PROCEEDINGS OF THE 24TH SYMPOSIUM ON RING THEORY

HELD AT OSAKA CITY UNIVERSITY, OSAKA

AUGUST 2 — 4, 1991

EDITED BY

Takeshi SUMIOKA

Osaka City University

1991

OKAYAMA, JAPAN

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PREFACE

The 24th Symposium on Ring Theory was held at Osaka City University, Osaka, on August 2–4, 1991. More than one hundred participants attended the symposium. This volume is the proceedings of the symposium, which consists of all the twelve articles given by the speakers.

The symposium and proceedings were financially supported by the Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture through the arrangements by Professor Y. Kitaoka, Nagoya University, whom we would like to thank for his kind arrangements.

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Osaka City University, Osaka, October 1991

Takeshi Sumioka

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SELF-INJECTIVITY OF SEMITRIVIAL EXTENSIONS [†]

Kazutoshi KOIKE

Let R be a ring with identity and M an (R,R)-bimodule. An (R,R)-bimodule homomorphism $M \otimes_R M \to R : m_1 \otimes m_2 \mapsto m_1 m_2$ is called a *pairing* in case $m_1(m_2 m_3) = (m_1 m_2)m_3$ for all $m_1, m_2, m_3 \in M$. Then the additive group $R \oplus M$ becomes a ring by defining a mulitiplication as $(a,m)(b,n) = (ab + mn, an + mb)$. This ring is called *the semitrivial extension of R by M*. Note that in case the pairing is zero, this ring is *the trivial extension.* Self-injectivity of trivial extension and semitrivial extension have been investigated by several authors. These results can be found in [6] and [7]. In this note, we characterize the self-injectivity of semitrivial extensions completely (Theorem 5).

Throughout this note, R is a ring with identity, M is an (R,R) -bimodule with a pairing $M \otimes_R M \to R : m_1 \otimes m_2 \mapsto m_1 m_2$ and T is the semitrivial extension of R by M. We denote the injective hull of a module X by $E(X)$.

Let X be a right T -module. Since R is isomorphic to a subring of T naturally, X becomes a right R -module. We put $xm = x(0,m)$ for all $x \in X$ and $m \in M$. For two right R-submodules X_1 and X_2 of X such that $X_2 M \leq X_1$, there exists a homomorphism $X_2 \to Hom(M_R, X_{1R}) : x_2 \mapsto x_2-$, where $(x_2-)(m) = x_2 m$ for all $m \in M$. We shall call such homomorphisms *canonical.*

For any right R-module X, we note that $Hom(M_R, X_R)$ and $Hom(T_R, X_R)$ become right R-module and right T-module by usual definitions respectively. We define a right T-module structure on $X \oplus Hom(M_R, X_R)$ by

$$(x,f)(a,m) = (xa + f(m), fa + xm-),$$

where $xm-$ is a homomorphism in $Hom(M_R, X_R)$ such that $(xm-)(m') = x(mm')$. Then $Hom(T_R, X_R) \cong X \oplus Hom(M_R, X_R)$ as right T-modules.

[†]The detailed version of this paper will be submitted for publication elsewhere.

The following proposition is easily verified by flatness of ${}_TT$.

Proposition 1 *Let X be a right T-module. Then X_T is injective if X_R has a decomposition $X = X_1 \oplus X_2$ satisfying the following conditions :*

(0) $X_1M \leq X_2$ *and* $X_2M \leq X_1$.

(1) X_{1R} *is injective.*

(2) $X_2 \cong Hom(M_R, X_{1R})$ *canonically.*

To show the converse of this proposition, we must compute the injective hulls of T-modules.

Let X be a right T-module. For any subset S of X, put

$$L(S) = \{s \in S \mid sM = 0\}.$$

If S is a right R-(or T-)submodule of X, so is $L(S)$. Assume that X has a decomposition $X_R = Y \oplus Z$ such that $YM \leq Z$ and $ZM \leq Y$. Put

$$Y' = L(Y) \leq Y_R, \quad Z' = L(Z) \leq Z_R$$

and choose extensions $\alpha : Y_R \to E(Y'_R)$ of the inclusion map $Y'_R \to E(Y'_R)$ and $\beta : Z_R \to E(Z'_R)$ of the inclusion map $Z'_R \to E(Z_R)$. Moreover put

$$Y'' = \{y \in Y \mid \alpha(y) = 0, \beta(yM) = 0\} \leq Y_R$$

and choose an extension $\gamma : Y_R \to E(Y''_R)$ of the inclusion map $Y''_R \to E(Y''_R)$. We define a T-homomorphism

$$\begin{aligned} \psi_X : X = Y \oplus Z \to \quad & E(Y') \oplus E(Z') \oplus E(Y'') \\ & \oplus H(E(Y')) \oplus H(E(Z')) \oplus H(E(Y'')) \end{aligned}$$

via

$$x = y + z \mapsto (\alpha(y), \beta(z), \gamma(y), \alpha(z-), \beta(y-), \gamma(z-)),$$

where $H = Hom(M_R, -)$ and $\alpha(z-)$ is the homomorphism defined by $(\alpha(z-))(m) = \alpha(zm)$ for all $m \in M$ ($\beta(z-)$ and $\gamma(y-)$ are defined similary).

Next proposition plays very important role in this note.

Proposition 2 *Let X_T be a right T-module with a decomposition $X_R = Y \oplus Z$ such that $YM \leq Z$ and $ZM \leq Y$. Then ψ_X is an essential T-monomorphism (i.e. the injective hull of X_T).*

Using this proposition, we obtain the following :

Theorem 3 *Let X be a right T-module with a decomposition $X_R = Y \oplus Z$ such that $YM \leq Z$ and $ZM \leq Y$. Then X_T is injective if and only if Y_R and Z_R have a decomposition $Y = Y_1 \oplus Y_2$ and $Z = Z_1 \oplus Z_2$ satisfying the following conditions :*

(0) *$Y_iM \leq Z_j, Z_iM \leq Y_j$ for $i \neq j$.*

(1) *Y_{1R} and Z_{1R} are injective.*

(2) *$Z_2 \cong Hom(M_R, Y_{1R})$ and $Y_2 \cong Hom(M_R, Z_{1R})$ canonically.*

Remark 4 Let $(A, B, {}_AU_B, {}_BV_A)$ be a Morita context. Put $R = A \times B$ the product ring and $M = U \oplus V$ the direct sum. Then M becomes an (R, R)-bimodule and a pairing of ${}_RM_R$ is induced naturally. In this case, the generalized matrix ring

$$\begin{pmatrix} A & U \\ V & B \end{pmatrix}$$

is isomorphic to the semitrivial extension of R by M. Therefore generalized matrix rings are paticular semitrivial extensions. In [4], Müller determined the injective hulls of modules over generalized matrix rings (See [4, Proposition 2.1]). However it is easy to see that all modules over generalized matrix ring have the decomposition which satisfies the assumption in Theorem 3. So we can regard Theorem 3 as a generalization of Müller's result.

Trivially, as a right R-module, semitrivial extension $T = R \oplus M$ satisfies the assumption in Theorem 3. So we can apply the result of Theorem 3 to T. But for the module T_T, the decompositions of R and M are determined by single idempotent of R.

Theorem 5 *Let T be a semitrivial extension of R by M. Then T is right self-injective if and only if there exists an idempotent e of R satisfying the following conditions :*

(1) eR_R *and* $(1-e)M_R$ *are injective.*

(2) $eM \cong Hom(M_R, eR_R)$ *and* $(1-e)R \cong Hom(M_R,(1-e)M_R)$ *canonically.*

Now we show the results for injectivity of semitrivial extensions by Sakano. To prove these results, we need the following lemma :

Lemma 6 *Let* X *be a right* T*-module with a decomposition* $X_R = Y \oplus Z$ *such that* $YM \leq Z$ *and* $ZM \leq Y$*. Assume that* $L(Z) = 0$*. Then* X_T *is injective if and only if the following conditions hold:*

(1) Y_R *is injective.*

(2) $Z \cong Hom(M_R, Y_R)$ *canonically.*

Recall that a pairing $M \otimes M \to R : m_1 \otimes m_2 \mapsto m_1m_2$ is *right nondegenerate* in case $mM = 0$ implies $m = 0$.

The next two results follows from Lemma 6 immediately.

Corollary 7 [6, Theorem 3.1] *Assume that the pairing is right nondegenerate. Then* T *is right self-injective if and only if the following conditions hold:*

(1) R *is right self-injective.*

(2) $M \cong Hom(M_R, R_R)$ *canonically.*

Corollary 8 [6, Theorem 3.2] *Assume that* $_RM$ *is faithful. Then* T *is right self-injective if and only if the following conditions hold.*

(1) M_R *is injective.*

(2) $R \cong Hom(M_R, M_R)$ *canonically.*

We provide examples which satisfy the conditions in Corollary 7 and 8 respectively.

Example 9 (a) Let p be a prime number and n a positive integer. Let R be the factor ring Z/p^nZ of the ring of integers Z and $M = Z/pZ$. Then R is a commutative local QF ring and M is a non-injective R-module. We define a pairing $M \otimes_R M \rightarrow R$ via

$$(a + pZ) \otimes (b + pZ) \mapsto abp^{n-1} + p^n Z.$$

Then we can easily verify the canonical homomorphism $M \rightarrow Hom(M_R, R_R)$ is isomorphic. So the semitrivial extension T of R by M is self-injective.

(b) Let M be an (R, R)-bimodules which defines a self-duality on R. Then for any pairing of M (e.g. zero pairing), the semitrivial extension of R by M is self-injective.

At the end of this note, we remark the injectivity of modules which are generated by a primitive idempotent. Let T be any ring and e a primitive idempotent of T. Then $(eTe, (1-e)T(1-e), eT(1-e), (1-e)Te)$ is a Morita context. So we obtain the following (See Remark 4).

Corollary 10 *Let T be any ring and e a primitive idempotent of T. Then eT_T is injective if and only if one of the following two conditions holds :*

(a) *eTe is right self-injective and*
$eT(1-e) \cong Hom((1-e)Te_{eTe}, eTe_{eTe})$ canonically.

(b) *$eT(1-e)_{(1-e)T(1-e)}$ is injective and*
$eTe \cong End(eT(1-e)_{(1-e)T(1-e)})$ canonically.

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Department of Computer Science
Ube College
5-40, Bunkyou-chou, Ube 755, Japan

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ON THE BICOMMUTATORS OF MODULES OVER H-SEPARABLE EXTENSION RINGS

Kozo SUGANO

1. Throughout this section A will be a ring with the identity 1, B a subring of A containing 1, C the center of A and $D = V_A(B)$, the centralizer of B in A. M will be a left A-module chosen arbitrarily, and $A^* = Bic({}_AM)$, $B^* = Bic({}_BM)$, the bicommutators of ${}_AM$ and ${}_BM$, respectively. Moreover C^* will be the center of A^* and $D^* = V_{A^*}(B^*)$. There exists a canonical ring homomorphism ι of A to A^*. We will write $\bar{a} = \iota(a)$ for each a in A and $\bar{X} = \iota(X)$ for any subset X of A.

We can easily see that $C^* = V_{A^*}(A)$ and $D^* = V_{A^*}(D)$. Furthermore,

Lemma 1. *If A is an H-separable extension of B, then we have $B^* = V_{A^*}(V_{A^*}(B^*))$ and $D \otimes_C C^* = D^*$ via $d \otimes c^* \mapsto \iota(d)c^*$ for $d \in D$ and $c^* \in C^*$.*

Theorem 1. *Let A be an H-separable extension of B. Then we have*

(1) *If A is left (resp. right) B-f.g. projective, A^* is an H-separable extension of B^* and left (resp. right) B^*-f.g. projective.*

(2) *If B is a left (resp. right) B-direct summand of A, B^* is a left (resp. right) B^*-direct summand of A^*.*

A is an H-separable extension of B if and only if $1 \otimes 1 \in (A \otimes_B A)^A D$ by Proposition 1 [4], where $(A \otimes_B A)^A = \{\alpha \in A \otimes_B A \mid a\alpha = \alpha a$ for each a in $A\}$. When we write $1 \otimes 1 = \sum x_{ij} \otimes y_{ij} d_i$ with $\sum x_{ij} \otimes y_{ij} \in (A \otimes_B A)^A$ and $d_i \in D$, we call $\{\sum x_{ij} \otimes y_{ij},\ d_i\}$ an H-system of A over B (See [2]).

This lecture is derived from [5], [6] and [7].

Theorem 1 (1) shows that it is natural for us to consider the case where both A^* and A are H-separable extensions of B^* and B, respectively. Under these conditions we have

Lemma 2. *Let A^* and A be H-separable extensions of B^* and B, respectively, and $\{\sum x_{ij} \otimes y_{ij},\ d_i\}$ an H-system of A over B. Then we have*

(1) $\{\sum \bar{x}_{ij} \otimes \bar{y}_{ij},\ \bar{d}_i\}$ is an H-system of A^* over B^*.

(2) $A^* \otimes_B A$ is isomorphic to $A^* \otimes_{B^*} A^*$ via $a^* \otimes b \mapsto a^* \otimes \bar{b}$ for $a^* \in A^*$ and $b \in A$. Similarly we have $A \otimes_B A^* \cong A^* \otimes_{B^*} A^*$.

Theorem 2. *Let A^* and A be H-separable extensions of B^* and B, respectively. Then if B^* is a left (resp. right) B^*-direct summand of A^*, we have $A^* = \bar{A}B^* \cong A \otimes_B B^*$ (resp. $A^* = B^*\bar{A} \cong B^* \otimes_B A$).*

2. We can apply the results in §1 to the H-separable extensions of strongly primitive rings. For any left (resp. right) A-module ${}_AX$ (resp. X_A) we will denote its socle by $s({}_AX)$ (resp. $s(X_A)$). A ring A is called to be strongly primitive if A has a faithful minimal left ideal. If A is strongly primitive, then A has also a faithful minimal right ideal, and we have $s({}_AA) = s(A_A)$, and consequently, we can call it simply the socle of A and denote it by $s(A)$. The socle of a strongly primitive ring A is contained in every non-zero ideal of A. In this section A and B will always be strongly primitive rings, and M will be a faithful minimal left ideal of A. For A, B and M we will use the same notation as §1. Therefore B will always be a subring of A. Moreover, $A^* = Bic({}_AM)$ and $B^* = Bic({}_BM)$. The next lemma has been shown in [5].

Lemma 3. *Let A and B be strongly primitive. If A is left or right B-projective. Then we have either $s(A) \cap B = 0$ or $s(A) \cap B = s(B)$ and $s(A) = As(B)A$.*

Theorem 3. *Let A and B be strongly primitive and A an H-separable extension of B. Assume furthermore that A is left B-finitely generated projective. Then we have*

(1) $s(A) \cap B = s(B)$ and $s(A) = As(B)A = s(B)A = s({}_BA)$.

(2) $B^* \cong Bic({}_BI)$, where I is a faithful minimal left ideal of B.

(3) A^* is an H-separable extension of B^* such that $V_{A^*}(V_{A^*}(B^*)) = B^*$.

(4) D^* is a simple C^*-algebra and isomorphic to $D \otimes_C C^*$.

(5) A^* is both left and right B^*-free, having both left and right B-bases consisting of $[D^* : C^*]$ elements.

(6) $A^* = B^*A = AB^*$.

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Department of Mathematics
Hokkaido University
060 Sapporo, Japan

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PRIME IDEALS IN POLYNOMIAL RINGS OVER HEREDITARY PI-RINGS

Hidetoshi MARUBAYASHI, Yang LEE and Jae Keol PARK

A ring is called left (resp. right) hereditary provided every left (resp. right) ideal is projective. A commutative hereditary domain is called a Dedekind domain.

It is a standard fact that every prime factor ring of a commutative hereditary ring is hereditary. This fact was extended by Armendariz and Hajarnavis [1] to hereditary rings, satisfying a polynomial identity, i.e., hereditary PI-rings. On the other hand, when R is a Dedekind domain, Hillman [6] gave a criterion for R-torsion-free prime factor rings of R[x] to be Dedekind.

Motivated by these results, in [10] and [11], Marubayashi, Lee and Park characterized hereditary prime factor rings of the polynomial ring over a hereditary PI-ring, thereby they could provide an answer to a question of Armendariz, that is, a characterization of prime ideals P of the polynomial ring A[x] over a hereditary PI-ring

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The final version of this paper will be submitted for publication elsewhere.

such that A[x]/P is a hereditary ring.

We introduce, in this expository article, some results on hereditary prime factor rings of the polynomial ring over a hereditary PI-ring, which were mainly observed in [10] and [11] related to the question of Armendariz.

First we begin with the following example which is essentially due to Hillman [6].

Example 1. For the ring Z of integers, the prime factor ring $Z[x]/(x^3-4)Z[x]$ of Z[x] is not Dedekind. Actually, $Z[x]/(x^3-4)Z[x] = Z[\sqrt[3]{4}]$ and $Q(\sqrt[3]{4})$ is the field of fractions of $Z[\sqrt[3]{4}]$, where Q is the rational numbers field. In this situation, $\sqrt[3]{2}$ in $Q(\sqrt[3]{4})$ is integral over $Z[\sqrt[3]{4}]$, but $\sqrt[3]{2}$ is not in $Z[\sqrt[3]{4}]$ and hence $Z[\sqrt[3]{4}]$ is not a Dedekind domain.

As we mentioned before, Hillman [6] provided the following criterion for prime factor rings of the polynomial ring over a Dedekind domain to be Dedekind.

Theorem 2 [6, Hillman]. Assume that R is a Dedekind domain and f(x) in R[x] such that f(x)R[x] is a non-maximal prime ideal in R[x]. Then R[x]/f(x)R[x] is a Dedekind domain if and only if f(x) is not contained in the square of any maximal ideal of R[x].

Actually in Example 1, the polynomial x^3-4 is contained in the square of the maximal ideal $2Z[x]+xZ[x]$ of Z[x].

Now for the sketch of a generalization of Theorem 2 to PI-ring case, we assume that a ring A is prime hereditary PI.

Theorem 3 [14, Robson and Small]. The center R of A is a Dedekind domain and A is a finitely generated R-module.

In this situation, the ring of quotients $Q(A)$ of A is simple Artinian and $Q(A) = AK$, where K is the field of fractions of R. Furthermore, the polynomial ring $A[x]$ is a tame $R[x]$-order in the sense of Silver-Fossum [4] and hence it is a v-HC order in the sense of Marubayashi [8 and 9]. We also have following nice properties:

$n = \text{gl.dim } A[x] = \text{gl.dim } R[x] = \text{cl.k.dim } A[x] = \text{cl.k.dim } R[x]$, where $n = 1$ if $A = Q(A)$ and $n = 2$ if $A \neq Q(A)$ by [2, Lemma 3.5] and [15, Theorem 1.3].

For a central polynomial $f(x)$ in $A[x]$ with $P = f(x)A[x]$ a prime ideal, if $A = Q(A)$, then of course $A[x]/P$ is simple Artinian. Also, if P is a maximal ideal, then $A[x]/P$ is a simple Artinian ring. Therefore to remove the triviality, we assume from now on that $A \neq Q(A)$ and $P = f(x)A[x]$ is a non-maximal prime ideal of $A[x]$. Observe that P is a minimal non-zero prime ideal of $A[x]$ and $p = P \cap R[x]$ is also a minimal non-zero prime ideal of $R[x]$. Moreover $A[x]/P$ is a finite centralizing extension of $R[x]/p$. Therefore for any maximal ideal M of $A[x]$ containing properly P, $m = M \cap R[x]$ is a maximal ideal of $R[x]$ which contains p properly. Conversely, for any maximal ideal m of $R[x]$ containing p properly, there is a maximal ideal M of $A[x]$ which contains P properly and $M \cap R[x] = m$.

Now let m be a maximal ideal of $R[x]$ which contains p properly. By localizing $R[x]$ and $A[x]$ with m, since the ring $A[x]_m$ is a finite centralizing extension of $R[x]_m$, we have $J(A[x]_m) \cap R[x]_m = J(R[x]_m) = m_m$, where $J(-)$ denotes the Jacobson radical. Thus $A[x]_m/J(A[x]_m)$ is also a finite centralizing extension of the field

$R[x]_m/m_m$ and hence we have that $A[x]_m/J(A[x]_m)$ is a semi-simple Artinian ring. Therefore there are only finitely many maximal ideals M_1', M_2', ..., M_k' of $A[x]_m$ such that $J(A[x]_m) = M_1' \cap M_2' \cap \ldots \cap M_k'$. Also in this case $M_i' \cap R[x]_m = m_m$ for $i = 1, 2, \ldots, k$.

Put $M_i = M_i' \cap A[x]$ for $i = 1, 2, \ldots, k$. Then it may be easily checked that M_1, M_2, ..., M_k are only maximal ideals of $A[x]$ such that $M_i \cap R[x] = m$. Also each M_i contains P properly and $M_i = M_m$. Furthermore for $i = 1,2, \ldots, k$, we have the fact that $f(x)$ is not contained in the square of M_i if and only if $f(x)$ is not contained in the square of ${M_i}_m$. Also each M_i contains M_i^2 properly for each i. Indeed, let $M = M_i$ and $m_0 = M \cap A$ a prime ideal. By observing that $Q(A) = AK$, it can be easily verified that $mK[x]$ is a nonzero proper maximal ideal of $K[x]$ if $m_0 = 0$. Thus $pK[x]$ is also a nonzero proper ideal of $K[x]$ and hence we have $mK[x] = pK[x]$. So it follows that $m = p$, which is a contradiction. Since m_0 is a non-zero prime ideal of the hereditary prime PI-ring A, A/m_0 is simple Artinian and thus it follows that $m_0[x]$ is a minimal non-zero prime ideal of $A[x]$. So in the principal ideal ring $A[x]/m_0[x]$, the ideal $M/m_0[x]$ contains $(M/m_0[x])^2$ properly and hence M contains M^2 properly.

On the other hand, since ${M_i}_m$ contains P_m properly and gl.dim $A[x]_m$ is less than or equal to gl.dim $A[x] = 2$, we have that gl.dim $A[x]_m = 2$. Obviously $(A[x]/P)_{m/p}$, which is the localization of $A[x]/P$ by m/p, is equal to $A[x]_m/P_m$.

Now for our convenience, for a moment, let $S = A[x]$. Then by previously mentioned facts we are now in the following situation:

1) S is a prime Noetherian ring with the non-zero Jacobson radical J.

2) S/J is a semi-simple Artinian ring such that the Jacobson radical J is the intersection of maximal ideals $N_1, N_2, \ldots$, and N_k.

3) gl.dim $S = 2$ and there is a central element c in J such that $P = cS$ is a prime ideal of S.

By modifying Kaplansky's method [7, Theorems 3 and 8, Part III], we have following

Lemma A. If c is not in N_i^2 for $i = 1, 2, \ldots, k$, then the ring S/P is a hereditary ring.

With Lemma A together with all our preparations so far we have done, now we can introduce following result, which is a generalization of Theorem 2 to PI-ring case.

