [14] and Lemma 2.2 of [12]. Then $(IT)_v = (X^nX^{-n}IT)_v =$ $(X^n(X^{-n}IT)_v)_v = (X^n(X^{-(n+1)}IT)_v)_v = (X^nX^{-(n+1)}IT)_v =$ $((X^nX^{-(n+1)}I)_vT)_v = ({}_v(X^nX^{-(n+1)}I)T)_v = ((X^{-1}I)_vT)_v = (X^{-1}IT)_v$ by using Lemma 2.3 of [12] (see, also the proof of Proposition 1.1 of [17]). To prove that $(IT)_v$ is a left ideal of T, let $y \in T$ and $x \in (IT)_v$, then we have $y \in Y^{-1}$ for some $Y \in S$ and so $yx \in$ $Y^{-1}(IT)_v \subseteq (Y^{-1}IT)_v = (IT)_v$. Hence $(IT)_v$ is an ideal of T.

Let R be a v-HC order in Q. We denote by M(R) the set of all maximal v-invertible ideals of R. Set $M(S) = \{P \in M(R) \mid P \supsetneq X$ for some $X \in S\}$ and set $M^*(S) = \{A \in M(R) \mid A \not\supseteq X$ for all $X \in S\}$. Then $M(R) = M(S) \cup M^*(S)$ and $M(S) \cap M^*(S) = \phi$. Note that T is also a v-HC order by Proposition 4.1 of [7].

Lemma 3.2. Under the same notation and assumptions as in Lemma 3.1, there is a one-to-one correspondence between $M^*(S)$

and $M(T)$, which is given by ; $A \longrightarrow (AT)_v = A'$, $A' \longrightarrow A = A' \cap R$, where $A \in M^*(S)$ and $A' \in M(T)$.

Proof. By the same way as in Proposition 2.11 of [14], we have $R = \cap R_P \cap T$, where $P \in M(S)$ and $I_v = \cap IR_P \cap (IT)_v$ for any right R-ideal I. Let M' be any maximal v-ideal of T. Then we shall prove that $M = M' \cap R$ is also a maximal v-ideal of R. Since M' is a prime ideal of T by Lemma 1.4 of [14], it follows from Lemma 3.1 that M is also a prime ideal of R. Assume that M is not a maximal v-ideal of R and let N be a maximal v-ideal containing M. If $(NT)_v \subsetneqq T$, then $(NT)_v = (MT)_v = M'$ and so $N \subseteq M$, a contradiction. Hence $(NT)_v = T$. This implies that N contains an element in S. But, since N is a prime ideal, N contains an element $P_0 \in M(S)$, because any v-invertible ideal of R is a finite product of maximal v-invertible ideals as v-ideals (see the proof of Proposition 2.1 of [5]). Then $NR_{P_0} \subsetneqq R_{P_0}$ by Lemma 2.3 of [8] and $N = NR_{P_0} \cap R$. Since M is a prime v-ideal and R satisfies the maximum condition on one-sided v-ideals of R, we also have $MR_{P_0} \cap R = M$. Hence NR_{P_0} and MR_{P_0} are both prime ideals of an HNP ring R_{P_0} and so $NR_{P_0} = MR_{P_0}$. This entails that $N = M$, a contradiction. Hence M is a maximal v-ideal. Assume that $MR_P \subsetneqq R_P$ for some $P \in M(S)$, then $(P^n)_v \subseteq MR_P \cap R = M$ for some n. Hence we have $T = (P^n T)_v \subseteq (MT)_v = M'$, a contradiction. Hence $MR_P = R_P$ for all P in $M(S)$. Now, let A' be an element in $M(T)$ and write $A' = M_1' \cap \cdots \cap M_n'$ as an intersection of a cycle (see Lemma 1.12 of [14] and also see Theorem 1.11 of [7]). Put $A = A' \cap R$ and $M_i = M_i' \cap R$. Then $O_r(M_i) = \cap O_r(M_i R_P) \cap O_r(M_i') = \cap R_P \cap O_r(M_i') = \cap R_P \cap O_l(M_{i+1}') = O_l(M_{i+1})$ for all i, where $i+1 = 1$ if $i = n$ ($O_r(M_i R_P) = R_P$ follows from $M_i R_P = R_P$). Furthermore, $(M_i^2)_v = \cap M_i^2 R_P \cap (M_i'^2)_v = \cap R_P \cap M_i' = M_i$ and so A is an element of $M(R)$ by Lemma 1.12 of [14] (also, see Lemmas 1.8 and 1.9 of [7]). It is clear that $A \in M^*(S)$ and $A' = (AT)_v$. Conversely, let A be any element in $M^*(S)$. Then it follows from the same way as in the above that $(AT)_v$ is an element in $M(T)$ and $(AT)_v \cap R = A$. This completes the proof.

Let R be a v-HC order with an automorphism σ. A finite set of distinct maximal σ-invariant v-ideals $M_1, \cdots, M_n$ of R which are v-idempotents is called a σ-v-cycle if $O_r(M_1) = O_l(M_2), \cdots, O_r(M_n) = O_l(M_1)$. A maximal σ-invariant v-ideal M which is v-invertible is also considered as a σ-v-cycle, because $O_r(M) = O_l(M)$.

Lemma 3.3. Let R be a v-HC order with an automorphism σ and let P be an ideal of R. Then P is a maximal σ-invariant v-invertible ideal of R if and only if it is an intersection of a σ-v-cycle.

Proof. As in Lemma 1.12 of [14] (also, see Lemmas 1.8, 1.9 of [7] and Proposition 1.6 of [12]).

Lemma 3.4. Let Λ be a tame D-order with an automorphism σ and let P' be an ideal of Λ[[x,σ]] with $P' \cap \Lambda \neq 0$. Then P' is a maximal v-invertible ideal of Λ[[x,σ]] if and only if P' = P[[x,σ]] for a maximal σ-invariant v-invertible ideal P of Λ.

Proof. By Theorem 2.11, Λ[[x,σ]] is a v-HC order with enough v-invertible ideals. Before proving the necessity, we note the following; let M be a v-ideal of Λ with σ(M) = M. Then it is not hard to prove that M is a σ-prime ideal if and only if M[[x,σ]] is a prime ideal, where M is said to be σ-prime if $AB \subseteq M$, where A and B are ideals of Λ such that $\sigma(A) \subseteq A$ and $\sigma(B) \subseteq B$, implies that either $A \subseteq M$ or $B \subseteq M$. Furthermore, in this case, M[[x,σ]] is a maximal v-ideal of Λ[[x,σ]] by Lemma 1.2 of [16]. To prove the necessity, let P' be a maximal v-invertible ideal of Λ[[x,σ]] with $P = P' \cap \Lambda \neq 0$. Write $P' = M_1' \cap \cdots \cap M_n'$ as an intersection of a cycle and put $M_i = M_i' \cap \Lambda$, a v-ideal of Λ by Lemmas 1.1 and 2.2. It is clear that M_i is σ-invariant by Lemma 1.2 and is a σ-prime ideal. Hence $M_i[[x,\sigma]] = M_i'$, and M_i is a maximal σ-invariant v-ideal of Λ and is v-idempotent if $n \geq 2$ (if n = 1, M_1 is a maximal σ-invariant v-invertible ideal of Λ).

Hence $M_1, \cdots, M_n$ is a σ-v-cycle and so $P = M_1 \cap \cdots \cap M_n$ is a maximal σ-invariant v-invertible ideal of Λ. Since $M_i[[x,\sigma]] = M_i'$, we have $P' = P[[x,\sigma]]$. To prove the sufficiency, let $P = M_1 \cap \cdots \cap M_n$ be an intersection of a σ-v-cycle. Then it follows that $O_r(M_i[[x,\sigma]]) = O_r(M_i)[[x,\sigma]] = O_l(M_{i+1})[[x,\sigma]] = O_l(M_{i+1}[[x,\sigma]])$ and so $P[[x,\sigma]]$ is a maximal v-invertible ideal by Lemma 1.12 of [14].

From Lemmas 3.2 and 3.4, we have

Proposition 3.5. Let Λ be a tame D-order with an automorphism σ and let $T = \bigcup B^{-1}[[x,\sigma]]$, where B runs over all σ-invariant, v-invertible ideals of Λ. Then

(1) There is a one-to-one correspondence between the set of all maximal v-invertible ideals P' of $\Lambda[[x,\sigma]]$ such that $P = P' \cap \Lambda \neq 0$ and the set of all maximal σ-invariant, v-invertible ideals P of Λ, which is given by $P' \longrightarrow P = P' \cap \Lambda$; $P \longrightarrow P' = P[[x,\sigma]]$.

(2) There is a one-to-one correspondence between the set of all maximal v-invertible ideals A of $\Lambda[[x,\sigma]]$ such that $A \cap \Lambda = 0$ and the set of all maximal v-invertible ideals A' of T, which is given by $A \longrightarrow A' = (AT)_v$ and $A' \longrightarrow A = A' \cap \Lambda[[x,\sigma]]$.

Let R be a v-HC order with enough v-invertible ideals. Then, D(R), the set of all v-invertible ideal, is a free abelian group generated by the maximal v-invertible ideals of R by Theorem 1.13 of [14].

Remark. (1) We have obtained in the forthcoming paper that T is also a v-HC order with enough v-invertible ideals, where T is one in the proposition. So we can restate the proposition as follows;

$D(\Lambda[[x,\sigma]]) \simeq D_\sigma(\Lambda) \oplus D(T)$,

where $D_\sigma(\Lambda)$ is the free subgroup of $D(\Lambda)$ generated by the maximal σ-invariant, v-invertible ideals of Λ.

(2) As it will be seen in the following example, D(T) is usually

not trivial. Let Z be the ring of integers. Then 2 + x, 2 + x + x^2, ..., 2 + x + x^2 + ... are all prime elements in Z[[x]] and so the principal ideals generated by these elements are all maximal v-ideals, respectively (see Theorems 5.3 and 2.1 of [23, p. 16 and p. 50]).

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Department of Mathematics
Naruto University of Teacher Education
Naruto, 772, Japan
and
Graduate School of Science and Technology
Kobe University
Kobe, 657, Japan

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PROCEEDINGS OF THE 20TH
SYMPOSIUM ON RING THEORY (1987)

HOPF-GALOIS EXTENSIONS OF ALGEBRAS, THE MIYASHITA-ULBRICH ACTION, AND AZUMAYA ALGEBRAS

Mitsuhiro TAKEUCHI and Yukio DOI

We work over a commutative ring R. Let $X = \mathrm{Spec}\ B$ be an affine R-scheme and let $G = \mathrm{Spec}\ A$ be an affine R-group. A scheme map

$$\psi : X \times G \to X$$

is represented by an R-algebra map

$$\rho : B \to B \otimes A$$

where $\otimes = \otimes_R$. To say G <u>acts on the right on</u> X <u>via</u> ψ means that B is a right A-comodule with structure ρ. In this case we say B is a right A-<u>comodule</u> R-<u>algebra</u>. The quotient affine R-scheme $Y = X/G$ is given by $Y = \mathrm{Spec}\ C$,

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This note is derived from the introduction of [0].

with

$$C = \{ b \in B \mid \rho(b) = b \otimes 1 \}$$

called the <u>invariants</u> in B. The most important condition in order for X/Y to be a <u>principal homogeneous space for</u> G is that ψ and the projection pr: X x G → X give rise to an isomorphism of affine R-schemes

$$(\mathrm{pr}, \psi): X \times G \to X \times_Y X$$

[3, 1.7, p.362], [2, p.154], [18, 18.3, p.142]. In the context of the A-comodule algebra B, this is equivalent to saying that the map

$$\beta: B \otimes_C B \to B \otimes A, \quad \beta(b' \otimes b) = (b' \otimes 1)\rho(b)$$

is bijective. If this is the case, we call B/C a <u>right A-Galois extension</u> of R-algebras.

Such a concept has been studied under the name 'Galois A-objects' [1, 7.3, p.56], or 'PPHS for G' [2, p.158] with some additional conditions including C = R.

The concept is meaningful without any commutativity of B or A, and such a non-commutative Hopf-Galois extension has been studied in [9], [16] with some finiteness assumption of A. We review the main results in §1.

In a series of papers [5, 6, 7] the first author Doi introduces two important concepts, (<u>total</u>) <u>integrals</u> and the category $\mathbf{M}_B^A$, to study the right A-Galois extension B/C. In §2 we study the structure of $\mathbf{M}_B^A$. We show (2.11),(a) if the left C-module B is <u>flat</u>, then we have the structure theorem

$$M \cong M_0 \otimes_C B, \quad M \in \mathbf{M}_B^A$$

where M_0 denotes the invariants in M. This covers the case A is finitely generated projective. Further we show (2.11), (b) if there is a total integral $\phi: A \to B^C$ with B^C the centralizer of C in B, then the category M_B^A is equivalent to the category of right C-modules. This strengthens our previous result [8, Theorem 9], where we assert the same conclusion when there is an invertible integral $\phi: A \to B$.

When B/C is a right A-Galois extension, there is a canonical action $B^C \otimes A \to B^C$ which makes B^C into a right A-module algebra with the invariants $Z(B)$, the center of B [16, II]. We call this the Miyashita-Ulbrich action since this gives rise to the so-called Miyashita automorphisms [15], [17] in case $A = R[G]$, a group algebra. We give a different approach based on the π-method to this action in § 3. We show for an algebra E and an algebra map $\alpha: B \to E$, there is an action $E^C \otimes A \to E^C$ such that E^C is a right A-module algebra with the invariants E^B. In § 6, for a given extension of R-algebras E/C, we ask whether every measuring action $E^C \otimes A \to E^C$ comes from a right A-Galois extension B/C and an algebra map $\alpha: B \to E$ over C. This question is solved affirmatively when E is an Azumaya algebra and the right C-module E is a progenerator (6.20). In fact, the pair $(B/C, \alpha)$ is determined uniquely up to isomorphisms from the action $E^C \otimes A \to E^C$.

In § 4 and § 5, we assume the Hopf R-algebra A is finitely generated projective. We denote by A^* the dual

Hopf R-algebra of A. Then, with the above notation, E^C is a left A^*-comodule algebra with the invariants E^B. We say E^C/E^B is a left A^*-extension (not necessarily Galois). In §4, we define the concept of (weak) Galois contexts $(B/C, C'/B', \alpha, \alpha')$ in E for A from the viewpoint of generalized #-products [14, §8]. Here B/C (resp. C'/B') is a right A- (resp. left A^*-) extension, and $\alpha : B \to E$, $\alpha' : C' \to E$ are R-algebra maps. When B/C and C'/B' are both Galois extensions, we say we have a Galois context. The construction of §3 can be rephrased so that if we have a right A-Galois extension B/C and an algebra map $\alpha : B \to E$, then we have a weak Galois context $(B/C, E^C/E^B, \alpha, \text{incl.})$ in E. We have a similar construction from a left A^*-Galois extension C'/B' and an algebra map $\alpha' : C' \to E$. Our concept of Galois contexts has a similar property as the context studied in [11]. We show (4.5) if we have a Galois context $(B/C, C'/B')$ in E, then the weak Galois contexts $(B/C, E^C/E^B)$ and $(E^{B'}/E^{C'}, C'/B')$ are Galois contexts, too, and we have the double centralizer property

$$E^{B'} \cong E^{C'} \otimes_C B \quad \text{and} \quad E^C \cong C' \otimes_{B'} E^B.$$

The structure theorem of $\mathbf{M}_B^A$ (§2) are used to prove this.

Let B/C be a right A-Galois extension. Since $Z(B)$ equals the invariants of the left A^*-comodule algebra B^C, if we have a Galois context of the form $(B/C, B^C/R)$, then B is obviously central. We ask whether this implies, or is implied from, that B is separable. Central separable algebras are called Azumaya algebras. We prove (5.2) that if there is a total integral $\phi: A \to B^c$ then B is an Azumaya algebra if and only if C is a separable algebra and B^c/R is a left A^*-Galois extension. If C is commutative, C is contained in B^c, and C is a subcomodule of the A^*-comodule B^c if in addition A is cocommutative. In such a case, we prove a more clear result (5.8): if we have a Galois context for A of the form $(B/C, C/R)$, then C is commutative and B is an Azumaya algebra containing C as a maximal commutative subalgebra.

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Institute of Mathematics
University of Tsukuba
Ibaraki 305, Japan

and

Department of Mathematics
Fukui University
Fukui 910, Japan

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MORITA CONTEXTS AND EQUIVALENCES II

Toyonori KATO

Throughout this summary, R and S are associative rings (not necessarily with identity), I an ideal of R, J an ideal of S, and $\langle {}_RU_S, {}_SV_R; I, J\rangle$ a Morita context consisting of bimodules ${}_RU_S$ and ${}_SV_R$ together with bimodule epimorphisms

$$U \otimes_S V \longrightarrow I \quad (u \otimes v \longmapsto uv)$$

and

$$V \otimes_R U \longrightarrow J \quad (v \otimes u \longmapsto vu)$$

satisfying $(uv)u' = u(vu')$ and $(vu)v' = v(uv')$ for $u, u' \in U$ and $v, v' \in V$.

Definition 1. Let M be a left R-module. For each $v \in V$ and $m \in M$, define $\langle v, m\rangle \in \mathrm{Hom}_R(U, M)$ (= $[U, M]$ for short) via

$$u\langle v, m\rangle = (uv)m \quad \text{for} \quad u \in U,$$

and $\langle V, M\rangle$ the submodule of $[U, M]$ generated by the homomorphisms $\langle v, m\rangle$, $v \in V$, $m \in M$. Similarly for a left S-module N, $n \in N$, and $u \in U$, define $\langle u, n\rangle \in \mathrm{Hom}_S(V, N)$ (= $[V, N]$ for short) via

$$v\langle u, n\rangle = (vu)n \quad \text{for} \quad v \in V,$$

The detailed version of this summary will be submitted for publication elsewhere.

and $\langle U, N\rangle$ the submodule of $[V, N]$ generated by the homomorphisms $\langle u, n\rangle$, $u \in U$, $n \in N$.

Let R-Mod denote the category of all left R-modules and S-Mod the category of all left S-modules. Then one obtains

Proposition 1. $\langle V, -\rangle : R\text{-Mod} \to S\text{-Mod}$ naturally turns out to be a subfunctor of the Hom-functor $[U, -]$ as well as a quotient functor of the tensor functor $(V \otimes_R -)$. Similar statement holds also for $\langle U, -\rangle : S\text{-Mod} \to R\text{-Mod}$.

Remark. The functors $\langle V, -\rangle$ and $\langle U, -\rangle$ have already appeared in Nicholson and Watters [7] or in Kyuno [4] in slightly different forms.

Definition 2. Let

$$_I\mathrm{M} = \{M \in R\text{-Mod} \mid IM = M \text{ and } \mathrm{Ann}_M(I) = 0\}$$

be the full subcategory of R-Mod, where $\mathrm{Ann}_M(I)$ denotes the annihilator of I in M. Similarly define $_J\mathrm{M}$ the full subcategory of S-Mod.

Theorem 1. Let $\langle {}_RU_S, {}_SV_R; I, J\rangle$ be a Morita context. Then the functors

$$\langle V, -\rangle : {}_I\mathrm{M} \rightleftarrows {}_J\mathrm{M} : \langle U, -\rangle$$

induce inverse equivalences.

Remark. Theorem 1 above has been proved in slightly different forms by many authors under some additional conditions; e.g., by Morita [5] under the condition that $1 \in I$ and $1 \in J$, by Nobusawa [8] under $I^2 = R$ and $J^2 = S$, by Kyuno [4] under $I = R$ and $J = S$, and by Nicholson and Watters [7] under $1 \in R$ and $1 \in S$.

Definition 3. Let

$${}_I\mathcal{C} = \{M \in R\text{-Mod} \mid I \otimes_R M \xrightarrow{\simeq} M \ \text{ via } \ a \otimes m \mapsto am \ \ (a \in I,\ m \in M)\}$$

be the full subcategory of R-Mod. Similarly define ${}_J\mathcal{C}$ the full subcategory of S-Mod.

Theorem 2. Let $\langle {}_RU_S,\ {}_SV_R;\ I,\ J\rangle$ be a Morita context. Then the functors

$$(V \otimes_R -) : {}_I\mathcal{C} \rightleftarrows {}_J\mathcal{C} : (U \otimes_S -)$$

induce inverse equivalences.

Remark. Theorem 2 has been obtained in a slightly different form by Kato and Ohtake [3] over unital rings R and S.

Definition 4. Let

$${}_I\mathcal{L} = \{M \in R\text{-Mod} \mid M \xrightarrow{\simeq} [I,\ M] \ \text{ via } \ m \mapsto \bar{m} \ \ (m \in M)\}$$

be the full subcategory of R-Mod, where $\bar{m}$ denotes the right multiplication by $m \in M$. Similarly define ${}_J\mathcal{L}$ the full subcategory of S-Mod.

Theorem 3. Let $\langle {}_RU_S,\ {}_SV_R;\ I,\ J\rangle$ be a Morita context. Then the functors

$$[U,\ -] : {}_I\mathcal{L} \rightleftarrows {}_J\mathcal{L} : [V,\ -]$$

induce inverse equivalences.

Remark. Theorem 3 has been proved in slightly different forms by several authors under some additional conditions; e.g., by Kato [1] for a derived Morita context over unital rings R and S, by Müller [6] over unital rings R and S, and by Nobusawa [9] for $I = R$ and $J = S$.

Definition 5. Let

$$I(U \otimes_S -) = \{M \in R\text{-Mod} \mid M \simeq U \otimes_S N \ \text{ for } \ N = JN\}$$

and

$$_J\mathcal{M}' = \{N \in S\text{-Mod} \mid JN = N \text{ and } A_N(U) = 0\}$$

respectively be the full subcategories of R-Mod and S-Mod, where

$$A_N(U) = \{n \in N \mid U \otimes n = 0 \text{ in } U \otimes_S N\}.$$

Theorem 4. Let $\langle {}_RU_S, {}_SV_R; I, J\rangle$ be a Morita context. Then

(1) $_I\mathcal{C} \subset I(U \otimes_S -)$ and $_J\mathcal{M} \subset {}_J\mathcal{M}'$.

(2) The functors

$$\langle V, -\rangle : I(U \otimes_S -) \rightleftarrows {}_J\mathcal{M}' : (U \otimes_S -)$$

induce inverse equivalences.

Definition 6. Let

$$I[V, -] = \{M \in R\text{-Mod} \mid M \simeq [V, N] \text{ for } \mathrm{Ann}_N(J) = 0\}$$

and

$$_J\mathcal{M}'' = \{N \in S\text{-Mod} \mid T_N(V) = N \text{ and } \mathrm{Ann}_N(J) = 0\}$$

respectively be the full subcategories of R-Mod and S-Mod, where $T_N(V)$ denotes the trace of V in N.

Theorem 5. Let $\langle {}_RU_S, {}_SV_R; I, J\rangle$ be a Morita context. Then

(1) $_I\mathcal{L} \subset I[V, -]$ and $_J\mathcal{M} \subset {}_J\mathcal{M}''$.

(2) The functors

$$\langle V, -\rangle : I[V, -] \rightleftarrows {}_J\mathcal{M}'' : [V, -]$$

induce inverse equivalences.

We end off this summary with the following examples (cf. Kato [2]).

Examples. Let $\langle {}_RU_S, {}_SV_R; I, J\rangle$ be a Morita context.

(1) If $a \in Ia$ for each $a \in I$, then we have

$${}_I\mathrm{M} = {}_IC = \{M \in R\text{-Mod} \mid IM = M\},$$

as well as

$$(U \otimes_S -) \simeq \langle U, -\rangle \quad \text{on} \quad \{N \in S\text{-Mod} \mid JN = N\}.$$

(2) If $I = Re$ with $e = e^2 \in R$, then we obtain

$${}_I\mathrm{M} = {}_IL = \{M \in R\text{-Mod} \mid \mathrm{Ann}_M(I) = 0\},$$

as well as

$$\langle U, -\rangle \simeq [V, -] \quad \text{on} \quad \{N \in S\text{-Mod} \mid \mathrm{Ann}_N(J) = 0\}.$$

(3) If $UJ = U$, then ${}_IC = \mathrm{I}(U \otimes_S -)$ and ${}_J\mathrm{M} = {}_J\mathrm{M}'$. It thus follows from Theorem 4 that the functors

$$\langle V, -\rangle : {}_IC \rightleftarrows {}_J\mathrm{M} : (U \otimes_S -)$$

induce inverse equivalences in case $UJ = U$.

(4) If $JV = V$, then ${}_IL = \mathrm{I}[V, -]$ and ${}_J\mathrm{M} = {}_J\mathrm{M}''$. It follows from Theorem 5 above that the functors

$$\langle V, -\rangle : {}_IL \rightleftarrows {}_J\mathrm{M} : [V, -]$$

induce inverse equivalences in case $JV = V$.

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Institute of Mathematics
University of Tsukuba
Ibaraki 305, Japan

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PRIMITIVE ELEMENTS OF CYCLIC EXTENSIONS OF COMMUTATIVE RINGS

Isao KIKUMASA, Kazuo KISHIMOTO and Takasi NAGAHARA

Throughout this paper, A will mean a commutative ring with identity element 1 which is an algebra over a finite prime field $GF(p)$, and all ring extensions of A will be assumed with identity element 1, the identity element of A. Moreover, B will mean a Galois extension of A with a cyclic Galois group $G = \langle\sigma\rangle$ generated by σ of order p^n, which will be called a cyclic p^n-extension of A (with a Galois group G). If B is generated by a single element z over A then B (resp. z) will be called a simple extension of A (resp. a primitive element of B over A).

One of the present authors made a study on primitive elements for cyclic 2^2-extensions in [2]. In this paper, we shall present a sharpening of [2] and some generalizations to cyclic p^n-extensions with $p \geq 2$ and $n \geq 1$.

In what follows, given a Galois extension S/R with a Galois group G, we shall use the following conventions: For any subring T of S and any subgroup H of G,

1) $\mathfrak{m}(T) = \{M ; M$ is a maximal ideal of $T\}$,
2) $G(T) = \{\sigma \in G ; \sigma(a) = a$ for all $a \in T\}$,
3) $S(H) = \{a \in S ; \sigma(a) = a$ for all $\sigma \in H\}$,
4) $t_H(a) = \sum_{\sigma\in H}\sigma(a)$ for each $a \in T$, which will be

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The detailed version of this paper will be submitted for publication elsewhere.

called the H-trace of a. Moreover, for any set V and its subset W,

5) $|V|$ = the cardinal number of V.

6) $V \backslash W$ = the complement of W in V.

