PROCEEDINGS OF THE 21ST SYMPOSIUM ON RING THEORY

HELD AT HIROSAKI UNIVERSITY, HIROSAKI

October 20—22, 1988

EDITED BY

Kaoru MOTOSE

Hirosaki University

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OKAYAMA, JAPAN

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PREFACE

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The meeting and the Proceedings were financially supported by the Scientific Research Grant of the Educational Ministry of Japan through the arrangements by Professor H. Hijikata at Kyoto University. We appreciate his arrangements.

We wish also to express our thanks to all speakers of the meeting, to staffs of Okayama University for the publication of the Proceedings, and to staffs and graduate students of Hirosaki University for their best help in the organization of the meeting.

Hirosaki University, December 1988

Kaoru Motose

PREFACE

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Kaoru Motose

ENDOMORPHISM RINGS OF SEMICOCRITICAL MODULES

Hiroshi YOSHIMURA

This is a summary of the author's paper [10], in which we investigate relationships between semicocritical modules and their endomorphism rings, and show that the compressibility of semicocritical modules is closely related to the property that their endomorphism rings are orders in semisimple rings (Theorems 2 and 4).

1. Preliminaries

Throughout this note R will always denote a ring with identity, all R-modules will be unital right R-modules and homomorphisms will be written as acting on the left. Torsion theories will always mean hereditary torsion theories for mod-R.

Let τ be a torsion theory. A submodule M' of an R-module M is <u>τ-closed</u> in M if M/M' is τ-torsionfree. An R-module is <u>τ-artinian</u> if it has DCC on τ-closed submodules. An R-module M is <u>τ-full</u> if M/M' is τ-torsion for any essential submodule M' of M. A non-zero R-module M is <u>τ-cocritical</u> if it is τ-torsionfree and M/M' is τ-torsion for any non-zero submodule M' of M. It is <u>τ-semicocritical</u> if there exists a finite set $\{M_1, \ldots, M_n\}$ of submodules of M such that $\bigcap_{i=1}^{n} M_i = 0$ and M/M_i is τ-cocritical for

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This note is a summary of [10].

each i=1,...,n. A non-zero R-module is called <u>cocritical</u> (resp. <u>semicocritical</u>) if it is τ-cocritical (resp. τ-semicocritical) for some torsion theory τ and R is called <u>τ-artinian</u> (resp. <u>τ-full</u>, <u>(τ-)cocritical</u>, <u>(τ-)semicocritical</u>) if it is τ-artinian (resp. τ-full, (τ-)cocritical, (τ-)semicocritical) as an R-module. Let M be a non-zero R-module. We will say an R-module <u>M-torsionless</u> if it can be embedded into a product of copies of M.

We will denote by E(M) the injective hull of an R-module M and by χ(M) the torsion theory cogenerated by E(M). The Jacobson radical of a ring S will be denoted by J(S).

The following two results will be used repeatedly in the next section. The first lemma is immediate from [4, Proposition 18.3] and [9, Proposition 1.1] and the second lemma is an analogue of a result for cocritical modules (cf. [4, Proposition 14.25]).

<u>Lemma A</u>. Let τ be a torsion theory. Then for a non-zero R-module M, the following conditions are equivalent:

(1) M is τ-semicocritical.

(2) (i) For every indexed set $(M_\lambda)_{\lambda\in\Lambda}$ of τ-torsionfree R-modules and every monomorphism $M \to \prod_{\Lambda} M_\lambda$, there is a finite set Λ' of Λ and a monomorphism $M \to \prod_{\Lambda'} M_\lambda$.

(ii) $\bigcap\{N \leq M \mid M/N \text{ is } \tau\text{-cocritical}\} = 0$.

<u>Lemma B</u>. For a non-zero R-module M, the following conditions are equivalent:

(1) M is semicocritical.

(2) M has finite Goldie dimension and every homomorphism $N \to M$ from a submodule N of M to M with essential kernel is a zero homomorphism.

(3) M has finite Goldie dimension and α(M) = 0 for any $\alpha \in J(\mathrm{End}_R(E(M)))$.

(4) M is χ(M)-semicocritical.

2. Endomorphism rings of semicocritical modules

We consider the relationships between the compressibility of semicocritical modules and the property that their endomorphism rings are orders in semisimple rings. From this point of view the following lemma is useful.

Lemma 1. If M is a semicocritical R-module, then $End_R(M)$ is a subring of a semisimple ring.

Following [6] we call an R-module M compressible if for any essential submodule N of M there exists a monomorphism $M \to N$.

Now we state our main theorem.

Theorem 2. For a semicocritical R-module M, the following conditions are equivalent:

(1) M is compressible.

(2) (i) $End_R(M)$ is a right order in a semisimple ring.

(ii) $Hom_R(M,N) \neq 0$ for any non-zero submodule N of M.

In order to give a two-sided version of Theorem 2, we need the following result.

Lemma 3. (1) Let τ be a torsion theory and M a non-zero R-module. Assume that $End_R(M)$ is a semiprime ring and $Hom_R(M,N) \neq 0$ for any non-zero submodule N of M. Then any τ-semicocritical submodule of a direct product of τ-cocritical submodules of M is compressible.

(2) If M is a compressible and semicocritical R-module, then any $\chi(M)$-semicocritical and M-torsionless R-module is compressible.

Theorem 4. Assume that M is a semicocritical R-module. Then the following conditions are equivalent:

(1) Every $\chi(M)$-torsionfree homomorphic image of a finite direct sum of copies of M is compressible.

(2) $M + f(M)$ is compressible for any $f \in \mathrm{End}_R(E(M))$.

(3) (i) $\mathrm{End}_R(M)$ is a two-sided order in a semisimple ring.

(ii) $\mathrm{Hom}_R(M,N) \neq 0$ for any non-zero submodule N of M.

Moreover if M satisfies the above equivalent conditions, then every $\chi(M)$-torsionfree homomorphic image of a finite direct sum of copies of M is finitely cogenerated by M, and if in addition, M is compressible, then the converse is also true.

We will denote by G the Goldie torsion theory for mod-R. It is immediate from [9, Proposition 1.1] and Lemma B that R is semicocritical if and only if it is G-semicocritical if and only if it is right non-singular right Goldie ring.

As a consequence of Theorem 4, we obtain the following result, a part of which has been shown in [6, Theorem 2.2.15] under the assumption that R is a right order in a semisimple ring.

Corollary 5. Assume that R is a semicocritical ring. Then the following conditions are equivalent:

(1) Every finitely generated non-singular R-module is compressible.

(2) $R + xR$ is compressible for any $x \in E(R_R)$.

(3) R is a two-sided order in a semisimple ring.

Moreover if R satisfies the above equivalent conditions, then every finitely generated non-singular R-module is finitely cogenerated by R, and if in addition, R_R is compressible, i.e., R is a right order in a semisimple ring, then the converse is also true.

The following result shows that in some particular situations the compressibility of semicocritical modules coincides with the property that their endomorphism rings are orders in semisimple rings.

Corollary 6. (1) For a G-semicocritical essential submodule M of a direct product of copies of R_R, the following conditions are equivalent:

(a) M is compressible (resp. M + f(M) is compressible for any f ε $End_R(E(M))$.

(b) $End_R(M)$ is a right (resp. two-sided) order in a semisimple ring.

(2) Let N be a non-zero R-module such that $End_R(N)$ is a semiprime ring and $Hom_R(N,K) \neq 0$ for any non-zero N-torsionless R-module K. Then for a semicocritical and N-torsionless R-module M, the above conditions (a) and (b) are equivalent.

Assume that R is a semiprime ring. Then it follows from Corollary 6(2) that for a semicocritical and R-torsionless R-module M, the conditions (a) and (b) of Corollary 6(1) are equivalent.

<u>Lemma</u> <u>7</u>. Let M be a semicocritical R-module such that M + f(M) is compressible for any f ε $End_R(E(M))$. If N is a $\chi(M)$-semicocritical and M-torsionless R-module, then N + g(N) is compressible for any g ε $End_R(E(N))$.

<u>Theorem</u> <u>8</u>. Let M be a semicocritical R-module such that M is compressible (resp. M + f(M) is compressible for any f ε $End_R(E(M))$). If N is a $\chi(M)$-full and M-torsionless R-module with finite Goldie dimension, then $End_R(N)$ is a right (resp. two-sided) order in a semisimple ring.

<u>Corollary</u> <u>9</u> (Zelmanowitz [<u>11</u>]). Assume that R is a right (resp. two-sided) order in a semisimple ring. If M is an R-torsionless R-module with finite Goldie dimension, then $End_R(M)$ is a right (resp. two-sided) order in a semisimple ring.

Finally, we give a sufficient condition for R-modules to satisfy the equivalent conditions of Theorem 4.

<u>Proposition</u> <u>10</u>. Assume that R is a left noetherian ring and τ is a torsion theory. If M is a finitely generated τ-

torsionfree and τ-full R-module such that $R/Ann_R(M)$ is a τ-artinian and semiprime ring, then M satisfies the equivalent conditions of Theorem 4.

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A COUNTEREXAMPLE TO TARSY'S CONJECTURE ON GLOBAL DIMENSION OF ORDERS*

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Let D be a local Dedekind domain with a unique maximal ideal πD and the quotient ring K. If a D-order Λ in the full $n \times n$ matrix ring $(K)_n$ contains n orthogonal idempotents then Λ is isomorphic to a subring $(\pi^{\lambda_{ij}}D)$ of $(D)_n$ where $\lambda_{ij} \geq 0$, $\lambda_{ii} = 0$ and $\lambda_{ij} + \lambda_{jk} \geq \lambda_{ik}$ for any $1 \leq i, j, k \leq n$ and so Λ is called a tiled D-order in $(K)_n$ (cf. [5]).

In [8], among other things, R. Tarsy posed the following conjecture:

(T) the maximum finite global dimension of a D-order in $(K)_n$ is $n - 1$.

For some classes of tiled D-orders, this is settled by V. A. Jategaonkar [4], [5] and Kirkman and Kuzmanovich [6]. As a strategy to prove (T) for tiled D-orders, Jategaonkar [5] conjectured

(J) if Λ is a basic tiled D-order of finite global dimension then there is a vertex i in the quiver of Λ such that $|i^+| = 1$ or $|i^-| = 1$.

Recently, in [6], Kirkman and Kuzmanovich obtained a counter-

*The results announced here and their related facts can be found in [1] and [2].

example to (J) (but not to (T)). In this summary, we announce a counterexample to (T), which was found in an attempt to obtain the list of tiled D-orders of finite global dimension.

We now recall some definitions and notation of tiled D-orders. Let $n \geq 2$ and let $\Lambda = (\pi^{\lambda_{ij}}D)$ be a basic tiled D-order in $(K)_n$, where λ_{ij}'s are non-negative integers, $\lambda_{ii} = 0$, $\lambda_{ij} + \lambda_{jk} \geq \lambda_{ik}$ and $\lambda_{ij} + \lambda_{ji} > 0$ (if $i \neq j$) for any $1 \leq i, j, k \leq n$. Put $m_{kij} = 1$ (if $i = j = k$) and λ_{ij} (otherwise) for any $1 \leq i, j, k \leq n$ and put $M_k = (\pi^{m_{kij}}D)$ for any $1 \leq k \leq n$. Then $M_1, \ldots, M_n$ are the maximal ideals of Λ and $J = M_1 \cap \ldots \cap M_n$ is the Jacobson radical of Λ. Let e_i be the matrix in $(K)_n$ whose (i,i)-entry is 1 and the others are 0. Following Wiedemann and Roggenkamp [9], we call $\mathcal{Q}(\Lambda) = (\mathcal{Q}(\Lambda)_0, \mathcal{Q}(\Lambda)_1, v)$ the <u>valued quiver</u> of Λ provided that $\mathcal{Q}(\Lambda)_0 = \{1, \ldots, n\}$ is the set of vertices, $\mathcal{Q}(\Lambda)_1$ is the set of arrows defined by $i \to j \in \mathcal{Q}(\Lambda)_1$ if $e_j(J/J^2)e_i \neq 0$ and that v is the mapping from $\mathcal{Q}(\Lambda)_1$ to non-negative integers such that for $\alpha: i \to j \in \mathcal{Q}(\Lambda)_1$, $v(\alpha) = \lambda_{ji}$ (if $i \neq j$) and 1 (if $i = j$). Let $p: x_0 \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_m} x_m$ be a path in $\mathcal{Q}(\Lambda)$. Then we put $v(p) = \sum_{i=1}^{m} v(\alpha_i)$ and we call p a $v(p)$-path. $\mathcal{P}(\Lambda)$ denotes the set of all λ_{ij}-paths from j to i in $\mathcal{Q}(\Lambda)$ where $i, j \in \mathcal{Q}(\Lambda)_0$ and $i \neq j$. We associate an order with $\mathcal{P}(\Lambda)$, i.e., for $p_1, p_2 \in \mathcal{P}(\Lambda)$, $p_1 \leq p_2$ if p_1 is a subpath of p_2. For $i \in \mathcal{Q}(\Lambda)_0$, i^+ (resp. i^-) denotes the set $\{x \in \mathcal{Q}(\Lambda)_0 \mid i \to x \in \mathcal{Q}(\Lambda)_1\}$ (resp. $\{y \in \mathcal{Q}(\Lambda)_0 \mid y \to i \in \mathcal{Q}(\Lambda)_1\}$).

We next define an idealizer with respect to a link between maximal ideals. Let S be an arbitrary ring with identity and let M_a and M_b be maximal ideals of S. When $M_a \cap M_b \supsetneqq M_bM_a$ and S/M_i is artinian ($i = a, b$), we say that there is a <u>link</u> from M_a to M_b and denote it by $M_a \rightsquigarrow M_b$. In this case, since M_b/M_bM_a is semisimple as a right S-module, there is a right ideal A of S such that $M_b/M_bM_a = A/M_bM_a \oplus (M_a \cap M_b)/M_bM_a$. Let R be the idealizer subring $\{s \in S \mid sA \subset A\}$ of S at A. We call R <u>an idealizer with respect to a link</u> $M_a \rightsquigarrow M_b$.