Theorem B. Let A be a hereditary prime PI-ring with the center R and let f(x) be a central polynomial of A[x] such that $P = f(x)A[x]$ is a non-maximal prime ideal. Then the ring A[x]/P is hereditary if and only if f(x) is not contained in the square of any maximal ideal of A[x].

By Lemma A and previously mentioned several facts, if f(x) is not in M^2 for any maximal ideal M of A[x], then we can show that $(A[x]/P)_{m/p} = A[x]_m/P_m$ is hereditary for any maximal ideal m of R[x] containing properly p. Thus we have that the ring A[x]/P is a hereditary ring. Conversely, assume that A[x]/P is hereditary. Now for any maximal ideal M of A[x], if f(x) is not in M, then we are done. So we may assume that f(x) is in M. By [12, claim 4], the center of A[x]/P is R[x]/p and hence R[x]/p is a Dedekind domain. Thus the localization $(R[x]/p)_{m/p}$ is a dicrete rank one valuation domain which is the center of the ring $(A[x]/P)_{m/p}$. So in this case, if we assume that f(x) is not in M^2, then we have either gl.dim $A[x]_m \leq 1$ or $M = M^2$, which is a contradiction. Therefore f(x) is not in M^2 for any maximal ideal M of A[x].

Theorem 4 [1, Armendariz and Hajarnavis]. Every prime factor ring of a hereditary PI-ring is hereditary.

When A is hereditary PI and $P = f(x)A[x]$ is a prime ideal of A[x], we may reduce this situation to hereditary prime PI case. Explicitly, $A_0 = A/(P \cap A)$ is a hereditary prime PI by Theorem 4 and A[x]/P is isomorphic to $A_0[x]/f_0(x)A_0[x]$ for some central polynomial $f_0(x)$ in $A_0[x]$.

In [11] by adopting v-HC orders and tame orders, we also can prove following result which is an answer to the question of Armendariz.

Theorem C. Let A be a hereditary PI-ring and let P be a prime ideal of A[x]. Set $P_0 = P \cap A$ a prime ideal of A. Then we have the following:

1) If $P = P_0[x]$, then A[x]/P is hereditary if and only if P_0 is a maximal ideal of A.

2) If P contains properly $P_0[x]$, then A[x]/P is hereditary if and only if $M^2 + P_0[x]$ does not contain P for any prime ideal M of A[x] which contains P properly.

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Naruto University of Education, Naruto, Tokushima 772, Japan

and

Busan National University, Busan 609-735, Korea

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PRIME IDEALS IN STRONGLY GRADED RINGS BY POLYCYCLIC-BY-FINITE GROUPS

Hidetoshi MARUBAYASHI and Haruo MIYAMOTO

§ Introduction. Let G be any group with identity e and let $R = \sum \oplus_{x \in G} R_x$ be a strongly G–graded ring. An ideal I of R_e is called G–stable if $I^x = R_{x^{-1}} I R_x = I$. I is called G–prime if I is G–stable and if $AB \subseteq I$ implies that either $A \subseteq I$ or $B \subseteq I$ for any G–stable ideals A and B of R_e. The ring R_e is called G–prime if 0 is a G–prime ideal. Lorenz and Passman proved the following

Proposition 1. [P] Let G be a finite group and let R be a crossed product of G over its base ring R_e which is G–prime. Then the number of the minimal prime ideals of R and the nilpotency of the prime radical of R are both less than or equal to $|G|$, the order of G.

They also obtained, by using Proposition 1, the relationship between the prime ideals of R and of R_e which are the classical properties known as *Lying over, Going up, Going down and Incomparability.* More precisely,

Proposition 2. [P] Let G be a finite group and let R be a crossed product of G over its base ring R_e. Then

(1) (Cutting down) If P is a prime ideal of R, then there exists a prime ideal $\wp$ of R_e, unique up to G–conjugation, with $\wp$ minimal over $P \cap R_e$. Indeed, we have $P \cap R_e = \cap_{x \in G} \wp^x$. When this occurs, we say that P lies over $\wp$.

(2) (Lying over) If $\wp$ is a prime ideal of R_e, then there are primes $P_1, P_2, \cdots, P_n$ of R with $n \leq |G|$ such that P_i lies over $\wp$.

The detailed version of this paper has been submitted for publication elsewhere.

(3) (Incomparability) Suppose that $P_1 \supseteq P_2$ are prime ideals of R and $\wp_1 \supseteq \wp_2$ are prime ideals of R_e such that P_i lies over $\wp_i$. If $P_1 \neq P_2$, then $\wp_1 \neq \wp_2$.
(4) (Going up) Let $\wp_1 \supseteq \wp_2$ are prime ideals of R_e and P_2 is a prime ideal of R lying over $\wp_2$. Then there exists a prime ideal $P_1 \supseteq P_2$ of R with P_1 lying over $\wp_1$.
Conversely, suppose that $P_1 \supseteq P_2$ are prime ideals of R and $\wp_2$ is a prime ideal of R_e such that P_2 lies over $\wp_2$. Then there exists a prime ideal $\wp_1 \supseteq \wp_2$ of R_e such that P_1 lies over $\wp_1$.
(5) (Going down) Let $\wp_1 \supseteq \wp_2$ are prime ideals of R_e and P_1 is a prime ideal of R lying over $\wp_1$. Then there exists a prime ideal $P_2 \subseteq P_1$ of R with P_2 lying over $\wp_2$.
Conversely, suppose that $P_1 \supseteq P_2$ are prime ideals of R and $\wp_1$ is a prime ideal of R_e such that P_1 lies over $\wp_1$. Then there exists a prime ideal $\wp_2 \subseteq \wp_1$ of R_e such that P_2 lies over $\wp_2$.

Let P and $\wp$ be prime ideals of R and R_e, respectively. Then we say that P lies over $\wp$ if $\bigcap_{x \in G} \wp^x = P \cap R_e$ and if $\wp$ is minimal over $\bigcap \wp^x$. If G is a finite group, then second condition is superfluous, and this condition is equivalent to "*lying over*" in [P].

In this paper, we will give the outline of the proofs of

Theorem 1. Let G be a polycyclic-by-finite group and let $R = \sum \oplus_{x \in G} R_x$ be a strongly G-graded ring with R_e G-prime and right Noetherian. Then the number of the minimal prime ideals of R and the nilpotency of the prime radical of R is less than or equal to $|\Delta^+(G)|$, where $\Delta^+(G) = \{ x \in G \mid x$ has a finite order and $|G{:}C_G(x)| < \infty \}$ is the unique maximal finite normal subgroup of G.

Theorem 2. Let G be a polycyclic-by-finite group and let $R = \sum \oplus_{x \in G} R_x$ be a strongly G-graded ring with R_e G- prime and right Noetherian. Then
(1) (Cutting down) Let P be a prime ideal of R. Then there exists a prime ideal $\wp$ of R_e, unique up to G-conjugation, such that $\wp$ is minimal over $P \cap R_e$ and $\bigcap \wp^x = P \cap R_e$.
(2) (Lying over) Let $\wp$ be a prime ideal of R_e, then there exist prime ideals $P_1, P_2, \cdots, P_n$ of R with $n \leq |\Delta^+(G)|$ such that P_i lies over $\wp$.
(3) (Incomparability) Let $\wp_1 \subseteq \wp_2$ be prime ideals of R_e, and let $P_1 \subseteq P_2$ be prime ideals lying over $\wp_1$ and $\wp_2$, respectively. If $P_1 \neq P_2$, then $\wp_1 \neq \wp_2$.

Theorem 3. Let R be a strongly G-graded ring whose base ring R_e is right Noetherian and let G be a polycyclic-by-finite group. Then
(1) (Going up) Let $\wp_1$ and $\wp$ be prime ideals of R_e with $\wp_1 \supseteq \wp$ and let P be a prime ideal of R lying over $\wp$. Then there exists a prime ideal P_1 of R such that P_1 lies over $\wp_1$

and $P_1 \supseteq P$.

(2) (Going down) Let $\wp_1$ and $\wp$ be prime ideals of R_e with $\wp \supseteq \wp_1$ and let P be a prime ideal of R lying over $\wp$. Then there exists a prime ideal P_1 of R such that P_1 lies over $\wp_1$ and $P \supseteq P_1$.

In Theorems 2 and 3, we give the classical properties known as *Lying over, Going down, Going up and Incomparability*. If G is a finite group, then Passman and Lorenz proved two different types of Going up theorem and Going down theorem, respectively. But if G is infinite, then one of them does not hold, respectively, in general. An easy example will be offered.

§ Proofs of Theorems. If R_e is a semi-prime right Goldie ring, then the set $\mathcal{C}_e = \mathcal{C}_{R_e}(0)$ is a regular right Ore set of R by Proposition 1.4 of [N. N. V], where $\mathcal{C}_{R_e}(A) = \{ c \in R_e \mid c \text{ is regular mod } A \}$ for any ideal A of R_e. The right quotient ring $Q^g = Q^g(R)$ with respect to $\mathcal{C}_e$ is also a strongly G-graded ring and we can write $Q^g = \sum \oplus_{x \in G} R_x Q_e$, where Q_e is a right quotient ring of R_e with respect to $\mathcal{C}_e$, and Q_e is a semi-simple Artinian ring.

Lemma 1. If R_e is G-prime, then R_e is semiprime and Q_e is G-simple, i.e., G-stable ideals of Q_e are trivial. In particular, Q_e is G-prime.

We write

$$Q^g(\Delta^+(G)) = \sum \oplus_{x \in \Delta^+(G)} R_x Q_e$$

and

$$S = \sum \oplus_{x \in \Delta^+(G)} R_x$$

<u>First step of the proof of Theorem 1.</u> Suppose that R_e is prime. Then Q_e is a simple Artinian ring and so

$$Q^g(\Delta^+(G)) \cong Q_e * \Delta^+(G).$$

By Proposition 1 there exist ℓ ($\leq |\Delta^+(G)|$) minimal prime ideals $\overline{P}_1, \cdots, \overline{P}_\ell$ of $Q^g(\Delta^+(G))$ such that $\overline{P}_i \cap Q_e = 0$. Further, for the prime radical $\overline{N}$ of $Q^g(\Delta^+(G))$ it holds that $\overline{N}^{|\Delta^+(G)|} = 0$.

Lemma 2. There exists a 1-1 correspondence between the minimal prime ideals $\overline{P}$

of $Q^g(\Delta^+(G))$ and the minimal prime ideals $\mathcal{P}$ of S in such a way that $\mathcal{P} = \overline{P} \cap S$ and $\overline{P} = \mathcal{P}Q^g(\Delta^+(G))$. Further, $\overline{N} = \mathcal{P}Q^g(\Delta^+(G))$ and $\mathcal{N} = \overline{N} \cap S$, where $\mathcal{N}$ is the prime radical of S.

Hence there exist ℓ $(\leq |\Delta^+(G)|)$ minimal prime ideals $\mathcal{P}_1, \cdots, \mathcal{P}_\ell$ of S such that $\mathcal{P}_i \cap R_e = 0$ and $\mathcal{N}^{|\Delta^+(G)|} = 0$.

Lemma 3. There exists a 1–1 correspondence between the minimal prime ideals P of R and the minimal G–prime ideals $\mathcal{P}^*$ of S in such a way that $\mathcal{P}^* = P \cap S$ and $P = \mathcal{P}^*R$. Further, if N is the prime radical of R, then $N = \mathcal{N}R = R\mathcal{N}$.

Lemma 4. Let $\mathcal{P}_i^* = \cap_{x \in G} \mathcal{P}_i^x$ and let $\mathcal{P}_1^*, \cdots, \mathcal{P}_k^*$ $(k \leq \ell)$ be distinct $\mathcal{P}_i^*$'s, renumbered if necessary, then they are the minimal G–prime ideals of S and $\mathcal{N} = \cap_{i=1}^{k} \mathcal{P}_i^*$.

Thus we get that Theorem 1 is true if R_e is prime.

<u>Second step of the proof of Theorem 1.</u> Suppose that R_e is G–prime but not prime. Then

Lemma 5. There exists a minimal prime ideal $\wp$ of R_e such that $\cap_{x \in G} \wp^x = 0$ and $H = \{ x \in G \mid \wp^x = \wp \}$ is a subgroup of finite index in G.

Lemma 6. There exists a 1–1 correspondence between the prime ideals P of R with $P \cap R_e = 0$ and the prime ideals L of $R(H) = \sum \oplus_{x \in H} R_x$ with $L \cap R_e = \wp$ in such a way that

$$P = \cap_{x \in G} \{LR\}^x = L^{|G} \text{ and } L = \{r \in R(H) \mid Ar \subseteq P\} = P_{|H},$$

where $A = \mathrm{Ann}(\wp)$.

Let $\tilde{R} = R(H)/\wp R(H)$. Then $\tilde{R}_e = R_e/\wp$ is prime, and so, there exist k $(\leq |\Delta^+(H)|)$ minimal prime ideals $\tilde{L}_1, \cdots, \tilde{L}_k$ of $\tilde{R}$ with $\tilde{L}_i \cap \tilde{R}_e = 0$ and it holds for the prime radical $\tilde{N}$ of $\tilde{R}$ that $\tilde{N}^{|\Delta^+(H)|} = 0$. We denote by L_i the canonical inverse image of $\tilde{L}_i$. L_i's are the minimal prime ideals of R(H) with $L_i \cap R_e = \wp$. Thus we get k $(\leq |\Delta^+(H)|)$ minimal prime ideals $P_1, \cdots, P_k$ with $P_i \cap R_e = 0$ by the correspondence in Lemma 6.

Further, put $J = L_1 \cap \cdots \cap L_k$. Then $\tilde{J} = \tilde{N}$ and so $J^{|\Delta^+(H)|} \subseteq \wp R(H)$.

Lemma 7. For any ideals A and B of R(H)

$$A^{|G}B^{|G} \subseteq (AB)^{|G} \quad \text{and} \quad (A \cap B)^{|G} = A^{|G} \cap B^{|G}.$$

Further,

$$(\wp R(H))^{|G} = 0.$$

Hence it holds for the prime radical N of R that

$$\begin{aligned} N^{|\Delta^+(H)|} &= (P_1 \cap \cdots \cap P_k)^{|\Delta^+(H)|} = \left\{\cap_{i=1}^{k} L_i^{|G}\right\}^{|\Delta^+(H)|} \\ &= \left\{(\cap L_i)^{|G}\right\}^{|\Delta^+(H)|} = \left(J^{|G}\right)^{|\Delta^+(H)|} = (J^{|\Delta^+(H)|})^{|G} \\ &\subseteq (\wp R(H))^{|G} = 0. \end{aligned}$$

Since H is a subgroup of finite index in G $\Delta^+(H) \subseteq \Delta^+(G)$. Thus we complete the proof of Theorem 1.

Proof of Theorem 2. (1) Since $P \cap R_e$ is G-prime, $\tilde{R}_e = R_e/(P \cap R_e)$ is a G-prime ring, and so, $\tilde{R}_e$ has a minimal prime ideal $\tilde{\wp}$ with $\cap_{x \in G} \tilde{\wp}^x = \tilde{0}$, by Lemma 5. Then $\wp$, the ideal of R_e whose canonical image in $\tilde{R}_e$ equals to $\tilde{\wp}$, is a prime ideal which is minimal over $P \cap R_e$ and $\cap_{x \in G} \wp^x = P \cap R_e$.

(2) Since $\cap_{x \in G} \wp^x$ is a G-prime ideal of R_e,

$$\tilde{R} = R/(\cap_{x \in G} \wp^x)R$$

satisfies the condition in Theorem 1, hence there exist the minimal prime ideals $\tilde{P}_1, \cdots, \tilde{P}_\ell$ with $\ell \leq |\Delta^+(G)|$ and $\tilde{P}_i \cap \tilde{R}_e = 0$ for all i. Hence P_i, the ideal of R whose canonical image in $\tilde{R}$ equals $\tilde{P}_i$, clearly lies over $\wp$ for each i.

Furthermore, $P_1, \cdots, P_\ell$ are incomparable since $\tilde{P}_1, \cdots, \tilde{P}_\ell$, are minimal primes. Hence (3) follows immediately.

Lemma 8. Let G be a finite group and let R be a ring such that R is the sum $\sum_{x \in G} R_x$ of (R_e,R_e)-bisubmodules R_x with $R_x \cdot R_y = R_{xy}$ for all $x, y \in G$. If I is an essential ideal of R_e, i.e., I intersects nontrivially all nonzero ideals of R_e then there exists a non-zero ideal J of R with $0 \neq J \cap R_e \subseteq I$.

Lemma 9. Let P be an ideal of R. Then P is minimal prime if and only if

(1) $P = (P \cap S)R$ with $P \cap S$ G-prime,

and

(2) $P \cap R_e = 0$.

<u>The proof of Theorem 3.</u> (1) Let $\mathcal{P} = P \cap S$, G–prime. Then $\mathcal{P} \cap R_e = \cap_{x \in G} \wp^x$. Let $\mathcal{P}_1$ be the maximal element of the set $\{ A :$ ideal of $S \mid A \cap R_e \subseteq \wp_1$ and $A \supseteq \mathcal{P} \}$. Then it is easily checked that $\mathcal{P}_1$ is a prime ideal of S since $\wp_1$ is prime. First, we will prove that $\wp_1$ is a minimal prime ideal over $\mathcal{P}_1 \cap R_e$. Let $\overline{S} = S/\mathcal{P}_1$ and $\pi : S \longrightarrow \overline{S}$ be the canonical mapping. Then the set $\{ \overline{R}_x = \pi(R_x) \mid x \in \Delta^+(G)\}$ satisfies the condition of Lemma 8 with $\overline{R}_e = R_e/(\mathcal{P}_1 \cap R_e)$. If $\overline{I}$ is an ideal of $\overline{S}$ with $\overline{I} \cap \overline{R}_e \subseteq \overline{\wp}_1$, then $I \cap R_e \subseteq \wp_1 + \mathcal{P}_1$, where I is the inverse image of $\overline{I}$ in S. Let $r = z + p$ be any element in $I \cap R_e$, where $z \in \wp_1$ and $p \in \mathcal{P}_1$. Then $p = r - z \in R_e \cap \mathcal{P}_1 \subseteq \wp_1$. Hence $I \cap R_e \subseteq \wp_1$ and so, by the choice of $\mathcal{P}_1$, $I = \mathcal{P}_1$, i.e., $\overline{I} = \overline{0}$. Thus, by Lemma 8, $\overline{\wp}_1$ is not essential. So there exists a nonzero ideal $\overline{J}$ of $\overline{R}_e$ with $\overline{\wp}_1 \cap \overline{J} = \overline{0}$. Since $\overline{R}_e$ is semi–prime, there exists a minimal prime ideal $\overline{\wp'}$ with $\overline{\wp'} \not\supseteq \overline{J}$. Thus $\overline{\wp'} \supseteq \overline{\wp}_1$ implies that $\overline{\wp'} = \overline{\wp}_1$. Hence $\wp_1$ is a minimal prime ideal over $\mathcal{P}_1 \cap R_e$ and so $\cap_{y \in \Delta^+(G)} \wp_1^y = \mathcal{P}_1 \cap R_e$. Put $P_1 = (\cap_{x \in G} \mathcal{P}_1^x)R$. Then P_1 is a prime ideal of R by Proposition 8.3 of [P] and $P_1 \supseteq P$, because $\cap \mathcal{P}_1^x \supseteq \mathcal{P}$ and $P = \mathcal{P}R$. Furthermore, $P_1 \cap S = \cap_{x \in G} \mathcal{P}_1^x$, G–prime with $P_1 = (P_1 \cap S)R$ and $P_1 \cap R_e = \cap \mathcal{P}_1^x \cap R_e = \cap (\mathcal{P}_1 \cap R_e)^x = \cap (\cap \wp_1^y)^x = \cap \wp_1^x$. Hence P_1 is a minimal prime ideal over $(\cap \wp_1^x)R$ by Lemma 9 and therefore P_1 lies over $\wp_1$ with $P_1 \supseteq P$.

(2) By Theorem 2 there exist a finite number of prime ideals $P_1, P_2, \cdots, P_r$ of R lying over $\wp_1$. Then, it is clear from Lemma 9 that P_i's are the full set of minimal prime ideals over $(\cap \wp_1^x)R$. Therefore for some integer n,

$$(P_1 \cap P_2 \cdots \cap P_r)^n \subseteq (\cap_{x \in G} \wp_1^x)R \subseteq (\cap_{x \in G} \wp^x)R \subseteq P,$$

and so $P_i \subseteq P$ for some i.

Another types of Going up and Going down Theorems of prime ideals do not hold; for example, let K be a field, G be an infinite cyclic group $< x >$, and R be the group ring K[G]. Then consider two prime ideals $P = (x - 1) \supsetneq 0$. Obviously 0 is a prime ideal over the ideal 0 of $K = R_e$ but P does not lie over any ideal of R_e.

To give a counter example for another *Going down* theorem, let D be a commutative unique factorization domain and let S = D[x], the polynomial ring over D in an indeterminate x. For any prime element p of D, put $\wp = pS + xS$, a prime ideal of S. Let $f(y) = xy + p \in S[y]$, the polynomial ring over S in an indeterminate y. Then by Eisenstein's theorem, f(y) is a prime element in S[y] with $f(y)S \cap S = 0$. Now let G be the infinite cyclic group generated by y and let R be the group ring S[G] with $R_e = S$. Then $P = \wp[G]$ and $P_1 = f(y)R$ are both prime ideals of R satisfying; $P \supsetneq P_1$, $P \cap R_e = \wp$ and $P_1 \cap R_e = 0$. Hence P lies over $\wp$ but P_1 does not lie over 0, because 0 is a prime ideal of R.

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Naruto University of Education
Takashima, Naruto, Tokushima, 772, JAPAN
and
Tokushima University
Tokushima, 770, JAPAN

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Modular representation of finite solvable groups (M_p-groups and related topics)

Akihide Hanaki
and
Akihiko Hida

Let (K,R,F) be a splitting p-modular system for all groups considered here, and let (π) be the maximal ideal of R. It is well known that a finite group G is called an M-group if any irreducible ordinary character of G is induced from a linear character of some subgroup of G. In [2], Okuyama generalized it in characteristic p. Namely, a finite group G is called an M_p-group if any irreducible FG-module is induced from a one-dimensional module of some subgroup of G.

It is well known that M-groups are solvable, and also M_p-groups are solvable [2, Corollary 3.8]. So M-groups are M_p-groups for any prime p, by Fong-Swan's theorem. In this paper, we assume that groups are finite solvable. In general, a subgroup of M_p-groups need not be an M_p-group. This fact makes the study of M_p-groups difficult. We call G an $\overline{M}_p$-group if all subsections of G are M_p-groups. $\overline{M}$-groups are defined similarly in characteristic 0.

In §1, we consider the property of $\overline{M}_p$-groups, and in §2, we consider what kind of groups cannot be a normal subgroup of any M_p-group.

§1. $\overline{M}_p$-groups and minimal non M_p-groups

In Price [3], $\overline{M}$-groups are determined. Namely,

Theorem 1 (Price). *Let G be a finite solvable group. Then G is an $\overline{M}$-group if and only if no subsection of G has a ramified chief section.*

A chief section A/B of G is called ramified if there exists a G-invariant irreducible ordinary character χ of A such that $\chi_B = e\theta$ for some irreducible character θ of B and $e^2 = | A : B |$. In characteristic p, the next theorem holds.