Now, we shall here consider a cyclic p^n-extension B/A with a Galois group $G = \langle\sigma\rangle$. Then, there exists an element a in B whose G-trace is 1 ([1, Lemma 1.6]). If, in particular, $|G| = p$ then there exists an element b in B such that $\sigma(b) = b + 1$. When this is the case, there holds that $t_G(b) = 0$ if $p > 2$ and $B = A[b]$ ([7, Theorem 1.2]). Such an element b will be called a σ-generator of B/A (cf. [2]). In case $|G| = 2$, an element c in B is a σ-generator of B/A if and only if $t_G(c) = 1$.

All the facts in this paper are contained in I. Kikumasa and T. Nagahara [*] and [**].

1. On primitive elements of cyclic 2^2-extensions. In this section, we shall discuss the case $p = 2$ and $n = 2$.

Throughout this section, H will mean a subgroup of G generated by σ^2, i.e., $H = \langle\sigma^2\rangle$. Moreover, we put $T = B(H)$ and $\sigma|T = \bar{\sigma}$.

First, we shall prove the following theorem which contains the result of K. Kishimoto [2, Lemma 1].

Theorem 1. The following conditions are equivalent.

(a) There exists a primitive element for B/A whose G-trace is zero.

(b) There exists an invertible element of T whose $\langle\bar{\sigma}\rangle$-trace is 1.

Proof. (a) ⟹ (b). Let $B = A[z]$ and $t_G(z) = 0$, and set $b = z + \sigma(z)$. Then, we have $\sigma^2(b) = b$, and so, $b \in T$. By [4, Theorem 3.3], b and $b + \sigma(b) = z + \sigma^2(z)$ are invertible in B. Hence $x = b(b + \sigma(b))^{-1}$ is an invertible element of T and $t_{\langle\bar{\sigma}\rangle}(x) = 1$.

(b) $\Rightarrow$ (a). Let x be an invertible element of T whose $\langle\bar{\sigma}\rangle$-trace is 1. Then, $\sigma(x) = x + 1$. Hence we have $T = A[x]$ by [7. Theorem 1.2]. Since B is a Galois extension of A, there exists an element y in B such that $t_G(y) = 1$. Put

$$b = x^2 + x \quad \text{and} \quad z = xy + x\sigma(y) + \sigma(xy + x\sigma(y)).$$

Then, since x is invertible, $\sigma(x) = x + 1$ is also invertible and so is $b = x\sigma(x)$. Moreover, since $t_G(y) = 1$, we have $\sigma^2(z) = z + 1$. Hence $B = T[z]$. Further,

$$z + \sigma(z) = xy + x\sigma(y) + \sigma^2(xy + x\sigma(y))$$
$$= xt_G(y) = x.$$

Hence we have $\sigma(z) = z + x$. Then we obtain $\sigma(z^2 + z + xb) = z^2 + z + xb$. Therefore, it follows that $c = z^2 + z + xb \in A$, and $x = (z^2 + z + c)b^{-1} \in A[z]$. This implies that $A[z] = A[z,x] = T[z] = B$. Moreover, noting $\sigma(z) = z + x$ and $\sigma(x) = x + 1$, we have $t_G(z) = 0$.

Corollary 2. Let x be an invertible element of T with $t_{\langle\bar{\sigma}\rangle}(x) = 1$ and y an element of B with $t_G(y) = 1$. Then,

$$z = xy + x\sigma^2(y) + \sigma(y) + \sigma^2(y)$$

is a primitive element for B/A whose G-trace is zero and so is $z + a$ for any $a \in A$. In particular,

$$z_1 = xy + x\,\sigma^2(y) + \sigma(y) + \sigma^2(y^2) + y + y^2$$

is also an element which has the property.

Proof. The first part is shown in the proof of Theorem 1. Moreover, it is clear that $A[z + a] = A[z] = B$ and $t_G(z + a) = t_G(z) = 1$ for any $a \in A$. Since $t_G(y) = 1$ and

$$z + z_1 = y + \sigma^2(y) + y^2 + \sigma^2(y^2),$$
$$\sigma(z + z_1) = (\sigma(y) + \sigma^3(y)) + (\sigma(y^2) + \sigma^3(y^2))$$
$$= (y + \sigma^2(y) + 1) + (y^2 + \sigma^2(y^2) + 1)$$
$$= z + z_1.$$

Hence, $z + z_1$ is in A and $z_1 = z + b$ for some $b \in A$. This shows the last part.

Remark 1. Assume that there is an invertible element x

in T whose $<\bar{\sigma}>$-trace is 1. Then, for any element y of B whose $<\sigma>$-trace is 1, we set

$b = x^2 + x$, $z = zy + x\sigma(y) + \sigma(xy + x\sigma(y))$, $c = z^2 + z + xb$

and $f = (X - z)(X - \sigma(z))(X - \sigma^2(z))(X - \sigma^3(z))$.

Then, noting $\sigma(z) = z + x$, we have

$$f = X^4 + (b + 1)X^2 + bX + (b^3 + bc + c^2)$$

and $B = A[z] \cong A[X]/(f)$ by [4, Theorem 3.3, 3.4]. Clearly $\{1, z, z^2, z^3\}$ is a linearly independent A-basis for B.

Next, for the z_1 in Corollary 2, we set $a = z_1 + z$ ($\in A$), and

$$f_1 = (X - z_1)(X - \sigma(z_1))(X - \sigma^2(z_1))(X - \sigma^3(z_1)).$$

Then

$$f_1 = X^4 + (b + 1)X^2 + bX + (b^3 + b(c + a^2 + a) + (c + a^2 + a)^2)$$

and $B = A[z_1] \cong A[X]/(f_1)$. This primitive element z_1 for B/A and the polynomial f_1 are in [2, Lemma 1].

Next, we shall present an alternative proof of [2, Lemma 2] which is simple.

Lemma 3. Assume that B has a primitive element. Then, for any $M \in \mathfrak{M}(A)$, if $A/M = GF(2)$ then $T/TM = GF(4)$.

Proof. Let $M \in \mathfrak{M}(A)$ and $A/M = GF(2)$. Moreover, let x and z be primitive elements for T/A and B/A, respectively. Then, B/BM is a cyclic 2^2-extension of A/M with a Galois group $<\rho>$ where ρ is an automorphism of B/BM induced by σ. We set $s = z + BM$ and $r = x + BM$ in B/BM. Then, $B/BM = GF(2)[s]$ and $(B/BM)(\rho^2) = T/TM = GF(2)[r]$. We shall here assume that $r^2 - r = 0$. Then, noting $[GF(2)[r]:GF(2)] = 2$, we have $T/TM = GF(2)r \oplus GF(2)(1 - r)$. Hence the units of T/TM are only 1. Clearly $s + \rho^2(s) \in T/TM$. By [4, Theorem 3.3], $s + \rho^2(s)$ is a unit in B/BM, and so is in T/TM. Hence $s + \rho^2(s) = 1$, which implies that $t_{<\rho>}(s) = 0$. Thus, by Theorem 1, there exists a unit t in T/TM such that $t + \rho(t) = 1$. For $t = 1$, we have $t + \rho(t) = 0$, and this is a contradiction. Hence $r^2 - r \neq 0$, and so, $r^2 - r = 1$. Since $f = X^2 + X + 1$ is irreducible over $GF(2)$,

$$GF(4) = GF(2)[X]/(f) \cong GF(2)[r].$$

Now, we define here the set $\mathfrak{m}_0$ as follows:

$$\mathfrak{m}_0 = \{M \in \mathfrak{m}(A);\ TM \in \mathfrak{m}(T)\}.$$

In the rest of this note, we will often use this set and, moreover, will omit the proofs of our results. The precise descriptions of these results are contained in [*].

Theorem 4. Assume that $|\mathfrak{m}(A)\setminus\mathfrak{m}_0|$ is finite and $T/TM = GF(4)$ for any $M \in \mathfrak{m}(A)$ such that $A/M = GF(2)$. Then, there exists an invertible element y in T with $t_{<\bar{\sigma}>}(y) = 1$. Therefore B has a primitive element for B/A.

Corollary 5. Assume that $|\mathfrak{m}(A)\setminus\mathfrak{m}_0|$ is finite. Then the following are equivalent.

(a) B/A has a primitive element.

(b) B/A has a primitive element whose trace is zero.

The following theorem contains the result of [2, Theorem 3].

Theorem 6. Assume that $|\{M \in \mathfrak{m}(A);\ A/M \neq GF(2)\}|$ is finite. Then, the following conditions are equivalent.

(a) B has a primitive element for B/A.

(b) $T/TM = GF(4)$ for any $M \in \mathfrak{m}(A)$ such that $A/M = GF(2)$.

2. On primitive elements of cyclic p^n-extensions. Set

$$B_i = B(\sigma^{p^i}) \quad (i = 0, 1, 2, \dots, n) \quad \text{and}$$

$$\mathfrak{m}_i = \{M \in \mathfrak{m}(B_i);\ B_{i+1}M \in \mathfrak{m}(B_{i+1})\} \quad (i = 0, 1, 2, \dots, n-1).$$

Then, obviously $B = B_n$ and $A = B_0$. Moreover, B_i is a cyclic p^{i-j}-extension of B_j with a Galois group $<\sigma^{p^j}|B_i>$.

Theorem 7. Assume that $p = 2$ and $|\mathfrak{M}(B_0)\setminus\mathfrak{M}_0|$ is finite. Then, the following conditions are equivalent.

(a) B_2/B_0 has a primitive element.

(b) B_{k+2}/B_k has a primitive element for any k $(0 \leqq k \leqq n-2)$.

Theorem 8. Assume that $p = 2$ and $|\{M \in \mathfrak{M}(A);\ A/M \not\cong GF(2)\}|$ is finite. Then, the following conditions are equivalent.

(a) B_2/B_0 has a primitive element.

(b) B_{k+2}/B_k has a primitive element for any k $(0 \leqq k \leqq n-2)$.

Corollary 9. When B/A is the situation of Theorem 7 or 8, this has a system of generating elements consisting of m elements where $m = n/2$ if n is an even number and $m = (n + 1)/2$ if n is an odd number.

Theorem 10. Assume that $p \geqq 2$ and $\mathfrak{M}(A) = \mathfrak{M}_0$. Then, B/A has a primitive element. Moreover, if $x \in B$ with $t_G(x) = 1$ then x is a primitive element for B/A and is invertible.

3. On primitive elements of cyclic p^n-extensions over fields. Let A be a field of characteristic $p \neq 0$. For any A-algebra S which is a finitely generated A-module, $\ell(S)$ will denote the length of the composition series of S. Then, we have

Theorem 11. Let B/A be a cyclic p^n-extension. Then, the following conditions are equivalent.

(a) B/A has a primitive element.

(b) $\ell(B) \geqq p^n([A:GF(p)]p^n - 1)/(np^n - 1)$.

Remark 2. The result of Theorem 11 can be generalized to any Galois extension B/A where A is a field (cf. [**]).

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Department of Mathematics
Okayama University,
Department of Mathematics
Shinshu University
and
Department of Mathematics
Okayama University

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AUTOMORPHISMS OF A CERTAIN SKEW POLYNOMIAL RING OF DERIVATION TYPE

Isao KIKUMASA

Throughout, p will represent a prime integer and F a field of characteristic p. For an element $\alpha \in F$, we set $A = F[x]/(x^p - \alpha)F[x]$. Moreover, D and $A[X;D]$ will denote the derivation of A induced by the ordinary derivation of $F[x]$ and the skew polynomial ring with $aX = Xa + D(a)$ where $a \in A$ (that is called "derivation type"), respectively.

In 1968, R. W. Gilmer [3] determined the B-automorphisms of the ordinary polynomial ring $B[X]$ for any commutative ring B. As to algebra automorphisms of the polynomial ring over any (not necessary commutative) ring, D. B. Coleman and E. E. Enochs [1] established a thorough result three years later. Since then, characterization of automorphisms of skew polynomial rings has come into question.

The answer to this question for the case of automorphism type was completely given by M. Rimmer [5]. Furthermore, in case of derivation type, M. Ferrero and K. Kishimoto got results in [2].

However, [2] is studied on B-automorphisms of $B[X;\delta]$ in case that B is a ring with a derivation δ satisfying the condition $\delta(N) \subset N$ where N is the union of all nilpotent

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The detailed version of this paper will be submitted for publication elsewhere.

ideals of B. Hence there are rings with derivations to which the results cannot be applied. In fact, the ring $F[x]/(x^p)F[x]$ with the derivation D does not fulfill the condition and so we can never apply the results to the ring with the derivation, though it plays an important role in studies of algebras.

On the other hand, for the algebra A, N. Jacobson [4] mentions a certain kind of A-automorphisms of $A[X;D]$ in case that $x^p - \alpha$ is irreducible in $F[x]$. However, if $x^p - \alpha$ is not irreducible then A is isomorphic to $F[x]/(x^p)F[x]$ and hence the problem to determine all the A-automorphisms of $A[X;D]$ has never been solved except the case that A is a field.

In this note, we will solve the problem and, as a result, we will obtain an automorphism whose type is quite different from ones in [1], [2], [3] and [5].

First, we shall state the following lemma which can characterize the A-endomorphisms of $A[X;D]$.

Lemma. Let B be a commutative algebra over the prime field $GF(p)$ and δ a derivation of B such that $\delta^p = 0$. Moreover, let $B[X;\delta]$ be the skew polynomial ring over B where $bX = Xb + \delta(b)$ $(b \in B)$. Assume that $\delta(z) = 1$ for some $z \in B$. Then the map $X \to \sum_{i=0}^{n} X^i b_i$ $(b_i \in B,\ n \geqq 1,\ b_n \neq 0)$ induces a B-endomorphism of $B[X;\delta]$ if and only if

(i) $b_1 = 1$ and

(ii) $b_i = 0$ for all $i \in \{j: 2 \leqq j \leqq n$ and $p \nmid j\}$.

When this is the case, the image of X takes the form

$$X + \sum_{i=0}^{s_0} X^{pi} b_{pi}$$

where s_0 is an integer such that $ps_0 = n$ if $n \geqq 2$, and $s_0 = 0$ if $n = 1$.

To determine all the A-automorphisms of $A[X;D]$, we shall consider the following conditions for the A-linear map ϕ of

$A[X;D]$ to itself defined by

(#) $X^k \to (\sum_{i=0}^{n} X^i a_i)^k$, $k = 0, 1, 2, \ldots$ $(n \geqq 2,\ a_n \neq 0)$.

(In case $n = 1$, see Remark 2).

(i) $a_1 = 1$.

(ii) $a_i = 0$ for all $i \in \{j\colon 2 \leqq j \leqq n$ and $p \nmid j\}$.

Assume that (ii) is fulfilled. Then, we have $p|n$ because $n \geqq 2$ and $a_n \neq 0$. Hence there exist integers s and t which satisfy $ps = n$ and $pt \leqq s < p(t + 1)$. Thus, the following conditions can be considered.

(iii) $D^{p-1}(a_p) + 1 \neq 0$.

(iv) $D^{p-1}(a_{p^2 i}) + a_{pi}^p = 0$ for all $i \in \{j\colon 1 \leqq j \leqq t\}$.

(v) $a_{pi}^p = 0$ for all $i \in \{j\colon t + 1 \leqq j \leqq s\}$.

(vi) $D^{p-1}(a_{pi}) = 0$ for all $i \in \{j\colon 2 \leqq j \leqq s$ and $p \nmid j\}$.

Now, using the above conditions, we state our main theorem.

Theorem. The map ϕ is an A-automorphism of $A[X;D]$ if and only if there hold (i) - (vi). Furthermore, in this case, the inverse map ϕ^{-1} of ϕ is induced by

$$X^k \longrightarrow (X + \sum_{j=0}^{s} X^{pj} b_{pj})^k, \quad k = 0, 1, 2, \ldots$$

where

$$b_{pj} = \sum_{i=j}^{s} (-1)^{i-j+1} \binom{i}{j} (D^{p-1}(a_0) + a_0^p)^{i-j} (D^{p-1}(a_p)+1)^{-i} a_{pi}$$

for each j.

In the rest of this note, let y be the image of x in A by the canonical homomorphism from $F[x]$ to A.

Remark 1. For $a_{pi} \in A = F[x]/(x^p - \alpha)F[x]$, put

$$a_{pi} = \sum_{k=0}^{p-1} y^k \gamma_{pi,k} \quad (\gamma_{pi,k} \in F).$$

Then, $a_{pi}^p = \sum_{k=0}^{p-1} \alpha^k \gamma_{pi,k}^p$ and, by Wilson's Theorem,

$$D^{p-1}(a_{pi}) = (p-1)!\gamma_{pi,p-1} = -\gamma_{pi,p-1}.$$

Hence, we can replace the conditions (iii) - (vi) with the following:

(iii)' $\gamma_{p,p-1} \neq 1$.

(iv)' $\gamma_{p^2i,p-1} = \sum_{k=0}^{p-1}\alpha^k\gamma_{pi,k}^p$ for all $i \in \{j: 1 \leq j \leq t\}$.

(v)' $\sum_{k=0}^{p-1}\alpha^k\gamma_{pi,k}^p = 0$ for all $i \in \{j: t+1 \leq j \leq s\}$.

(vi)' $\gamma_{pi,p-1} = 0$ for all $i \in \{j: 2 \leq j \leq s$ and $p \nmid j\}$.

Obviously, in case $\alpha = 0$, these relations show that whether or not the A-endomorphism ϕ of $A[X;D]$ is an A-automorphism depends only on the coefficients $\gamma_{pi,p-1}$ of y^{p-1} and constant terms $\gamma_{pi,0}$ of a_{pi} $(1 \leq i \leq s)$. Therefore, the coefficients $\gamma_{pi,k}$ of intermediate terms y^k $(1 \leq k \leq p-2)$ can be taken freely, and hence, if $p \neq 2$ then we can easily make different A-automorphisms of $A[X;D]$ from any given A-automorphism of $A[X;D]$.

This also means that there exist at least $|F|^{(p-2)s}$ A-automorphisms of $A[X;D]$ whose image of X is of degree $n = ps$, where $|F|$ is the cardinal number of the field F.

Remark 2. In case $n = 1$, M. Ferrero and K. Kishimoto [2, Lemma 2] have shown that if B is a ring and δ is a derivation of B, then the map $X \to b_0 + Xb_1$ induces a B-automorphism of $B[X;\delta]$ if and only if b_1 is a central unit and $b_0b - bb_0 = \delta(b)(b_1 - 1)$ for all $b \in B$. Noting $D(y) = 1$, we can easily see that the map $X \to a_0 + Xa_1$ induces an A-automorphism of $A[X;D]$ if and only if $a_1 = 1$. Thus, we can consider our theorem to contain the case $n = 1$.

Finally, we shall present some interesting results which can be obtained by our theorem.

Examples. Let $\alpha = 0$ i.e. $A = F[x]/(x^p)F[x]$.

1. Suppose that $p = 2$. Let maps ϕ_1 and ϕ_2 be

A-endomorphisms of $A[X;D]$ induced by

$$X \longrightarrow X + X^2y \quad \text{and} \quad X \longrightarrow X + X^2y\beta \quad (\beta \neq 1 \in F),$$

respectively. Then, ϕ_1 is not an A-automorphism by the condition (iii) (or (iii)'). But ϕ_2 is an A-automorphism of $A[X;D]$ by our theorem. When this is the case,

$$\phi_1(X^2) = 0 \quad \text{and} \quad \phi_2^{-1}(X) = X - X^2y(\beta + 1)^{-1} .$$

2. It is easily seen from the condition (iv) that a_{pi} don't have to be nilpotent for all $i \geqq 2$, though the map is an A-automorphism of $A[X;D]$. Actually, by our theorem, we see that the map

$$X \longrightarrow X + X^p + X^{p^2}y^{p-1}$$

induces an A-automorphism of $A[X;D]$, though $a_p = 1$ is not nilpotent. This shows that there exists an automorphism whose form is quite different from ones known by now, because all results in [1] - [3] and [5] show that a_i $(i \geqq 2)$ must be nilpotent in case the map $X \to \sum_{i=0}^n X^i a_i$ induces a B-automorphism of a commutative or skew polynomial ring over a ring B.

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Department of Mathematics
Okayama University

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COMMUTATIVE SEMIGROUP RINGS

Ryûki MATSUDA

1. Introduction. The commutative semigroup ring $A[S]$ is a ring which reflects properties of the semigroup S and the ring of coefficients A. These rings may be studied for their own sake or as a tool for tackling other problems. There are many results by many people in the theory of commutative semigroup rings. We will choose two topics among them. One is the divisor class groups of commutative semigroup rings and the other is the Kronecker function rings of semigroups. The former is mainly by R. Gilmer, T. Parker, D.F. Anderson and L. Chouinard.

Let S be a torsion-free cancellative commutative semigroup $\supsetneqq \{0\}$. The semigroup operation is denoted by "+". If $ns_1 = ns_2$ where $s_1, s_2 \in S$ and $n \in N$ implies $s_1 = s_2$, then S is called torsion-free. The group $\{s_1 - s_2; s_i \in S\}$ is denoted by $q(S)$, and is called the quotient group of S. $q(S)$ is a torsion-free abelian group. Let A be a (commutative) integral domain $\ni 1$. We set $A[X;S] = A[S] = \{\sum_{\text{finite}} a_s X^s;\ a_s \in A,\ s \in S\}$.

2. The divisor class groups of commutative semigroup rings.

Theorem 1([9, Gilmer-Parker]). $A[X;S]$ is a UFD(i.e. unique factorization domain) if and only if A is a UFD, S

is a UFS(i.e. unique factorization semigroup) and q(S) satisfies a.c.c.c. (i.e. ascending chain condition for cyclic subgroups).

If each element of S is expressed as a sum of irreducible elements uniquely up to associates, then S is called a UFS. q(S) satisfies a.c.c.c. if and only if each nonzero element has the type (0,0,0,...).(For the definition we refer to [7, § 85].)

Corollary 2([8, Gilmer]). Let p be a prime number or zero, and d a natural number with $d \geqq 2$. Then there exists a non-Noetherian UFD of characteristic p and of (Krull) dimension d.

This is the answer for David's conjecture([6]). Cor. 2 is proved by Th. 1 and by a result on torsion-free abelian groups by L. Pontryagin([14] or [7, § 88]). A UFD of dimension 1 is a PID(i.e. principal ideal domain), and hence a Noetherian ring.

Let K be the quotient field q(A) of A. If there exists a family $\{ V_\lambda \ ; \ \lambda \in \Lambda \}$ of DVR's(i.e. discrete valuation rings) of K such that (1) $A = \bigcap_\lambda V_\lambda$, and (2) each non-zero element of A is a unit of V_λ for almost all λ , then A is called a Krull ring. A UFD is a Krull ring.

Corollary 3. Suppose that S is a group G. Then A[X;G] is a Krull ring if and only if A is a Krull ring and G satisfies a.c.c.c.

Proof of the sufficiency. Set q(A) = K. Then K[X;G] is a UFD by Th. 1. Let $\{ w_\sigma \ ; \ \sigma \in \Sigma \}$ be a family of discrete valuations of q(K[X;G]) under which K[X;G] is a Krull ring. Let $\{ v_\lambda \ ; \ \lambda \in \Lambda \}$ be a set of discrete valuations of K under which A is a Krull ring. If we set $v_\lambda^*(\sum a_s X^s) = \inf_s v_\lambda(a_s)$, then v_λ^* is a discrete valuation of q(K[X;G]). (v_λ^* is called the natural extension of v_λ .) Then A[X;G] is a Krull ring under $\{ w_\sigma \ , \ v_\lambda^* \ ; \ \sigma, \lambda \}$.

Let $F(A)$ be the set of fractional ideals $\neq (0)$ of A. Set $q(A) = K$. We set $A:(A:I) = I^{v}$ for each $I \in F(A)$, where $A:I$ denotes $\{x \in K;\ xI \subset A\}$. If $I_1^{v} = I_2^{v}$ for $I_i \in F(A)$, we set $I_1 \sim I_2$. We set $F(A)/\sim\ = D(A)$. The equivalence class containing I is denoted by $\mathrm{div}(I)$, and is called a divisor of A. If we set $\mathrm{div}(I_1) + \mathrm{div}(I_2) = \mathrm{div}(I_1 I_2)$, then $D(A)$ is a semigroup. $D(A)$ is a group if and only if A is completely integrally closed. If A is a Krull ring, then A is completely integrally closed. We set $D(A)/\{\mathrm{div}\ (x);\ 0 \neq x \in K\} = C(A)$, and is called the semigroup of divisor classes of A.

Proposition 4. Suppose that S is a group G and $A[X;G]$ is a Krull ring. Then we have $C(A[X;G]) \cong C(A)$ canonically.

Set $q(A) = K$. Prop. 4 is proved by Nagata's Theorem ([13]) and by the fact that $K[X;G]$ is a UFD.

Serre's Conjecture([18]). Let k be a field. Then each finitely generated projective module over $k[X_1, \ldots, X_n]$ is free.

This has been solved affirmatively by D. Quillen([15]) and A. Suslin([19]). Relating to the conjecture we have

Proposition 5([1, Anderson]). Let A be a Krull subring of $k[X_1, X_2]$ generated by monomials. Then each projective module over A is free.

The statement of [1] is seemingly different from Prop. 5. But they are essentially the same. Subrings of $k[X_1, \ldots, X_n]$ generated by monomials has been studied besides by various authors; for example [10].

Problem(Murthy). Let A be a Krull subring of $k[X_1, \ldots, X_n]$ generated by monomials. Calculate $C(A)$ explicitly.

Let $\{ X_1^{e_{i(1)}} \dots X_n^{e_{i(n)}} ;\ i \in I \}$ be the set of monomials of $k[X_1, \dots, X_n]$ which generates A. Then $\{ (e_{i(1)}, \dots, e_{i(n)}) ;\ i \}$ generates a subsemigroup S of Z_0^n. And A is the semigroup ring $k[X;S]$ of S over k.

Anderson([2]) and Chouinard([3]) solved the above problem by studying,so to speak,"the Krull ring theory of a semigroup S". They also solved a Fossum's problem (which will be mentioned later). Their Krull ring theory of semigroups is as follows.

Let G be the quotient group of S. A non-empty subset I of G is called a fractional ideal of S if (1) $S + I \subset I$ and (2) $s + I \subset S$ for some $s \in S$. Let $F(S)$ be the set of fractional ideals of S. We set $S:(S:I) = I^v$ for each $I \in F(S)$, where $S: I$ denotes $\{ \alpha \in G;\ \alpha + I \subset S \}$. If $I_1^v = I_2^v$ for $I_i \in F(S)$, we set $I_1 \sim I_2$. We set $F(S)/\!\sim\, = D(S)$. The equivalence class containing I is denoted by $\mathrm{div}(I)$. $D(S)$ is an additive semigroup. Then $D(S)$ is a group if and only if S is completely integrally closed. If each element $\alpha \in G$ belongs to S whenever there exists $s \in S$ such that $s + n\alpha \in S$ for each $n \in N$, then S is called completely integrally closed. We set $D(S)/ \{ \mathrm{div}\ (\alpha);\ \alpha \in G \} = C(S)$. $C(S)$ is called the semigroup of divisor classes. Next, an additive homomorphism v of G into Z is called a discrete valuation of G. And the subsemigroup $\{ \alpha \in G;\ v(\alpha) \geqq 0 \}$ of G is called a DVS (i.e. discrete valuation semigroup). If there exists a family $\{ V_\lambda ;\ \lambda \in \Lambda \}$ of DVS's such that (1) $S = \bigcap_\lambda V_\lambda$, and (2) each $s \in S$ is a unit of V_λ for almost all λ, then S is called a Krull semigroup. If S is a Krull semigroup, then S is completely integrally closed.