It is analogous to the case when A is semimaximal (Robson [8]) that S and R have closely related structure. An essential difference of ours from the semimaximal case is that A is not generative (i.e., $SA = S$).

The following lemma is an origin of an idealizer w.r.t. a link between maximal ideals.

Lemma 1. Let $\Lambda = (\pi^{\lambda_{ij}}D)$ be a basic tiled D-order in $(K)_n$. For $1 \le a, b \le n$ with $a \ne b$, put $\gamma_{ij} = \lambda_{ij} + 1$ (if $(i,j) = (b,a)$) and λ_{ij} (otherwise) and put $\Gamma = (\pi^{\gamma_{ij}}D)$. Then Γ is also a basic tiled D-order if and only if $M_a \rightsquigarrow M_b$. In this case, Γ is obtained as an idealizer w.r.t. $M_a \rightsquigarrow M_b$.

Remarks. (1) In the proof of Lemma 1, we put $A = (\pi^{a_{ij}}D)$ where $a_{ij} = \lambda_{ij} + 1$ (if $(i,j) = (b,a)$ or (b,b)) and λ_{ij} (otherwise) and show that Γ is the idealizer of A in Λ. In the definition of an idealizer w.r.t. a link, A is not uniquely determined by the link. So, when we consider tiled D-orders, we suppose that the idealizer w.r.t. a link is taken as above.

(2) For a basic tiled D-order Λ, $M_a \rightsquigarrow M_b$ if and only if $a \to b \in Q(\Lambda)_1$.

Conversely to Lemma 1, we obtain the following

Lemma 2. Let $\Gamma = (\pi^{\gamma_{ij}}D)$ be a basic tiled D-order in $(K)_n$. Suppose that $\gamma_{ab} + \gamma_{ba} \ge 2$ and $\gamma_{ba} \ge 1$ for some $1 \le a, b \le n$ with $a \ne b$. Put $\lambda_{ij} = \gamma_{ij} - 1$ (if $(i,j) = (b,a)$) and γ_{ij} (otherwise) and put $\Lambda = (\pi^{\lambda_{ij}}D)$. Then Λ is also a basic tiled D-order if and only if any γ_{ba}-path form a to b in $Q(\Gamma)$ is maximal in $P(\Gamma)$.

By Lemmas 1 and 2, we obtain the following

Proposition 3. Any tiled D-order is obtained by iterating the idealizers with respect to links of maximal ideals from a hereditary order.

By the examples of [4] and [5], we cannot expect to compare global dimension of an idealizer w.r.t. a link with that of a given ring without any assumption. Utilizing tame subidealizers (Goodearl [3]), we can obtain bounds of global dimension of an idealizer w.r.t. a certain link.

Theorem 4. Let S be a ring and M_a, M_b maximal ideals of S such that $M_a \rightsquigarrow M_b$ and $M_b \not\rightsquigarrow M_a$, and let R be an idealizer w.r.t. $M_a \rightsquigarrow M_b$. Suppose that there exists a ring T such that S is contained in T as a subring and that M_b is a generative right ideal of T (or M_a is a generative left ideal of T). Then

$$\text{r.gl.dim } S - 1 \leq \text{r.gl.dim } R \leq \text{r.gl.dim } S + 1$$

and

$$\ell\text{.gl.dim } S - 1 \leq \ell\text{.gl.dim } R \leq \ell\text{.gl.dim } S + 1.$$

When S is a noetherian prime ring, the theorem is rephrased as follows.

Corollary 5. Let S be a noetherian prime ring and M_a, M_b maximal ideals of S with $M_a \rightsquigarrow M_b$ and $M_b \not\rightsquigarrow M_a$, and let R be an idealizer w.r.t. $M_a \rightsquigarrow M_b$. If M_b is idempotent and left S-projective (or M_a is idempotent and right S-projective). Then

$$\text{gl.dim } S - 1 \leq \text{gl.dim } R \leq \text{gl.dim } S + 1.$$

According to the corollary, we call a link between maximal ideals <u>projective</u> if it satisfies the hypotheses of the theorem.

Remarks. (1) In [8], Tarsy also conjectured that if $\Lambda \supset \Gamma$ are successive D-orders of finite global dimension then their global dimensions differ at most one. A counterexample to this conjecture is given by Jategaonkar [4]. However, if Λ and Γ are basic tiled D-orders and if the associated link is projective then this conjecture follows from our theorem.

(2) The hypothesis '$M_b \not\rightsquigarrow M_a$' is necessary.

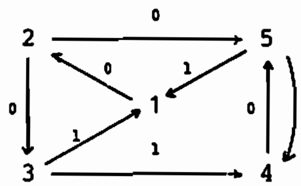
(3) In the theorem, $\text{gl.dim } S - 1$ and $\text{gl.dim } S + 1$ are

the best bounds of gl.dim R.

Example 6. Let Λ be the basic tiled D-order

$$\begin{pmatrix} D & \pi D & \pi D & \pi D & \pi D \\ D & D & \pi D & \pi D & \pi D \\ D & D & D & \pi D & \pi D \\ \pi D & \pi D & \pi D & D & \pi D \\ D & D & \pi D & D & D \end{pmatrix}.$$

Then gl.dim $\Lambda = 3$ and the valued quiver of Λ is given by



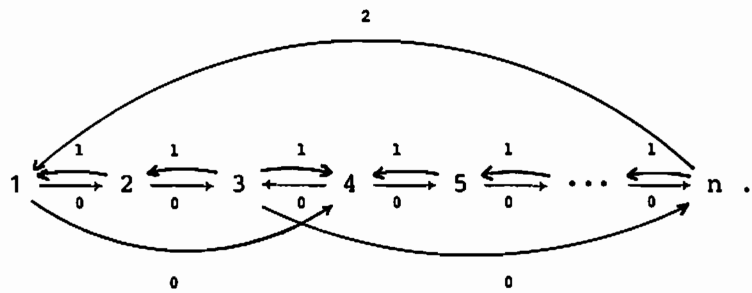
Note that M_1 and M_4 are left Λ-projective. Let $\alpha_1, \ldots, \alpha_4$ be the links $M_3 \rightsquigarrow M_4$, $M_3 \rightsquigarrow M_1$, $M_5 \rightsquigarrow M_1$ and $M_5 \rightsquigarrow M_4$, respectively, and let Γ_i be the idealizer w.r.t. α_i $(1 \le i \le 4)$. Then α_1, α_2 and α_3 are projective links and gl.dim $\Gamma_1 = 2$, gl.dim $\Gamma_2 = 3$ and gl.dim $\Gamma_3 = 4$. But since $M_4 \rightsquigarrow M_5$, α_4 is not a projective link, and gl.dim $\Gamma_4 = \infty$.

Theorem 7. Suppose that $2 \le n \le 5$. Let Λ be a basic tiled D-order in $(K)_n$ and gl.dim $\Lambda < \infty$. Then there exists a chain of basic tiled D-orders $\Lambda = \Lambda_0 \subset \cdots \subset \Lambda_t$ such that Λ_t is hereditary and that Λ_{i-1} is the idealizer w.r.t. a projective link in Λ_i where $i = 1, \ldots, t$. If $n \ge 6$ then there exists a tiled D-order of finite global dimension in $(K)_n$ which does not have the above property.

Remark. Making use of this theorem, for $2 \le n \le 5$, a list of the representatives of isomorphism classes of basic tiled D-orders of finite global dimension in $(K)_n$ is obtained.

The following example is one of the basic tiled D-orders in $(K)_n$ $(n \ge 6)$ of finite global dimension those are not obtained by iterating the idealizers w.r.t. projective links from hereditary orders. This is also a counterexample to Tarsy's conjecture.

Example 8. Let Λ be the basic tiled D-order in $(K)_n$ whose valued quiver is given by



where $n \geq 6$. Then $gl.dim\ \Lambda = n$ and (J) does not hold.

Remarks. (1) This example is something like Tarsy's [8, Theorem 11].

(2) It is shown in [5] that for a fixed integer n, there are only finitely many isomorphism classes of tiled D-orders of finite global dimension in $(K)_n$. Hence there is the maximum finite global dimension of a tiled D-order in $(K)_n$. Although, at present, n is the maximum among known examples, we do not have enough reason to conjecture that n is the maximum.

Proposition 9. Let Λ be a basic tiled D-order in $(K)_n$ and $gl.dim\ \Lambda = 2$. Then there exist $a, b \in Q(\Lambda)_0$ such that $|a^-| = 1$ and $|b^+| = 1$. Hence (J) holds. Moreover, there exists a chain of tiled D-orders $\Lambda = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_m$ such that Γ_m is a maximal order, Γ_{i-1} is obtained as an idealizer in Γ_i $(1 \leq i \leq m)$ and that for some $1 \leq \ell \leq m$, $gl.dim\ \Gamma_i = 1$ (if $\ell \leq i \leq m$) and 2 (if $0 \leq i \leq \ell$).

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ON PRIMITIVE ELEMENTS OF GALOIS EXTENSIONS OF COMMUTATIVE RINGS

Isao KIKUMASA and Takasi NAGAHARA

Throughout, all rings will be assumed commutative, and all Galois extensions will mean Galois extensions in the sense of [1]. Moreover, A will mean a field, and all ring extensions of A will be assumed with identity element 1, the identity element of A. A ring extension B/A will be called to be simple if B is generated by a single element over A, that is, B/A has a primitive element.

The purpose of this note is to present some arithmetical conditions of the simplicity for Galois extensions.

In what follows, given a set S, an A-module M and a ring B, we shall use the following conventions:

$|S|$ = the cardinal number of S,

$[M:A]$ = the rank of A-module M, and

$\ell(B)$ = the length of composition series of B-module B.

The detailed version of this paper will be submitted for publication elsewhere.

First, we shall state the following lemma which is fundamental.

Lemma 1. Let B/A be a Galois extension.

1) If $|A| = \infty$ then B/A is simple.

2) If A is of characteristic 0 then B/A is simple.

By the lemma, we may assume that A is finite, namely

$$|A| = q = p^m \quad (p\text{: a prime, } m \geqq 1),$$

in the rest of this paper.

Next, we shall present a theorem which plays an important role on our study.

Theorem 2. Let $a = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$ where $n, \alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{N}$ and $p_1, p_2, \dots, p_n$ are prime integers with $p_1 < p_2 < \dots < p_n$, and let $b \in \mathbb{N}$ with $b \geqq a$. Moreover, set

$$f(x) = \sum_{\substack{0 \leqq i \leqq n \\ 1 \leqq e_1 < e_2 < \cdots < e_i \leqq n}} (-1)^{n-i} x^{p_{e_1} p_{e_2} \cdots p_{e_i}}$$

where $p_{e_1} p_{e_2} \cdots p_{e_i} = 1$ when $i = 0$, and put

$$g(x) = x^{p_1 p_2 \cdots p_n} - f(x).$$

Then, the equation

$$(*) \qquad f(x) = b \quad (x > 0)$$

has the unique solution.

Furthermore, let $x_0(a,b)$ denote the solution of the equation (*). Then there holds

$$1 < x_0(a,b) \leq g(x_0(a,b)) \leq b.$$

In particular, if $n = 1$ then $x_0(a,b) = g(x_0(a,b))$.

Now, for the $g(x)$ and $x_0(a,b)$ in the above theorem, we put

$$\varepsilon(a,b) = g(x_0(a,b))/b.$$

Then, the ε is an arithmetic function on the set $\{(a,b);\ a,\ b \in \mathbb{N},\ 2 \leq a \leq b\}$ such that

$$0 < \varepsilon(a,b) \leq 1.$$

Moreover

$$\varepsilon(a,b_1) > \varepsilon(a,b_2) \quad \text{if} \quad b_1 < b_2.$$

For examples,

$$\varepsilon(2,2) = 1,\quad \varepsilon(2,10) = 0.370\cdots,\quad \varepsilon(2,100) = 0.105\cdots,$$

$$\varepsilon(10,10) = 0.415\cdots,\quad \varepsilon(100,100) = 0.115\cdots,\quad \text{and}$$

$$\varepsilon(1000,1000) = 0.0341\cdots.$$

In virtue of this function, we shall state one of our main theorem.

Theorem 3. Let B/A be a G-Galois extension and $b = |G|$.

1) In case $\ell(B) = b$, B/A is simple if and only if

$$\ell(B) \leq q.$$

2) In case $\ell(B) \neq b$, for $a := b/\ell(B)$, the following conditions are equivalent.

i) B/A is simple.

ii) $\ell(B) \leq bm/(\log_p b + \log_p(1 + \varepsilon(a,b)))$

where $m = [A:GF(p)]$.

If, in particular, the rank of B/A in Theorem 3 is a power of a single prime then we have the following

Theorem 4. Let B/A be a G-Galois extension and $|G| = r^k$ where r is a prime integer and $k \in \mathbb{N}$. Assume that $\ell(B) \neq r^k$. Then the following conditions are equivalent.

i) B/A is simple.

ii) $\ell(B) \leqq r^k m/(k \cdot \log_p r + \log_p(1 + x_0(r^k, r^k)/r^k))$.

Combining Theorems 3 1) and 4 with Theorem 2, we readily obtain

Corollary 5. Let B/A be a G-Galois extension and $|G| = r^k$ where r is a prime integer and $k \in \mathbb{N}$. Then

1) If B/A is simple then

$$\ell(B) < mr^k/(k \cdot \log_p r).$$

2) If $\ell(B) \leqq mr^k/(k \cdot \log_p r + \log_p 2)$ then B/A is simple.

In case $r = p$, the following theorem is a main result.

Theorem 6. Let B/A be a G-Galois extension and $|G| = p^k$ $(k \in \mathbb{N})$. Then the following conditions are equivalent.

i) B/A is simple.

ii) $\ell(B) \leqq p^k(mp^k - 1)/(kp^k - 1)$.

