PROCEEDINGS OF THE 22ND SYMPOSIUM ON RING THEORY

HELD AT HOKKAIDO UNIVERSITY, SAPPORO

August 2–4, 1989

EDITED BY

Kozo SUGANO

Hokkaido University

1989

OKAYAMA, JAPAN

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PREFACE

The 22nd Symposium on Ring Theory was held at Hokkaido University, Sapporo, on August 2–4, 1989, immediately after the 35th Symposium on Algebra, which was held at the same university.

The Proceedings contain twelve articles presented at the Symposium including the one given by a special guest, Prof. Yao Musheng, China. We desire earnestly that many more foreign ring-theorists will take part in this Symposium hereafter.

The meeting and the Proceedings were financially supported by the Grant-in-Aid for Scientific Research from the Ministry of Education through the arrangements by Prof. H. Hijikata. We appreciate his arrangements.

We wish also to express our thanks to all speakers of the meeting and to staffs and graduate students of Hokkaido University for their help in the organization of the meeting.

October 31, 1989

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SYMPOSIUM ON RING THEORY (1989)

ON STRONGLY SEPARABLE EXTENSIONS

Yasukazu YAMASHIRO

E.McMahon and A.C.Mewborn introduced a type of separable extensions in [4], which is called strongly separable extension. A ring Λ is a strongly separable extension of a subring Γ if and only if the commutor ring Δ of Γ in Λ is C-f.g.projective, where C is the center of Λ, and a map $\varphi:\Lambda\otimes_\Gamma\Lambda \longrightarrow \mathrm{Hom}_C(\Delta,\Lambda)$ given by $\varphi(\lambda\otimes\lambda')(\delta)=\lambda\delta\lambda'$ for λ, $\lambda'\in\Lambda$ and $\delta\in\Delta$ is a Λ-Λ-split epimorphism. In this paper, we shall study some properties of strongly separable extensions corresponding to H-separable extensions. In § 1, we give some equivalent conditions (1.4) and in § 2, we give the commutor theorem for strongly separable extensions (2.5).

1. Strongly separable extensions

Let R be a ring and M and N left R-modules. We shall denote $M \rightsquigarrow N$ if M is a direct sum of submodules S and K such that ${}_RS<\oplus_R(N\oplus\cdots\oplus N)$ and $\mathrm{Hom}({}_RK,{}_RN)=0$. It is easy to see that K coincides with the reject of N in M (cf.[1]), which is defined by

$$\mathrm{Rej}_M(N)=\cap\{\ \ker f \mid f\in\mathrm{Hom}({}_RM,{}_RN)\}.$$

Using this notation, we can state that a ring Λ is a strongly separable extension of a subring Γ if and only if $\Lambda\otimes_\Gamma\Lambda \rightsquigarrow \Lambda$

The final version of this paper will be submitted for Hokkaido Math. J.

as Λ-Λ-modules.

Lemma 1.1. Let R be a ring and M and N left R-modules such that $M \rightsquigarrow N$. Then for every R-direct summand L_1 of M, $L_1 \rightsquigarrow N$.

Proof. We can write $M=L_1\oplus L_2$ and $M=S\oplus K$ with

$$_RS<\oplus_R(N\oplus\cdots\oplus N),\ \mathrm{Hom}(_RK,{}_RN)=0.$$

Let π_1 and π_2 be projections of M to L_1 and L_2, respectively, and p_K the projection of M to K. By (8.18) in [1], we have $K=\pi_1(K)\oplus\pi_2(K)$. Then the restriction of π_ip_K to L_i is the projection of L_i to $\pi_i(K)$ $(i=1,2)$. Hence we can write $L_1=S_1\oplus\pi_1(K)$ and $L_2=S_2\oplus\pi_2(K)$. Then we have $M=S\oplus K=S_1\oplus S_2\oplus K$ and $S\simeq M/K\simeq S_1\oplus S_2$. Hence $S_1<\oplus S<\oplus(N\oplus\cdots N)$. Since $\pi_1(K)<\oplus K$, $\mathrm{Hom}(_R\pi_1(K),{}_RN)=0$. Then $L_1\rightsquigarrow N$.

Let $\Gamma\subset B\subset\Lambda$ be rings. In case the map $B\otimes_\Gamma\Lambda\longrightarrow\Lambda$ such that $b\otimes\lambda\longmapsto b\lambda$ for $b\in B$ and $\lambda\in\Lambda$ splits as a B-Λ-map, we shall call briefly that $B\otimes_\Gamma\Lambda\longrightarrow\Lambda$ splits. In this case, by tensoring on the left with Λ over B, $\Lambda\otimes_B\Lambda<\oplus\ \Lambda\otimes_\Gamma\Lambda$ as Λ-Λ-modules. So, from the above lemma, we obtain

Proposition 1.2. Let Λ be a strongly separable extension of Γ. Then for every subring B of Λ such that $\Gamma\subset B$ and $B\otimes_\Gamma\Lambda\longrightarrow\Lambda$ splits, Λ is strongly separable over B.

Corollary 1.3. Let Λ be a strongly separable extension of Γ. Then for every separable subextension B of Λ over Γ, Λ is strongly separable over B.

For any Λ-Λ-module M, we denote by M^Λ the subset $\{m\in M\mid\lambda m=m\lambda \text{ for all } \lambda\in\Lambda\}$ of M, and for any subring A of Λ, we denote by $V_\Lambda(A)$ the commutor ring of A in Λ.

Let $\Gamma\subset\Lambda$ be arbitrary rings C the center of Λ and $\Delta=V_\Lambda(\Gamma)$. Then we always have a Λ-Λ-map $\varphi:\Lambda\otimes_\Gamma\Lambda\longrightarrow\mathrm{Hom}_C(\Delta,\Lambda)$

defined by $\varphi(\lambda\otimes\lambda')(\delta)=\lambda\delta\lambda'$ for $\lambda,\lambda'\in\Lambda$ and $\delta\in\Delta$. we shall denote its kernel by $R_\Gamma(\Lambda)$. Since $\mathrm{Hom}({}_\Lambda\Lambda\otimes_\Gamma\Lambda_\Lambda,{}_\Lambda\Lambda_\Lambda)\simeq\Delta$ by the map $f\longmapsto f(1\otimes 1)$ for $f\in\mathrm{Hom}({}_\Lambda\Lambda\otimes_\Gamma\Lambda_\Lambda,{}_\Lambda\Lambda_\Lambda)$, $R_\Gamma(\Lambda)$ coincides with the reject of Λ in $\Lambda\otimes_\Gamma\Lambda$ as a Λ-Λ-module. In particular, if Λ is strongly separable over Γ then we can write

$$\Lambda\otimes_\Gamma\Lambda\simeq\mathrm{Hom}_C(\Delta,\Lambda)\oplus R_\Gamma(\Lambda)$$

as Λ-Λ-modules.

The next theorem is a generalization of Theorem 1.2 in [6].

Theorem 1.4. Let $\Gamma\subset\Lambda$ be rings, C the center of Λ and $\Delta=V_\Lambda(\Gamma)$. Then the following statements are equivalent.

(1) Λ is a strongly separable extension of Γ.

(2) For every Λ-Λ-module M,

$$M^\Gamma=\Delta M^\Lambda\oplus X$$

such that the map $g:\Delta\otimes_C M^\Lambda\longrightarrow\Delta M^\Lambda$ defined by $g(\delta\otimes m)=\delta m$ for $\delta\in\Delta$ and $m\in M^\Lambda$ is an isomorphism and $X\subset\mathrm{Rej}_M(\Lambda)$.

(3) $(\Lambda\otimes_\Gamma\Lambda)^\Gamma=\Delta(\Lambda\otimes_\Gamma\Lambda)^\Lambda\oplus X$

such that the map g for $M=\Lambda\otimes\Lambda$ is an isomorphism and $X\subset R_\Gamma(\Lambda)$.

Proof. Assume (1). By (3.10) in [4],

$$M^\Gamma\simeq(\Delta\otimes_C M^\Lambda)\oplus\mathrm{Hom}({}_\Lambda R_\Gamma(\Lambda)_\Lambda,{}_\Lambda M_\Lambda).$$

In this case, the injection $\mathrm{Hom}({}_\Lambda R_\Gamma(\Lambda)_\Lambda,{}_\Lambda M_\Lambda)\longrightarrow M^\Gamma$ is given by $f\longmapsto f(k)$ for $f\in\mathrm{Hom}({}_\Lambda R_\Gamma(\Lambda)_\Lambda,{}_\Lambda M_\Lambda)$, where k is the image of $1\otimes 1$ in $R_\Gamma(\Lambda)$ by the projection $p:\Lambda\otimes_\Gamma\Lambda\longrightarrow R_\Gamma(\Lambda)$. For any $g\in\mathrm{Hom}({}_\Lambda M_\Lambda,{}_\Lambda\Lambda_\Lambda)$, $g\circ f\circ p$ is a map $\Lambda\otimes\Lambda\longrightarrow\Lambda$. Since $k\in R_\Gamma(\Lambda)$, the reject of Λ in $\Lambda\otimes\Lambda$, $g(f(k))=g\circ f\circ p(k)=0$. Then $g(X)=0$ and $X\subset\mathrm{Rej}_M(\Lambda)$. Hence (2) holds. If we put $M=\Lambda\otimes_\Gamma\Lambda$ then (2) implies (3). Assume (3). We can write

$$1\otimes 1=\Sigma_{ij}\delta_i x_{ij}\otimes y_{ij}+k$$

for some $\delta_i\in\Delta$, $\Sigma_j x_{ij}\otimes y_{ij}\in(\Lambda\otimes\Lambda)^\Lambda$ and $k\in X$. By the definition of $R_\Gamma(\Lambda)$,

$$\delta=\varphi(1\otimes 1)(\delta)=\Sigma_{ij}\delta_i x_{ij}\delta y_{ij} \qquad \text{for all } \delta\in\Delta.$$

Hence Λ is strongly separable over Γ by (3.5)(2) in [4]. This completes the proof.

2. Commutor theorem

Throughout this section, whenever we denote a ring and its subring by Λ and Γ, respectively, we denote the center of Λ by C and $V_\Lambda(\Gamma)=\Delta$.

Let $\mathscr{B}_l$ be the set of subrings B of Λ such that $\Gamma\subset B$, $B\otimes_\Gamma\Lambda \longrightarrow \Lambda$ splits and there exists a B-Γ-projection $p_B:\Lambda \longrightarrow B$ such that $(1_\Lambda\otimes p_B)(R_B(\Lambda))=0$, where 1_Λ is the identity map of Λ and $1_\Lambda\otimes p_B$ is the map of $\Lambda\otimes_B\Lambda$ to Λ given by $(1_\Lambda\otimes p_B)(\lambda\otimes\lambda')$ $=\lambda p_B(\lambda')$ for $\lambda,\lambda'\in\Lambda$, and $\mathscr{D}_l$ the set of C-subalgebras D of Δ such that ${}_D D<\oplus {}_D\Delta$ and $D\otimes_C\Delta \longrightarrow \Delta$ splits. $\mathscr{B}_r$ and $\mathscr{D}_r$ are defined similarly. Furthermore, let $\mathscr{B}$ be the set of subrings B of Λ such that B is a separable extension of Γ and there exists a B-B-projection $p_B:\Lambda \longrightarrow B$ such that $(1_\Lambda\otimes p_B)(R_\Gamma(\Lambda))=0$ and $\mathscr{D}$ the set of separable C-subalgebras of Δ.

Firstly, we prove

Proposition 2.1. Let Λ be a strongly separable extension of Γ, D a C-subalgebra of Δ such that $D\otimes_C\Delta \longrightarrow \Delta$ splits, and $B=V_\Lambda(D)$. Then there exists a B-Γ-projection $p_B:\Lambda \longrightarrow B$ such that $(1_\Lambda\otimes p_B)(R_\Gamma(\Lambda))=0$ and the map $\psi_B:B\otimes_\Gamma\Lambda \longrightarrow \mathrm{Hom}({}_D\Delta,{}_D\Lambda)$ defined by $\psi_B(b\otimes\lambda)(\delta)=b\delta\lambda$ for $b\in B$, $\lambda\in\Lambda$ and $\delta\in\Delta$ is a split epimorphism as a B-Λ-map. If furthermore ${}_D D<\oplus {}_D\Delta$, then $B\otimes_\Gamma\Lambda \longrightarrow \Lambda$ splits.

Proof. Let $\Sigma_i d_i\otimes\delta_i\in(D\otimes_C\Delta)^D$ such that $\Sigma_i d_i\delta_i=1$. If we put $p_B:\Lambda \longrightarrow B$ by $p_B(\lambda)=\Sigma_i d_i\lambda\delta_i$ for $\lambda\in\Lambda$, and $\pi_D:\mathrm{Hom}_C(\Delta,\Lambda) \longrightarrow \mathrm{Hom}({}_D\Delta,{}_D\Lambda)$ by $\pi_D(f)(\delta)=\Sigma_i d_i f(\delta_i\delta)$ for $\delta\in\Delta$ and $f\in\mathrm{Hom}_C(\Gamma,\Lambda)$ then these maps are split epimorphisms as a B-Γ-map and a B-Λ-map, respectively. Now, consider the commutative diagram

$$\begin{array}{ccc}
\Lambda\otimes_\Gamma\Lambda & \xrightarrow{\ \ \varphi\ \ } & \mathrm{Hom}_C(\Delta,\Lambda) \\
{\scriptstyle p_B\otimes 1_\Lambda}\downarrow & & \downarrow{\scriptstyle \pi_D} \\
B\otimes_\Gamma\Lambda & \xrightarrow{\ \ \psi_B\ \ } & \mathrm{Hom}({}_D\Delta,{}_D\Lambda).
\end{array}$$

Since φ is a split epimorphism, ψ_B is a split epimorphism. If we put $\eta:\mathrm{Hom}_C(D',\Lambda)\longrightarrow\Lambda$ by $\eta(f)=\Sigma f(d_i)\delta_i$ for $f\in\mathrm{Hom}_C(D',\Lambda)$, where $D'=V_\Lambda(B)$, we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & R_B(\Lambda) & \longrightarrow & \Lambda\otimes_B\Lambda & \xrightarrow{\ \varphi_B\ } & \mathrm{Hom}_C(D',\Lambda) \\ & & & & {\scriptstyle 1_\Lambda\otimes p_B}\searrow & & \swarrow{\scriptstyle\eta} \\ & & & & & \Lambda & \end{array}$$

where the row is exact. Then we have

$$(1_\Lambda\otimes p_B)(R_\Gamma(\Lambda))=\eta\circ\varphi_B(R_\Gamma(\Lambda))=0.$$

Consider the commutative diagram

$$\begin{array}{ccc} B\otimes_\Gamma\Lambda & \xrightarrow{\ \psi_B\ } & \mathrm{Hom}({}_D\Delta,{}_D\Lambda) \\ & \searrow \quad \swarrow{\scriptstyle\alpha} & \\ & \Lambda & \end{array}$$

where α is the map given by $\alpha(f)=f(1)$ for $f\in\mathrm{Hom}({}_D\Delta,{}_D\Lambda)$. If ${}_DD<\oplus{}_D\Delta$, then α is a split epimorphism and $B\otimes_\Gamma\Lambda\longrightarrow\Lambda$ splits.

Proposition 2.2. Let Λ be a strongly separable extension of Γ. Then for every $B\in\mathcal{B}_1$, $V_\Lambda(B)\in\mathcal{D}_1$.

Proof. Since $B\otimes_\Gamma\Lambda\longrightarrow\Lambda$ splits, we have ${}_DD<\oplus{}_D\Delta$, where $D=V_\Lambda(B)$. By (1.2), Λ is strongly separable over B. Then we have the following isomorphisms

$$\begin{aligned} &\mathrm{Hom}({}_B\Lambda_\Gamma,{}_B\Lambda_\Gamma)\simeq\mathrm{Hom}({}_\Lambda\Lambda\otimes_B\Lambda_\Gamma,{}_\Lambda\Lambda_\Gamma) \\ &\simeq\mathrm{Hom}({}_\Lambda\mathrm{Hom}_C(D,\Lambda)_\Gamma,{}_\Lambda\Lambda_\Gamma)\oplus\mathrm{Hom}({}_\Lambda R_B(\Lambda)_\Gamma,{}_\Lambda\Lambda_\Gamma) \\ &\simeq(D\otimes_C\Delta)\oplus\mathrm{Hom}({}_\Lambda R_B(\Lambda)_\Gamma,{}_\Lambda\Lambda_\Gamma). \end{aligned}$$

In the above direct decomposition, the injection $\psi_D:D\otimes_C\Delta\longrightarrow\mathrm{Hom}({}_B\Lambda_\Gamma,{}_B\Lambda_\Gamma)$ is given by $\psi_D(d\otimes\delta)(\lambda)=d\lambda\delta$ for $d\in D$, $\delta\in\Delta$ and $\lambda\in\Lambda$. Clearly ψ_D is the D-Δ-homomorphism. In this case, the action of D and Δ to $\mathrm{Hom}({}_B\Lambda_\Gamma,{}_B\Lambda_\Gamma)$ is given by $(df)(\lambda)=df(\lambda)$ and $(f\delta)(\lambda)=f(\lambda)\delta$ for $d\in D$, $\delta\in\Delta$, $\lambda\in\Lambda$ and $f\in\mathrm{Hom}({}_B\Lambda_\Gamma,{}_B\Lambda_\Gamma)$. Let $\alpha:\mathrm{Hom}({}_B\Lambda_\Gamma,{}_B\Lambda_\Gamma)\longrightarrow\mathrm{Hom}({}_\Lambda R_B(\Lambda)_\Gamma,{}_\Lambda\Lambda_\Gamma)$ be the projection in the above decomposition, and M the map $\Lambda\otimes_B\Lambda\longrightarrow\Lambda$ given by $M(\lambda\otimes\lambda')=\lambda\lambda'$ for $\lambda,\lambda'\in\Lambda$. Then $\alpha(f)(x)=M(1_\Lambda\otimes f)(x)$ for $f\in\mathrm{Hom}({}_B\Lambda_\Gamma,{}_B\Lambda_\Gamma)$ and $x\in R_\Gamma(\Lambda)$. Since $\alpha(p_B)=0$ by the definition of $\mathcal{B}_1$, we have $p_B\in\psi_D(D\otimes_C\Delta)$. Hence there exists $\Sigma d_i\otimes\delta_i\in D\otimes_C\Delta$ such that $p_B=\psi_D(\Sigma d_i\otimes\delta_i)$. Then we have

$$\Sigma d_i\delta_i=\psi_D(\Sigma\, d_i\otimes\delta_i)(1)=p_B(1)=1$$

and for any $d\in D$,

$$\psi_D(\Sigma dd_i\otimes\delta_i)=d\psi_D(d_i\otimes\delta_i)=dp_B=p_Bd=\psi_D(\Sigma d_i\otimes\delta_i)d$$
$$=\psi_D(\Sigma d_i\otimes\delta_i d)$$

as the image of p_B is B. Since ψ_D is a monomorphism, $\Sigma dd_i\otimes\delta_i=\Sigma d_i\otimes\delta_i d$. Then $\Sigma d_i\otimes\delta_i\in(D\otimes_C\Delta)^D$ and this implies $D\otimes_C\Delta\longrightarrow\Delta$ splits.

As a generalization of Proposition 1.2 in [6], we have the next lemma.

Lemma 2.3. Let $\Gamma\subset\Lambda$ be rings and there exists a left Γ-projection $p:\Lambda\longrightarrow\Gamma$ such that $(1_\Lambda\otimes p)(R_\Gamma(\Lambda))=0$, then $V_\Lambda(V_\Lambda(\Gamma))=\Gamma$.

Proof. Let $x\in V_\Lambda(V_\Lambda(\Gamma))$. By definition of $R_\Gamma(\Lambda)$, $x\otimes 1-1\otimes x\in R_\Gamma(\Lambda)$. By hypothesis, we have $x-p(x)=0$ and $x\in\Gamma$.

Lemma 2.4. Let Λ be a strongly separable extension of Γ. Then for every $D\in\mathcal{D}_1$, $V_\Lambda(V_\Lambda(D))=D$.

