PROCEEDINGS OF THE 25TH SYMPOSIUM ON RING THEORY

HELD AT SHINSHU UNIVERSITY, MATSUMOTO

AUGUST 1-3, 1992

EDITED BY

Yukio TSUSHIMA — Osaka City University

Yutaka WATANABE — Osaka Women's University

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OKAYAMA, JAPAN

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PREFACE

This is the proceedings of the 25th Symposium on Ring Theory held at Shinshu University on August 1 – 3, 1992. The symposium and these proceedings were financially supported by the Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture through the arrangements by Prof. Y. Morita of Tohoku University. We should like to thank him for his nice arrangements.

We should also like to thank Prof. K. Kishimoto and his group of Shinshu University for their hospitality, and Prof. H. Tominaga and Dr. H. Komatsu of Okayama University for making possible the publication of the proceedings in these beautiful forms.

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PROCEEDINGS OF THE 25TH
SYMPOSIUM ON RING THEORY, 1992

REPRESENTATION THEORY OF q-SCHUR ALGEBRAS AND HECKE ALGEBRAS

Mitsuhiro TAKEUCHI

The Schur algebra $S_K(d,n)$ is associated with polynomial representations of GL_d of degree n. If E is a d-dimensional K-vector space, the symmetric group $W = S_n$ acts naturally on $E^{\otimes n}$ on the right. Schur's reciprocity law tells that the Schur algebra $S_K(d,n)$ is isomorphic to the endomorphism algebra of the right KW module $E^{\otimes n}$. Here, the base field K is arbitrary and KW denotes the group algebra. This gives a direct relationship between degree n polynomial representations of GL_d (for various d) and representations of the group algebra KW. This classical theory was reviewed in my talk of the first day and will be omitted in this report. See Green's book [1].

Throughout the report, we fix the base field K, a non-zero element q in K, and the number n. We put $W = S_n$ the symmetric group on n letters. The q-Schur algebra $S_{K,q}(d,n)$, a q-analogue of $S_K(d,n)$, has been introduced by Dipper-James [2] to study unipotent representations of the finite general linear group $GL_n(q)$ with q a power of a prime. Later, it has been combined with degree n polynomial representations of some quantum GL_d [3] and its representation theory has been deduced from the representation theory of the Hecke algebra to establish a complete analogue of the classical theory [4,5].

The purpose of this note is to give an elementary introduction

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This expository paper is in final form, but there is some possibility for some material to overlap with the author's other papers.

to the representation theory of q-Schur algebras and Hecke algebras. I will also talk about my own contributions [6,7].

1. Quantum GL_d

Various versions of quantum GL_d have appeared. I consider it is most convenient to use the 2-parameter quantization of the author [6].

Take two non-zero elements α, β in K such that $\alpha\beta = q$. Define the K-algebra $A_{K,\alpha,\beta}(d)$ by generators $x_{11},\dots,x_{dd}$ and the following relations:

$$x_{ik}x_{ij} = \alpha x_{ij}x_{ik} \quad \text{if} \quad j < k,$$
$$x_{jk}x_{ik} = \beta x_{ik}x_{jk} \quad \text{if} \quad i < j,$$
$$x_{jk}x_{i\ell} = \beta\alpha^{-1}x_{i\ell}x_{jk}, \quad x_{j\ell}x_{ik} - x_{ik}x_{j\ell} = (\beta-\alpha^{-1})x_{i\ell}x_{jk} \quad \text{if} \quad i < j,\ k < \ell.$$

This algebra is a polynomial algebra in $x_{11},\dots,x_{dd}$. This means if we give an arbitrary total ordering on these generators, then the set of all monomials (relative to the given ordering) forms a basis. This algebra has the following bialgebra structure:

$$\Delta(x_{ik}) = \textstyle\sum_{j=1}^{d} x_{ij} \otimes x_{jk}, \quad \varepsilon(x_{ij}) = \delta_{ij}.$$