Example 1) Let $A = GF(5)$ and B' a field which is a Galois extension of A with Galois group G' of order 6. Moreover, let

$$B = B' \oplus B' \oplus \dots \oplus B' \quad (12 \text{ copies}).$$

Then the ring B is a G-Galois extension of A in the sense of [1], and $q = 5$, $|G| = 72$ and $\ell(B) = 12$. Putting $b = |G|$ and $a = b/\ell(B)$, we see that $b = 2^3 \cdot 3^2$ and $a = 2 \cdot 3$. In this case, the equation (*) in Theorem 2 is

$$x^6 - x^3 - x^2 + x - 72 = 0 \quad (x > 0).$$

For the solution $x_0(6,72)$ of this equation, as is easily seen, there holds

$$2.09 < x_0(6,72) < 2.1.$$

Using this fact, we know that the right-hand side in the inequality of Theorem 3 ii) is more than 26.1 and less than 26.2. Hence B/A is simple by Theorem 3.

2) Let $q = 19$, $|G| = 3^8$ and $\ell(B) = 3^7$ (of course, as in the above, there exists a Galois extension B/A satisfying this condition). Then, in Theorem 4, the right-hand side is 2197.3.... Hence, by the theorem, we see that B/A is simple in this case because $\ell(B) = 2187$. But we find that there does not hold the inequality of Corollary 5 2). Indeed,

$$\begin{aligned} & mr^n/(n \cdot \log_p r + \log_p 2) \\ = \ & 6561/\log_{19} 13122 \\ = \ & 2037.3\dots \\ < \ & 2187 = \ell(B). \end{aligned}$$

3) Let $q = 3^3$, $|G| = 3^9$ and $\ell(B) = 3^8$ (one can

easily see the existence of such a Galois extension). Then, the right-hand side in ii) of Theorem 6 is less than $\ell(B)$. In truth,

$$\begin{aligned} & p^n(mp^n - 1)/(np^n - 1) \\ = & \ 6560.9... \\ < & \ 6561 = \ell(B). \end{aligned}$$

This implies that B/A is not simple, that is B/A has no primitive element because of Theorem 6.

The following references [1]-[12] are used to the proofs of the results in this note and other results.

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SOME RESULTS ON ADDITIVE GROUPS OF RINGS

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There are many works on the additive groups of rings. Many of the results are found in Chapter 17 of L. Fuchs' book [6] and S. Feigelstock's books [3] and [4]. In this note, we deal with some problems in [1], [4] and [7].

Notation.

R	a ring
R^+	the additive group of R
$C(R)$	the commutator ideal of R
(a)	the principal ideal generated by $a \in R$
R_t	the torsion part of R^+
R_p	the p-primary component of R^+, p a prime
Z	the ring of integers
Q	the field of rational numbers
$Z(n)$	cyclic group of order n

1. On a problem of Szász. In Problem 84 in [7], F. A. Szász asks: In which rings R has the additive group Z^+ of the centre Z, a finite group-theoretic index with respect to R^+?

We begin with the following

Proposition 1.1. If Z^+ has finite index in R^+, then $C(R)$ is finite.

The detailed version of this note will appear elsewhere.

Proposition 1.2. Assume that $C(R)$ is finite. Then there exists a finite nilpotent ideal N of R such that R/N is the direct sum of a finite semisimple ring and a commutative ring.

As an immediate consequence of Propositions 1.1 and 1.2, we have the following

Corollary 1.3. Let R be a semiprime ring with centre Z. Then the following statements are equivalent:

(1) Z^+ has finite index in R^+.

(2) $C(R)$ is finite.

(3) R is the direct sum of a finite ring and a commutative ring.

The following is our main theorem.

Theorem 1.4. Let R be a ring with centre Z. Then the following statements are equivalent:

(1) Z^+ has finite index in R^+.

(2) R has an ideal I contained in Z such that R/I is a finite ring.

2. The additive group of a non-periodic C-ring. Following H. E. Bell [1], we call a ring R quasi-periodic if for each $x \in R$ there exist integers k, n, m, all depending on x, such that $n > m > 0$ and $x^n = kx^m$. Among the quasi-periodic rings which have been previously studied are the periodic rings - those such that for each $x \in R$, there exist distinct positive integers n, m for which $x^n = x^m$. Others to which less attension has been paid are the C-rings discussed by S. Feigelstock in [2]; specifically, a ring R is a C-ring if for each $x \in R$, there exists an integer $n > 1$ and an integer k such that $x^n = kx$. If in a C-ring we can take $n = 2$ for all $x \in R$, then R is called a C_2-ring. Clearly, a ring R is a C_2-ring if and only if every cyclic subgroup of R^+ is a subring of R. Clearly every abelian group G can be provided with a periodic ring structure in a trivial way by defining all products to be 0. However the additive groups of non-periodic, quasi-periodic rings are rather restrictive.

Theorem 2.1. Let G be an abelian group. Then the following statements are equivalent:

(1) $G \cong Z^+ \oplus H$ for some abelian group H.

(2) There is a non-periodic C_2-ring R such that $R^+ = G$.

(3) There is a non-periodic C-ring R such that $R^+ = G$.

(4) There is a non-periodic, quasi-periodic ring R such that $R^+ = G$.

Theorem 2.2. Let G be an abelian group. Then the following statements are equivalent:

(1) $G \cong Z^+ \oplus H$ for some torsion abelian group H.

(2) There is a non-periodic, quasi-periodic ring R with identity such that $R^+ = G$.

3. When can a ring be embedded in a direct sum of simple rings.

At the beginning of this section, we give a counter-example to [4, Corollary 1.4.5].

Example 3.1. Let R be the additive abelian group $Z \oplus Z/2Z$ and define multiplication by $(a,b)(c,d) = (ab, ad + bc)$, where $a, c \in Z$ and $b, d \in Z/2Z$. Then R is a non-fissible ring. However it is easily checked that R can be embedded in the direct sum $Q \oplus M_2(Z/2Z)$.

The following is an answer to [4, Question 1.4.6].

Theorem 3.2. The following statements are equivalent:

(1) R can be embedded in a direct sum of simple rings.

(2) R is a subdirect sum of a torsion-free ring F and a torsion ring T with T^+ square-free.

The following is the correct form of [4, Corollary 1.4.5].

Corollary 3.3. Let R be a ring with identity. Then the following conditions are equivalent:

(1) R can be embedded in a direct sum of simple rings with identity.

(2) R^+ is square-free, or equivalently, $R_t^+ = \bigoplus_{p \in P} \bigoplus_{\alpha_p} Z(p)$, with P a finite set of primes, α_p an arbitrary cardinal for each $p \in P$.

4. Fully idempotent rings are strongly p-fissible. A ring R is said to be fissible if the torsion part R_t is a ring direct summand of R, and p-fissible if the p-primary component R_p is a ring direct summand of R. A fissible ring R is strongly fissible if every group decomposition $R^+ = R_t \oplus H$ is a ring decomposition. A ring R is said to be strongly p-fissible if R is p-fissible and every group decomposition $R^+ = R_p \oplus H$ is a ring decomposition. A ring R is said to be fully idempotent if $I^2 = I$ for every ideal I of R.

Proposition 4.1. Every biregular ring R is fully idempotent. Hence $R/R_t = \bigoplus_{\alpha} Q$ for some cardinal α.

Remark 4.2. Let R be a biregular ring. By Proposition 4.1, for every $a \in R/R_t$, $t(a) = (\infty, \ldots, \infty, \ldots)$ (cf. [4, Question 3.3.15]).

The following theorem generalizes [5, Theorems 7 and 8] (see also [4, Chapter 3, §2]).

Theorem 4.3. Let R be a fully idempotent ring. Then every ring S with $S^+ = R^+$ is strongly p-fissible for every prime p. If $S^+ = S_t \oplus F$, then S is strongly fissible.

5. Quotient no-zero-divisor rings. A ring R is called a (proper) quotient no-zero-divisor ring if every (proper) quotient ring of R has no zero-divisor. In this section, we answer to [4, Question 4.1.13].

Theorem 5.1. The following statements are equivalent:

(1) R is a quotient no-zero-divisor ring.

(2) R is a chain ring satisfying $(a) = (a^2)$ for every $a \in R$.

Corollary 5.2. Let R be a quotient no-zero-divisor ring. Then either $R^+ = \bigoplus_\alpha Q^+$ or $R^+ = \bigoplus_\beta Z(p)$ where p is a prime and α, β are cardinals.

We give some examples of quotient no-zero-divisor rings. We begin with the following

Proposition 5.3. Let R be a quotient no-zero-divisor ring. Suppose that R satisfies a polynomial identity. Then R is a division ring.

Obviously, every simple domain is a quotient no-zero-divisor ring. Let R be a simple domain with identity which is not a division ring, and let K be the centre of R. Then K is a field. Take a non-unit $x \neq 0$ of R, and consider the subring $S = K + xR$ of R. Then 0 and xR are the only proper ideals of S, and these are completely prime. Thus S is a qoutient no-zero-divisor ring.

More generally, we have the following

Theorem 5.4. Let D be a simple domain with identity which is not a division ring, and K the centre of D. Suppose that $D \otimes_K D \otimes_K \cdots \otimes_K D$, n times, is a domain for every positive integer n. Take a non-unit $x \in D$, and let $R = R^{(1)} = K + xD$. Define the subalgebra $R^{(n)}$ of $D \otimes_K \cdots \otimes_K D$ (n times) by $R^{(n)} = K + xD \otimes_K R^{(n-1)}$, inductively. Then, for any positive integer n, $R^{(n)}$ is both a quotient no-zero-divisor ring and a $(n + 1)$-chain ring.

Let K be a field of characteristic 0, and $A_1(K)$ be the algebra generated by x and y over K with relation $xy - yx = 1$, that is, $A_1(K)$ be the Weyl algebra on x and y over K. Then $A_1(K)$ satisfies the hypotheses of Theorem 5.4.

6. Proper quotient no-zero-divisor rings. If R has a completely prime ideal, then R has a minimal completely prime ideal. Using this fact, we have the following

Theorem 6.1. Let R be a proper quotient no-zero-divisor ring. Then one of the following holds:

a) R is a quotient no-zero-divisor ring.

b) R has a unique minimal nonzero ideal P and R/P is a quotient no-zero-divisor ring.

c) R has the only two minimal prime ideals P_1, P_2, R/P_i is a quotient no-zero-divisor ring for each $i = 1, 2$, and the lattice of ideals of R is the following form:

$$0 \begin{matrix} \subset P_1 \\ \subset P_2 \end{matrix} \subset P_1 + P_2 \subset \cdots \subset R.$$

Corollary 6.2. Let R be a proper quotient no-zero-divisor ring. Then one of the following holds:

i) $R^+ = \bigoplus_{\alpha} Q^+$ where α is a cardinal.

ii) $R^+ = \bigoplus_{\alpha} Z(p)$ where p is a prime and α is a cardinal.

iii) $R^+ = \bigoplus_{\alpha} Q^+ \oplus \bigoplus_{\beta} Z(p)$ where p is a prime and α, β are cardinals.

iv) $R^+ = \bigoplus_{\alpha} Z(p) \oplus \bigoplus_{\beta} Z(q)$ where p, q are two distinct primes and α, β are cardinals.

v) $R^+ = \bigoplus_{\alpha} Z(p) \oplus \bigoplus_{\beta} Z(p^2)$ where p is a prime and α, β are cardinals.

Example 6.3. Let K be a field of characteristic 0, and $A_1(K)$ the Weyl algebra on x, y over K. Let $R^{(n)}$ be the algebra defined in Theorem 5.4 (with $R = A_1(K)$), and P the minimum nonzero ideal of $R^{(n)}$. Then the trivial extension

$R^{(n)} \bowtie P$ is a proper quotient no-zero-divisor ring satisfying the condition b) in Theorem 6.1. The subring $S = K + M_n(xA_1(K))$ of $M_n(A_1(K))$ also satisfies b) in Theorem 6.1. Next, consider the subring $T = K + xA_1(K) \oplus yA_1(K)$ of $A_1(K) \oplus A_1(K)$. Then T satisfies the condition c) in Theorem 6.1. The lattice of ideals of T is the following: $0 \subset \begin{matrix} xA_1(K) \\ yA_1(K) \end{matrix} \subset xA_1(K) \oplus yA_1(K) \subset T$.

7. Rings whose additive endomorphisms are multiplicative. A ring R is said to be an AE-ring, (additive endomorphism), if every $f \in \mathrm{End}(R^+)$ is a ring endomorphism of R. The structure of the torsion AE-rings was completely described in [4, Theorem 6.3.6]. In [4, Question 6.3.7], Feigelstock asked whether [4, Theorem 6.3.6] remains true for an arbitrary AE-ring or not. Now we give a partial solution.

Theorem 7.1. The following statements are equivalent:

(1) R is an AE-ring with $R_tR \neq 0$.

(2) $R = (a) \oplus S \oplus T$ with $|a| = 2^n$, n a positive integer, $2^{n-1}S = 0$, T is 2-torsion-free and 2-divisible, and multiplication in R is defined by $(m_1a + s_1 + t_1)(m_2a + s_2 + t_2) = 2^{n-1}m_1m_2a$ for all integers m_i, all $s_i \in S$, and all $t_i \in T$, $i = 1, 2$.

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THE CHARACTER RINGS OF FINITE GROUPS AND GENERALIZED DEDEKIND SUMS

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§1. **Introduction.** In the paper [6], T.Yoshida defined a function $\Delta[\lambda] = \Delta_A[\lambda]$ on a finite abelian group A as

$$\Delta_A[\lambda] = \sum_{j=1}^{|A|-1} \left(\left(\frac{j}{|A|}\right)\right)\lambda^j,$$

where λ is a linear character of A, and for $x \in \mathbb{R}$,

$$((x)) = x - [x] - \frac{1}{2}\Big(1 - \delta(x)\Big), \quad \delta(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

For a positive integer k and an integer h, the Dedekind sum is defined by

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The detailed version of this paper has been submitted for publication elsewhere.

$$s(h,k) = \sum_{j=1}^{k-1} \left(\left(\frac{j}{k}\right)\right)\left(\left(\frac{hj}{k}\right)\right).$$

Yoshida has shown that if the order of λ is k, then for any integer h,

(1) $$\langle \Delta_A[\lambda], \Delta_A[\lambda^h] \rangle_A = s(h,k),$$

where $\langle\ ,\ \rangle_A$ is a usual scalar product on the space of complex valued functions on A. For the properties of the function $\Delta_A[\lambda]$ and their applications to the group theory, see [6].