Proof. Since $D\otimes_C\Delta\longrightarrow\Delta$ splits and Δ is C-f.g.projective, Δ is left D-f.g.projective. Let $B=V_\Lambda(D)$ and $D'=V_\Lambda(B)$. By (2.1), ${}_B\mathrm{Hom}({}_D\Delta,{}_D\Lambda)_\Lambda<\oplus\,{}_BB\otimes_\Gamma\Lambda_\Lambda$. Then we have

$$D'\otimes_D\Delta\simeq\mathrm{Hom}({}_B\Lambda_\Lambda,{}_B\Lambda_\Lambda)\otimes_D\Delta\simeq\mathrm{Hom}({}_B\mathrm{Hom}({}_D\Delta,{}_D\Lambda)_\Lambda,{}_B\Lambda_\Lambda)$$
$$<\oplus\mathrm{Hom}({}_BB\otimes_\Gamma\Lambda_\Lambda,{}_B\Lambda_\Lambda)\simeq\mathrm{Hom}({}_BB_\Gamma,{}_B\Lambda_\Gamma)\simeq\Delta.$$

Hence the map $D'\otimes_D\Delta\longrightarrow\Delta$ given by $d'\otimes\delta\longmapsto d'\delta$ is injective. Since this map is always surjective, $D'\otimes_D\Delta\simeq\Delta$. Then $D'=D$, since ${}_DD<\oplus_D\Delta$.

Now, we can obtain the commutor theorem for strongly separable extensions, which is a generalization of (1.3) in [9].

Theorem 2.5. Let Λ be a strongly separable extension of Γ, and consider the correspondence $V:A\rightsquigarrow V_\Lambda(A)$ for a subring A of Λ. Then we have

(1) V yields a one to one correspondence between $\mathscr{B}_l$ and $\mathscr{D}_l$ (resp. $\mathscr{B}_r$ and $\mathscr{D}_r$) such that V^2=identity.

(2) V yields a one to one correspondence between $\mathscr{B}$ and $\mathscr{D}$ such that V^2=identity.

Proof. (1) For any $B\in\mathscr{B}_l$, $V_\Lambda(B)\in\mathscr{D}_l$ by (2.2) and $V_\Lambda(V_\Lambda(B))=B$ by (2.3). For any $D\in\mathscr{D}_l$, $V_\Lambda(D)\in\mathscr{B}_l$ by (2.1) and $V_\Lambda(V_\Lambda(D))=D$ by (2.4).

(2) Since $\mathscr{B}\subset\mathscr{B}_l$, for any $B\in\mathscr{B}$, $V_\Lambda(V_\Lambda(B))=B$ and $V_\Lambda(B)=D\in\mathscr{D}_l$. Since $B\otimes_\Gamma B \longrightarrow B$ splits, ${}_D D_D <\oplus {}_D\Delta_D$. Hence D is a C-separable algebra by (1.4) in [9].

By (1.1) in [9], $\mathscr{D}\subset\mathscr{D}_l$. Then for any $D\in\mathscr{D}$, $V_\Lambda(V_\Lambda(D))=D$ and $V_\Lambda(D)=B\in\mathscr{B}_l$. Since $D\otimes_C D \longrightarrow D$ splits, ${}_B B_B <\oplus {}_B\Lambda_B$. Hence B is separable over Γ by (1.4) in [9].

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GROUP RINGS WHICH ARE V-HC ORDERS AND KRULL ORDERS

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Let R be a prime Goldie ring with quotient ring Q and let G be a polycyclic-by-finite group. In this note, we shall characterize those group rings R[G] which are Krull orders and v-HC orders with enough v-invertible ideals

1. A ring R is an <u>order</u> in a quotient ring Q provided
(i) R is a subring of Q,
(ii) every regular element of R is a unit of Q, and
(iii) for every element q of Q there exist elements $r_i, c_i \in R$ with c_i regular (i=1,2) such that $q = c_1^{-1}r_1 = r_2c_2^{-1}$.

Orders R,S in Q are called <u>equivalent</u>, written $R \sim S$, provided there exist q_i of Q ($1 \leq i \leq 4$) such that $q_1Rq_2 \subseteq S$ and $q_3Sq_4 \subseteq R$. An order R in Q is <u>maximal</u> if the only order S in Q such that $R \sim S$ and $R \subseteq S$ is R = S.
Let $F(\tau)$ ($F'(\tau)$) be a right(left) Gabriel topology on R

The detailed version of this note will appear elsewhere.

corresponding to the torsion theory cogenerated by the right (left) injective hull $E(Q/R)$ $(E'(Q/R))$ of a right (left) R-module Q/R. Then $F(\tau) = \{ H : \text{right ideal of } R \mid (R:r^{-1}H)_1 = R \text{ for any } r \in R \}$, where $r^{-1}H = \{ x \in R \mid rx \in H \}$ (see [15]). If I is a right ideal of R, then we write $cl(I) = \{ r \in R \mid rH \subseteq I \text{ for some } H \in F(\tau) \}$, and if $I = cl(I)$, then we say that I is <u>τ-closed</u>. Similarly, we can define <u>τ-closed left ideals</u>. R is called <u>τ-Noetherian</u> if R satisfies the a.c.c. on τ-closed right ideals as well as τ-closed left ideals.

Following [6], R is called a <u>Krull order</u> if R is a maximal order in Q and is τ-Noetherian. In [11, p.181, problem 7], they pose the following question: Let R be a ring and G a group such that the group ring R[G] is an order in a quotient ring Q(R[G]); When is R[G] a maximal order? This problem was first attacked in [7], [8], [14] and in [1], he obtained the following result:

Theorem 1.1 ([1]). Let R be a Noetherian commutative domain and let G be a polycyclic-by-finite group. Then the group ring R[G] is a prime maximal order if and only if

(i) R is integrally closed,

(ii) $\Delta^+(G) = 1$, and

(iii) G is dihedral-free.

Here $\Delta^+(G) = \{ x \in G \mid |G:C_G(x)| < \infty$ and x has a finite order }. Note that R[G] is a prime ring if and only if $\Delta^+(G) = 1$ and R is a prime ring by Theorem 4.2.10 of [13]. A subgroup H of G is called <u>G-orbital</u> if $|G:N_G(H)| < \infty$. Let $D = \langle a,b \mid a^2 = 1, aba = b^{-1} \rangle$ be an infinite dihedral group. Following [1], G is <u>dihedral-free</u> provided G does not have any G-orbital infinite dihedral subgroup.

<u>From now on, let R be an order in a simple Artinian ring Q and let G be a polycyclic-by-finite group</u>.

Proposition 1.2. The group ring R[G] is a Krull order in a simple Artinian ring Q(R[G]) if and only if

(i) R is a Krull order in Q,

(ii) $\Delta^+(G) = 1$, and

(iii) Q[G] is a Krull order.

This is proved by using localization (see [10]). Write $Q = (F)_n$, the nxn matrix ring over a division ring F, and let K be the center of F. Then we have $Q[G] \cong (F[G])_n$ and $F[G] \cong F \otimes_K K[G]$. Hence Q[G] is a maximal order if and only if F[G] is a maximal order. Furthermore we see from the following lemma that F[G] is a maximal order if and only if K[G] is a maximal order.

Lemma 1.3. Assume that $\Delta^+(G) = 1$. Then there is a one-to-one correspondence between the set of all ideals A of F[G] and the set of all ideals $\mathcal{A}$ of K[G] given by $A \longrightarrow A \cap K[G]$, $\mathcal{A} \longrightarrow F \otimes \mathcal{A}$.

From (1.1), (1.2) and (1.3), we have

Theorem 1.4. The Group ring R[G] is a Krull order in a simple Artinian ring Q(R[G]) if and only if

(i) R is a Krull order in Q,

(ii) $\Delta^{+}(G) = 1$, and

(iii) G is dihedral-free.

2. In the passing thirty years, the theory of hereditary rings (especially HNP rings) is one of the most successful subjects in non-commutative ring theory. But as it is easily obtained, some important ring extensions of hereditary rings are not necessary to be hereditary; for examples, polynomial ring, formal power series ring and graded ring (including group ring) extension of hereditary. But some such extension rings mostly inherit the ideal theory broadly obtained in HNP rings. Furthermore, hereditary rings are those rings which have global dimension one. But, from the point of view of the ideal theory, there are some important classes of rings which do not have global dimension one (even not necessary to have a finite global dimension); for examples, local rings having

finite global dimensions and Krull orders in the sense of Chamarie. These aspects led us to define the concept of v-HC orders which was a Krull type generalization of hereditary rings. Let A be an R-ideal. Then we use the notations;
$(R:A)_l = \{ q \in Q \mid qA \subseteq R \}$ and $(R:A)_r = \{ q \in Q \mid Aq \subseteq R \}$.
We set $A_v = (R:(R:A)_l)_r$ and ${}_vA = (R:(R:A)_r)_l$. If ${}_vA = A = A_v$, then we say that A is a <u>v-ideal</u>. If A is a right projective, then we have from the dual basis theorem that $A(R:A)_l = O_l(A)$ $= \{ q \in Q \mid qA \subseteq A \}$. Now consider the following condition;

(P): ${}_v(A(R:A)_l) = O_l(A)$ for any ideal $A = {}_vA$, and $(B(R:B)_r)_v$
$= O_r(B)$ for any ideal $B = B_v$.

R is called a <u>v-HC order</u> if R satisfies the condition (P) and is τ-Noetherian. This concept was first introduced in [9]. A v-ideal A of R is called <u>v-invertible</u> if there exists an R-ideal A^{-1} with $(AA^{-1})_v = R = {}_v(A^{-1}A)$. We say that <u>R has enough v-invertible ideals</u> if any v-ideal of R contains a v-invertible ideal. In this section, we shall characterize those group rings R[G] which are v-HC orders with enough v-invertible ideals. First of all, as in the case of Krull orders, we have

Proposition 2.1. The group ring R[G] is a v-HC order with enough v-invertible ideals if and only if

(i) R is a v-HC order with enough v-invertible ideals,

(ii) $\Delta^+(G) = 1$, and

(iii) K[G] is a v-HC order.

A plinth in G is a torsion-free Abelian G-orbital subgroup A of G such that $A \otimes_Z Q$ is an irreducible Q[T]-module for every subgroup T of a finite index in $N_G(A)$, where Z is the ring of integers and Q is the field of rationals. We denote by P(G) the subgroup generated by all plinths of G. It is clear that P(G) is a characteristic subgroup of G. The group S(G) is the isolator of the plinth socle P(G), i.e., S(G) is the largest normal subgroup of G containing P(G) as a subgroup of finite index. By[12], S(G) is a characteristic Abelian-by-finite subgroup of G.

Lemma 2.2. Assume that $\Delta^+(G) = 1$ and let P be a prime ideal of K[G] with ht(P) = 1. Then $P = (P \cap K[S(G)])K[G]$ (cf.[1]).

The first statement in the following proposition follows from Lemma 2.2.

Proposition 2.3. Assume that $\Delta^+(G) = 1$. Then
(i) If K[S(G)] is a v-HC order, then so is K[G].
(ii) If char(K) = 2 and G is dihedral-free, then K[G] is not a v-HC order.
(iii) If char(K) $\neq$ 2, then K[G] is a v-HC order.

The second statement essentially follows from the technique used in [3]. We use some results in [3] and [4] to prove the third statement. From Propositions 2.1 and 2.3, we have

Theorem 2.4. The group ring R[G] is a v-HC order with enough v-invertible ideals in a simple Artinian ring Q(R[G]) if and only if

(i) R is a v-HC order with enough v-invertible ideals,

(ii) $\Delta^{+}(G) = 1$, and

(iii) either G is dihedral-free or char(R) $\neq$ 2.

The applications of Theorems 1.4 and 2.4 will appear in the forthcoming papers.

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NON-RATIONALITY OF ALGEBRAIC TORI OF NORM TYPE AND ITS APPLICATION TO GENERIC DIVISION RINGS

Shizuo ENDO

1. Let G be a finite group. A G-module is called a permutation module if it is isomorphic to $\oplus_{i=1}^{r} ZG/G_i$, where G_i, $1 \leq i \leq r$, are subgroups of G. A G-module M is called a quasi-permutation module if there exists an exact sequence

$$0 \longrightarrow M \longrightarrow S \longrightarrow S' \longrightarrow 0,$$

where S and S' are permutation G-modules. The dual module $\mathrm{Hom}_Z(M, Z)$ of a G-module M is denoted by M^*.

Let k be a field and let F be an extension field of k. F is said to be rational over k if it is generated by a finite number of algebraically independent elements over k. F is said to be stably rational over k if there exists an extension field F' of F which is rational over each of k and F. Further, F is said to be retract rational over k if it is the quotient field of an integral domain B such that B satisfies the following condition: There exist a localized polynomial ring $A = k[x_1, x_2, \ldots, x_n][1/s]$, where $x_1, x_2, \ldots, x_n$ are variables and $0 \neq s \in k[x_1, x_2, \ldots, x_n]$, and k-algebra homomorphisms $\phi : B \longrightarrow A$ and $\psi : A \longrightarrow B$ such that $\psi \cdot \phi$ is the identity on B. It is easy to see that 'rational' $\Longrightarrow$ 'stably rational' $\Longrightarrow$ 'retract rational'.

Let G be a finite group and let M be a G-module with a Z-free basis $u_1, u_2, \ldots, u_n$. Define the action of G on the rational function field $k(x_1, x_2, \ldots, x_n)$ with variables $x_1, x_2, \ldots, x_n$ over a field k as follows: for each $\sigma \in G$,

$$\sigma(a) = a, \quad a \in k,$$

$$\sigma(x_i) = \Pi_{j=1}^{n} x_j^{m_{ij}}, \quad 1 \leq i \leq n,$$

when $\sigma \cdot u_i = \Sigma_{j=1}^{n} m_{ij} u_j$, $m_{ij} \in Z$, and denote $k(x_1, x_2, \ldots, x_n)$ with this action

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The detailed version of this note will appear elsewhere.

of G by k(M).

Further, let K be a Galois extension of k with group G. Define the action of G on the rational function field $K(x_1, x_2, \ldots, x_n)$ with variables $x_1, x_2, \ldots, x_n$ over K, as an extension of the action of G on K, as follows: for each $\sigma \in G$,

$$\sigma(x_i) = \Pi_{j=1}^{n} x_j^{m_{ij}}, \quad 1 \leq i \leq n,$$

when $\sigma \cdot u_i = \Sigma_{j=1}^{n} m_{ij} u_j$, $m_{ij} \in Z$, and denote $K(x_1, x_2, \ldots, x_n)$ with this action of G by K(M).

As is well known, there is an algebraic torus T defined over k and split over K such that the character group of T is isomorphic to M as G-modules, and the invariant subfield $K(M)^G$ of K(M) can be identified with the function field of T over k.

Proposition. Let G be a finite group and let k be a field. Let K be a Galois extension of k with group G and let M be a Z-free G-module.

(i) (e.g. [4, 1.6]) M is a quasi-permutation G-module if and only if $K(M)^G$ is stably rational over k.

(ii) ([10, 3.14]) M is a direct summand of a quasi-permutation G-module if and only if $K(M)^G$ is retract rational over k.

2. Let p be a prime, and let P be an elementary abelian p-group of order p^m, $m \geq 1$. Let P_i, $1 \leq i \leq r$, be distinct subgroups of index p in P, and let

$$\varepsilon_i \ : \ ZP/P_i \longrightarrow Z$$

be the augmentation epimorphism. Further, for $h_1, h_2, \ldots, h_r \geq 1$, let

$$\Phi = (\varepsilon_1^{h_1}, \varepsilon_2^{h_2}, \ldots, \varepsilon_r^{h_r}) \ : \ \oplus_{i=1}^{r} [ZP/P_i]^{h_i} \longrightarrow Z$$

and put $L = \operatorname{Ker} \Phi$.

Main result of this note is the following

Theorem 1. (i) In case of p = 2, L^* is a quasi-permutation P-module if and only if r = 1, 2.

(ii) In case of $p \neq 2$, L^* is a quasi-permutation P-module if and only if r = 1.

In order to prove this, we need to consider a more general situation. Let P, P_i and ε_i, $1 \leq i \leq r$, be as above, and define the homomorphism $\delta : Z \longrightarrow Z$ by $\delta(1) = p$. For $h_1, h_2, \ldots, h_r \geq 1$ and $h \geq 0$, let

$$\tilde{\Phi} = (\varepsilon_1^{h_1}, \varepsilon_2^{h_2}, \ldots, \varepsilon_r^{h_r}, \delta^h) \; : \; \oplus_{i=1}^{r} [ZP/P_i]^{h_i} \oplus Z^h \longrightarrow Z,$$

and put $\tilde{\Gamma}$ = Ker $\tilde{\Phi}$. Further, let

$$\Phi_1 = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r) \; : \; \oplus_{i=1}^{r} ZP/P_i \longrightarrow Z,$$

and put L_1 = Ker Φ_1. Then the key lemma is given as follows:

Lemma. $\tilde{\Gamma} \cong L_1 \oplus \oplus_{i=1}^{r} [ZP/P_i]^{h_i - 1} \oplus Z^h$.

By this lemma we have only to consider the case where $h_1 = h_2 = \ldots = h_r = 1$. However, in this case, using again the lemma, it suffices to show the following facts:

(1) L^* is a quasi-permutation P-module if one of the following conditions is satisfied: (a) $r = 1$; (b) $p = 2$ and $r = 2$.

(2) L^* is not a quasi-permutation P-module if one of the following conditions is satisfied: (a) $p = 2$, $m = 2$ and $r = 3$; (b) $p = 2$, $m = 3$ and $r = 4$; (c) $p \neq 2$, $m = 2$ and $r \geq 2$.

(1) is well known. (2), (a) was shown by T. Miyata in 1974 (unpublished), and this is also given in [8]. Accordingly, we have only to show (2), (b) and (c). Using repeatedly the key lemma and the exact sequences of cohomology groups, they can be shown by direct computations.

By Proposition, Theorem 1 can be restated as follows:

Theorem 1'. Let P, L, ... be as above, and let K be a Galois extension of a field k with group P. Then:

(i) In case of $p = 2$, $K(L^*)^P$ is stably rational (retract rational) over k if and only if $r = 1, 2$.

(ii) In case of $p \neq 2$, $K(L^*)^P$ is stably rational (retract rational) over k if and only if $r = 1$.

It is noted that the algebraic torus corresponding to L^*, defined over k and split over K, is of norm type.

The part (i) of Theorem 1' is an answer to the question asked T. Miyata by A. Merkurjev in 1982. It should be noted that Theorem 1 was obtained in 1982 (unpublished).

3. Assume that k is a field of characteristic 0. For m, $n \geq 2$, let $x_{ij}^{(r)}$, $1 \leq i, j \leq n$, $1 \leq r \leq m$, be variables and consider the $n \times n$ matrices

$$X_r = [x_{ij}^{(r)}], \quad 1 \leq r \leq m,$$

which are called generic matrices over k, and denote by $k\{X\}$ the k-algebra generated by X_r, $1 \leq r \leq m$. Then $k\{X\}$ has the quotient division ring $q(k\{X\})$ (S.A. Amitsur, e.g. [9, II, 1.3]). The quotient ring $q(k\{X\})$ is called a generic division ring over k. The center of $q(k\{X\})$ is denoted by $Z_n(m)$.

For generic division rings, the following problem is basic and open.

Problem. Is $Z_n(m)$ rational over k?.

For this the following fact is known.

(1) ([9, IV, 6.4 and 6.5]) For $m \geq 2$, $Z_n(m)$ is rational over $Z_n(2)$, and $Z_2(2)$ is rational over k.

By this it suffices to consider the case of $m = 2$ and $n \geq 3$.

Let S_n be the symmetric group on n letters, and let S_{n-1} identify with the subgroup $\{\sigma \in S_n | \sigma(n) = n\}$ of S_n. Define the epimorphism

$$\varepsilon \ : \ \mathbb{Z}S_n/S_{n-1} \longrightarrow \mathbb{Z}$$

by $\varepsilon(\sigma S_{n-1}) = 1$, $\sigma \in S_n$, and put $I_n = \operatorname{Ker} \varepsilon$. Further, define the epimorphism

$$\eta \ : \ \mathbb{Z}S_n/S_{n-1} \otimes_{\mathbb{Z}} \mathbb{Z}S_n/S_{n-1} \longrightarrow I_n$$

by $\eta(\sigma S_{n-1} \otimes \tau S_{n-1}) = \sigma S_{n-1} - \tau S_{n-1}$, $\sigma, \tau \in S_n$, and put $J_n = \operatorname{Ker} \eta$. Then we have

(2) ([6, Theorem 3]) $Z_n(2) = k(\mathbb{Z}S_n/S_{n-1} \oplus J_n)^{S_n}$

From (1) and (2) it follows that

(3) ([6], [7]) For each of $n = 3, 4$, $Z_n(m)$ is rational over k.