The detailed version of this paper will be submitted for publication elsewhere.

Theorem 2. *Let G be a finite solvable group. Then G is an M_p-group if no subsection of G has a p'-ramified chief section.*

In Theorem 2, a p'-ramified chief section means a ramified chief section of p'-order. Now we need the next lemma, since a ramified chief section is defined in characteristic 0 only.

Lemma 3. *Let A/B be a chief section of G . Assume that there exists a G-invariant irreducible FA-module V such that $V_B \cong eW$ for some irreducible trivial source FB-module W and $e > 1$. Then A/B is a p'-ramified chief section of G.*

Proof. Let A/B be not a ramified chief section. Since W is a trivial source module, there exists a trivial source RB-module W_0 such that $W_0/\pi W_0 \cong W$, and it is uniquely determined. Now $W_0 \otimes_R K$ is a G-invariant irreducible KB-module. Since A/B is not ramified, there exists an irreducible KA-module U which is an extension of $W_0 \otimes_R K$. Let U_0 be an R-form of U. Then $(U_0/\pi U_0)_B \cong W$, and $V \cong (U_0/\pi U_0) \otimes_F X$, for some irreducible $F[A/B]$-module X. Since A/B is abelian, $\dim_F X = 1$, and so $V_B \cong W$. This contradicts to $e > 1$. If $| A/B |$ is p-power then clearly $e = 1$. The proof is completed.

Now we can prove Theorem 2, using the similar argument as the proof of Theorem 1.

Proof of Theorem 2. By induction on $|G|$, we can assume that all proper subsection of G are M_p-groups. Hence we may assume that there exists an irreducible FG-module V which is faithful and primitive. By primitivity of V, V_N is homogeneous for any normal subgroup N of G. If $V_N \cong eW$ and $e > 1$ for some normal subgroup N of G, then some chief section satisfies the hypothesis of Lemma 3. So it cannot occur. Especially $V_{\{1\}} \cong F_{\{1\}}$ and so $\dim_F V = 1$. Now V is monomial. The proof is completed.

We do not know whether the converse of Theorem 2 is true or not. Is there an $\overline{M}_p$-group which has a p'-ramified chief section ?

Next we consider the structure of minimal non M_p-groups. A finite solvable group G is called a minimal non M_p-group if G is not an M_p-group and all proper subsections are M_p-groups. Minimal non M-groups are defined similarly in characteristic 0. Clearly, G is an $\overline{M}_p$-group if and only if no subsection of G is isomorphic to any minimal non M_p-group. So study of minimal non M_p-groups is strongly related to that of $\overline{M}_p$-groups. The structure of minimal non M-groups is completely determined by Waall [4]. We could not determine the complete structure of minimal non M_p-groups, but we have the following.

Proposition 4. *Let G be a minimal non M-group. If $O_p(G) = 1$, then G is a minimal non M_p-group.*

This is easy since we know the structure of minimal non M-groups. We do not know the other example of minimal non M_p-groups. In general, the next proposition holds.

Proposition 5. *Let G be a minimal non M_p-group. Then $G = EH$, $E \triangleleft G$ and,*

(a) *E is an extraspecial q-group, for $q \neq p$, and exponent q if q odd, not dihedral if q=2.*

(b) *H acts trivially on $Z(E)$ and irreduciblly on $E/Z(E)$.*

(c) *Either,*

  (1) *H is a p'-group, or*

  (2) *$q = 2$, and $H/O_{2'}(H)$ is a cyclic 2-group.*

(d) $O_{q'}(G) = 1$.

(e) *If H is of odd order then H is of prime order.*

This theorem is proved by the similar argument as Price [3, Theorem1.4] and Theorem 2.

§2. Groups which cannot be normal subgroups of any M_p-groups

In this section, we shall consider normal subgroups of M-groups or M_p-groups. Normal subgroups of M-groups, M_p-groups, need not be M-groups, M_p-groups, respectively. The example of Dade [1] is well known. In general, it is very difficult to consider that what kind of groups can be normal subgroups of some M_p-groups. So we shall consider what kind of groups cannot be normal subgroups of any M_p-groups.

First, we note the following.

Remark. (a) *Monomial modules are trivial source modules.*

(b) *Let N be a normal subgroup of G. If all irreducible FG-modules are trivial source modules, then all irreducible FN-modules have trivial sources.*

(c) *If there exists an irreducible FG-module which is not a trivial source module then G cannot be a normal subgroup of any M_p-groups.*

Now we consider groups which have non trivial source irreducible modules. The next lemma holds.

Lemma 6. *Let G be a semi-direct product of Q by P, where P is a p-group and Q is an extraspecial q-group of order q^{2n+1}, p and q are distinct prime numbers. Assume that $N_G(P) = Z(Q) \times P$ and $Z(Q) = Z(G)$. If $q^n \not\equiv 1 \pmod{p}$ then there exists an irreducible FG-module whose source is not a trivial module, and so G cannot be a normal subgroup of any M_p-groups.*

The hypothesis of this lemma is very strong. But if G is a minimal non M-group and $O^2(G) = G$, then G satisfies the hypothesis for a prime p such that $O^p(G) \neq G$.

Proposition 7. *Let G be a minimal non M-group. If $O^2(G) = G$ and $O^p(G) \neq G$ then G cannot be a normal subgroup of any M_p-group.*

We shall consider more general situation.

If G is a minimal non M-group such that $O^2(G) = G$ then G is a semi-direct product of Q by C_p, where Q is an extraspecial q-group and C_p is a cyclic group of order p, $p \neq q$. Put

$$H_n = (Q_1 \times \cdots \times Q_n) \rtimes C_p$$

where $Q_i \cong Q$, $i = 1, \ldots, n$ and C_p acts on each Q_i the same as on Q. Let z_i be a generater of $Z(Q_i)$. Assume that all z_i are corresponding to each other by C_p-isomorphisms. Let K be a subgroup of H_n generated by $z_1 z_2^{-1}$, $z_1 z_j$, $j = 3, \ldots, n$. Put $G_n = H_n / K$. Then we have

Proposition 8. *G_n is an M-group if n is even and G_n cannot be a normal subgroup of any M_p-group if n is odd.*

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Department of Mathematics, Mathematics and Physical Science,
Graduate School of Science and Technology,
Chiba University, Yayoi cho, Chiba city, 260, Japan

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G-extensions and Galois theory of semi-connected rings

Takasi NAGAHARA and Kazuo KISHIMOTO

This is an expository talk on our paper [1], and we will discuss on the number of idempotents of a commutative G-extension ring and Galois theory of a G-Galois extension ring over a semi-connected ring.

Throughout this talk, we assume that A is a ring with an identity 1 and B is an extension ring of A with common identity 1. For a finite group G of automorphisms of B, B is said to be a *G-extension* of A (or B/A is a *G-extension*) if a fixed ring $B^G = \{b \in B \mid \sigma(b) = b \text{ for all } \sigma \in G\} = A$. For a ring $R \ni 1$, $I(R)$ (resp. $PI(R)$) denotes the set of all central idempotents (resp. central primitive idempotents) of R. Further R is said to be *semi-connected* (resp. *connected*) if $|I(R)| < \infty$ (resp. $|I(R)| = 2$), where $| * |$ denotes the cardinality of $*$, a set.

1. Commutative case. In this section, we assume that B/A is a commutative G-extension. Further, we put

$I(B : G) = \{e \in I(B) \mid \text{there holds either } \sigma(e) = e \text{ or } e\sigma(e) = 0 \text{ for each } \sigma \in G\}$.

Since $1 \in I(B : G)$, $I(B : G)$ is non-empty. By $m(I(B : G))$, we denote $\max_{e\in I(B:G)} |G\{e\}|$, where $G\{e\}=\{\sigma(e) \mid \sigma \in G\}$.

Remark: Let $B = A^{(3)} = A \oplus A \oplus A$, and let σ be an automorphism of B such that $\sigma(a_1, a_2, a_3) = (a_3, a_1, a_2)$. Then $B^\sigma = \{(a, a, a) \mid a \in A\} \cong A$, and so, we may understand that B/A is a G-extension. For $e_1 = (1, 0, 0)$ and $e_2 = (0, 1, 0)$, $e = e_1 + e_2$ is a central idempotent and e is neither $\sigma(e) = e$ nor $e\sigma(e) = 0$. Thus $e \notin I(B : \sigma)$.

If $e \in PI(B)$, we can see that $e \in I(B : G)$. More precisely, we have the following

Lemma 1. Assume A is connected. Then $e \in PI(B)$ if and only if $e \in I(B:G)$ and $|G\{e\}| = m(I(B:G))$.

In virtue of Lemma 1, we can obtain the following

Theorem 2. (i) $|PI(A)| \leq |PI(B)| \leq |PI(A)||G|$.

(ii) $|I(A)| \leq |I(B)| \leq 2^{|PI(A)||G|}$ if either $|I(A)| < \infty$ or $|I(B)| < \infty$.

(iii) $PI(B) \neq \emptyset$ if and only if $PI(A) \neq \emptyset$.

(iv) B is semi-connected if and only if A is semi-connected.

2. Galois theory. In this section, we assume that B is semi-connected and B/A will mean a G-extension which is not necessarily commutative.

Let $PI(B) = \{e_1, e_2, \dots, e_n\}$, $S_i = Be_i$ and $H_i = G(\{e_i\})|S_i$ where $G(\{e_i\}) = \{\sigma \in G \mid \sigma(e_i) = e_i\}$. Then $B = \sum_{i=1}^{n} \oplus S_i$ is clear.

By G^*, we denote the set of all automorphisms σ of B such that $\sigma|S_i = g_i|S_i$ for some $g_i \in G$, $i = 1, 2, \dots, n$. Then G^* becomes a group, and G is said to be *a fat group* if

$G = G^*$.

Moreover, if

$H_i(S_i^N) = N$ for every subgroup N of H_i and $i = 1, 2, \dots, n$,

then B/A is called *a strong G-extension.*

Finally, an intermediate ring T of B/A is said to be G^**-subfixed* if every

$e \in PI(B)$, $Be(G^*(Te)) = Te$ and $\sum_{e' \in G^*(T)\{e\}} e' \in T$.

The following theorem is a fundamental theorem of Galois theory of a strong G-extension.

Theorem 3. Let B/A be a strong G-extension. Then, there exists a 1-1 dual correspondence between the set of intermediate G^*-subfixed subring T of B/A and the set of fat subgroups K of G^* in the usual sense of Galois theory : $T \longleftrightarrow K$ with $G^*(T) = K$ and $B(K) = T$.

For a commutative case, O. E. Villamayor and D. Zelinsky proved the following theorem [2].

Theorem. Let S be a commutative ring with an identity element 1 which is a G-extension of a semi-connected ring R such that S is projective and separable over R. Let

H be the group of all R-algebra homomorphisms of S. Then, S is semi-connected, $G^* = H$ and there exists a 1-1 dual correspondence between the set of separable R-subalgebras of S and the set of fat subgroups of H in the usual sense of Galois theory.

In the theorem, we can see that

(1) S/R is a strong G-extension, $G^* = H$, and

(2) for an intermediate ring T of S/R, T is separable over R if and only if T is H-subfixed.

Thus Theorem 3 is a slight generalization of the above Theorem.

Corollary 4. Let B_i $(i = 1, 2, \ldots, t)$ be semi-connected rings, and each B_i a G_i-Galois extension of a subring A_i in the sense of [2]. Let $B = \sum_{i=1}^{t} \oplus B_i$, $A = \sum_{i=1}^{t} \oplus A_i$ and $G = G_1 \times G_2 \times \cdots \times G_t$ which is an automorphism group of B by the composition

$$(\sigma_1, \sigma_2, \ldots, \sigma_t)(b_1 + b_2 + \cdots + b_t) = \sum_{i=1}^{t} \sigma_i(b_i)$$

where $\sigma_i \in G_i$ and $b_i \in B_i$ $(i = 1, 2, \ldots, t)$. Then B/A is a strong G-extension to which Theorem 3 applies.

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Okayama University
Okayama 700, Japan
and
Shinshu University
Matsumoto 390, Japan

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ON THE FINITISTIC DIMENSION CONJECTURE

Mitsuo HOSHINO

Throughout this note, all rings are associative rings with identity, and all modules are unitary modules. Given a ring A, we denote by Mod A the category of all left A-modules and by mod A the category of all coherent left A-modules. We consider right A-modules as left A^{op}-modules, where A^{op} denotes the opposite ring of A.

There is a long standing open question: Let A be a finite dimensional algebra over a field. Does there always exist an integer $n \geqq 0$ such that $\text{proj dim}_A M \leqq n$ for all $M \in \text{mod} A$ with $\text{proj dim}_A M < \infty$? (See e.g. [4], [5], [7] and so on for references and recent progress on the question).

Let us consider the same problem as above for various abelian categories $\mathcal{A}$ with enough projectives. Then we have negative answers in the following two cases:

(1) $\mathcal{A} = \text{mod} A$ with A noetherian. Let A be commutative, noetherian and Cohen-Macaulay. Then, for each maximal ideal $\underline{m}$, $\text{Ext}_A^i(A/\underline{m}, A) = 0$ for $0 \leqq i < \text{ht}(\underline{m})$, which implies the existence of an $M \in \text{mod} A$ with $\text{proj dim}_A M = \text{ht}(\underline{m})$. On the other hand, as given by Nagata [8, Appendix, Example 1], there exists a regular ring of infinite dimension.

The final version of this note will be submitted for publication elsewhere.

(2) $\mathcal{A}$ = Mod A with A semiprimary. (See e.g. an example given by Kirkman and Kuzmanovich [7]).

In case $\mathcal{A}$ = mod A or Mod A with A artinian, the question is still open.

Let A be a left and right artinian ring. We will show that, for a suitable idempotent $e \in A$, the colocalization

$$\mathrm{Hom}_A(Ae,-) : \mathrm{mod}\, A \longrightarrow \mathrm{mod}\, eAe$$

reduces the question for mod A to that for mod eAe.

In the first three sections, we will make some general remarks on the question. In the final section, we will summarize the author's work [6].

1. Localization and injective dimensions. We refer to Gabriel [3], Popescu [9] and Swan [12] for localization in abelian categories.

Let $\mathcal{A}$, $\mathcal{B}$ be abelian categories. Let

$$Q : \mathcal{A} \to \mathcal{B} \quad \text{and} \quad F : \mathcal{B} \to \mathcal{A}$$

be additive covariant functors and let

$$\varepsilon : \mathbb{1}_{\mathcal{A}} \to FQ \quad \text{and} \quad \eta : QF \to \mathbb{1}_{\mathcal{B}}$$

be natural transformations. Suppose the following conditions:

(a) $\eta_Q \circ Q\varepsilon = \mathrm{id}_Q$ and $F\eta \circ \varepsilon_F = \mathrm{id}_F$.
(b) Q is exact.
(c) η is an isomorphism.

Then $FQ : \mathcal{A} \to \mathcal{A}$ is called a localization functor. Let us recall several basic facts: By the condition (a), for $M \in \mathcal{A}$ and

$X \in \mathcal{B}$, there exists a natural bijection

$$\alpha_{M,X} : \mathrm{Hom}_{\mathcal{A}}(M,FX) \longrightarrow \mathrm{Hom}_{\mathcal{B}}(QM,X)$$

such that $\alpha_{M,X}(f) = \eta_X \cdot Qf$ for $f \in \mathrm{Hom}_{\mathcal{A}}(M,FX)$. Namely F is right adjoint to Q, which implies F left exact. Then by the condition (c) for X, $Y \in \mathcal{B}$ we get a natural bijection

$$\beta_{X,Y} = \mathrm{Hom}_{\mathcal{B}}(\eta_X{}^{-1},Y) \circ \alpha_{FX,Y} : \mathrm{Hom}_{\mathcal{A}}(FX,FY) \longrightarrow \mathrm{Hom}_{\mathcal{B}}(X,Y) .$$

Thus F is fully faithful. Let Ker Q denote the full subcategory of $\mathcal{A}$ consisting of all $N \in \mathcal{A}$ with $QN = 0$. Then by the condition (b) Ker Q makes a Serre class in $\mathcal{A}$ and is called the localizing subcategory. The quotient category $\mathcal{A}/\mathrm{Ker}\,Q$ exists and is equivalent to $\mathcal{B}$. Since by the conditions (a) and (c) $Q\varepsilon$ is an isomorphism, we may consider Ker ε, Cok ε as functors from $\mathcal{A}$ to Ker Q. Then Ker ε is right adjoint to the inclusion $\mathrm{Ker}\,Q \to \mathcal{A}$ and thus left exact. Finally, for an $M \in \mathcal{A}$, ε_M is monic if and only if $\mathrm{Hom}_{\mathcal{A}}(-,M)$ vanishes on Ker Q, and ε_M is an isomorphism if and only if for i = 0, 1 $\mathrm{Ext}^i_{\mathcal{A}}(-,M)$ vanishes on Ker Q.

Lemma 1.1. Let $X \in \mathcal{B}$. Let $f : FX \to M$ be monic in $\mathcal{A}$. Then f splits if and only if $Qf \circ \eta_X{}^{-1} : X \to QM$ does. In particular, X is injective if and only if FX is.

Proof. We have the following commutative square:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{A}}(M,FX) & \xrightarrow{\alpha_{M,X}} & \mathrm{Hom}_{\mathcal{B}}(QM,X) \\ \Big\downarrow {\scriptstyle \mathrm{Hom}_{\mathcal{A}}(f,FX)} & & \Big\downarrow {\scriptstyle \mathrm{Hom}_{\mathcal{B}}(Qf \cdot \eta_X{}^{-1},X)} \\ \mathrm{End}_{\mathcal{A}}(FX) & \xrightarrow{\beta_{X,X}} & \mathrm{End}_{\mathcal{B}}(X) \end{array} .$$

Since $\beta_{X,X}(\mathrm{id}_{FX}) = \mathrm{id}_X$, the assertion follows.

Lemma 1.2. Let $f : X \to Y$ be monic in $\mathcal{B}$. Suppose that Cok Ff is injective. Then Cok f is injective and Cok Ff $\xrightarrow{\sim}$ FCok f.

Proof. Since F is left exact, we have a monic $g : \mathrm{Cok}\,Ff \to F\mathrm{Cok}\,f$. Also, since Q is exact, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & QFX & \longrightarrow & QFY & \longrightarrow & Q\mathrm{Cok}\,Ff & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \eta_X} & & \downarrow{\scriptstyle \eta_Y} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & \mathrm{Cok}\,f & \longrightarrow & 0 . \end{array}$$

Put $E = \mathrm{Cok}\,Ff$. Then $g = Fh \circ \varepsilon_E$. Thus, since h is an isomorphism, ε_E is monic and splits. Hence $\mathrm{Hom}_{\mathcal{A}}(\mathrm{Cok}\,\varepsilon_E, FQE) = 0$ implies $\mathrm{Cok}\,\varepsilon_E = 0$. Therefore ε_E and thus g are isomorphisms. Now, since FQE is injective, so is $QE \simeq \mathrm{Cok}\,f$ by Lemma 1.1.

In the following, we assume that both $\mathcal{A}$ and $\mathcal{B}$ have enough injectives. For $i \geqq 0$, we denote by F^i the i^{th} right derived functor of F. Also, for $M \in \mathcal{A}$, we define

$$d(M) = \sup\{\mathrm{inj\,dim}\,L \mid L = \mathrm{Ker}\,\varepsilon_M,\ \mathrm{Cok}\,\varepsilon_M \text{ or } F^iQM\ (i \geqq 1)\}.$$

<u>Lemma 1.3</u>. Let $E \in \mathcal{A}$ be injective. Suppose that $\mathrm{inj\,dim}\,\mathrm{Ker}\,\varepsilon_E \leqq 1$. Then QE is injective.

Proof. Note that $\mathrm{Im}\,\varepsilon_E$ is injective. Thus the canonical exact sequence $0 \to \mathrm{Im}\,\varepsilon_E \to FQE \to \mathrm{Cok}\,\varepsilon_E \to 0$ splits and, as in the proof of Lemma 1.2, $\mathrm{Cok}\,\varepsilon_E = 0$. Hence FQE is injective and by Lemma 1.1 so is QE.

<u>Proposition 1.4</u>. Let $X \in \mathcal{B}$. Suppose that $\mathrm{inj\,dim}\,FX \leqq 1$. Then $\mathrm{inj\,dim}\,X = \mathrm{inj\,dim}\,FX$.

Proof. By Lemma 1.1 we may assume that $\mathrm{inj\,dim}\,FX = 1$ and that $\mathrm{inj\,dim}\,X \geqq 1$. Let $f : X \to I$ be monic in $\mathcal{B}$ with I injective. Since by Lemma 1.1 FI is injective, so is $\mathrm{Cok}\,Ff$. Thus by Lemma 1.2 $\mathrm{Cok}\,f$ is injective. Namely $\mathrm{inj\,dim}\,X \leqq 1$.

Lemma 1.5. Let $d \geqq 0$ and $X \in \mathcal{B}$. Suppose that $\operatorname{inj\,dim} X < \infty$ and that $\operatorname{inj\,dim} F^i X \leqq d$ for all $i \geqq 1$. Then $\operatorname{inj\,dim} FX \leqq \operatorname{inj\,dim} X + d + 1$.

Proof. By Lemma 1.1 we may assume that $n = \operatorname{inj\,dim} X \geqq 1$. Let

$$0 \to X \to I^0 \to \ldots \to I^n \to 0$$

be an injective resolution and put

$$B^i = \operatorname{Im}(FI^i \to FI^{i+1}) \quad \text{and} \quad Z^{i+1} = \operatorname{Cok}(FI^i \to FI^{i+1})$$

for $i \geqq 0$. Then for each $i \geqq 1$ we have exact sequences

$$(a_i) : 0 \longrightarrow F^i X \longrightarrow Z^i \longrightarrow B^i \longrightarrow 0 \,,$$

$$(b_i) : 0 \longrightarrow B^{i-1} \longrightarrow FI^i \longrightarrow Z^i \longrightarrow 0 \,.$$

We will show by induction that $\operatorname{inj\,dim} B^{n-i} \leqq i + d$ for $1 \leqq i \leqq n$. Note that by Lemma 1.1 the FI^j are injective. Since $Z^n \simeq F^n X$, the exact sequence (b_n) yields $\operatorname{inj\,dim} B^{n-1} \leqq 1 + d$. Let $i \geqq 2$ and assume that $\operatorname{inj\,dim} B^{n-i+1} \leqq i - 1 + d$. Then the exact sequence (a_{n-i+1}) yields $\operatorname{inj\,dim} Z^{n-i+1} \leqq i - 1 + d$ and then the exact sequence (b_{n-i+1}) yields $\operatorname{inj\,dim} B^{n-i} \leqq i + d$. Now, since $\operatorname{inj\,dim} B^0 \leqq n + d$, the exact sequence $0 \to FX \to FI^0 \to B^0 \to 0$ yields $\operatorname{inj\,dim} FX \leqq n + d + 1$, as required.