The purpose of this report is to state the applications of the character theory to the properties of Dedekind sums. By using (1), Yoshida proved some formulas for Dedekind sums, especially the reciprocity formula for Dedekind sums;

(2) $$s(h,k) + s(k,h) = \frac{k^2 + h^2 + 1 - 3kh}{12kh},$$

provided that $(k,h) = 1$. In §2 we apply the methods used in [6] to the proof of the reciprocity formula for generalized Dedekind sums defined as follows(see [1]).

$$s(h,k,r) = \sum_{j=1}^{k-1} \left(\left(\frac{j}{k}\right)\right)\left(\left(\frac{hj+r}{k}\right)\right),$$

where k is a positive integer, h is an integer and r is a real number. If λ is a linear character of an abelian group A and the order of λ is k, then for any integers h and r, we have

$$(3) \qquad \langle \Delta_A[\lambda], \Delta_A[\lambda^h]\cdot\lambda^r \rangle_A = s(h,k,r)$$

([4]). In §3 we state some congruences of Dedekind sums proved by using the function Δ([5]). These results are related to Rademacher's three-term relations for Dedekind sums(§4).

§2. The reciprocity formula for generalized Dedekind sums. Let M and N be cyclic groups of order k and h, and put $A = M \times N$. Following to [6], we define the function θ on A as

$$\theta = \Delta_A[\lambda\times 1_N]\cdot\Delta_A[1_M\times\mu] - \Delta_A[\lambda\times\mu](\Delta_A[\lambda\times 1_N] + \Delta_A[1_M\times\mu]),$$

where λ and μ are generators of the linear characters of M and N respectively. Yoshida proved (2) by calculating $\langle\theta, 1_A\rangle_A$. We consider $\langle\theta, \lambda^c\times\mu^d\rangle_A$ where c and d are any integers such that $1 \le c < k$, $1 \le d < h$, and get the reciprocity formula for the generalized Dedekind sums due to Knuth([1]).

Theorem 1([1],[4]). Let k and h be positive integers. If $(k,h) = 1$, then for a real number r such that $-h < r < k$, we have

$$s(h,k,r) + s(k,h,r) = \frac{k^2 + h^2 + 1}{12kh} + \frac{[r]([r]+1-\delta(r))}{2kh}$$

$$+ \frac{1}{2}\left(\left[\frac{r}{k}\right] - \left[\frac{r}{h}\right]\right) - \frac{1}{4}(e(r,k) + e(r,h) - 1),$$

where for $z \in \mathbb{Z}$, $e(r,z) = \begin{cases} 1 & \text{if } r = 0 \text{ or } \\ & \quad r \neq 0,\ r \not\equiv 0 \pmod{z}, \\ 0 & \text{if } r \neq 0,\ r \equiv 0 \pmod{z}. \end{cases}$

Note that $s(h,k,r) = s(h,k,-r)$. We prove the theorem following to the paper [4]. If we can prove the theorem in the case where r is an integer, then we can extend the result to the general case. To prove the theorem, we calculate $\langle \theta, \lambda^c \times \mu^d \rangle_A$.

Let k' and h' be integers satisfying $kk' + hh' = 1$. From (3), we have

$$\langle \theta, \lambda^c \times \mu^d \rangle_A = s(hh', kh, hh'c+kk'd) + s(kk', kh, hh'c+kk'd)$$

$$+ \langle \Delta_A[\lambda \times 1_N] \cdot \Delta_A[1_M \times \mu], \lambda^c \times \mu^d \rangle_A .$$

By the character theory of finite groups, we can easily see that

$$\langle \Delta_A[\lambda \times 1_N] \cdot \Delta_A[1_M \times \mu], \lambda^c \times \mu^d \rangle_A = \left\{ \sum_{i=0}^{h-1} \left(\left(\frac{ik+c}{kh}\right)\right) \right\} \left\{ \sum_{j=0}^{k-1} \left(\left(\frac{jh+d}{kh}\right)\right) \right\}$$

$$= \left(\left(\frac{c}{k}\right)\right) \left(\left(\frac{d}{h}\right)\right).$$

Here we use the formula; $((n \cdot x)) = \sum_{i=0}^{n-1} \left(\left(\frac{i}{n} + x\right)\right)$ for $x \in \mathbb{R}$, $n \in \mathbb{N}$. Furthermore we have the following.

Lemma 2([4]).

$$s(hh',kk',hh'c+kk'd) = s(h,k,d-c) - \frac{1}{2}\left(\left(\frac{d-c}{k}\right)\right)\left(1 - \delta\left(\frac{d}{h}\right)\right).$$

On the other hand, following the definition of the product $<\ ,\ >_A$, we have

$$<\theta,\lambda^c\times\mu^d>_A = (k<\theta_{|M},\lambda^c>_M + h<\theta_{|N},\mu^d>_N$$

$$+ \sum_{i=1}^{k-1}\sum_{j=1}^{h-1}\theta(m^i\times n^j)\cdot\lambda(m^{ci})\cdot\mu(n^{dj}))/kh,$$

where m and n are the generators of cyclic groups M and N respectively. Note that $\theta(1_A) = 0$. From (3) we get

$$<\theta_{|M},\lambda^c>_M = <\Delta_M[\lambda],\Delta[\lambda]\cdot\lambda^c>_M = s(1,k,c).$$

Furthermore we have the following.

Lemma 3([4]).

$$s(1,k,c) = \frac{(k-1)(k-2)}{12k} + \frac{c^2}{2k} - \frac{c}{2} + \frac{1}{4}\left(1 - \delta\left(\frac{c}{k}\right)\right).$$

The key result is the following.

Lemma 4([6]). $\theta(m^i\times n^j) = -\frac{1}{4}$ for $1 \le i < k,\ 1 \le j < h.$

Furthermore we have

$$\sum_{i=1}^{k-1}\sum_{j=1}^{h-1}\lambda(m^{ci})\cdot\mu(n^{dj}) = \left(k\delta\left(\frac{c}{k}\right) - 1\right)\left(h\delta\left(\frac{d}{h}\right) - 1\right).$$

Combining the above results, we have

$$s(h,k,d-c) + s(k,h,c-d) = \frac{k^2 + h^2 + 1}{12kh} + \frac{(c-d)^2}{2kh}$$

$$- \frac{1}{2}\left(\left[\frac{d-c}{k}\right] + \left[\frac{c-d}{h}\right]\right) - \frac{1}{4}\left(\delta\left(\frac{d-c}{k}\right) + \delta\left(\frac{c-d}{h}\right) - 3\right).$$

Replacing $c-d$ with r, we obtain the required reciprocity formula in the case where r is an integer.

§3. The congruences of Dedekind sums.

Theorem 5([5]). Let p, q and r be positive integers such that $(p,qr) = 1$. Let r' be an integer satisfying $rr' \equiv 1 \pmod{p}$. Then we have

$$(4) \quad p\cdot s(qr,p) \equiv p\cdot s(qr',p) \pmod{2}.$$

If neither 2 nor 3 divide (q,r), then we have

$$(5) \quad qr\cdot s(qr,p) \equiv qr\cdot s(qr',p) + \frac{(q^2-1)(r^2-1)}{12p} \left(\mathrm{mod}\ \frac{2}{(2,qr)}\right).$$

We prove this theorem following to [5]. To prove Theorem 5, we define

$$\sigma(q,r;p) = \langle \Delta[\lambda^q] - \frac{1}{q}\cdot\Delta[\lambda], \Delta[\lambda^r] - \frac{1}{r}\cdot\Delta[\lambda] \rangle$$

where p, q and r are positive integers, and λ is a generator of the linear characters of the cyclic group A of order p. If $(p,qr) = 1$, then we have

$$(6) \quad \sigma(q,r;p) = s(qr',p) + \frac{1}{qr}\cdot s(1,p) - \frac{1}{q}\cdot s(r,p) - \frac{1}{r}\cdot s(q,p).$$

The above theorem is based on the following proposition.

Proposition 6([5]). If $(qr,p) = 1$ and 2 does not divide (q,r), then we have $qr\cdot\sigma(q,r;p) \equiv 0 \left(\bmod \frac{2}{(2,qr)}\right)$.

This result is given by the equality;

$$\sigma(q,r;p) = \frac{1}{qr}\cdot \sum_{\substack{i,j=1 \\ qj\equiv ri \pmod p}}^{p-1} \left[\frac{qj}{p}\right]\cdot\left[\frac{ri}{p}\right] - \frac{(p-1)(q-1)(r-1)}{4qr}.$$

Combining Proposition 6 with (6), we get

$$(7) \quad qr\cdot s(qr',p) + \frac{(p-1)(p-2)}{12p} \equiv q\cdot s(q,p) + r\cdot s(r,p) \quad \left(\bmod \frac{2}{(2,qr)}\right).$$

In this congruence, we can exchange r for r', because we may suppose that (q,r') is not divisible by 2. These two congruences implies (4)(see [5]).

From (2), (4) and (7), we have

$$pqr\cdot s(p,qr) + \frac{p(q-1)(r-1)}{4} \equiv pq\cdot s(p,q) + pr\cdot s(p,r) + \frac{(q^2-1)(r^2-1)}{12} \left(\mathrm{mod}\ \frac{2}{(2,qr)}\right).$$

This implies the following.

Proposition 7([5]). Under the assumptions of Proposition 6, we have

$$3qr\cdot s(p,qr) + \frac{3(q-1)(r-1)}{4} \equiv 3q\cdot s(p,q) + 3r\cdot s(p,r) \left(\mathrm{mod}\ \frac{6}{(2,qr)\cdot\varepsilon(q,r)}\right),$$

where

$$\varepsilon(q,r) = \begin{cases} 1 & \text{if } q \text{ or } r \text{ is not divisible by } 3, \\ 3 & \text{otherwise.} \end{cases}$$

Now combining this with (2) and (7), we can conclude (5). We note that the above result is concerned with the generalized Gauss' lemma of the Jacobi symbol([5]).

§4. Three-term relations for Dedekind sums. Let p, q and r be pairwise coprime positive integers. Let p', q' and r' be integers such that $pp' \equiv 1 \pmod{qr}$, $qq' \equiv 1 \pmod{rp}$

and $rr' \equiv 1 \pmod{pq}$. Then the Rademacher's three-term relation, which is a generalization of (2), is the following.

Theorem 8([3]).

$$s(qr',p) + s(rp',q) + s(pq',r) = \frac{1}{12}\left(\frac{p}{qr} + \frac{q}{rp} + \frac{r}{pq}\right) - \frac{1}{4}.$$

As corollary to this theorem, we have the following three-term relation for $\sigma(q,r;p)$.

Corollary 9([5]).

$$\sigma(q,r;p) + \sigma(r,p;q) + \sigma(p,q;r) = -\frac{(p-1)(q-1)(r-1)}{4pqr}.$$

There exists one more three-term relation for Dedekind sums due to Rademacher, as follows.

Theorem 10([2]).

$$\left(s(qr,p) - \frac{qr}{12p}\right) + \left(s(rp,q) - \frac{rp}{12q}\right) + \left(s(pq,r) - \frac{pq}{12r}\right)$$

$$\equiv -\frac{1}{4} - \frac{pqr}{12} + \frac{1}{12pqr} \pmod{2}.$$

This theorem is related to the number of lattice points in a tetrahedron([2]).

Now, by Theorem 5, we can see that Theorem 10 is deduced from Theorem 8.

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ON TRACE FORMS OF ALGEBRAS

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For an algebra A over a field F, we denote by Tr_A the trace of the regular representation of A. A bilinear form t_A on A, called the <u>trace form</u> of A, is defined by $t_A(x, y) = Tr_A(xy)$. t_A is symmetric, and is non-degenerate if A is separable. When an F-linear base $\{v_1,\ldots,v_n\}$ of A is fixed, t_A is represented by the matrix $(t_A(v_i, v_j)) = (Tr_A(v_i v_j))$. By our definition, the (i, j)-entry of this matrix is given by $\sum_m c^{ij}_{mm}$ where $(v_i v_j)v_m = \sum_k c^{ij}_{km} v_k$.

For F-algebras A and B, the trace form $t_{A\otimes B}$ of tensor product $A\otimes B$ is isometric to the tensor product of bilinear forms $t_A \otimes t_B$. So, if A is a full matrix algebra $M_n(D)$ over D, t_A is isometric to $t_{M_n(F)} \otimes t_D$.

When two forms f and f' over F give rise to the same element in the Witt ring $W(F)$, we use the notation $f \sim f'$; i.e., $f \sim f'$ if and only if $f \perp H \cong f' \perp H'$ for some hyperbolic forms H and H'.

<u>Proposition 1.</u> There is an isometry

$$t_{M_n(F)} \cong \langle \overbrace{n, n, \ldots\ldots, n}^{n\text{-times}} \rangle \perp \text{(hyperbolic)}$$

The detailed version of this note will be submitted for publication elsewhere.

i.e., $t_{M_n(F)} \sim \langle n, n, \ldots\ldots, n \rangle$

hence, $t_{M_n(D)} \sim nt_D \perp nt_D \perp \cdots \perp nt_D$

We denote by $Q = \left(\frac{-1,\ -1}{F}\right)$ the quaternion algebra over F, that is, $Q = F \oplus Fi \oplus Fj \oplus Fij$ ($i^2 = j^2 = -1$, $ij = -ji$). By the standard base $\{ 1, i, j, ij \}$ we get a diagonalization

$$t_Q \cong \langle 4, -4, -4, -4 \rangle \sim \langle -4, -4 \rangle.$$

Hence, for $A = M_n(Q)$ we get

$$t_A \sim \langle \overbrace{-4n, -4n, \ldots, -4n}^{2n\text{-times}} \rangle$$

by Proposition 1.

Let F be a formally real field and P be an ordering of F. For a bilinear form f over F, the Sylvester signature of f according to P is denoted by $\mathrm{sgn}_P f$. If F is real closed, an Azumaya algebra over F is either isomorphic to $M_n(F)$ or to $M_n(Q)$ for some n. Since the Sylvester signature is invariant under any coefficient field extention, we get,

<u>Theorem 2.</u> Let F and P be as above. The real closure of F according to P is denoted by $\overline{F}_P$. For an Azumaya algebra A with $\dim_F A = n^2$, $\mathrm{sgn}_P t_A$ is equal to n or $-n$. And $\mathrm{sgn}_P t_A = n$ if and only if A is split by $\overline{F}_P$. The case $\mathrm{sgn}_P t_A = -n$ (in this case $\overline{F}_P$ does not split A) occurs only when n is even.