There is a group-like element g called the quantum determinant. It is defined by

$$g = \textstyle\sum_\sigma (-\alpha)^{-\ell(\sigma)} x_{1,\sigma(1)}\cdots x_{d,\sigma(d)} = \sum_\sigma (-\beta)^{-\ell(\sigma)} x_{\sigma(1),1}\cdots x_{\sigma(d),d}$$

the sum over all σ in S_d. The quantum determinant g satisfies

$$x_{ij}g = (\beta\alpha^{-1})^{i-j}gx_{ij}$$

so that the localization $A_{K,\alpha,\beta}(d)[g^{-1}]$ is defined by Ore's method. If we extend the bialgebra structure to this localization by making g^{-1} into a group-like element, then this becomes a Hopf algebra, i.e., a bialgebra with antipode. The two-parameter quantization of GL_d relative to (α,β) is defined to be the quantum group associated with the Hopf algebra $A_{K,\alpha,\beta}(d)[g^{-1}]$.

By a polynomial representation of the quantum GL_d, we mean a right comodule for $A_{K,\alpha,\beta}(d)$. As an algebra, it is graded:

$$A_{K,\alpha,\beta}(d) = \oplus_{n=0} A_{K,\alpha,\beta}(d,n)$$

where $A_{K,\alpha,\beta}(d,n)$ is the n-th component which is a subcoalgebra of finite dimension. Right comodules for $A_{K,\alpha,\beta}(d,n)$ are identified with polynomial representations of degree n.

The coalgebra structure of $A_{K,\alpha,\beta}(d)$ and $A_{K,\alpha,\beta}(d,n)$ depend on the product $\alpha\beta = q$. This means if we are given another pair of parameters α', β', then $\alpha\beta = \alpha'\beta'$ implies that

$$A_{K,\alpha,\beta}(d) \simeq A_{K,\alpha',\beta'}(d),\ A_{K,\alpha,\beta}(d,n) \simeq A_{K,\alpha',\beta'}(d,n)$$

as coalgebras. This property is called the hyperbolic invariance [8]. This yields we can put

$$\underline{\underline{\mathrm{com}}}\ A_{K,\alpha,\beta}(d) = \underline{\underline{M}}_{K,q}(d)$$

the category of all (finite dimensional) right $A_{K,\alpha,\beta}(d)$ comodules or the category of all (finite dimensional) polynomial modules for the quantum GL_d. We can also define the dual algebra

$$S_{K,q}(d,n) = A_{K,\alpha,\beta}(d,n)^*$$

which is called the q-Schur algebra. The category of all (finite dimensional) left modules for $S_{K,q}(d,n)$

$$\underline{\underline{\mathrm{mod}}}\ S_{K,q}(d,n) = \underline{\underline{M}}_{K,q}(d,n)$$

is identified with the category of (finite dimensional) polynomial modules of degree n.

The quantization associated with $\alpha = \beta$ is the most standard one , while Dipper-Donkin [3] uses the pair $(\alpha,\beta) = (1,q)$.

2. The Hecke algebra, a q-analogue of KW

We define the Hecke algebra $\mathcal{H} = \mathcal{H}_{K,q}(W)$ by generators $T_1,\ldots,T_{n-1}$ and the following defining relations

i) $(T_i-q)(T_i+1) = 0$,

ii) $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$,

iii) $T_iT_j = T_jT_i$ if $|i-j| > 1$.

A transposition of the form $s_a = (a,a+1)$ is called basic. We can write any permutation π in W as a product of $\ell = \ell(\pi)$ number of basic transpositions $\pi = s_{i_1}\ldots s_{i_\ell}$. Such an expression is called reduced. Then, the product $T_\pi = T_{i_1}\ldots T_{i_\ell}$ is well-defined, and the set T_π, π in W forms a base of the Hecke algebra $\mathcal{H}$. If $q = 1$, the Hecke algebra reduces to the group algebra KW. If q is a power of a prime number, then $\mathcal{H}_{K,q}(W)$ coincides with the Iwahori-Hecke algebra $H_K(G,B)$ where $G = GL_n(q)$ and B the upper Borel subgroup.