Lemma 1.6. Let $X \in \mathcal{B}$. Suppose that $\operatorname{inj\,dim} F^i X < \infty$ for all $i \geqq 0$ and that $F^i X = 0$ for $i \gg 0$. Then $\operatorname{inj\,dim} X < \infty$.

Proof. Let $n \geqq 0$ and assume that $F^i X = 0$ for $i > n$. Let

$$0 \to X \to I^0 \to I^1 \to \ldots$$

be an injective resolution and put $X^i = \operatorname{Ker}(I^i \to I^{i+1})$ for $i \geqq 0$.

Then for each $i \geq 1$ we have an exact sequence

$$0 \to FX^{i-1} \to FI^{i-1} \to FX^{i} \to F^{i}X \to 0 .$$

Note that, since $FX^0 \simeq F^0X$, $\operatorname{inj\,dim} FX^0 < \infty$ and that by Lemma 1.1 the FI^j are injective. Thus, by induction, we conclude that $\operatorname{inj\,dim} FX^i < \infty$ for all $i \geq 0$. Now, since we have an injective resolution

$$0 \to FX^n \to FI^n \to FI^{n+1} \to \dots$$

with $FX^i \simeq \operatorname{Ker}(FI^i \to FI^{i+1})$ for $i \geq n$, by Lemma 1.1 $\operatorname{inj\,dim} FX^n < \infty$ implies $\operatorname{inj\,dim} X^n < \infty$. Therefore $\operatorname{inj\,dim} X < \infty$.

Proposition 1.7. Let $M \in \mathcal{A}$ with $d(M) < \infty$. Then the following statements hold:

(1) $\operatorname{inj\,dim} M \leq \operatorname{inj\,dim} QM + d(M) + 1$.

(2) If $\operatorname{inj\,dim} M < \infty$ and if $F^iQM = 0$ for $i >> 0$, then $\operatorname{inj\,dim} QM < \infty$.

Proof. (1) We may assume that $\operatorname{inj\,dim} QM < \infty$. Then by Lemma 1.5 $\operatorname{inj\,dim} FQM \leq \operatorname{inj\,dim} QM + d(M) + 1$. Thus the canonical exact sequence

$$0 \to \operatorname{Ker} \varepsilon_M \to M \to FQM \to \operatorname{Cok} \varepsilon_M \to 0$$

yields the desired inequality.

(2) Note that, by the above exact sequence, $\operatorname{inj\,dim} M < \infty$ implies $\operatorname{inj\,dim} FQM < \infty$. Thus by Lemma 1.6 the assertion follows.

Theorem 1.8. Let $d, n \geq 0$. Suppose the following conditions:

(1) $\operatorname{inj\,dim} N \leq d$ for all $N \in \mathcal{A}$ with $QN = 0$.

(2) For each injective $E \in \mathcal{A}$, $F^iQE = 0$ for $i >> 0$.

(3) $\operatorname{inj\,dim} X \leq n$ for all $X \in \mathcal{B}$ with $\operatorname{inj\,dim} X < \infty$.

Then $\operatorname{inj\,dim} M \leq n + d + 1$ for all $M \in \mathcal{A}$ with $\operatorname{inj\,dim} M < \infty$.

Proof. Note that by the condition (1) $d(M) \leqq d$ for all $M \in \mathcal{A}$. Thus by the condition (2) and Proposition 1.7(2) $\operatorname{inj\,dim} QE < \infty$ for all injective $E \in \mathcal{A}$. Hence, since Q is exact, $\operatorname{inj\,dim} QM < \infty$ and thus by the condition (3) $\operatorname{inj\,dim} QM \leqq n$ for all $M \in \mathcal{A}$ with $\operatorname{inj\,dim} M < \infty$. Now by Proposition 1.7(1) the assertion follows.

2. Colocalization and projective dimensions. For later use, we will dualize the statements of the preceding section.

Let $\mathcal{A}$, $\mathcal{B}$ be abelian categories. Let

$$Q : \mathcal{A} \to \mathcal{B} \quad \text{and} \quad G : \mathcal{B} \to \mathcal{A}$$

be additive covariant functors and let

$$\delta : GQ \to 1_{\mathcal{A}} \quad \text{and} \quad \theta : 1_{\mathcal{B}} \to QG$$

be natural transformations. Suppose the following conditions:

(a)* $Q\delta \cdot \theta_Q = \mathrm{id}_Q$ and $\delta_G \cdot G\theta = \mathrm{id}_G$.
(b)* Q is exact.
(c)* θ is an isomorphism.

Then $GQ : \mathcal{A} \to \mathcal{A}$ is called a colocalization functor.

Remark. Consider the case $\mathcal{A} = \operatorname{Mod} A$ in the above. Then it is well known and easily checked that there exists an idempotent ideal I of A such that $\operatorname{Mod} A/I \xrightarrow{\sim} \operatorname{Ker} Q$ canonically. Moreover, the colocalization functor GQ is isomorphic to $I \otimes_A I \otimes_A -$ and the counit δ is induced by the canonical map $I \otimes_A I \to A$ (see e.g. Sato [10]). Suppose further that A is semiprimary. Then $I = AeA$ for some idempotent $e \in A$. Thus $\mathcal{B}$ is equivalent to $\operatorname{Mod} eAe$ and Q is of the form:

$$eA \otimes_A - \simeq \operatorname{Hom}_A(Ae, -) : \operatorname{Mod} A \longrightarrow \operatorname{Mod} eAe \ ,$$

because both $I \otimes_A I \otimes_A -$ and $Ae \otimes_{eAe} eA \otimes_A -$ are colocalization functors with the same colocalizing subcategory Mod A/I .

In the following, we assume that both $\mathcal{A}$ and $\mathcal{B}$ have enough projectives. For $i \geqq 0$, we denote by G_i the i^{th} left derived functor of G. Also, for $M \in \mathcal{A}$, we define

$$d^*(M) = \sup\{\text{proj dim } L \mid L = \text{Cok } \delta_M, \text{ Ker } \delta_M \text{ or } G_iQM \ (i \geqq 1)\}.$$

<u>Lemma 2.1</u>. Let $X \in \mathcal{B}$. Then X is projective if and only if GX is.

<u>Lemma 2.2</u>. Let $f : Y \to X$ be epic in $\mathcal{B}$. Suppose that Ker Gf is projective. Then so is Ker f .

<u>Lemma 2.3</u>. Let $P \in \mathcal{A}$ be projective. Suppose that proj dim Cok $\delta_P \leqq 1$. Then QP is projective.

<u>Proposition 2.4</u>. Let $X \in \mathcal{B}$. Suppose that proj dim GX $\leqq 1$. Then proj dim X = proj dim GX.

<u>Lemma 2.5</u>. Let $d \geqq 0$ and $X \in \mathcal{B}$. Suppose that proj dim $X < \infty$ and that proj dim $G_iX \leqq d$ for all $i \geqq 1$. Then proj dim GX $\leqq$ proj dim X + d + 1.

<u>Lemma 2.6</u>. Let $X \in \mathcal{B}$. Suppose that proj dim $G_iX < \infty$ for all $i \geqq 0$ and that $G_iX = 0$ for $i >> 0$. Then proj dim $X < \infty$.

<u>Proposition 2.7</u>. Let $M \in \mathcal{A}$ with $d^*(M) < \infty$. Then the following statements hold:

(1) proj dim M $\leqq$ proj dim QM + $d^*(M)$ + 1.

(2) If proj dim $M < \infty$ and if $G_iQM = 0$ for $i >> 0$, then proj dim QM $< \infty$.

<u>Theorem 2.8</u>. Let $d, n \geqq 0$. Suppose the following conditions:

(1) proj dim $N \leqq d$ for all $N \in \mathcal{A}$ with $QN = 0$.

(2) For each projective $P \in \mathcal{A}$, $G_i QP = 0$ for $i >> 0$.

(3) proj dim $X \leqq n$ for all $X \in \mathcal{B}$ with proj dim $X < \infty$.

Then proj dim $M \leqq n + d + 1$ for all $M \in \mathcal{A}$ with proj dim $M < \infty$.

3. Idempotent ideals. We will make some other remarks on colocalization in abelian categories. However, in this section, we deal only with the case where $\mathcal{A} = \operatorname{Mod} A$.

Let A be a ring and $I = AeA$ with e an idempotent. We assume that $e \neq 0$ or 1. Also, we consider Mod A/I as a full subcategory of Mod A consisting of all N with $IN = 0$.

Let us observe the following exact sequence:

$$(e) : 0 \longrightarrow K \longrightarrow Ae \otimes_{eAe} eA \xrightarrow{\delta} A \longrightarrow A/I \longrightarrow 0$$

where δ denotes a bilinear map induced by multiplication. Note that $IK = KI = 0$. Also, since both $Ae \otimes_{eAe} eA \otimes_A -$ and $I \otimes_A I \otimes_A -$ are colocalization functors of Mod A with the same colocalizing subcategory Mod A/I, we have the following

Proposition 3.1. $Ae \otimes_{eAe} eA \xrightarrow{\sim} I \otimes_A I$ naturally.

Lemma 3.2. For $i = 0, 1$ $\operatorname{Ext}^i_A(Ae \otimes_{eAe} eA, -)$ vanishes on Mod A/I and $\operatorname{Tor}^A_i(-, Ae \otimes_{eAe} eA)$ vanishes on $\operatorname{Mod}(A/I)^{op}$.

Proof. Since $Ae \otimes_{eAe} - : \operatorname{Mod} eAe \to \operatorname{Mod} A$ is right exact, we have a projective presentation $P_1 \to P_0 \to Ae \otimes_{eAe} eA \to 0$ in Mod A with the P_i direct sums of copies of Ae. Since $\operatorname{Hom}_A(Ae, -)$ vanishes on Mod A/I, the first assertion follows. Also, since $- \otimes_A Ae$ vanishes on $\operatorname{Mod}(A/I)^{op}$, the last assertion follows.

Lemma 3.3. $K \simeq \operatorname{Tor}^A_2(A/I, A/I)$ as a left A-module.

Proof. Since $IK = 0$, applying $A/I \otimes_A -$ to the exact sequence (e), we conclude by Lemma 3.2 that

$$\begin{aligned} K &\simeq A/I \otimes_A K \\ &\simeq \operatorname{Tor}_1^A(A/I, I) \\ &\simeq \operatorname{Tor}_2^A(A/I, A/I) . \end{aligned}$$

Proposition 3.4. Suppose that proj dim $_A A/I \leqq 2$. Then $\operatorname{Tor}_2^A(A/I, A/I)$ is projective in Mod A/I.

Proof. Let $N \in$ Mod A/I and apply $\operatorname{Hom}_A(-, N)$ to the exact sequence (e). Then by Lemma 3.2

$$\begin{aligned} \operatorname{Hom}_{A/I}(K, N) &= \operatorname{Hom}_A(K, N) \\ &\simeq \operatorname{Ext}_A^1(I, N) \\ &\simeq \operatorname{Ext}_A^2(A/I, N) . \end{aligned}$$

Thus proj dim $_A A/I \leqq 2$ implies $\operatorname{Hom}_{A/I}(K, -)$ exact. Now by Lemma 3.3 the assertion follows.

Proposition 3.5. Suppose that proj dim $_A A/I \leqq 2$ and that $\operatorname{Tor}_2^A(A/I, A/I) = 0$. Then proj dim $_{eAe} eA \leqq 1$.

Proof. By Lemma 3.3 proj dim $_A Ae \otimes_{eAe} eA \leqq 1$ and thus by Proposition 2.4 proj dim $_{eAe} eA \leqq 1$.

In case A is perfect, the above proposition is generalized as follows:

Proposition 3.6. Let A be perfect and J denote the Jacobson radical of A. Suppose that proj dim $_A A/I < \infty$, that $\operatorname{Tor}_2^A(A/I, A/I) = 0$ and that $\operatorname{Tor}_i^A((1-e)A/(1-e)J, A/I) = 0$ for $i \geqq 3$. Then proj dim $_{eAe} eA < \infty$.

Proof. Let

$$\cdots \to P_1 \to P_0 \to {}_A Ae \otimes_{eAe} eA \to 0$$

be a minimal projective resolution. It suffices to show that ${}_{eAe}eA \otimes_A P_i$ is projective for all $i \geqq 0$. In the proof of Lemma 3.2, we have seen that ${}_{eAe}eA \otimes_A P_i$ is projective for $i = 0, 1$. Let $i \geqq 2$. Note that by Lemma 3.3 ${}_A Ae \otimes_{eAe} eA$ is a first syzygy of ${}_A A/I$. Thus

$$\begin{aligned} &\operatorname{Tor}_i^A((1-e)A/(1-e)J, Ae \otimes_{eAe} eA) \\ \simeq\ &\operatorname{Tor}_{i+1}^A((1-e)A/(1-e)J, A/I) \\ =\ &0 , \end{aligned}$$

which implies $(1-e)A/(1-e)J \otimes_A P_i = 0$. Hence P_i is a direct summand of a direct sum of copies of ${}_A Ae$, which implies ${}_{eAe}eA \otimes_A P_i$ projective.

4. Finitistic homological dimension of modules. We will provide another reduction which is more effective than Theorem 2.8. In this section, we restrict ourselves to the case where $\mathcal{A} = \operatorname{mod} A$ with A artinian.

Let A be a left and right artinian ring. We define

$$\operatorname{fin\,dim} A = \sup\{\operatorname{proj\,dim} {}_A M \mid M \in \operatorname{mod} A \text{ with } \operatorname{proj\,dim} {}_A M < \infty\}.$$

We evaluate fin dim A at each $X \in \operatorname{mod} A^{op}$. For $X \in \operatorname{mod} A^{op}$ and $M \in \operatorname{mod} A$, we define

$$t(X,M) = \inf\{n \geqq -1 \mid \operatorname{Tor}_i^A(X,M) = 0 \text{ for } i > n\}$$

and then for $X \in \operatorname{mod} A^{op}$ we define

$$p(X_A) = \sup\{t(X,M) \mid M \in \operatorname{mod} A \text{ with } \operatorname{proj\,dim} {}_A M < \infty\} .$$

Lemma 4.1. Let A be a left and right artinian ring and $X \in \text{mod}\, A^{op}$. Then the following statements hold:

(1) $p(X_A) \leqq \text{proj dim}\, X_A$.

(2) $p(X_A) \leqq \text{fin dim}\, A$.

(3) If every simple right module appears as a direct summand of X, then $p(X_A) = \text{fin dim}\, A$.

(4) Let $\cdots \to P_1 \to P_0 \to X_A \to 0$ be a minimal projective resolution and put $X_i = \text{Cok}(P_{i+1} \to P_i)$ for $i \geqq 0$. Let $n \geqq 0$. Suppose that every indecomposable direct summand of X_n appears as a direct summand of some X_i $(i > n)$. Then, for an $M \in \text{mod}\, A$, $\text{Tor}_i^A(X,M) = 0$ for $i >> 0$ implies $\text{Tor}_i^A(X,M) = 0$ for $i > n$. In particular $p(X_A) \leqq n$.

Theorem 4.2. Let A, Λ be left and right artinian rings. Let $e \in A$ be an idempotent. Let ${}_\Lambda X_A$ be a Λ-A-bimodule such that

(1) X_A is finitely generated,

(2) ${}_\Lambda X$ is finitely generated projective,

(3) $X(\text{rad}\, A) \subset (\text{rad}\, \Lambda)X$ and

(4) $AeA \subset \text{ann}(X_A) \subset AeA + \text{rad}\, A$.

Then the following inequality holds:

$$\text{fin dim}\, A \leqq \text{fin dim}\, \Lambda + \text{fin dim}\, eAe + p(X_A) + 1 \ .$$

Remark. Consider the case $e = 0$ in the above theorem. Then we have the following inequality:

$$\text{fin dim}\, A \leqq \text{fin dim}\, \Lambda + p(X_A) \ ,$$

which is due essentially to Small [11, Theorem 1].

In the following, A is a left and right artinian ring, $J = \text{rad}\, A$, $e \in A$ an idempotent and $f = 1 - e$.

Corollary 4.3. The following inequality holds:

$$\text{fin dim}\, A \leqq \text{fin dim}\, eAe + p(fA/fJ_A) + 1 \ .$$

Remark. The above corollary together with Lemma 4.1(2) yields that the following statements are equivalent:

(1) fin dim A < ∞ for all artinian rings A.

(2) Every artinian ring A has a simple right module X with $p(X_A) < \infty$.

Corollary 4.4. The following inequality holds:

$$\text{fin dim}\, A \leq \text{fin dim}\, A/AeA + \text{fin dim}\, eAe + p(A/AeA_A) + 1 .$$

Remark. The above corollary generalizes a result of Auslander, Platzeck and Todorov [1, Theorem 5.4]. Also, since fAeAf = 0 implies A/AeA $\cong$ fAf as a ring, we get a generalization of Fossum, Griffith and Reiten [2, Corollary 4.21].

Corollary 4.5. Let A be an artin algebra and $X \in \text{mod}\, A^{op}$ nonzero. Then the following inequality holds:

$$\text{fin dim}\, A \leq \text{fin dim}\, \text{End}(A_A \oplus X_A) + p(X_A) .$$

We end with raising the following

Question. Let A be a left and right artinian ring. For $X \in \text{mod}\, A^{op}$, let us define

$$\hat{p}(X_A) = \sup\{t(X,M) \mid M \in \text{mod}\, A \text{ with } t(X,M) < \infty\} .$$

Does it always hold that $\hat{p}(X_A) < \infty$? Though very strong, this question seems to be quite natural.

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Institute of Mathematics
University of Tsukuba
Ibaraki, 305 Japan

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An Application of Intersection Theory to Commutative Algebra

Kazuhiko KURANO

1 Introduction

In recent years P. Roberts has solved several problems in commutative algebra using intersection theory. Roberts proved the following theorems. (Assume that all rings are commutative with unit.)

Theorem 1.1 *([3], [6]) Let M, N be finitely generated modules over a regular local ring A such that $\ell_A(M \otimes_A N) < \infty$ and $\dim M + \dim N < \dim A$. Then $\sum_i (-1)^i \ell_A(\mathrm{Tor}_i^A(M, N)) = 0$.*

Here $\ell_A(\ )$ means the length of the given A-module and $\dim(\ )$ is the Krull dimension.

Theorem 1.2 *(New intersection theorem [9]) Let A be a Noetherian local ring and*

$$\mathsf{F}. \ : \ 0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to 0$$

a complex of A-modules satisfying the following three conditions:

- *F_i is a finitely generated free A-module for each i.*
- *$\mathsf{F}.$ is not exact.*
- *$\ell_A(\mathrm{H}_i(\mathsf{F}.)) < \infty$ for each i.*

Then $n \geq \dim A$ holds.

The above theorem has many applications. For instance,

Corollary 1.3 *Let A be a Noetherian local ring. Then A is Cohen-Macaulay if and only if A has a finitely generated module of finite injective dimension.*

The detailed version of this paper has been submitted for publication elsewhere.

Corollary 1.4 *Let A be a Noetherian local ring and M, N finitely generated A-modules such that $pd_A M < \infty$ and $\ell_A(M \otimes_A N) < \infty$. Then $pd_A M \geq \dim N$ holds.*

Roberts succeeded in proving above theorems by using intersection theory. Especially local Chern characters and the singular Riemann-Roch theorem ([2]) play important roles in his proofs. The basic tool to define local Chern characters is MacPherson's graph construction (see Chapter 18 of [2]). It is a purely geometric argument. But in the case of characteristic $p > 0$, we can interpret local Chern characters and the singular Riemann-Roch theorem in an algebraic method. In the next section we will study local Chern characters and singular Riemann-Roch theorem in the case of positive characteristic. The final section is devoted to an application of intersection theory to commutative algebra.

2 Intersection theory of characteristic $p > 0$

In this section we will construct the singular Riemann-Roch formula and local Chern characters in the case of characteristic $p > 0$ by using the Frobenius map.

Throughout this section all rings are equi-characteristic complete Noetherian local rings whose residue class field k is algebraically closed of characteristic $p > 0$ unless otherwise specified. $\mathbb{Z}$ (resp. $\mathbb{Q}$) denotes the ring of integers (resp. rational numbers).

Definition 2.1 For a Noetherian local ring $(A, \mathfrak{m})$ we denote by $\mathrm{K}_0 A$ (resp. $\mathrm{K}_0 A_{\mathbb{Q}}$) the Grothendieck group (resp. the rational Grothendieck group) of finitely generated A-modules, i.e.,

$$\mathrm{K}_0 A = \left(\bigoplus_{M\ :\ \text{f. g. } A\text{-module}} \mathbb{Z}\cdot[M] \right) \Big/ \langle [M] - [N] - [L] \mid 0 \to L \to M \to N \to 0\ :\ \text{exact} \rangle$$
$$\mathrm{K}_0 A_{\mathbb{Q}} = \mathrm{K}_0 A \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Let $f^e : A \to A$ be the e-th iteration of the Frobenius map, i.e., $f(x) = x^{p^e}$ for any $x \in A$. Since A is a commutative ring of characteristic p, f is a ring homomorphism. We sometimes denote it by $f^e : A \to {}^eA$ to distiguish A of both sides.

Lemma 2.2 *Let $f^e : A \to {}^eA$ be the Frobenius map.*

1. *f^e is finite, i.e., ${}^eA$ is a finitely generated A-module.*
2. *If A is an integral domain of dimension d, then $\mathrm{rank}_A {}^eA = p^{de}$ holds.*

Proof. Since A is complete, A contains the residue class field k. Hence there exist a formal power series ring $k[[x_1, \ldots, x_n]]$ and its ideal I such that $A \simeq k[[x_1, \ldots, x_n]]/I$. Let

$\{y_1, \ldots, y_d\}$ be a system of parameters of A. Then $i : k[[y_1, \ldots, y_d]] \hookrightarrow A$ is finite, i.e., $k[[y_1, \ldots, y_d]]$ is a Noether normalization of A.

It is easy to check that $f^e : k[[y_1, \ldots, y_d]] \to k[[y_1, \ldots, y_d]]$ is finite of rank p^{de}. As $i \circ f^e = f^e \circ i$, the assertions are obvious. **Q.E.D.**

If $g : A \to R$ is finite, we get the induced homomorphism between rational Grothendieck groups $g^* : \mathrm{K}_0 R_{\mathbb{Q}} \to \mathrm{K}_0 A_{\mathbb{Q}}$.

Definition 2.3 For $j = 0, 1, 2, \ldots$ we define the $\mathbb{Q}$-vector subspaces of $\mathrm{K}_0 A_{\mathbb{Q}}$ by

$$\mathrm{L}_j \mathrm{K}_0 A_{\mathbb{Q}} = \left\{ c \in \mathrm{K}_0 A_{\mathbb{Q}} \mid f^*(c) = p^j c \right\} \subseteq \mathrm{K}_0 A_{\mathbb{Q}}.$$

The decomposition of the rational Grothendieck group as in the next proposition is the algebraic interpretation of the singular Riemann-Roch theorem (Theorem 18.3 in [2]).