On the other hand, the following result was obtained by a quite different way,

(4) ([10, 5.3]) For any square-free integer $n \geq 2$, $Z_n(m)$ is retract rational over k.

We easily see that

(5) J_n is (a direct summand of) a quasi-permutation S_n-module if and only if I_n^* is (a direct summand of) a quasi-permutation S_n-module.

Put $E_n = k(\mathbb{Z}S_n/S_{n-1})^{S_n}$. Then, by Proposition, (1) and (2), J_n is (a direct summand of) a quasi-permutation S_n-module if and only if $Z_n(m)$ is stably rational (retract rational) over E_n. Note here that the action of S_n on $\mathbb{Z}S_n/S_{n-1}$ is standard so that E_n is rational over k.

Now, we have

Theorem 2. J_n is (a direct summand of) a quasi-permutation S_n-module if and only if n = 2, 3.

Equivalently, we have

Theorem 2'. $L_n(m)$ is stably rational (retract rational) over E_n if and only if n = 2, 3.

The if part of Theorem 2 is shown in [9] and [6]. The only if part can be shown by dividing it into the following two cases:

(i) n is not square-free; (ii) n is square-free.

For the case (i), this follows directly from (5) and non-rationality of the Chevalley module ([1, V], [5, 1.5]). It is noted that this was also shown by D. J. Saltman (unpublished). It was remarked by R. L. Snider ([7, p.319]) that J_4 is not a quasi-permutation S_4-module, and a proof of it is given in [3, 9.9]. On the other hand, for the case (ii), this follows from (5) and Theorem 1.

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RING EPIMORPHISMS AND TORSION THEORIES

YAO MUSHENG

Let R, S be rings (associative and with identity). A ring homomorphism $\varphi: R \to S$ is called an epimorphism iff for any ring homomorphisms $\alpha, \beta: S \to C$, $\alpha\varphi = \beta\varphi$ always implies $\alpha = \beta$. Surjective homomorphisms are surely epimorphisms but epimorphisms need not be surjective. If $\varphi: R \to S$ is an epimorphism which makes S to be a flat left R-module, we call φ a right perfect epimorphism (or simply perfect map). Given a ring R, Mod-R will denote the category of right R-modules. A torsion theory τ defined on Mod-R is a pair $(\mathcal{T}, \mathcal{F})$ of classes of modules in Mod-R such that

i. $Hom_R(T,F) = 0$ for all $T \in \mathcal{T}$, $F \in \mathcal{F}$;
ii. if $Hom_R(C,F) = 0$ for all $F \in \mathcal{F}$, then $C \in \mathcal{T}$;
iii. if $Hom_R(T,C) = 0$ for all $T \in \mathcal{T}$, then $C \in \mathcal{F}$.

$\mathcal{T}$ is called the torsion class of τ, while $\mathcal{F}$ is called the torsionfree class of τ. If $\mathcal{T}$ is closed under taking submodules, then τ is called a hereditary torsion theory. The family of all torsion theories (resp. hereditary torsion theories) defined on Mod-R will be denoted by R-Tors (resp. R-tors). Let σ and τ be torsion theories on Mod-R. Then $\sigma \le \tau$ iff $\mathcal{T}_\sigma \subseteq \mathcal{T}_\tau$, where $\mathcal{T}_\sigma$ is the torsion class of σ and $\mathcal{T}_\tau$ is the torsion class of τ. For further notations and terminologies we refer to Stenström [6].

Let $\varphi: R \to S$ be a ring homomorphism. Then every right S-module M can be regarded as a right R-module: for any $r \in R$, $x \in M$, $x \cdot r = x\varphi(r)$. From this, we can define a canonical map $\varphi_\#$ from R-Tors to S-Tors (or R-tors to S-tors) as follows: for each $\sigma \in$ R-Tors, $\varphi_\#(\sigma) = \tau$ is defined by the condition that a right S-module N is τ-torsion iff N is σ-torsion as a right

This paper is in final form and no version of it will be submitted for publication elsewhere.

R-module (see [1]). When φ is a right perfect epimorphism, $\varphi_{\#}$ is a surjective map and relations between R-Tors and S-Tors have been extensively studied. But if φ is only an epimorphism, we merely know a little. In [2], Golan raised a problem: if φ is surjective, is the map $\varphi_{\#}$ surjective? He conjectured that $\varphi_{\#}$ is surjective for every surjective homomorphism from R if and only if R is weakly regular. But this conjecture is false. Mr. Sen Daching [5] proved that $\varphi_{\#}$ is surjective whenever φ is surjective. A more interesting problem is: if φ is an epimorphism, is $\varphi_{\#}$ surjective? In this paper, we shall show that under a bit stronger assumption the answer is yes and the known results can be regarded as corollaries. We shall also investigate relations between R-tors and S-tors, and generalize some known results.

(I)

Given a ring homomorphism $\varphi: R \to S$, we can define maps from S-tors to R-tors or from S-Tors to R-Tors. When φ makes S a flat left R-module, one can define a map $\varphi^{\#}$ from S-tors to R-tors which assigns to each torsion theory τ on Mod-S the torsion theory $\sigma = \varphi^{\#}(\tau)$ defined by the condition that a right R-module M is σ-torsion iff $M \otimes_R S$ is τ-torsion. Obviously, $\varphi^{\#}$ preserves orders. When φ is a right perfect epimorphism, $\varphi_{\#}\varphi^{\#}$ is the identity map of S-tors. Therefore $\varphi_{\#}$ is surjective. This result can be found in [1].

The second mapping φ^{g} from S-Tors to R-Tors is defined as follows: for any $\tau \in$ S-Tors, $\varphi^{g}(\tau)$ is the torsion theory on Mod-R generated by the torsion class T_{τ} of τ. It can be shown that $\varphi^{g}(\tau)$ is hereditary for any $\tau \in$ S-tors when φ is surjective. In this case, Sen proved that $\varphi_{\#}\varphi^{g}(\tau) = \tau$ for any $\tau \in$ S-tors (see [5]).

The third mapping φ^{t} from S-tors to R-tors is defined as follows: Let F_{τ} be the Gabriel filter of S which corresponds to the hereditary torsion theory τ on Mod-S, and let L = $\{I \leq R_R \mid I \supseteq \varphi^{-1}(J)$ for some $J \in F_{\tau}\}$. It is not difficult to verify that L is a linear topology on R but in general it is not a Gabriel filter. Let F be the Gabriel filter generated by L, and let σ be the corresponding hereditary torsion theory. We define $\varphi^{t}(\tau) = \sigma$.

We are now going to define the fourth mapping φ^e from S-tors to R-tors, which plays an important role in this paper. Let $\tau \in$ S-tors. Then τ can be cogenerated by an injective S-module E_0. Let E be the injective hull of E_0 regarded as a right R-module. Then E cogenerates a hereditary torsion theory σ on Mod-R. We put $\varphi^e(\tau) = \sigma$.

Lemma 1. Let $\varphi: R \to S$ be a ring epimorphism. Then for any right S-modules M and N, $\mathrm{Hom}_R(M,N) = \mathrm{Hom}_S(M,N)$.

This is well-known (see [6]).

Definition. Let $\varphi: R \to S$ be a ring homomorphism. If for any $s \in S$, there exist $a_1, \ldots, a_n \in R$ and $b_1, \ldots, b_n \in S$ such that $s\varphi(a_i) \in \varphi(R)$ for all i and $\sum_{i=1}^{n} \varphi(a_i)b_i = 1$, then φ is called a strong epimorphism.

Remark 1. Every perfect epimorphism is a strong epimorphism (see [6]), but the converse is not true. For example, every surjective ring epimorphism is surely a strong epimorphism, but it need not be flat, so need not be perfect.

Remark 2. Every strong epimorphism is a ring epimorphism (see [6]).

Remark 3. In the above definition, a_i, b_i and n may depend on s.

Lemma 2. Let $\varphi: R \to S$ be a strong epimorphism, and M a right S-module. Let E_0 be an injective S-module, and E the injective hull of E_0 as R-module. Then $\mathrm{Hom}_R(M,E) = 0$ if and only if $\mathrm{Hom}_R(M,E_0) = 0$, or equivalently $\mathrm{Hom}_S(M,E_0) = 0$.

Proof. The only if part is obvious. We now assume that $\mathrm{Hom}_R(M,E) \neq 0$. Then there is a non-zero element $\alpha \in \mathrm{Hom}_R(M,E)$ such that $\mathrm{Im}\,\alpha \cap E_0 \neq 0$, and so $0 \neq \alpha(x) \in E_0$ with some $x \in M$. Since M is a right S-module, xS is an S-submodule as well as an R-submodule of M. The restriction α' of α on xS is then an R-homomorphism from xS into E. We want to show that $\alpha'(xs) = \alpha'(x)s$ for each $s \in S$. Since φ is a strong epimorphism, there exist $a_1, \ldots, a_n \in R$ and $b_1, \ldots, b_n \in S$ such that $s\varphi(a_i) \in \varphi(R)$ and $\sum \varphi(a_i)b_i = 1$. Therefore, $s = s\sum\varphi(a_i)b_i$ and $\alpha'(x)s =$

$\alpha(x)s\sum\varphi(a_i)b_i = \sum\alpha(x)s\varphi(a_i)b_i = \sum\alpha(xs\varphi(a_i))b_i = \sum\alpha(xs)\varphi(a_i)b_i =$ $\alpha(xs)\sum\varphi(a_i)b_i = \alpha'(xs)$. This shows that the image of α' belongs to E_0, for E_0 is an S-module and $\alpha(x) \in E_0$. By Lemma 1, α' is also an S-homomorphism from xS into E_0. Since E_0 is S-injective, α' can be extended to an S-homomorphism of M into E_0, that is, $\mathrm{Hom}_S(M,E_0) \neq 0$, or equivalently $\mathrm{Hom}_R(M,E_0) \neq 0$.

Theorem 3. Let $\varphi: R \to S$ be a strong epimorphism. Then $\varphi_{\#}\varphi^e(\tau) = \tau$ for any $\tau \in$ S-tors, and therefore $\varphi_{\#}$ is surjective.

Proof. Let $\sigma = \varphi^e(\tau)$, and $\tau' = \varphi_{\#}(\sigma)$. The torsion classes corresponding to τ and τ' are denoted by T_τ and $T_{\tau'}$, respectively. We want to show that $T_\tau = T_{\tau'}$. Assume that τ is cogenerated by an S-injective module E_0. If E is the R-injective hull of E_0, then σ is cogenerated by E. Let M be a right S-module with $M \in T_{\tau'}$. By the definition of $\varphi_{\#}$, $M \in T_\sigma$ as right R-module, and so $\mathrm{Hom}_R(M,E) = 0$. Hence $\mathrm{Hom}_S(M,E_0) = 0$ by Lemma 2. This means that $M \in T_\tau$, so $T_{\tau'} \subseteq T_\tau$. Conversely, let $M \in T_\tau$. Then $\mathrm{Hom}_S(M,E_0) = 0$. Obviously, $\mathrm{Hom}_R(M,E) = 0$, and so $M \in T_\sigma$ as R-module. Then $M \in T_{\tau'}$ as right S-module. This proves that $T_\tau \subseteq T_{\tau'}$.

We are now going to study the relations among the four maps $\varphi^{\#}$, φ^t, φ^g and φ^e. For convenience, they are all regarded as maps from S-tors to R-Tors.

Proposition 4. Let $\varphi: R \to S$ be a strong epimorphism. Then for any $\tau \in$ S-tors, $\varphi^g(\tau) \le \varphi^t(\tau) \le \varphi^e(\tau) \le \varphi^{\#}(\tau)$. If φ is surjective then $\varphi^g(\tau) = \varphi^t(\tau)$; if φ is perfect then $\varphi^e(\tau) = \varphi^{\#}(\tau)$.

Proof. (i) First, we have to verify that $T_{\varphi^g(\tau)}$ (abbrev. T_g) $\subseteq T_{\varphi^t(\tau)}$ (abbrev. T_t). Since T_g is generated by T_τ (as R-module), it is enough to show that $T_\tau \subseteq T_t$. Let $M \in T_\tau$. Then, for any $x \in M$, the annihilator $(0:x)_S$ of x in S is in the Gabriel filter F_τ of S corresponding to τ. But, as R-module, the annihilator $(0:x)_R$ of x in R contains $\varphi^{-1}((0:x)_S)$. In fact, if $r \in \varphi^{-1}((0:x)_S)$, i.e., $\varphi(r) \in (0:x)_S$, then $x\varphi(r) = 0$, so $r \in (0:x)_R$. By the definition of $\varphi^t(\tau)$, $(0:x)_R$ belongs to the Gabriel filter F_t of R corresponding to $\varphi^t(\tau)$. This shows that M is $\varphi^t(\tau)$-torsion, and therefore $T_g \subseteq T_t$.

(ii) In order to see that $\varphi^t(\tau) \leq \varphi^e(\tau)$, it suffices to show that F_t is contained in the Gabriel filter F_e of R corresponding to $\varphi^e(\tau)$. Since F_t is generated by L, it is enough to show that $L \subseteq F_e$. Let $I \in L$. Then $I \supseteq \varphi^{-1}(J)$ for some $J \in F_\tau$. Since S/J is τ-torsion, we get $\mathrm{Hom}_S(S/J,E_0) = 0$, which implies that $\mathrm{Hom}_R(S/J,E) = 0$ (Lemma 2). But there is an embedding $0 \to R/\varphi^{-1}(J) \to S/J$ and E is R-injective, so every homomorphism from $R/\varphi^{-1}(J)$ to E can be extended to a homomorphism from S/J to E. Therefore $\mathrm{Hom}_R(R/\varphi^{-1}(J),E) = 0$. This shows that $\varphi^{-1}(J) \in F_e$, so $I \in F_e$.

(iii) Next, we shall show that T_e is contained in the torsion class $T_\#$ of $\varphi^\#(\tau)$. Let $M \in T_e$. Then $\mathrm{Hom}_R(M,E) = 0$, so $\mathrm{Hom}_R(M,E_0) = 0$. Hence $\mathrm{Hom}_S(M \otimes_R S,E_0) \cong \mathrm{Hom}_R(M,\mathrm{Hom}_S(S,E_0)) \cong \mathrm{Hom}_R(M,E_0) = 0$. Therefore $M \otimes_R S \in T_\tau$, which means that $M \in T_\#$.

(iv) Now assume that φ is surjective. (Then $\varphi^g(\tau)$ is hereditary. See [5].) We want to show that $\varphi^g(\tau) = \varphi^t(\tau)$. To see this, it suffices to show that $L \subseteq F_g$. For any $I \in L$, there exists a right ideal J of S such that $J \in F_\tau$ and $I \supseteq \varphi^{-1}(J)$. Since φ is surjective, $R/\varphi^{-1}(J) \cong S/J$, which implies that $R/\varphi^{-1}(J)$ is τ-torsion. (Obviously, $R/\varphi^{-1}(J)$ can be regarded as a right S-module.) Then $R/\varphi^{-1}(J)$ is $\varphi^g(\tau)$-torsion, because T_g is generated by T_τ as R-module. This shows that $\varphi^{-1}(J) \in F_g$, and so $I \in F_g$.

(v) Finally, assume that φ is perfect. In order to see that $\varphi^e(\tau) = \varphi^\#(\tau)$, it suffices to show that $T_\# \subseteq T_e$. Let $M \in T_\#$. Then $M \otimes_R S \in T_\tau$ and $\mathrm{Hom}_S(M \otimes_R S,E_0) = 0$. Noting that E_0 is also R-injective and $E = E_0$, we see that $\mathrm{Hom}_R(M,E) = \mathrm{Hom}_R(M,E_0) \cong \mathrm{Hom}_R(M,\mathrm{Hom}_S(S,E_0)) \cong \mathrm{Hom}_S(M \otimes_R S,E_0) = 0$. This proves that $M \in T_e$.

Corollary 5. If $\varphi: R \to S$ is a right perfect homomorphism, then $\varphi_\#\varphi^\#(\tau) = \tau$ for any $\tau \in$ S-tors.

Proposition 6. If $\varphi: R \to S$ is a strong epimorphism, then $\varphi_\#\varphi^g(\tau) = \tau$ for any $\tau \in$ S-Tors. In particular, if φ is surjective then φ^g is a map from S-tors to R-tors and $\varphi_\#\varphi^g(\tau) = \tau$ for any $\tau \in$ S-tors.

Proof. By definition, $T_\tau \subseteq T_{\varphi^g(\tau)}$, where modules in T_τ are regarded as R-modules. So, $T_\tau \subseteq T_{\varphi_\# \varphi^g(\tau)}$ as S-module. Now, let $0 \neq M \in T_{\varphi_\# \varphi^g(\tau)}$, i.e., $M \in T_{\varphi^g(\tau)}$ as R-module, and suppose that $M \bar{\in} T_\tau$. Then we may assume that M is τ-torsionfree. But $T_{\varphi^g(\tau)}$ is generated by T_τ, and $\mathrm{Hom}_R(T,M) = 0$ $(T \in T_\tau)$ yields a contradiction that M is $\varphi^g(\tau)$-torsionfree. Therefore $T_{\varphi_\# \varphi^g(\tau)} = T_\tau$, i.e., $\varphi_\# \varphi^g(\tau) = \tau$.

(II)

Let $\varphi: R \to S$ be a ring homomorphism and let σ be a torsion theory on Mod-R with $\tau = \varphi_\#(\sigma)$. It is not necessarily the case that a right S-module M is τ-torsionfree iff it is σ-torsionfree as right R-module. If this condition happens to hold, we say that σ is compatible with φ.

Proposition 7. If $\varphi: R \to S$ is a strong epimorphism and $\tau \in$ S-tors, then $\varphi^e(\tau)$ is compatible with φ.

Proof. Let $\sigma = \varphi^e(\tau)$. Then $\tau = \varphi_\#(\sigma)$, by Theorem 3. If N is a τ-torsionfree S-module, then N can be embedded in E_0^A with some index set A. Now, in view of Lemma 1, this embedding is also an R-module embedding. Moreover, E_0 is an R-submodule of E, so N can be embedded in E^A. This shows that N is σ-torsionfree. Conversely, if M is a right S-module which is σ-torsionfree as R-module, then $\mathrm{Hom}_R(T,M) = 0$ for any $T \in T_\sigma$, especially for any $T \in T_\tau$. This means that M is τ-torsionfree. We have thus seen that σ is compatible with φ.

Proposition 8. Let $\varphi: R \to S$ be a strong epimorphism, $\tau \in$ S-tors, $\sigma = \varphi^e(\tau)$, and M a right S-module. Then there hold the following:

(i) An S-submodule N of M is τ-dense iff N is σ-dense in M, where M and N are regarded as R-modules.

(ii) If M is σ-injective, then M is τ-injective. In particular, if M is σ-closed then it is τ-closed.

Proof. Noting that M/N is τ-torsion iff M/N is σ-torsion, we can easily see (i). In order to see (ii), it suffices to recall that every τ-dense right ideal of S is σ-dense.

Let $\sigma \in$ R-tors. A left R-module N is called σ-flat if for each exact sequence $0 \to K \to M$ of right R-modules such that K

is σ-dense in M, the sequence $0 \to K \otimes_R N \to M \otimes_R N$ is still exact [3]. The following proposition sharpens a result in [1].

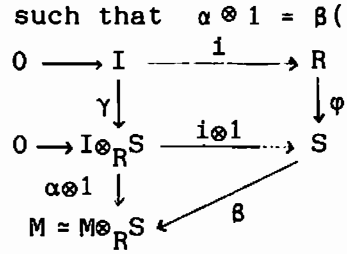
Proposition 9. Let $\varphi: R \to S$ be a strong epimorphism, $\tau \in$ S-tors, and $\sigma = \varphi^e(\tau)$. Then the following are equivalent:

(i) ${}_RS$ is σ-flat.

(ii) A right S-module M is τ-injective iff it is σ-injective as R-module.

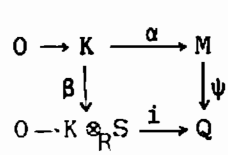
When this is the case, a right S-module M is τ-closed iff it is σ-closed.