For example, let $F = \bigoplus_{\lambda \in \Lambda} Ze_\lambda$ be a free abelian group with a basis $\{ e_\lambda ;\ \lambda \}$. Set $\bigoplus_\lambda Z_0 e_\lambda = F_+$. Then F_+ is a Krull semigroup. In fact the projection mapping $pr_\lambda : F \longrightarrow Z$ is a discrete valuation of F. And F_+ is a Krull semigroup under the family $\{ pr_{\lambda} \ ; \lambda \}$ of discrete valuations.

Theorem 6. (1) A[X;S] is a Krull ring if and only if A is a Krull ring, S is a Krull semigroup and q(S) satisfies a.c.c.c.

(2) Assume that A[X;S] is a Krull ring. Then we have $C(A[S]) \cong C(A) \oplus C(S)$ canonically.

Proof. Set q(A) = K and q(S) = G. (1) The sufficiency. Let $\{ w_{\sigma} \ ; \sigma \in \Sigma \}$ be the family of discrete valuations of q(A[X;G]) under which A[X;G] is a Krull ring. Let $\{ v_{\lambda} \ ; \lambda \in \Lambda \}$ be the family of discrete valuations of G under which S is a Krull semigroup. If we set $v_{\lambda}^{*}(\sum a_s X^s) = \inf_s v_{\lambda}(s)$, then v_{λ}^{*} is a discrete valuation of q(A[X;S]). (v_{λ}^{*} is called the natural extension of v_{λ}.) Then A[X;S] is a Krull ring under $\{ w_{\sigma}, v_{\lambda}^{*} \ ; \sigma, \lambda \}$. (2) We have canonical homomorphisms : $C(A) \longrightarrow C(A[S]) \longrightarrow C(K[S])$. We may consider an analogy: $A \longleftrightarrow S$ and $K \longleftrightarrow G$. Then we have a canonical isomorphism: $C(K[S]) \cong C(S)$ analogously to Prop. 4. Moreover, there exist canonical homomorphisms: $C(S) \longrightarrow C(A[S]) \longrightarrow C(A[G])$ corresponding to the sequence $C(A) \longrightarrow C(A[S]) \longrightarrow C(K[S])$. The proof is complete.

Proposition 7. Let S be a Krull semigroup only unit of which is 0. Set q(S) = G. Then S is embedded in a free abelian group $F = \bigoplus_{\lambda \in \Lambda} Z e_{\lambda}$ such that (1) $G \cap F_+ = S$ (F_+ denotes $\bigoplus_{\lambda} Z_0 e_{\lambda}$), (2) each projection $pr_{\lambda} : G \longrightarrow Z$ is a surjection, (3) if $\lambda \neq \lambda'$, we have $pr_{\lambda}(s) = 0 < pr_{\lambda'}(s)$ for some $s \in S$. Moreover, if a free abelian group F satisfies these conditions, then C(S) is canonically isomorphic with F/G.

Proof. In fact, let $\{ P_{\lambda} \ ; \lambda \in \Lambda \}$ be the set of minimal prime ideals of S. Then D(S) is a free abelian group F with a basis $\{ \operatorname{div}(P_{\lambda}); \lambda \}$ as the case of rings. We may identify $\alpha \in G$ with div (α). Moreover we have $C(S) \cong F/G$ for this F by the definition of $C(S)$.

Each Krull semigroup S is of the form $H \oplus S_1$, where H is a group and only units of S_1 is 0. In fact, let $\{ v_\lambda ; \lambda \in \Lambda \}$ be a family of discrete valuations of G under which S is a Krull semigroup. Let $F = \bigoplus_\lambda Ze_\lambda$ be a free abelian group with basis $\{ e_\lambda ; \lambda \}$. Set $\phi(\alpha) = \sum v_\lambda(\alpha)e_\lambda$ for each $\alpha \in G$. Then ϕ is a homomorphism of G into F, and induces the decomposition $G \cong H \oplus \phi(G)$.

Example 8([16, Samuel]). Let k be a field of characteristic p.

(1) Assume that n is a natural number such that $(p, n) = 1$ if $p > 0$, and assume that k contains a primitive n-th root of 1. Then $C(k[X^n, XY, Y^n]) \cong Z/nZ$;

(2) Assume that $p > 0$, then $C(k[X^p, XY, Y^p]) \cong Z/pZ$.

The cohomology of finite groups is used for the proof. (1) relies on Galois descent. (2) relies on a radical descent of a derivation and its logarithmic derivative.

Assume that the characteristic $p > 0$ and n is a power of p. Then W. Waterhouse([20]) used the cohomology theory of Hopf algebras to prove $C(k[X^n, XY, Y^n]) \cong Z/nZ$.

But, if we use Th.6 and Prop. 7, simple calculations show the following,

Example 9. Let A be a Krull ring. Then we have $C(A[X^n, XY, Y^n]) \cong C(A) \oplus Z/nZ$.

Proof. Set $(1, 0) = e_1$, $(0, 1) = e_2$, $Ze_1 + Ze_2 = F$ and $nZ_0e_1 + Z_0(e_1 + e_2) + nZ_0e_2 = S$. Let $q(S) = G$. We have $G = nZe_1 + Z(e_1 + e_2)$ and $G \cap F_+ = S$. Then we have $C(S) \cong F/G \cong Z/nZ$ by Prop. 7. We have $C(A[X;S]) \cong C(A) \oplus Z/nZ$ by Th.6.

Fossum's Problem. Is every abelian group H is isomorphic with the divisor class group $C(R)$ of a quasi-local Krull ring

R?

L. Claborn([5]) proved that every abelian group is isomorphic with the divisor class group of a Krull ring.

To solve the problem, set $Z/2Z \oplus H = \Lambda$. Let F be a free abelian group with a basis $\{ e_{(\varepsilon, h)};\ (\varepsilon, h) \in \Lambda \}$. If we set $\phi(e_{(\varepsilon, h)}) = h$, then ϕ is a homomorphism of F onto H. We set $\mathrm{Ker}(\phi) \cap F_+ = S$. Then we see $q(S) = \mathrm{Ker}(\phi)$. We have $C(S) \cong F/\mathrm{Ker}(\phi) \cong H$ by Prop. 7. Let k be a field and let $k[X;S] = A$. We set $\{ \sum n_\lambda e_\lambda \in S;\ \sum n_\lambda = n \} = S_n$ and $\{ \sum a_s X^s;\ s \in S_n \} = A_n$ for each $n \in Z_0$; $A_0 = k$. Then A is a graded Krull ring : $A = \bigoplus_{n \geqq 0} A_n$. If we set $\bigoplus_{n \geqq 1} A_n = M$, then we have $C(A_M) \cong C(A)$ by [17, Prop. 7. 4]. It follows $C(A_M) \cong C(k[X;S]) \cong H$.

Chouinard([4]) proved that if S is a Krull semigroup with $C(S)$ torsion, then each finitely generated projective module over $k[X;S]$ is free.

3. Kronecker function rings of semigroups. Details of this section appear on [12]. Let K be the quotient field of an integral domain A. A mapping $I \mapsto I^*$ of $F(A)$ into $F(A)$ is called a $*$-operation on A if the following conditions hold for each $x \in K - \{0\}$ and all I, J in $F(A)$: (1) $(x)^* = (x)$; $(xI)^* = xI^*$; (2) $I \subset I^*$; if $I \subset J$, then $I^* \subset J^*$; (3) $(I^*)^* = I^*$. For example the mapping $I \mapsto I^v$ in Section 2 is a $*$-operation. If $(IJ)^* \subset (IK)^*$ implies $J^* \subset K^*$ for each finitely generated I, J, K of $F(A)$, then $*$ is called an e.a.b. $*$-operation. For an e.a.b. $*$-operation $*$ we set $\{ f/g;\ f, g \in A[X] - \{0\},\ c(f)^* \subset c(g)^* \} \cup \{0\} = A_*$; where $c(f)$ denotes the ideal of A generated by the coefficients of f. Then A_* is a subring of $q(A[X])$, and is called the Kronecker function ring of A with respect to $*$. The ring A_* has various interesting properties.

We found that we are able to define Kronecker function rings of semigroups and that the analogous results with those for Kronecker function rings of integral domains hold for Kronecker function rings of semigroups.

First, a mapping $I \mapsto I^*$ of $F(S)$ into $F(S)$ is called a $*$-operation on S if the following conditions hold for each $\alpha \in G$ and all I, J of $F(S)$: (1) $(\alpha)^* = (\alpha)$; $(\alpha + I)^* = \alpha + I^*$; (2) $I \subset I^*$; if $I \subset J$, then $I^* \subset J^*$; (3) $(I^*)^* = I^*$. We define an e.a.b. $*$-operation on S similarly. Let k be a field. For an e.a.b. $*$-operation $*$ on S we set $\{ f/g;\ f, g \in k[X;S] - \{0\},\ e(f)^* \subset e(g)^* \} \cup \{0\} = S_*^k$; where $e(f)$ denotes the ideal of S generated by the exponents appearing in f. We call S_*^k the Kronecker function ring of S with respect to $*$(and k). We set $\{ I^*;\ I$ is a finitely generated fractional ideal of $S \} = D_f^*(S)$. If $D_f^*(S)$ is a group under the natural addition, we call S a Prüfer $*$-multiplication semigroup. We set $\{ f \in k[X;S];\ e(f)^* = S \} = U^*$. U^* is a multiplicative subset of $k[X;S]$. We have $S \subset k[X;S] \subset k[X;S]_{U^*} \subset S_*^k$. The following result is one of semigroup versions of results on Kronecker function rings of integral domains.

Theorem 10. Let $*$ be an e.a.b. $*$-operation on S, and k a field. Then the following conditions are equivalent:

(1) S is a Prüfer $*$-multiplication semigroup;

(2) $k[X;S]_{U^*} = S_*^k$;

(3) $k[X;S]_{U^*}$ is a Prüfer ring;

(4) S_*^k is a quotient ring of $k[X;S]$;

(5) Each prime ideal of $k[X;S]_{U^*}$ is the contraction of a prime ideal of S_*^k ;

(6) Each prime ideal of $k[X;S]_{U^*}$ is the extension of a prime ideal of S;

(7) Each valuation ring of $q(S_*^k)$ containing S_*^k is the natural extension of an essential valuation semigroup of S;

(8) S_*^k is a flat $k[X;S]$-module.

An additive homomorphism v of $q(S)$ into a totally ordered abelian group is called a valuation of $q(S)$. If we set $v^*(\sum a_s X^s) = \inf_s v(s)$, then v^* is a valuation of $q(k[X;S])$ (which is called the natural extension of v). On [11] we stated that conditions (1), (2), (3), (4), (7) and (8) of Th. are equivalent, and posed a question if 8 conditions of Th. are equivalent or not.

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Department of Mathematics
Ibaraki University
Mito, Ibaraki 310, Japan

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ON LOEWY SERIES OF GROUP ALGEBRAS OF SOME SOLVABLE GROUPS

Kaoru MOTOSE

Loewy series of projective indecomposable modules of group algebras of some solvable groups have been studied by several authors (see [1,2,4,5,6]). The purpose of this paper is to calculate Loewy series of projective indecomposable modules of the following group G (see [3]).

Let p be a fixed prime, let $X = \{ 0,1, \dots , p-1 \}$, let F be a finite field of order q^p where $q = p^r$ and let α be an element in F of order $h = (q^p - 1)/(q - 1)$. We consider the permutation group G on F such that

$$G = \{ x \to \beta x^{q^s} + \gamma \mid \beta \in \langle\alpha\rangle, \gamma \in F, s \in X \}$$

and the group algebra KG of G over a field K containing F.

For integers s and t, we define $s \sim t$ if and only if $s \equiv tq^k \mod h$ for some $k \in X$. Since the subgroup $\{ x \to \beta x^{q^s} \mid \beta \in \langle\alpha\rangle, s \in X \}$ is a Frobenius group and the full set of irreducible $K\langle\alpha\rangle$-modules is equal to the set $A = \{ 0,1, \dots , h - 1 \}$, the set $A^\sim$ of representatives of classes with respect to the equivalence relation $\sim$ in A is just equal to the set of irreducible KG-modules and of these projective covers.

The final version of this paper will be submitted for publication elsewhere.

For $s \in B = \{ 0,1, \ldots , q^p - 1 \}$, let $s^{(k)} \in B$ such that $s^{(k)} \equiv sq^k \mod q^p - 1$. Set $s = i_0 + i_1 p + \ldots + i_{rp-1} p^{rp-1}$ and $t = j_0 + j_1 p + \ldots + j_{rp-1} p^{rp-1}$ where $i_k, j_k \in X$ for all k. Then we define

$$s \# t = \begin{cases} s + t & \text{if } i_k + j_k < p \text{ for all } k \\ 0 & \text{otherwise} \end{cases}$$

and $s^* = \sum_{k=0}^{rp-1} i_k$.

In the remainder of this paper, we shall fix the numbers m *and* n *such that* $m \in A^{\sim}$ *and* $0 \leq n \leq (rp + 1)(p - 1)$. We shall use some notations for our main theorem.

$C = \{ j \in B \mid j \equiv -m \mod h \}$.

$T_{ij} = (i \# j, i \# j^{(1)}, \ldots , i \# j^{(p-1)})$ for $i \in B$ and $j \in C$.

$D_t = \{ T_{ij} \mid i^* + j^* = t$ and all T_{ij} are distinct $\}$ where $t \leq n$.

$E_t = \{ s \in B \mid s^* = t \}$ where $0 \leq t \leq rp(p - 1)$.

$H = \{ s + m \mid s \in B, s^* = n, s \neq i \# j$ for all $T_{ij} \in D_n \}$.

$I = \{ s + m \mid s \in B, s^* = n, s = i \# j$ for some $T_{ij} \in D_n$ and T_{ij} has no zero components $\}$.

$J_k = \{ i \mid T_{ij} \in D_{n-k}$ and T_{ij} has at most k zero components $\}$ where $0 < k < p$ and $k \leq n$.

Let a_{ts}, b_s, c_s and d_{ks} be the numbers of elements in the classes of $s \in A^{\sim}$ in the sets E_t, H, I and J_k, respectively. We replace a_{t0}, b_s, c_0 and d_{k0} by pa_{t0}, pb_s, pc_0 and pd_{k0}, respectively. Of course these numbers may be often zero. Let g_s be the multiplicities of irreducible components M_s ($s \in A^{\sim}$) of the Loewy factor $N^nP_m/N^{n+1}P_m$ of the projective indecomposable module P_m where N is the radical of KG.

We can now state our main theorem which contains [1, p.214, 15.10 Examples] and [4, Theorem] (see also [6, p.65, (4.3) Bemerkungen]).

Theorem. In case $m = 0$, $g_s = (\sum_{k=0}^{p-1} a_{n-k\ s})/p$ where $0 \leq n - k \leq rp(p - 1)$.

In case $m \neq 0$, $g_s = (b_s + c_s + \sum_{k=1}^{p-1} d_{ks})/p$.

In virtue of this theorem and a computer, we can calculate Loewy series of projective indecomposable KG-modules. This theorem has been proved by using Jacobi sums and a nice basis of the radical of KG. We shall have the analogous results on the group obtained by replacing $\langle\alpha\rangle$ by a subgroup of F^* containing $\langle\alpha\rangle$ (see [5]).

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Department of Mathematics
Faculty of Science
Hirosaki University
Hirosaki 036, Japan

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EMBEDDING THE FRAME OF TORSION THEORIES IN A LARGER CONTEXT--SOME CONSTRUCTIONS

Jonathan S. GOLAN

0. Introduction. One of major means of studying the structure of the ring R is to make use of the set of all ideals of R. This set has the order structure of a complete modular lattice and also a related algebraic structure of a semiring. The corresponding means for studying the structure of the category R–mod of left modules over R is to make use of the frame (= complete brouwerian lattice) R–tors of all torsion theories on R–mod. This lattice does not have a corresponding semiring structure but, as we will show, can be embedded in a canonical way into the set of idempotent elements of at least two different semirings, each of which can be used to provide information on the structure of R–mod. In what follows, all rings R are associative with multiplicative identity and all modules are unital. Notation and terminology concerning torsion theories will always follow [Golan, 1986].

1. Torsion theories and linear topologies. One method of embedding R–tors in a semiring has been discussed in detail in [Golan, 1987]. A topology on R which turns R into a topological ring is uniquely determined by the family of neighborhoods of 0 in it. Such a topology is *linear* if this family of neighborhoods of 0 has a base consisting of left ideals of R. The family κ of all left ideals which are neighborhoods of 0 in a given linear topology satisfies the following three conditions:

An expanded version of this paper will appear elsewhere.

(1) If $I \in \mathcal{K}$ then any left ideal of R containing I also belongs to $\mathcal{K}$;

(2) If $I, H \in \mathcal{K}$ then $I \cap H \in \mathcal{K}$;

(3) If $I \in \mathcal{K}$ and if $r \in R$ then $(I:r) \in \mathcal{K}$.

Conversely, any family $\mathcal{K}$ of left ideals of R satisfying these three conditions is a base for the family of neighborhoods of 0 in a linear topology of R. Such a family is called a *(topologizing) filter* of left ideals of R. We will denote the set of all topologizing filter of left ideals of R by R–fil. If τ is a torsion theory on R–mod then the set $\mathcal{L}_\tau$ of all left ideals I of R such that R/I is τ–torsion is a topologizing filter of left ideals which, in fact, uniquely determines τ. Thus the map $\tau \mapsto \mathcal{L}_\tau$ defines an embedding of R–tors into R–fil.

The set R–fil has the structure of a complete lattice the meet in which is defined by intersection. The unique minimal element of R–fil is $\eta[R] = \{R\}$ and the unique maximal element of R–fil is the set $\eta[0]$ of all left ideals of R. This lattice is not distributive in general, as has been pointed out by Katayama [1986]. However, it does have other nice properties. See [Golan, 1987] for details. More importantly, there is another operation on R–fil with which we can work, namely that of multiplication. Following [Gabriel, 1962], we define the product $\mathcal{K}\mathcal{K}'$ of filters $\mathcal{K}$ and $\mathcal{K}'$ to be the set of all left ideals I of R satisfying the condition that there exists a left ideal H of $\mathcal{K}'$ such that $I \subseteq H$; and $(I:a) \in \mathcal{K}$ for all a in H. This operation is associative but is not, in general, commutative. The filter $\eta[R]$ acts as a multiplicative identity while the filter $\eta[0]$ acts as a zero–element: $\mathcal{K}\eta[0] = \eta[0] = \eta[0]\mathcal{K}$ for all $\mathcal{K}$ in R–fil. No nontrivial element of R–fil has a multiplicative inverse.

Multiplication distributes over intersection differently on the left and on the right. If $\mathcal{K} \in$ R–fil and if Y is a nonempty subset of R–fil then $\mathcal{K}(\cap Y) = \cap\{\mathcal{K}\mathcal{K}' \mid \mathcal{K}' \in Y\}$ and $(\cap Y)\mathcal{K} \subseteq \cap\{\mathcal{K}'\mathcal{K} \mid \mathcal{K}' \in Y\}$, with equality holding when Y is finite (and in certain infinite cases as well). In particular, this shows us that (R–fil, $\cap$, $\cdot$) has the structure of a semiring with zero element. Semirings were first studied implicitly by Dedekind and explicitly by Vandiver [1934]; they are described in detail in [Almeida Costa, 1963; 1974].

They have in recent years been used considerably in automata theory, optimization, and theoretical computer science and so the interest in them has revived.

How does the image of R-tors fit into this structure? If κ is an arbitrary element of R-fil then $\kappa^2 \supseteq \kappa$. Such a filter is *idempotent* if and only if $\kappa^2 = \kappa$. As Gabriel already noted, the idempotent elements of R-fil are precisely those which come from R-tors under the above-mentioned embedding. Thus we see that the frame R-tors can be considered as the set of idempotent elements of the semiring R-fil. This set is not, in general, closed under multiplication in R-fil and, indeed, the product of two idempotent filters is again idempotent if and only if they commute. If $\kappa \in$ R-fil then there is a unique minimal idempotent filter κ^* containing κ. This is the unique minimal solution of the equation $X = \kappa X$ and of the equation $X = X\kappa$ in R-fil. The map $\kappa \mapsto \kappa^*$ is a closure operator on R-fil satisfying $(\kappa \cap \kappa')^* = \kappa^* \cap \kappa'^*$ for all κ and κ' in R-fil. Such a closure operator on a lattice is called a *nucleus* or a *modal operator*. In [Fuchs, 1963] such operators are called *linear closure operators*.

2. Nuclei on R-tors. We now turn to a different way of embedding the frame R-tors into a larger structure, which was developed by Simmons in a series of papers [1978, 1980, 1982, 1986a, 1986b, 1986c, 1986d], building on work of Beazer and Macnab [1979] and Isbell [1972, 1975]. Their work was done for arbitrary frames, but here we will concentrate on the special case of the frame R-tors.

As defined above, a nucleus on R-tors is a closure operator f:R-tors → R-tors satisfying the additional condition that $f(\tau \wedge \sigma) = f(\tau) \wedge f(\sigma)$. Let us denote the set of all nuclei on R-tors by N(R-tors). We note several examples of such operators:

(A) If $\tau \in$ R-tors let $\mathbb{P}(\tau)$ be the set of all prime torsion theories greater than or equal to τ. Then the map $\tau \mapsto \wedge\mathbb{P}(\tau)$ is a nucleus on R-tors. More generally, if M is a left R-module and if $\mathrm{pinv}_\tau(M)$ is the set of τ-pseudoinvariants of M relative to a torsion theory τ then the map $p_M{:}\tau \mapsto \wedge\mathrm{pinv}_\tau(M)$ is a nucleus on

R-tors for each left R-module M.

(B) A left R-module M is *decisive* if and only if, for each torsion theory τ on R-mod, M is either τ-torsion or τ-torsionfree. Each such module M defines a nucleus q_M on R-tors defined by $q_M(\tau) = \chi$ if M is τ-torsion and $q_M(\tau) = \chi(M)$ if M is τ-torsionfree. If M is simple then $q_M = p_M$.

(C) An arbitrary torsion theory τ on R-tors defines three nuclei in **N**(R-tors):

(1) $u_\tau:\sigma \mapsto \sigma \vee \tau$;

(2) $v_\tau:\sigma \mapsto (\sigma:\tau)$; and

(3) $w_\tau:\sigma \mapsto (\tau:(\tau:\sigma))$.

In particular, $w_\xi:\sigma \mapsto \sigma^{\perp\perp}$ is a nucleus on R-tors.

If U is a set of nuclei on R-tors then the function $\wedge U:\sigma \mapsto \wedge\{f(\sigma) \mid f \in U\}$ is again a nucleus on R-tors. Thus **N**(R-tors) has the structure of a complete lattice, in which $f \leq g$ if and only if $f(\tau) \leq g(\tau)$ for all τ in R-tors. Indeed, **N**(R-tors) is a frame in which, for nuclei f and g, $(g:f) = \wedge\{v_{f(\tau)}gu_\tau \mid \tau \in \text{R-tors}\}$. The map $\tau \mapsto u_\tau$ is a natural embedding of R-tors into **N**(R-tors) in the category of frames. It is an isomorphism precisely when **N**(R-tors) is boolean.

<u>3. Derivatives and filtrations on R-tors</u>. A *derivative* on R-tors is a function d from R-tors to itself satisfying the following conditions:

(1) $\tau \leq d(\tau)$ for all τ in R-tors;

(2) If $\sigma \leq \tau$ in R-tors then $d(\sigma) \leq d(\tau)$.

Thus, for example, any nucleus on R-tors is a derivative. Many of the most important examples of derivatives are not, however, nuclei. For example:

(A) We have the well-known *Gabriel derivative* on R–tors defined by $d_g(\tau) = \tau \vee [\vee\{\xi(M) \mid M$ is τ-cocritical$\}]$. This is the same as $\vee\{\xi(M) \mid M$ is τ-artinian$\}$. A related derivative on R–tors is given by $d_n(\tau) = \vee\{\xi(M) \mid M$ is τ-noetherian$\}$.

(B) If $\tau \leq \sigma$ in R–tors then we write $\tau << \sigma$ if and only if $\sigma \wedge \sigma' > \tau$ whenever $\sigma' > \tau$ in R–tors. The *Cantor-Bendixson-Simmons derivative* on R–tors is defined by $d_{cb} = \wedge\{\sigma \mid \tau << \sigma\}$. Note that it is not necessarily the case that $\tau << d_{cb}(\tau)$ for all torsion theories τ.

(C) The *Boyle derivative* on R–tors is defined by $d_b(\tau) = \tau \vee [\vee\{\xi(M) \mid M$ is τ-full$\}]$, where a left R–module M is *τ-full* if and only if it is τ-torsionfree and a submodule of M is τ-dense in M when and only when it is large there.

(D) The *socle derivative* on R–tors is defined by $d_s(\tau) = \tau \vee \{\sigma \in$ R–tors $\mid \sigma$ is an atom over $\tau\}$.

(E) The *jansian hull derivative* on R–tors is defined by $d_{jh}(\tau) = \wedge\{\sigma \geqq \tau \mid \sigma$ jansian$\}$. Similarly, the *stable hull derivative* is defined by $d_{sh}(\tau) = \wedge\{\sigma \geqq \tau \mid \sigma$ stable$\}$. Note that both of these functions are closure operators on R–tors, but do not, in general, satisfy the linearity condition.

Derivatives can be transfinitely iterated. If d is a derivative on R–tors and if i is an ordinal then we define the derivative d^i inductively as follows:

(1) $d^0(\tau) = \tau$;

(2) If $i > 0$ is not a limit ordinal then $d^i(\tau) = d(d^{i-1}(\tau))$;

(3) If $i > 0$ is a limit ordinal then $d^i(\tau) = \vee\{d^h(\tau) \mid h < i\}$.