Proposition 2.4 $\mathrm{K}_0 A_{\mathbb{Q}} = \bigoplus_{j=0}^{\dim A} \mathrm{L}_j \mathrm{K}_0 A_{\mathbb{Q}}$.

Proof. First note that $\sum_{j=0}^{\dim A} \mathrm{L}_j \mathrm{K}_0 A_{\mathbb{Q}} = \oplus_{j=0}^{\dim A} \mathrm{L}_j \mathrm{K}_0 A_{\mathbb{Q}}$. Hence we have only to prove that $\mathrm{K}_0 A_{\mathbb{Q}} \subseteq \sum_{j=0}^{\dim A} \mathrm{L}_j \mathrm{K}_0 A_{\mathbb{Q}}$.

Any finitely generated A-module M has a filtration

$$M = M_n \supseteq M_{n-1} \supseteq \cdots \supseteq M_0 = (0)$$

such that $M_i/M_{i-1} \simeq A/\mathfrak{q}_i$ and $\mathfrak{q}_i$ is a prime ideal with $\dim A/\mathfrak{q}_i \leq \dim M$ for $i = 1, \ldots, n$. Therefore we have $[M] = \sum_{i=1}^{n} [A/\mathfrak{q}_i]$ in $\mathrm{K}_0 A_{\mathbb{Q}}$ and $\mathrm{K}_0 A_{\mathbb{Q}} = \sum_{\mathfrak{q} \in \mathrm{Spec}(A)} \mathbb{Q} \cdot [A/\mathfrak{q}]$.

We will prove $[A/\mathfrak{q}] \in \sum_{j=0}^{\dim A/\mathfrak{q}} \mathrm{L}_j \mathrm{K}_0 A_{\mathbb{Q}}$ by induction on $\dim A/\mathfrak{q}$ for $\mathfrak{q} \in \mathrm{Spec}(A)$.

If $\dim A/\mathfrak{q} = 0$, $A/\mathfrak{q} = k$. Then $f^*([k]) = [k]$ as k is algebraically closed. Therefore $[k] \in \mathrm{L}_0 \mathrm{K}_0 A_{\mathbb{Q}}$.

Suppose $\dim A = n > 0$. Since $rank_{A/\mathfrak{q}}{}^1(A/\mathfrak{q}) = p^n$ (see Lemma 2.2), we obtain $f^*([A/\mathfrak{q}]) = [^1(A/\mathfrak{q})] p^n [A/\mathfrak{q}] + \sum_{i=0}^{n-1} c_i$ ($c_i \in \mathrm{L}_i \mathrm{K}_0 A_{\mathbb{Q}}$) by induction on n. Then we have

$$\begin{aligned} & f^*([A/\mathfrak{q}] + \sum_{i=0}^{n-1} \frac{1}{p^n - p^i} c_i) \\ = \; & p^n [A/\mathfrak{q}] + \sum_{i=0}^{n-1} c_i + \sum_{i=0}^{n-1} \frac{p^i}{p^n - p^i} c_i \\ = \; & p^n [A/\mathfrak{q}] + \sum_{i=0}^{n-1} \frac{p^n}{p^n - p^i} c_i \\ = \; & p^n ([A/\mathfrak{q}] + \sum_{i=0}^{n-1} \frac{1}{p^n - p^i} c_i). \end{aligned}$$

Therefore $[A/\mathfrak{q}] + \sum_{i=0}^{n-1} \frac{1}{p^n - p^i} c_i \in \mathrm{L}_n \mathrm{K}_0 A_{\mathbb{Q}}$. **Q.E.D.**

Remark 2.5 Let $g : A \to R$ be a finite morphism. Because of $f \circ g = g \circ f$, it holds that $g^* \circ f^* = f^* \circ g^*$. Then it is easy to check that $g^*(c) \in \mathrm{L}_j\mathrm{K}_0A_{\mathbb{Q}}$ for any $c \in \mathrm{K}_0R_{\mathbb{Q}}$. Therefore the decomposition $\mathrm{K}_0A_{\mathbb{Q}} = \oplus_{i=0}^{\dim A}\mathrm{L}_j\mathrm{K}_0A_{\mathbb{Q}}$ is compatible with finite morphisms. (This corresponds to the fact that the Riemann-Roch map is compatible with proper push-forwards.)

Definition 2.6 Let A be a Noetherian local ring of dimension d. For $i = 0, 1, \ldots, d$ we denote by τ_i the projection $\mathrm{K}_0A_{\mathbb{Q}} \to \mathrm{L}_j\mathrm{K}_0A_{\mathbb{Q}}$. (Therefore $[M] = \tau_0([M]) + \tau_1([M]) + \cdots + \tau_d([M])$.)

Remark 2.7

1. By definition of τ_i, we have $(f^*)^e[M] = \tau_0([M]) + p^e\tau_1([M]) + \cdots + p^{de}\tau_d([M])$. Therefore
$$\tau_d([M]) = \lim_{e\to\infty} \frac{1}{p^{de}}(f^*)^e([M])$$
in $\mathrm{K}_0A_{\mathbb{Q}}$.

2. (a) Suppose that A is a regular local ring. Then A is isomorphic to a formal power series ring $k[[x_1, \ldots, x_d]]$. It is easy to see that ${}^1A$ is a free A-module of rank p^d. Hence we have $f^*([A]) = [{}^1A] = p^d[A]$ and $[A] = \tau_d([A])$.

   (b) More generally, if A is a complete intersection, we can prove $[A] = \tau_d([A])$ (by using Corollary 18.1.2 in [2]).

   (c) If A is Gorenstein of dimension d, we have $\tau_{d-1}([A]) = \tau_{d-3}([A]) = \cdots = 0$. (See Example 18.1.2 and Theorem 18.2 in [2].)

   (d) If A is Cohen-Macaulay of dimension d, we have $[K_A] = \tau_d([A]) - \tau_{d-1}([A]) + \tau_{d-2}([A]) \cdots + (-1)^d\tau_0([A])$. (See Example 18.1.2 and Theorem 18.2 in [2].)

Definition 2.8 Let $\mathsf{F}. : 0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to 0$ be a complex of A-modules such that F_i is a finitely generated free A-module and $\ell_A(\mathrm{H}_i(\mathsf{F}.)) < \infty$ for each i. We define the map $\chi(\mathsf{F}.) : \mathrm{K}_0A \to \mathbb{Z}$ by $\chi(\mathsf{F}.)([M]) = \sum_i(-1)^i\ell_A(\mathrm{H}_i(\mathsf{F}. \otimes_A M))$. Put $\chi(\mathsf{F}.)_{\mathbb{Q}} = \chi(\mathsf{F}.) \otimes_{\mathbb{Z}} \mathbb{Q} : \mathrm{K}_0A_{\mathbb{Q}} \to \mathbb{Q}$. Then we define the j-th local Chern character of $\mathsf{F}.$ as the restriction $\chi(\mathsf{F}.)_{\mathbb{Q}}|_{\mathrm{L}_j\mathrm{K}_0A_{\mathbb{Q}}} : \mathrm{L}_j\mathrm{K}_0A_{\mathbb{Q}} \to \mathbb{Q}$ and denoted by $\mathrm{ch}_j(\mathsf{F}.)$. We set $\mathrm{ch}_j(\mathsf{F}.) \cap c = \mathrm{ch}_j(\mathsf{F}.)(c)$ for $c \in \mathrm{L}_j\mathrm{K}_0A_{\mathbb{Q}}$.

Of course local Chern characters in the above definition coincide with local Chern characters defined by MacPherson's graph construction in [2]. (In [2] local Chern characters are defined over an arbitrary scheme.)

Remark 2.9 Let $\dim A = d$ and $\mathsf{F}.$ a complex as in the previous definition. By definition of local Chern characters, we have

$$\begin{aligned} & \sum_i (-1)^i \ell_A(\mathrm{H}_i(\mathsf{F}. \otimes_A M)) \\ = \ & \chi(\mathsf{F}.)_{\mathbb{Q}}([M]) \\ = \ & \chi(\mathsf{F}.)_{\mathbb{Q}}(\tau_0([M]) + \tau_1([M]) \cdots + \tau_d([M])) \\ = \ & \mathrm{ch}_0(\mathsf{F}.) \cap \tau_0([M]) + \mathrm{ch}_1(\mathsf{F}.) \cap \tau_1([M]) \cdots + \mathrm{ch}_d(\mathsf{F}.) \cap \tau_d([M]). \end{aligned}$$

(See Example 18.3.12 in [2].)

Furthermore we have

$$\begin{aligned} & \mathrm{ch}_d(\mathsf{F}.) \cap \tau_d([M]) \\ = \ & \chi(\mathsf{F}.)_{\mathbb{Q}}(\tau_d([M])) \\ = \ & \sum_i (-1)^i \ell_A(\mathrm{H}_i(\mathsf{F}. \otimes_A \left[\lim_{e \to \infty} \frac{1}{p^{de}} (f^*)^e([M])\right])) \\ = \ & \lim_{e \to \infty} \frac{1}{p^{de}} \sum (-1)^i \ell_A(\mathrm{H}_i(\mathsf{F}. \otimes_A (f^*)^e([M]))) \in \mathbb{Q} \end{aligned}$$

(See [9].)

3 An application of intersection theory to commutative algebra

In this section we assume that all rings are homomorphic image of a regular local ring (to use intersection theory). We do not have to assume that given rings contain a fields.

Our first aim is to prove the following theorem:

Theorem 3.1 *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension d and $\mathsf{F}. : 0 \to F_d \to F_{d-1} \to \cdots \to F_0 \to 0$ a complex of finitely generated A-free modules such that $\ell_A(\mathrm{H}_i(\mathsf{F}.)) < \infty$ for each i. Suppose that one of the following conditions is satisfied:*

(0) *$(A, \mathfrak{m})$ is a Gorenstein ring.*

(1) *$d \leq 2$ and A is equi-dimensional.*

(2) *$(A, \mathfrak{m})$ is normal with $d \leq 4$ and the canonical class $\mathrm{cl}(K_A)$ is torsion in the divisor class group $\mathrm{Cl}(A)$.*

(3) *There exist a regular local ring $(T, \mathfrak{n})$ and a complex $\mathsf{G}.$ of fnitely generated T-free modules such that A is a homomorphic image of T and $\mathsf{G}. \otimes_T A$ is isomorphic to $\mathsf{F}.$.*

Then $\sum_{i=0}^{d}(-1)^i\ell_A(\mathrm{H}_i(\mathsf{F}.)) = \sum_{i=0}^{d}(-1)^i\ell_A(\mathrm{H}_i(\mathsf{F}.^*[-d]))$ *holds.*

Here $\mathsf{F}.^*$ is the dual complex with degree $F_t^* = \mathrm{Hom}_A(F_{-t}, A)$ and $\mathsf{F}.^*[-d]$ is the shifted complex, i.e., $(\mathsf{F}.^*[-d])_i = F_{-d+i}^*$.

Before proving this theorem we have

Remark 3.2 With notation as above, put $M = \mathrm{H}_0(\mathsf{F}.)$. When A is Cohen-Macaulay, we have

$$\sum_{i=0}^{d}(-1)^i\ell_A(\mathrm{H}_i(\mathsf{F}.)) = \ell_A(M)$$

$$\sum_{i=0}^{d}(-1)^i\ell_A(\mathrm{H}_i(\mathsf{F}.^*[-d])) = \ell_A(\mathrm{Ext}_A^d(M, A)) = \ell_A(M \otimes_A K_A)$$

by the local duality theorem ([4]) and the depth sensitivity for $\mathsf{F}.$ (see Remark 2.2 in [5]).

Proof of Theorem 3.1. By Remark 3.2, it is obvious that $\sum_{i=0}^{d}(-1)^i\ell_A(\mathrm{H}_i(\mathsf{F}.))$ is equal to $\sum_{i=0}^{d}(-1)^i\ell_A(\mathrm{H}_i(\mathsf{F}.^*[-d]))$ when A is a Gorenstein ring.

Suppose $d \leq 2$.

It is trivial when $d = 0$.

We can prove $\ell_A(\mathrm{H}_0(\mathsf{F}.)) - \ell_A(\mathrm{H}_1(\mathsf{F}.)) = \ell_A(\mathrm{H}_0(\mathsf{F}.^*[-1])) - \ell_A(\mathrm{H}_1(\mathsf{F}.^*[-1]))$ by an elementary method in the case of $d = 1$ (for example, see Appendix A in [2]).

Next assume $d = 2$. Then we have

$$\sum_{i=0}^{2}(-1)^i\ell_A(\mathrm{H}_i(\mathsf{F}.)) - \sum_{i=0}^{2}(-1)^i\ell_A(\mathrm{H}_i(\mathsf{F}.^*[-2])) = 2\cdot\mathrm{ch}_1(\mathsf{F}.) \cap \tau_1([A])$$

from Example 18.1.2 in [2]. By the vanishing theorem of the first local Chern characters([7]), we get $\mathrm{ch}_1(\mathsf{F}.) = 0$ since A is equi-dimensional.

When A is normal, it is easy to see that there exists a natural isomorphism $\mathrm{Cl}(A)\otimes_{\mathbb{Z}}\mathbb{Q} \simeq \mathrm{L}_{d-1}\mathrm{K}_0 A_{\mathbb{Q}}$ such that

$$\mathrm{cl}(K_A) = 2\cdot\tau_{d-1}([A]) \in \mathrm{Cl}(A) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Suppose $(A, \mathfrak{m})$ is a normal local ring of dimension 3. Then we obtain

$$\sum_{i=0}^{3}(-1)^i\ell_A(\mathrm{H}_i(\mathsf{F}.)) - \sum_{i=0}^{3}(-1)^i\ell_A(\mathrm{H}_i(\mathsf{F}.^*[-3])) = 2\cdot\mathrm{ch}_2(\mathsf{F}.) \cap \tau_2([A]) + 2\cdot\mathrm{ch}_0(\mathsf{F}.) \cap \tau_0([A]).$$

It is obvious that $\tau_0([A]) = 0$ since $\mathrm{L}_0\mathrm{K}_0 A_{\mathbb{Q}} = (0)$. Therefore $\sum_{i=0}^{d}(-1)^i\ell_A(\mathrm{H}_i(\mathsf{F}.)) = \sum_{i=0}^{d}(-1)^i\ell_A(\mathrm{H}_i(\mathsf{F}.^*[-d]))$ if the canonical class $\mathrm{cl}(K_A)$ is torsion in $\mathrm{Cl}(A)$. Next suppose $(A, \mathfrak{m})$ is a normal local ring of dimension 4. Then

$$\sum_{i=0}^{4}(-1)^i\ell_A(\mathrm{H}_i(\mathsf{F}.)) - \sum_{i=0}^{4}(-1)^i\ell_A(\mathrm{H}_i(\mathsf{F}.^*[-4])) = 2\cdot\mathrm{ch}_3(\mathsf{F}.) \cap \tau_3([A]) + 2\cdot\mathrm{ch}_1(\mathsf{F}.) \cap \tau_1([A])$$

holds. By the assumption that $\tau_3([A]) = 0$ and the vanishing theorem of the first local Chern character [7], we get $\sum_{i=0}^{4}(-1)^i \ell_A(\mathrm{H}_i(\mathbf{F}.)) = \sum_{i=0}^{4}(-1)^i \ell_A(\mathrm{H}_i(\mathbf{F}.^{\bullet}[-4]))$ immediately.

Finally assume that the condition (3) is satisfied. We define a closed subset of $\mathrm{Spec}(T)$ by $\mathrm{supp}(\mathbb{G}.) = \{\mathfrak{p} \mid \mathbb{G}. \otimes_T T_\mathfrak{p} \text{ is not exact}\}$. Since $\mathrm{supp}(\mathbb{G}.) \cap \mathrm{Spec}(A) = \{\mathfrak{m}\}$, $\dim \mathrm{Spec}(A) + \dim \mathrm{supp}(\mathbb{G}.) \leq \dim \mathrm{Spec}(T)$ holds by [10]. For $j < \dim \mathrm{Spec}(A)$, we obtain

$$\mathrm{ch}_j(\mathbf{F}.) = \mathrm{ch}_j(\mathbb{G}. \otimes_T A) = 0$$

by [8] since $j < \dim \mathrm{Spec}(T) - \dim \mathrm{supp}(\mathbb{G}.)$. Therefore

$$\begin{aligned} & \sum_{i=0}^{d}(-1)^i \ell_A(\mathrm{H}_i(\mathbf{F}.)) - \sum_{i=0}^{d}(-1)^i \ell_A(\mathrm{H}_i(\mathbf{F}.^{\bullet}[-d])) \\ = \; & 2\{\mathrm{ch}_{d-1}(\mathbf{F}.) \cap \tau_{d-1}([A]) + \mathrm{ch}_{d-3}(\mathbf{F}.) \cap \tau_{d-3}([A]) + \cdots\} \\ = \; & 0 \end{aligned}$$

is satisfied. **Q.E.D.**

The following proposition claims that $\sum_{i=0}^{d}(-1)^i \ell_A(\mathrm{H}_i(\mathbf{F}.))$ does not always coincide with $\sum_{i=0}^{d}(-1)^i \ell_A(\mathrm{H}_i(\mathbf{F}.^{\bullet}[-d]))$ even if $(A, \mathfrak{m})$ is a Cohen-Macaulay normal ring.

Proposition 3.3 *Let k be a field and put*

$$A = \left(k[x_0, x_1, x_2, y_0, y_1]/(x_0x_2 - x_1^2, x_0y_1 - x_1y_0, x_1y_1 - x_2y_0)\right)_{(x_0,x_1,x_2,y_0,y_1)}.$$

Then there exists a finitely generated A-module M such that $pd_A M < \infty$ and $\infty > \ell_A(M) \neq \ell_A(\mathrm{Ext}_A^3(M, A))$.

Proof. First note that A is a Cohen-Macaulay normal domain of dimension 3.
Put

$$S = (k[\alpha, \beta, \gamma, \delta]/(\alpha\delta - \beta\gamma))_{(\alpha,\beta,\gamma,\delta)}.$$

We define a ring homomorphism $\phi : S \to A$ with $\phi(\alpha) = x_0$, $\phi(\beta) = x_2$, $\phi(\gamma) = y_0^2$ and $\phi(\delta) = y_1^2$. Then it is easy to check that A is isomorphic to $S^3 \oplus (\beta, \delta)S$ as an S-module. Due to the famous example constructed by Dutta, Hochster and MacLaughlin ([1]), there exists an S-module N such that $\ell_S(N) = 15$, $pd_S N = 3$ and $\sum_{i=0}^{3}(-1)^i \ell_S(\mathrm{Tor}_i^S(N, S/(\beta, \delta)S)) = -1$. Put $M = N \otimes_S A$ and let

$$\mathbf{F}. \; : \; 0 \to F_3 \to F_2 \to F_1 \to F_0 \to 0$$

be the minimal S-free resolution of N. Since A is a maximal Cohen-Macaulay S-module,

$$\mathbf{H}. = \mathbf{F}. \otimes_S A \; : \; 0 \to F_3 \otimes_S A \to F_2 \otimes_S A \to F_1 \otimes_S A \to F_0 \otimes_S A \to 0$$

is the minimal A-free resolution of M. Hence we have $\ell_A(M) < \infty$ and $pd_A M = 3$.

Then we obtain

$$\begin{aligned} \mathrm{Ext}^3_A(M, A) &= \mathrm{Ext}^3_A(N \otimes_S A, A) \\ &\simeq \mathrm{Ext}^3_S(N, S) \otimes_S A \\ &\simeq \left(\mathrm{Ext}^3_S(N, S)\right)^3 \oplus \left(\mathrm{Ext}^3_S(N, S) \otimes_S (\beta, \delta)S\right) \\ M &= N \otimes_S A \\ &\simeq N^3 \otimes (N \otimes_S (\beta, \delta)S). \end{aligned}$$

Since S is a Gorenstein ring, $\ell_S(N) = \ell_S(\mathrm{Ext}^3_S(N, S))$ (see Remark 3.2). Therefore we have only to show that $\ell_S(N \otimes_S (\beta, \delta)S) \neq \ell_S(\mathrm{Ext}^3_S(N, S) \otimes_S (\beta, \delta)S)$.

For finitely generated S-modules P and Q, put $\chi_S(P, Q) = \sum_i (-1)^i \ell_S(\mathrm{Tor}^S_i(P, Q))$.

Then we obtain

$$\begin{aligned} \chi_S(N, (\beta, \delta)S) &= \ell_S(N \otimes_S (\beta, \delta)S) \\ \chi_S(\mathrm{Ext}^3_S(N, S), (\beta, \delta)S) &= \ell_S(\mathrm{Ext}^3_S(N, S) \otimes_S (\beta, \delta)S) \end{aligned}$$

because $(\beta, \delta)S$ is a maximal Cohen-Macaulay S-module.

By the exact sequence $0 \to (\beta, \delta)S \to S \to S/(\beta, \delta)S \to 0$,

$$\begin{aligned} \chi_S(N, (\beta, \delta)S) &= \chi_S(N, S) - \chi_S(N, S/(\beta, \delta)S) \\ \chi_S(\mathrm{Ext}^3_S(N, S), (\beta, \delta)S) &= \chi_S(\mathrm{Ext}^3_S(N, S), S) - \chi_S(\mathrm{Ext}^3_S(N, S), S/(\beta, \delta)S) \end{aligned}$$

are satisfied.

Because of $\chi_S(N, S) = \ell_S(N) = \ell_S(\mathrm{Ext}^3_S(N, S)) = \chi_S(\mathrm{Ext}^3_S(N, S), S)$, it is sufficient to prove $\chi_S(\mathrm{Ext}^3_S(N, S), S/(\beta, \delta)S) \neq \chi_S(N, S/(\beta, \delta)S)$. Remember that

$$\chi_S(N, S/(\beta, \delta)S) = \sum_i (-1)^i \ell_S(\mathrm{Tor}^S_i(N, S/(\beta, \delta)S)) = -1$$

by [1]. On the other hand, the following lemma guarantees $\chi_S(\mathrm{Ext}^3_S(N, S), S/(\beta, \delta)S) = 1$.

Lemma 3.4 *Let B be a Cohen-Macaulay Noetherian local ring of dimension d and I an ideal of B such that B/I is a Gorenstein ring of dimension $d - i$. Then for a finitely generated B-module L such that $\ell_B(L) < \infty$ and $pd_B L < \infty$,*

$$\chi_B(L, B/I) = (-1)^i \chi_B(\mathrm{Ext}^d_B(L, B), B/I)$$

holds.

Proof. Let $\mathbf{F}_\cdot$: $0 \to F_d \to \cdots \to F_0 \to 0$ be the minimal free B-resolution of L. Then $\mathbf{F}_\cdot^*[-d]$ is the minimal free B-resolution of $\mathrm{Ext}_B^d(L,B)$. Put $\mathbb{G}_\cdot = \mathbf{F}_\cdot \otimes_B B/I$. Then we have

$$\begin{aligned} \chi_B(L, B/I) &= \sum_i (-1)^i \ell_B(\mathrm{H}_i(\mathbb{G}_\cdot)), \\ \chi_B(\mathrm{Ext}_B^d(L,B), B/I) &= \sum_i (-1)^i \ell_B(\mathrm{H}_i(\mathbb{G}_\cdot^*[-d])). \end{aligned}$$

Furthermore let

$$\mathbf{D}^\cdot \ : \ 0 \to D^0 \to \cdots \to D^{d-i} \to 0$$

be the dualizing complex of B/I, i.e., the minimal injective B/I-resolution of B/I.