Proof. (i) $\Rightarrow$ (ii). Suppose that ${}_RS$ is σ-flat. In view of Proposition 8, it remains only to prove the only if part. Let M be a τ-injective S-module, and I a σ-dense right ideal of R. Since ${}_RS$ is σ-flat, the sequence $0 \to I \otimes_R S \xrightarrow{i\otimes 1} R \otimes_R S \cong S$ is exact. Given an R-homomorphism $\alpha: I \to M$, we define $\gamma: I \to I \otimes_R S$ by $\gamma(a) = a \otimes 1$. Since R/I is σ-torsion and σ is compatible with φ (Proposition 7), $R/I \otimes_R S$ is τ-torsion (see [1, Prop. 47.2]). Now, it is easy to see that $R/I \otimes_R S \cong S/\varphi(I)S$. Moreover, ${}_RS$ being σ-flat, $\varphi(I)S \cong I \otimes_R S$. Thus $I \otimes_R S$ is σ-dense as well as τ-dense in S. Now, recalling that M is τ-injective, we can find an S-homomorphism $\beta: S \to M$ such that $\alpha \otimes 1 = \beta(i \otimes 1)$.



Then $(\beta\varphi)i = \alpha$ and M is σ-injective.

(ii) $\Rightarrow$ (i). Let $0 \to K \xrightarrow{\alpha} M$ be an exact sequence of R-modules such that K is σ-dense in M. Let Q be the τ-injective hull of $K \otimes_R S$. Then Q is σ-injective as R-module by hypothesis, and there exists an R-homomorphism $\psi: M \to Q$ such that the following diagram commutes:



where i is the inclusion map and $\beta(k) = k \otimes 1$ ($k \in K$). Apply the functor $\sim \otimes_R S$ to the diagram. Noting that every right S-module W is isomorphic to $W \otimes_R S$ and that φ is an epimorphism (see [1]), we get the following commutative diagram:

$$\begin{array}{ccccc} & K\otimes_R S & \xrightarrow{\alpha\otimes 1} & M\otimes_R S \\ & {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle \psi\otimes 1} \\ 0 \longrightarrow & K\otimes_R S & \xrightarrow{\quad i \quad} & Q \end{array}$$

Clearly, $\alpha \otimes 1$ is monic.

Since σ is compatible with φ, the latter assertion is an easy consequence of (ii).

Proposition 10. Let $\varphi\colon R \to S$ be a strong epimorphism, $\tau \in$ S-tors, and $\sigma = \varphi^e(\tau)$. Then there hold the following:

(i) If σ is Noetherian, then so is τ.

(ii) If σ is of finite type, then so is τ.

(iii) If σ is stable, then so is τ.

If, furthermore, ${}_R S$ is σ-flat, then there hold the following:

(iv) If σ is exact, then so is τ.

(v) If σ is perfect, then so is τ.

Proof. (i) Let $J_1 \subseteq J_2 \subseteq \dots$ be an ascending chain of right ideals of S such that $J = \bigcup J_i$ is τ-dense in S. Putting $I_i = \varphi^{-1}(J_i)$, we get $I = \varphi^{-1}(J) = \bigcup I_i$. Since σ is Noetherian, I_k is σ-dense in R for some k. By [1, Prop. 47.2], I_k is τ-dense in S.

(ii) Let J be a τ-dense right ideal of S. Then $\varphi^{-1}(J)$ is σ-dense in R. Since σ is of finite type, there is a finitely generated σ-dense right ideal I of R. Then $\varphi(I)S$ is a finitely generated right ideal of S contained in J, and it is τ-dense by $\varphi^{-1}(\varphi(I)S) \supseteq I$. Hence τ is of finite type.

(iii) Let M be a τ-torsion S-module. We want to show that the S-injective hull Q_0 of M is also τ-torsion. Let Q be the R-injective hull of M. Then Q is σ-torsion, since σ is stable. Consider the following diagram:

$$\begin{array}{ccccc} 0 \longrightarrow & M & \xrightarrow{\quad i \quad} & Q_0 \\ & {\scriptstyle j}\downarrow & \swarrow{\scriptstyle \alpha} & \\ & Q & & \end{array}$$

where i and j are injections. By the injectivity of the R-module Q, there is an R-homomorphism $\alpha: Q_0 \to Q$ which makes the diagram commutative. Obviously, α is an embedding, and Q_0 is σ-torsion as R-module, i.e., Q_0 is τ-torsion as S-module.

Henceforth, we assume further that ${}_RS$ is σ-flat. Then the τ-localization and σ-localization of any right S-module coincide.

(iv) Recall that a hereditary torsion theory τ is exact iff the localizing functor corresponding to τ is exact. We denote by Q_τ and Q_σ the localizing functors corresponding to τ and σ, respectively. Since Q_σ is exact, it suffices to show that $Q_\tau(M) = Q_\sigma(M)$ for any right S-module M. Since $\tau = \varphi_\#(\sigma)$ and σ is compatible with φ, the set of τ-torsion submodules of M and the set of σ-torsion submodules of M coincide. Hence, by Proposition 9, $Q_\tau(M) = Q_\sigma(M)$.

(v) This is an easy combination of (iii) and (iv).

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ON SEPARABLE POLYNOMIALS

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Throughout this report, B will represent a ring with 1, ρ an automorphism of B, and D a ρ-derivation of B (i.e. an additive endomorphism of B such that $D(ab) = D(a)\rho(b) + aD(b)$ for all $a, b \in B$). Let $R = B[X;\rho,D]$ be the skew polynomial ring in which the multiplication is given by $aX = X\rho(a) + D(a)$ $(a \in B)$. By $R_{(0)}$ we denote the set of all monic polynomials g in R with $gR = Rg$. We shall use the following conventions: $U(A)$ = the set of all invertible elements of a ring A; $B^{\rho,D} = \{a \in B \mid \rho(a) = a, D(a) = 0\}$. A ring extension A/B is called a separable extension if the A-A-map $A \otimes_B A \to A$ defined by $x \otimes y \mapsto xy$ $(x, y \in A)$ splits. A polynomial g in $R_{(0)}$ is called a separable polynomial if R/fR is a separable extension of B. Moreover, a ring extension A/B is called to be G-Galois if there exists a finite group G of automorphisms of A such that $B = A^G$ (the fixed ring of G in A) and $\Sigma_i x_i\sigma(y_i) = \delta_{1,\sigma}$ $(\sigma \in G)$ for some finite $x_i, y_i \in A$.

In the rest of this report, we assume that $\rho D = D\rho$. Let f be in $R_{(0)} \cap B^{\rho,D}[X]$ and degree of f is m. As was shown in [1, Lemma 1.2], f is in $C(B^{\rho,D})[X]$, where $C(B^{\rho,D})$ is the center of $B^{\rho,D}$. The $C(B^{\rho,D})$-module $C(B^{\rho,D})[X]/fC(B^{\rho,D})[X]$ has a free basis $\{1, x,\cdots,x^{m-1}\}$ where $x = X + fC(B^{\rho,D})[X]$.

The detailed version of this paper will be submitted for publication elsewhere.

Let π_i be the projection of $C(B^{\rho,D})[X]/fC(B^{\rho,D})[X]$ onto the coefficients of x^i. The trace map t is defined by $t(z) = \Sigma_{i=0}^{m-1} \pi_i(zx^i)$ $(z \in C(B^{\rho,D})[X]/fC(B^{\rho,D})[X])$. Then the discriminant $\delta(f)$ is defined by $\delta(f) = \det\|t(x^k x^\ell)\|$ $(0 \leq k, \ell \leq m-1)$.

Now, we shall introduce here the following definition.

Definition. If a ring extension of B is generated by a set $\{\alpha_1, \cdots, \alpha_m\}$ such that $\alpha_i\alpha_j = \alpha_j\alpha_i$ and $b\alpha_i = \alpha_i\rho(b) + D(b)$ for all i, j and $b \in B$ then it will be denoted by $B[\alpha_1, \cdots, \alpha_m; \rho, D]$. Let f be a polynomial in $R_{(0)} \cap B^{\rho,D}[X]$ of degree m. If $S = B[\alpha_1, \cdots, \alpha_m; \rho, D]$ and $f = (X - \alpha_1) \cdots (X - \alpha_m)$ in $B^{\rho,D}[\alpha_1, \cdots, \alpha_m][X]$ then S will be called a splitting ring of f over B. Moreover, a splitting ring $A = B[x_1, \cdots, x_m; \rho, D]$ of f is said to be universal if for any splitting ring $S = B[\alpha_1, \cdots, \alpha_m; \rho, D]$ of f, there exists a B-ring homomorphism of $A \to S$ mapping x_i into α_i for $i = 1, \cdots, m$.

Now, let $f = X^m + X^{m-1}a_{m-1} + \cdots + Xa_1 + a_0 \in R_{(0)} \cap B^{\rho,D}[X]$ and $R_m = B[X_1, \cdots, X_m; \rho, D]$ where $X_1, \cdots, X_m$ are indeterminates which are independent. Moreover, for elementary symmetric polynomials s_i of $X_1, \cdots, X_m$ (deg $s_i = i$, $i = 1, \cdots, m$), we set $t_i = a_{m-i} - (-1)^i s_i$ and $N_f = \Sigma_{i=1}^m R_m t_i$. Then $t_i b = \Sigma_{j=m-i}^m \binom{j}{m-i} \rho^{-i} D^{j-(m-i)}(b) t_{m-j}$ $(b \in B)$ and $t_i X_j = X_j t_i$ $(1 \leq i, j \leq m)$. Hence N_f is an ideal of R_m. By R_f, we denote the factor ring R_m/N_f.

Under this situation, we have the following

Proposition 1. Let f be a polynomial in $R_{(0)} \cap B^{\rho,D}[X]$ of degree m. Then R_f is a universal splitting ring of f. Moreover, for any universal splitting ring $A = B[x_1, \cdots, x_m; \rho, D]$ of f, there exists a B-ring isomorphism of $A \to R_f$ mapping

x_i into $X_i + N_f$ for $i = 1, \cdots, m$.

Let $f \in R_{(0)} \cap B^{\rho,D}[X]$ and $B[\alpha_1, \cdots, \alpha_m; \rho, D]$ a splitting ring of f. Then $f \in C(B^{\rho,D})[X]$ and $C(B^{\rho,D})[\alpha_1, \cdots, \alpha_m]$ is a splitting ring of f over $C(B^{\rho,D})$. Then we obtain the following

Proposition 2. Let f be a polynomial in $R_{(0)} \cap B^{\rho,D}[X]$ of degree m, and $B[\alpha_1, \cdots, \alpha_m; \rho, D]$ any splitting ring of f. Then

(1) $\delta(f) = \prod_{i<j}(\alpha_i - \alpha_j)^2$,

(2) $b\delta(f) = \delta(f)\rho^{m(m-1)}(b)$ for all $b \in B$. (Cf. [2, Lemma 1.1].)

In [3], [4] and [5], T. Nagahara and A. Nakajima presented a theory of splitting rings of separable polynomials over a commutative ring, and in [7] T. Nagahara studied splitting rings of some type of separable polynomials in a skew polynomial ring of automorphism type $B[X;\rho]$ ($= B[X;\rho,0]$).

The present report is a study about splitting rings of separable polynomials in the skew polynomial ring $B[X;\rho,D]$ with $\rho D = D\rho$. We shall generalize the results of T. Nagahara [7] and obtain some other related results.

Next, we shall state one of our main theorems.

Let f be a polynomial in $R_{(0)} \cap B^{\rho,D}[X]$ of degree m and $A = B[x_1, \cdots, x_m; \rho, D]$ be a universal splitting ring of f. Let S_m be the symmetric group of the set $\{1, \cdots, m\}$. Then, for every $\sigma \in S_m$, we have a B-ring automorphism σ^* of A mapping x_i into $x_{\sigma(i)}$ for $i = 1, \cdots, m$. Obviously, the mapping $(^*) : \sigma \mapsto \sigma^*$ is a group homomorphism of S_m into the group of B-ring automorphisms of A. In the following theorem, the image of $(^*)$ will be denoted by S_V where $V = \{x_1, \cdots, x_m\}$.

In case $m > 2$, we see that $(*)$ is a monomorphism, that is, $S_m \cong S_V$ (Cf. [3, Remark 1.1]).

Theorem (Cf. [7, Theorem 4]). Let f be a polynomial in $R_{(0)} \cap B^{\rho,D}[X]$ of degree m, and $A = B[V;\rho,D]$ ($V = \{x_1,\cdots,x_m\}$) be a universal splitting ring of f. Then the following conditions are equivalent.

(1) $\delta(f) \in U(B)$.

(2) $A/B[V\backslash W]$ is S_W-Galois for every subset W of V.

(3) $A/B[V\backslash\{x_1, x_2\}]$ is $S_{\{x_1, x_2\}}$-Galois.

(4) $x_1 - x_2 \in U(A)$.

Remark. Let g be a polynomial in $B[X;\rho]_{(0)} \cap B^{\rho}[X]$ of degree 2. By [6, Theorem 2.5] we know g is Galois if and only if $\delta(g) \in U(B)$. Therefore if $R = B[X;\rho]$, the condition (3) may be replaced with (3)'

(3)' $A/B[V\backslash\{x_1, x_2\}]$ is Galois.

We obtained some other related results concerning the splitting rings and the details will be appear in the forthcoming paper.

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ON ARTINIAN RINGS OF FINITE GLOBAL DIMENSION

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In this talk, I summarize my joint work with Y. Yukimoto [5] and raise a question for artinian rings of finite global dimension.

1. Introduction. Throughout, A stands for any basic left and right artinian ring, J its Jacobson radical and $\{e_1,\ldots,e_n\}$ the complete set of orthogonal primitive idempotents in A. Let c_{ij} denote the composition length of e_iAe_j over e_iAe_i for $1 \leqq i, j \leqq n$. The matrix $C(A) = (c_{ij})$ is called the left Cartan matrix of A. Does $\mathrm{gl\,dim}\, A < \infty$ imply $\det C(A) = 1$? This has been partially answered by several authors(e.g., Zacharia [7], Wilson [6], Burgess et al. [1], Fuller and Zimmermann-Huisgen [4] and so on), but is still open.

The above problem would be a consequence of the following

<u>Question</u>. Does $\mathrm{gl\,dim}\, A < \infty$ ensure the existence of a torsionless left A-module Q such that

(a) $D = \mathrm{End}_A(Q)$ is a division ring,

(b) the evaluation map $Q \otimes_D \mathrm{Hom}_A(Q,A) \to A$ is monic and

(c) $\mathrm{Tor}_k^A(\mathrm{Tr}Q,Q) = 0$ for $k \geqq 2$, where Tr is the transpose?

In case $\mathrm{proj\,dim}_A Q \leqq 1$, the condition (c) is automatically satisfied. Thus, this question is affirmative if $\mathrm{gl\,dim}\, A \leqq 3$, because $\mathrm{gl\,dim}\, A < \infty$ ensures the existence of a torsionless left A-module which satisfies the conditions (a) and (b) and has projective dimension $\leqq$ gl dim A - 2 (if gl dim A $\geqq$ 2).

2. A generalization of heredity ideals. The next theorem shows that, if our question is always affirmative, so is the Cartan determinant problem.

<u>Theorem</u>([5, Theorem]). Let Q be a torsionless left A-module and I its trace ideal. Suppose the following conditions:

(a) $D = \mathrm{End}_A(Q)$ is a division ring,

(b) the evaluation map $Q \otimes_D \mathrm{Hom}_A(Q,A) \to A$ is monic,

(c) $\mathrm{Tor}_k^A(\mathrm{Tr}Q,Q) = 0$ for $k \geqq 2$, where Tr is the transpose,

and

(d) $\mathrm{proj\,dim}_A Q < \infty$.

Then we have

(1) $\mathrm{gl\,dim}\, A/I \leqq \mathrm{gl\,dim}\, A + \mathrm{proj\,dim}_A Q$,

(2) $\mathrm{gl\,dim}\, A \leqq \mathrm{gl\,dim}\, A/I + \max\{2, \mathrm{proj\,dim}_A Q + 1\}$ and

(3) $\det C(A/I) = \det C(A)$.

In case ${}_AQ$ is projective, the ideal I is just a heredity ideal (see Dlab and Ringel [3]), the notion of which was first introduced by Cline, Parshall and Scott [2], and the statements (1) and (2) have been known.

Dlab and Ringel [3, Theorem 2] showed that, if gl dim A $\leqq$ 2, there always exists a projective left A-module which satisfies the conditions (a) and (b). Thus, one can apply the above theorem repeatedly to conclude that gl dim A $\leqq$ 2 implies $\det C(A) = 1$. This induction is different from Zacharia's one [7]. Another example is the case of A being left serial. In that case, gl dim A $< \infty$ ensures the existence of a simple torsionless left A-module Q with $\mathrm{proj\,dim}_A Q \leqq 1$ (cf. Burgess et al. [1, Lemma 2]).

The next proposition shows that our question is affirmative if gl dim A $\leqq$ 3.

<u>Proposition</u> 1([5, Proposition 1]). Suppose $2 \leqq \mathrm{inj\,dim}_A A = m < \infty$. Let Q be minimal with respect to inclusions in the

class of all non-zero torsionless left A-modules X with $\mathrm{Ext}_A^k(X,A) = 0$ for $k \geqq m - 1$. Then Q satisfies the conditions (a) and (b).

Remark. In case $\mathrm{gl\,dim}\, A < \infty$, $\mathrm{inj\,dim}\, {}_A A = \mathrm{gl\,dim}\, A$ and for any left A-module X $\mathrm{proj\,dim}\, {}_A X \leqq r$ if and only if $\mathrm{Ext}_A^k(X,A) = 0$ for $k > r$.

3. Zacharia's reduction. There is a way to reduce the size of the matrix $C(A)$. Namely, we have the following

<u>Proposition</u> 2(e.g. [5, Proposition 4]). Suppose that $\mathrm{proj\,dim}\, {}_A Ae_1/Je_1 < \infty$ and that $\mathrm{Ext}_A^k(Ae_1/Je_1, Ae_1/Je_1) = 0$ for $k > 0$. Then we have

(1) $\mathrm{gl\,dim}\,(1 - e_1)A(1 - e_1) \leqq \mathrm{gl\,dim}\, A + \mathrm{proj\,dim}\, {}_A Je_1$ and

(2) $\det C((1 - e_1)A(1 - e_1)) = \det C(A)$.

This reduction was first used by Zacharia [7] to show that $\mathrm{gl\,dim}\, A \leqq 2$ implies $\det C(A) = 1$ (see also Burgess et al. [1]). If $\mathrm{gl\,dim}\, A \geqq 3$, as shown by the next example, this reduction is not necessarily available.

Example. Let A be a subalgebra of $(F)_8$, the 8×8 matrix algebra over a field F, with basis elements

$$e_1 = \sum_{i=1}^{5} e_{ii}, \quad e_2 = \sum_{i=6}^{8} e_{ii}, \quad a = e_{26},$$

$$e_{36} + e_{47} + e_{58},\ e_{41} + e_{52},\ e_{71} + e_{82},\ e_{56} \text{ and } e_{86},$$

where e_{ij} are matrix units. Then $\mathrm{gl\,dim}\, A = 3$ and for $i = 1$ and 2 $\mathrm{Ext}_A^2(Ae_i/Je_i, Ae_i/Je_i) \neq 0$. On the other hand, one can take Ae_1/Je_1 or Ae_2/Aa as a torsionless left A-module which satisfies all the conditions in Theorem. Also, A does not have any heredity ideal.

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BLOCKS OF p-SOLVABLE GROUPS WITH TWO OR THREE SIMPLE MODULES

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1. Introduction. Let G be a finite p-solvable group, k a splitting field for G which has characteristic p and let B denote a block (ideal) of the group algebra kG. The number of non-isomorphic simple B-modules will be denoted by $\ell(B)$. In the present paper, we consider the Cartan matrix C_B of B with $\ell(B) = 2$. If $B = B_0$, the principal block, then, as is well known, $B_0 \simeq kG/O_{p'}(G)$. Therefore $\ell(B_0)$ is the number of p-regular classes in $G/O_{p'p}(G)$. In Section 2, we shall give the structure of $G/O_{p'p}(G)$ which has exactly two or three p-regular classes. In Section 3, we shall show that if $\ell(B) = 2$, C_B can be almost determined by means of the dimensions of simple B-modules, and in particular we shall show that if $\ell(B_0) = 2$, C_{B_0} is completely determined. In Section 4, We shall make a conjecture that if $\ell(B) = 2$ then the dimensions of two simple B-modules have the identical p'-parts, and state that this is true in certain special cases.