<u>Corollary.</u> For any ordering P, an odd dimensional Azumaya algebra A is split by $\overline{F}_P$. In this case we have $\mathrm{sgn}_P t_A = \sqrt{\dim_F A}$.

Here we shall show some computations of trace forms of Azumaya algebras which are written as crossed products. Let L/F be a Galois extension of degree n with Galois group G and f be a 2-cocycle in $Z^2(G, L^*)$. The crossed product

$A = \triangle(f, L/F)$ is defined by the usual manner. We suppose that 2-cocycle f is normalized. Let $\{w_1, \ldots, w_n\}$ be a linear base of L over F. For elements g, h in G, we denote by $X_{g,h}$ an $n \times n$-matrix whose (i, j)-entry is $Tr_A(f(g, h)w_i w_j^g)$. If $g = h = e$ (the unit of G), $X_{e,e}$ is equal to the matrix $(Tr_A(w_i w_j))$ since f is normalized. But $Tr_A = nTr_L$ holds on L, so the matrix $X_{e,e}$ represents the form nt_L.

In the following proposition, $\langle X \rangle$ denotes the bilinear form represented by the matrix X.

<u>Proposition 3.</u> For a crossed product $A = \triangle(f, L/F)$,

$$t_A \sim \perp_{g^2=e} \langle X_{g,g} \rangle$$

$$\sim nt_L \perp (\perp_{g:\text{order } 2} \langle X_{g,g} \rangle)$$

<u>Corollary.</u>

$$t_A \sim nt_L$$

if $|G| = (L : F) = n$ is odd. So (if F is formally real) $sgn_P t_L$ is necessarily equal to n.

The remainder problem is in the case n is even. For this sake we shall restrict our situation to the cyclic case.

Let L/F be a cyclic extension of even degree $n = 2m$ with cyclic Galois group G, and g be a generator of G.

An element $a \in F$, $a \neq 0$, defines a cyclic algebra $A = \triangle(a, L/F)$; that is, $A = L \oplus Lu \oplus \cdots \oplus Lu^{n-1}$, $u^n = a$, $us = s^g u$ ($s \in L$). We put $h = g^m = g^{\frac{n}{2}}$. h is the unique element of order 2 in G. So the fixed field M of h is the unique mid-field of $[L : M] = 2$. We choose and fix an element $b \in L$ such that $L = M(\sqrt{b})$.

To state the following theorem, we have to introduce one more notation; for a commutative F-algebra C and for $b \in C$, $t_C b$ denotes the form defined by $(t_C b)(x, y) = Tr_C(bxy)$.

$t_C b$ is also symmetric and bilinear.

<u>Theorem 4.</u> Let L/F, a, M, b be as above. For the trace form t_A of a cyclic algebra $A = \triangle(a, L/F)$, we have

$$t_A \sim 2n(t_M \perp at_M \perp t_M b \perp -at_M b)$$

Using this expression of t_A, we get a theorem about the "real splitting" of cyclic algebras.

<u>Theorem 5.</u> Let L be a cyclic extension of even degree $n = 2m$ of a formally real field F with an ordering P. For a cyclic algebra $A = \triangle(a, L/F)$, $a \in F^*$,

1) if $\mathrm{sgn}_P t_L = n$, A is split by $\overline{F}_P$,

2) if $\mathrm{sgn}_P t_L \neq n$ ($\mathrm{sgn}_P t_L = 0$ necessarily holds in this case), A is split by $\overline{F}_P$ when $a > 0$ and not split when $a < 0$.

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REPRESENTATIONS OF HECKE ALGEBRAS AND SYMMETRIC SPACES

———— IN CASE OF ALTERNATING MATRICES

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§1. Introduction

In the present note we report some results on the structure of certain function spaces on the set of non-degenerate alternating matrices over $\mathbb{Q}$ or p-adic number fields, as modules of the Hecke algebra of $GL(2n)$. A similar problem has been recently solved for alternating matrices over $\mathbb{F}_q$ by Bannai-Kawanaka-Song[1]. As an introduction, we briefly recall their result over $\mathbb{F}_q$.

Let G be a finite group and we identify the group algebra $\mathbb{C}G$ with the set of $\mathbb{C}$-valued functions on G, with the element $\Sigma a_x x \in \mathbb{C}G$ corresponding to the function $f: G \longrightarrow \mathbb{C}$ defined by $f(x) = a_x$, $x \in G, a_x \in \mathbb{C}$. Then the product in $\mathbb{C}G$ carries over to the convolution product:

$$(f*g)(x) = \sum_{y \in G} f(y)g(y^{-1}x), \quad x \in G.$$

The final version of this paper will be submitted for publication elsewhere.

For a subgroup K of G, we identify the subalgebra

$$\mathcal{H}(G,K) = e_K \mathbb{C}G e_K, \quad e_K = |K|^{-1}\sum_{y\in K} y$$

with the set $\{f\in\mathbb{C}G \mid f(yxy') = f(x),\ y,y'\in K,\ x\in G\}$.
It is easily seen that $\mathcal{H}(G,K)$ becomes a subalgebra of $\mathbb{C}G$, and this is called the Hecke algebra of G with respect to K. It is known[2,(11.25)] that the restriction of representations of G to those of $\mathcal{H}(G,K)$ gives a bijective correspondence between the set

$$\{\xi\in \mathrm{Irr}(G) \mid \langle \xi, \mathrm{Ind}_K^G(1_K)\rangle \neq 0\}$$

and the set $\mathrm{Irr}(\mathcal{H}(G,K))$, and $\mathcal{H}(G,K)$ is commutative if and only if $\mathrm{Ind}_K^G(1_K)$ is multiplicity-free.

Let $G = GL(2n,\mathbb{F}_q)$ and $K = Sp(2n,\mathbb{F}_q)$. Then $\mathcal{H}(G,K)$ is commutative. According to the result of [1], every irreducible character of $\mathcal{H}(G,K)$ can be obtained by replacing q by q^2 in the values of an irreducible character of G whose degree is prime to q. In this way, they have determined irreducible characters α_i, $1\le i\le m(=\dim_{\mathbb{C}}\mathcal{H}(G,K))$, for which

$$he_i = \alpha_i(h)e_i, \quad 1\le i\le m, \quad h\in\mathcal{H}(G,K),$$

where $\mathcal{H}(G,K) = \sum_{i=1}^{m}{}^{\oplus}\, \mathbb{C}e_i$ and $e_ie_j = \delta_{ij}e_i$.

In the following, we will consider a similar problem for algebraic groups defined over $\mathbb{Q}$ or p-adic number fields. Though the problem can be formulated in a more general setting of symmetric spaces, we treat here only the case of alternating matrices to make the exposition simpler.

§2. Hecke algebras over p-adic number field

Let k be a p-adic number field with the ring of integers $\mathcal{O}$ and the maximal ideal $\mathfrak{p}=(\pi)$. Denote by q the cardinality of $\mathcal{O}/\mathfrak{p}$. Let $G = GL(2n,k)$ and $H = Sp(2n,k)$. Then the symmetric space $X = G/H$ can be identified with the set $\{x \in GL(2n,k) : {}^t x = -x\}$, and G acts on X by $g \cdot x = g x {}^t g$, $g \in G$, $x \in X$. Let $K = GL(2n,\mathcal{O})$ and let

$$\mathcal{H}(G,K) = \{f : G \longrightarrow \mathbb{C} \mid \begin{array}{l} f \text{ is compactly supported} \\ f(ygy') = f(g),\ y,y' \in K,\ g \in G \end{array}\}.$$

Then $\mathcal{H}(G,K)$ is a commutative algebra (called the Hecke algebra of G with respect to K) by the convolution product

$$(f_1 * f_2)(x) = \int_G f_1(g) f_2(g^{-1}x)\,dg,$$

where dg is a normalized Haar measure on G. The algebra $\mathcal{H}(G,K)$ acts on the space

$$\mathcal{C}^\infty(K\backslash X) = \{\varphi : X \longrightarrow \mathbb{C} \mid \varphi(k \cdot x) = \varphi(x),\ x \in X,\ k \in K\}$$

by $(f * \varphi)(x) = \int_G f(g)\varphi(g^{-1} \cdot x)\,dg$.

We want to determine $\mathcal{H}(G,K)$-common eigenfunctions in $\mathcal{C}^\infty(K\backslash X)$, which are called spherical functions on X, i.e. functions φ in $\mathcal{C}^\infty(K\backslash X)$ satisfying $f * \varphi = \omega(f)\varphi$ with some algebra homomorphism $\omega : \mathcal{H}(G,K) \longrightarrow \mathbb{C}$ (compare with the result of [1] explained in §1). To describe our result, it needs to prepare some notations.

Let $\Lambda_n = \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \cdots \geq \lambda_n\}$. For $\lambda \in \Lambda_n$, put

$$\pi^\lambda = \begin{pmatrix} 0 & \pi^{\lambda_1} \\ -\pi^{\lambda_1} & 0 \end{pmatrix} \perp \cdots \perp \begin{pmatrix} 0 & \pi^{\lambda_n} \\ -\pi^{\lambda_n} & 0 \end{pmatrix} \in X,$$

then $\{\pi^\lambda \mid \lambda \in \Lambda_n\}$ is a complete set of representatives of $K\backslash X$.

For $\lambda\in\Lambda_n$, put

$$P_\lambda(x_1,\dots,x_n;t) = \sum_{\sigma\in\mathfrak{S}_n/\mathfrak{S}_n^\lambda} x_{\sigma(1)}^{\lambda_1}\cdots x_{\sigma(n)}^{\lambda_n} \prod_{\lambda_i>\lambda_j} \frac{x_{\sigma(i)}-tx_{\sigma(j)}}{x_{\sigma(i)}-x_{\sigma(j)}},$$

where $\mathfrak{S}_n$ denotes the symmetric group on n letters and $\mathfrak{S}_n^\lambda = \{\sigma\in\mathfrak{S}_n \mid \sigma(\lambda) = \lambda\}$ (Hall-Littlewood polynomial);

$$w_\lambda(t) = \prod_i w_{\ell_i(\lambda)}(t),$$

where $\ell_i(\lambda) = \#\{j \mid \lambda_j = i\}$ and $w_\ell(t) = \prod_{i=1}^{\ell}(1-t^i)$.

Define $\Psi_z(x)$ in $\mathscr{C}^\infty(K\backslash X)$ by

$$\Psi_z(\pi^\lambda) = q^{-\langle\rho,\lambda\rangle} \frac{w_\lambda(q^{-2})}{w_n(q^{-2})} P_\lambda(q^{z_1},\dots,q^{z_n};q^{-2}),$$

where π^λ is as above, $\rho=(n-1,n-3,\dots,1-n)$ and $\langle\rho,\lambda\rangle = \sum_{i=1}^{n}\rho_i\lambda_i$.

Theorem[3,Th.2,Th.3]. *Every $\mathscr{H}(G,K)$-common eigenfunction in $\mathscr{C}^\infty(K\backslash X)$ is a constant multiple of $\Psi_z(x)$ for some* $z\in\mathbb{C}^n/\mathfrak{S}_n$.

Remark. Recall some facts on the zonal spherical functions of $G_0 = GL(n,k)$ with respect to $K_0 = GL(n,\mathcal{O})$ ([4,Chap.5] or [5,§8]).

A zonal spherical function $\omega(g)$ is characterized by

1) $\omega: G_0 \longrightarrow \mathbb{C}$ and $\omega(kgk') = \omega(g)$, $g\in G_0$, $k,k'\in K_0$,
2) ω is an $\mathscr{H}(G_0,K_0)$-common eigenfunction,
3) $\omega(1)=1$.

All zonal spherical functions are completely determined as follows: $\omega = \omega_z$, for some $z\in\mathbb{C}^n/\mathfrak{S}_n$, where

$$\omega_z\left(\begin{pmatrix}\pi^{\lambda_1} & & 0\\ & \ddots & \\ 0 & & \pi^{\lambda_n}\end{pmatrix}\right) = q^{-(1/2)\langle\rho,\lambda\rangle} \frac{w_\lambda(q^{-1})}{w_n(q^{-1})} P_\lambda(q^{z_1},\dots,q^{z_n};q^{-1}).$$

The relation between $\mathcal{H}(G,K)$-common eigenfunctions in $\mathcal{C}^{\infty}(K\backslash X)$ and zonal spherical functions of G_0 with respect to K_0 is analogous to the case over $\mathbb{F}_q$ explained in §1.

Let $\mathcal{S}(K\backslash X)=\{\varphi\in\mathcal{C}^{\infty}(K\backslash X) \mid \varphi$ is compactly supported$\}$, this is an $\mathcal{H}(G,K)$-submodule of $\mathcal{C}^{\infty}(K\backslash X)$. We define the spherical transform on $\mathcal{S}(K\backslash X)$ as follows: for $\varphi\in\mathcal{S}(K\backslash X)$, let

$$\hat{\varphi}(z) = \int_X \varphi(x)\Psi_z(x^{-1})dx \quad (\in\mathbb{C}(q^{z_1},\ldots,q^{z_n})),$$

where dx is a normalized G-invariant measure on X. The Satake transform $\hat{f}(\nu)$ for $f\in\mathcal{H}(G,K)$ is given by

$$\hat{f}(\nu) = \int_G f(g)\prod_{i=1}^{2n}|a_i(g)|_p^{\nu_i-(n-i+1/2)}dg$$
$$(\in\mathbb{C}[q^{\pm\nu_1},\ldots,q^{\pm\nu_{2n}}]^{\mathfrak{S}_{2n}}),$$

where $g=ka$, $k\in K$ and $a=\begin{pmatrix} a_1(g) & & * \\ & \ddots & \\ 0 & & a_{2n}(g)\end{pmatrix}$. Define $\tilde{f}(z)$ for $f\in\mathcal{H}(G,K)$ by

$$\tilde{f}(z) = \hat{f}(z_1+\tfrac{1}{2},z_1-\tfrac{1}{2},\ldots,z_n+\tfrac{1}{2},z_n-\tfrac{1}{2}) \quad (\in\mathbb{C}[q^{\pm z_1},\ldots,q^{\pm z_n}]^{\mathfrak{S}_n}),$$

and give an $\mathcal{H}(G,K)$-module structure on $\mathbb{C}[q^{\pm z_1},\ldots,q^{\pm z_n}]^{\mathfrak{S}_n}$ through the epimorphism $f \longmapsto \tilde{f}(z)$.