3. The reciprocity law, or the double centralizer property

Let E be a d dimensional K vector space with a base $e_1,\ldots,e_d$. The K space $E^{\otimes n}$ has a base

$$e_i = e_{i_1} \otimes \ldots \otimes e_{i_n},\ i = (i_1,\ldots,i_n) \text{ in } I(d,n)$$

where $I(d,n)$ denotes the set of all sequences i with $1 \leq i_1,\ldots,i_n \leq d$. This space has the following right $A_{K,\alpha,\beta}(d,n)$ comodule structure

$$e_j \mapsto \textstyle\sum_{i \text{ in } I(d,n)} e_i \otimes x_{ij}$$

where

$$x_{ij} = x_{i_1 j_1}\ldots x_{i_n j_n}.$$

Hence it is an object in $\underline{\underline{M}}_{K,q}(d,n)$ and has a left $S_{K,q}(d,n)$ module structure.

On the other hand, there is a right $\mathcal{H}$ module structure on $E^{\otimes n}$ described as follows: Let i be in $I(d,n)$ and let $s = (a,a+1)$ a basic transposition. We define

$$e_i T_s = \begin{cases} q e_{is} & \text{if } i_a = i_{a+1}, \\ \beta e_{is} & \text{if } i_a < i_{a+1}, \\ (q-1)e_i + \alpha e_{is} & \text{if } i_a > i_{a+1}. \end{cases}$$

One checks that this makes $E^{\otimes n}$ into a right $\mathcal{H}$ module whose structure also depends only on the product $q = \alpha\beta$.

The above structures make $E^{\otimes n}$ into a bimodule (left $S_{K,q}(d,n)$, right $\mathcal{H}$). The canonical algebra map

$$S_{K,q}(d,n) \to \mathrm{End}_{\mathcal{H}}(E^{\otimes n})$$

is an isomorphism. This is a q-analogue of Schur's reciprocity law [1,p.28].(See [3,8] for the proof). This characterizes the q-Schur algebra as the endomorphism algebra of some $\mathcal{H}$ module. It follows immediately that $S_{K,q}(d,n)$ is semi-simple if $\mathcal{H}$ is semi-simple. This occurs when $[n][n-1]\ldots[1] \neq 0$, where $[i] = 1 + q + \ldots + q^{i-1}$.

4. Compositions and partitions

A composition of n means a sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of integers ≥ 0 whose sum is n. Let $\mathcal{C}(n)$ be the set of all compositions of n which is the union of $\Lambda(d,n)$ for $d \geq 1$, where $\Lambda(d,n)$ denotes the subset of all compositions λ with $\lambda_a = 0$ if $a > d$.

A partition of n means a composition λ of n such that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots$ Let $\mathcal{P}(n)$ be the set of all partitions of n which is the union of $\Lambda^+(d,n) = \mathcal{P}(n) \cap \Lambda(d,n)$, $d \geq 1$. It coincides with $\Lambda^+(n,n)$. Let λ, μ be two compositions of n. We write $\lambda \sim \mu$ if there is a permutation σ such that $\lambda_{\sigma(i)} = \mu_i$ for all i. Obviously, the set $\mathcal{P}(n)$ forms a complete set of representatives for the equivalence relation $\sim$.

Each composition λ has the dual partition λ' defined by

$$\lambda'_a = \text{the number of } b \text{ with } \lambda_b \geq a.$$

For example, we have $(2,0,1,3,3,0,0,\ldots)' = (4,3,2,0,0,\ldots)$. We have $\lambda \sim \mu$ if and only if $\lambda' = \mu'$.

Some ordering, called the dominance ordering, is defined on $\mathcal{P}(n)$. Let λ, μ be two partitions. We write $\lambda \trianglelefteq \mu$ if

$$\lambda_1 + \ldots + \lambda_a \leq \mu_1 + \ldots + \mu_a \quad \text{for all } a \geq 1.$$

The largest (resp. smallest) element is (n) (resp. $(1,1,\ldots,1)$). We have $\lambda \trianglelefteq \mu$ if and only if $\mu' \trianglelefteq \lambda'$.