The transfinite ascending chain $\tau \leqq d(\tau) \leqq d^2(\tau) \leqq \dots$ is called the *filtration* of the torsion theory τ with respect to the given derivative. There must be a least ordinal i such that $d^i(\tau) = d^k(\tau)$ for all $k \geq i$. This ordinal i is called the *d-length* of τ. Since R-tors is a set, there is a least ordinal h greater than or equal to the d-length of every torsion theory. We denote the derivative d^h by d^∞.

Let $\mathbb{D}$(R-tors) be the set of all derivatives on R-tors. If W is a nonempty subset of $\mathbb{D}$(R-tors) then we set $\wedge W$ be the function $\tau \mapsto \wedge\{d(\tau) \mid d \in W\}$. This is again a derivative on R-tors and so $\mathbb{D}$(R-tors) is a complete lattice, containing $\mathbb{N}$(R-tors) as a subset (but not a sublattice since the joins are different). In particular, $\mathbb{D}$(R-tors) is a partially-ordered set.

Simmons and I have spent considerable time looking at the structure of the lattice $\mathbb{D}$(R-tors). For example, it is always true that $d_{cb} \geq d_b \geq d_g$ and that $d_g^\infty \geq d_n \geq d_g$. If R is a left semistable ring (i.e., if every indecomposable injective left R-module is decisive), then $d_s \geq d_g$. If R is left semistable and left seminoetherian then $d_s = d_g = d_b = d_{cb}$. One of Simmons' major results is that $d_g = d_g^\infty \wedge d_{cb}$, from which we can conclude that $d_g = d_n \wedge d_{cb}$.

The main use of derivatives is for defining dimension functions on R-mod. Indeed, if $d \in \mathbb{D}$(R-tors) and if $\tau \in$ R-tors then a left R-module M is said to have (τ,d)-*dimension* i if and only if M is $d^i(\tau)$-torsion but not $d^h(\tau)$-torsion for all $h < i$. Such dimension functions are studied in detail in [Golan, 1977], and we will go into them no further here, except to note that every "reasonable" notion of dimension in module categories seems to indeed arise in this manner.

<u>4. Prenuclei on R-tors</u>. If d is a derivative on R-tors satisfying the condition that $d(\tau \wedge \sigma) = d(\tau) \wedge d(\sigma)$ for all torsion theories τ and σ then d is not necessarily a nucleus, since it still may not be a closure operator. We will call such derivatives *prenuclei* on R-tors, and denote the set of all prenuclei on R-tors by $\mathbb{P}$(R-tors). This is a subset of $\mathbb{D}$(R-tors) containing $\mathbb{N}$(R-tors) which is closed under taking arbitrary meets and, more importantly, closed under composition as well. The derivatives d_b, d_g, and d_n are all prenuclei. The Cantor-Bendixson derivative is not,

nor are the derivatives d_{jh} and d_{sh}.

Since $\mathbb{P}$(R–tors) is closed under composition, it is easily verified that $\mathbb{P}$(R–tors) is a semiring, addition in which is $\wedge$ and multiplication in which is composition. This semiring has a zero element as well, namely the prenucleus d_χ defined by $d_\chi:\tau \mapsto \chi$ for all τ in R–tors. The nuclei on R–tors are precisely the idempotent elements of this semiring. Moreover, as we have already seen, there is a canonical embedding $\tau \mapsto u_\tau$ of R–tors into $\mathbb{P}$(R–tors). This map can be extended to a map from R–fil to $\mathbb{P}$(R–tors) in the following manner: for each $\kappa \in$ R–fil, let u_κ be the map from R–tors to itself given by $u_\kappa:\tau \mapsto (\kappa\tau)^*$. (Here we are identifying R–tors with the set of all idempotent elements of R–fil!) If $\tau \in$ R–tors then $\tau \subseteq \kappa\tau \subseteq (\kappa\tau)^*$ in R–fil by Proposition 3.6 of [Golan, 1987] and so $\tau \leq u_\kappa(\tau)$. Moreover, if $\tau \leq \sigma$ in R–tors then $\kappa\tau \subseteq \kappa\sigma$ in R–fil by Corollary 3.14 of [Golan, 1987] and so $u_\kappa(\tau) \leq u_\kappa(\sigma)$. Therefore $u_\kappa \in \mathbb{D}$(R–tors). Finally, if $\sigma, \tau \in$ R–tors then $[\kappa(\sigma \cap \tau)]^* = (\kappa\sigma \cap \kappa\tau)^*$ $= (\kappa\sigma)^* \cap (\kappa\tau)^*$ in R–fil by Propositions 3.13 and 5.20 of [Golan, 1987] and so $u_\kappa(\sigma \wedge \tau) = u_\kappa(\sigma) \wedge u_\kappa(\tau)$, showing that $u_\kappa \in \mathbb{P}$(R–tors).

If κ and κ' are elements of R–fil and if $\tau \in$ R–tors then, by Propositions 3.13 and 5.20 of [Golan, 1987] we have $[(\kappa \cap \kappa')\tau]^* = (\kappa\tau \cap \kappa'\tau)^* = (\kappa\tau)^* \cap (\kappa'\tau)^*$ and so $(u_{\kappa\cap\kappa'})(\tau) = u_\kappa(\tau) \wedge u_{\kappa'}(\tau)$. This shows that $u_{\kappa\cap\kappa'} = u_\kappa \wedge u_{\kappa'}$ in $\mathbb{P}$(R–tors). Moreover, by Propositions 3.9 and 5.25 of [Golan, 1987] we have $[(\kappa\kappa')\tau]^* = [\kappa(\kappa'\tau)]^* =$ $[\kappa(\kappa'\tau)^*]^* = [\kappa u_{\kappa'}(\tau)]^* = u_\kappa u_{\kappa'}(\tau)$ and so $u_{\kappa\kappa'} = u_\kappa u_{\kappa'}$ in $\mathbb{P}$(R–tors). Finally, it is clear that $u_\chi = d_\chi$, and so we have shown that the function $\kappa \mapsto u_\kappa$ is a homomorphism in the category of semirings with zero element. It is not monic, however, since, by Corollary 5.24 of [Golan, 1987], we have $u_\kappa(\tau) = u_{\kappa^*}(\tau)$ for all torsion theories τ on R–mod and so $u_\kappa = u_{\kappa^*}$. Indeed, the kernel of this homomorphism is precisely $\{\kappa \in \text{R–fil} \mid \kappa^* = \chi\}$.

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University of Haifa
31999 Haifa, Israel

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ALMOST M-PROJECTIVES AND NAKAYAMA RINGS

Anri TOZAKI

Throughout this note, R is a ring with identity, and J is the Jacobson radical of R. Further we assume that every module is a unitary right R-module. Consider the following condition (D_1) on a module M which is in Oshiro's definition of quasi-semiperfect modules [3].

(D_1) For any submodule N of M, there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq N$, and $N \cap M_2$ is small in M_2.

Note that if a module M satisfies (D_1), then so does every direct summand of M. Further a module is indecomposable and satisfies (D_1) if and only if M is hollow. Hence if a module satisfying (D_1) has an indecomposable decomposition, then it is decomposed as a direct sum of hollow modules. On the other hand as shown by Oshiro, every quasi-semiperfect module is decomposed as a direct sum of hollow modules. Therefore we consider (D_1) for a direct sum of hollow modules. In particular, we deal with a finite direct sum of hollow modules which are LE(= with local endomorphism ring). Then for such a module $M = \bigoplus_{i=1}^{n} M_i$ with each M_i hollow and LE, (D_1) is equivalent both to (D_1') and to $(1\text{-}D_1)$ stated below which are more useful to treat than (D_1).

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(D_1'): If $N \leqq M$, and $\pi_j(N) = M_j$ for some j $(1 \leqq j \leqq n)$ where π_j is the canonical projection from M to M_j, then there exists some direct summand N_1 $(\neq 0)$ of M with $N_1 \leqq N$.

We remark that the validity of (D_1') does not depend on the choice of a hollow and LE decomposition.

$(1\text{-}D_1)$: For every non-small submodule N of M, there exists some direct summand M_1 $(\neq 0)$ of M with $M_1 \leqq N$.

Note that for an infinite direct sum of hollow modules, $(1\text{-}D_1)$ (or (D_1')) does not imply (D_1) in general. We can characterize the condition (D_1) by a new concept of "an almost M-projective module" (which is a homomorphic notion).

<u>Definition</u>. Let M and M' be modules. We say that M is <u>almost M'-projective</u> in case given a diagram below, either (i) or (ii) holds.

$$\begin{array}{ccccc} M' & \xrightarrow{\varphi} & H & \longrightarrow & 0 \text{ (exact)} \\ & & \uparrow h & & \\ & & M & & \end{array}$$

(i) There exists some $\tilde{h}: M \longrightarrow M'$ such that $\varphi\tilde{h} = h$.

(ii) There exists some direct summand M_0' $(\neq 0)$ of M' which is not contained in $\ker \varphi$ and some homomorphism $\tilde{h}: M_0' \longrightarrow M$ such that $h\tilde{h} = \varphi|_{(M_0')}$.

We usually make use of this definition when M' is a direct sum of hollow and LE modules and M is an indecomposable module. We remark that M is said to be M'-projective when (i) always holds for any diagram above [1]. This implies that an M'-projective module is an almost M'-projective module. The condition (D_1) is related to the concept of an almost M-projective module as follows.

<u>Theorem 1</u>. Assume that each M_i $(1 \leq i \leq n)$ is a hollow and LE module. Then the following are equivalent.

(1) $M = \bigoplus_{i=1}^{n} M_i$ satisfies the condition (D_1').

(2) For each i $(1 \leq i \leq n)$, M_i is almost $\bigoplus_{j \neq i} M_j$-projective.

Now we study the almost M-projectivity. As for the M-projectivity, the following holds: if M is both M_1- and M_2-projective, then M is $(M_1 \oplus M_2)$-projective [1]. Does the similar assertion hold as to the almost M-projectivity? The almost M-projectivity is difficult to examine in general even for a hollow module. So we study this property for a uniserial module of finite length to get the following result.

<u>Proposition 2</u>. Let e_i be a primitive idempotent with e_iR a uniserial module, and $M_i \simeq e_iR/A_i$ be of finite length with $A_i \leq e_iR$ for each i $(1 \leq i \leq n)$. Then the following are equivalent.

(1) M_1 is almost M_j-projective for all j $(2 \leq j \leq n)$.

(2) M_1 is almost $\bigoplus_{j=2}^{n} M_j$-projective.

Over a semiperfect ring R, the following result about the M-projectivity is known: M_1 is M_2-projective for any two hollow R-modules M_1 and M_2 if and only if R is semisimple. Here we examine what condition on a ring is necessary and sufficient for the corresponding assertion about the almost projectivity to hold. As easily seen, if M_1 (resp. M_2) is almost M_2- (resp. M_1-) projective, then $(M_1 \oplus M_2)$ has the lifting property of simple modules. In addition, Harada [2] has shown that if every direct sum of two hollow modules satisfies this lifting property, then any projective indecomposable module is uniserial. Therefore one of the necessary condition required above is that R is a right Nakayama ring. Thus the result for uniserial modules (in Proposition 2) is not too special. Consequently, we have an answer to the problem as follows.

__Theorem 3__. Let R be a semiperfect ring. A necessary and sufficient condition for any hollow module M_1 to be almost M_2-projective for any hollow module M_2 is that R is a right Nakayama ring with $J^2 = 0$. When this is the case, any hollow module M_1 is almost $\bigoplus_{i=2}^{n} M_i$-projective for any finitely many hollow modules $M_2, \cdots, M_n$.

In Theorem 3, suppose that $M_1/M_1J \simeq M_2/M_2J$, then we have the following theorem.

__Theorem 4__. Suppose that a ring R is semiperfect. Then the following four statements are equivalent.

(1) For each primitive idempotent e in R, eR/A_1 is almost (eR/A_2)-projective, where A_1 and A_2 are any submodules of eR.

(2) R is a right Nakayama ring. Further for each primitive idempotent e in R, $eR/eJ \simeq eJ^j/eJ^{j+1}$ $(j \geqq 1)$ implies that $eJ^{j+1} = 0$.

(3) R is a right Nakayama ring. Further for each primitive idempotent e in R, eR/eJ is almost (eR/A)-projective, where A is any submodule of eR.

(4) For each primitive idempotent e in R and for any natural number n, eR/A_1 is almost $(\bigoplus_{i=2}^{n} eR/A_i)$-projective, where A_i are any submodule of eR.

Suppose that R is a ring with the conditions in Theorem 4. Then the socle $Soc(eR)$ of a projective indecomposable module eR can be isomorphic to only one composition factor $(\neq Soc(eR))$ of eR.

We say that a submodule Y of a direct sum $X = \bigoplus_{i=1}^{n} X_i$ is __standard__ in case Y is described as $Y = \bigoplus_{i=1}^{n} Y_i$ for some submodules Y_i of X_i $(1 \leqq i \leqq n)$. Suppose that for a primitive idempotent e in R, any submodule of $eR \oplus eR$ is a standard submodule of some isomorphic decomposition of $eR \oplus eR$. Then clearly eR satisfies the condition (A) defined in [4]. Further for hollow modules M_i $(1 \leqq i \leqq n)$, assume that any submodule of $M = \bigoplus_{i=1}^{n} M_i$ is standard with respect to some decomposition isomorphic to the given one. Then an easy consideration shows that M satisfies the condition (D_1). These facts are included in the following theorem.

Theorem 5. Let R be a semiperfect ring. Then the following statements are equivalent.

(1) For each primitive idempotent e in R, the following holds : any submodule of $eR/A_1 \oplus eR/A_2$ (A_1 and A_2 are any submodules of eR) is standard with respect to some isomorphic decomposition of $eR/A_1 \oplus eR/A_2$.

(1') For each primitive idempotent e in R and any natural number, the following holds : any submodule of $\bigoplus_{i=1}^{n} (eR/A_i)$ (each A_i is any submodule of eR) is standard with respect to some isomorphic decomposition of $\bigoplus_{i=1}^{n} (eR/A_i)$.

(2)(i) For each primitive idempotent e in R, $eR/A_1 \oplus eR/A_2$ satisfies the condition (D_1) for any submodules A_i $(i = 1, 2)$.

(ii) For each primitive idempotent e in R, any submodule of $eR \oplus eR$ is standard with respect to some isomorphic decomposition of $eR \oplus eR$.

(2')(i) = (2)(i).

(ii) For each primitive idempotent e in R and any natural number n, any submodule of $eR^{(n)}$ is standard with respect to some isomorphic decomposition of $eR^{(n)}$.

(3)(i)=(2)(i)

(ii) For each primitive idempotent e in R, the following holds. Suppose that L and K be submodules of eR with $eR \geqq L > K$. Any automorphism h of L/K is extended and lifted to some endomorphism of eR (or $h^{-1}$is extended and lifted to some endomorphism of eR).

(4)(i)=(2)(i)

(ii) If a module satisfies that $M/MJ \simeq (eR/eJ)^n$ for some primitive idempotent e and some natural number n, then $M \simeq \bigoplus_{i=1}^{n} (eR/B_i)$ for some submodules B_i of eR.

Remark. By Theorem 1, the condition (2)(i) in Theorem 5 is equivalent to each of the conditions in Theorem 4 in particular to Theorem 4(2) which is useful to verify.

Theorem 6. Assume that R is a finite dimensional algebra over a field k. In case R satisfies the condition (2)(i) in Theorem 5, the following two conditions are necessary and sufficient for R to satisfy the condition (2)(ii).

(1) If primitive idempotents e and f in R satisfy $eJf \neq 0$, then we have $[\overline{eRe} : k] = [\overline{fRf} : k]$, where $\overline{eRe} = eRe/eJe$.

(2) If a primitive idempotent e satisfies that $eJ^n = Soc(eR) \simeq eJ^i/eJ^{i+1}$ for some $i \neq n$, then $i = 0$.

Remark. When R is a finite dimensional algebra, Remark and Theorem above show that each of the conditions in Theorem 5 holds if and only if Theorem 4(2) and Theorem 6(1),(2) hold. Note that the latter is useful to verify, and that the second half of Theorem 4(2) and the condition Theorem 6(2) are symmetric in a sense.

Example 1. Let R be an algebra defined by the following quiver with a relation:

$$1 \xrightarrow{\alpha} 2 \circlearrowleft \beta, \qquad \beta^2 = 0.$$

Then R satisfies the condition (2)(i) in Theorem 5, but not the condition (2)(ii) in Theorem 5 because $Soc(e_1R) \simeq e_2R/e_2J \simeq e_1J/e_1J^2$ and $e_1J^2 \neq 0$.

<u>Example 2</u>. Let R be an algebra defined by the following quiver with relations:

$$1 \longrightarrow 2 \rightleftarrows 3 \ , \quad \text{all paths of length 3 are zero.}$$

Then R satisfies both the condition (2)(i) and (2)(ii) in Theorem 5.

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Department of Mathematics
Osaka City University
Sumiyoshi-ku, Osaka 558, Japan

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ENVELOPING ALGEBRAS

Fujio KUBO

1. Introduction. There are many works on the enveloping algebras of Lie algebras. Some of them are concerned with the representations of semisimple Lie algebras, the ring theoretic structures on the enveloping algebras. These works are found in the Dixmier's book[7], the Borho's article[6] and other many books. In this talk we present the other works, which are on the Poisson Lie structures and the finiteness conditions on the enveloping algebras, given by Amayo[2], Bakhturin[4], Kubo & Mimura[10].

2. A Lie structure on $S(G), S^{-1}(S(G))$. Let G be a Lie algebra over a field k of characteristic zero with a basis $\{x_1, \cdots, x_n\}$. Let $U(G)$ be the universal enveloping algebra of G, $U_n(G)$ the vector space spanned by products $y_1 \cdots y_p$, where $y_1, \cdots, y_p \in G$ and $p \leqq n$. Let $S(G)$ be the symmetric algebra of the vector space G and $S^n(G)$ the set of elements of $S(G)$ which are homogeneous of degree n. The *Poisson Lie structure* on $S(G)$ is given by

$$[f,g]=\Sigma_{i,j}[x_i,x_j]\partial f/\partial x_i \partial g/\partial x_j \quad (f,g \in S(G)).$$

This Lie structure is the same as one on $S(G)$ given ,by use of the

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Lie structure on $U(G)$, as follows ([7;p97]): Let π_n be a canonical map of $U_n(G)$ onto $S^n(G)$. Let $p \in S^m(G)$ and $q \in S^n(G)$, and take the elements $p^\wedge \in U_m(G)$ and $q^\wedge \in U_n(G)$ such that $\pi_m(p^\wedge)=p$, $\pi_n(q^\wedge)=q$. Then the Lie product of p and q is given by

$$\pi_{m+n-1}([p^\wedge, q^\wedge]).$$

The concept of this Lie structure is extended to that of the Lie structure on some commutative associative algebras ([10]). We here present one of the results in [10] about the localization of Lie algebras.

Let A be a commutative associative algebra over k with an identity 1. Assume that A is integral and has a Lie structure whose product [,] satisfies the condition $[ab,c]=[a,c]b+a[b,c]$ $(a,b,c \in A)$. Consider A as an associative algebra, take a multiplicatively closed subset S of A containing 1, and denote the localization of A by $S^{-1}A$. Then we can extend the Lie structure on A to $S^{-1}A$ as follows.

THEOREM. Let A,S be given above. Then the localization $S^{-1}A$ is a Lie algebra with the product

$$[f/s,g/t]=([f,g]st+[g,s]ft+[s,t]fg+[t,f]gs)/s^2t^2$$

$(f,g \in A, s,t \in S)$.

EXAMPLE. Let A be the polynomial algebra $k[x_1,\cdots,x_n]$ and [,] the Poisson Lie product given above. Let S be the set of all monomials of A. Then we can construct a Lie algebra $S^{-1}A$. This Lie algebra is the same as the Poisson Lie algebra $k\langle x_1,\cdots,x_n\rangle$ of the Laurent polynomial algebras.

3. Finiteness conditions on U(G). Let G be a Lie algebra over a field k of characteristic zero and $U(G)$ the universal enveloping algebra of G. In this section ,according to Amayo[1] and Bakhturin

[4], we present the properties of Lie algebras G such that U(G) is finitely generated over a finitely generated polynomial algebra in the centre of U(G),and such that U(G) satisfies the polynomial identity.

We first state some examples and results which are used in proving the theorems given later,and found in Amayo & Stewart[3] and Herstein[8].

EXAMPLE 1. Let $A=\langle x,y\rangle$ with $[x,y]=x$ be the 2-dimensional non-abelian Lie algebra over *k*. Then the centre Z(U(A)) of U(A) is *k*. Let M be the polynomial algebra *k*[t] and regard M as an A-module by $f(t)x=tf(t)$, $f(t)y=f(t+1)$ for each $f(t)\in M$.Then M is an infinite -dimensional simple A-module.

EXAMPLE 2. Let M(n) be the polynomial algebra $k[t_1,\cdots,t_n]$ and let the Heisenberg Lie algebra $B(n)=\langle x_1,\cdots,x_n,y_1,\cdots,y_n;z\rangle$ act on M(n) by $fx_i=t_if$, $fy_i=\partial f/\partial t_i$, $fz=f$,for each $f=f(t_1,\cdots,t_n)\in M(n)$. Then M(n) is an infinite-dimensional simple B(n)-module.

LEMMA A ([3;pp.225-232]).Every simple module of a Curtis algebra is finite-dimensional (The definition of the Curtis algebras is given later).

LEMMA B ([8;Chap.6]). Let R be a primitive algebra over *k* satisfying an identity of degree d. Then $\dim_{Z(R)}R\leqq[d/2]^2$, where Z(R) is the centre of R.

3-1. Chain conditions on U(G). Let us first recall the definitions of several classes of Lie algebras over *k*.

$G\in$ Max-cu ; There exists a finitely generated polynomial subring R of the centre of U(G) such that U(G) is finitely generated R-module.

$G\in$ Max-u ; U(G) is right noetherian.

$G\in$ Max ; Every subalgebra of G is finitely generated.

$G\in$ **A** ; G is an abelian Lie algebra.

$G \in F$; G is a finite-dimensional Lie algebra.
Then we have the following inclusions among these classes.

$$\begin{array}{ccccc} \text{Max-cu} & \subset & \text{Max-u} & \underset{(c)}{\subset} & \text{Max} \\ \cup_{(*)} & & \cup_{(b)} & & \cup_{(a)} \\ F \cap A & \subset & F & = & F \end{array}$$

It has been shown that the inclusion (a) is strict in Amayo[1] and Kubo[9]. Let $G = \langle w(i); i \in N \rangle$ be the Witt algebra whose Lie product is given by $[w(i), w(j)] = (i-j)w(i+j)$ for each $W(i), w(j) \in G$. Then $G \in \text{Max} \setminus F$. To know the strictness of (b) and (c), it is one way to see whether the enveloping algebra of this Witt algebra is right noetherian or not.

Lie algebra G is called a *Curtis algebra* if $G \in$ Max-cu. Amayo proved that the inclusion (*) is not strict in [2].

THEOREM. Max-cu $\subset F \cap A$.

Sketch of the proof. Let $G \in$ Max-cu. The proof may be reduced to the case that G is a finite-dimensional Lie algebra over an algebraically closed field of characteristic zero. Assume that G is finite-dimensional and nonabelian. Let I be a maximal abelian ideal of G. Then the Lie algebra $G^{\wedge} = G/I$ is simple of dimension at least 3 ,A or B(n) for some n (see Example 1,2).If $G^{\wedge} = A$,then $U(G^{\wedge})$ is infinite-dimensional over the centre of $U(G^{\wedge})$. This shows that $G^{\wedge}$ can not be a Curtis algebra,and G is not in Max-cu. If $G^{\wedge} = B(n)$, then $G^{\wedge}$ has an infinite-dimensional simple $G^{\wedge}$-module M(n). It is well known that a simple Lie algebra has an infinite-dimensional representation. Whence in these cases $G^{\wedge}$ can not be a Curtis algebra by Lemma A,and G is not in Max-cu.

REMARK. In [2] ,Amayo gives a class Y containing Max-cu such that Y $\subset F \cap A$.

3-2.PI-algebras. An associative algebra R over k is called a PI-algebra if there exists a polynomial $f(x_1,\cdots,x_n)$ without constant term such that $f(r_1,\cdots,r_n)=0$ for any $r_1,\cdots,r_n\in R$.

The following theorem is due to Latyshev[11].

THEOREM. Let G be a Lie algebra over k. Then U(G) is a PI-algebra if and only if G is finite-dimensional and abelian.

Sketch of the proof. Here we sketch the proof according to Bakhturin[4]. The proof may be reduced to the case that G is finite-dimensional and k is algebraically closed. Assume that G is finite-dimensional and nonabelian. Then G contains A or B(1) (see Examples 1,2). Let us denote B(1),M(1) in Example 2 by B,N respectively. Assume that U(G) is a PI-algebra. Then U(A) and U(B) are PI-algebras. Let $I=\mathrm{Ann}_{U(A)}(M)$ and $J=\mathrm{Ann}_{U(B)}(N)$. Then M(resp.N) is a faithful simple U(A)/I(resp.U(B)/J)-module of infinite dimension. By Lemma B, U(A)/I and U(B)/J are finite-dimensional over their centres. But since $\mathrm{End}_{U(A)}(M)=\mathrm{End}_{U(B)}(N)=k\supset$ the centres of U(A)/I and U(B)/J, U(A)/I and U(B)/J are finite-dimensional. This shows that M and N are finite-dimensional,which is a contradiction.

LIE SUPERALGEBRAS. A Z_2-graded k-algebra $G=G_0\oplus G_1$ is called a Lie superalgebra if the following conditions hold in G ; $x_ix_j+(-1)^{ij}x_jx_i=0$, $x_i(x_jy)=((x_ix_j)y)+(-1)^{ij}(x_j(x_iy))$, $x_i\in G_i$, $x_j\in G_j$, $y\in G$. For Lie superalgebras,Bakhturin proved ,in [5],the following

THEOREM. Let $G=G_0\oplus G_1$ be a Lie superalgebra, and U(G) be its universal enveloping algebra.Then U(G) is a PI-algebra if and only if G_0 is an abelian Lie algebra,G_1 contains a G_0-submodule M of finite codimension such that $[M,M]=0$,and $\dim[G_0,M]<\infty$.

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Department of Mathematics,
Kyushu Institute of Technology,
Tobata,Kitakyushu 804

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STRONG ALGEBRA DIAGRAMS AND FROBENIUS DIAGRAM ALGEBRAS

Koichiro OHTAKE

In [2] Fuller defined a module diagram, an algebra diagram and a diagram algebra. We were interested in characterizing algebra diagrams which define Frobenius algebras. To do that we had a useful tool [2, Proposition 4.6]. To use this result we had to exploit a tool to check that the given algebra diagram is strong. By using matrices we managed to give a necessary and sufficient condition in order that the given algebra diagram is strong. Moreover from the idea of this result we got a necessary and sufficient condition that the algebra diagram defines a Frobenius algebra. For example let $\mathcal{R}=\{e_1, e_2, a, b, c, d, e, f, g, y, z, 0\}$ be a semigroup with the following multiplication table.