Compute the homologies of the double complex $\mathrm{Hom}_{B/I}(\mathbb{G}_\cdot, \mathbf{D}^\cdot)$ by using the argument on the spectral sequences.

$$\begin{array}{ccccccc}
\mathrm{Hom}_{B/I}(G_0, D^0) & \to & \mathrm{Hom}_{B/I}(G_0, D^1) & \to & \cdots & \to & \mathrm{Hom}_{B/I}(G_0, D^{d-i}) \\
\downarrow & & \downarrow & & & & \downarrow \\
\mathrm{Hom}_{B/I}(G_1, D^0) & \to & \mathrm{Hom}_{B/I}(G_1, D^1) & \to & \cdots & \to & \mathrm{Hom}_{B/I}(G_1, D^{d-i}) \\
\downarrow & & \downarrow & & & & \downarrow \\
\vdots & & \vdots & & & & \vdots \\
\downarrow & & \downarrow & & & & \downarrow \\
\mathrm{Hom}_{B/I}(G_d, D^0) & \to & \mathrm{Hom}_{B/I}(G_d, D^1) & \to & \cdots & \to & \mathrm{Hom}_{B/I}(G_d, D^{d-i})
\end{array}$$

Then we will obtain $\chi_B(L, B/I) = (-1)^i \chi_B(\mathrm{Ext}_B^d(L,B), B/I)$ immediately. **Q.E.D.**

We have completed the proof of Proposition 3.3.

Remark 3.5 From Theorem 3.1 we will see that the complex $\mathbf{H}_\cdot = \mathbf{F}_\cdot \otimes_S A$ defined in the proof of Proposition 3.3 never lift to complexes over regular local rings. But it is easy to check that it lifts to a complex over a hypersurface.

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DEPARTMENT OF MATHEMATICS, TOKYO METROPOLITAN UNIVERSITY,
MINAMI-OHSAWA 1-1, HACHIOJI, TOKYO, 192-03, JAPAN

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THE STABLE CATEGORY OF MODULES OF FINITE PROJECTIVE DIMENSION

Yuji YOSHINO

§0. Introduction

Let A be a Noetherian (associative) ring with unity. We denote by $\mathcal{F}(A)$ the category of finitely generated right A-modules of finite projective dimension. We show in the first section of this note that the stable category of $\mathcal{F}(A)$ can be embedded into the derived category of complexes of left A-modules in a certain way. See Theorem 1 in §1. In the previous work we proved the same theorem for commutative Noetherian rings, but the proof of this can be easily generalized to non-commutative rings.

We develop this embedded theorem for non-commutative case, because it enables us to classify certain modules over A. In fact, we will define the notion of maximal quasi-Buchsbaum modules on A and classify these modules on orders of global dimension two.

To be more precise, let A be an order of Krull dimension and global dimension two. Then a finitely generated right A-module M is said to be a maximal quasi-Buchsbaum module if $\mathrm{Ext}^i(M, A)$ $(i = 1, 2)$ are semi-simple. Using Theorem 1, we can classify all such modules on A. In fact, we show in Theorem 2 that there are only a finite number of indecomposable maximal quasi-Buchsbaum modules over A. Those indecomposable modules actually correspond to the indecomposable representations of a certain quiver.

These results are already known by [3] for commutative case, and the whole discussion in this note is merely a generalization of that in [3].

The final version of this paper will be submitted for publication elsewhere.

§1. An equivalence theorem

Let A be a Noetherian associative ring with unity and let mod-A be the category of finitely generated right A-modules. We denote by $\mathcal{F}(A)$ the full subcategory of mod-A consisting of all right modules of finite projective dimension. Recall the definition of the stable category $\underline{\mathcal{F}(A)}$ associated with $\mathcal{F}(A)$. Objects of $\underline{\mathcal{F}(A)}$ are the same as those of $\mathcal{F}(A)$ and if M and N are two objects in $\underline{\mathcal{F}(A)}$, then the set of morphisms from M to N is defined to be:

$$\underline{\mathrm{Hom}}(M_A, N_A) = \mathrm{Hom}(M_A, N_A)/\mathfrak{P}(M, N),$$

where $\mathfrak{P}(M, N)$ denotes the set of right A-homomorphisms $f : M \to N$ which factor as $M \to P \to N$ with P projective. Note that, in $\underline{\mathcal{F}(A)}$, projective modules are isomorphic to the zero module. Furthermore, if $\mathcal{F}(A)$ satisfies the Krull-Schmidt theorem, then that two modules M and N in $\mathcal{F}(A)$ are isomorphic in $\underline{\mathcal{F}(A)}$ means exactly that there are projective right A-modules P and Q so that $M \oplus P$ is isomorphic to $N \oplus Q$ in $\mathcal{F}(A)$. Our aim in this section is to construct a certain full embedding of $\underline{\mathcal{F}(A)}$ into the derived category of complexes of left A-modules.

Let us fix some notation for complexes. When we write $X^{\cdot}$, it means the complex of left A-modules:

$$\cdots \xrightarrow{d_X^{n-1}} X^n \xrightarrow{d_X^n} X^{n+1} \xrightarrow{d_X^{n+1}} X^{n+2} \xrightarrow{d_X^{n+2}} \cdots .$$

Now we denote by $D(A)$ the derived category of complexes of left A-module, that is, an object of $D(A)$ is a complex of left A-modules and if $X^{\cdot}$ and $Y^{\cdot}$ are two complexes, a morphism $\varphi : X^{\cdot} \to Y^{\cdot}$ is represented by a diagram:

$$X^{\cdot} \xleftarrow{f} Z^{\cdot} \xrightarrow{g} Y^{\cdot},$$

where the both f and g are chain homomorphisms and f is a quasi-isomorphism. See [2] for more detail.

To state the theorem we need the notion of the truncation functor τ_0 which is defined in such a way that, for any complex $X^{\cdot} \in D(A)$, $\tau_0(X^{\cdot})$ is the complex:

$$0 \longrightarrow \mathrm{Ker}(d_X^0) \longrightarrow X^0 \xrightarrow{d_X^0} X^1 \xrightarrow{d_X^1} X^2 \xrightarrow{d_X^2} X^3 \longrightarrow \cdots .$$

Definition. $E(A)$ is defined to be the full subcategory of $D(A)$ consisting of the complexes $X^{\cdot}$ satisfying the following conditions:

(a) $X^{\cdot}$ is bounded above, i.e. $X^i = 0$ for large i,

(b) $H^i(X^{\cdot}) = 0$ for any $i \leq 0$, and

(c) $H^j(\mathrm{RHom}(X^{\cdot}, A)) = 0$ for any $j < 0$.

(Here RHom denotes the right derived functor of Hom.)

Now we can state our main theorem in this section.

Theorem 1. *The additive contravariant functor $\tau_0 \mathrm{RHom}(\ \ , A)$ induces an equivalence of the categories:*

$$\rho : \underline{\mathcal{F}(A)}^{op} \longrightarrow E(A).$$

The proof of this theorem is seen in our previous paper [3], where the ring is assumed to be commutative but the proof works well even in the non-commutative case. For the reader's convenience, we give an outline of the proof below. First of all it is easily seen that $\tau_0 \mathrm{RHom}(M, A) \in E(A)$ for any $M \in \mathcal{F}(A)$, thus it is a well-defined functor from $\mathcal{F}(A)$ to $E(A)$. Secondly we notice that $\tau_0 \mathrm{RHom}(P, A) = 0$ for any projective module P. Hence the functor $\tau_0 \mathrm{RHom}(\ \ , A) : \mathcal{F}(A)^{op} \to E(A)$ yields the functor $\rho : \underline{\mathcal{F}(A)}^{op} \to E(A)$. Thirdly we can show that ρ gives a surjective map from the set of isomorphism classes of objects of $\underline{\mathcal{F}(A)}$ to that of $E(A)$. In fact, for any $X^{\cdot} \in E(A)$, taking a complex $P^{\cdot}$ of projective left A-modules that is quasi-isomorphic to $X^{\cdot}$, we define a right A-module M to be the cokernel of the map:

$$\mathrm{Hom}(d_P^0, A) : \mathrm{Hom}(P^1, A) \longrightarrow \mathrm{Hom}(P_0, A).$$

Then it is easy to see that M belongs to $\underline{\mathcal{F}(A)}$ and $\rho(M) \simeq X^{\cdot}$ in $E(A)$. Finally we have to show that, for any two objects $M, N \in \underline{\mathcal{F}(A)}$, ρ gives an isomorphism:

$$\underline{\mathrm{Hom}}(M, N) \simeq \mathrm{Hom}_{E(A)}(\rho(N), \rho(M)).$$

But this is quite easy to show using only a definition of ρ, and we leave the proof of this to the reader.

Remark. (a) From the definition of ρ, it is clear that, for any $M \in \underline{\mathcal{F}(A)}$, $H^i(\rho(M))$ is isomorphic to $\mathrm{Ext}^i(M, A)$ if $i > 0$, and 0 otherwise.

(*b*) In general, there always exists a natural functor from the category A-mod of left modules to $D(A)$:

$$\nu : A\text{-mod} \longrightarrow D(A),$$

which sends a module M to a complex $\cdots \to 0 \to M \to 0 \to \cdots$.

In case that A has global dimension one, we can describe the difference of this natural functor from our functor ρ. In this case, $\mathcal{F}(A)$ is equal to the whole category mod- A, thus the theorem says the equivalence:

$$(\underline{\text{mod-}A})^{op} \simeq E(A).$$

under the functor ρ. Let tr : $(\underline{\text{mod-}A})^{op} \to A$-mod be the Auslander's transpose that is equivalent to the functor $\text{Ext}^1(\ ,A)$, because the global dimension of A is one. Then we can easily see that there is a commutative diagram:

$$\begin{array}{ccc} (\underline{\text{mod-}A})^{op} & \xrightarrow{\rho} & E(A) \\ {\scriptstyle tr}\downarrow & & \| \\ A\text{-mod} & \xrightarrow{\nu} & E(A). \end{array}$$

For the general cases where A may not have global dimension one, ρ contains all the information about $\text{Ext}^i(\ ,A)$ $(i \geq 1)$. And we may sometimes regard ρ as a kind of generalization of the Auslander's transpose.

§2. Definition of a maximal quasi-Buchsbaum module

In this section we always assume that A is a Noetherian ring of finite global dimension.

DEFINITION. A finitely generated right A-module M is said to be a maximal quasi-Buchsbaum module if

(*) $\text{Ext}^i(M,A)$ is a semi-simple left A-module for any $i > 0$.

Note that the condition (*) is equivalent to saying that all the homologies of the complex $\rho(M)$ are semi-simple. (cf. §1, Remark (a).)

REMARK. (*a*) For commutative rings, the notion of maximal quasi-Buchsbaum modules is widely used. And we must mention that the above definition is slightly different from that for commutative rings.

When A is a commutative *Gorenstein* local ring of positive Krull dimension with maximal ideal $\mathfrak{m}$, a finite A-module M is usually said to be a maximal quasi-Buchsbaum module if it satisfies the condition (*) and if M has maximal Krull dimension (i.e. the Krull dimension of the ring A). Notice that it is an easy exercise to see that, under the condition (*), that M has maximal dimension is equivalent to that M is not an Artinian module. It is also easy to see that any Artinian module satisfying the condition (*) is a direct sum of copies of $A/\mathfrak{m}$. (Note that $A/\mathfrak{m}$ is the unique simple A-module). Furthermore, if M satisfies the condition (*), then M is a direct sum of M_0 with a sum of copies of $A/\mathfrak{m}$, where M_0 satisfies (*) and has maximal dimension. Thus the above definition equals to the usual one up to a direct summand of a semi-simple module.

(*b*) As in (*a*), let A be a commutative Gorenstein local ring with maximal ideal $\mathfrak{m}$. In this case, M is called a *maximal Buchsbaum module* if $\rho(M)$ is isomorphic to a complex of $A/\mathfrak{m}$-modules and if M has maximal dimension.

The notion of a maximal Buchsbaum module is, of course, stronger than that of a maximal quasi-Buchsbaum module. In fact, if A is a complete regular local ring (i.e. a commutative complete local ring of finite global dimension), then Goto's theorem says that any indecomposable Buchsbaum module is isomorphic to one of the i-th syzygies ($1 \leq i \leq d$) of $A/\mathfrak{m}$. On the other hand, there are much more indecomposable quasi-Buchsbaum modules over A.

To explain this more closely, assume that A is a complete regular local ring of dimension two (i.e. a commutative complete local ring of global dimension two). Let L^i be the i-th syzygy module of $A/\mathfrak{m}$. Note that $L^0 = A/\mathfrak{m}$, $L^1 = \mathfrak{m}$ and $L^2 = A$. Then Goto's theorem says that L^1 and L^2 are the all of the indecomposable maximal Buchsbaum modules over A. For any module M satisfying (*), we denote

$$h(M) = (\dim_{A/\mathfrak{m}} \mathrm{Ext}^1(M,A), \dim_{A/\mathfrak{m}} \mathrm{Ext}^2(M,A)) \in \mathbf{N}^2.$$

Then clearly, $h(L^1) = (1,0)$ and $h(L^2) = (0,0)$. On the other hand, there are four indecomposable maximal quasi-Buchsbaum modules over A: L^0, L^1, L^2 and M, where M is the indecomposable module with $h(M) = (1,1)$. (Notice that L^0 is an Artinian module but satisfies the condition (*) and $h(L^0) = (0,1)$.) Thus the set $\{h(M) \in \mathbf{N}^2 | \ M \text{ is an indecomposable maximal quasi-Buchsbaum module}\}$ forms the set of positive roots of the root system of type A_2. It is observed that similar phenomena occur even when A has higher dimension than two. (See [3].)

Our aim of the next section is to extend this to the case for non-commutative rings.

§3. Maximal quasi-Buchsbaum modules over orders of Krull dimension and global dimension two

In this section, R denotes a (commutative) normal complete local domain of dimension two with unique maximal ideal $\mathfrak{m}$, and A is always an R-order of global dimension two. Note that, in this case, we have $\underline{\mathcal{F}(A)} = \underline{\text{mod-}A}$. Note also that such orders are completely classified by M. Artin [1]. But we are interested in maximal quasi-Buchsbaum modules over A and would like to make a classification of those modules.

For this purpose, let $\underline{\mathcal{B}}$ denote the full subcategory of $\underline{\text{mod-}A}$ consisting of all maximal quasi-Buchsbaum right A-modules and let $|\underline{\mathcal{B}}|$ be the set of isomorphism classes of objects of $\underline{\mathcal{B}}$. Recall that an object of $\underline{\mathcal{B}}$ is a right module M satisfying that $\mathrm{Ext}^1(M, A)$ and $\mathrm{Ext}^2(M, A)$ are semi-simple. So, given semi-simple left A-modules T_1 and T_2, we can consider the subcategory of $\underline{\mathcal{B}}$ whose objects are the modules M with $\mathrm{Ext}^i(M, A) \simeq T_i$, $i = 1, 2$. We denote this category by $\underline{\mathcal{B}}(T_1, T_2)$ and likewise as above, we denote by $|\underline{\mathcal{B}}(T_1, T_2)|$ the set of classes of objects of $\underline{\mathcal{B}}(T_1, T_2)$.

By the equivalence theorem in §1, $(\underline{\text{mod-}A})^{op}$ is equivalent to $E(A)$ that is a full subcategory of $D(A)$. Under this equivalence, any $M \in \underline{\text{mod-}A}$ corresponds to a complex $X^{\cdot} = \rho(M)$ which has the property that $\mathrm{H}^i(X^{\cdot}) \simeq T_i$, $i = 1, 2$ with T_i some semi-simple left modules. Thus there is an exact sequence:

$$(1) \qquad 0 \longrightarrow T_1[-1] \longrightarrow X^{\cdot} \longrightarrow T_2[-2] \longrightarrow 0,$$

in the category $E(A)$ and this extension determines an element of $\mathrm{Ext}^1(T_2[-2], T_1[-1]) = \mathrm{Ext}^{1-1-(-2)}(T_2, T_1) = \mathrm{Ext}^2(T_2, T_1)$.

On the other hand, if there is an exact sequence (1), then the complex $X^{\cdot}$ appearing in the middle is easily seen to belong to $E(A)$ so that there is a right module M with $\rho(M) \simeq X^{\cdot}$. Clearly this M is a maximal quasi-Buchsbaum right A-module and belongs to $\underline{\mathcal{B}}(T_1, T_2)$. We can show:

LEMMA 1. *There is a bijective map:*

$$|\underline{\mathcal{B}}(T_1, T_2)| \longrightarrow \mathrm{Ext}^2(T_2, T_1) / \mathrm{Aut}(T_2)^{op} \times \mathrm{Aut}(T_1).$$

If A is commutative, then the proof of this lemma can be shown in [3], and for non-commutative case, that proof is still valid. (Only one should become aware of that modules are one-sided.)

Now let us assume that the residue field $k = R/\mathfrak{m}$ of the center R is an algebraically closed field. And let $\{S_1, S_2, \dots, S_\ell\}$ be all of the classes of simple left A-modules. Note that $\mathrm{Hom}(S_i, S_j) = k$ if $i = j$, and otherwise 0. We can write:

$$T_1 = S_1^{a_1} \oplus S_2^{a_2} \oplus \cdots \oplus S_\ell^{a_\ell},$$
$$T_2 = S_1^{b_1} \oplus S_2^{b_2} \oplus \cdots \oplus S_\ell^{b_\ell},$$

for some integers a_i, b_i $(i = 1, 2, \dots, \ell)$. Thus we have:

$$\mathrm{Aut}(T_1) = \mathrm{GL}(a_1) \times \mathrm{GL}(a_2) \times \cdots \times \mathrm{GL}(a_\ell),$$
$$\mathrm{Aut}(T_2) = \mathrm{GL}(b_1) \times \mathrm{GL}(b_2) \times \cdots \times \mathrm{GL}(b_\ell).$$

Furthermore, letting

$$c_{ij} = \dim_k \mathrm{Ext}^2(S_i, S_j),$$

we have an isomorphism of $\mathrm{Aut}(T_1)^{op} \times \mathrm{Aut}(T_2)$-modules:

$$\mathrm{Ext}^2(T_2, T_1) \simeq \prod_{i,j} \prod^{c_{ij}} \mathrm{Hom}(k^{a_i}, k^{b_j}).$$

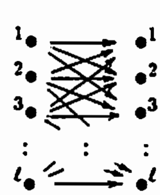
Thus, under the assumption that k is algebraically closed, the above mentioned lemma becomes:

COROLLARY. *There is a bijective map:*

$$|\underline{B}(T_1, T_2)| \longrightarrow \prod_{i,j} \prod^{c_{ij}} \mathrm{Hom}(k^{a_i}, k^{b_j}) / \prod_i \mathrm{GL}(a_i)^{op} \times \prod_j \mathrm{GL}(b_j).$$

By the usual argument of the theory of representations of quivers, this corollary leads to the following:

PROPOSITION. *Let Q be a quiver:*



where there are c_{ij} arrows form the i-th vertex of the left side to the j-th one of the right side. Then there is a bijective map from $|\underline{\mathcal{B}}|$ to the set of classes of representations of the quiver Q, which preserves direct sums.

EXAMPLE. Let $R = k[[x,y]]$ be a formal power series ring over an algebraically closed field k and let $A = R < u, v > /(u^n - x, v^n - y, uv - \zeta vu)$, where ζ is a primitive n-th root of unity. This A is known to be the order of type I of global dimension two, which is one of the orders M. Artin classified. In this case, letting $J = (u,v)A$ be the Jacobson radical, we see that $S = A/J$ is the unique simple right (and left) module over A. And an easy computation shows that $\mathrm{Ext}^2(S,S) = k$, hence the quiver Q in the proposition looks like $\bullet \longrightarrow \bullet$. As a result, there are only three indecomposable objects in $\underline{\mathcal{B}}$. They are J, S and M, where M is the module defined by the following exact sequence of right A-modules:

$$0 \longrightarrow A \xrightarrow{\psi} A^2 \xrightarrow{\varphi} A^2 \longrightarrow M \longrightarrow 0,$$

where $\psi(a) = (va, -ua)$ and $\varphi(b,c) = (-\zeta^{-1}uvb - \zeta v^2 c, u^2 b + \zeta uvc)$ for any $a, b, c \in A$. Actually there is a nonsplit exact sequence

$$0 \longrightarrow S[-1] \longrightarrow \rho(M) \longrightarrow S[-2] \longrightarrow 0.$$

Since all the orders A of global dimension and Krull dimension two are completely classified by the work of Artin, we can determine the numbers c_{ij} from the classification table. However we can show directly that they are 1 or 0.

LEMMA 2. *Let $\{S_1, S_2, \ldots, S_\ell\}$ be the set of simple left A-modules as above. Then there is a permutation τ on the set $\{1, 2, \ldots, \ell\}$ such that $\mathrm{Ext}^2(S_i, S_j) = k$ if $j = \tau(i)$ and otherwise 0.*

This lemma says that the quiver Q is a disjoint union of copies of the quiver of type A_2: $\bullet \longrightarrow \bullet$. Combining this with the proposition, we finally get the following theorem.

THEOREM 2. *Let R be a normal complete local domain of dimension two and let A be an R-order of global dimension two. Then there is a bijective map from $|\underline{\mathcal{B}}|$ to the set of classes of representations of a quiver that is a disjoint union of ℓ copies of $\bullet \longrightarrow \bullet$, where ℓ is the number of nonisomorphic simple left modules over A. In particular there are only 3ℓ indecomposable objects in $\underline{\mathcal{B}}$.*

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Yuji Yoshino
Math. Inst.
Yoshida College, Kyoto Univ.
Yoshida-Nihonmatsu, Sakyo-ku
Kyoto 606, Japan

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BLOCKS OF MACKEY FUNCTORS

Fumihito ODA

Recently,in their paper [4],J.Thévenaz and P.Webb described the division of the simple Mackey functors $S_{H,V}$ into blocks,using some general results giving information about the existence of non-trivial extensions of simple Mackey functors. The result is that the blocks of Mackey functors biject with the ordinary blocks of finite group G together with the blocks of certain sections of G.

Now we define a Mackey functor over a field k of characteristic $p > 0$. (in general we work with Mackey functors over a commutative ring.) Let Mod_{kG} be the category of kG-modules and the category of Mackey functors over k for G denoted by $Mack(G)$.