Theorem [3,Th.4]. *The spherical transform* $\varphi\longmapsto\hat{\varphi}(z)$ *induces an* $\mathcal{H}(G,K)$*-module isomorphism*

$$\mathcal{S}(K\backslash X) \simeq \mathbb{C}[q^{\pm z_1},\ldots,q^{\pm z_n}]^{\mathfrak{S}_n}.$$

§3. Hecke algebra over $\mathbb{Q}$

Let $G = GL(2n,\mathbb{Q})$, $H = Sp(2n,\mathbb{Q})$, $\Gamma = GL(2n,\mathbb{Z})$ and

$$P = \left\{ \begin{pmatrix} g_1 & & 0 \\ & \ddots & \\ * & & g_n \end{pmatrix} \in G \mid g_i \in GL(2,\mathbb{Q}), 1 \le i \le n \right\}.$$

Denote by $\tilde{G}$ the completion of G with respect to $\{\Gamma_N \mid N \in \mathbb{N}\}$, where $\Gamma_N = \{g \in \Gamma \mid g \equiv 1 (\mathrm{mod} N)\}$, and for a subgroup S denote by $\bar{S}$ the closure of S in $\tilde{G}$. As in the local case, the symmetric space $X = G/H$ can be identified with the set $\{x \in GL(2n,\mathbb{Q}) \mid {}^t x = -x\}$ and G acts on X by $g \cdot x = gx{}^tg$, $g \in G$, $x \in X$. Let $\Omega = \{x \in X \mid \prod_{i=1}^{n} \mathrm{Pf}_i(x) \neq 0 \}$, where $\mathrm{Pf}_i(x)$ is the Pfaffian of the upper left $2i$ block of x. Let $\tilde{X} = \tilde{G}/\bar{H}$ and $\tilde{\Omega} = \bar{P}/\overline{P \cap H}$.

As in the local case we define

$$\mathcal{H}(G,\Gamma) = \left\{ f : G \longrightarrow \mathbb{C} \;\middle|\; \begin{array}{l} f(\gamma g \gamma') = f(g),\ g \in G,\ \gamma,\gamma' \in \Gamma \\ f = 0 \text{ outside a finite union of} \\ \text{double } \Gamma\text{-cosets} \end{array} \right\},$$

$$\mathcal{C}^\infty(\Gamma \backslash X) = \{\varphi : X \longrightarrow \mathbb{C} \mid \varphi(\gamma \cdot x) = \varphi(x),\ x \in X,\ \gamma \in \Gamma \},$$

$$\mathcal{S}(\Gamma \backslash X) = \left\{ \varphi \in \mathcal{C}^\infty(K \backslash X) \;\middle|\; \begin{array}{l} f = 0 \text{ outside a finite union of} \\ \Gamma\text{-orbits} \end{array} \right\},$$

and an $\mathcal{H}(G,\Gamma)$-action on $\mathcal{C}^\infty(\Gamma \backslash X)$ and $\mathcal{S}(\Gamma \backslash X)$ is given as follows: for the characteristic function f_g of $\Gamma g \Gamma$ and $\varphi \in \mathcal{C}^\infty(\Gamma \backslash X)$,

$$(f_g * \varphi)(x) = \sum_{i=1}^{t} \varphi(g_i^{-1} \cdot x), \quad \text{where } \Gamma g \Gamma = \bigcup_{i=1}^{t} g_i \Gamma \text{ (disjoint)}.$$

The function spaces above can be naturally identified with $\mathcal{H}(\tilde{G},\bar{\Gamma})$, $\mathcal{C}^\infty(\bar{\Gamma} \backslash \tilde{X})$ and $\mathcal{S}(\bar{\Gamma} \backslash \tilde{X})$, respectively.

We define the Eisenstein series on X as follows: for $x \in X$ and $s = (s_1, \ldots, s_n) \in \mathbb{C}^n$, put

$$(*) \qquad E(x;s) = \sum_{y \in \Gamma \cap P \backslash \Omega \cap (\Gamma \cdot x)} \mu(x) \prod_{i=1}^{n} |\mathrm{Pf}_i(x)|^{-s_i},$$

where $|\ |$ is the absolute value and $\mu(x) = vol((\overline{\Gamma \cap P}) \cdot x)$

with respect to a suitable $\bar{P}$-invariant measure on $\hat{\Omega}$. The right hand side of (*) is convergent if $\mathrm{Re}(s_i)>0$, $1\le i\le n-1$. Transforming the variables $s_1,\ldots,s_n$ into $z_1,\ldots,z_n$ by

$$\begin{cases} s_i = -z_i + z_{i+1} - 2, & 1\le i\le n-1 \\ s_n = -z_n + n - 1 \end{cases},$$

we obtain the following explicit formula

$$E(x;z) = E(J;z)\cdot \prod_p \Psi_z^{(p)}(x), \quad E(J;z) = c\cdot \prod_{1\le i<j\le n} \frac{\zeta(z_j-z_i-1)}{\zeta(z_j-z_i+1)},$$

where $J = \pi^{(0,\ldots,0)}$, p runs over all prime numbers, $\Psi_z^{(p)}(x)$ is $\Psi_z(x)$ in §2 for $k = \mathbb{Q}_p$, c is a positive constant and $\zeta(z)$ is the Riemann zeta function. Hence we may regard $E(x;z)$ as an element of $\mathcal{E}^\infty(\Gamma\backslash X)$, further we can prove

Theorem. *Every $\mathcal{H}(G,K)$-common eigenfunction is a constant multiple of $E(x,z)$ for some $z\in\mathbb{C}^n/\mathfrak{S}_n$.*

Note that $E(x,z)$ has an Euler product expansion. As we shall see in the theorem below, it reflects the fact that $\mathcal{S}(\Gamma\backslash X)$ has a restricted tensor product over prime numbers.

Now we define the Fourier-Eisenstein transform (an analogue of the spherical transform) on $\mathcal{S}(\Gamma\backslash X)$ as follows: for $\varphi\in\mathcal{S}(\Gamma\backslash X)$, let

$$\hat{\varphi}(z) = \int_{\hat{\Omega}} \varphi(x)\frac{E(x^{-1};z)}{E(J;z)}\, d_{\hat{\Omega}}(x) \quad \left(\in \bigotimes_p{}' \mathbb{C}(q^{z_1},\ldots,q^{z_n})\right),$$

where $d_{\hat{\Omega}}(x)$ is a $\bar{P}$-invariant measure on $\hat{\Omega}$ and $\bigotimes_p{}'$ means the restricted tensor product over prime numbers p. On the other hand, we obtain

$$\mathcal{H}(G,\Gamma) \simeq \mathcal{H}(\tilde{G},\bar{\Gamma}) \simeq \bigotimes_p{}' \mathcal{H}(GL(2n,\mathbb{Q}_p),GL(2n,\mathbb{Z}_p))$$

$$\simeq \bigotimes_p{}' \mathbb{C}[p^{\pm\nu_1},\ldots,p^{\pm\nu_{2n}}]^{\mathfrak{S}_{2n}}, \text{ by the Satake transform}$$

$$\longrightarrow \bigotimes_p{}' \mathbb{C}[p^{\pm z_1},\ldots,p^{\pm z_n}]^{\mathfrak{S}_n}, \text{ cf. §2,}$$

and through this epimorphism, we regard $\bigotimes_p{}' \mathbb{C}[p^{\pm z_1},\ldots,p^{\pm z_n}]^{\mathfrak{S}_n}$ as an $\mathcal{H}(G,\Gamma)$-module.

Theorem. *The Fourier-Eisenstein transform* $\varphi \longmapsto \hat{\varphi}(z)$ *induces an* $\mathcal{H}(G,\Gamma)$*-module isomorphism*

$$\mathcal{G}(\Gamma\backslash X) \simeq \bigotimes_p{}' \mathbb{C}[p^{\pm z_1},\ldots,p^{\pm z_n}]^{\mathfrak{S}_n}.$$

Remark. This investigation is an analogy of the theory of spherical functions of symmetric spaces over $\mathbb{R}$. It is interesting to study representations of Hecke algebras of reductive algebraic groups over $\mathbb{Q}$ or p-adic number fields employing suitable function spaces on symmetric spaces as representation spaces. In general, Eisenstein series $E(x;z)$ has no Euler product, e.g. $X = GL(n,\mathbb{Q})/O(n,\mathbb{Q})$. A general theory for Eisenstein series and its Euler product will be developed in a forthcoming paper.

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ON INTERIOR G-ALGEBRAS WITH THE TRIVIAL DEFECT GROUP

Tadashi IKEDA

1. Introduction.

Let G be a finite group, p a prime number, k an algebraically closed field of characteristic p and $k[G]$ a group algebra of G over k. Whenever $B = k[G]b$ is a block of $k[G]$ with trivial defect group (b is a central primitive idempotent of $k[G]$), then the block B has unique simple $k[G]$-module S which also is a projective indecomposable $k[G]$-module and the block algebra B is isomorphic to the full matrix algebra over k with degree $\dim_k S$. See [1] p.103 Corollary 6. The purpose of this paper is to extend this fact for interior G-algebras using $k[G\times G]$-modules.

2. General properties of interior G-algebras.

In this paper, a k-algebra is a k-finite dimensional k-algebra with the unit element 1_A and a module over a k-algebra is a finite k-dimensional left module. For a k-algebra A the Jacobson radical is denoted by $J(A)$. Whenever H is a subgroup of G, the set $[G/H]$ is a complete set of representatives of the left cosets G/H. For $k[H]$-module W and $k[G]$-module V, we denote the induced module by $\mathrm{Ind}_H{}^G(W)$ and the restricted module $\mathrm{Res}^G{}_H(V)$. The k-subspace V^G

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is $\{ v \in V : gv = v$ for any element $g \in G \}$.

Let A be a k-algebra. An interior G-algebra is the pair (A, ρ) where ρ is a unitary k-algebra homomorphism of $k[G]$ to A. Then the k-algebra A is a G-algebra by the action $ga = \rho(g)a\rho(g^{-1})$,where $g \in G$ and $a \in A$. Also the k-algebra A is a $k[G\times G]$-module by the action $(g,h)a = \rho(g)a\rho(h^{-1})$,where $(g,h) \in G\times G$ and $a \in A$. Whenever H is a subgroup of G, the subalgebra A^H is $\{ a \in A : ha = a$ for any element $h \in H \}$ and the k-linear map Tr_H^G of A^H to A^G is defined by $Tr_H^G(a) = \Sigma_{g \in [G/H]} ga$ and A_H^G is the image $Tr_H^G(A^H)$. Then A_H^G is a two-sided ideal of A^G. An interior G-algebra (A, ρ) is indecomposable if the subalgebra A^G is local. In this case, the minimal subgroup H of G satisfying $A^G = A_H^G$ is called a defect group of (A, ρ). The defect group is a p-subgroup of G and determined under G-conjugate. See [4] p.138. Whenever B is a block of $k[G]$ and $\pi : k[G] \longrightarrow B$ is the projection $x \longmapsto xb$, where $x \in k[G]$, then the pair (B, π) is an indecomposable G-algebra. The defect group of interior G-algebra (B, π) equals the defect group in Brauer's sense. Let V be a k[G]-module, $End_k(V)$ the k-endomorphism ring and ρ_V the representation of $k[G]$. Then the pair $(End_k(V), \rho_V)$ is an interior G-algebra. By Higamann's criteria for relative projectivity, the k[G]-module V is H-projective if and only if $End_k(V)_H^G = End_k(V)^G$. See [2] Ch.2 Th.3.8. In particular, if V is an indecomposable k[G]-module, the interior G-algebra $(End_k(V), \rho_V)$ is an indecomposable interior G-algebra and the defect group of $(End_k(V), \rho_V)$ equals the vertex $vtx_G(V)$ of V. Whenever (A, ρ) is an indecomposable interior G-algebra and W is an A-module, then W becomes k[G]-module through the k-algebra homomorphism ρ.

<u>Definition</u>. Let (A, ρ) be an indecomposable interior G-algebra and V a k[G]-module. Then V belongs to (A, ρ) if there exists an indecomposable A-module W such that the k[G]-module V is a direct summand of the k[G]-module W.

Lemma 1. Let (A, ρ) be an indecomposable interior G-algebra and V a k[G]-module belonging to (A, ρ). Assume that the defect group of (A, ρ) is D. Then the k[G]-module V is D-projective.

Proof. Since V belongs to (A, ρ) there exists an indecomposable A-module W such that the k[G]-module V is a direct summand of the k[G]-module W. Let ϕ be the representation of A to $End_k(W)$ introduced from the A-module W and $\tau = \phi \cdot \rho$. Then the interior G-algebra $(End_k(W), \tau)$ equals the interior G-algebra introduced from the k[G]-module W. Then it is easily checked that $\phi(A_H{}^G) \subset End_k(W)_H{}^G$ for a subgroup H of G. By definition of defect groups and Higmann's criteria for relative projectivity, the k[G]-module W is D-projective. Since the k[G]-module V is a direct summand of the k[G]-module W the k[G]-module V is D-projective and proved lemma.

Definition. Whenever (A, ρ) and (A', ρ') are interior G-algebras, (A, ρ) and (A', ρ') are isomorphic if there exists a k-algebra isomorphism ϕ such that $\rho' = \phi \cdot \rho$.

3. Interior G-algebras with the trivial defect group.

In this section, we shall characterize indecomposable interior G-algebras with the trivial defect group <1> using k[G]-modules belonging to the interior G-algebras.