This is a joint work with Prof. K.R. Fuller of the University of Iowa. The detailed version of this paper has been submitted for publication elsewhere.

	e_1	e_2	a	b	c	d	e	f	g	y	z
e_1	e_1	0	0	b	0	d	0	0	g	y	0
e_2	0	e_2	a	0	c	0	e	f	0	0	z
a	a	0	0	e	0	f	0	0	z	0	0
b	0	b	d	0	g	0	g	y	0	0	0
c	0	c	f	0	0	0	z	0	0	0	0
d	d	0	0	g	0	y	0	0	0	0	0
e	0	e	f	0	z	0	z	0	0	0	0
f	f	0	0	z	0	0	0	0	0	0	0
g	0	g	y	0	0	0	0	0	0	0	0
y	y	0	0	0	0	0	0	0	0	0	0
z	0	z	0	0	0	0	0	0	0	0	0 .

Let K be any field. Then we shall show how to check that the semigroup algebra $K\mathfrak{R}$ is Frobenius.

1. Module diagrams. A <u>module diagram</u> $\mathfrak{M}$ is a finite directed graph with distinguished node 0 such that:

(1) There is at most one arrow between two nodes.

(2) The only oriented cycle in $\mathfrak{M}$ is an arrow beginning and ending at 0.

(3) $x \to 0$ in $\mathfrak{M}$ iff there is no arrow $x \to y \neq 0$ in $\mathfrak{M}$.

(4) $\mathfrak{M}$ entails a function $\lambda : \mathfrak{M}\setminus\{0\} \to \{1,2,..,n\}$ which we call labels of $\mathfrak{M}$.

A <u>subdiagram</u> $\mathfrak{U}$ of a module diagram $\mathfrak{M}$ is a subgraph such that if $x \in \mathfrak{U}$ and $x \to y$ in $\mathfrak{M}$ then $x \to y$ in $\mathfrak{U}$. A quotient diagram of $\mathfrak{M}$ by $\mathfrak{U}$, denoted by $\mathfrak{M}/\mathfrak{U}$, consists of the nodes in $\mathfrak{M}\setminus\mathfrak{U}$ together with all arrows of $\mathfrak{M}$ between them and arrows $x \to 0$ for each $x \in \mathfrak{M}\setminus\mathfrak{U}$ such that every arrow in $\mathfrak{M}$ begins at x and ends in $\mathfrak{U}$. If $a_1,..,a_r \in \mathfrak{M}$ then $\mathfrak{U}(a_1,..,a_r)$ denotes the smallest subdiagram of $\mathfrak{M}$ which contains $a_1,..,a_r$. $\mathcal{L}(\mathfrak{M})$ denotes the set of all subdiagrams of $\mathfrak{M}$. Then $\mathcal{L}(\mathfrak{M})$ is a complete lattice under union and intersection, i.e. every subset of $\mathcal{L}(\mathfrak{M})$ has both an inf and a sup. $\mathrm{Rad}\,\mathfrak{M}$ is the intersection of all maximal subdiagrams of $\mathfrak{M}$ and $\mathrm{Soc}\,\mathfrak{M}$ is the union of all minimal subdiagrams of $\mathfrak{M}$. More explicitly the following hold.

$x \in \text{Rad}\,\mathfrak{M} \Longleftrightarrow \exists u \in \mathfrak{M}$ such that $u \to x$ in $\mathfrak{M}$.

$x \in \text{Soc}\,\mathfrak{M} \Longleftrightarrow x \to 0$ in $\mathfrak{M}$.

Let $\mathfrak{M}$ and $\mathfrak{N}$ be both module diagrams with label functions λ and λ' respectively. Then a homomorphism $\phi: \mathfrak{M} \to \mathfrak{N}$ is a function on nodes satisfying

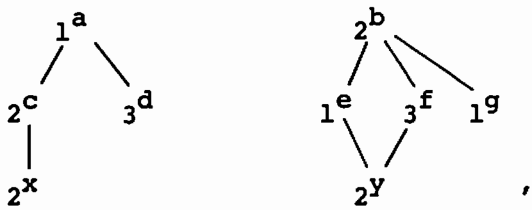
(a) If $\phi(x) \to v$ in $\mathfrak{N}$ then there exists a $y \in \mathfrak{M}$ such that $x \to y$ in $\mathfrak{M}$ and $\phi(y) = v$.

(b) If $x \to y$ in $\mathfrak{M}$ and $\phi(y) \neq 0$ then $\phi(x) \to \phi(y)$ in $\mathfrak{N}$.

(c) If $\phi(x) \neq 0$ then $\lambda'(\phi(x)) = \lambda(x)$.

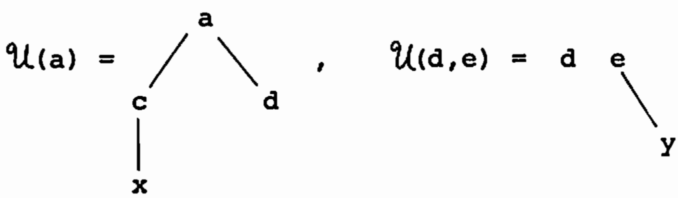
To illustrate a module diagram we usually omit the arrows from nodes to 0.

Example. (1) A module diagram $\mathfrak{M}$. $\mathfrak{M}$ can be illustrated as follows.

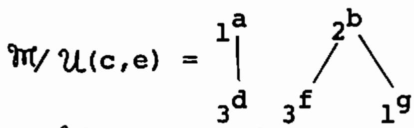


where headless arrows go down and left index numbers of nodes indicate their labels, for example $\lambda(a) = 1$, $\lambda(b) = 2$, etc.

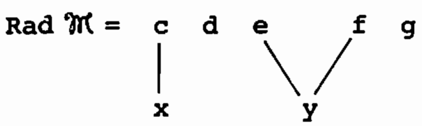
(2) Subdiagrams of $\mathfrak{M}$.



(3) Quotient diagrams of $\mathfrak{M}$.



(4) Rad $\mathfrak{M}$ and Soc $\mathfrak{M}$.



Soc $\mathfrak{M}$ = $\{d, g, x, y, 0\}$

(5) A diagram homomorphism $\phi: \mathfrak{M} \to \mathfrak{M}$. Let ϕ be a function such that $\phi(a) = e$, $\phi(c) = y$ and $\phi(t) = 0$ for all $t \neq a, c$. Then ϕ is a diagram homomorphism.

2. Diagram algebras. Let $\mathfrak{R}$ be a module diagram which satisfies the following conditions.

(1) $\mathfrak{R} = \mathfrak{P}_1 \cup \cdots \cup \mathfrak{P}_n$ with $\mathfrak{P}_i \cap \mathfrak{P}_j = \{0\}$ if $i \neq j$.

(2) $\mathfrak{P}_i = \mathfrak{U}(e_i)$ for each i.

(3) If $e_i \to a \neq 0$ then there is no other arrow $x \to a$, and there is an epimorphism $\phi_a: \mathfrak{P}_{\lambda(a)} \to \mathfrak{U}(a)$.

ϕ_a's are naturally considered as endomorphisms of $\mathfrak{R}$. Let $\phi_{e_i}: \mathfrak{R} \to \mathfrak{P}_i$ be the natural projections.

(4) If $a_1,\ldots,a_r;\ b_1,\ldots,b_s \in \mathfrak{R}\setminus \mathrm{Rad}^2\mathfrak{R}$ and $\phi_{a_r}\cdots\phi_{a_1}(e_{\lambda(a_1)}) = \phi_{b_s}\cdots\phi_{b_1}(e_{\lambda(a_1)}) \neq 0$ then $\phi_{a_r}\cdots\phi_{a_1} = \phi_{b_s}\cdots\phi_{b_1}$ holds.

In this case $\mathfrak{R}$ is called an <u>algebra diagram</u>. Suppose $\mathfrak{R}$ is an algebra diagram. If $0 \neq x = \phi_{a_r}\cdots\phi_{a_1}(e_{\lambda(a_1)})$ then we can define $\phi_x = \phi_{a_r}\cdots\phi_{a_1}$ by (4). Then for any $y \in \mathfrak{R}$, define

$$y \cdot x = \phi_x(y).$$

By this definition, $(\mathfrak{R}, \cdot)$ is a semigroup. Let $\mathfrak{J} = \mathrm{Rad}\,\mathfrak{R}$. Then the diagram $\mathfrak{R}$ can be characterized as follows.

(a) $\mathfrak{R} = \{e_1,\ldots,e_n\} \cup \mathfrak{J}$, where $e_1,\ldots,e_n$ are orthogonal idempotents and $\mathfrak{J}$ is a nilpotent ideal of $\mathfrak{R}$.

(b) $x \to y \neq 0$ if and only if $\exists\, a \in \mathfrak{J}\setminus\mathfrak{J}^2$ such that $y = ax$.

(c) $X \to 0$ if and only if $\mathfrak{J}x = 0$.

Conversely if a semigroup $\mathfrak{R}$ satisfies the condition (a) then we can construct a module diagram $\mathfrak{R}_\ell$ whose nodes are elements of $\mathfrak{R}$ and arrows are defined in terms of (b) and (c). Moreover the associated semigroup of $\mathfrak{R}_\ell$ coincides with $\mathfrak{R}$. We call $\mathfrak{R}$ an <u>algebra semigroup</u>. We can say that $\mathfrak{R}_\ell$ is the associated algebra diagram for $\mathfrak{R}$.

Let K be any field and $R = K\mathfrak{R}$ the semigroup algebra. Then R is called a <u>diagram algebra</u>.

Let $\delta_\ell: \mathcal{L}(\mathfrak{R}_\ell) \to \mathcal{L}({}_R R)$ be an injective lattice homomorphism defined by $\delta_\ell(\mathfrak{U}) = K\mathfrak{U}$. Then $\delta_\ell(\mathrm{Rad}\,\mathfrak{U}) = \mathrm{Rad}\,\delta_\ell(\mathfrak{U})$ holds for any $\mathcal{U} \in \mathcal{L}(\mathcal{R}_\ell)$. For $\mathcal{U} \in \mathcal{L}(\mathcal{R}_\ell)$, the Cosoc $\mathcal{U}$ is defined by

$$\text{Cosoc}\,\mathcal{U}/\mathcal{U} = \text{Soc}(\mathcal{R}_\ell/\mathcal{U}).$$

If δ_ℓ preserves cosocles then $\mathcal{R}_\ell$ is said to be strong for ${}_RR$.

We should define another diagram $\mathcal{R}_r$ as the following way.

(a) The nodes of $\mathcal{R}_r$ are the elements of $\mathcal{R}$.

(b) $x \to y \neq 0$ in $\mathcal{R}_r$ if and only if $\exists\, a \in \mathcal{J}\setminus\mathcal{J}^2$ such that $y = xa$.

(c) $x \to 0$ in $\mathcal{R}_r$ if and only if $x\mathcal{J} = 0$.

The lattice homomorphism $\delta_r: \mathcal{L}(\mathcal{R}_r) \to \mathcal{L}(R_R)$ is defined similarly as in the case of δ_ℓ.

$\mathcal{R}_\ell$ is said to have cancellation (or $\mathcal{R}$ is left cancellable) provided $ax = ay \neq 0$ for some $a \in \mathcal{J}\setminus\mathcal{J}^2$ implies $x = y$. Fuller [2] proved that if $\mathcal{R}_\ell$ has cancellation then $\mathcal{R}_\ell$ is strong for ${}_RR$ for any field K. $\mathcal{R}$ is said to be cancellable if both of $\mathcal{R}_\ell$ and $\mathcal{R}_r$ have cancellation. We are interested in the case of Frobenius algebras. We have the following criterion.

Proposition 2.1. (See [2, Proposition 4.6]). Let $\mathcal{R}$ be an algebra semigroup and K any field. Let $\mathcal{R} = \{e_1,..,e_n\} \cup \mathcal{J}$ be the decomposition such that $e_1,..,e_n$ are orthogonal idempotents and $\mathcal{J} = \text{Rad}\,\mathcal{R}$. Then $R = K\mathcal{R}$ is Frobenius if and only if the following conditions are satisfied.

(1) δ_ℓ and δ_r preserve socles.

(2) Let $\mathcal{R}_\ell = \mathcal{P}_1 \cup .. \cup \mathcal{P}_n$ with $\mathcal{P}_i = \mathcal{U}(e_i)$. Then $\text{Soc}\,\mathcal{P}_i = \{x_i, 0\}$ for some $x_i \in \mathcal{P}_i$ and $\text{Soc}\,\mathcal{P}_i = \text{Soc}\,\mathcal{P}'_{\lambda(x_i)}$ holds for each i, where $\mathcal{P}'_{\lambda(x_i)}$ denotes the subdiagram of $\mathcal{R}_r$ generated by $e_{\lambda(x_i)}$.

Proof. Suppose R is Frobenius. Since Re_i is indecomposable, Soc Re_i is simple. Thus we may assume $\text{Soc}\,\mathcal{P}_i = \{x_i, 0\}$ since $0 \neq \delta_\ell(\text{Soc}\,\mathcal{P}_i) \subseteq \text{Soc}\,Re_i$. Let $j = \lambda(x_i)$, i.e. $e_j x_i = x_i$. Then we have to show that $\text{Soc}\,\mathcal{P}_i = \text{Soc}\,\mathcal{P}'_j$. Suppose $\text{Soc}\,\mathcal{P}_i \neq \text{Soc}\,\mathcal{P}'_j$. Since Soc e_jR is simple, we may assume $\text{Soc}\,\mathcal{P}'_j = \{y, 0\}$ for some y. Since $x_i \in \mathcal{P}'_j$, there exists $g \in \mathcal{J}$ such that $y = x_i g$. Since $(Re_i)^* \cong e_jR$ ($()^*$ denotes the K-dual of a module), we have $\lambda(g) = \lambda'(g) = i$. Thus $y \in \mathcal{P}_i$ and there exists $h \in \mathcal{J}$ such that $hx_ig = x_i$. This is impossible since $\mathcal{J}$ is nilpotent.

The converse has been proved in [2,Proposition 4.6].

By this result if we know that $\mathfrak{R}_\ell$ and $\mathfrak{R}_r$ are strong for $\mathfrak{R}$ then it is easy to check that whether R is Frobenius. Our first result is the following.

Theorem 2.2. Let $\mathfrak{R}$ be an algebra semigroup with $R = K\mathfrak{R}$ Frobenius. Then $\mathfrak{R}$ is left cancellable if and only if $\mathfrak{R}$ is right cancellable. Moreover if $\mathfrak{R}$ is cancellable then the Nakayama automorphism of R is induced by a semigroup automorphism of $\mathfrak{R}$.

3. Strong algebra diagrams. In this section we will introduce a matrix test for strength of an algebra diagram. Let $\mathfrak{R}$ be an algebra semigroup. Let $\mathfrak{J}\setminus\mathfrak{J}^2 = \{a_1,\ldots,a_m\}$. For each $0 \neq y \in \mathfrak{J}$ let $\{x_1,\ldots,x_{w(y)}\}$ denote the set of x in $\mathfrak{R}_\ell$ such that $x \to y$ and $\mathfrak{R}_\ell$ contains no other path from x to y. Then we define a $m \times w(y)$ matrix $B(y) = (b_{ij})$, where

$$b_{ij} = \begin{cases} 1, & \text{if } a_i x_j = y \\ 0, & \text{otherwise.} \end{cases}$$

Then we obtain

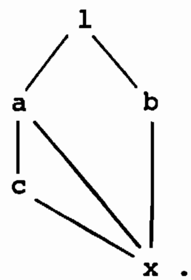
Theorem 3.1. Let $\mathfrak{R}$ be an algebra semigroup with $R = K\mathfrak{R}$. Then $\mathfrak{R}_\ell$ is strong for ${}_RR$ if and only if $\operatorname{rank} B(y) = w(y)$ in K for each $0 \neq y \in \mathfrak{R}$.

Example. Let $\mathfrak{R}$ be an algebra semigroup with the following multiplication table.

1	a	b	c	x
a	c	x	x	0
b	x	x	0	0
c	x	0	0	0
x	0	0	0	0

Then $\mathfrak{R}$ is commutative and

$\mathcal{R}_\ell = \mathcal{R}_r =$



For example $w(x) = 2$ and $B(x) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Thus rank $B(x) = 2 = w(x)$. It is clear that rank $B(a)$=rank $B(b)$ = rank $B(c) = 1$ (it is convenient to define $B(1) = (0)$ since $w(1) = 0$). Thus $\mathcal{R}_\ell = \mathcal{R}_r$ are strong. Moreover in this case the algebra $K\mathcal{R}$ is Frobenius.

We conclude this section by proposing a problem. Suppose $K\mathcal{R}$ is Frobenius and $\mathcal{R}_\ell$ is strong for $K\mathcal{R}$. Then is $\mathcal{R}_r$ strong too?

4. Frobenius algebras. In this section we modify Theorem 3.1 in order to determine that δ_ℓ preserves socles. In this case the matrix will have very big size.

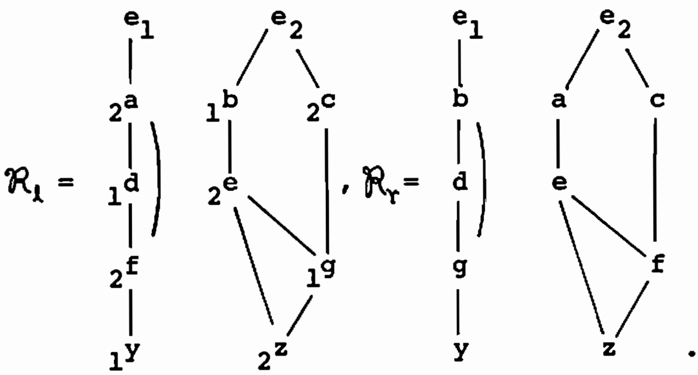
Proposition 4.1. Let $\mathcal{R}$ be an algebra semigroup, $\mathcal{J}\backslash\mathcal{J}^2 = \{a_1, \ldots, a_m\}$, $\mathcal{J}\backslash \mathrm{Soc}\,\mathcal{J} = \{x_1, \ldots, x_s\}$ and $\mathcal{J}\backslash\{0\} = \{y_1, \ldots, y_t\}$, where $\mathrm{Soc}\,\mathcal{J}$ denotes the socle of the subdiagram $\mathcal{J}$ of $\mathcal{R}_\ell$. Then δ_ℓ preserves socles if and only if rank $B = s$, where $B = \begin{bmatrix} B_1 \\ \cdot \\ \cdot \\ B_t \end{bmatrix}$ is the $mt \times s$ matrix with $B_\ell = (b^\ell_{ij})$ and

$$b^\ell_{ij} = \begin{cases} 1, & \text{if } a_i x_j = y_\ell \\ 0, & \text{otherwise.} \end{cases}$$

Now by combining Propositions 2.1 and 4.1 we got a test to check that a diagram algebra $K\mathcal{R}$ is Frobenius.

Example. $K\mathcal{R}$ is Frobenius and $\mathcal{R}_\ell$, $\mathcal{R}_r$ are not strong. Let

$\mathfrak{R}$ be the algebra semigroup in the introduction of this paper. Then



Thus the condition (2) in Proposition 2.1 is satisfied. Since rank $B(g)=1 < w(g) = 2$, $\mathfrak{R}_l$ is not strong. Thus we need to use Proposition 4.1. $\mathfrak{J}\backslash\mathfrak{J}^2 = \{a,b,c\}$, $\mathfrak{J}\backslash \mathrm{Soc}\,\mathfrak{J} = \{a,b,c,d,e,f,g\}$ and $\mathfrak{J}^2\backslash\{0\} = \{d,e,f,g,y,z\}$. We always omit zero rows because there appear a lot of such rows. $B_d = [1\ 0\ 0\ 0\ 0\ 0\ 0]$, $B_e = [0\ 1\ 0\ 0\ 0\ 0\ 0]$, $B_f = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $B_g = [0\ 0\ 1\ 0\ 1\ 0\ 0]$, $B_x = [0\ 0\ 0\ 0\ 0\ 1\ 0]$, $B_y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$. Thus

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

It is easy to see that rank $B = 7$. Therefore δ_l preserves socles. Similarly δ_r preserves socles. Thus the diagram algebra $K\mathfrak{R}$ is Frobenius for any field K.

Suppose an algebra semigroup $\mathfrak{R}$ satisfies the condition (2) of Proposition 2.1. Then is it true that δ_l preserves socles if and only if δ_r preserves socles? This is open. But it is easy to see that if $\mathfrak{J}^3 = 0$ then the problem is affirmative.

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Department of Mathematics
Faculty of Education
Gunma University
Maebashi, 371, JAPAN

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ON F_R^1-TOPOLOGY OF KRULL-TYPE RINGS

Ahmad HAGHANY

1. Introduction. Rings are assumed associative with unit elements. Let R be an order in a simple Artinian quotient ring Q. We say R is a Krull-type ring if there exists a family of two-sided hereditary partial quotient rings R_λ satisfying the following three conditions:

i) $R = \cap R_\lambda$ (The intersection property).

ii) Each non-zero non-unit element x of R survives in only finitely many R_λ, that is, $R_\lambda x = xR_\lambda = R_\lambda$ for almost all λ (The finite character property).

iii) For any two distinct members of the family, $R_\lambda \otimes R_\mu \simeq Q$.

Bounded Krull rings [8], certain UFR [2], some bounded v-HC orders [9], simple Ore extensions of commutative Noetherian domains by derivations, and the Weyl algebras A_n are some examples of rings satisfying i) and ii). In fact most of these are both Noetherian and Krull-type. Denoting the canonical topology of a ring R by F_R^1 (see [11] and [5]). We show that when R is Noetherian Krull-type F_R^1 is completely determined by the injective module $Q \oplus \sum Q/R_\lambda$. Next we consider a Noetherian ring R with only the intersection property and prove that if P is a two-sided ideal maximal with respect to $P^* \neq R$ then P is completely prime, and if R_P exists, it is a bounded right and left principal ideal ring.

This paper is in final form and no version of it will be submitted for publication elsewhere.

2. We begin with a lemma:

2.1. Lemma. Let τ be a hereditary torsion theory over an arbitrary ring R, and let

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

be an exact sequence of left R-modules. If B is τ-torsionfree and C is τ-torsion then f is an essential extension of A.

Proof. We shall identify A with a submodule of B. Consider a non-zero element b of B in order to show that $A \cap Rb \neq 0$. Let I and J be the left annihilators of b and bg respectively. Clearly J contains I. Since R/I is isomorphic to a non-zero submodule of B, it is τ-torsionfree. But R/J, being isomorphic to a submodule of C, is τ-torsion. Therefore J properly contains I which means that $A \cap Rb \neq 0$.

For the rest of this section R will denote a Noetherian Krull-type ring. If M is a left R-module there is a canonical map of left R-modules $M \to \prod R_\lambda \otimes M$ given by $m \to (\ldots, 1 \otimes m, \ldots)$. Notice that if M is a torsion R-module (in the sense that any element is annihilated by a regular member of R) then by the finite character property of R this maps M into the direct sum $\sum R_\lambda \otimes M$. In the following τ will denote the torsion theory cogenerated by $Q \oplus E(Q/R)$.

2.2. Proposition. Let the torsion R-module M be τ-torsionfree. Then the canonical map $M \to \sum R_\lambda \otimes M$ defines an essential extension of M and its cokernel is τ-torsion.

Proof. First note that the canonical map is a monomorphism. For, let m be in the kernel; then $R_\lambda \otimes Rm = 0$ for all λ. Hence by [5, Proposition 2] Rm is τ-torsion. But M is assumed τ-torsionfree, so Rm = 0. Next, let N be the cokernel of the canonical map; so there is an exact sequence

$$0 \to M \to \sum R_\lambda \otimes M \to N \to 0. \qquad (1)$$

Each $R_\lambda \otimes M$ is τ_λ-torsionfree where τ_λ is the torsion theory corresponding to the perfect topology F_λ consisting of left ideals I in R such that $R_\lambda I = R_\lambda$. It is immediate from [5, Proposition 2] that the canonical topology $F_R^1 = \{I: R_\lambda I = R_\lambda$ for all $\lambda\}$. Hence $F_R^1 \subseteq F_\lambda$ for each λ. But then by [4, (7.1) Proposition] any τ_λ-torsionfree module is τ-torsionfree. Since the class of τ-torsionfree modules is closed under direct product we deduce that $\sum R_\lambda \otimes M$ is τ-torsionfree. Thus if we show that in (1), N is τ-torsion then the proof will be complete by Lemma. Now upon tensoring with R_λ, we obtain from (1) the following exact sequence.

$$0 \longrightarrow R_\lambda \otimes M \longrightarrow R_\lambda \otimes \sum R_\mu \otimes M \longrightarrow R_\lambda \otimes N \longrightarrow 0 \qquad (2)$$

We have the isomorphisms

$$R_\lambda \otimes \sum R_\mu \otimes M \simeq \sum_\mu R_\lambda \otimes (R_\mu \otimes M) \simeq \sum_\mu (R_\lambda \otimes R_\mu) \otimes M$$

If $R_\lambda \neq R_\mu$, by assumption $R_\lambda \otimes R_\mu \simeq Q$, the total quotient ring of R. Hence in this case $(R_\lambda \otimes R_\mu) \otimes M \simeq Q \otimes M = 0$ since M is assumed to be a torsion R-module. On the other hand $R_\lambda \otimes R_\lambda \simeq R_\lambda$ as R_λ is a partial quotient ring of R. It follows that $\sum_\mu (R_\lambda \otimes R_\mu) \otimes M \simeq R_\lambda \otimes M$, and now from (2) it is thus clear that $R_\lambda \otimes N = 0$. This being true for all λ, we get N to be a τ-torsion module.

2.3. Theorem. $E(Q/R) \simeq \sum Q/R_\lambda$, that is, the injective envelope of Q/R is isomorphic to the direct sum of all Q/R_λ.

Proof. The torsion R-module Q/R is certainly τ-torsionfree, so by the Proposition, the canonical map $Q/R \longrightarrow \sum R_\lambda \otimes Q/R$ is an essential extension. But $\sum R_\lambda \otimes Q/R \simeq \sum Q/R_\lambda$; and Q/R_λ is R_λ-injective, hence it is R-injective. Thus $\sum Q/R_\lambda$ is an injective R-module (R is Noetherian). Therefore $E(Q/R) \simeq \sum Q/R_\lambda$.

2.4. Corollary. If further R is of finite global dimension then R is hereditary if and only if $Q/R \simeq \sum Q/R_\lambda$.