2. p-solvable groups with two or three p-regular classes. In this section we shall give the structure of p-solvable group G with $O_p(G) = \langle 1 \rangle$ which has two or three p-regular classes. At first assume that G has exactly two p-regular classes. If

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G is a p'-group then clearly p is odd and $|G| = 2$. On the other hand, if $|G|$ is divisible by p then a Sylow p-subgroup P of G acts transitively on $O_{p'}(G) - \{1\}$. In this case the order of $O_{p'}(G)$ and the structure of P are completely determined by Passman [8]. Hence we have the following

Theorem 1. *Let G be a p-solvable group with $O_p(G) = \langle 1\rangle$. Suppose that G has exactly two p-regular classes. Then G is either a p′-group or a p-nilpotent group; and*

(1) *if G is a p′-group then p is odd and $G \simeq Z_2$, and*

(2) *if G is a p-nilpotent group then one of the following holds:*

(a) $p = 2$ *and* $G \simeq E_{3^2} \rtimes Z_8$.

(b) $p = 2$ *and* $G \simeq E_{3^2} \rtimes Q_8$.

(c) $p = 2$ *and* $G \simeq E_{3^2} \rtimes S_{16}$.

(d) $p = 2$ *and* $G \simeq Z_q \rtimes Z_{2^n}$, *where* $q = 2^n + 1$ *is a Fermat prime.*

(e) $p = 2^n - 1$ *(a Mersenne prime) and* $G \simeq E_{2^n} \rtimes Z_p$.

We therefore see that if $\ell(B_0) = 2$ then $G/O_{p'p}(G)$ is isomorphic to one of the groups mentioned in the theorem and $B_0 \simeq kG/O_{p'}(G)$. The notation used in Theorem 1 will be introduced just after Theorem 2. By making use of this result, we can give the structure of p-solvable groups which have exactly three p-regular classes, that is, we have

Theorem 2. *Let G be a p-solvable group with $O_p(G) = \langle 1\rangle$. Suppose that G has exactly three p-regular classes. Then the p′-length of G is at most 2, and one of the following holds:*

(1) $p \neq 3$ *and* $G \simeq Z_3$.

(2) $p \neq 2, 3$ *and* $G \simeq \Sigma_3$.

(3) $p = 2$ *and* $G \simeq M(3) \rtimes P$, *where P is Z_8 or S_{16}.*

(4) $p = 3$ *and* $G \simeq Q_8 \rtimes Z_3$ $(\simeq SL(2,3))$.

(5) $p = 2$ *and* $G \simeq E_{3^2} \rtimes P$, *where P is Z_4 or D_8.*

(6) $p = 2$ *and* $G \simeq Z_q \rtimes Z_{2^n}$, *where* $q = 2^{n+1} + 1$ *is a Fermat prime.*

(7) $p \neq 2$ and $G \simeq Z_q \rtimes Z_{p^n}$, where $q = 2p^n + 1$ is a prime.

(8) $p \neq 2, 3$ and $G \simeq E_{3^\ell} \rtimes Z_{p^n}$, where $3^\ell = 2p^n + 1$.

(9) $p = 2$ and $G \simeq E_{3^4} \rtimes P$, where P is a 2-group which contains a normal subgroup R of index 2 satisfying one of the following conditions:

(a) $|R| = 2^5$ and $R = Z_8 \times_s Z_8$, $Q_8 \times_s Q_8$ or $S_{16} \times_s S_{16}$.

(b) $|R| = 2^6$ and $R = Z_8 \times Z_8$, $Q_8 \times Q_8$ or $S_{16} \times_s S_{16}$.

(c) $|R| = 2^7$ and $R = S_{16} \times_s S_{16}$.

(d) $|R| = 2^8$ and $R = S_{16} \times S_{16}$.

(10) $p = 2$ and $G \simeq Z_{q^2} \rtimes P$, where q is a Fermat prime greater than 3 and P is either

(a) a Sylow 2-subgroup of $GL(2,q)$, or

(b) a 2-group defined by

$$\langle x, y \mid x^{2^e} = 1,\ x^{2^{e-1}} = y^{2^e},\ x^y = x^{-1} \rangle,$$

where $2^e = q - 1$.

(11) $p = 2$ and $G \simeq E_{7^2} \rtimes T$, where T is a group generated by a normal subgroup R isomorphic to Q_8 and two elements w, x with the following properties:

$$w^3 = 1,\ x^2 \in R,\ x^4 = 1,\ w^x = w^{-1}.$$

(12) $p = 2$ and $G \simeq E_{5^2} \rtimes T$, where T is a group generated by a normal subgroup R isomorphic to $\mathcal{T}_0(5)$ and two elements w, x with the following properties:

$$w^3 = 1,\ x^2 \in R,\ x^8 = 1,\ w^x = w^{-1}.$$

We then see that if $\ell(B_0) = 3$ then $G/O_{p'p}(G)$ is isomorphic to one of the groups mentioned in the theorem and $B_0 \simeq kG/O_{p'}(G)$. The notation in the above theorems is as follows:

- Z_n the cyclic group of order n,
- E_{p^n} the elementary abelian group of order p^n,
- Σ_3 the symmetric group of degree 3,
- Q_8 the quaternion group of order 8,
- D_8 the dihedral group of order 8,
- S_{16} the semi-dihedral group of order 16,
- $M(3)$ the nonabelian 3-group which is of order 3^3

and has exponent 3,

that is, $H(3)$ is a group given by

$$\langle a,b,c \mid a^3 = b^3 = c^3 = 1,\ b^a = bc,\ c^a = c,\ c^b = c\rangle.$$

Further, following [8], p.229, we denote by $\mathcal{T}_0(5)$ the subgroup of $GL(2,5)$ consisting of the matrices

$$\begin{pmatrix} a & 0 \\ 0 & \pm a^{-1} \end{pmatrix},\quad \begin{pmatrix} 0 & a \\ \pm a^{-1} & 0 \end{pmatrix},\quad a \in GF(5),\quad a \neq 0.$$

Given two groups H and K, $H \rtimes K$ denotes a semidirect product of H by K, namely, H is normal in $H \rtimes K$ and $(H \rtimes K)/H \simeq K$; and $H \times_s K$ denotes a subdirect product of H and K, namely, $H \times_s K$ is a subgroup of the direct product $H \times K$ which satisfies

$$\varphi_H(H \times_s K) = H,\qquad \varphi_K(H \times_s K) = K,$$

where φ_H and φ_K are canonical homomorphisms of $H \times K$ onto H and K respectively.

The construction of Theorem 2 is as follows: It is easy to see that the p'-length of G is at most 2. At first, part (1) or (2) holds if G is a p'-group. Next, if G is p-nilpotent then we can see that $O_{p'}(G)$ is a q-group for some prime q. Part (3) or (4) holds for the case where $O_{p'}(G)$ is nonabelian. Now suppose $O_{p'}(G)$ is abelian. If a Sylow p-subgroup of G acts $\frac{1}{2}$-transitively on $O_{p'}(G)^{\#} = O_{p'}(G) - \{1\}$, then part (5), (6),(7) or (8) holds. On the other hand, if a Sylow p-subgroup of G does not act $\frac{1}{2}$-transitively on $O_{p'}(G)^{\#}$, then part (9) or (10) holds. Finally, the case $G = O_{p'pp'}(G)$ does not occur and part (11) or (12) holds for the case $G = O_{p'pp'p}(G)$.

3. The Cartan matrix of a block with two simple modules. Suppose $\ell(B) = 2$ and let S_1, S_2 be non-isomorphic simple B-modules. We denote by U_i a projective cover of S_i. Then by [3], the p-part of $\dim_k U_i$ is equal to p^a, the order of a Sylow p-subgroup of G, and the p'-part of $\dim_k U_i$ coincides with that of $\dim_k S_i$. Since B has a simple module whose vertex is equal to a defect group of B, we may assume that a vertex of S_1 is a defect group of B. Now let V_i be a vertex of

S_i. Then by [5], the p-part of $[G : V_i]$ is equal to that of $\dim_k S_i$. Set $f_i = \dim_k S_i$ and $u_i = \dim_k U_i$, and denote by f'_i the p'-part of f_i. If B is of defect d, then by the above, we can write as follows:

$$f_1 = p^{a-d} f'_1, \quad f_2 = p^{a-d+e} f'_2, \quad u_i = p^a f'_i = |V_i| f_i,$$

where $p^{d-e} = |V_2|$.

Now let $c_{ij} (1 \le i, j \le 2)$ be the Cartan invariants of B. By [5], we have $c_{ii} < |V_i|$ for $i = 1,2$. So we set $c_{ii} = |V_i| - q_i$, $0 < q_i < |V_i|$. Then from the equalities

$$c_{11} f_1 + c_{12} f_2 = |V_1| f_1, \quad c_{21} f_1 + c_{22} f_2 = |V_2| f_2,$$

we have

$$c_{12} f_2 = q_1 f_1, \quad c_{21} f_1 = q_2 f_2.$$

But $c_{12} = c_{21}$. Hence $q_1 f_1 / f_2 = q_2 f_2 / f_1$ and so $q_2 = q_1 (f_1/f_2)^2$. Since $|V_2| = p^{d-e}$, setting $q = q_1$, we have

$$(*) \qquad C_B = \begin{pmatrix} p^d - q & q f_1/f_2 \\ q f_1/f_2 & p^{d-e} - q(f_1/f_2)^2 \end{pmatrix}.$$

We are now in a position to state our theorem.

Theorem 3. *Let G be a p-solvable group and B a block of kG. If $\ell(B) = 2$, then $2e < d$ and the Cartan matrix of B is of the form*

$$C_B = \begin{pmatrix} p^d - p^{2e+\gamma} h & p^{e+\gamma} h f'_1/f'_2 \\ p^{e+\gamma} h f'_1/f'_2 & p^{\gamma}(1 + hp^e) \end{pmatrix},$$

where γ and h are integers with $0 \le \gamma < d - 2e$, $1 \le h < p^{d-2e-\gamma}$ and $(h, p) = 1$. Further, concerning the integers h and γ, we have

(i) *h satisfies the equality*

$$p^{d-e-\gamma} - h(p^e + (f'_1/f'_2)^2) = 1.$$

(ii) *p^{γ} is an elementary divisor of C_B.*

Proof. By (*), we have $c_{22} = p^{d-e} - (q/p^{2e})(f_1'/f_2')^2$, which shows that $q \equiv 0 \pmod{p^{2e}}$. Suppose now $2e \geq d$. Then $q \equiv 0 \pmod{p^d}$. This contradicts the fact that $0 < q < |V_1| = p^d$. Hence $2e < d$. Since $q \equiv 0 \pmod{p^{2e}}$, we may write $q = p^{2e+\gamma}h$, where $\gamma \geq 0$ and $(h, p) = 1$. Then from the form of C_B in (*), we have

$$\det C_B = p^d(p^{d-e} - p^\gamma h(p^e + (f_1'/f_2')^2)).$$

Noting that $\det C_B > 0$, we have $p^{d-e} > p^{e+\gamma}h$ and hence $\gamma < d-2e$ and $h < p^{d-2e-\gamma}$. We now claim that $f_1'^2 + p^e f_2'^2$ is not divisible by p. In fact, it is easy to see that

$$p^{2a-d}(f_1'^2 + p^e f_2'^2) = u_1 f_1 + u_2 f_2 = \dim_k B.$$

But the p-part of $\dim_k B$ is p^{2a-d}([2]). Hence $f_1'^2 + p^e f_2'^2$ is the p'-part of $\dim_k B$. Since

$$\det C_B = p^{d+\gamma}(p^{d-e-\gamma} - h(p^e + (f_1'/f_2')^2)),$$

and $\det C_B$ is a power of p, from the above, we see that the integer h satisfies the equality stated in (i). Combining this equality and (*), we see immediately that C_B is of the form as required. It is well known that the largest elementary divisor of C_B is p^d and $\det C_B$ is a product of elementary divisors. But $\det C_B = p^{d+\gamma}$. Thus (ii) follows, and we complete the proof of Theorem 3.

Remark 1. If $p = 2$ and $\ell(B) = 2$ then $e > 0$. In fact, we pointed out, in the proof of Theorem 3, that $f_1'^2 + 2^e f_2'^2$ is odd, and so clearly $e \neq 0$.

Corollary 1. *Let $p \neq 2$ and G a semidirect product of a p-group P of order p^a by a group $\langle x \rangle$ of order 2. We set $|C_P(x)| = p^\gamma (\gamma < a)$. Then the Cartan matrix of kG is given by*

$$\begin{pmatrix} p^\gamma(p^{a-\gamma} + 1)/2 & p^\gamma(p^{a-\gamma} - 1)/2 \\ p^\gamma(p^{a-\gamma} - 1)/2 & p^\gamma(p^{a-\gamma} + 1)/2 \end{pmatrix}.$$

From Theorem 3, we see that the possibilities for the Cartan matrix are remarkably restricted, and, in particular, the following corollary suggests that if the order of a defect group of G is small there is only a small number of possibilities for the Cartan matrix.

Corollary 2. *Let $p = 2$ and D a defect group of B. Suppose $\ell(B) = 2$. Then the following hold.*

(1) *If* $|D| = 2^3$, *then* $(e, \gamma, h) = (1, 0, 1)$, $f_1' = f_1'$ *and* $C_B = \begin{pmatrix} 4 & 2 \\ 2 & 3 \end{pmatrix}$.

(2) *If* $|D| = 2^4$, *then* $(e, \gamma, h) = (1, 1, 1)$, $f_1' = f_2'$ *and* $C_B = \begin{pmatrix} 8 & 4 \\ 4 & 6 \end{pmatrix}$.

(3) *If* $|D| = 2^5$, *then either* $(e, \gamma, h) = (1, 0, 5)$, $f_1' = f_2'$, $C_B = \begin{pmatrix} 12 & 10 \\ 10 & 11 \end{pmatrix}$ *or* $(e, \gamma, h) = (1, 2, 1)$, $f_1' = f_2'$, $C_B = \begin{pmatrix} 16 & 8 \\ 8 & 12 \end{pmatrix}$.

Example. We give some examples for Corollary 2. Let $p = 2$, and Σ_4 the symmetric group of degree 4. Then $k\Sigma_4$ is an example for (1). For case (2), we give three examples. One of them is the group algebra of $\Sigma_4 \times Z_2$. As is well known, Σ_4 has two representation groups. One of them is $GL(2,3)$ and the other one is the binary octahedral group G_{48} of order 48 (see [1], 5.6 Definition and [7], Ⅲ, Definition XⅡ.8.4). The group algebras of these groups are also examples for case (2). We note that G_{48} is a group T given in Theorem 2(11). If G is a semidirect product $(E_{2^2} \times E_{2^2}) \rtimes \Sigma_3$, where Σ_3 acts on each E_{2^2} as an automorphism group, then kG is an example for the first case of (3). For the second case of (3), we give five examples. Clearly the group algebras of $\Sigma_4 \times Z_4$, $\Sigma_4 \times E_{2^2}$, $GL(2,3) \times Z_2$, $G_{48} \times Z_2$ are examples for this case. Let T be a group given in Theorem 2(12). Then kT is also an example for this case.

Remark 2. By the same argument as in the proof of Corollary 2, we have the following:

(1) If $|D| = 2^6$, then $(e, \gamma, h) = (1, 1, 5)$, $(1, 3, 1)$ or $(2, 0, 3)$.

(2) If $|D| = 2^7$, then $(e,\gamma,h) = (1,2,5)$, $(1,4,1)$, $(2,1,3)$ or $(1,0,21)$,

(3) If $|D| = 2^8$, then $(e,\gamma,h) = (1,3,5)$, $(1,5,1)$, $(2,2,3)$ or $(1,1,21)$.

Further, in these cases it holds that $f_1' = f_2'$.

We next determine the Cartan matrix of the principal block B_0 with $\ell(B_0)= 2$. Since $B_0 \simeq kG/O_{p'}(G)$, to accomplish our end, it will suffice to determine the Cartan matrix of $kG/O_{p'}(G)$. So we asuume that $O_{p'}(G) = \langle 1\rangle$. Then $G/O_p(G)$ is one of the groups mentioned in Theorem 1. If p is odd and G satisfies condition (1) in the theorem, then as stated in Corollary 1, the Cartan matrix of kG is easily obtained. Accordingly, in the rest of this section, we concentrate our concern to obtain the Cartan matrix of kG for cases (a), (b), (c), (d) and (e) in Theorem 1(2).

Now let S be a non-trivial simple kG-module. Set $H = G/O_p(G)$ and $V = O_{p'}(H)$. Then S is a non-trivial simple kH-module. If case (a), (b), (d) or (e) occurs, then H is a complete Frobenius group with complement T, a Sylow p-subgroup of H. Therefore $S \simeq L^{\uparrow H}$ where L is a non-trivial simple kV-module. On the other hand, if case (c) occurs, then the inertial group $I_H(L)$ of a non-trivial simple kV-module L is isomorphic to $E_{3^2} \rtimes Z_2$, and there exists a simple $kI_H(L)$-module $\hat{L}$ such that $\hat{L}_{\downarrow V} \simeq L$ and $\hat{L}^{\uparrow H} \simeq S$. Further we can show that $k_T^{\uparrow H} \simeq k_H \oplus S$, where k_T and k_H are the trivial simple kT- and kH-modules respectively. From this we see that S is realizable in the field $GF(p)$. Hence we may assume $k = GF(p)$. Since S appears in the second Loewy layer of a projective cover of k_G, by Gaschütz's theorem ([7], II, Theorem VII.15.5), S is isomorphic to a complemented p-chief factor of G. Thus we obtain the following

Proposition 1. *Let G be a p-solvable group and B_0 the principal block of kG. Suppose $\ell(B_0) = 2$ and let S be a non-trivial simple B_0-module. If G has p-length 2, then the* *following hold:*

(1) $\dim_k S = \begin{cases} 8 & \textit{for case (a), (b) or (c).} \\ 2^n & \textit{for case (d).} \\ p & \textit{for case (e).} \end{cases}$

(2) *$GF(p)$ is a splitting field for $G/O_{p'}(G)$.*

(3) *We may regard, by (2), S as a G-module over $GF(p)$. Then S is isomorphic to some complemented p-chief factor of $O_{p'p}(G)$ in G.*

Again, let G be a p-solvable group with $O_{p'}(G) = \langle 1 \rangle$ which satisfies the condition stated in (a), (b), (c), (d) or (e). At first we consider the case where $O_p(G)$ is a minimal normal subgroup. We call such a group a group of type (a), of type (b),... or of type (e). We set $Q = O_p(G)$. We denote by H a complement of Q in G (see [7], II, Lemma VII.15.4), by T a Sylow p-subgroup of H, and by V the subgroup $O_{p'}(H)$. In order to determine the Cartan invariants of kG we have to calculate the number p^γ in Theorem 3. Since p^γ is an elementary divisor of the Cartan matrix of kG, it is the p-part of $|C_G(v)|$, where v is a non-trivial element of V. Hence

$$p^\gamma = \begin{cases} |C_Q(v)| & \text{if } G \text{ is of type (a),(b),(d) or (e),} \\ 2|C_Q(v)| & \text{if } G \text{ is of type (c).} \end{cases}$$

Since $Q \simeq S$ as kG-modules, the action of H on Q is induced from that of H on S. But we already know that $k_T^{\uparrow H} \simeq k_H \oplus S$, which implies that $S \simeq I_V kT \otimes_{kT} k\hat{T}$, where I_V is the augmentation ideal of kV and $\hat{T} = \sum_{t\in T} t$. From this we can obtain the value of p^γ. Hence we have the following:

Type of G	p^d	p^e	p^γ
(a),(b)	2^{11}	2^3	2^2
(c)	2^{12}	2^3	2^3
(d)	2^{2^n+n}	2^n	1
(e)	p^{p+1}	p	$p^{(p-1)/2}$

Combined with Theorem 3, we obtain the following

Theorem 4. *The Cartan invariants for the groups G of type* (a)~(e) *are as follows:*

Type of G	c_{11}	c_{12}	c_{22}
(a), (b)	2^8	$2^5\cdot 7$	$2^2\cdot 3\cdot 19$
(c)	2^9	$2^6\cdot 7$	$2^3\cdot 3\cdot 19$
(d)	$\dfrac{2^{2n}(2^{2^n-n}+1)}{2^n+1}$	$\dfrac{2^n(2^{2^n}-1)}{2^n+1}$	$\dfrac{2^{2^n+n}+1}{2^n+1}$
(e)	$\dfrac{p^{\frac{p+3}{2}}(p^{\frac{p-1}{2}}+1)}{p+1}$	$\dfrac{p^{\frac{p+1}{2}}(p^{\frac{p+1}{2}}-1)}{p+1}$	$\dfrac{p^{\frac{p-1}{2}}(p^{\frac{p+3}{2}}+1)}{p+1}$

Let G be a p-solvable group with $O_{p'}(G) = \langle 1\rangle$. Suppose $\ell(kG) = 2$. Set $P = O_p(G)$. In order to determine the Cartan invariants of kG, we have to calculate the order of $C_P(v)$, where v is a non-trivial p'-element of G. Since, by a theorem of Cossey, Fong and Gaschütz ([7], II, Theorem VII.13.9), any p-chief factor X of G is isomorphic to a simple kG-module, where $k = GF(p)$, it is isomorphic either to a trivial simple module or to a non-trivial simple module S. In the former case, $|X| = p$, while in the latter case, $|X| = p^{\lambda}$, where $\lambda = \dim_k S$. Assume that in a chief series of P in G, m factor groups are of order p and n factor groups are of order p^{λ}. Now set $p^{\gamma'} = |C_X(v)|$, where X is a chief factor of P in G whose order is p^{λ}. Then by [4], Theorem 5.3.15, we obtain $|C_P(v)| = p^{m+n\gamma'}$. We already know the value of $p^{\gamma'}$. Therefore, once we know the numbers m and n, we can obtain exactly the Cartan invariants of kG.