For interior G-algebra (A, ρ) and the factor algebra $A/J(A)$ we set $\pi : A \longrightarrow A/J(A)$ the canonical homomorphism and $\tau = \pi \cdot \rho$. Then the pair $(A/J(A), \tau)$ is an interior G-algebra. If the number of isomorphism classes of simple A-modules is more than one, the interior G-algebra $(A/J(A), \tau)$ is not indecomposable.

Our purpose is to prove the following theorem.

Theorem 2. Let (A, ρ) be an indecomposable interior G-algebra. Then the following statements are equivalent.

(1). The interior G-algebra (A, ρ) has the trivial defect group $\langle 1 \rangle$.

(2). Any $k[G]$-module belonging to (A, ρ) is projective $k[G]$-module.

(3). There exists a projective indecomposable $k[G]$-module U such that the interior G-algebra $(End_k(U), \rho_U)$ introduced from the $k[G]$-module U is isomorphic to the interior G-algebra $(A/J(A), \tau)$. In particular, the interior G-algebra $(A/J(A), \tau)$ is indecomposable interior G-algebra.

Proof. (1)$\Longrightarrow$(2). Obvious from lemma 1.

(2)$\Longrightarrow$(3). Let $\Delta G = \{ (g,g) \in G \times G : g \in G \}$. Since $(A/J(A), \tau)$ is interior G-algebra we have a $k[G \times G]$-module $A/J(A)$. Whenever $S_1, S_2, \cdots, S_r$ are all simple A-modules and $S = S_1 \oplus S_2 \oplus \cdots \oplus S_r$, then the Wedderburn's structure theorem implies that the interior G-algebra $(A/J(A), \tau)$ is isomorphic to the interior G-algebra $(End_k(S), \rho_S)$. Therefore we have $Res^G_{\Delta G}(A/J(A)) \simeq Res^G_{\Delta G}(End_k(S)) \simeq S \otimes_k S^*$, S^* is the contragradient $k[G]$-module. By assumption, the $k[G]$-module S is projective and $S \otimes_k S^*$ is projective. Thus the restriction $Res^G_{\Delta G}(A/J(A))$ is projective $k[\Delta G]$-module.

Since the projection $\pi: A \longrightarrow A/J(A)$ is a surjective homomorphism as $k[G \times G]$-modules and $J(A)$ is a $k[G \times G]$-submodule of A, the projectivity of $Res^G_{\Delta G}(A/J(A))$ implies that

$$Res^G_{\Delta G}(A) = Res^G_{\Delta G}(A/J(A)) \oplus Res^G_{\Delta G}(J(A))$$

, as $k[\Delta G]$-module. This equation implies that

$$A^G = (A/J(A))^G \oplus (J(A))^G$$

, as k-space, and we have

$$\dim_k A^G = \dim_k (A/J(A))^G + \dim_k (J(A))^G.$$

(*) Let $\pi|A^G: A^G \longrightarrow (A/J(A))^G$ be the restriction map of π to the subalgebra A^G. Because that the kernel of $\ker(\pi|A^G)$ equals $(J(A))^G$, the factor k-space $A^G/(J(A))^G$ is embedded into the k-space $(A/J(A))^G$ by $a + J(A)^G \longmapsto a + J(A)$, where

$a \in A^G$.But the equation (*) implies that this embedding is isomorphism. In particular, the homomorphism $\tau|A^G$ is surjective. Since the k-algebra A^G is local the factor ring $A^G/(J(A))^G$ is local, and therefore the k-algebra $(A/J(A))^G$ is local. Since the interior G-algebra $(A/J(A),\tau)$ is isomorphic to the interior G-algebra $(End_k(S),\rho_S)$, the locality of $(A/J(A))^G$ implies that the k-algebra A has unique simple A-module U. Of course, the k[G]-module U is projective k[G]-module and $End_k(U)^G = End_{k[G]}(U)$ is a local ring. Thus there exists a projective indecomposable k[G]-module U such that the interior G-algebra $(End_k(U),\rho_U)$ introduced from the k[G]-module U is isomorphic to the interior G-algebra $(A/J(A),\tau)$ and we have proved (2)$\Longrightarrow$ (3).

(3)$\Longrightarrow$ (1). By assumption, there exists a projective indecomposable k[G]-module U such that the interior G-algebra $(End_k(U),\rho_U)$ introduced from the k[G]-module U is isomorphic to the interior G-algebra $(A/J(A),\tau)$. So we have $(A/J(A))_{\langle 1\rangle}{}^G = (A/J(A))^G$. And there exists an element a of A such that $1_A + J(A) = Tr_{\langle 1\rangle}{}^G(a) + J(A)$. This implies that $1_A - Tr_{\langle 1\rangle}{}^G(a) \in J(A)$. But 1_A and $Tr_{\langle 1\rangle}{}^G(a) \in A^G$, and so we have $1_A - Tr_{\langle 1\rangle}{}^G(a) \in J(A)^G$. Thus $1_A \in A_{\langle 1\rangle}{}^G + J(A)^G$. Because A^G is a local ring and the k-subspaces $A_{\langle 1\rangle}{}^G$ and $J(A)^G$ are two sided ideals of A^G, the Rosenberg's lemma (see [3] p.109) implies that $1_A \in A_{\langle 1\rangle}{}^G$ or $1_A \in J(A)^G$. But since $J(A)^G$ is a nilpotent ideal we have $1_A \in A_{\langle 1\rangle}{}^G$ and so the defect group of the interior G-algebra (A,ρ) is trivial.

<u>Corollary</u> 3. There exists infinitely many isomorphism classes of indecomposable interior G-algebras with the trivial defect group.

Proof. Under the notation of theorem 2, by the condition of (3), the interior G-algebra $(End_k(U),\rho_U)$ has the trivial defect group. So there exists an indecomposable interior G-algebra (A,ρ) such that the defect group of (A,ρ) is the trivial defect group. We shall construct a new indecomposable interior G-algebra with the trivial defect group using (A, ρ) and (A, A)-bimodule M. Let $A_M = A \oplus M$ as k-space. Define the product on A_M by

$$(a, m)(a', m') = (aa', am' + ma')$$

, where $a, a' \in A$ and $m, m' \in M$. Then the k-space A_M is a k-algebra with the unit element $(1_A, 0)$. Define the k-linear map ρ_M of $k[G]$ to A_M by

$$\rho_M : x \longmapsto (\rho(x), 0)$$

, where $x \in k[G]$. Then the map ρ_M is a k-algebra homomorphism of $k[G]$ to A_M, and the pair (A_M, ρ_M) is an interior G-algebra. Since the subset $\{ (0, m) \in A_M : m \in M \}$ is a nilpotent ideal of A_M the interior G-algebra (A_M, ρ_M) is an indecomposable interior G-algebra and the interior G-algebra $(A_M/J(A_M), \pi \cdot \rho_M)$ is isomorphic to the interior G-algebra $(A/J(A), \pi \cdot \rho)$. By the condition (3) of theorem 2, the defect group of the interior G-algebra (A_M, ρ_M) is trivial. Because there exists infinitely many finite dimensional (A, A)-bimodule, we obtain the corollary.

<u>Corollary</u> 4. Let (A, ρ) be an indecomposable interior G-algebra with defect group D of order p. Then there exists an indecomposable $k[G]$-module V such that the vertex $vtx_G(V)$ equals D.

Proof. By the condition (2) of theorem 2, there exists an indecomposable $k[G]$-module V belonging to (A, ρ) such that V is not a projective $k[G]$-module. But by lemma 1, V is a D-projective $k[G]$-module and D is a cyclic group of order p. Thus the vertex of the indecomposable $k[G]$-module V equals D.

<u>Remark</u> 5. The argument of this paper can use for $\mathcal{O}$-free $\mathcal{O}$ -algebra , where $\mathcal{O}$ is a complete discrete valuation ring of characteristic 0 with unique maximal ideal $J(\mathcal{O})$ such that

the factor field $\mathcal{O}/J(\mathcal{O})$ is k.Let K is the quotient field of $\mathcal{O}$. In this case, we do not know the structure of the factor K-algebra $K \otimes_{\mathcal{O}} A/J(K \otimes_{\mathcal{O}} A)$,where (A, ρ) is an indecomposable interior G-algebra with the trivial defect group.

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NOTE ON ALMOST M-INJECTIVES

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Recently, in [2], Harada and Tozaki defined 'almost M-projectives' which are generalized from the concept 'M-projectives' due to Azumaya. In this article we shall define a dual concept 'almost M-injectives' and generalize the following Azumaya's theorem concerning to M-injectives: N is M_1- and M_2-injective if and only if N is $M_1 \oplus M_2$-injective for modules N, M_1 and M_2, to a case of 'almost M-injectives'. An easy example shows that the theorem can not be modified as the same form.

Let R be an associative ring with identity and every module be a unitary right R-module. For modules M and N with $N \subseteq M$, $N \subsetneq_e M$ denotes that M is an essential extension of N. For modules M, N and a homomorphism $f: M \to N$, $M(f)$ denotes $\{m+f(m) \mid m \in M\}$. For a module M, $|M|$ denotes its composition length. If for each simple submodule S of M there is a direct summand M' of M such that $S \subsetneq_e M'$, we say that M is extending for simple modules.

The detailed version will appear elsewhere.

Definition. Let M and N be R-modules. We say that N is almost M-injective if at least one of the following conditions holds for each submodule L of M and each homomorphism $f:L\to N$:

(1) There exists a homomorpnism $\tilde{f}:M\to N$ such that $\tilde{f}\cdot i=f$,

(2) There exists a non-zero direct summand M_1 of M and a homomorphism $\tilde{f}:N\to M_1$ such that $\tilde{f}\cdot f=\pi\cdot i$, where $\pi:M\to M_1$ is a projection of M onto M_1.

In this definition, for a given diagram:

$$(*)\qquad \begin{array}{ccccc} 0 & \longrightarrow & L & \xrightarrow{\ i\ } & M \\ & & f\downarrow & & \\ & & N & & \end{array}$$

we call that the first (respectively, second) case occurs in the diagram(*) if the condition (1) (respectively, (2)) holds in the diagram.

Lemma A. Let U be a uniform module and X an indecomposable module. If U is almost X-injective and $|U|\geqq|X|$, U is X-injective.

Lemma B. Let M and N be R-modules. Consider a diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & L & \xrightarrow{\ i\ } & M \\ & & f\downarrow & & \\ & & N & & \end{array}$$

and put $K:=\mathrm{Ker}(f)$. Then if the second case occurs in this diagram, there is a proper direct summand M' of M which contains K.

In particular, if $K\subsetneq_e M$, then the first case occurs.

We prepare for Lemma C below. Let N, M_1 and M_2 be modules, and put

$M := M_1 \oplus M_2$. Consider a diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & L & \overset{i}{\dashrightarrow} & M \\ & & f\downarrow & & \\ & & N & & \end{array} \qquad (1)$$

From this diagram we induce the following for $k=1,2$:

$$\begin{array}{ccccc} 0 & \longrightarrow & L_k & \overset{i_k}{\longrightarrow} & M_k \\ & & f|_{L_k}\downarrow & & \\ & & N & & \end{array} \qquad (2\text{-}k)$$

where $L_k := L \cap M_k$. Moreover when the first case occurs in both diagrams (2-1) and (2-2) (let $\tilde{f}_k : M_k \to N$ be homomorphisms such that $f|_{L_k} = \tilde{f}_k \cdot i_k$ for $k=1,2$), we shall consider the following for $k=1,2$:

$$\begin{array}{ccccc} 0 & \longrightarrow & L^k & \overset{i^k}{\longrightarrow} & M_k \\ & & f'_k\downarrow & & \\ & & N & & \end{array} \qquad (3\text{-}k)$$

where, letting $\pi_k : M\ (=M_1 \oplus M_2) \to M_k$ be the projection, $L^k := \pi_k(L)$ and the homomorphisms f'_k is defined as follows : Put $f_0 := f - (\sum_{k=1}^{2} \tilde{f}_k \cdot \pi_k|_L) : L \to N$. Since $f_0(L_1 \oplus L_2) = 0$ (from the definition of $\tilde{f}_k$), the canonical map $\bar{f}_0 : L/(L_1 \oplus L_2) \to N$ is induced. We let $f'_k : L^k \to N$ be the composite map: $L^k \xrightarrow{\text{natural epi.}} L^k/L_k \xrightarrow[\text{natural iso}]{\approx} L/(L_1 \oplus L_2) \xrightarrow{\bar{f}_0} N$.

Lemma C. Assume that N is almost M_1- and M_2-injective. Consider a diagram (1) and induce the above diagrams. If the first case occurs in both diagrams (2-1) and (2-2) and does in either (3-1) or (3-2), then so does in the diagram (1).

Corollary 1. [Azumaya] Let N, M_1 and M_2 be modules. If N is M_1- and M_2-injective, then N is $M_1 \oplus M_2$-injective.

Corollary 2. Let N, M_1 and M_2 be modules, and let N be almost M_1- and M_2-injective. Consider a diagram:

$$
\begin{array}{lccccc}
 & 0 & \longrightarrow & L & \xrightarrow{\ i\ } & M:=M_1 \oplus M_2 \\
(*) & & & f\downarrow & & \\
 & & & N & &
\end{array}
$$

and put $K:=\mathrm{Ker}(f)$. Then if $K \subsetneq M$, the first case occurs in the diagram (*).

Theorem. Let U_k be a uniform module of finite composition length for $k=0,1,2,\cdots,n$. Then the following two conditions are equivalent:

(1) U_0 is almost $\sum_{k=1}^{n}\oplus U_k$-injective.

(2) U_0 is almost U_k-injective for every $k=1,2,\cdots,n$ and if $\mathrm{Soc}(U_0)\approx \mathrm{Soc}(U_k)\approx \mathrm{Soc}(U_l)$ (any $k,l \in \{1,2,\cdots,n\}$, $k\neq l$) then (i) U_0 is U_k- and U_l-injective or (ii) $U_k \oplus U_l$ is extending for simple modules.

Definition. Let R be an artinian ring. We say that R is <u>right Co-Nakayama</u> if every indecomposable injective right R-module E is uniserial (i.e. E has a unique composition series.).

Corollary. The following two conditions are equivalent:

(1) R is right Co-Nakayama.