Proof. If R is hereditary then Q/R being the factor module of an injective module is itself injective. Thus $Q/R = E(Q/R) \simeq \sum Q/R_\lambda$. Conversely assume that $Q/R \simeq \sum Q/R_\lambda$. Then Q/R is an injective R-module, hence the exact sequence

$$0 \longrightarrow R \longrightarrow Q \longrightarrow Q/R \longrightarrow 0$$

shows that the injective dimension of R is at most 1. As R is a Noetherian prime ring of finite global dimension we deduce that R is hereditary.

3. Throughout this section R is assumed Noetherian with intersection property, that is, $R = \bigcap R_\lambda$ where R_λ are hereditary partial quotient rings of R. As before τ is the hereditary torsion theory corresponding to F_R^1.

3.1. Lemma. A finite direct sum of finitely generated R-modules is reflexive if and only if each direct summand is reflexive.

Proof. Let A_i, i = 1, ..., n be finitely generated R-modules. Then for any (τ-torsion) module N we have

$$\mathrm{Ext}_R^1(N, \sum A_i) \simeq \sum \mathrm{Ext}_R^1(N, A_i).$$

Hence $\sum A_i$ is τ-injective if and only if each A_i is τ-injective. Also $\sum A_i$ is torsionless if and only if each A_i is torsionless. The lemma is now clear by [5, Proposition 7].

If S is a Noetherian prime ring with unique maximal ideal J such that S/J is Artinian then S is called local and it is hereditary if and only if J is invertible [6, Proposition 1.3]. Adopting the proof of this result, we can prove the following.

3.2. Proposition. Assume that S is a partial quotient ring of R. Suppose further that S is local. Then S is hereditary if and only if the Jacobson radical J of S is a reflexive left ideal.

Proof. We only need to prove that if J is reflexive then S is hereditary. Let M be a maximal left ideal of S. Then since S/J is simple Artinian, there is a left ideal N of S with $M \cap N = J$ and $M + N = S$. The map $M \oplus N \to S$ sending (m,n) to m − n is a homomorphism with kernel $M \cap N = J$. Hence

$$0 \to J \to M \oplus N \to S \to 0$$

is a split exact sequence. That is, $M \oplus N \simeq J \oplus S$, hence since J is assumed reflexive $M \oplus N$ is reflexive by 3.1 (note that S is also an intersection of hereditary localizations). Thus M is reflexive.

We now assume that S is not hereditary. Since every non-essential left ideal is a direct summand of an essential left ideal we can choose an essential left ideal K which is maximal in the set of non-reflexive left ideals. Let $S \supseteq I_0 \supsetneq I_1 \supsetneq \ldots \supseteq K$ be a strictly descending chain of left ideals. Then either $K = I_n$ for some n or the chain is infinite. Let c be a regular element of K. Then $S \leq I_1^* \leq \ldots \leq c^{-1}S$, and since $c^{-1}S$ is a Noetherian S-module $I_m^* = I_{m+1}^*$ for some m. By the maximality of K in the set of non-reflexive left ideals, I_m and I_{m+1} are reflexive. Hence $I_m = I_{m+1}$, not allowed. Thus $K = I_n$ is the only possibility, and we can suppose that I_{n-1}/K is a simple module. Hence there exists a maximal left ideal M_1 such that $I_{n-1}/K \cong S/M_1$. By the first part of the proof M_1 is reflexive, so S/M_1 by [5, Proposition 4] is σ-torsionfree, where σ is the torsion theory cogenerated by $Q \oplus E(Q/S)$. Thus I_{n-1}/K is also σ-torsionfree. Consider the exact sequence

$$0 \to I_{n-1}/K \to S/K \to S/I_{n-1} \to 0$$

in which I_{n-1}/K and S/I_{n-1} are both σ-tosionfree. Since the class of σ-torsionfree modules is closed under extension we deduce that S/K is σ-torsionfree. This means that K is a reflexive left ideal of S. This contradiction establishes the result.

3.3 Theorem. Let I be a left ideal of R maximal in the set of non-τ-dense left ideals. Then I is proper, reflexive and irreducible. If further I is two-sided then I is completely prime and in case R_I exists it is a bounded left and right principal ideal ring.

Proof. By assumption $I^* \neq R$, and if J is a left ideal which properly contains I then $J^* = R$. Now I is clearly proper and reflexive, for otherwise $I^{***} = R$ that is $I^* = R$. Thus R/I is τ-torsionfree but any proper homomorphic image of it is τ-torsion.

This implies that I is irreducible.

We now assume further, that I is two-sided. Let $a, b \in R$ with $ab \in I$ and $a \notin I$. Then $J = Ra + I$ properly contains I, and as such R/J is τ-torsion. Hom(R/J, R/I) is thus zero, so in particular the map defined by $r + J \to rb + I$ must be the zero map. This implies that $b \in I$. Thus I is completely prime. By previous result if $S = R_I$ exists then it is hereditary, hence a bounded principal left and right ideal ring.

3.4. Remark. If we assume that R is also a maximal order in Q then R is a Krull ring in the sense of Chamarie, hence by [1, Proposition 2.5] R_I exists whenever I is an ideal maximal in the set of non-τ-dense regular left ideals.

Let **A** = {I: I is a regular non-τ-dense left ideal} and
B = {I: I is a regular proper reflexive left ideal}.

3.5. Proposition. **B** ⊆ **A**, and these sets have the same maximal elements.

Proof. That **B** is a subset of **A** is evident. By 3.3, any maximal element of **A** belongs to **B**. Now let P be a maximal member of **B**. Then $P = P^{**} \neq R$, giving $P^{*} \neq R$. Let I be a proper left ideal of R which properly contains P. Then $I \neq I^{**}$ by maximality of P in **B**. If I^{**} is proper then $P \subsetneq I^{**}$, and $I^{**} \in$ **B**, not possible. Thus $I^{**} = R$, hence $I^{*} = R$, that is $I \notin$ **A**. This proves that P is a maximal element of **A**.

3.6. Remark. If R is hereditary then **A** = **B**. But **B** in general is a proper subset of **A**. For example if R = k[X,Y] the commutative polynomial ring over a field k then the ideal $I = RX^2 + RXY$ has the property that $I^{*} = X^{-1}R \neq R$, while $I^{**} = RX$ properly contains I. It would be interesting to characterize rings for which **A** = **B**.

It is well-known that torsionless modules are torsionfree, and in fact commutative Noetherian rings over which torsionfree finitely generated modules are torsionless have been characterized by Vasconcelos as rings without embedded primes which are Gorenstein at minimal primes. In the non-commutative theory the question of when "torsionfree implies torsionless" seems to have received little attention. Malliavin has proved this implication for enveloping algebras of solvable Lie algebras finite dimensional over fields of characteristic zero [7, Theorem 1.1].

3.7. Proposition. Let M be a finitely generated left R-module. Then M is torsionless if and only if M is torsionfree.

Proof. Let N be the kernel of the natural map of M into its double dual, and suppose that M is torsionfree. Since $R_\lambda \otimes M$ is a finitely generated torsionfree R_λ-module and R_λ is hereditary by [3, Theorem 2.1] $R_\lambda \otimes M$ is projective, hence reflexive as an R_λ-module. But the double dual of $R_\lambda \otimes M$ is R_λ-isomorphic to $R_\lambda \otimes M^{**}$ since R_λ is a partial quotient ring of R. It follows that $R_\lambda \otimes N = 0$ for all λ, hence N is τ-torsion. Because the singular submodule of M is zero, M is τ-torsionfree, hence N = 0. This means that M is torsionless.

3.8. Proposition. If R is a domain and B any hereditary two-sided partial quotient ring of R then Q/B is τ-torsionfree.

Proof. Let $q = ac^{-1} \in Q$, and I a τ-dense left ideal of R with $Iq \subseteq B$. Then $Ia \subseteq Bc$, so $BIa \subseteq Bc$. Since BIa is a left ideal in B we have $BIa = (BIa)^{**}$. But $(BIa)^* = a^{-1}I^*B = a^{-1}RB$. Hence $(BIa)^{**} = Ba$, so $Ba \subseteq Bc$ that is $q = ac^{-1} \in B$.

Finally we specialize R, and let $R = A_n$ the n-th Weyl algebra over a field of characteristic zero. As is well-known A_n has two-sided hereditary partial quotient rings B_i with $A_n = \bigcap B_i$; see [10, Corollary 3.4]. In fact A_n is a Noetherian Krull-type domain.

3.9. Proposition. Let E be a non-zero τ-torsionfree indecomposable injective A_n-module. Then E is isomorphic either to $Q(A_n)$ or to a submodule of Q/B for some hereditary partial quotient ring B of A_n.

Proof. There is an irreducible proper left ideal I such that $E = E(A_n/I)$. If I = 0 then $E \simeq Q$, so assume that $I \neq 0$. Since A_n/I is τ-torsionfree I is reflexive, thus by [10, Theorem 3.2] there exists a (hereditary) partial quotient ring B and an element $c \in I$ such that $I = Bc \cap A_n$. Now

$$A_n/I = A_n/Bc \cap A_n \simeq Bc + A_n/Bc \leq B/Bc$$

and B/Bc is isomorphic to a submodule of Q/B. By 3.8, Q/B is τ-torsionfree, and it is B-injective, hence R-injective. It follows that E embedds into Q/B.

3.10. Remark. In [10] Stafford shows that one can take B to be B_n defined in termes of new generators. If we knew that Q/B_n were indecomposable then $E \simeq Q/B_n$ or $E \simeq Q$ hence Q and Q/B_i were the only indecomposable τ-torsionfree A_n-modules.

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Department of Mathematics
Isfahan University of Technology
Isfahan, IRAN

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STRUCTURE OF NAKAYAMA RINGS

Kiyoichi OSHIRO

0. Introduction. In 1940, Nakayama ([11]) introduced generalizd uniserial rings which are today called (artinian) serial or Nakayama rings. Since then, many authors studied these rings.

'Kuppisch series' due to Kuppisch ([6]) is one of important tools for the study of these rings. Using the Kuppisch series, Murase ([8] - [10]) showed that a certain type of serial rings may be represented as a quasi-matrix ring over a division ring. Fuller ([3]) completely determined the global dimension of serial rings using the Kuppisch series. Even now, serial rings are well studied. The self-duality of serial rings was recently studied by Haack ([5]), Mano ([7]) and Dischinger and Müller ([2]). However, Waschbüsch ([15]) pointed out an interesting fact that the self-duality of these rings are already shown by Amdal and Ringdal ([1]) in 1968, and he himself gave a proof.

Now, recently, Harada rings appeared as those artinian rings which contains serial rings and also quasi-Frobenius (QF) rings. So, results on these rings are applied to QF-rings or serial rings, as the following show for examples:

Theorem A ([13]). We can construct all left Harada rings from QF-rings by taking their suitable extensions and factors of the

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The final version of this paper will be submitted for publication elsewhere.

extensions. In particular, we can construct all serial rings by QF serial rings.

Theorem B ([13],[14]). Left Harada rings have the self-duality iff basic indecomposable QF serial rings have the Nakayama automorphisms. In particular, serial rings have the self-duality iff basic indecomposable QF serial rings have Nakayama automorphisms.

Here, the word 'Nakayama automorphism' is due to Yamagata ([16]):

Definition. Let R be a basic indecomposable QF ring and E = $\{e_1, \ldots, e_n\}$ a complete set of orthogonal primitive idempotents. For each e_i, there exists unique f_i in E such that $S(e_iR) \cong f_iR/J(f_iR)$, where S( ) and J( ) denote the socle and the Jacobson radical. The permutation $\{f_1, \ldots, f_n\}$ is called the Nakayama permutation of $\{e_1, \ldots, e_n\}$. If there exists a ring automorphism τ such that $\tau(e_i) = f_i$ for all i, then τ is called a Nakayama automorphism of R. If the Nakayama permutation is identity, then the identity map of R is a Nakayama automorphism of R.

Now, in the present paper, we further study the structures of serial rings as applications of Harada rings. Before we start, we again look at Theorem B. As serial rings have the self-duality, this theorem says that basic indecomposable QF serial rings have Nakayama automorphisms. Although Haack did not succeed to give a proof of the self-duality of serial rings, his [5, Proposition 3.2] is just that basic indecomposable QF serial rings have Nakayama automorphisms (cf. (3.1) later). Therefore, combining Theorem B with the Haack's result, we can confirm the self-duality of serial rings as an affirmative answer of a subproblem of the problem whether left Harada rings have the self-duality or not.

In view of Theorems A and B, in order to study serial rings, we may restrict our attention to basic indecomposable QF serial rings, and then Nakayama automorphisms are deeply concerned in the structure of these rings. We develop our study along this line and completely determine the structure of serial rings. Roughly speaking, we can construct all serial rings from skew-matrix rings over uniserial (i.e., local serial) rings by taking suitable extensions of these rings and factors of the extensions. Though we only give sketch,detail will appear elsewhere.

Notation. Throughout this paper all rings considered are associative rings with identity and all R-modules are unitary. The notation M_R (resp. $_RM$) is used to denote that M is a right (resp. left) R-module. For an R-module M, J(M) and S(M) mean its Jacobson radical and socle, respectively, and $\{S_i(M)\}$ denotes its ascending Loewy chain, For R-modules M and N, we put

$$(M,N) = \mathrm{Hom}_R(M,N)$$

and in particular,

$$(e,f) = (eR,fR) = \mathrm{Hom}_R(eR,fR)$$

for idempotents e, f in R.

Let R be a ring which is represented as a matrix form;

$$R = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ & \cdots & \\ A_{n1} & \cdots & A_{nn} \end{bmatrix}$$

Then we use $\langle a\rangle_{ij}$ to denote the matrix of R whose (i,j)-position is a but other positions zero. Consider another ring which is also represented as a matrix form;

$$T = \begin{bmatrix} B_{11} & \cdots & B_{1n} \\ & \cdots & \\ B_{n1} & \cdots & B_{nn} \end{bmatrix}$$

In this paper, when we say ' $\tau = \begin{bmatrix} \tau_{11}\cdot\cdot\tau_{1n} \\ \cdot\cdot \\ \tau_{n1}\cdot\cdot\tau_{nn} \end{bmatrix}$ is a map from R to T'

this word means that τ_{ij} is a map from A_{ij} to B_{ij} and $\tau(\langle a\rangle_{ij})$ =

$<\tau_{ij}(a)>_{ij}$ for all ij.

Let R be an artinian ring and $E = \{e_1, \ldots, e_n\}$ a complete set of orthogonal primitive idempotents of R. The following basic result due to Fuller ([3]) is very useful: Let f be in E. ${}_R Rf$ is injective iff there exists e in E such that (eR;Rf) is an injective pair, that is

$${}_R Re/J({}_R Re) \cong {}_R S({}_R Rf) \text{ and } fR_R/J(fR_R) \cong S(eR_R)_R$$

When this is so, eR_R is also injective.

We note that if R is basic and (eR;Rf) is an injective pair, then $S({}_{eRe}eRf) = S(eRf_{fRf})$ and

$$S(eR_R) = \begin{bmatrix} & 0 & \\ 0 & S(eRf) & 0 \\ & 0 & \end{bmatrix} = S({}_R Rf)$$

For a ring R, End(R) and Aut(R) stand for the set of all ring endomorphisms of R and one of all automorphisms of R, respectively.

1. Skew matrix ring. Let Q be a ring and let $c \in Q$ and $\sigma \in$ End(Q) such that

$$\sigma(c) = c \quad \text{and} \quad \sigma(q)c = cq \quad \forall q \in Q$$

We put

$$R = \begin{bmatrix} Q & \cdots & Q \\ & \cdots & \\ Q & \cdots & Q \end{bmatrix}$$

We define a multiplication in R which depends on σ and c as follows: For (x_{ik}), (y_{ik}) in R,

$$(z_{ik}) = (x_{ik})(y_{ik})$$

where z_{ik} is defined as follows:

(1) If $i \le k$,

$$z_{ik} = \sum_{j<i} x_{ij}\sigma(y_{jk})c + \sum_{i\le j\le k} x_{ij}y_{jk} + \sum_{k<j} x_{ij}y_{jk}c$$

(2) If $k < i$,

$$z_{ik} = \sum_{j\le k} x_{ij}\sigma(y_{jk}) + \sum_{k<j<i} x_{ij}\sigma(y_{jk})c + \sum_{i\le j} x_{ij}y_{jk}$$

We may understand this multiplication as follows:

$$\langle a\rangle_{ij}\langle b\rangle_{jk} = \begin{cases} \langle a\sigma(b)\rangle_{ik} & (j\le k<i) \\ \langle a\sigma(b)c\rangle_{ik} & (k<j<i \text{ or } j<i\le k) \\ \langle ab\rangle_{ik} & (i=j) \\ \langle abc\rangle_{ik} & (i\le k<j) \\ \langle ab\rangle_{ik} & (k<i<j \text{ or } i<j\le k) \end{cases}$$

Now, it is straightforward to check the associative law:

$$(\langle x\rangle_{ij}\langle y\rangle_{jk})\langle z\rangle_{kl} = \langle x_{ij}\rangle(\langle y\rangle_{jk}\langle z\rangle_{kl}), \quad \forall\ ij,\ jk,\ kl$$

Therefore R becomes a ring by this multiplication together with the usual sum of matrix. We call R the skew matrix ring of degree n over Q, and denote it by

$$\begin{bmatrix} Q & \cdots & Q \\ & \cdots & \\ Q & \cdots & Q \end{bmatrix}_{\sigma,c}$$

When n = 2, the multiplication is that

$$\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \begin{bmatrix} y_1 & y_2 \\ y_3 & y_4 \end{bmatrix} = \begin{bmatrix} x_1y_1+x_2y_3c & x_1y_2+x_2y_4 \\ x_3\sigma(y_1)+x_4y_3 & x_3\sigma(y_2)c+x_4y_4 \end{bmatrix}$$

When n = 3,

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{bmatrix} =$$

$$\begin{bmatrix} x_{11}y_{11}+x_{12}y_{21}c+x_{13}y_{31}c & x_{11}y_{12}+x_{12}y_{22}+x_{13}y_{32}c & x_{11}y_{13}+x_{12}y_{23}+x_{13}y_{33} \\ x_{21}\sigma(y_{11})+x_{22}y_{21}+x_{23}y_{31} & x_{21}\sigma(y_{12})c+x_{22}y_{22}+x_{23}y_{32}c & x_{21}\sigma(y_{13})c+x_{22}y_{23}+x_{23}y_{33} \\ x_{31}(y_{11})+x_{32}\sigma(y_{21})c+x_{33}y_{31} & x_{31}\sigma(y_{12})+x_{32}\sigma(y_{22})+x_{33}y_{32} & x_{31}\sigma(y_{13})c+x_{32}\sigma(y_{23})+x_{33}y_{33} \end{bmatrix}$$

Now, we put $e_i = \langle 1 \rangle_{ii}$, $i = 1, \ldots, n$. Then $e_1, \ldots, e_n$ are orthogonal idempotents, $1 = e_1 + \ldots + e_n$ and

$$e_iR = \begin{bmatrix} & 0 & \\ \Omega & \ldots & \Omega \\ & 0 & \end{bmatrix} (i$$

$$Re_j = \begin{bmatrix} & \Omega & \\ 0 & \vdots & 0 \\ & \Omega & \end{bmatrix}$$
$$\hat{j}$$

Note that each e_i is a primitive idempotent if Ω is a local ring.

We put

$$W_i = \begin{bmatrix} & & & 0 & & & \\ \Omega & \ldots & \Omega & \Omega c & \Omega & \ldots & \Omega \\ & & & 0 & & & \end{bmatrix} (i \quad , \quad i = 1, \ldots, n.$$
$$\hat{i}$$

Then W_i is a right submodule of e_iR_R. For $i = 2, \ldots, n$, let $\phi_i : e_iR \longrightarrow W_{i-1}$ be a map given by

$$\begin{bmatrix} & & 0 & & \\ x_1 & \ldots & x_{i-1} \; x_i & \ldots & x_n \\ & & 0 & & \end{bmatrix} \rightarrow \begin{bmatrix} & & 0 & & \\ x_1 & \ldots & x_{i-1}c \; x_i & \ldots & x_n \\ & & 0 & & \end{bmatrix}$$

and let $\phi_1 : e_1R \longrightarrow W_n$ be a map given by

$$\begin{bmatrix} x_1 \ \cdots \ x_n \\ \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} & 0 & \\ \\ \sigma(x_1) \ .. \ \sigma(x_{n-1}) & \sigma(x_n)c \end{bmatrix}$$

Then each ϕ_i is a homomorphism. In particular, if $\sigma \in \mathrm{Aut}(Q)$, then each ϕ_i is an onto homomorphism and

$$\mathrm{Ker}\phi_1 = \begin{bmatrix} 0 \ \cdots \ 0 \ (0:c) \\ \\ 0 \end{bmatrix}$$

$$\mathrm{Ker}\phi_i = \begin{bmatrix} & 0 & \\ 0 & (0:c) & 0 \\ & 0 & \end{bmatrix} (i \quad , \qquad i = 2, \ldots, n$$
$$\overset{\frown}{i-1}$$

where (0:c) is a right (or left) annihilator ideal of c.

(1.1). The mapping $\tau: R \longrightarrow R$ given by

$$\begin{bmatrix} x_{11} \ x_{12} \ \cdots \ x_{1n} \\ x_{21} \ x_{22} \ \cdots \ x_{2n} \\ \cdots \\ x_{n1} \ x_{n2} \ \cdots \ x_{nn} \end{bmatrix} \longrightarrow \begin{bmatrix} x_{nn} & x_{n1} & \cdots\cdots & x_{n,n-1} \\ \sigma(x_{1n}) & \sigma(x_{11}) & \cdots\cdots & \sigma(x_{1,n-1}) \\ & \cdots\cdots\cdots & & \\ \sigma(x_{n-1,n}) & \sigma(x_{n-1,1}) & \cdots & \sigma(x_{n-1,n-1}) \end{bmatrix}$$

is a ring homomorphism; so if $\sigma \in \mathrm{Aut}(Q)$, then $\tau \in \mathrm{Aut}(R)$.

(1.2). If Q is a local QF-ring, $\sigma \in \mathrm{Aut}(Q)$ and $c \in J(Q)$, then R is a basic indecomposable QF-ring and $\{e_n, e_1, \ldots, e_{n-1}\}$ is the Nakayama permutation of $\{e_1, e_2, \ldots, e_n\}$.

Proof (sketch). We can see that

$$S(e_1R) = \begin{bmatrix} 0 & \cdots & 0 & X \\ & & & \\ & 0 & & \end{bmatrix} = S(Re_n)$$

$$S(e_iR) = \begin{bmatrix} 0 & & & \overset{i-1}{\smile} & & & & \\ 0 & \cdot & & & & & 0 & \\ & \cdot & \cdot & & & & & \\ & & \cdot & \cdot & & & & \\ & & & 0 & \cdot & & & \\ & & & & X & 0 & & \\ & & & & & 0 & \cdot & \\ & 0 & & & & & \cdot & \cdot \\ & & & & & & 0 & 0 \end{bmatrix} \textcircled{i} = S(Re_{i-1})$$

for $i = 2, \ldots, n$, where $X = S(Q_Q) = S({}_QQ)$. Hence it follows that $(e_1R;Re_n)$, $(e_2R;Re_1)$, ..., $(e_nR;Re_{n-1})$ are injective pairs; So, (1.2) is proved.

(1.3). If Q is a local QF-ring, $\sigma \in \mathrm{Aut}(Q)$ and $c \in J(Q)$, then R is a basic indecomposable QF-ring which has a Nakayama automorphism.

(1.4). If Q is a uniserial ring, $\sigma \in \mathrm{Aut}(Q)$ and $cQ = J(Q)$, then R is a basic indecomposable serial ring such that $\{e_nR, e_{n-1}R, \ldots, e_1R\}$ is a Kuppisch series and $\{e_n, e_1, e_2, \ldots, e_{n-1}\}$ is the Nakayama permutation of $\{e_1, e_2, \ldots, e_n\}$.

2. We shall consider those basic indecomposable QF serial rings which satisfy the condition

(*) If $\{e_nR, e_{n-1}R, \ldots, e_1R\}$ is a Kuppisch series, then $\{e_n, e_1, e_2, \ldots, e_{n-1}\}$ is the Nakayama permutation of $\{e_1, e_2, \ldots, e_n\}$.
where $\{e_i\}$ is a complete set of orthogonal primitive idempotents of R.

Now our purpose of this section is to show the following:

(2.1). If R is a basic indecomposable QF serial ring with the condition (*), then R is represented as a skew matrix ring

$$R = \begin{bmatrix} Q & \cdots & Q \\ & \cdots & \\ Q & \cdots & Q \end{bmatrix}_{\sigma,c}$$

where Q is a uniserial ring and $\sigma \in \mathrm{Aut}\,(Q)$ and $cQ = J(Q)$.

Proof (sketch). Putting $A_{ij} = e_i R e_j$ $(i \neq j)$ and $Q_i = e_i R e_i$, we shall represent R as

$$R = \begin{bmatrix} Q_1 & A_{12} & \cdots\cdots & A_{1n} \\ A_{21} & Q_2 & A_{23} \cdots & A_{2n} \\ & & \cdots\cdots & \\ A_{n1} & \cdots & A_{n,n-1} & Q_n \end{bmatrix}$$

We put $Q = Q_1$. Q is a uniserial ring.

First we shall consider the case n = 2. Put $A = A_{12}$, $B = A_{21}$ and $T = Q_2$, and put $e = e_1$ and $f = e_2$. Then

$$R = \begin{bmatrix} Q & A \\ B & T \end{bmatrix}$$

Since {f, e} is the Nakayama permutation of {e, f}, we see

$$S(eR) = S(Rf) = \begin{bmatrix} 0 & S(A) \\ 0 & 0 \end{bmatrix}$$

$$S(fR) = S(Re) = \begin{bmatrix} 0 & 0 \\ S(B) & 0 \end{bmatrix}$$

We denote the factor ring R/S(fR) by $\overline{R} = \begin{bmatrix} Q & A \\ \overline{B} & T \end{bmatrix}$ and r + S(fR) by $\overline{r}$. Then $\overline{R}$ is a serial ring and $\overline{e}$, $\overline{f}$ are orthogonal primitive idempotents. We see that $\bar{e}\bar{R}_{\bar{R}}$ is injective and $J(\bar{e}\bar{R}) \simeq \bar{f}\bar{R}_{\bar{R}}$. So, here, using the matrix representation of left Harada rings (cf. [13], [14]), we can get an isomorphism

$$\tau = \begin{bmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{bmatrix} : \begin{bmatrix} Q & A \\ \bar{B} & T \end{bmatrix} \longrightarrow \begin{bmatrix} Q & \Omega \\ J(\Omega) & Q \end{bmatrix}$$

Put $\langle\alpha\rangle_{12} = \tau^{-1}(\langle 1\rangle_{12})$. Then $A = Q\alpha = \alpha T$, ${}_{Q}Q\alpha \simeq {}_{Q}\Omega$, $\alpha T_{T} \simeq T_{T}$ and $q\alpha = \alpha\tau_{22}(q)$ for all q in Ω.