(1) (Dress [2]) A *Mackey functor* M consists of a contravariant functor $M^*$from the category of finite G-sets to kG-module with $M^*(X) = M_*(X)(=: M(X))$ for any finite G-set X. For any G-map $f : X \longrightarrow Y$, we put

$$f^* := M^*(f) : M(Y) \longrightarrow M(X)$$

$$f_* := M_*(f) : M(X) \longrightarrow M(Y).$$

The pair (M^*, M_*) must satisfy the following two axioms:
(M.1) M^* maps finite coproducts to finite products.
(M.2) For every pullback diagram,

$$\begin{array}{ccc} X_1 & \xrightarrow{\alpha} & X_2 \\ \beta\downarrow & & \downarrow\gamma \\ X_3 & \xrightarrow{\delta} & X_4 \end{array}$$

in G-sets we have $M^*(\delta)M_*(\gamma) = M_*(\beta)M^*(\alpha)$.

(2)(Yoshida[5])A *Mackey functor* is a k-additive contravariant functor from the Mackey category to Mod_k. Here the Mackey category $M_c(G)_k$ of G over k is constructed as follows: An object is a finite G-sets. The hom-set Hom(Y, X) from Y to X is k-module generated by the symbols of the form

$$[X \xleftarrow{f'} A \xrightarrow{f''} Y],$$

where A is a finite G-set and f' and f'' are G-map, with relations

$$[X \xleftarrow{f'} A \xrightarrow{f''} Y] = [X \xleftarrow{g'} B \xrightarrow{g''} Y]$$

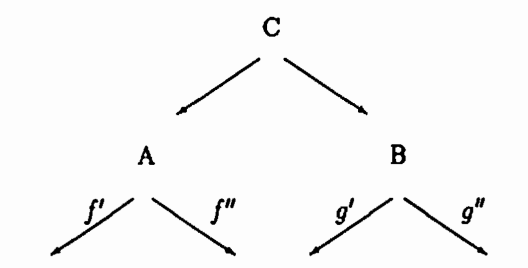
whenever there is an isomorphism $h : A \longrightarrow B$ such that $f' = g'h, f'' = g''h$, and

$$[X \longleftarrow A \longrightarrow Y] + [X \longleftarrow B \longrightarrow Y] = [X \longleftarrow A + B \longrightarrow Y].$$

The composition is defined by

$$[X \xleftarrow{f'} A \xrightarrow{f''} Y] \circ [Y \xleftarrow{g'} B \xrightarrow{g''} Z] = [X \longleftarrow C \longrightarrow Z],$$

where C is the pullback of f'' and g':



(3)(Yoshida [5])A *Mackeyfunctor* is a module over the k-algebra $\mu_k(G)$. Let $\mu_k(G)$ be the k-module generated by the symboles of the form

$$[H, A, x, K]$$

where $H, K \leq G, A \leq H \cap^x K$, and following relation holds:

$$[H, A, x, K] = [{}^gH, {}^{gh}A, hxkg'^{-1}, {}^{g'}K], g, g' \in G, h \in H, k \in K.$$

(Here, ${}^gH := gHg^{-1}$.)Then $\mu_k(G)$ becomes a k-algebra by the multiplication:

$$[H, A, x, K] \circ [K, B, y, L] := \sum_{k \in x^{-1}Ax \cap B \backslash K / B} [H, A, \cap^{xk} B, xky, L].$$

This k-algebra is isomorphic to the path algebra of the subcategory consisting of transitive G-sets in the Mackey category $Mc(G)_k$, and it is Morita equivalent to the path algebra of $Mc(G)_k$.

(4)(Green[3]) Assume that

$$(\mathrm{a}, \tau, \rho, \sigma)$$

be a system consisting of k-modules $\mathrm{a}(H), H \leq G$,and k-linear maps for

$$H \leq K \leq G$$

and $g \in G$:

$$\tau_H^K : \mathrm{a}(H) \longrightarrow \mathrm{a}(K) : \alpha \longmapsto \alpha \uparrow^K;$$

$$\rho_H^K : \mathrm{a}(K) \longrightarrow \mathrm{a}(H) : \beta \longmapsto \beta \downarrow_H;$$

$$\sigma_H^g : \mathrm{a}(H) \longrightarrow \mathrm{a}({}^gH) : \alpha \longmapsto {}^g\alpha$$

which satisfies the following axiom (where $D, H, K, L \leq G, g, g' \in G, \alpha \in \mathrm{a}(H), \beta \in \mathrm{a}(K)$) :

(G1)$\alpha \uparrow^H = \alpha, (\alpha \uparrow^K) \uparrow^L = \alpha \uparrow^L$ if $H \leq K \leq L$;

(G2)$\beta \downarrow_H = \beta, (\beta \downarrow_H)_D = \beta \downarrow_D$ if $D \leq H \leq K$;

(G3)$\alpha^{gg'} = (\alpha^g)^{g'}, \alpha^h = \alpha$ if $h \in H$;

(G4)${}^g(\alpha \uparrow^K) = ({}^g\alpha) \uparrow^{{}^gK}, {}^g(\alpha \downarrow_H) = ({}^g\alpha) \downarrow_{{}^gH}$;

(G5)(Mackey decomposition)If $H, K \leq L$,then

$$\alpha \uparrow^L \downarrow_K = \sum_{KgH \in K \backslash L / H} ({}^g\alpha) \downarrow_{{}^gH \cup K} \uparrow^K .$$

For our example we set $FP_V(H) := V^H$ which is the all H-fixed points in left kG- module V for subgroup H of G.We put the morphisms by

$$\tau_H^K : V^H \longrightarrow V^K : v \longmapsto \sum_{h \in K/H} hv,$$

$$\rho_H^K : V^K \hookrightarrow V^H : v \longmapsto v,$$

$$\sigma_H^g : V^H \longrightarrow V^H : v \longmapsto gv (H \leq K \leq G, g \in G).$$

Thus FP_V is a Mackey functor which is called fixed point Mackey functor. The simple Mackey functors $S_{P,V}$ were constructed by FP_V. (see [4])

We are now prepared to analyze the Burnside algebra $B(G)$ which is free k- module with basis the G-sets G/H ,for each conjugacy class of subgroups $H \leq G$, and giving the multiplication in terms of Cartesian products. The identity element of this algebra corresponds to one point set with trivial action , and the zero element corresponds to the empty set.

PROPOSITION 1 ([4](9.2)PROPOSITION)

There is a central subring of $\mu_k(G)$ isomorphic to the Burnsidealgebra $B(G)$ and which contains the identity. Specifically,this subalgebra has a basis consisting of elements

$$b_H = \sum_{K \leq G} \sum_{x \in K \backslash G/H} \tau^K_{K \cap {}^x H} \rho^K_{K \cap {}^x H}$$

Thus every idempotent of Burnside algebra gives a central idempotent of Mackey algebra,and hence a decomposition of the Mackey algebra into ring direct summands.But in general we do not obtain primitive central idempotents of Mackey algebra from it .So we studying the unions of blocks for using next theorem.

PROPOSITION 2 ([4](9.3)THEOREM)

In the Burnside algebra $B(G)$ over k we have

$$1 = \sum_{\substack{J \leq G \text{ up to conjugacy} \\ J:\, p-\text{perfect}}} f_J$$

where the f_J are primitive orthogonal idempotents in bijection with the conjugacy classes of p-perfect subgroups of G. For each p-perfect subgroup J let $P \leq G$ be such that $J \lhd P$ and P/J is a Sylow p-subgroup of $N_G(J)/J$.Then f_J is a linear combination of elements G/K with $K \leq P$. If L contains no conjugate of J then $f_J \cdot G/L = 0$,and $f_J \downarrow_L^G = 0$.

For each p-perfect subgroup J of G we define the $M_c(G, J)_k$ to the full subcategory of $Mack(G)$ whose object are the Mackey functors M for which $f_J \cdot M = M$.

PROPOSITION 3 ([4] (9.6)PROPOSITION)
Let $S_{K,W}$ be a simple Mackey functor over k.The simple Mackey functor s in $M(G, J)_k$ are precisely the $S_{K,W}$ with $J = O^p(K)$.

This propostion and next theorem shows that the determination of the block of Mackey functors of $M(G, J)_k$: it is enough to answer the question for the case $J = 1$.

THEOREM 4 ([4] (10.1) THEOREM)
The categories $M(N_G(J)/J, 1)_k$ and $M(G, J)_k$ are equivalent.

We mention that a simple Mackey functor $S_{K,W}$ in $M(G, 1)_k$ iff $K$is a p-subgroup of G. We prepare the definition of the Brauer morphism and the block of Mackey functors for report the most important theorem.

For every p-subgroup P of G,the Brauer morphism is a ring homomorphism.

$$Br_P : Z(kG) \longrightarrow Z(kC_G(P))^{N_G(P)} : \sum_{x \in G} a_x x \longmapsto \sum_{x \in C_G(P)} a_x x$$

where $Z(kG)$ is center of kG and $Z(kC_G(P))^{N_G(P)}$ is the subring of $N_G(P)$-fixed points in center of $kC_G(P)$. If e is a block of $N_G(P)$ then lies in $Z(kC_G(P))^{N_G(P)}$,where it is the sum of conjugate primitive idempotents, and so there is a unique block b of G such that $Br_P(b) \cdot e = e$. This unique block will be denoted by $b = e^G$.

Let

$$1 = \sum_{J: p-perfect, b: block\ of\ N_G(J)/J} e_{J,b}$$

be a sum where each $e_{J,b}$ is a central primitive idempotent in Mackey algebra $\mu_k(G)$. If M is a Mackey functor in $M(G,J)_k$ we denoted by M belngs to a block corresponding $e_{J,b}$ when $e_{J,b}$ acts trivial on M.

THEOREM 5 ([4](17.1) THEOREM)
Let $S_{P,V}$ and $S_{Q,W}$ be two simple Mackey functors, where P and Q are p-subgroups of G. Let e be the block of $N_G(P)$ which V belongs to and f the block of $N_G(Q)$ which W belongs to. Then $S_{P,V}$ and $S_{Q,W}$ belongs to the same block of Mackey functors if and only if the corresponding blocks e^G and f^G of G are equal. Thus the blocks of $M(G,1)_k$ are in bijection with the blocks of kG, via the map sending the block containing $S_{P,V}$ to the block e^G of kG.

The last remark is a basic for the block theory of Mackey functors. Let **B** be a block of $M(G,1)_k$ corresponding a block B of G the sense of Theorem 5 . We also let $\delta(\mathbf{B})$ denote the maximal p-subgroup of G, unique up to conjugacy in G,such that the simple Mackey functor $S_{P,V}$ in **B**.

REMARK 6
$\delta(\mathbf{B})$ is equal to the defect group of B.

proof
Let D be a defect group of B. So there is an indecomposable kG-module U lying in B such that D is a vertex of U. If V is a Green correspondent of U then there is a block b of $N_G(D)$ such that B is a Brauer correspondent of b which contains V. Let T be a composition factor of V so T lies in b ,hence simple Mackey functor $S_{D,T}$ belongs to **B**,by theorem 5. It suffices to show the maximality of D.

For each simple Mackey functor $S_{P,V}$ in **B**, there is a block e of $N_G(P)$ such that simple $kN_G(P)$-module V lies in e and $e^G = B$ is defined, by theorem 5. Hence the defect group D of B contains in defect group Δ of e.Since $\Delta$is the largest normal p-subgroup of $N_G(P)$,the remark is proved.

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Department of Mathematics
Hokkaido University
Sapporo 060, Japan

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ON FINITE INDECOMPOSABLE RINGS

Yasuyuki HIRANO and Takao SUMIYAMA

In what follows, when S is a finite set, $|S|$ denotes the number of elements of S.

Let R be a finite ring whose order is $|R| = p_1^{e_1}p_2^{e_2}\cdots p_t^{e_t}$, where $p_1, p_2, \cdots, p_t$ are distinct primes. Then R is uniquely decomposable as the direct sum of ideals $I_1, I_2, \cdots, I_t$ of orders $p_1^{e_1}, p_2^{e_2}, \cdots, p_t^{e_t}$. So, if R is a finite indecomposable ring, then $|R|$ is a power of a prime. A ring R is called a finite p-ring if $|R|$ is a power of a prime p.

Let R be a finite indecomposable ring with identity. Let R^* denote the group of units of R. If $|R|$ is not a power of 2, then $|R| \le |R^*|^2$ by [3, Lemma 2.3]. If $|R|$ is a power of 2, then $|R| \le 2|R^*|^2$ by [1, Proposition]. Let $Rad(R)$ denote the Jacobson radical of R. As $1 + Rad(R)$ is a subgroup of R^*, $|Rad(R)|$ is a divisor of $|R^*|$. The first purpose of this note is to estimate $|R|$, $|R^*|$ and $\delta(R) = |R^*|/|R|$ using $|Rad(R)|$.

First of all, we shall state a lemma, which is easy, but plays an important role to prove our theorems. In what follows, a graph means a finite undirected graph without loops. The edge which joins two vertices x and y is denoted by (x, y) or (y, x).

The detailed version of this paper will be submitted for publication elsewhere.

Lemma 1. Let $G = (V, E)$ be a non-trivial connected graph, where V is the set of vertices of G and E is the set of edges of G. Let u be a vertex in V. Then there exists an injective mapping $\varphi : V \setminus \{u\} \longrightarrow E$ which satisfies (i) for any $v \in V \setminus \{u\}$, one of the endpoints of $\varphi(v)$ is v, and (ii) $\varphi(w) = (w, u)$ for some $w \in V \setminus \{u\}$.

Before stating our theorems, we shall define some rings. Let e_{ij} denote the (i, j) matrix unit of the $(m + 1) \times (m + 1)$ matrix ring $M_{m+1}(GF(p))$ over $GF(p)$. The subring $GF(p)e_{11} + \Sigma_{i=2}^{m+1}\Sigma_{j=1}^{m+1} GF(p)e_{ij}$ of $M_{m+1}(GF(p))$ will be denoted by $A_{m+1}(p)$.

Let σ be a fixed automorphism of $GF(p^m)$. The ring $\left\{\begin{pmatrix} a & b \\ 0 & \sigma(a)\end{pmatrix} \mid a, b \in GF(p^m)\right\}$ will be denoted by $B_\sigma(p^m)$.

For a prime p and positive integers n and t, there exists uniquely a Galois extension of $\mathbb{Z}/\mathbb{Z}p^n$ of degree t (see [2, p. 307]), which we denote by $GR(p^n, t)$.

Let R be a finite indecomposable ring with identity. Let $R/\mathrm{Rad}(R) = M_{n_1}(K_1) \oplus \cdots \oplus M_{n_s}(K_s)$, where $K_i = GF(p^{k_i})$ $(1 \leq i \leq s)$. Suppose that $s \geq 2$. Let e_i denote the identity of $M_{n_i}(K_i)$. Then R contains orthogonal idempotents $f_1, f_2, \ldots, f_s$ whose sum is 1 such that $\pi(f_i) = e_i$ $(1 \leq i \leq s)$, where $\pi : R \longrightarrow R/\mathrm{Rad}(R)$ is the natural projection. We shall define a graph $G = (V, E)$ as follows. Let $V = \{1, 2, \ldots, s\}$. Two distinct vertices $i, j \in V$ are joined by an edge (i, j) if either $f_iRf_j \neq 0$ or $f_jRf_i \neq 0$. Let E be the set of such edges (i, j). Then we get

$$\Sigma_{(i,j)\in E}\, n_in_j(\mathrm{l.c.m.}\{k_i, k_j\}) \leq m .$$

This equality together with Lemma 1 yields the following theorems.

Theorem 2. Let p be a prime, and R a finite indecomposable p-ring with identity. Suppose $|\mathrm{Rad}(R)| = p^m$ $(m \geq 1)$. Then

$$p^{m+1} \leq |R| \leq p^{m^2+m+1} .$$

The first equality holds if and only if $R/\mathrm{Rad}(R) = GF(p)$. The second equality holds if and only if R is isomorphic to either $A_{m+1}(p)$ or its opposite ring $A_{m+1}(p)^{op}$.

Theorem 3. Let p be a prime, and R a finite indecomposable p-ring with identity. Suppose $|\mathrm{Rad}(R)| = p^m$ $(m \geq 1)$.

(I) If $p = 2$ and $m = 2$, then

$4 \leq |R^*| \leq 36$.

The first equality holds if and only if $R/\mathrm{Rad}(R) = GF(2) \oplus \cdots \oplus GF(2)$. The second equality holds if and only if R is isomorphic to $\begin{pmatrix} GF(4) & GF(4) \\ 0 & GF(4) \end{pmatrix}$.

(II) If $p \neq 2$ or $m \neq 2$, then

$$p^m(p-1) \leq |R^*| \leq p^{\frac{1}{2}m(m+1)}(p-1)(p-1)(p^2-1)\cdots(p^m-1) .$$

When $p = 2$, the first equality holds if and only if $R/\mathrm{Rad}(R) = GF(2) \oplus \cdots \oplus GF(2)$. When $p \neq 2$, the first equality holds if and only if $R/\mathrm{Rad}(R) = GF(p)$. The second equality holds if and only if R is isomorphic to either $A_{m+1}(p)$ or $A_{m+1}(p)^{op}$.

Theorem 4. Let p be a prime, and R a finite indecomposable p-ring with identity. Suppose $|\mathrm{Rad}(R)| = p^m$ $(m \geq 1)$. Let $\delta(R) = |R^*|/|R|$. Then

$$(1 - 1/p)^{m+1} \leq \delta(R) \leq 1 - 1/p^m .$$

The first equality holds if and only if R is isomorphic to either $GR(p^2, m)$ or $B_\sigma(p^m)$. The second equality holds if and only if R is an algebra over $GF(p)$ such that $R/\mathrm{Rad}(R) = \oplus_{m+1} GF(p)$.

Example 5. The subring $\Sigma_{i=1}^{m+1} GF(p)e_{1i} + \Sigma_{j=2}^{m+1} GF(p)e_{jj}$ of $M_{m+1}(GF(p))$ satisfies the second equality of Theorem 4.

By Theorem 2, we see that the order of a finite indecomposable ring R with identity is limited under the condition $|\mathrm{Rad}(R)| = p^m$ $(m \geq 1)$. We want to know how many finite indecomposable rings of order p^e there are. In [4], Wiesenbauer has established the algorism to determine all finite rings which have the same abelian group as their additive group.

Let p be a prime, and $1 \leq e_1 \leq e_2 \leq \cdots \leq e_n$ be a nondecreasing sequence of positive integers. Let $S_n = \{(a_{ij}) \in M_n(\mathbb{Z}) \mid a_{ij} \equiv 0 \mod p^{e_j - e_i} \text{ if } i < j\}$. Clearly S_n is a subring of $M_n(\mathbb{Z})$. For (a_{ij}) , $(b_{ij}) \in S_n$, we shall write $(a_{ij}) \equiv (b_{ij})$ if $a_{ij} \equiv b_{ij} \mod p^{e_j}$ $(1 \leq i, j \leq n)$. Let $I = \{(a_{ij}) \in S_n \mid (a_{ij}) \equiv 0\}$, which is an ideal of S_n . Let $\bar{S}_n = S_n/I$. Let $\pi : S_n \longrightarrow \bar{S}_n$ be the natural projection. An element $(a_{ij}) \in S_n$ is called non-singular if $\pi((a_{ij}))$ is an invertible element of $\bar{S}_n$.

Let $C(p^m)$ denote the cyclic group of order p^m . An abelian group A will be called of type $(p^{e_1}, p^{e_2}, \cdots, p^{e_n})$ if $A \simeq C(p^{e_1}) \oplus C(p^{e_2}) \oplus \cdots \oplus C(p^{e_n})$. Let $\langle a \rangle$ denote the cyclic group generated by a .

Theorem 6 ([4, Satz 2]). Let R be a finite p-ring whose additive group is $R^+ = \langle a_1 \rangle \oplus \langle a_2 \rangle \oplus \cdots \oplus \langle a_n \rangle$, where $\langle a_i \rangle \simeq C(p^{e_i})$ $(1 \leq i \leq n)$ and $1 \leq e_1 \leq e_2 \leq \cdots \leq e_n$. Let us write

(1) $\quad a_i a_k = \Sigma_{j=1}^{n} \alpha_{ijk} a_j$,

where α_{ijk} are integers such that

(2) $\quad 0 \leq \alpha_{ijk} \leq p^{e_j} - 1 \quad (1 \leq i, j, k \leq n)$.

Then it holds that

(3) $\quad \alpha_{ijk} \equiv 0 \mod p^{e_j - e_k}$ if $j > k$,

(4) $\quad \alpha_{ijk} \equiv 0 \mod p^{e_j - e_i}$ if $i < j$

and

(5) $\quad \Sigma_{k=1}^{n} \alpha_{rki}\alpha_{kjs} \equiv \Sigma_{k=1}^{n} \alpha_{iks}\alpha_{rjk} \mod p^{e_j}$

for every $1 \leq i, j, r, s \leq n$.

Conversely, let

(6) $\quad A = \langle a_1 \rangle \oplus \langle a_2 \rangle \oplus \cdots \oplus \langle a_n \rangle$

$\quad\quad (\langle a_i \rangle \simeq C(p^{e_i})$, $1 \leq e_1 \leq e_2 \leq \cdots \leq e_n)$

be an abelian group. If α_{ijk} $(1 \leq i, j, k \leq n)$ are integers which satisfy (2), (3), (4) and (5), then we can make A into a ring by defining the multiplication by (1). By this manner we can construct all rings which have A as their additive group.

By Theorem 6 we can get all rings of order p^e, since the additive group of any ring of order p^e is of type $(p^{e_1}, p^{e_2}, \cdots, p^{e_n})$, where $e = e_1 + e_2 + \cdots + e_n$ and $1 \leq e_1 \leq e_2 \leq \cdots \leq e_n$.

When α_{ijk} $(1 \leq i, j, k \leq n)$ are integers which satisfy (2), (3), (4) and (5), we call $\{\alpha_{ijk}\}_{i,j,k=1}^{n}$ structure constants for the abelian group (6).

Let $\{\alpha_{ijk}\}_{i,j,k=1}^{n}$ and $\{\beta_{ijk}\}_{i,j,k=1}^{n}$ be two sets of structure constants for the abelian group (6). We shall say that $\{\alpha_{ijk}\}_{i,j,k=1}^{n}$ and $\{\beta_{ijk}\}_{i,j,k=1}^{n}$ are equivalent if there exists a non-singular element $(t_{ij}) \in S_n$ such that
$\Sigma_{j=1}^{n} \beta_{ijk}t_{js} \equiv \Sigma_{j=1}^{n}\Sigma_{r=1}^{n} t_{ij}t_{kr}\alpha_{jsr}$ mod p^{e_s} $(1 \leq i, k, s \leq n)$.

Theorem 7 ([4, Satz 5]). Let $\{\alpha_{ijk}\}_{i,j,k=1}^{n}$ and $\{\alpha'_{ijk}\}_{i,j,k=1}^{n}$ be two sets of structure constants for (6). Let R be the ring whose additive group is (6) and whose multiplication is defined by

$$a_i a_k = \Sigma_{j=1}^{n} \alpha_{ijk} a_j .$$

Let R' be the ring whose additive group is (6) and whose multiplication is defined by

$$a_i \circ a_k = \Sigma_{j=1}^{n} \alpha'_{ijk} a_j .$$

Then R and R' are isomorphic if and only if $\{\alpha_{ijk}\}_{i,j,k=1}^{n}$ and $\{\alpha'_{ijk}\}_{i,j,k=1}^{n}$ are equivalent.