(2) For any uniform modules U^i and U_j $(i=1,\cdots,m; j=1,\cdots,n)$ of finite composition length, $\bigoplus_{i=1}^{m} U^i$ is almost $\bigoplus_{j=1}^{n} U_j$-injective if U^i is almost U_j-injective for all i and j. (i.e.The almost injectivity among uniform modules which have finite composition length is closed under finite direct sums.)

Example. There is an example which shows that the Azumaya's Theorem is not able to be extended without an additional condition.

Let K be a field and

$$R = \begin{pmatrix} K & 0 & K \\ & K & K \\ 0 & & K \end{pmatrix}$$

Then, $e_{33}R$ is almost $e_{11}R$-and $e_{22}R$-injective, but not almost $e_{11}R \oplus e_{22}R$-injective, where e_{kk} are matrix units.

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AUSLANDER-REITEN QUIVERS AND GREEN CORRESPONDENCE

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1. Introduction

Let G be a finite group and k be a field of characteristic $p>0$. Let Θ be a connected component of the stable Auslander-Reiten quiver $\Gamma s(kG)$ of the group algebra kG and set $V(\Theta) = \{ vx(M) \mid M$ is an indecomposable kG-module in $\Theta \}$, where $vx(M)$ denotes the vertex of M. As we shall see in Proposition 2 below, if Q is a minimal element in $V(\Theta)$, then $Q \leq_G H$ for all $H \in V(\Theta)$. In particular Q is uniquely determined up to conjugation in G.

Let $N = N_G(Q)$ and let f be the Green correspondence with respect to (G,Q,N). Choose an indecomposable kG-module M_0 in Θ with Q its vertex. Let Δ be the connected component of $\Gamma s(kN)$ containing fM_0. The purpose is to show that there is a subquiver Λ of Δ and a graph isomorphism $\psi : \Lambda \to \Theta$ such that ψ^{-1} behaves like the Green correspondence f as a bijective map between modules in Λ and those in Θ.

The notation is almost standard. All the modules considered here are finite dimensional over k. We write $W|W'$ for kG-modules W and W', if W is isomorphic to a direct summand of W'. $(W, W')^G$ denotes the k-space $\mathrm{Hom}_{kG}(W,W')$.

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For a set $\mathfrak{Z}$ of subgroups of G, write $(W, W')^G_{\mathfrak{Z}} = \{\varphi \in (W, W')^G \mid \varphi$ factors through a $\mathfrak{Z}$-projective module$\}$ and $(W, W')^{\mathfrak{Z}, G} = (W, W')^G / (W, W')^G_{\mathfrak{Z}}$. For an indecomposable non-projective kG-module M, we write $\mathcal{A}(M)$ to denote the Auslander-Reiten sequence terminating at M. A sequence $M_0 - M_1 - \cdots - M_t$ of indecomposable kG-modules M_i $(0 \leq i \leq t)$ is said to be a walk if there exists either an irreducible map from M_i to M_{i+1} or an irreducible map from M_{i+1} to M_i for $0 \leq i \leq t-1$. Concerning some basic facts and terminologies used here, we refer to [1],[3] and [5].

2. Minimal element in $V(\Theta)$

Let Ξ be a subgraph of the stable Auslander-Reiten quiver $\Gamma_s(kG)$ and set $V(\Xi) = \{ vx(M) \mid M \in \Xi \}$. Note that every element in $V(\Xi)$ is a non-trivial p-subgroup of G since every M is non-projective.

<u>Lemma 1</u>. Let Ξ be a subgraph of $\Gamma_s(kG)$. Assume that Ξ is connected. Take any $Q \in V(\Xi)$ with the smallest order among those p-subgroups in $V(\Xi)$. Then for any indecomposable module $N \in \Xi$, N_Q has an indecomposable direct summand whose vertex is Q. ▯

Lemma 1 implies that the minimal element with respect to the partial order $\leq_G$ are those that have the smallest order. Thus the following holds.

<u>Proposition 2</u>. Let Θ be a connected component of $\Gamma_s(kG)$. Let Q be an element of $V(\Theta)$ which is minimal with respect to the partial order $\leq_G$. Then for any $H \in V(\Theta)$, we have $Q \leq_G H$. In particular Q is uniquely determined up to conjugation in G. ▯

3. Module correspondence

Now returning the situation of the introduction, let Q be a minimal element in $V(\Theta)$ throughout this section. Let Λ be the subquiver of Δ consisting of those kN-modules L in Δ such that there exists a walk $fM_0 = L_0 - L_1 - \cdots - L_t = L$ with $Q \leq_G vx(L_i)$ $(i = 0, 1, \cdots, t)$.

The following fact is immediate from Lemma 1.

<u>Lemma 3</u>. Let L be an indecomposable kN-module in Δ. Then $Q \leq vx(L)$. ▯

Let $\mathfrak{X}$ be the set of all p-subgroups of N of order smaller than $|Q|$. Also let $\mathfrak{Y} = \{ N \cap Q^g \mid g \in G \setminus N \}$.

<u>Lemma 4</u>. Let W be an indecomposable kG-module in Θ. Then there exists a kN-module T satisfying the following two conditions:

(i) $(T^G)_N \simeq T \oplus T'$, where T' is $\mathfrak{Y}$-projective.

(ii) $(W_N, T)^{\mathfrak{X}, N} \neq 0$.

Proof. By Lemma 1, W_Q has an indecomposable direct summand S whose vertex is Q. Let $T = S^N$. Then T satisfies the above two conditions. ▯

<u>Lemma 5</u>. Let T be a kN-module satisfying the condition (i) of Lemma 4. Let L be an indecomposable kN-module in Λ. Then the following k-isomorphism holds:

$$((L^G)_N, T)^{\mathfrak{X}, N} \simeq (L, T)^{\mathfrak{X}, N}.$$

Proof. It follows that $((L^G)_N, T)^{\mathfrak{X}, N} \simeq (L, (T^G)_N)^{\mathfrak{X}, N}$ from [5,Cor.5.4]. Let $(T^G)_N \simeq T \oplus (\bigoplus_i X_i)$. It is enough to show that $(L, X_i)^{\mathfrak{X}, N} = 0$ for all X_i. By the condition (i) of Lemma 4, each X_i is $\mathfrak{Y}$-projective. This implies that any $\alpha \in (L, X_i)^N$ is $\mathfrak{X}$-projective, i.e., $\alpha \in (L, X_i)^N_{\mathfrak{X}}$. ▯

Lemma 6. Let L be an indecomposable kN-module in Λ. Then L^G has a unique indecomposable direct summand M whose vertex contains Q, and we have

(1) L is a direct summand of M_N, and

(2) M lies in Θ.

Moreover letting T be a kN-module satisfying the conditions (i) and (ii) in Lemma 4 for M, we have;

$$((L^G)_N, T)^{\mathfrak{X},N} \simeq (M_N, T)^{\mathfrak{X},N} \simeq (L, T)^{\mathfrak{X},N} \neq 0.$$

In particular L is a direct summand of $(L^G)_N$ with multiplicity one.

Proof. Since $L \mid (L^G)_N$, L^G has an indecomposable direct summand M such that $L \mid M_N$. Therefore the vertex of M contains Q and L^G has at least one indecomposable direct summand whose vertex contains Q.

Let $fM_0 = L_0 - L_1 - \cdots - L_t = L$ be a walk with $Q \leq_G vx(L_i)$ $(i = 0,1,\cdots,t)$. We prove the assertion by induction on the "distance" t.

If $t = 0$, i.e., $L \simeq fM_0$, then the assertion follows since f is the Green correspondence.

Suppose the assertion holds for L_{t-1}. We shall derive a contradiction assuming that L^G has two indecomposable direct summand M and W whose vertices contain Q. Let $L^G \simeq M \oplus W \oplus W'$. We may assume that $L \mid M_N$. By [2, Lemma 1.5], $\mathcal{A}(L_{t-1})^G \simeq \mathcal{A}(M_{t-1}) \oplus \mathcal{E}$, where M_{t-1} is the unique indecomposable direct summand of $L_{t-1}{}^G$ whose vertex contains Q and $\mathcal{E}$ is a split sequence. Let Y (resp. Y') be the middle term of $\mathcal{A}(M_{t-1})$ (resp. $\mathcal{A}(\Omega^{-2}M_{t-1})$). Since L is a direct summand of the middle term of $\mathcal{A}(L_{t-1})$ or $\mathcal{A}(\Omega^{-2}L_{t-1})$, it follows that $M \oplus W \mid Y$ or $M \oplus W \mid Y'$. In particular both M and W lie in Θ.

Let T and U be kN-modules satisfying the conditions (i) and (ii) for M and W respectively in Lemma 4 and put $T' = T \oplus U$. Then $((L^G)_N, T')^{\mathfrak{X},N}$

$$\simeq (M_N, T')^{\mathfrak{X},N} \oplus (W_N, T')^{\mathfrak{X},N} \oplus (W'_N, T')^{\mathfrak{X},N}$$

$\simeq (L, T')^{\mathfrak{X},N} \oplus (Z, T')^{\mathfrak{X},N} \oplus (W_N, T')^{\mathfrak{X},N} \oplus (W'_N, T')^{\mathfrak{X},N}$
where $M_N = L \oplus Z$. But by Lemma 5, $((L^G)_N, T')^{\mathfrak{X},N} \simeq (L, T')^{\mathfrak{X},N}$. This implies that $(W_N, U)^{\mathfrak{X},N} \subset (W_N, T')^{\mathfrak{X},N} = 0$, which is a desired contradiction. Thus L^G has a unique indecomposable direct summand M whose vertex contains Q, and the statements (1) and (2) hold. Moreover we obtain that

$$((L^G)_N, T)^{\mathfrak{X},N} \simeq (M_N, T)^{\mathfrak{X},N} \simeq (L, T)^{\mathfrak{X},N} \neq 0,$$

since $M | L^G$ and $L | M_N$. Hence L is a direct summand of $(L^G)_N$ with multiplicity one; for otherwise

$$(L, T)^{\mathfrak{X},N} \oplus (L, T)^{\mathfrak{X},N} \subset ((L^G)_N, T)^{\mathfrak{X},N} \simeq (L, T)^{\mathfrak{X},N} \neq 0,$$

a contradiction. ▯

For an indecomposable kN-module L in Λ, let ψL be a unique indecomposable direct summand of L^G whose vertex contains Q. It follows that $\mathcal{A}(L)^G \simeq \mathcal{A}(\psi L) \oplus \mathcal{E}$, where $\mathcal{E}$ is a split sequence by Lemma 6 and [2,Lemma 1.5]. As the Auslander-Reiten quiver is the graph which is constructed by joining Auslander-Reiten sequences, ψ induces a graph homomorphism from Λ to Θ. Also we have the following fact.

<u>Lemma 7</u>. Let L and L' be indecomposable kN-modules in Λ. Then $\psi L \simeq \psi L'$ if and only if $L \simeq L'$. ▯

It is a direct consequence of Lemmas 6 and 7 that ψ is a graph monomorphism. Moreover we have

<u>Lemma 8</u>. ψ is an epimorphism.

Proof. Let M be an arbitrary element of Θ and let $M_0 - M_1 - \cdots - M_t = M$ be a walk. If $t = 0$, i.e., $M = M_0$, then $M_0 = f^{-1}(fM_0) = \psi L_0$. Now suppose then that there exists an element L_{t-1} in Λ such that $M_{t-1} = \psi L_{t-1}$. By Lemma 6 we have. $\mathcal{A}(L_{t-1})^G \simeq \mathcal{A}(M_{t-1}) \oplus \mathcal{E}$ and $\mathcal{A}(\Omega^2 L_{t-1})^G \simeq \mathcal{A}(\Omega^2 M_{t-1}) \oplus \mathcal{E}'$, where $\mathcal{E}$ and $\mathcal{E}'$ are split sequences. Recall that M_t

is a direct summand of the middle term of $\mathscr{A}(M_{t-1})$ or $\mathscr{A}(\Omega^2 M_{t-1})$. Therefore there exists some direct summand L of the middle term of $\mathscr{A}(L_{t-1})$ or $\mathscr{A}(\Omega^2 L_{t-1})$ such that $M | L^G$. Since $Q \leq_G vx(M) \leq_G vx(L)$, L lies in Λ. Hence $M = \psi L$ and ψ is an epimorphism. []

We are now ready to state the main theorem.

__Theorem__. ψ induces a graph isomorphism from Λ onto Θ which preserves edge-multiplicity and direction. Also ψ gives rise to a one-to-one correspondence between indecomposable modules in Θ and those in Λ and the following hold:

(1) Let M be an indecomposable kG-module in Θ. Then $M_N \simeq \psi^{-1}M \oplus (\bigoplus_i W_i)$, where $W_{i\,Q}$ is $\mathfrak{X}$-projective for all i.

(2) Let L be an indecomposable kN-module in Λ. Then $L^G \simeq \psi L \oplus (\bigoplus_i V_i)$, where V_i is $\mathfrak{X}$-projective for all i. []

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A REMARK ON MODULAR REPRESENTATION THEORY

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Here I just add one remark to my previous result [1,Theorem]. In order to describe it, we fix an algebraically closed field F of nonzero characteristic p, a finite group G and a p-block B of G with its block idempotent e_B, that is, B is a block ideal of the group algebra FG such that $B = FGe_B = e_BFG$. We write $J(R)$ for the Jacobson radical for a ring R. Other notation is the same as in [1].

Proposition (see [1,p.152 Theorem]). With the same notation as the above the following conditions are equivalent to (1)-(7) in [1, Theorem].

(8) The correspondence $b \to B$ given by $x \to xe_B$ $(x \in b)$ is an F-algebra-isomorphism.

(9) $J(FD)B \subseteq J(B)$.

(10) $G = N \cdot N_B^* = N \cdot N_B$.

(11) Every simple FG-module in B has D as its vertex, and the FG-module $F_D{}^{\uparrow G} \cdot e_B$ is semi-simple (completely reducible), where $F_D{}^{\uparrow G}$ is the induced FG-module from the trivial FD-module F_D.

The final detailed version of this note may perhaps be submitted for publication elsewhere.

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