Now, by $Q \overset{\tau_{22}^{-1}}{\simeq} T$, we exchange T by Q in $\begin{bmatrix} Q & A \\ B & T \end{bmatrix}$; so

$$R = \begin{bmatrix} Q & A \\ B & Q \end{bmatrix}$$

and then $q\alpha = \alpha q$ for all q in Q.

By the same argument, we can get $\beta \in B$ such that $B = \Omega\beta = \beta\Omega$, ${}_{\Omega}\Omega\beta \simeq {}_{\Omega}Q$ and $\beta Q_{\Omega} \simeq Q_{Q}$. We define a mapping $\sigma: \Omega \to \Omega$ by the rule: $\sigma(q) = q$ if $q\alpha = \alpha q$. Then σ is an automorphism. Putting $c = \alpha\beta$, we see the following

(1) $c = \alpha\beta = \beta\alpha$

(2) $\sigma(q)c = cq$, $\forall q \in Q$

(3) $\sigma(c) = c$

(4) $c\Omega = J(Q)$

Thus the skew matrix $\begin{bmatrix} Q & \Omega \\ \Omega & \Omega \end{bmatrix}_{\sigma,c}$ is considered. For $\begin{bmatrix} x_1 & x_2\alpha \\ x_3\beta & x_4 \end{bmatrix}$ and $\begin{bmatrix} y_1 & y_2\alpha \\ y_3\beta & y_4 \end{bmatrix} \in R$, their multiplication is

$$\begin{bmatrix} x_1y_1+x_2y_3c & (x_1y_2+x_2y_4)\alpha \\ (x_3\sigma(y_1)+x_4y_3)\beta & x_3\sigma(y_2)c+x_4y_4 \end{bmatrix}$$

Therefore, R is isomorphic to the skew matrix ring $\begin{bmatrix} \Omega & \Omega \\ \Omega & \Omega \end{bmatrix}_{\sigma,c}$ as desired.

Next, we shall consider the case n = 3. We may denote the factor ring $R/S(e_3R)$ by

$$\overline{R} = \begin{bmatrix} \Omega & A_{12} & A_{13} \\ A_{21} & \Omega_2 & A_{23} \\ A_{31} & \overline{A}_{32} & \Omega_3 \end{bmatrix}$$

and the element $r + S(e_3R)$ by $\overline{r}$. We see that $\overline{R}$ is a serial ring, $\overline{e}_1$, $\overline{e}_2$, $\overline{e}_3$ are orthogonal primitive idempotents and

$\overline{e}_i\overline{R}_{\overline{R}}$ is injective for i = 1,2,

$J(\overline{e}_2\overline{R})_{\overline{R}} \simeq \overline{e}_3\overline{R}_{\overline{R}}$.

Hence, by using the matrix representation of left Harada rings, we can obtain an isomorphism

$$\tau = \begin{bmatrix} \tau_{11}\tau_{12}\tau_{13} \\ \tau_{21}\tau_{22}\tau_{23} \\ \tau_{31}\tau_{32}\tau_{33} \end{bmatrix} : \begin{bmatrix} \Omega & A_{12} & A_{13} \\ A_{21} & \Omega_2 & A_{23} \\ A_{31} & \overline{A}_{32} & \Omega_3 \end{bmatrix} \longrightarrow \begin{bmatrix} \Omega & A_{12} & A_{12} \\ A_{21} & \Omega_2 & \Omega_2 \\ A_{21} & J & \Omega_2 \end{bmatrix}$$

where $J = J(\Omega_2)$, $\begin{bmatrix} \tau_{11}\tau_{12} \\ \tau_{21}\tau_{22} \end{bmatrix} : \begin{bmatrix} \Omega & A_{12} \\ A_{21}\Omega_2 \end{bmatrix} \longrightarrow \begin{bmatrix} \Omega & A_{12} \\ A_{21}\Omega_2 \end{bmatrix}$ is the identity map.

Here, applying the result above of the case n = 2 to $\begin{bmatrix} \Omega & A_{12} \\ A_{21}\Omega_2 \end{bmatrix}$, we can exchange Q_2 by Ω and get $\alpha_{12} \in A_{12}$, $\alpha_{21} \in A_{21}$, $\sigma \in \mathrm{Aut}(\Omega)$ and $c \in J(\Omega)$ such that

$$A_{12} = \alpha_{12}\Omega = \Omega\alpha_{12},\ A_{21} = \alpha_{21}\Omega = \Omega\alpha_{21},$$

$$\alpha_{12}\Omega_\Omega \simeq \Omega_\Omega \simeq \alpha_{21}\Omega_\Omega,\ {}_\Omega\Omega\alpha_{12} \simeq {}_\Omega\Omega \simeq {}_\Omega\Omega\alpha_{21},$$

$$c\Omega = J(\Omega),\ c = \alpha_{21}\alpha_{12} = \alpha_{12}\alpha_{21}$$

$$\sigma(c) = c,\ \sigma(q)c = cq,\ \forall q \in \Omega$$

We put $\langle\alpha_{13}\rangle_{13} = \tau^{-1}(\langle\alpha_{12}\rangle_{13})$, $\langle\alpha_{31}\rangle_{31} = \tau^{-1}(\langle\alpha_{21}\rangle_{31})$, $\langle\alpha_{23}\rangle_{23} = \tau^{-1}(\langle 1\rangle_{23})$ and $\alpha_{32} = \alpha_{31}\alpha_{12}$. Then we see that

$$A_{ij} = \alpha_{ij}\Omega = \Omega\alpha_{ij},\ {}_\Omega\Omega \simeq {}_\Omega\Omega\alpha_{ij},\ \Omega_\Omega \simeq \alpha_{ij}\Omega_\Omega$$

for $ij \in \{13, 31, 23, 32\}$. And we obtain the following:

(a) if $j < i$,

$$\sigma(q)\alpha_{ij} = \alpha_{ij}q, \ \forall q \in Q$$

$$\alpha_{ij}\alpha_{jk} = \begin{cases} \alpha_{ik} & (j \leq k < i) \\ \alpha_{ik}c & (k < j \text{ or } i \leq k) \end{cases}$$

(b) if $i = j$,

$$q\alpha_{ij} = \alpha_{ij}q, \ \forall q \in Q$$

$$\alpha_{ij}\alpha_{jk} = \alpha_{ik}$$

(c) if $i < j$,

$$q\alpha_{ij} = \alpha_{ij}q, \ \forall q \in Q$$

$$\alpha_{ij}\alpha_{jk} = \begin{cases} \alpha_{ik}c & (i \leq k < j) \\ \alpha_{ik} & (k < i \text{ or } j \leq k) \end{cases}$$

Comparing these (a), (b), (c) with the definition of the multiplication of the skew matrix ring of degree 3 over Q with respect to σ and c, we see that the mapping

$$\begin{bmatrix} x_{11} & x_{12}\alpha_{12} & x_{13}\alpha_{13} \\ x_{21}\alpha_{21} & x_{22} & x_{23}\alpha_{23} \\ x_{31}\alpha_{31} & x_{32}\alpha_{32} & x_{33} \end{bmatrix} \longrightarrow \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$$

gives an isomorphism from R to $\begin{bmatrix} Q & Q & Q \\ Q & Q & Q \\ Q & Q & Q \end{bmatrix}_{\sigma,c}$

Our argument works on all n; so (2.1) is proved.

3. In this section, we shall show that every basic indecomposable QF serial ring with non-identity Nakayama permutation can be constructed by a basic indecomposable QF serial ring with (*) mentioned in the section 2.

Henceforth R is a basic indecomposable QF serial ring with non-identity Nakayama permutation, and let $E = \{e_1, \ldots, e_n\}$ a complete set of orthogonal primitive idempotents. We put $J = J(R)$ and $A_{ij} = e_iRe_j$ for all i, j.

Now, we may assume that

$$e_nR,\ e_{n-1}R,\ \ldots,\ e_1R$$

is a Kuppisch series. So, there exists an onto homomorphisms:

$$\theta_{i,i+1}: e_{i+1}R \longrightarrow e_iJ,\ i = 1, \ldots, n-1$$

$$\theta_{n1}: e_1R \longrightarrow e_nJ\ (\neq 0)$$

We put

$$\theta_{ij} = \theta_{i,i+1}\theta_{i+1,i+2} \cdots \theta_{j-1,j} \qquad (i < j)$$

$$\theta_{ii} = \text{the identity map of } e_iR$$

Then we note that

$$\operatorname{Ker} \theta_{ij} = S_{j-i}(e_jR), \quad \operatorname{Im}\theta_{ij} = e_iJ^{j-i} \qquad (J^0 = R) .$$

So, θ_{ij} induces an isomorphism $\overline{\theta}_{ij}: e_jR/S_{j-i}(e_jR) \cong e_iJ^{j-i}$.

If $\alpha \in (e_tR,\ e_jR/S_k(e_jR))$, then there exists $\beta \in (e_tR, e_jR)$ satisfying $\alpha = \eta\beta$, where η is the canonical homomorphism of e_jR to $e_jR/S_k(e_jR)$. If $(e_tR, S_k(e_jR)) = 0$, we see that such β is unique; we denote it by

$$[\alpha]$$

The following (A) and (B) play important roles for calculation of this section:

(A) If $\gamma \in (e_tR, e_iJ^{j-i})$ $(i \leq j)$ and $(e_tR, S_{j-i}(e_jR)) = 0$, then $\theta_{ij}[\overline{\theta}_{ij}^{-1}\gamma] = \gamma$.

(B) Let $\alpha, \beta \in (e_tR, e_jR)$ and let $(e_tR, S_{j-i}(e_jR)) = 0$. If $\theta_{ij}\alpha = \theta_{ij}\beta$, then $\alpha = \beta$.

We need the following (perhaps well known):

(3.1). There exists $s \neq n$ such that $\{e_{n-s+1}, e_{n-s+2}, \ldots, e_n, e_1, e_2, \ldots, e_{n-s}\}$ is the Nakayama permutation of $\{e_1, \ldots, e_n\}$.

We observe the structure of R for each case of the following:

(I) $s > n - s$

(II) $n = sq$

(III) $n = sq + r, \quad 0 < r < s$

Case (I). Put $\ell = n - s + 1$, $w = \ell - 1$ and $t = n - w + 1$. R is represented as

$$\begin{bmatrix}
A_{11} & \cdots & A_{1w} & A_{1\ell} & \cdots & & \cdots & A_{1n} \\
 & \cdots & & \ddots & \cdots & & \cdots & \\
A_{w1} & \cdots & A_{ww} & A_{w\ell} & \cdots & & \cdots & A_{wn} \\
A_{\ell 1} & \cdots & A_{\ell w} & A_{\ell\ell} & \cdots & & \cdots & A_{\ell n} \\
 & \cdots & & & \cdots & \ddots & \cdots & \\
A_{t-1,1} & & \cdots & & & A_{t-1,t} & \cdots & A_{t-1,n} \\
A_{t1} & & \cdots & & & A_{tt} & \cdots & A_{tn} \\
 & \ddots & \cdots & & & & & \\
A_{n1} & \cdots & A_{nw} & A_{n\ell} & \cdots & & \cdots & A_{nn}
\end{bmatrix}$$

We put

$$P = \begin{bmatrix} A_{11} & A_{1\ell} \\ A_{t1} & A_{\ell\ell} \end{bmatrix}$$

and define a multiplication in P as follows:

$$\begin{bmatrix} x_{11} & x_{1\ell} \\ x_{t1} & x_{\ell\ell} \end{bmatrix} \begin{bmatrix} y_{11} & y_{1\ell} \\ y_{t1} & y_{\ell\ell} \end{bmatrix}$$

$$= \begin{bmatrix} x_{11}y_{11}+x_{1\ell}\theta_{\ell t}x_{t1} & x_{11}y_{1\ell}+x_{1\ell}y_{\ell\ell} \\ x_{t1}y_{11}+[\theta^{-1}_{\ell t}x_{\ell\ell}\theta_{\ell,\ell+1}]\theta_{\ell+1,t}y_{t1} & \theta_{\ell t}x_{t1}y_{1\ell}+x_{\ell\ell}y_{\ell\ell} \end{bmatrix}$$

$[\theta^{-1}_{\ell t}x_{\ell\ell}\theta_{\ell,\ell+1}]$ is well defined since $(e_{\ell+1}R, S_{t-\ell}(e_tR)) = 0$. Put

$$\langle x\rangle_{11} = \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}, \quad \langle x\rangle_{1\ell} = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}, \quad \langle x\rangle_{t1} = \begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix}, \quad \langle x\rangle_{\ell\ell} = \begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}$$

Using (A) and (B), we can calculate that

$$(\langle x\rangle_{ij}\langle y\rangle_{jk})\langle z\rangle_{k\ell} = \langle x\rangle_{ij}(\langle y\rangle_{jk}\langle z\rangle_{k\ell})$$

for $ij, jk, k\ell \in \{11, 1\ell, t1, \ell\ell\}$. So, by this multiplication together with the usual matrix sum, P becomes a ring. Put $f_1 = \langle 1\rangle_{11}$ and $f_2 = \langle 1\rangle_{\ell\ell}$. Then $\{f_1, f_2\}$ are orthogonal primitive idempotents and $1 = f_1 + f_2$. We obtain

(3.2). P is a basic indecomposable QF serial ring such that $\{f_2, f_1\}$ is the Nakayama permutation of $\{f_1, f_2\}$.

Now, we put

$$R_{11} = \begin{bmatrix} A_{11} & \cdots & A_{1w} \\ & \cdots & \\ A_{w1} & \cdots & A_{ww} \end{bmatrix} \qquad R_{12} = \begin{bmatrix} A_{1\ell} & \cdots\cdots\cdots & A_{1n} \\ & \cdots\cdots\cdots & \\ A_{w\ell} & \cdots\cdots\cdots & A_{wn} \end{bmatrix}$$

$$R_{21} = \begin{bmatrix} A_{\ell 1} & \cdots & A_{\ell w} \\ & \cdots & \\ A_{t1} & & \\ \vdots & \ddots & \\ A_{n1} & \cdots & A_{nw} \end{bmatrix} \qquad R_{22} = \begin{bmatrix} A_{\ell\ell} & \cdots\cdots\cdots & A_{\ell n} \\ & \cdots\cdots\cdots & \\ A_{n\ell} & \cdots\cdots\cdots & A_{nn} \end{bmatrix}$$

So,

$$R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}$$

Corresponding to each R_{ij}, we make P_{ij} of the same type as follows:

$$P_{11} = \begin{bmatrix} A_{11} & \cdots & A_{11} \\ & \ddots & \vdots \\ J(A_{11}) & & A_{11} \end{bmatrix} \qquad P_{21} = \begin{bmatrix} A_{1\ell} & \cdots\cdots\cdots & A_{1\ell} \\ & \cdots\cdots\cdots & \\ A_{1\ell} & \cdots\cdots\cdots & A_{1\ell} \end{bmatrix}$$

$$P_{21} = \begin{bmatrix} A_{21} & \cdots & A_{21} \\ & \cdots & \\ A_{21} & & \\ \vdots & \ddots & \\ A_{21} & \cdots & A_{21} \end{bmatrix} \qquad P_{22} = \begin{bmatrix} A_{\ell\ell} & \cdots\cdots\cdots & A_{\ell\ell} \\ & \ddots & \vdots \\ J(A_{\ell\ell}) & & A_{\ell\ell} \end{bmatrix}$$

Then

$$\Omega = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

is an extension ring of P. We define a map $\tau = \begin{bmatrix} \tau_{11} & \cdots & \tau_{1n} \\ & \cdots & \\ \tau_{n1} & \cdots & \tau_{nn} \end{bmatrix}$: $\Omega \longrightarrow R$ as follows:

(a) $1 \le i \le w$, $1 \le j \le w$;

$$\tau_{ij}(\alpha_{11}) = \begin{cases} \alpha_{11}\theta_{1j} & (i=1) \\ [\theta^{-1}_{1i}\alpha_{11}\theta_{1j}] & (1<i\le j) \\ [\theta^{-1}_{1i}\alpha_{11}]\theta_{1j} & (1\le j<i) \end{cases}$$

(b) $1 \le i \le w < j \le n$;

$$\tau_{ij}(\alpha_{1\ell}) = [\theta^{-1}_{1i}\alpha_{1\ell}]\theta_{\ell j}$$

(c) $t \le i \le n$, $1 \le j \le w$;

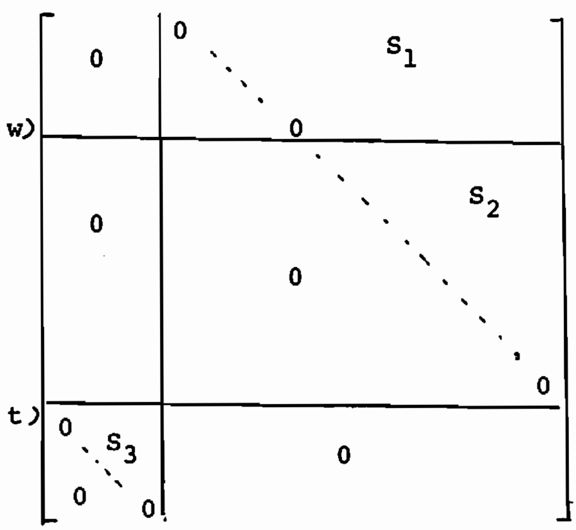
$$\tau_{ij}(\alpha_{t1}) = [\theta^{-1}_{ti}\alpha_{t1}]\theta_{1j}$$

(d) $\ell \le i \le n$, $\ell \le j \le n$;

$$\tau_{ij}(\alpha_{\ell\ell}) = \begin{cases} \alpha_{\ell\ell}\theta_{\ell j} & (i=\ell) \\ [\theta^{-1}_{\ell i}\alpha_{\ell\ell}\theta_{\ell,\ell+1}]\theta_{\ell+1,j} & (\ell<i\le j) \\ [\theta^{-1}_{\ell i}\alpha_{\ell\ell}]\theta_{\ell j} & (\ell\le j<i) \end{cases}$$

Of course τ is well defined. And we have

(3.3). τ is an onto ring homomorphism and its kernel is



$S_1 = S(A_{1\ell})$

$S_2 = S(A_{\ell\ell})$

$S_3 = S(A_{t1})$

Thus R is indeed constructed by the basic indecomposable QF serial ring P with (*).

Case (II): $n = sq$, $2 \leq q$. In this case, we shall consider the partition

$$\{1, 2, \ldots, s\} \cup \{s+1, \ldots, 2s\} \cup \ldots \cup \{(q-1)s+1, \ldots, n\}.$$

Put $t_1 = 1$, $s_1 = s$, $t_2 = s+1$, $s_2 = 2s$, ..., $t_q = (q-1)s+1$, $s_q = n$. Consider the subring of R:

$$P = \begin{bmatrix} A_{t_1t_1} & A_{t_1t_2} & \cdots & A_{t_1t_q} \\ A_{t_2t_1} & A_{t_2t_2} & \cdots & A_{t_2t_q} \\ & \cdots\cdots\cdots & & \\ A_{t_qt_1} & A_{t_qt_2} & \cdots & A_{t_qt_q} \end{bmatrix}$$

and put $f_i = \langle 1 \rangle_{ii}$, $i = 1, 2, \ldots, q$. Then $\{f_1, \ldots, f_q\}$ is a complete set of orthogonal primitive idempotents. We obtain

(3.2′). P is a basic indecomposable QF serial ring such that $\{f_nP, f_{n-1}P, \ldots, f_1P\}$ is a Kuppisch series and $\{f_n, f_1, \ldots, f_{n-1}\}$ is the Nakayama permutation of $\{f_1, f_2, \ldots, f_n\}$.

Now we define (s,s) matrix Q_{ij} ($1 \leq i \leq q$, $1 \leq j \leq q$) as follows:

$$Q_{ii} = \begin{bmatrix} A_{t_it_i} & \cdots & A_{t_it_i} \\ & \ddots & \vdots \\ J(A_{t_it_i}) & & A_{t_it_i} \end{bmatrix}$$

$$Q_{ij} = \begin{bmatrix} A_{t_it_j} & \cdots & A_{t_it_j} \\ & \cdots & \\ A_{t_it_j} & \cdots & A_{t_it_j} \end{bmatrix} \qquad (i \neq j)$$

and put

$$Q = \begin{bmatrix} Q_{11} & \cdots & Q_{1q} \\ & \cdots & \\ Q_{q1} & \cdots & Q_{qq} \end{bmatrix}$$

Then Q is an extension ring of P. We define $X_{1q}, X_{21}, \ldots, X_{q,q-1}$ as follows:

$$X_{1q} = \begin{bmatrix} 0 & & \\ & \ddots & S_1 \\ 0 & & \ddots \; 0 \end{bmatrix} \subseteq Q_{11} \qquad (S_1 = S(A_{t_1 t_q}))$$

$$X_{i,i-1} = \begin{bmatrix} 0 & & \\ & \ddots & S_i \\ 0 & & \ddots \; 0 \end{bmatrix} \subseteq Q_{i,i-1} \qquad (S_i = S(A_{t_i t_{i-1}}))$$

for $i = 2, \ldots, q$. Then

$$X = \begin{bmatrix} 0 & \cdots\cdots & 0 & X_{1q} \\ X_{21} & 0 \cdots\cdots & & 0 \\ 0 & X_{32} & 0 \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & 0 \; X_{q,q-1} & 0 \end{bmatrix}$$

is an ideal of Q. We obtain the following:

(3.3′). There is an onto ring homomorphism $\tau = \begin{bmatrix} \tau_{11} & \cdots & \tau_{1n} \\ & \cdots & \\ \tau_{n1} & \cdots & \tau_{nn} \end{bmatrix}$ from Q to R with the kernel $= X$; so, Q/X is the representative matrix ring which is constructed by the basic indecomposable QF serial ring with (*).

Case (III): $n = sq + r$, $0 < r < s$. In this case, we shall consider the partition

$$\{1, \ldots, s\} \cup \{s+1, \ldots, 2s\} \cup \ldots \cup \{(q-2)s+1, \ldots, (q-1)s\} \cup$$
$$\{(q-1)s+1, \ldots, (q-1)s+r\} \cup \{(q-1)s+r+1, \ldots, n\}$$

and put

$$t_1 = 1,\ s_1 = s,\ t_2 = s+1,\ s_2 = 2s,\ \ldots,\ t_{q-2} = (q-3)s+1,$$
$$s_{q-2} = (q-2)s,\ t_{q-1} = (q-2)s+1,\ s_{q-1} = (q-2)s+r,$$
$$t_q = (q-2)s+r+1,\ s_q = n.$$

Put

$$R_{ij} = \begin{bmatrix} A_{t_i t_j} & A_{t_i t_{j+1}} & \cdots & A_{t_i s_j} \\ A_{t_i+1, t_j} & A_{t_i+1, t_j+1} & \cdots & A_{t_i+1, s_j} \\ & \cdots\cdots\cdots\cdots\cdots & & \\ A_{s_i t_j} & A_{s_i, t_j+1} & \cdots & A_{s_i s_j} \end{bmatrix}$$

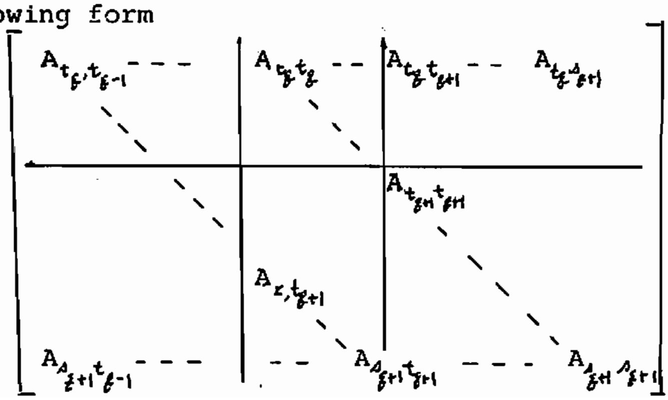
Then

$$R = \begin{bmatrix} R_{11} & \cdots & R_{1,q+1} \\ & \cdots & \\ R_{q+1,1} & \cdots & R_{q+1,q+1} \end{bmatrix}$$

In particular,

$$\begin{bmatrix} R_{q,q-1} & R_{qq} & R_{q,q+1} \\ R_{q+1,q-1} & R_{q+1,q} & R_{q+1,q+1} \end{bmatrix}$$

is the following form



where $x = n - r + 1$. Here we shall consider the set

$$P = \begin{bmatrix} A_{t_1t_1} & A_{t_1t_2} & \cdots & A_{t_1t_{q-1}} & A_{t_1t_q} & A_{t_1t_{q+1}} \\ A_{t_2t_1} & A_{t_2t_2} & \cdots & A_{t_2t_{q-1}} & A_{t_2t_q} & A_{t_2t_{q+1}} \\ & & \cdots\cdots\cdots & & & \\ A_{t_qt_1} & A_{t_qt_2} & \cdots & A_{t_qt_{q-1}} & A_{t_qt_q} & A_{t_qt_{q+1}} \\ A_{t_{q+1}t_1} & \cdots & & A_{t_{q+1}t_{q-1}} & A_{xt_q} & A_{t_{q+1}t_{q+1}} \end{bmatrix}$$

In P, we define a sum by the usual matrix sum and a multiplication by following relations:

a) $<\alpha>_{t_it_j}<\beta>_{t_jt_k} = <\alpha\beta>_{t_it_j} \qquad ((t_i,t_j,t_k) \neq (t_{q+1},t_1,t_q))$

b) $<\alpha>_{t_{q+1}t_1}<\beta>_{t_1t_q} = <[\bar{\theta}^{-1}_{t_{q+1}x}\alpha]\beta>_{xt_q}$

c) $<\alpha>_{xt_q}<\beta>_{t_qt_i} = \begin{cases} <\theta_{t_{q+1}x}\alpha\beta>_{t_{q+1}t_i} & (i \neq q) \\ <\alpha\beta>_{xt_q} & (i = q) \end{cases}$

d) $<\alpha>_{t_it_{q+1}}<\beta>_{xt_q} = \begin{cases} <\alpha\theta_{t_{q+1}x}\beta>_{t_it_q} & (i \neq q+1) \\ <[\bar{\theta}^{-1}_{t_{q+1}x}\alpha]\beta>_{xt_q} & (i = q+1) \end{cases}$

where $<\ >_{t_it_j} = <\ >_{ij}$ and $<\ >_{xt_q} = <\ >_{q+1,q}$ in P. It is easily checked that P becomes a ring by these operations and $\{f_i = <1>_{t_it_i} \mid i = 1, \ldots, q+1\}$ is a complete set of orthogonal primitive idempotents. We obtain

(3.2''). P is a basic indecomposable QF serial ring such that $\{f_{q+1}P, f_qP, \ldots, f_1P\}$ is a Kuppisch series and $\{f_{q+1}, f_1, \ldots, f_q\}$ is the Nakayama permutation of $\{f_1, \ldots, f_{q+1}\}$.