4. Dimensions of simple modules. Let B be a block of a p-solvable group G, and suppose $\ell(B) = 2$. As we saw in Proposition 1, it holds that $f_1' = f_2'$ for the principal block B_0. Furthermore, by Corollary 2 and Remark 2, if $p = 2$ and

$|D| \leq 2^8$ then $f_1' = f_2'$. Then we are inclined to believe that the following is true.

Conjecture. *If* G *is a p-solvable group and* B *is a block of* kG *with* $\ell(B) = 2$, *then* $f_1' = f_2'$.

If our conjecture is affirmative, then C_B is determined only by the values of e and γ. By making use of Fong reduction theorem and a result of Higgs [6], we see that the following hold:

Proposition 2. *Let* G *be a p-solvable group and* B *a block of* kG *with* $\ell(B) = 2$. *If a defect group of* B *is abelian, then* $f_1 = f_2$.

Proposition 3. *Let* G *be a p-solvable group and* B *a block of* kG *with* $\ell(B) = 2$. *If* p *is odd and* $|D| \leq p^3$, *then* $f_1 = f_2$.

Remark 3. If $|D| = 3^4$ and $e > 0$, then we have $f_1' = f_2'$. Hence in this case we have $(e,\gamma,h) = (1,1,2)$ and $C_B = \begin{pmatrix} 27 & 18 \\ 18 & 21 \end{pmatrix}$. Since a subgroup of $GL(3,3)$ generated by

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

is isomorphic to the alternating group A_4 of degree 4, we get a semidirect product $G = (Z_9 \times Z_9 \times Z_9) \rtimes A_4$. Then kG is an example for this case. We note that this group G is a group of type (e) in Theorem 1.

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PROJECTIVE MODULES OVER REGULAR RINGS OF BOUNDED INDEX

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In this paper, we shall study directly finite projective modules over regular rings of bounded index. In [3], we have shown that a directly finite regular ring R satisfying the comparability axiom has the property that the direct sum of two directly finite projective R-modules is directly finite. We shall call this property (DF). It is natural to ask which kind of regular rings have (DF). However, we see (Example) that there exists a commutative regular ring which does not have (DF). Therefore we shall study the property (DF) for regular rings of bounded index. First, we give a criterion of the directly finiteness of projective modules over these rings (Theorem 2), and, using this criterion, we show that all direct sums of finite copies of directly finite projective modules over these rings are directly finite (Theorem 4). In main Theorem 8, we characterize the property (DF) for regular rings R of bounded index.

Throughout this paper, R is a ring with identity and R-modules are unitary right R-modules.

1. Preliminaries. For two R-modules P and Q, we use $P \lesssim Q$ (resp. $P \lesssim^{\oplus} Q$) to mean that P is isomorphic to a submodule of Q (resp. a direct summand of Q). For a submodule P of an R-

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This note is a summary of [4].

module Q, $P <^{\oplus} Q$ means that P is a direct summand of Q. For a cardinal number α and an R-module P, αP denotes the direct sum of α-copies of P.

Definition. An R-module P is directly finite provided that P is not isomorphic to a proper direct summand of itself. If P is not directly finite, then P is said to be directly infinite.

Definition. The index of a nilpotent element x in a ring R is the least positive integer n such that $x^n = 0$. (In particular, 0 is nilpotent of index 1.) The index of a regular ring R is the supremum of the indices of all nilpotent elements of R. If this supremum is finite, then R is said to have bounded index.

Note thet a regular ring R is abelian (i.e. all idempotents in R are central) if and only if it has index 1.

We shall recall the following basic properties, which need for section 2.

(1) If P is a projective module over a regular ring, then all finitely generated submodules of P are direct summands of P ([1, Theorem 1.11]).

(2) Every projective modules over regular rings have the exchange property (see [3] and [4]).

(3) If R is a regular ring of bounded index, then it is unit-regular, and so all finitely generated projective R-modules have the cancellation property ([1, Theorem 4.14 and Corollary 7.11]).

(4) Let R be a regular ring, and let n be a positive integer. Then R has index at most n if and only if R contains no direct sums of n+1 nonzero pairwise isomorphic right ideals ([1, Theorem 7.2]).

(5) Let R be a regular ring of bounded index, and let P

be a finitely generated projective R-module. Then $End_R(P)$ has bounded index ([1, Corollary 7.13]).

2. Directly finite projective modules.

Lemma 1. Let R be a regular ring of bounded index at most n for some positive integer n, and let B, A_1, ..., A_k be projective R-modules such that each A_i is cyclic. Let

$$
\begin{aligned}
& A_{11} \oplus \dots \oplus A_{1k} \\
\cong\ & A_{21} \oplus \dots \oplus A_{2k} \oplus B_2 \\
\cong\ & \dots\dots \\
\cong\ & A_{s_k 1} \oplus \dots \oplus A_{s_k k} \oplus B_{s_k} \\
\lesssim\ & A_1 \oplus \dots \oplus A_k \oplus B,
\end{aligned}
$$

and let

$$B_2 \oplus \dots \oplus B_{s_k} <\oplus\ B \text{ and}$$

$$A_{1i} \oplus \dots \oplus A_{s_k i} <\oplus\ A_i \text{ for } i = 1,\dots,k,$$

where $s_1 = 1+n$ and $s_k = 1+ns_{k-1}$ for $k > 1$. Then $A_{11} \oplus \dots \oplus A_{1k} \lesssim\oplus\ B_2 \oplus \dots \oplus B_{s_k} <\oplus\ B$.

Theorem 2. Let R be a regular ring of bounded index. For a projective R-module P with a cyclic decomposition $P = \oplus_{\lambda\in\Lambda} P_\lambda$, the following conditions (a)-(d) are equivalent:

(a) P is directly infinite.

(b) There exists a nonzero cyclic projective R-module X such that $\aleph_0 X \lesssim P$.

(c) There exists a nonzero cyclic projective R-module X such that $X \lesssim \oplus_{\lambda\in\Lambda-\{\lambda_1,\dots,\lambda_n\}} P_\lambda$ for all finite subsets $\{\lambda_1,\dots,\lambda_n\}$ of Λ.

(d) There exists a nonzero cyclic projective R-module X such that $\aleph_0 X \lesssim_{\oplus} P$.

Therefore, for a projective R-module P with a cyclic decomposition as above, the following conditions (e)-(h) are equivalent:

(e) P is directly finite.

(f) P contains no infinite direct sums of nonzero pairwise isomorphic submodules.

(g) Every submodule of P is directly finite.

(h) For each nonzero cyclic projective R-module X, there exists a finite subset $\{\lambda_1,\ldots,\lambda_n\}$ of Λ such that $X \not\lesssim \oplus_{\lambda\in\Lambda-\{\lambda_1,\ldots,\lambda_n\}} P_\lambda$.

Using the basic property (4), we obtain the following.

Lemma 3. Let R be a regular ring of index at most n, and let I_1, I_2, ... be a sequence of cyclic right ideals of R such that $I_i \gtrsim 2I_{i+1}$ for i = 1,2,... . Then we have that $I_k = 0$ for all positive integers k satisfying $2^{k-1} \geq n+1$.

Theorem 4. Let R be a regular ring which has bounded index, and let k be a positive integer. If P is a directly finite projective R-module, then so is kP.

For regular rings R of bounded index, it does not hold that the direct sum of two directly finite projective R-modules is directly finite in general, as later Example shows. Therefore we shall investigate the condition for a regular ring R of bounded index that the direct sum of two directly finite projective R-modules is directly finite.

Let R be a regular ring. For a given nonzero finitely generated projective R-module P, we consider the following condition:

(#) For each nonzero finitely generated submodule Q of P and each family $\{A_1, B_1, A_2, B_2, \ldots\}$ of submodules of Q with decompositions

$$Q = A_1 \oplus B_1,$$
$$A_i = A_{2i} \oplus B_{2i} \text{ and}$$
$$B_i = A_{2i+1} \oplus B_{2i+1} \text{ for } i=1,2,\ldots,$$

there exists a nonzero projective R-module X such that $X \lesssim \oplus_{i=m}^{\infty} A_i$ or $X \lesssim \oplus_{i=m}^{\infty} B_i$ for each positive integer m.

Remark. 1) We can take above X as a finitely generated submodule of Q. 2) If P is a nonzero finitely generated projective R-module which satisfies the condition (#), then any nonzero direct summand of P satisfies (#).

Definition. Let P be a finitely generated projective module over a regular ring R. We use L(P) to denote the lattice of all finitely generated submodules of P, partially ordered by inclusion.

Lemma 5. (cf. [1, Proposition 2.4]). Let P be a finitely generated projective module over a regular ring R, and set $T = \mathrm{End}_R(P)$.

(a) There is a lattice isomorphism $\phi: L(T_T) \to L(P)$, given by the rule $\phi(J) = JP$. For $A \in L(P)$, we have $\phi^{-1}(A) = \{f \in T \mid fP \leq A\}$.

(b) For $J, K \in L(T_T)$, we have $J \cong K$ if and only if $\phi(J) \cong \phi(K)$.

(c) For $J, K \in L(T_T)$, we have $J \lesssim K$ if and only if $\phi(J) \lesssim \phi(K)$.

(d) For $J, K \in L(T_T)$ such that $J \oplus K \in L(T_T)$, we have that

$\phi(J \oplus K) = \phi(J) \oplus \phi(K)$. For $A, B \in L(P)$ such that $A \oplus B \in L(P)$, we have that $\phi^{-1}(A \oplus B) = \phi^{-1}(A) \oplus \phi^{-1}(B)$.

The following is immediate from Lemma 5.

Lemma 6. Let P be a nonzero finitely generated projective module over a regular ring R, and set $T = End_R(P)$. Then the following are equivalent:

(a) P satisfies the condition (#).

(b) T satisfies the condition (#) as T-module.

Lemma 7. Let R be a regular ring. Then the following are equivalent:

(a) R satisfies the condition (#) as R-module.

(b) All nonzero finitely generated projective R-modules satisfy the condition (#).

(c) For any positive integer k, kR satisfies the condition (#).

(d) There exists a positive integer k such that kR satisfies the condition (#).

Theorem 8. Let R be a regular ring of bounded index. Then the following are equivalent:

(a) R has the property (DF).

(b) R satisfies the condition (#) as R-module.

(c) For any nonzero finitely generated projective R-module P, $End_R(P)$ has the property (DF).

(d) For any positive integer k, $M_k(R)$ has the property (DF).

(e) There exists a positive integer k such that $M_k(R)$ satisfies the property (DF).

Corollary 9. The property (DF) for regular rings of bounded index is Morita invariant.

Corollary 10. If R is a regular ring of bounded index with the nonzero essential socle of R, then R has the property (DF).

Example. There exists a regular ring of bounded index which does not have the property (DF).

Proof. Choose a field F, and set $R_{2^n} = \oplus_{i=1}^{2^n} F_i$, where F_i = F for each i. Map each $R_{2^{n-1}} \to R_{2^n}$, given by the rule $x \to (x,x)$, and set $R = \lim_{\to} R_{2^n}$. This R is desired one.

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THE MAXIMAL QUOTIENT RING OF A LEFT H-RING

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In [2], M.Harada has introduced two new artinian rings which are closely related to QF-rings; one is a left artinian ring whose non-small left module contains a non-zero injective submodule and the other is a left artinian ring whose non-cosmall left module contains a non-zero projective summand. K.Oshiro called the first ring a *left H-ring* and the second one a *left co-H-ring* ([3]). However, later in [5], he showed that a ring R is a left H-ring if and only if it is a right co-H-ring. QF-rings and Nakayama (artinian serial) rings are left and right H-rings ([3]). As the maximal quotient rings of Nakayama rings are Nakayama, it is natural to ask whether the maximal quotient rings of left H-rings are left H-rings. In this note, we show that this problem is affirmative, by determining the structure of the maximal quotient rings of left H-rings.

Preliminaries. Throughout this paper, we assume that all rings R considered are associative rings with identity and all R-modules are unitary. Let M be an R-module. We use J(M) and S(M) to denote its Jacobson radical and its socle, respectively.

Definition [3]. A module is a *small* module if it is a

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small submodule of its injective hull and a module is a *non-small* module if it is not a small module. We say that a ring R is a *left H-ring* if R is a left artinian ring satisfying the condition that every non-small left R-module contains a non-zero injective submodule. Dually M is a *cosmall* module if, for any projective module P and any epimorphism $f:P \longrightarrow M$, kerf is an essential submodule of P, and M is a *non-cosmall* module if it is not a cosmall module. We call a ring R a *right co-H-ring* if R satisfies the ACC for right annihilator ideals and the condition that every non-cosmall right R-module contains a non-zero projective summand.

Definition [3,4]. A module M is an *extending* module if, for any submodule A of M, there exists a direct summand A^* of M such that A^* is an essential extension of A. Dually, M is a *lifting* module provided that, for any submodule A of M, there exists a direct summand A^* of M which is a co-essential submodule of A in M, i.e., $A^* \subset A$ and A/A^* is small in M/A^*.

First we shall refer to equivalent conditions for a left H-ring and a right co-H-ring.

Theorem A[2,3]. The following conditions are equivalent for a given ring R:

(1) R is a left H-ring.

(2) Every injective left R-module is a lifting module.

(3) R is a left perfect ring with the property that the family of all injective left R-module is closed under taking small covers.

(4) Every left R-module is expressed as a direct sum of an injective module and a small module.

(5) R is a left artinian ring with the condition: For any primitive idempotent e in R with Re non-small, there exists an integer t satisfying (a) $Re/S_k(Re)$ is

injective for all $0 \leq k \leq t$, and (b) $Re/S_{t+1}(Re)$ is a small module.

Theorem A'[2,3]. The following conditions are equivalent for a given ring R:

(1) R is a right co-H-ring.

(2) Every projective right R-module is an extending module.

(3) The family of all projective right R-modules is closed under taking essential extensions.

(4) Every right R-module is expressed as a direct sum of a projective module and a singular module.

(5) R is a left artinian ring with the condition: For a complete set $\{ e_i \} \cup \{ f_j \}$ of orthogonal primitive idempotent of R such that each e_iR is non-small and each f_jR is small,

(a) each e_iR is injective,

(b) for each e_iR, there exists an integer $t_i \geq 0$ such that $J_t(e_iR)$ is projective for all $0 \leq t \leq t_i$ and $J_{t_i+1}(e_iR)$ is a singular module, and

(c) for each f_jR, f_jR is isomorphic to a submodule of some e_iR.

In [5], K.Oshiro has shown that a ring R is a left H-ring if and only if it is a right co-H-ring. Moreover he has shown that a left H-ring (right co-H-ring) is also a right artinian ring [7, Th.3]. Therefore we have the following theorem ,by using the condition (5) of Theorem A': a ring R is a left H-ring if and only if it is left artinian and its complete set E of orthogonal primitive idempotents is arranged as E = $\{e_{11}, \ldots, e_{1n(1)}, \ldots, e_{m1}, \ldots, e_{mn(m)}\}$ for which

(1) each $e_{i1}R$ is injective,

(2) for each i, $e_{ik-1}R \cong e_{ik}R$ or $J(e_{ik-1}R) \cong e_{ik}R$ for $k=2, \ldots, n(i)$, and

(3) $e_{ik}R \not\cong e_{jt}R$ if $i \neq j$.

As a left H-ring is a QF-3 ring by [4], the maximal left quotient ring and the maximal right quotient ring of a left H-ring coincide by [9,Th.1.4]. From now on, let Q be the maximal quotient ring of a left H-ring R. We shall study the structure of Q. Since maximal quotient rings and left H-rings are Morita-invariant[7], in order to investigate the problem whether Q is a left H-ring or not, we may restrict our attention to basic left H-rings. Therefore, hereafter, we assume that R is a basic left H-ring and E is a complete set of orthogonal primitive idempotents of R. Then E is arranged as

$E = \{e_{11},\dots,e_{1n(1)},\dots,e_{m1},\dots,e_{mn(m)}\}$ for which

(1) each $e_{i1}R$ is injective,

(2) for each i, $J(e_{ik-1}R) \cong e_{ik}R$ for $k=2,\dots,n(i)$.

Definition [10,p.153]. A primitive idempotent e is called *S-primitive* if the simple module $eR/eJ(R)$ is isomorphic to a minimal right ideal.

We shall use the H.H.Storrer's characterization of the maximal quotient ring of a perfect ring [10].

Since each $e_{i1}R$ $(i=1,\dots,m)$ is injective, there exists a unique g_i in E such that $(e_{i1}R;Rg_i)$ is an *injective pair*, that is, $S(e_{i1}R) \cong g_iR/J(g_iR)$ and $S(Rg_i) \cong Re_{i1}/J(Re_{i1})$ (cf. K.R.Fuller [1,Th.3.1]). Each pair $\{e_{i1}, g_i\}$ $(i=1,\dots,m)$ is very important for studying left H-rings.

Now we shall determine all S-primitive idempotents in E. Let e be an idempotent in E. It is known that e is S-primitive if and only if $S(R_R)e \neq 0$ [10,Lemma 2.3]. Since $S(R_R) = \oplus_{i=1,k=1}^{m,\ n(m)} S(e_{ik}R)$, $S(e_{ik}R) \not\cong S(e_{jt}R)$ for $i \neq j$ and $S(e_{ik}R) \cong S(e_{it}R)$, we have $S(R_R)e \neq 0$ if and only if $S(e_{ik}R)e \neq 0$ for a unique i. Therefore e is an S-primitive idempotent if and only if $e = g_i$ for some i. Then $E' = \{g_1,\dots,g_m\}$ is the set of all S-primitive idempotents in E. Put $g = g_1+\ldots+g_m$ and $D = RgR$. Storrer has shown that $D = RgR$ is the minimal dense ideal of R and Q is isomorphic to $\mathrm{Hom}_R(D_R,D_R) = \mathrm{Hom}_R(D_R,R_R)$ by [10, Prop.1.2 and Th.2.5]. Since R is a two-sided artinian ring, Q is a left artinian ring by [10, Prop.3.1].

We shall prove that Q is a left H-ring by showing that E satisfies the conditions (1),(2) and (3) of left H-rings. We again note that left H-rings are also right artinian by [7,Th.3] and the maximal quotient ring Q of R is a left artinian ring.

Proposition 1. $E = \{e_{11},\ldots,e_{1n(1)},\ldots,e_{m1},\ldots,e_{mn(m)}\}$ is also a complete set of orthogonal primitive idempotents of the maximal quotient ring Q . In Q, $(e_{i1}Q;Qg_i)$ is an injective pair for $i = 1,\ldots,m$. Consequently $e_{i1}Q$ and Qg_i are injective Q-modules.

Next we shall study isomorphisms among the indecomposable right ideals $e_{ik}Q$. Let f_1, f_2 be idempotents in E and we assume that there exists a monomorphism $\theta:f_1R \longrightarrow f_2R$ such that $\mathrm{Im}\theta = J(f_2R)$. Then by [10, Prop.4], θ can be uniquely extended to a Q-homomorphism $\theta^*:f_1Q \longrightarrow f_2Q$. We have the following result.

Proposition 2. (1) If f_2 is not S-primitive, then the extension $\theta^*:f_1Q \longrightarrow f_2Q$ is an isomorphism.

(2) If f_2 is S-primitive, then $\theta^*:f_1Q \longrightarrow f_2Q$ is a monomorphism such that $\mathrm{Im}\theta^* = J(f_2Q)$.

Now we shall prove our main theorem.