Next we shall consider the algorism to see whether a ring has identity and whether a ring is indecomposable. In the following, δ_{ij} is the Kronecker's delta.

Theorem 8 (cf. [4, Satz 6]). Let $\{\alpha_{ijk}\}_{i,j,k=1}^{n}$ be structure constants for (6). Let R be the ring whose additive group is (6) and whose multiplication is defined by

$$a_i a_k = \Sigma_{j=1}^{n} \alpha_{ijk} a_j .$$

Then R has a left identity if and only if there exist integers $c_1, c_2, \cdots, c_n$ such that $0 \leq c_i \leq p^{e_i} - 1$ $(1 \leq i \leq n)$ and $\Sigma_{i=1}^{n} c_i\alpha_{ijk} \equiv \delta_{jk}$ mod p^{e_j} $(1 \leq j, k \leq n)$. Also, R has a right identity if and only if there exist integers $c_1, c_2, \cdots, c_n$ such that $0 \le c_i \le p^{e_i} - 1$ $(1 \le i \le n)$ and $\Sigma_{k=1}^{n} c_k \alpha_{ijk} \equiv \delta_{ij} \mod p^{e_j}$ $(1 \le i, j \le n)$.

Let $\{\alpha_{ijk}\}_{i,j,k=1}^{n}$ be structure constants for (6). We shall call $\{\alpha_{ijk}\}_{i,j,k=1}^{n}$ decomposable if there exists a partition $\{1, 2, \cdots, n\} = I_1 \cup I_2$ such that (i) $I_1 \cap I_2 = \phi$, (ii) $I_1 \ne \phi$, $I_2 \ne \phi$, (iii) if $i \in I_1$, $j \in I_2$ and $e_i = e_j$, then $i < j$, and (iv) if $i \in I_1$, $j \in I_2$ or $i \in I_2$, $j \in I_1$ or $j \in I_1$, $k \in I_2$ or $j \in I_2$, $k \in I_1$, then $\alpha_{ijk} = 0$.

Theorem 9. Let $\{\alpha_{ijk}\}_{i,j,k=1}^{n}$ be structure constants for (6). Let R be the ring whose additive group is (6) and whose multiplication is defined by

$$a_i a_k = \Sigma_{j=1}^{n} \alpha_{ijk} a_j \quad .$$

Then R is indecomposable if and only if the following condition is satisfied: if $\{\beta_{ijk}\}_{i,j,k=1}^{n}$ are structure constants for (6) which are equivalent to $\{\alpha_{ijk}\}_{i,j,k=1}^{n}$, then $\{\beta_{ijk}\}_{i,j,k=1}^{n}$ are not decomposable.

Example 10. Theorems 7, 8 and 9 enable us to determine all rings of order $2^3 = 8$. By computer, we see that there are 52 mutually nonisomorphic rings of order 8. Among them 7 rings are indecomposable and with identity.

Number of rings of order 8.

type of the additive group	number of rings	number of indecomposable rings	number of rings with 1	number of indecomposable rings with 1
(p^3)	4	4	1	1
(p, p^2)	20	14	3	2
(p, p, p)	28	14	7	4

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Department of Mathematics
Faculty of Science
Okayama University
Okayama 700, Japan

Department of Mathematics
Aichi Institute of Technology
Yakusa-chō, Toyota 470-03, Japan

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Remarks on a family of algebras with the same number of simple modules*

Kunio YAMAGATA

Dedicated to Professor Manabu Harada on the occasion of his 60th birthday

In [Y] we constructed from a given algebra A a family of algebras $A_n(n \geq 0, A_0 = A)$ with the same number of simple modules as the number of simple A-modules. This family has the following properties: the Cartan determinant of A_n is the same as the Cartan determinant of A, and $\operatorname{gl\,dim} A_n < \operatorname{gl\,dim} A_{n+1}$ for all n if $\operatorname{gl\,dim} A$ is finite. This construction generalizes the family of algebras $A_n(n \geq 0)$ with two simple modules given by Green [G] satisfying the property that $\operatorname{gl\,dim} A_n = n$ for all n. Happel [H] studied homological properties of the algebras by Green and proved the property on the global dimension, and Sikko and Smalø[SS] also gave a simple homological proof. In this note, we shall show a "splitting property" of morphisms over the algebras, with two simple modules, constructed by the rule in [Y], and we shall give another simple proof to the above fact on the global dimensions of algebras by Green.

Throughout this paper, all rings are semi-primary basic rings with identity, and modules are left modules . By M^n we denote a direct sum of n copies of a module M.

1 Splitting property of morphisms

1.1 We recall from [Y] the construction condition of a family of algebras which implies a construction of algebras with the properties mentioned in introduction.

Let $A_n(n \geq 0)$ be semi-primary basic rings, $A := A_0$, and $p_n : A_n \to A_{n-1}$ ring homomorphisms such that, for a fixed number $m \geq 2$, every A_n is the direct sum of projective modules $P_{n,1}, \ldots, P_{n,m}$, and $p_n(P_{n,i}) = P_{n-1,i}$ for $n \geq 1$ and all i. Let $p_{n,i} : P_{n,i} \to P_{n-1,i}$ be the restriction of p_n. Note that every $P_{n-j,i}$ is considered as an A_n-module by the composite $p_{n-j+1} \cdots p_n : A_n \to A_{n-j}$. We consider the following two conditions.

*This paper is in final form and no version of it will be submitted for publication elsewhere.

(C1) For every $n > 0$, there are exact sequences of A_n-modules:

$$\begin{array}{cccccccc} 0 \longrightarrow & P_{n-1,1}^{\alpha_{n,m}} & \xrightarrow{q_{n,m}} & P_{n,m} & \xrightarrow{p_{n,m}} & P_{n-1,m} & \longrightarrow 0 \\ 0 \longrightarrow & P_{n,m}^{\alpha_{n,m-1}} & \xrightarrow{q_{n,m-1}} & P_{n,m-1} & \xrightarrow{p_{n,m-1}} & P_{n-1,m-1} & \longrightarrow 0 \\ & & & \cdots\cdots\cdots & & & \\ 0 \longrightarrow & P_{n,2}^{\alpha_{n,1}} & \xrightarrow{q_{n,1}} & P_{n,1} & \xrightarrow{p_{n,1}} & P_{n-1,1} & \longrightarrow 0 \end{array}$$

where every $\alpha_{n,i}$ is positive integer.

(C2) Every $p_{n,i} : P_{n,i} \to P_{n-1,i}$ is a projective cover in the category of A_n-modules, and the restriction $p_{n,i} \mid \operatorname{rad} P_{n,i} \to \operatorname{rad} P_{n-1,i}$ is a splittable epimorphism.

Note that it follows from (C2) that

$$\operatorname{rad} P_{n,i} \xrightarrow{\sim} P_{n,i+1}^{\alpha_{n,i}} \oplus \operatorname{rad} P_{n-1,i} \quad \textit{for } 1 \le i < m,$$

$$\operatorname{rad} P_{n,m} \xrightarrow{\sim} P_{n-1,1}^{\alpha_{n,m}} \oplus \operatorname{rad} P_{n-1,m}.$$

In the sequel the indecies of the morphisms $p_{n,i}$ and $q_{n,i}$ are omitted for simplicity unless it implies confusion.

1.2 Let I be the left submodule $q_{1,m}(P_{0,1}^{\alpha_{1,m}})$ of $P_{1,m}$. From now on we fix a positive integer n and consider in the category of left A_n-modules.

For a finitely generated projective A_n-module P, let IP be the submodule of P generated by $f(I)$ for all morphisms f from $P_{1,m}$ to P. Since $q(P_{0,1}^{\alpha_{1,m}})$ is contained in $\operatorname{Ker}(A_1 \xrightarrow{p} A_0)$, note that $IA_0 = 0$ and so $f(IP_{1,i}) = 0$ for $f \in \operatorname{Hom}_{A_n}(P_{1,i}, P_{0,j})$. Moreover, $IP_{n,i}$ belongs to $\operatorname{add}_{A_n}(I)$. Because, let $(n,i) \ne (1,m)$ [otherwise the assertion is trivial]. Then $\operatorname{Im} f \subset \operatorname{rad} P_{n,i}$ for any morphism $f : P_{1,m} \to P_{n,i}$ since $P_{n,i}$ has no summands isomorphic to summands of $P_{1,m}$. Hence, in the case when $i = m$, we have a morphism $\binom{f_1}{f_1'} : P_{1,m} \xrightarrow{f} \operatorname{rad} P_{n,m} \xrightarrow{\sim} P_{n-1,1}^{\alpha_{n,m}} \oplus \operatorname{rad} P_{n-1,m}$ so that $IP_{n,m} \xrightarrow{\sim} IP_{n-1,1}^{\alpha_{n,m}} \oplus IP_{n-1,m}$. In case $i < m$, we also have a morphism $\binom{f_1}{f_1'} : P_{1,m} \xrightarrow{f} \operatorname{rad} P_{n,i} \xrightarrow{\sim} P_{n,i+1}^{\alpha_{n,i}} \oplus \operatorname{rad} P_{n-1,i}$ so that $IP_{n,i} \xrightarrow{\sim} IP_{n,i+1}^{\alpha_{n,i}} \oplus IP_{n-1,i}$. In consequence, by induction we conclude that $IP_{n,i} \in \operatorname{add}_{A_n}(I)$.

Lemma *Let $f : P_{n,i} \to P_{n-t,j}$ be a morphism with $f(P_{n,i}) \subset \operatorname{rad} P_{n-t,j}$. Then $f(IP_{n,i}) = 0$ in the case when $(1)\, t \ge 1$ or $(2)\, t = 0$ and $i \le j$.*

Proof. (1) It suffices to show that $fg(I) = 0$ for $g \in \operatorname{Hom}(P_{1,m}, P_{n,i})$. Since $f(P_{n,i}) \subset \operatorname{rad} P_{n-t,j}$, by applying the condition (C2) repeatedly we know that f is a sum of morphisms through $P_{0,k} (1 \le k \le m)$, say $f = \sum_k f_k' f_k$ where $f_k \in \operatorname{Hom}(P_{n,i}, A_0)$ and $f_k' \in \operatorname{Hom}(A_0, P_{n-t,j})$. Hence, for $g \in \operatorname{Hom}(P_{1,m}, P_{n,i})$, we have that $fg \in \operatorname{Hom}(A_0, P_{n-t,j}) \circ \operatorname{Hom}(P_{1,m}, A_0)$ so that $fg(I) = 0$ because, from the above note $h(I) = 0$ for any $h \in \operatorname{Hom}(P_{1,m}, A_0)$.

(2) This follows from (1) because by (C2) f is a sum of morphisms through some objects in $\mathrm{add}_{A_n}(A_{n-1})$.

1.3 For a morphism $f : X \to Y$, let $f_1 : X \to f(X), f_2 : f(X) \to Y$ be the canonical morphisms with $f = f_2 f_1$. Then f is said to be *splittable* if f_1 is a splittable epimophism and f_2 is a splittable monomorphism.

Lemma *Suppose that every $P_{0,i}$ $(1 \le i \le m)$ is indecomposable and let u, v be positive integers. Then, for a morphism $f : P^u_{n,i} \to P^v_{n,j}$ with $f(P^u_{n,i}) \subset \mathrm{rad}\, P^v_{n,j}$, the restriction $f | IP^u_{n,i} \to IP^v_{n,j}$ is zero or splittable.*

Proof. Suppose that $f(IP^u_{n,i})$ is not zero. In case $i \le j$, $f(IP^u_{n,i}) = 0$ by Lemma 1.2, so we assume that $j < i$. By the condition (C2), the morphism $f, P^u_{n,i} \to \mathrm{rad}\, P^v_{n,j}$ induced from f canonically is isomorphic to a sum of morphisms $f_1 : P^u_{n,i} \to (P^{\alpha_{n,j}}_{n,j+1})^v$ and $f_1' : P^u_{n,i} \to \mathrm{rad}\, P^v_{n-1,j}$

$$
\begin{array}{ccccccc}
 & & & P^u_{n,i} & & & \\
 & & \swarrow^{f_1} & \downarrow^{f_0} & \searrow^{f_1'} & & \\
0 \to (P^{\alpha_{n,j}}_{n,j+1})^v & & \xrightarrow{\oplus q_{n,j}} & P^v_{n,j} & \xrightarrow{\oplus p_{n,j}} & & P^v_{n-1,j} \to 0,
\end{array}
$$

where $f_1(IP^u_{n,i}) \xrightarrow{\sim} f(IP^u_{n,i}) \ne 0$ since $f_1'(IP^v_{n,i}) = 0$ by Lemma 1.2, and the restriction of $\oplus q_{n,j}$, $I(P^{\alpha_{n,j}}_{n,j+1})^v \to IP^v_{n,j}$, is a splittable monomorphism. Let $f_0 = f$, then the above procedure for f_0 to obtain the morphism f_1 is valid for f_1 in the case when $\mathrm{Im}\, f_1$ is contained in the radical of the range. So, by repeating it, we finally have the morphisms $f_k : P^u_{n,i} \to P^\alpha_{n,j+k}, q' : P^\alpha_{n,j+k} \to P^v_{n,j}$ for some integer α and $k < i - j$ such that $f_k(IP^u_{n,i}) \xrightarrow{\sim} f(IP^u_{n,i})$, the morphism $IP^\alpha_{n,j+k} \to IP^v_{n,j}$ induced from q' is a splittable monomorphism and $f_k(P^u_{n,i}) \not\subset \mathrm{rad}\, P^\alpha_{n,j+k}$. Then we have that $P_{n,i} \simeq P_{n,j+k}$ because they are indecomposable projectives. Considering f_k as a matrix over the local ring $\mathrm{End}\,(P_{n,i})$, by elementary operations we know the existence of the morphism $g := \begin{pmatrix} g_1 & g_{12} \\ 0 & g_2 \end{pmatrix}$: $P_1 \oplus P_2 \longrightarrow Q_1 \oplus Q_2$ isomorphic to f such that $g_1 : P_1 \to Q_1$ is an isomorphism, $g_2(P_2) \subset \mathrm{rad}\, Q_2$ and $g_{12}(P_2) \subset \mathrm{rad}\, Q_1$. Here note that $g_{12}(IP_2) = g_2(IP_2) = 0$ by Lemma 1.2. Therefore, we have a restriction g' of g such that $g' = \begin{pmatrix} g'_{IP_1} & 0 \\ 0 & 0 \end{pmatrix}$: $IP_1 \oplus IP_2 \longrightarrow IQ_1 \oplus IQ_2$, where g'_{IP_1} is an isomorphism. Consequently, g' is splittable and so is f.

1.4 Proposition *Assume that $m = 2$ and $P_{0,1}, P_{0,2}$ are indecomposable A-modules. Let $f : P \to Q$ be a morphism between finitely generated projective A_n-modules. Then $f(IP) = 0$ or $f \mid IP \to IQ$ is splittable.*

Proof. Suppose that $f(IP) \ne 0$, and let $P = P^u_{n,1} \oplus P^v_{n,2}$, $Q = Q^s_{n,1} \oplus Q^t_{n,2}$ for some non-negative integers u, v, s, t. [Here X^k stands for the zero module for a module X and $k = 0$.] According to this decomposition, put $f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$: $P^u_{n,1} \oplus P^v_{n,2} \longrightarrow Q^s_{n,1} \oplus Q^t_{n,2}$. Then, since $f_{ji}(IP_{n,i}) = 0$ for $i \le j$ by Lemma 1.2, the restriction $f_{12} \mid$

$IP^v_{n,2} \to IQ^s_{n,1}$ is not zero so that it is splittable by Lemma 1.3. Since $IP = IP^u_{n,1} \oplus IP^v_{n,2}$ and $IQ = IQ^s_{n,1} \oplus IQ^t_{n,2}$, we therefore conclude that f is splittable.

Note. Let R be a semi-primary ring with a heredity ideal $ReR\,(e^2 = e)$. Let $\mathrm{P}(R)$ be the category of finitely generated projective R-modules and $I = Re$. Then $IP = ReP$ for $P \in \mathrm{P}(R)$ and, in this case, any morphism in $\mathrm{P}(R)$ satisfes the splitting property in the above Proposition. But, it should be noted that $\mathrm{End}_{A_n}(I)$ is not in general a division ring.

2 Global dimensions

In this section we consider the global dimension of $A_n (n \geq 0)$ in the case when $m = 2$. The result is contained in [Y], but we give a direct and shorter proof. Now we assume that $m = 2$, but not assume that $P_{0,1}$, $P_{0,2}$ are indecomposable. All modules are considered in the category of left A_n-modules for a fixed integer $n \geq 0$.

2.1 Lemma (1) $\mathrm{pd}\,P_{n-i,1} = 2i - 1$, $\mathrm{pd}\,P_{n-i,2} = 2i$ *for* $i \geq 1$.
(2) *For* $0 \leq i < n$, $\mathrm{pd}\,(\mathrm{rad}\,P_{n-i,1}) = \max\{2(n-1),\ \mathrm{pd}\,(\mathrm{rad}\,P_{0,1})\}$
$\mathrm{pd}\,(\mathrm{rad}\,P_{n-i,2}) = \max\{2n - 1,\ \mathrm{pd}\,(\mathrm{rad}\,P_{0,2})\}$.

Proof. (1) For $i = 1$, the assertion is trivial by (C1). We prove by induction on i. Since $\mathrm{pd}\,P_{n-i,1} < \mathrm{pd}\,P_{n-i,2}$ by induction hypothesis, it follows from the short exact sequence

$$0 \longrightarrow P_{n-i,2}^{\alpha_{n-i,1}} \xrightarrow{\ q\ } P_{n-i,1} \xrightarrow{\ p\ } P_{n-(i+1),1} \longrightarrow 0,$$

that $\mathrm{pd}\,P_{n-(i+1),1} = 1 + \mathrm{pd}\,P_{n-i,2} = 2(i+1) - 1$. Moreover, since $\mathrm{pd}\,P_{n-i,2} < \mathrm{pd}\,P_{n-(i+1),1}$ in particular, by the exact sequence

$$0 \longrightarrow P_{n-(i+1),1}^{\alpha_{n-i,2}} \xrightarrow{\ q\ } P_{n-i,2} \xrightarrow{\ p\ } P_{n-(i+1),2} \longrightarrow 0,$$

we have that $\mathrm{pd}\,P_{n-(i+1),2} = 1 + \mathrm{pd}\,P_{n-(i+1),1} = 2(i+1)$.

(2) It follows from (C2) that $\mathrm{rad}\,P_{n-i,1} \xrightarrow{\sim} P_{n-i,2}^{\alpha_{n,i}} \oplus \mathrm{rad}\,P_{n-(i+1),1}$ for $i \geq 1$. Hence, by (1), $\mathrm{pd}\,(\mathrm{rad}\,P_{n-i,1}) = \max\{2i,\ \mathrm{pd}\,(\mathrm{rad}\,P_{n-(i+1),1})\}$. Similarly we have that $\mathrm{pd}\,(\mathrm{rad}\,P_{n-i,2}) = \max\{2i+1,\ \mathrm{pd}\,(\mathrm{rad}\,P_{n-(i+1),2})\}$. Thus the assertion is an easy consequence of these equalities.

From this lemma we have the following fact for $n > 0$.

Corollary $\mathrm{pd}\,(\mathrm{top}\,P_{n,1}) = \max\{2n-1,\ 1 + \mathrm{pd}\,(\mathrm{rad}\,P_{0,1})\}$
$\mathrm{pd}\,(\mathrm{top}\,P_{n,2}) = \max\{2n,\ 1 + \mathrm{pd}\,(\mathrm{rad}\,P_{0,2})\}$
$\mathrm{gl\,dim}\,A_n = \max\{2n,\ 1 + \mathrm{pd}\,(\mathrm{rad}\,A)\}$.

2.2 Proposition (1) *If A is semisimple, then* $\mathrm{gl\,dim}\,A_n = 2n$ *for* $n \geq 0$. (2) *If $P_{0,2}$ is semisimple and* $\mathrm{add}(\mathrm{rad}\,P_{0,1}) = \mathrm{add}(P_{0,2})$, *then* $\mathrm{gl\,dim}\,A_n = 2n + 1$.

Proof. (1) is trivial. (2) Since $\mathrm{rad}\, P_{0,2} = 0$, it holds that $\mathrm{pd}\,(\mathrm{top}\, P_{n,2}) = 2n$. Moreover $P_{0,2} = \mathrm{top}\,(P_{0,2}) \simeq \mathrm{top}\,(P_{n,2})$, and $\mathrm{pd}\,(\mathrm{rad}\, P_{0,1}) = \mathrm{pd}\,(P_{0,2})$ since $\mathrm{add}(\mathrm{rad}\, P_{0,1}) = \mathrm{add}(P_{0,2})$. Therefor we have that $\mathrm{pd}\,(\mathrm{rad}\, P_{0,1}) = 2n$.

2.3 Let $\Lambda_n\,(n \geq 1)$ be the algebra over an algebraic closed field with two vertices $\{1, 2\}$ and n arrows : a_{2i} from 2 to 1, a_{2i-1} from 1 to 2 for $i \geq 1$. The relations are $\alpha_j\alpha_i = 0$ for $j \leq i$. Let Λ_0 be the algebra with vertices $\{1, 2\}$ but without arrows ([G], [H]). Then we can apply the above Corollary to the algebras Λ_n as follows .

Let $f_k : \Lambda_k \rightarrow \Lambda_{k-2}\,(k \geq 2)$ be algebra homomorphism such that $f_k(a_1) = f_k(a_2) = 0$ and $f_k(a_{i+2}) = a_i$ for $i > 0$. Let e_i be the idempotent corresponding to the vertex $i(= 1, 2)$, then $a_{2i} = e_1 a_{2i} e_2$, $a_{2i-1} = e_2 a_{2i-1} e_1$, and $\Lambda_k = \Lambda_k e_1 \oplus \Lambda_k e_2$. For the restrictions $f_{k,i} : \Lambda_k e_i \rightarrow \Lambda_{k-2} e_i$ induced from f_k naturally, it is easily seen that $\mathrm{Ker}\, f_{k,2} \simeq \Lambda_{k-2} e_1$ and $\mathrm{Ker}\, f_{k,1} \simeq \Lambda_{k-2} e_2$. In consequence, the families $\{A_n = \Lambda_{2n}, P_{n,i} = \Lambda_{2n} e_i, p_n = f_{2n}\}$ and $\{A_n = \Lambda_{2n+1}, P_{n,i} = \Lambda_{2n+1} e_i, p_n = f_{2n+1}\}$ satisfy the conditions (C1), (C2) with all $\alpha_{n,i} = 1$. Thus it follows from Proposition 2.2 that $\mathrm{gl\,dim}\, \Lambda_n = n$ for $n \geq 0$.

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Institute of Mathematics, University of Tsukuba,
Tsukuba, Ibaraki 305, Japan
E-mail: a906025@sakura.cc.tsukuba.ac.jp