Now, corresponding to each R_{ij}, we make matrix Ω_{ij} of the same type as follows:

$$\Omega_{ii} = \begin{bmatrix} A_{t_it_i} & \cdots & A_{t_it_i} \\ & & \vdots \\ J(A_{t_it_i}) & & A_{t_it_i} \end{bmatrix} \qquad \Omega_{q+1,q} = \begin{bmatrix} A_{xt_q} & \cdots & A_{xt_q} \\ & \cdots & \\ A_{xt_q} & \cdots & A_{xt_q} \end{bmatrix}$$

$$\Omega_{ij} = \begin{bmatrix} A_{t_it_j} & \cdots & A_{t_it_j} \\ & \cdots & \\ A_{t_it_j} & \cdots & A_{t_it_j} \end{bmatrix} \qquad (\, i \neq j,\ (i,j) \neq (q+1,q))$$

and put

$$\Omega = \begin{bmatrix} \Omega_{11} & \cdots & \Omega_{1,q+1} \\ & \cdots & \\ \Omega_{q+1,1} & \cdots & \Omega_{q+1,q+1} \end{bmatrix}$$

Then Ω is a canonical ring extension of P. We define $X_{1,q+1}$, X_{21}, X_{32}, ..., $X_{q+1,q}$ as follows:

$$X_{1,q+1} = \begin{bmatrix} 0 & \ddots & S_1 \\ 0 & & \ddots \\ & & 0 \end{bmatrix} \subseteq \Omega_{1,q+1} \qquad (S_1 = S(A_{t_1t_{q+1}}))$$

$$X_{i,i-1} = \begin{bmatrix} 0 & \ddots & S_i \\ 0 & & \ddots \\ & & 0 \end{bmatrix} \subseteq \Omega_{i,i-1} \ (2 \le i \le q-1) \qquad (S_i = S(A_{t_it_{i-1}}))$$

$$X_{q,q-1} = \begin{bmatrix} 0 & \ddots & S_q \\ 0 & \ddots\ 0 & \end{bmatrix} \subseteq \Omega_{q,q-1} \qquad (S_q = S(A_{t_qt_{q-1}}))$$

$$X_{q+1,q} = \begin{bmatrix} & S^* \\ 0 & \\ & 0 \end{bmatrix} \subseteq Q_{q+1,q} \qquad (S^* = S(A_{t_{q+1},t_q}))$$

Then

$$X = \begin{bmatrix} 0 & \cdots\cdots & 0 & X_{1,q+1} \\ X_{21} & 0 & \cdots\cdots & 0 \\ 0 & X_{31} & 0 \ \cdots\cdots & 0 \\ & & \cdots & \\ 0 & \cdots & 0 \ X_{q+1,q} & 0 \end{bmatrix}$$

is an ideal of Q. We obtain

(3.3″). There is an onto ring homomorphism $\tau = \begin{bmatrix} \tau_{11} \cdots \tau_{1,q+1} \\ \cdots \\ \tau_{q+1,1} \cdots \tau_{q+1,q+1} \end{bmatrix}$ from Q to R, whose kernel is X. So, R is constructed by the basic indecomposable QF serial ring P with (*).

4. In this section, we shall study those basic indecomposable QF serial rings whose Nakayama permutations are identity (s = 1 in § 3).

Example 1. Let Q be a uniserial ring, $\sigma \in \mathrm{Aut}(Q)$ and $c \in Q$ such that $cQ = J(Q)$, $\sigma(c) = c$ and $\sigma(q)c = cq$ $\forall q \in Q$. Then, as we saw in § 1,

$$R = \begin{bmatrix} Q & \cdots & Q \\ & \cdots & \\ Q & \cdots & Q \end{bmatrix}_{\sigma,c}$$

is a basic indecomposable QF serial ring. We see that

$$I = \begin{bmatrix} 0 & & S \\ & \ddots & \\ S & & 0 \end{bmatrix} \qquad (S = S(Q))$$

is an ideal of R and R/I is a basic indecomposable QF serial ring with the identity Nakayama permutation.

Example 2. Let $Q = Q_1, Q_2, \ldots Q_n$ be uniserial rings. Let $c \in Q$ with $cQ = Qc = J(Q)$ and let σ be an automorphism of $\overline{Q} = Q/S(Q)$ satisfying $\sigma^*(\overline{q})\overline{c} = \overline{c}\overline{q}$ for all $\overline{q} = q + S(Q)$ and $\sigma(\overline{c}) = \overline{c}$. (cf. such c, σ^* always exist). Further assume that there exist isomorphisms $\tau_i: \overline{Q}_i = Q_i/S(Q_i) \simeq Q/S(Q)$, $i = 1, \ldots, n$, where τ_1 is the identity map. Choose $c_i \in Q_i$ such that $\tau_i(\overline{c}_i) = \overline{c}$. Consider the set

$$T = \begin{bmatrix} Q & \overline{Q} & \ldots\ldots & \overline{Q} \\ \overline{Q} & Q_2 & \overline{Q} \ldots & \overline{Q} \\ & & \ldots\ldots & \\ \overline{Q} & \ldots\ldots & \overline{Q} & Q_n \end{bmatrix}$$

In T, we define a sum by the usual matrix sum and a multiplication by the following relations:

$$\langle\overline{a}\rangle_{ij}\langle\overline{b}\rangle_{jk} = \begin{cases} \langle\overline{a}\sigma^*(\overline{b})\rangle_{ik} & (j<k<i) \\ \langle\overline{a}\sigma^*(\overline{b})\overline{c}\rangle_{ik} & (k<j<i \text{ or } j<i<k) \end{cases}$$

$$\langle\overline{a}\rangle_{ij}\langle b\rangle_{jj} = \langle\overline{a}\tau_i(\overline{b})\rangle_{ij} \quad (j<i)$$

$$\langle\overline{a}\rangle_{ij}\langle\overline{b}\rangle_{ji} = \langle xc_i\rangle_{ii} \quad (j<i), \text{ where } \overline{x} = \tau_i^{-1}(\overline{a}\sigma^*(\overline{b}))$$

$$\langle a\rangle_{ii}\langle\overline{b}\rangle_{ik} = \langle\tau_i(\overline{a})\overline{b}\rangle_{ik} \quad (i \neq k)$$

$$\langle a\rangle_{ii}\langle b\rangle_{ii} = \langle ab\rangle_{ii}$$

$$\langle\overline{a}\rangle_{ij}\langle\overline{b}\rangle_{jk} = \begin{cases} \langle\overline{a}\overline{b}\overline{c}\rangle_{ik} & (i<k<j) \\ \langle\overline{a}\overline{b}\rangle_{ik} & (k<i<j \text{ or } i<j<k) \end{cases}$$

$$\langle\overline{a}\rangle_{ij}\langle\overline{b}\rangle_{ji} = \langle xc_i\rangle_{ii} \quad (i<j), \text{ where } \overline{x} = \tau_i^{-1}(\overline{a}\overline{b})$$

$$\langle\overline{a}\rangle_{ij}\langle b\rangle_{jj} = \langle\overline{a}\tau_i(\overline{b})\rangle_{ij} \quad (i<j)$$

Then we see that T becomes a ring and moreover T is a basic indecomposable QF serial ring with identity Nakayama permutation.

Now, if R is a basic indecomposable QF serial ring with the identity Nakayama permutation, then R/S(Q) is a basic indecomposable QF serial ring with (*). Noting this fact, we can obtain

(4.1). Every basic indecomposable QF serial ring with the identity Nakayama permutation is represented as such a ring T in Example 2.

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Faculty of Science
Yamaguchi University

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ON H-SEPARABLE EXTENSIONS IN AZUMAYA ALGEBRAS

Hiroaki OKAMOTO

Throughout this report, A/B will represent a ring extension with common identity 1, C the center of A, and $V_A(B)$ the centralizer of B in A. Let M, N be A-B-bimodules. If M is A-B-isomorphic to some A-B-direct summand of a finite direct sum of copies of N, we write ${}_AM_B \mid {}_AN_B$. Needless to say, ${}_AM \mid {}_AA$ means that ${}_AM$ is finitely generated projective. Also, it is easy to see that ${}_BB_B \mid {}_BA_B$ (resp. ${}_BB \mid {}_BA$) if and only if B is B-B-isomorphic (resp. B-isomorphic) to a direct summand of ${}_BA_B$ (resp. ${}_BA$). An extension A/B is called a separable extension if the A-A-map $A \otimes_B A \to A$ defined by $x \otimes y \mapsto xy$ $(x, y \in A)$ splits. It is clear that A/B is separable if and only if ${}_AA_A \mid {}_AA \otimes_B A_A$. An extension A/B is called an H-separable extension if ${}_AA \otimes_B A_A \mid {}_AA_A$. It is well known that any H-separable extension is a separable extension (see, e.g., [2, Theorem 2.2]), and that if A is an Azumaya C-algebra then A/C is an H-separable extension (see, e.g., [6, Proposition 1.1]).

The main purpose of this report is to prove the following theorem.

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Theorem. Let A be an Azumaya C-algebra, B a C-subalgebra of A. Then

(1) B is a separable C-algebra if and only if ${}_B B_B \mid {}_B A_B$.

(2) B is an Azumaya C-algebra if and only if ${}_B A_B \mid {}_B B_B$.

In preparation for proving Theorem, we state first the next lemma.

Lemma. Let A be an Azumaya C-algebra, and B a C-subalgebra of A. Then A/B is H-separable if and only if $(A \otimes_B A)^A = \{\Sigma_i a_i \otimes b_i \in A \otimes_B A \mid a \Sigma_i a_i \otimes b_i = \Sigma_i a_i \otimes b_i a$ for all $a \in A\}$ is a projective C-module.

Proof. Assume that A/B is H-separable: ${}_A A \otimes_B A_A \mid {}_A A_A$. Then $(A \otimes_B A)^A \cong \mathrm{Hom}({}_A A_A, {}_A A \otimes_B A_A) \mid \mathrm{Hom}({}_A A_A, {}_A A_A) \cong C$ as C-module, that is, $(A \otimes_B A)^A$ is a finitely generated projective C-module. Conversely assume that $(A \otimes_B A)^A$ is a projective C-module. Since A/C is H-separable, $(A \otimes_C A)^A$ is a finitely generated projective C-module. In virtue of [1, p.52, Theorem 3.4], $(A \otimes_C A)^A \cong \mathrm{Hom}({}_A A_A, {}_A A \otimes_C A_A) \longrightarrow \mathrm{Hom}({}_A A_A, {}_A A \otimes_B A_A) \cong (A \otimes_B A)^A$ is a C-epimorphism. Hence $(A \otimes_B A)^A$ is a finitely generated projective C-module. Since $A \otimes_B A \cong A \otimes_C (A \otimes_B A)^A$ as A-A-bimodule by [1, p.54, Corollary 3.6], we get ${}_A A \otimes_B A_A \mid {}_A A_A$.

As a direct consequence of Lemma, we have the following corollary which is interesting in itself.

Corollary. If A is an Artinian semisimple ring and ${}_C A$ is finitely generated (In this case, A is an Azumaya C-algebra), then A/B is an H-separable extension for every C-subalgebra B of A. In particular, if A is a finite dimensional central simple C-algebra, then A/B is an

H-separable extension for every C-subalgebra B of A.

Recently, K. Hirata proved the following ([4, Proposition 6]): Let A be the group ring $K[G]$ of a finite group G with a coefficient field K whose characteristic does not divide the order of G. Let H be a subgroup of G, and $B = K[H]$. Then $A/V_A(V_A(B))$ is an H-separable extension. As a matter of fact, this is immediate by Corollary.

We are now ready to complete the proof of Theorem.

Proof of Theorem. (1) The only if part has been proved in [7, Proposition 1.5]. Assume now that ${}_B B_B \mid {}_B A_B$. Let m be an arbitrary maximal ideal of C, $\bar{A} = A/mA$, $\bar{B} = B/mB$, and $\bar{C} = C/m$. Then $\bar{A}$ is a finite dimensional central simple $\bar{C}$-algebra, and $\bar{B}$ is a $\bar{C}$-subalgebra of $\bar{A}$ such that ${}_{\bar{B}}\bar{B}_{\bar{B}} \mid {}_{\bar{B}}\bar{A}_{\bar{B}}$. Hence by Corollary, $\bar{A}/\bar{B}$ is an H-separable extension. Then $\bar{D} = V_{\bar{A}}(\bar{B})$ is a separable $\bar{C}$-algebra by [3, Proposition 4.7], and $V_{\bar{A}}(\bar{D}) = \bar{B}$ by [6, Proposition 1.2]. By [7, Proposition 1.5], we have ${}_{\bar{D}}\bar{D}_{\bar{D}} \mid {}_{\bar{D}}\bar{A}_{\bar{D}}$. Since $\bar{A}/\bar{D}$ is an H-separable extension (Corollary), we see that $V_{\bar{A}}(\bar{D}) = \bar{B}$ is a separable $\bar{C}$-algebra by [3, Proposition 4.7]. Since ${}_C B \mid {}_C A \mid {}_C C$, ${}_C B$ is finitely generated. Hence by [1, p.72, Theorem 7.1], B is a separable C-algebra.

(2) Assume that B is an Azumaya C-algebra. Then by [1, p.57, Theorem 4.3], $A = B \otimes_C V_A(B)$ and $V_A(B)$ is an Azumaya C-algebra. Hence, we see that ${}_B A_B = {}_B B \otimes_C V_A(B)_B \mid {}_B B \otimes_C C_B \cong {}_B B_B$. Conversely, we assume that ${}_B A_B \mid {}_B B_B$. It is well known that ${}_B A_B \mid {}_B B_B$ implies ${}_B B_B \mid {}_B A_B$ (see, e.g., [3, Proposition 5.6]). Hence ${}_C B$ is finitely generated and ${}_B B \otimes_C B_B \mid {}_B A \otimes_C A_B \mid {}_B A_B \mid {}_B B_B$. Therefore B is an Azumaya C-algebra by [6, Corollary 1.2].

The following proposition may be regarded as a sharpening of [4, Proposition 6].

Proposition. Let A be a separable (faithful) algebra over a commutative ring R, B a separable R-subalgebra of A, $D = V_A(B)$ and $V = V_A(V_A(B))$. Then A/V and A/D are H-separable extensions, ${}_V V_V \mid {}_V A_V$ and ${}_D D_D \mid {}_D A_D$.

Proof. By [1, p.55, Theorem 3.8], A is an Azumaya C-algebra and C is a separable R-algebra. Then BC is a separable R-algebra as a homomorphic image of $B \otimes_R C$ by [1, p.43, Proposition 4.6]. Then BC is a separable C-algebra by [1, p.46, Proposition 1.12]. Since $V_A(BC) = V_A(B) = D$ and $V_A(V_A(BC)) = V$, D and V are separable C-subalgebra of A by [1, p.57, Theorem 4.3]. Hence ${}_V V_V \mid {}_V A_V$ and ${}_D D_D \mid {}_D A_D$ by Theorem, A/V and A/D are H-separable extensions by [3, Proposition 4.3].

We shall conclude this report with giving four examples of H-separable extensions.

Examples. Let K be a field.

(1) Let $A = M_3(K)$, and $B = \begin{pmatrix} K & O & O \\ K & K & O \\ K & O & K \end{pmatrix}$. Then A/B is an H-separable extension (Corollary), and $V_A(B) = K$. As is easily seen, ${}_B A$ is projective, but A_B is not projective. Needless to say, both ${}_B A$ and A_B are finitely generated.

(2) Let $A = M_4(K)$, and $B = \begin{pmatrix} K & O & K & K \\ O & K & O & K \\ O & O & K & O \\ O & O & O & K \end{pmatrix}$. Then both ${}_B A$ and A_B are finitely generated and A/B is an H-separable extension with $V_A(B) = K$. But neither ${}_B A$ nor A_B is projective.

(3) Let $A = M_4(K)$, and

$$B = \left\{ \begin{pmatrix} a & o & b & c \\ o & a & d & e \\ o & o & a & o \\ o & o & o & a \end{pmatrix} \middle| \; a, b, c, d, e \in K \right\}.$$

Then both ${}_B A$ and A_B are finitely generated and A/B is an H-separable extension with $V_A(B) = B$. But, neither ${}_B A$ nor A_B is projective.

(4) Let $A = M_n(K)$ $(n \geq 3)$, $B = \left\{ \begin{pmatrix} a & & * \\ & a & \\ 0 & & \ddots \\ & & & a \end{pmatrix} \middle| \, a \in K \right\}$.
Then both ${}_BA$ and A_B are finitely generated and A/B is an H-separable extension. But, neither ${}_BA$ nor A_B is projective.

In [8], H. Tominaga proved that if A/B is an H-separable extension and ${}_BA$ is projective, then ${}_BA$ is finitely generated. These examples show that the converse need not be true.

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Department of Mathematics
Okayama University
Okayama 700, Japan

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EXAMPLE OF A D.F. RIGHT SELF-INJECTIVE REGULAR RING WHICH IS NOT LEFT SELF-INJECTIVE

Hikoji KAMBARA

Our first example answers negatively the well-known question whether every directly finite, regular right self-injective ring is necessarily left self-injective. We form a simple regular ring with the maximal right quotient ring which is directly finite and not left self-injective. Second example is a simple regular ring with continuous elements. Such a ring is given in [3] by von Neumann, but we form an algebraic example which is a direct limit of artinian simple rings.

1. Right self-injective ring. For a ring R with identity, R is called directly finite if $xy = 1$ implies $yx = 1$ for all x, $y \in R$. A ring R which is not directly finite is called directly infinite. A rank function on a (von Neumann) regular ring R is a map $N: R \longrightarrow [0,1]$ such that

(a) $N(1) = 1$,

(b) $N(xy) \leq N(x)$ and $N(xy) \leq N(y)$ for all x, $y \in R$,

(c) $N(e + f) = N(e) + N(f)$ for all orthogonal idempotents e, $f \in R$,

(d) $N(x) > 0$ for all nonzero $x \in R$.

If R is a regular ring with rank function N, then $\delta(x,y) = N(x - y)$ defines a metric on R and this metric δ is called the N-metric.

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For a field F (characteristic of F $\neq$ 2) and a positive integer n, $M_n(F)$ denote the ring of $n \times n$ matrices over F. We denote by $\{e_{ij}^{(n)}\}_{i,j=1}^{2^n}$ the matrix units of $M_{2^n}(F)$. By the homomorphism $M_{2^n}(F) \longrightarrow M_{2^{n+1}}(F)$ $(e_{ij}^{(n)} \longrightarrow e_{2i,2j}^{(n+1)} + e_{2i-1,2j-1}^{(n+1)})$, we consider the direct limit $S = \varinjlim M_{2^n}(F)$. Then the following are known:

(1) S is a directly finite regular simple ring with a rank function N.

(2) The completion $\bar{S}$ of S in the N-metric is a directly finite regular, right and left self-injective simple ring.

(3) The natural map $S \longrightarrow \bar{S}$ is a ring monomorphism.

(4) The maximal right quotient ring of S is directly infinite.

(5) (Theorem 5.5 [1]) Let R be a prime regular ring with rank function N. Then $Q(R) \subseteq \bar{R}$ as subring if and only if $\sup\{N(x) \mid x \in I\} = 1$ for all essential right ideals I of R, where Q(R) is the maximal right quotient ring of R and $\bar{R}$ is the completion of R in the N-metric.

From (3) we consider the ring S as a subring of $\bar{S}$. We denote by $\{e_{ij}^{(n)}\}_{i,j=1}^{2^n}$ the matrix units in S induced by the matrix units $\{e_{ij}^{(n)}\}$ in $M_{2^n}(F)$. Then from [1] S has an orthogonal subset $\{e_{i_t i_t}^{(n_t)}\}_{t=1}^{\infty}$ of $\{e_{ii}^{(n)}\}_{n,i}$ such that $\sum_t \oplus\, e_{i_t i_t}^{(n_t)} S$ is an essential right ideal and $1 \neq \sum_{t=1}^{\infty} N(e_{i_t i_t}^{(n_t)})$. Thus we get (4) by (5). So we will form a subring R ($\supseteq$ S) of $\bar{S}$ which satisfies $1 = \sup\{N(x) \mid x \in I\}$ for all essential right ideals I of R. We denote by T_0 the subring of $\bar{S}$ generated by $\{F, \{e_{ii}^{(n)}\}_{n,i}\}$ and by T the completion of T_0 in the N-metric. Then T_0, T are strongly regular rings, that is, every idempotent of T_0, T is central. For a ring T, B(T) represents the Boolean ring consisting of all central idempotents.

Definition of the ring R: In the ring $\bar{S}$, R is the subring generated by $\{S, B(T)\}$.

In this paper we give a brief sketch of the proof that the maximal right quotient ring Q of R is directly finite and not left self-injective.

First step: For every essential right ideal I of R, we have a set $\{f_t\}_t$ of idempotents in I which satisfies the following conditions;

(a) $f_t = a_t g_t$ for some $a_t \in S_{n_t}$, $g_t \in B(T)$, for all $t = 1,2,...$

(b) $a_t e_{ii}^{(n)} = a_t$ $(n = n_t)$, $e_{ii}^{(n)} a_t = e_{ii}^{(n)}$, $g_t e_{ii}^{(n)} = g_t$ for some i,

(c) $\{g_t\}$ is orthogonal in B(T),

(d) $g_t f_t = g_t$, $g_t f_{t'} = 0$ for all $t < t'$,

(e) $\sum\oplus f_t R$ is essential in I_R,

where S_n is the subring of $\bar{S}$ induced by $M_{2^n}(F)$.

Then by the above set $\{f_t\}$, we have $\sup\{N(x) \mid x \in I\} = \sup\{N(x) \mid x \in \sum\oplus f_t R\} = 1$. From (5) in the first note, we have that Q is directly finite and a subring of $\bar{S} = \bar{R}$.

Second step: We show that there is an element q in $\bar{S}$ satisfying the following conditions;

(i) $(R.\cdot q) = \{x \in R \mid xq \in R\}$ is an essential left ideal of R,

(ii) $qR \cap R = 0$.

Then we have $Q \neq \bar{S}$, so we have that Q is not left self-injective. For if Q is left self-injective, then Q is complete in the rank-metric and $S \subseteq Q$. This contradicts to $Q \neq \bar{S}$. Furthermore by Kobayashi's result [4], we have that $\bar{S}$ is the maximal left quotient ring of Q.

2. **Regular ring with a continuous element.** In [3] von Neumann defined a continuous element and showed an example of a regular ring with it. For a left and right self-injective regular simple ring R, $\{e_{ij}^{(n)}\}_{n,i,j}$ $(\subseteq R)$ is called a continuous set of matrix units of R if it satisfies the following conditions; (1) $e_{ij}^{(n)} = e_{2i-1,2j-1}^{(n+1)} + e_{2i,2j}^{(n+1)}$, (2) $1 = \sum_{i=1}^{2^n} e_{ii}^{(n)}$,

(3) $e^{(n)}_{ij} e^{(n)}_{k\ell} = \begin{cases} e^{(n)}_{i\ell} & (j = k) \\ 0 & (j \neq k) \end{cases}$, for all $i,j,k,\ell = 1,2,\ldots,2^n$, $n = 1,2,\ldots$. For the above ring R with the field of complex numbers C as the center of R, an element x in R is called a continuous element with respect to the continuous set $\{e^{(n)}_{ij}\}$ of matrix units, if there is a set $\{\rho^n_i \mid \rho^n_i \neq \rho^n_j \ (i \neq j)\}^{2^n}_{i=1}$ of rational numbers for all $n = 1,2,\ldots$, such that $e^{(n)}_{i1}(x + \rho^n_i)e^{(n)}_{1i} = xe^{(n)}_{ii} = e^{(n)}_{ii}x$ for any i, n.

Then for the above ring R, the following are known:

(1) R has a unique rank function N and is complete in the N-metric [2].

(2) For a continuous element x with respect to $\{e^{(n)}_{ij}\}$ in R, f(x) is invertible in R for all $f \in C[Y]$, where C[Y] is the polynomial ring over C [3].

(3) $1 = N(x - s)$ for all $s \in S$, where S is the direct union of subrings S_n generated by $\{C, \{e^{(n)}_{ij}\}^{2^n}_{i,j=1}\}$ [3].

For a field F with characteristic 0, we denote by F(x) the rational function field in one variable over F. For any n, we define a twisted matrix ring $M^*_{2^n}(F(x))$ by the following conditions;

$$f(x)e^{(n)}_{ij} = e^{(n)}_{ij} f\left(x + \frac{j-i}{2^n}\right)$$

$$\sum^{2^n}_{i,j} f_{ij}e^{(n)}_{ij} + \sum^{2^n}_{i,j} g_{ij}e^{(n)}_{ij} = \sum^{2^n}_{i,j} (f_{ij} + g_{ij})e^{(n)}_{ij}$$

$$\left(\sum^{2^n}_{i,j} f_{ij}e^{(n)}_{ij}\right)\left(\sum^{2^n}_{i,j} g_{ij}e^{(n)}_{ij}\right) = \sum^{2^n}_{i,j} \left(\sum_k f_{ik}(x)g_{kj}\left(x + \frac{k-j}{2^n}\right)\right)e^{(n)}_{ij}$$

where $\{e^{(n)}_{ij}\}^{2^n}_{i,j=1}$ is the matrix units of $M^*_{2^n}(F(x))$, and $f_{ij}, g_{ij} \in F(x)$.

Then $M^*_{2^n}(F(x))$ is an artinian simple ring. For any n, we define a ring homomorphism $\phi_n: M^*_{2^n}(F(x)) \longrightarrow M^*_{2^{n+1}}(F(x))$ by $\phi_n(f_{ij}e^{(n)}_{ij}) = f_{ij}(e^{(n+1)}_{2i-1,2j-1} + e^{(n+1)}_{2i,2j})$. By ϕ_n, we consider the direct limit $S = \varinjlim M^*_{2^n}(F(x))$, then S is a simple regular ring with a rank function N. Let R be the completion of S in the N-metric and $\{e_{ij}^{(n)}\}_{n,i,j}$ the matrix units in R induced by $\{e_{ij}^{(n)}\}$ in $M_{2^n}^*(F(x))$ for all n. Then $\{e_{ij}^{(n)}\}_{n,i,j}$ is a continuous set of matrix units in R, and x is a continuous element with respect to $\{e_{ij}^{(n)}\}$.

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Department of Mathematics
Osaka City University