Theorem 3. Let R be a left H-ring. Then the maximal quotient ring Q of R is also a left H-ring.

Proof. Let $E = \{e_{11},\ldots,e_{1n(1)},\ldots,e_{m1},\ldots,e_{mn(m)}\}$ be a complete set of orthogonal primitive idempotents of R such that

(1) each $e_{i1}R$ is injective,

(2) for each i, $J(e_{ik-1}R) \cong e_{ik}R$ for $k=2,\dots,n(i)$.
We have already known that Q is a left artinian ring and E is also a complete set of orthogonal primitive idempotents of Q.
By Proposition 1, each $e_{i1}Q$ is an injective Q-module and by Proposition 2, $e_{ik}Q \cong e_{ik-1}Q$ or $e_{ik}Q \cong J(e_{ik-1}Q)$ $k = 2,\dots,n(i)$ for each i. We shall show that $e_{ik}Q \not\cong e_{jt}Q$ if $i \neq j$. If $e_{ik}Q \cong e_{jt}Q$ for some $i \neq j$, k, t, then $S(e_{ik}Q) \cong S(e_{jt}Q)$. Since $S(e_{ik}Q)= S(e_{ik}R)Q$ and $S(e_{jt}Q)= S(e_{jt}R)Q$, we have $S(e_{ik}R) \cong S(e_{jt}R)$ as R-modules by [10, Th.4.5]. This contradicts the assumption of E.

We recall that g_i is the element of E such that $(e_{i1}R;Rg_i)$ is an injective pair for $i = 1,\dots,m$. Here we define two mappings

$$\sigma : \{1,\dots,m\} \longrightarrow \{1,\dots,m\}$$

$$\rho : \{1,\dots,m\} \longrightarrow \{1,\dots,n(1)\} \cup \dots \cup \{1,\dots,n(m)\}$$

by the rule $\sigma(i) = k$ and $\rho(i) = t$ if $g_i = e_{kt}$.
We note that $\{\sigma(1),\dots,\sigma(m)\} \subseteq \{1,\dots,m\}$ and $1 \leq \rho(i) \leq n(\sigma(i))$. Here we shall define a special left H-ring.

Definition [7,p.94]. A left H-ring is *Type (*)* if $\{\sigma(1),\dots,\sigma(m)\}$ is a permutation of $\{1,\dots,m\}$ and $\rho(i) = n(\sigma(i))$ for all $i = 1,\dots,m$.

Corollary. Let R be a left H-ring. Then the maximal quotient ring Q of R is a QF-ring if and only if R is Type (*).

Example. Let T be a local QF-ring , $J = J(T)$ and $S = S(T)$.
Put $V = \begin{pmatrix} T & T & T \\ J & T & T \\ J & J & T \end{pmatrix}$ and $W = \begin{pmatrix} 0 & 0 & S \\ 0 & 0 & S \\ 0 & 0 & S \end{pmatrix}$. The factor ring $R = V/W$ is a left H-ring such that e_1R is injective , $J(e_1R) \cong e_2R$ and $J(e_2R) \cong e_3R$, where e_i is the matrix such that its (i,i)-position is 1 and all other entries are zero. R is represented as follows: $\begin{pmatrix} T & T & \tilde{T} \\ J & T & \tilde{T} \\ J & J & \tilde{T} \end{pmatrix}$,

where $\tilde{T} = T/S$.

Since $(e_1R;Re_2)$ is injective pair by [8,§2], the minimal dense ideal is Re_2R. Therefore the maximal quotient ring Q of R is a left H-ring such that e_1Q is an injective module ,$e_1Q \cong e_2Q$ and $J(e_2Q) \not\cong e_3Q$. Since $e_1Q/J(e_1Q) \cong S(e_1Q)$, we have that $Hom_Q(e_1Q,J(e_1Q)) = J(e_1Qe_1)$, $Hom_Q(J(e_1Q),e_1Q) = e_1Qe_1/S(e_1Qe_1)$, $Hom_Q(J(e_1Q),J(e_1Q)) = e_1Qe_1/S(e_1Qe_1)$. Moreover, since $e_1Qe_1 = e_1Re_1 = T$ by [10, Lemma 4.2], Q is represented as a matrix ring $\begin{pmatrix} T & T & \tilde{T} \\ T & T & \tilde{T} \\ J & J & \tilde{T} \end{pmatrix}$.

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LOCALIZATION OF DERIVED CATEGORIES

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Introduction.

The notion of quotient and localization of abelian categories by dense subcategories (i.e. Serre classes) was introduced by Gabriel, and is useful in the ring theory [5], [11]. The notion of triangulated categories was introduced by Grothendieck and developed by Verdier [8], [14], and is recently useful in the representation theory [7], [3], [12]. The quotient of triangulated categories by épaisse subcategories was constructed there. Both constructions were indicated by Grothendieck, and resemble each other. In this paper, we will consider triangulated categories and derived categories from the point of view of localization of abelian categories. Verdier showed the equivalent condition that a quotient functor has a right adjoint, and considered the relation between épaisse subcategories [14]. We show that localization of triangulated categories is similarly defined, and have a relation between localizations and épaisse subcategories. Beilinson-Bernstein-Deligne introduced the notion of t-structure similar to

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torsion theory in abelian categories [2]. We, in particular, consider a stable t-structure, which is an épaisse subcategory, and deal with a correspondence between localizations of triangulated categories and stable t-structures. And then recollement, in the sense of [2], is equivalent to bilocalization. Next, we show that quotient and localization of abelian categories induce quotient and localization of its derived categories. Furthermore, we apply it to derived categories of modules over finite dimensional algebras.

1. Preliminaries.

In this section, we recall standard notations and terminologies of quotient and localization of abelian categories. Let A be an abelian category. A collection S of arrows of A is called a multiplicative system if it satisfies the following conditions:

(FR-1) If $f, g \in S$, and $f \circ g$ exists, then $f \circ g \in S$. For any $X \in A$, $1_X \in S$.

(FR-2) In A, any diagram:

$$\begin{array}{ccc} & & Y \\ & & \downarrow s \\ Z & \xrightarrow{f} & X \end{array} ,$$

with $s \in S$, can be completed to a commutative diagram:

$$\begin{array}{ccc} W & \xrightarrow{g} & Y \\ t \downarrow & & \downarrow s \\ Z & \xrightarrow{f} & X \end{array} ,$$

with $t \in S$. Ditto with the arrows reversed.

(FR-3) If f and g are morphisms in A, the following properties are equivalent.

(i) There exists $s \in S$ such that $s \circ f = s \circ g$.

(ii) There exists $t \in S$ such that $f \circ t = g \circ t$.

A full subcategory C of A is called dense if for every exact sequence $0 \to X \to Y \to Z \to 0$ in A, the following condition holds: $X, Z \in C$ if and only if $Y \in C$. We denote by $\phi(C)$ the set of morphisms f such that Ker f and Coker f are in C. Then $\phi(C)$ is a multiplicative system. And then C is an abelian category and the quotient category A/C is defined. In this case, we call it that $0 \to C \to A \xrightarrow{Q} A/C \to 0$ is exact. A section functor S is the right adjoint of Q. If there exists a section functor, then $\{A/C ; Q, S\}$ is called a localization of A. In this case, C is called a localizing subcategory of A. Then S is fully faithful. On the other hand, if $T: A \to B$ is an exact functor between abelian categories which has the fully faithful right adjoint $S: B \to A$, then Ker T is a localizing subcategory of A, and T induces that $A/\mathrm{Ker}T$ is equivalent to B. Colocalization of C is also defined, and similar results hold [5], [11]. We apply these ideas to triangulated categories in the next section.

2. Localization of Triangulated Categories.

A triangulated category is an additive category D, endued with:

(a) An autofunctor $T: D \to D$, is called a translation functor, and

(b) A family of triangles $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$, is called a family of distinguished triangles, satisfies the following conditions:

(TR1) Every triangle isomorphic to a distinguished triangle is distinguished. Every morphism $u:X \to Y$ is contained in a distinguished triangle (X,Y,Z,u,v,w). For every object X of D , $(X,X,0,1_X,o,o)$ is distinguished.

(TR2) (X,Y,Z,u,v,w) is distinguished if and only if $(Y,Z,TX,v,w,-Tu)$ is distinguished.

(TR3) Given two distinguished triangles (X,Y,Z,u,v,w) and (X',Y',Z',u',v',w'), for every morphism $(f,g):u \to u'$, there exists a morphism $h:Z \to Z'$ such that (f,g,h) is a morphism of triangles.

(TR4) Given two morphisms $u:X \to Y$ and $v:Y \to Z$, there exists a following diagram such that the first two rows and the middle two columns are distinguished:

$$
\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{i} & Z' & \to & TX \\
\| & & v\downarrow & & \downarrow & & \| \\
X & \to & Z & \to & Y' & \to & TX \\
 & & \downarrow & & \downarrow & & \downarrow Tu \\
 & & X' & = & X' & \xrightarrow{j} & TY \\
 & & j\downarrow & & \downarrow & & \\
 & & TY & \xrightarrow[Ti]{} & TZ' & &
\end{array}
$$

Given two triangulated categories D and D' , a grade functor from D to D' is an additive functor $F: D \to D'$ and an isomorphism $\Phi:FT \to T'F$. A grade functor (F,Φ) is called a ∂-functor if for every distinguished triangle (X,Y,Z,u,v,w) in D , $(X,Y,Z,Fu,Fv,\Phi_X \circ Fw)$ is distinguished in D' (we often simply write F unless it confounds us) [8], [14]. If F has a right or left adjoint G, then G is a ∂-functor, also [9].

A subcategory U of D is called épaisse if U is a full triangulated subcategory and if U satisfies the following condition: For any $f:X \to Y$, which factors through an object in U and which has a mapping cone in C , X and Y are objects in U . We denote by $\phi(U)$ the set of morphisms f which is contained in a distinguished triangle (X,Y,Z,f,g,h) where Z is an object of U . Then $\phi(U)$ is a multiplicative system which satisfies the following conditions:

(FR-4) $s \in \phi(U)$ if and only if $Ts \in \phi(U)$, where T is the translation functor.

(FR-5) Given distinguished triangles (X,Y,Z,u,v,w), (X',Y',Z', u',v',w'), if f and g are morphisms in $\phi(U)$ such that $u' \circ f = g \circ u$, then there exists a morphism h in $\phi(U)$ such that (f,g,h) is a morphism of distinguished triangles.

And the quotient category D/U is defined. In this case, we will call it that $0 \to U \xrightarrow{K} D \xrightarrow{Q} D/U \to 0$ is exact (see [2], [14] for details). Let $0 \to U \to D \to D/U \to 0$ be an exact sequence of triangulated categories. A section functor S is the right adjoint of Q. If there exists a section functor, then we will call $\{D/U ;Q,S\}$ a localization of D .

Let $\Phi:QS \to 1_{D/U}$ and $\Psi:1_D \to SQ$ be adjunction arrows.

Proposition 2.1. *Let $\{D/U ;Q,S\}$ be a localization of D .*

(a) Φ is an isomorphism (i.e. S is fully faithful).

(b) For every object $X \in D$, U_X belongs to U , where $U_X \to X \to SQX \to TU_X$ is the distinguished triangle determined by Ψ_X .

Proposition 2.2. *Let D and E be triangulated categories, T: D*

→ E a ∂-functor which has the fully faithful right adjoint S: E → D . Then T induces that D/KerT is equivalent to E .

Let U and V be full subcategories of D such that: (a) U and V are stable for translations; (b) Hom_D(U , V) = 0; (c) For every X ∈ D , there exists a distinguished triangle (U,X,V) with U ∈ U and V ∈ V . Then U and V are épaisse subcategories of D , and (U , V) is t-structure in the sense of Beilinson-Bernstein-Deligne. We will call (U , V) a stable t-structure. Moreover, there exist exact sequences $0 \to U \overset{K}{\to} D \overset{Q}{\to} V \to 0$ and $0 \to V \overset{R}{\to} D \overset{Q'}{\to} U \to 0$ such that Q is the left adjoint of R and that Q' is the right adjoint of K, where K and R are natural inclusions (see [2] for details). Namely, { V ;Q,R} is a localization of D , and { U ;K,Q'} is a colocalization of D . By Proposition 2.2 and [14, 6-6 Proposition], and their duals, D/U is a localization of D if and only if U is a colocalization of D , and D/U is a colocalization of D if and only if U is a localization of D . We later see that recollement, in the sense of [2], is equivalent to bilocalization.

<u>*Theorem 2.3.*</u> *Let D be a triangulated category. If { V ;Q,R} is a localization of D , then R is fully faithful, and (KU,RV) is a stable t-structure, where U = Ker Q and K is a natural inclusion. Conversely, if (U , V) is a stable t-structure in D , then a natural inclusion R: V → D has a left adjoint Q such that { V ;Q,R} is a localization.*

We have the same result of Cline-Parshall-Scott [4] under the weak conditions.

<u>*Proposition 2.4.*</u> *Let F: D → E be a ∂-functor of triangulated*

categories. Assume that F has a fully faithful right adjoint G: E → D . If F has a left adjoint H: E → D , then H is a fully faithful ∂-functor. In this case, (KerF , D , E) is a recollement.

3. Localization of Derived Categories.

Let A be an additive category, K(A) a homotopy category of A , and $K^+(A)$, $K^-(A)$ and $K^b(A)$ full subcategories of K(A) generated by the bounded below complexes, the bounded above complexes and the bounded complexes, respectively. For an abelian category A , a derived category D(A) (resp., $D^+(A)$, $D^-(A)$ and $D^b(A)$) of A is a quotient of K(A) (resp., $K^+(A)$, $K^-(A)$ and $K^b(A)$) by a multiplicative set of quasi-isomorphisms. Then $K^*(A)$ and $D^*(A)$ are triangulated categories, where * = nothing,+,- or b [8], [14]. In general, we denote by $K^*(A)$ a localizing subcategory of K(A) (i.e. $K^*(A)$ is a full triangulated subcategory of K(A) and $D^*(A) \to D(A)$ is a fully faithful ∂-functor, where $D^*(A)$ is a quotient of $K^*(A)$ by a multiplicative set of quasi-isomorphisms) [8], [14]. For a thick abelian subcategory C of A (i.e. C is extension closed in A), we denote by $D^*_C(A)$ a full subcategory of $D^*(A)$ generated by complexes of which all homologies are in C [8].

Let ∂($D^*(A)$, D(B)) be a category of ∂-functors from $D^*(A)$ to D(B) and $Hom_\partial(F,G)$ the set of morphisms from F to G for F, G ∈ ∂($D^*(A)$, D(B)). Given a ∂-functor F: $K^*(A) \to K(B)$, we obtain a right derived functor $R^*F : D^*(A) \to D(B)$ when there exists an object R^*F in ∂($D^*(A)$, D(B)) such that $Hom_\partial(R^*F ,?) \simeq Hom_\partial(Q^*\circ F, ?\circ Q)$, where $Q^*_A: K^*(A) \to D^*(A)$,$Q_B: K(B) \to D(B)$ are natural quotients, [8], [14]. When $R^+F : D^+(A) \to D(B)$ exists, we say F has right homological dimension $\leq n$ on A if $R^iF(X) = 0$ for all $X \in A$ and for all $i > n$. And an object X in A is called a right F-acyclic object if $R^iF(X) = 0$ for all $i > 0$. We also denote by $R^{*}F$ a right derived functor of an induced ∂-functor from F: A → B [8].

Let F: A → B be a left exact additive functor between abelian categories. If A has enough injectives, and F has finite right homological dimension on A , then RF , $R^{+}F$, $R^{-}F$ and $R^{b}F$ exist, and $RF|_{D^{*}(A)} \simeq R^{*}F$, and moreover, $R^{*}F$ has image in $D^{*}(B)$, where $* = +, -$ or b [8, I , §5]. We often denote by $R^{*,\#}F$, $R^{*}F|_{D^{\#}(A)}$ when $D^{\#}(A)$ is a full subcategory of $D^{*}(A)$. On the other hand, if A and B have enough injectives and projectives, respectively, and if the derived functor $R^{+,b}F : D^{b}(A) \to D(B)$ has image in $D^{b}(B)$ and $R^{+,b}F : D^{b}(A) \to D^{b}(B)$ has a left adjoint, then F has a left adjoint G: B → A and the derived functor $L^{-,b}G : D^{b}(B) \to D(A)$ has image in $D^{b}(A)$, and which is the left adjoint of $R^{+,b}F$ [3, (3.1) Lemma].

Theorem 3.1. Let $0 \to C \to A \xrightarrow{Q} A/C \to 0$ be an exact sequence of abelian categories. Then $0 \to D^{*}_{C}(A) \to D^{*}(A) \xrightarrow{Q^{*}} D^{*}(A/C) \to 0$ is an exact sequence of triangulated categories, where $* = +, -$ or b.

Corollary 3.2. Let $0 \to C \to A \to A/C \to 0$ be a localization $\{ A/C ; Q, T \}$ of A . Assume that A/C has enough injectives. Then $0 \to D^{+}_{C}(A) \to D^{+}(A) \to D^{+}(A/C) \to 0$ is a localization $\{ D^{+}(A) ; Q^{+}, R^{+}T \}$ of $D^{+}(A)$. In particular, $(D^{+}_{C}(A) , R^{+}T(D^{+}(A/C)))$ is a stable t-structure.

4. Localization of Derived Categories of Modules.

Equivalences of derived categories of modules was considered in [7], [3], [12]. For a finite dimensional algebra Λ, we denote by mod Λ the category of finitely presented right Λ-modules. According to Theorem 3.1, for a finitely generated projective Λ-module P, we have $0 \to D^{*}_{\text{mod } A}(\text{mod } \Lambda) \to D^{*}(\text{mod } \Lambda) \xrightarrow{Q^{*}} D^{*}(\text{mod } B) \to 0$ is exact, where $B = \text{End}_{\Lambda}(P)$, $Q = \text{Hom}_{\Lambda}(P,?)$, $QP\otimes_{\Lambda}\Lambda \to \Lambda \to A \to 0$ (exact) and $* = +$, $-$ or b. Moreover, Q^{+} (resp., Q^{-}) is a localization (resp., a colocalization). In this section, we consider only case of finite dimensional algebras over a fixed field k. For a finite dimensional algebra A, we know that Grothendieck groups of mod A is isomorphic to a free abelian group which has the complete set of non-isomorphic indecomposable projective A-modules as basis. We denote by Grot(A) a Grothendieck group of A , where A is an abelian category or a triangulated category. Here, we use Grot(mod A) $\simeq$ Grot(D^{b}(mod A)) and Proposition of Grothendieck (see [6] for details).

<u>*Proposition 4.1.*</u> *Let T be a right finitely generated A-module, $B := \text{End}_{A}(T)$ and $F := \text{Hom}_{A}(T,?)$: mod A $\to$ mod B. Assume that T satisfies the following conditions:*

(a) $\text{Ext}^{i}_{A}(T,T) = 0$ for all $i > 0$.

(b) pdim $T_{A} < \infty$.

Then $0 \to \text{Ker } R^{-}F \to D^{-}(\text{mod } A) \xrightarrow{R^{-}F} D^{-}(\text{mod } B) \to 0$ is exact.

<u>*Proposition 4.2.*</u> *Under the conditions of Proposition 4.1, if pdim $T_{A} < 1$, then*

$0 \to \text{Ker } R^{b}F \to D^{b}(\text{mod } A) \xrightarrow{R^{b}F} D^{b}(\text{mod } B) \to 0$ is exact.

Theorem 4.3. *Let A and B be artin algebras, F:mod $A \to$ mod B a left exact additive functor. Then $R^{+,b}F$ has image in D^b(mod B) and $R^{+,b}F : D^b(\text{mod } A) \to D^b(\text{mod } B)$ is a colocalization if and only if there exists a finitely generated B-A-bimodule T such that:*

(a) $F \simeq Hom_A(T, ?)$,

(b) $B \simeq End_A(T)$,

(c) $Ext_A^i(T, T) = 0$ *for all* $i > 0$,

(d) *pdim* T_A , *pdim* $_BT < \infty$.

Corollary 4.4. *Under the condition of Theorem 4.3, we have* $gl\ dim\ B \leq gl\ dim\ A + pdim\ _BT$.

For a finitely generated A-module M, Let n(M) be a number of non-isomorphic indecomposable modules which are direct summands of M.

Corollary 4.5. *Let T be a finitely generated right A-module such that: a)* $Ext_A^i(T, T) = 0$ $(i \geq 1)$; *b) pdim* T_A , *pdim* $_BT < \infty$, *where* $B = End_A(T)$. *Then we have* $n(T) \leq n(A)$.

Remarks. (1) Under the conditions of Theorem 4.3, global dimensions of A or B are not necessarily finite. Indeed, let A be a finite dimensional algebra over a field k with the following quiver

with relations: $a \xleftarrow{\delta} b \circlearrowleft^{\alpha} \xleftarrow{\beta} c \xleftarrow{\gamma} d$ with $\delta \circ \alpha = \alpha^2 = \delta \circ \beta = \beta \circ \gamma = 0$. Then gl dim A = ∞. Let T :=I(c)⊕(I(c)/S(c)), where S(c)is a simple right A-module corresponding with a vertex c , and

I(c) is an injective hull of S(c). Then pdim T_A = 2 and $\mathrm{Ext}_A^i(T,T)$ = 0 for all i > 0. Next, B := $\mathrm{End}_A(T)$ have a quiver with a relation: e ⟶ f ↻ ζ with $\zeta^2 = 0$. Then we have gl dim B = ∞ and $\mathrm{pdim}_B T = 1$. Hence $R^b\mathrm{Hom}_A(T,?) : D^b(\mathrm{mod}\ A) \longrightarrow D^b(\mathrm{mod}\ B)$ is a colocalization functor which has $L^b(?\otimes_B T)$ as a cosection functor.

(2) Under the conditions of Theorem 4.3, when we know if Ker R^bF is not zero, then Grot(Ker R^bF) is not zero (for example, A is hereditary), $D^b(\mathrm{mod}\ A)$ is equivalent to $D^b(\mathrm{mod}\ B)$ if and only if n(T) = n(A).

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STREB'S RESULTS AND COMMUTATIVITY THEOREMS

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Recently, in [8], W. Streb gave a classification of non-commutative rings. H. Tominaga and the author extended that to algebras in [5], and applied those to study algebras with some commutativity conditions in [3], [4] and [5]. In this paper, we shall introduce a classification of non-commutative algebras and pick up the results for algebras A satisfying the following condition:

(H) For each $x, y \in A$ there exist positive integers m and n such that $x^m y^n = y^n x^m$.

1. Classification of non-commutative algebras. Throughout this paper, A will denote an algebra (not necessarily with 1) over a commutative ring R with 1. If A has the smallest non-zero algebra ideal, it is called the heart of A. A factor algebra of a subalgebra of A is called a factorsubalgebra of A. As usual we define the commutator $[x,y] = xy-yx$ for $x, y \in A$, and D denotes the ideal of A generated by all commutators. We put $\mathrm{Ann}(D) = \{a \in A \mid aD = Da = 0\}$.

Theorem 1.1. Every non-commutative R-algebra has a factorsubalgebra of type a)$_\ell$, a)$_r$, b), c), d), e) or f):

a)$_\ell$ $\begin{pmatrix} R/\mathfrak{m} & R/\mathfrak{m} \\ 0 & 0 \end{pmatrix}$, where $\mathfrak{m}$ is a maximal ideal of R.

a)$_r$ $\begin{pmatrix} 0 & R/\mathfrak{m} \\ 0 & R/\mathfrak{m} \end{pmatrix}$, where $\mathfrak{m}$ is a maximal ideal of R.

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b) A non-commutative trivial extension $T \ltimes M$, where T is an R-algebra generated by one element without non-zero zero-divisors, and M is an irreducible bimodule over the R-algebra T and a faithful left and right T-module.

c) A non-commutative division R-algebra.

d) A simple radical R-algebra without non-zero zero-divisors.

e) An R-algebra A generated by two elements such that D is the heart of A and $A = \mathrm{Ann}(D)$.

f) An R-algebra A generated by two elements such that D is central and is the heart of A and $\mathrm{Ann}(D)$ is a commutative maximal ideal of A.

Theorem 1.2. Every non-commutative R-algebra with 1 has a factorsubalgebra of type $a)^1$, $b)^1$, c), $d)^1$, $e)^1$ or $f)^1$:

$a)^1$ $\begin{pmatrix} R/m & R/m \\ 0 & R/m \end{pmatrix}$, where m is a maximal ideal of R.

$b)^1$ A non-commutative trivial extension $T \ltimes M$, where T is an integral domain which is an R-algebra generated by one element together with 1, and M is an irreducible bimodule over the R-algebra T and a faithful left and right T-module.

c) A non-commutative division R-algebra.

$d)^1$ A domain which is an R-algebra generated by 1 and a simple radical subalgebra.

$e)^1$ An R-algebra A with 1 generated by two elements x, y together with 1 such that D is the heart of A and both x and y belong to $\mathrm{Ann}(D)$.

$f)^1$ An R-algebra A with 1 generated by two elements x, y together with 1 such that D is central and is the heart of A and $\mathrm{Ann}(D)$ is a commutative maximal ideal of A.

The proof of Theorems 1.1 and 1.2 can be reduced to the following two propositions.

Proposition 1.3. Let A be a non-commutative R-algebra.

(1) If D is central, then A has a factorsubalgebra of type e) or f).

(2) If $xy \neq 0 = yx$ for some $x, y \in A$, then A has a factorsubalgebra of type $a)_\ell$, $a)_r$, e) or f).

(3) If A has a non-central ideal I with $I^2 = 0$, then A has a factorsubalgebra of type $a)_\ell$, $a)_r$, b), e) or f).

Proposition 1.4. Let A be an R-algebra with 1.

(1) If A has a factorsubalgebra of type $a)_\ell$ or $a)_r$, then A has a factorsubalgebra of type $a)^1$.

(2) If A has a factorsubalgebra of type b), then A has a factorsubalgebra of type $a)^1$, $b)^1$, $e)^1$ or $f)^1$.

(3) If A has a factorsubalgebra of type d), then A has a factorsubalgebra of type $d)^1$.

(4) If A has a factorsubalgebra of type e), then A has a factorsubalgebra of type $e)^1$.

(5) If A has a factorsubalgebra of type f), then A has a factorsubalgebra of type $e)^1$ or $f)^1$.

Let R be a commutative ring with 1. We shall call R an <u>N-ring</u> if R is either a finitely generated ring or a finitely generated algebra over a commutative ring S such that $S/\mathfrak{p}$ is an algebraically closed field for any prime ideal $\mathfrak{p}$ of S. We shall call R an <u>S-ring</u> if R is a finitely generated algebra over a commutative ring S such that the quotient field of $S/\mathfrak{p}$ is a perfect field for any prime ideal $\mathfrak{p}$ of S. Obviously, every N-ring is an S-ring. In [6], T. Nakayama proved that an algebra A over an N-ring R is commutative if A satisfies the following condition:

(N) For each $x \in A$ there exists $f(X) \in X^2R[X]$ such that $x - f(x)$ is central.

More generally, in [7], W. Streb studied algebras over an S-ring R satisfying the following condition:

(S) For each $x, y \in A$ there exists $f(X,Y) \in R\langle X,Y\rangle[X,Y]R\langle X,Y\rangle$ such that $[x,y] - f(x,y) = 0$ and each monomial term of $f(X,Y)$ has degree ≥ 3, where $R\langle X,Y\rangle$ is the polynomial ring over R in the non-commuting indeterminates X and Y.

As is well-known, $R\langle X,Y\rangle[X,Y]R\langle X,Y\rangle$ consists of $f(X,Y)\in R\langle X,Y\rangle$ such that every commutative R-algebra satisfies the identity $f(X,Y) = 0$. By this reason, the condition (S) is natural as a commutativity condition for R-algebras.

Proposition 1.5. (1) Suppose that R is a finitely generated ring. If an R-algebra A is of type b), then A is isomorphic to $M_\sigma(K) = \left\{\begin{pmatrix}\alpha & \beta \\ 0 & \sigma(\alpha)\end{pmatrix} \,\middle|\, \alpha,\ \beta\in K\right\}$, where K is a finite field with a non-trivial $R\cdot 1$-automorphism σ.

(2) If R is an N-ring and is not a finitely generated ring, then no R-algebra is of type b).

Proposition 1.6. There exists no algebra of type f) over an S-ring.

Here, we make a remark concerning the proof of a theorem of Nakayama stated above. Let A be an algebra over an N-ring satisfying the condition (N). Then it is easy to see that A has no factorsubalgebra of type $a)_\ell$, $a)_r$, b), d) or e). Further, by Proposition 1.6, A has no factorsubalgebra of type f). Hence, in order to prove the commutativity of A, it suffices to consider the case that A is a division algebra.

2. Application 1. The next theorem is an easy application of Streb's results. We can prove more general results (see [4]).

Theorem 2.1. Let A be a ring with 1. Suppose that for each $x,\ y\in A$ there exist positive integers n_i $(i = 1, \dots, r)$ such that $(n_1,\dots,n_r) = 1$ and $[x^{n_i},y^{n_i}] = 0$ for $i = 1, \dots,$ r, where $(n_1,\dots,n_r)$ is the greatest common divisor of n_i (i = 1, ... , r). Then A is commutative.

Proof. In view of Theorem 1.1, it suffices to show that A has no factorsubring of type $a)_\ell$, $a)_r$, b), c), d), e) or f) as $\mathbb{Z}$-algebra.

(1) For any positive integer n, we see that $\left[\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}^n, \begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}^n\right] = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \neq 0$, and so A has no factorsubring of type a)$_\ell$; and similarly A has no factorsubring of type a)$_r$.

(2) Since D is nil by [1, Theorem], A has no factorsubring of type c) or d).

(3) Proposition 1.6 shows that A has no factorsubring of type f).

(4) Suppose, to the contrary, that A has a factorsubring B of type b). By Proposition 1.5 (1), B is isomorphic to some $M_\sigma(K)$, where K is a finite field with a non-trivial automorphism σ. Choose $\gamma \in K$ with $\sigma(\gamma) \neq \gamma$, and put $x = \begin{pmatrix}\gamma & 1\\ 0 & \sigma(\gamma)\end{pmatrix}$ and $y = \begin{pmatrix}\gamma & 0\\ 0 & \sigma(\gamma)\end{pmatrix}$. Then there exist positive integers n_i ($i = 1, \dots, r$) such that $(n_1, \dots, n_r) = 1$ and $[x^{n_i}, y^{n_i}] = 0$ for $i = 1, \dots, r$. We can see that $0 = [x^{n_i}, y^{n_i}] = (\sigma(\gamma^{n_i}) - \gamma^{n_i})^2(\sigma(\gamma)-\gamma)^{-1}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$, and so $\sigma(\gamma^{n_i}) = \gamma^{n_i}$ for $i = 1, \dots, r$. Hence $\sigma(\gamma) = \gamma$, a contradiction.

(5) Let B be a ring of type e)^1. By definition, B contains non-commuting elements x and y such that $x[x,y] = y[x,y] = [x,y]x = [x,y]y = 0$. It is easy to see that $[x^m, y^n] = 0$ for any positive integers m and n with $mn > 1$. Now, suppose that there exist positive integers n_i ($i = 1, \dots, r$) such that $(n_1, \dots, n_r) = 1$ and $[(1+x)^{n_i}, (1+y)^{n_i}] = 0$ for $i = 1, \dots, r$. Then, we can see that $0 = [(1+x)^{n_i}, (1+y)^{n_i}] = n_i^2[x,y]$ for $i = 1, \dots, r$, and hence $[x,y] = 0$. This contradiction shows that A has no factorsubring of type e)^1. By Proposition 1.4 (4), A has no factorsubring of type e).

3. Application 2. Y. Kobayashi, in [2], determined the structure of a ring A with 1 such that A satisfies the identity $[X^n, Y^n] = 0$ and the additive group $[A,A]$ is n-torsion free for some positive integer n. Such a ring satisfies (S) as $\mathbb{Z}$-algebra. From this viewpoint, we applied Streb's results to generalize the proof of [2, Theorem], and obtained some results in [5]. We shall state those without proof.

Theorem 3.1. Let A be an algebra over an S-ring R, and n a positive integer. Then the following conditions are equivalent:

1) A satisfies the identity $[X-X^m, Y-Y^m] = 0$ for some integer $m > 1$, and satisfies the identity $[X^n, Y^n] = 0$.

2) A satisfies (S) and the identity $[X^n, Y^n] = 0$.

3) A is a subdirect sum of R-algebras each of which has one of the following types:

i) A commutative algebra.

ii) $M_\sigma(K)$, where K is a finite field with a non-trivial $R \cdot 1$-automorphism σ and $(|K|-1)/(|K^\sigma|-1)$ divides n.

Theorem 3.2. Let A be a ring, and $m > 1$ an integer. Then the following conditions are equivalent:

1) A satisfies the identity $[X-X^m, Y-Y^m] = 0$ and the identity $[X^n, Y^n] = 0$ for some positive integer n.

2) A satisfies the identity $[X-X^m, Y-Y^m] = 0$ and satisfies (H).

3) A is a subdirect sum of rings each of which has one of the following types:

i) A commutative ring.

ii) $M_\sigma(K)$, where K is a finite field with a non-trivial automorphism σ such that $|K|-1$ divides $m-1$.

iii) $M_\sigma(K)$, where K is a finite field of characteristic 2 with an automorphism σ of order 2 such that $|K|-1$ divides $m-|K^\sigma|$.

Corollary 3.3. Let A be a ring, and let $m > 1$ and $n > 0$ be integers. Then the following conditions are equivalent:

1) A satisfies the identities $[X-X^m, Y-Y^m] = 0$ and $[X^n, Y^n] = 0$.

2) A is a subdirect sum of rings each of which has one of the following types:

i) A commutative ring.

ii) $M_\sigma(K)$, where K is a finite field with a non-trivial automorphism σ such that $|K|-1$ divides $m-1$ and $(|K|-1)/(|K^\sigma|-1)$ divides n.

iii) $M_\sigma(K)$, where K is a finite field of characteristic 2 with an automorphism σ of order 2 such that $|K|-1$ divides $m-|K^\sigma|$ and $|K^\sigma|+1$ divides n.

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GROWTH OF ALGEBRAS AND SUPER GELFAND-KIRILLOV DIMENSION

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§1 Introduction.

Let S be a set of sequences whose terms are non-negative real numbers, i.e.

$S = \{ s : N \to R_+ \}$

where N and R_+ denote the set of non-negative integers and real numbers.

Let S_1 denote a subset of S consisting of elements whose values are equal to or greater than 1, i.e.

$S_1 = \{ s \in S \mid s(n) \geqq 1 \}$

Let S_0 denote a subset of S consisting of non-decreasing sequences, i.e.

$S_0 = \{ s \in S \mid s(n) \leqq s(m) \text{ if } n \leqq m \}$

We consider four types of growth orders for elements of S.

<u>Definitions</u>

For $s \in S$, the following numbers are defined ;

(1) $d_+(s) = \limsup_{n \to \infty} \{ \log s(n) / \log n \}$

(2) $d_-(s) = \liminf_{n \to \infty} \{ \log s(n) / \log n \}$

For $s \in S$,

(3) $D_+(s) = \limsup_{n \to \infty} \{ \log (\log s(n)) / \log n \}$

(4) $D_-(s) = \liminf_{n \to \infty} \{ \log (\log s(n)) / \log n \}$

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Note that these limits exist and satisfy inequalities such as $d_-(s) \leqq d_+(s)$ and $D_-(s) \leqq D_+(s)$.

Next we define an order on S. For $a, b \in S$, set $a \leqq^{\cdot} b$ if and only if there exists a positive integer L such that $a(n) \leqq b(Ln)$ for sufficient large positive integers n and $a \sim b$ if and only if $a \leqq^{\cdot} b$ and $b \leqq^{\cdot} a$. Clearly $\sim$ is an equivalence relation on S.

We denote the equivalence class of a by $G(a)$ and denote the partial ordering on the set $S/\sim$ induced by $\leqq^{\cdot}$ as $\leqq$.

For $s \in S$, $d_+(s)$ (resp. $D_+(s)$) is called Gelfand-Kirillov (resp. super Gelfand-Kirillov) dimension of s. And the growth type of s is defined as follows,

(1) s has polynomial growth if and only if $G(s) \leqq G(\{n^d\})$ for some positive real number d.

(2) s has exponential growth if and only if $G(\{\exp(n)\}) \leqq G(s)$.

(3) s has subexponential growth if and only if for any positive real number d, $G(\{n^d\}) \leqq G(s)$, but $G(s) \leqq G(\{\exp(n)\})$.

In this paper, we shall calculate Gelfand-Kirillov (resp. super Gelfand-Kirillov) dimension for Lie algebras (resp. the universal enveloping algebras of Lie algebras) and give a relation between Lie algebras and their universal enveloping algebras.

The results of this paper are joint work with Manabu Sanami (Kobe Univ.).

§2 Results.

Let A be a (not necessarily associative) algebra over the field K. For a finite subset X of A, we denote $A(X;n)$ the K-vector subspace of A spanned by all monomials of length less than or equal to n in the element of X and denote $A(X;0) = K$. We set $d_n(X) = \dim_K A(X;n)$. Here the Gelfand-Kirillov dimension (GKdim) and the super Gelfand-Kirillov dimension (s-GKdim) of A are defined as follows,

$$\mathrm{GKdim}(A) = \sup_X d_+(\{d_n(X)\})$$

$$\text{s-}\mathrm{GKdim}(A) = \sup_X D_+(\{d_n(X)\})$$

where supremums are taken over all finite subset X of A.

If A is generated by a finite subset X, then $\mathrm{GKdim}(A) = d_+(\{d_n(X)\})$ and $\text{s-}\mathrm{GKdim}(A) = D_+(\{d_n(X)\})$.

A map $\delta : S_0 \to S$ is defined as follows ; for $s \in S_0$, $\delta(s)(0) = s(0)$ and $\delta(s)(n) = s(n) - s(n-1)$ for $n > 0$.

Theorem 1. Let g be a finitely generated Lie algebra over K and X be a finite generating subset of g and $U(g)$ be the universal enveloping algebra of g. We set $\gamma_n = \dim_K g(X;n)$, $\alpha = d_-(\delta(\{\gamma_n\}))$ and $\beta = d_+(\delta(\{\gamma_n\}))$. Then

$1 - 1/\alpha+1 \leqq D_-(\{\dim_K U(g)(X;n)\}) \leqq D_+(\{\dim_K U(g)(X;n)\}) = \text{s-}\mathrm{GKdim}(U(g)) \leqq 1 - 1/\beta$.

Theorem 2. Let g be a Lie algebra over K. If there exists a finitely generating subspace X of g such that the limit $\gamma = d(\{\dim_K g(X;n)\}) = \lim (\log (\dim_K g(X;n)) / \log n)$ exists. Then $\text{s-}\mathrm{GKdim}(U(g)) = 1 - 1/\gamma+1$.

§ 3 Examples.

In the following, we assume that K is a algebraically closed field.

(1) Let g be a finite dimensional Lie algebra over K. Then $\mathrm{GKdim}(g) = 0$ and $\mathrm{GKdim}(U(g)) = \dim_K g$. Thus $U(g)$ has polynomial growth.

(2) Let g be a Lie algebra over K with basis $\{ x, y_1, y_2, y_3, \cdots\}$ and satisfies the following relations,

$[x, y_i] = y_{i+1}$, $[y_i, y_j] = 0$.
Then $GKdim(g) = 2$ and $s\text{-}GKdim(U(g)) = 2/3$ by Theorem 2.
Thus $U(g)$ has subexponential growth.

(3) Let g be a finite dimensional semisimple Lie algebra over K and R be a commutative finitely generated K-algebra. We define the Lie algebra $L(g:R) = g \otimes R$ with relation $[x_1 \otimes r_1, x_2 \otimes r_2] = [x_1, x_2] \otimes r_1 r_2$ for $x_1, x_2 \in g$ and $r_1, r_2 \in R$. In this case, $s\text{-}GKdim(U(L(g:R))) = 1 - 1/(\text{Krull dim } R + 1)$. In particular, if g is a affine Kac-Moody Lie algebra, then $s\text{-}GKdim(U(g)) = 1 - 1/1+1 = 1/2$.

(4) Let M be a finitely generated subgroup of K as additive group. We define the Lie algebra $W(M) = \sum_{m \in M} Kx_m$, $[x_m, x_n] = (n-m)x_{n+m}$. Then $s\text{-}GKdim(U(W(M))) = 1 - 1/(\text{rank}(M)+1)$.

(5) Let g be a hyperbolic Kac-Moody Lie algebra, then $GKdim(g) = \infty$ and $s\text{-}GKdim(U(g)) = 1$. In this case g and $U(g)$ has exponential growth.

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