Some standard λ-tableau t^λ is associated with each composition λ as shown by the following example:

$$t^{2,1,3,4} = \begin{matrix} 1 & 2 & & \\ 3 & & & \\ 4 & 5 & 6 & \\ 7 & 8 & 9 & 10 \end{matrix}$$

(In general, compositions are denoted by finite sequences $(\lambda_1, \ldots, \lambda_d)$ for sufficiently large d.) Let $Y_\lambda = R(t^\lambda)$ the row stabilizer of t^λ. This subgroup of W is called the Young subgroup. A permutation d is distinguished relative to λ if the λ-tableau $d^{-1}(t^\lambda)$ is row-standard, i.e., if d^{-1} is increasing on each row of t^λ. Let $\mathcal{D}_\lambda$ be the set of all permutations distinguished relative to λ. Then we have

$$W = Y_\lambda \times \mathcal{D}_\lambda.$$

Let x_λ (resp. y_λ) be the sum of all T_π (resp. $(-q)^{-\ell(\pi)}T_\pi$) for π in Y_λ. These are q-analogues of the Young symmetrizer (resp. anti-symmetrizer).

The right ideal $x_\lambda\mathcal{H}$ has a K-basis $x_\lambda T_d$, d in $\mathcal{D}_\lambda$. For two compositions λ, μ, $\lambda \sim \mu$ implies $x_\lambda\mathcal{H} \simeq x_\mu\mathcal{H}$ and $y_\lambda\mathcal{H} \simeq y_\mu\mathcal{H}$ as right $\mathcal{H}$ modules.

5. Generalization of the q-Schur algebra

Let Λ be a finite subset of $\mathcal{C}(n)$. Let M_Λ be the direct sum of right $\mathcal{H}$ modules $x_\lambda\mathcal{H}$ for λ in Λ. We put

$$S_\Lambda = \mathrm{End}_{\mathcal{H}}(M_\Lambda).$$

The right $\mathcal{H}$ module $E^{\otimes n}$ in 3. is isomorphic to $M_{\Lambda(d,n)}$ [3,8]. It follows from the q-Schur reciprocity law that

$$S_q(d,n) = S_{\Lambda(d,n)}.$$

In this sense, our algebras S_Λ are generalizations of the q-Schur algebras $S_q(d,n)$.

The representation theory of the Hecke algebra $\mathcal{H}$ has been established in [4], and the representation theory of q-Schur algebras are deduced from it in [5]. We can apply this technique to deduce the representation theory of algebras S_Λ quite parallel to [5] (see [7]). The main results will be reviewed in the following.

For each λ in Λ, let $\xi_\lambda\colon M_\Lambda \to x_\lambda\mathcal{H}$ be the projection. We have orthogonal idempotents ξ_λ (λ in Λ) in S_Λ whose sum is 1. If V is a left S_Λ module, it is the direct sum of K subspaces $V^\lambda = \xi_\lambda V$, the λ-weight space. If two compositions λ, μ in Λ are equivalent under $\sim$, there are f, g in S_Λ such that $\xi_\lambda = fg$ and $\xi_\mu = gf$. This implies $\dim_K V^\lambda = \dim_K V^\mu$.

(In case $\Lambda = \Lambda(d,n)$, this means the formal character

$$\phi_V = \sum_{\lambda \text{ in } \Lambda(d,n)} \dim_K V^\lambda x_1^{\lambda_1} \ldots x_d^{\lambda_d}$$

is a symmetric function.)

The algebra S_Λ is the direct sum of subspaces $\xi_\lambda S_\Lambda \xi_\mu$, λ,μ in Λ and we can identify

$$\xi_\lambda S_\Lambda \xi_\mu = \mathrm{Hom}_{\mathcal{H}}(x_\mu \mathcal{H}, x_\lambda \mathcal{H})$$

which has the following K basis (independently of q). Let $\mathcal{J}(\mu,\lambda)$ be the set of all μ-tableaux of type λ. For example

$$A = \begin{matrix} 1 & 3 & \\ 3 & 1 & \\ 1 & 2 & 1 \end{matrix}$$

is a (2,2,3)-tableau of type (4,1,2). There is a bijection $A \leftrightarrow d_A$, $(\mu,\lambda) \simeq \mathcal{D}_\lambda$, where d_A maps every entry of t^μ which appears at $A^{-1}(a)$ to an entry of t^λ in row a, preserving the order. In case of the above example, we have

$$d_A: \begin{matrix} 1 & 4 & 5 & 7 \\ 6 & & & \\ 2 & 3 & & \end{matrix} \quad \to \quad \begin{matrix} 1 & 2 & 3 & 4 \\ 5 & & & \\ 6 & 7 & & \end{matrix} = t^\lambda.$$

For each μ-tableau A of type λ, there is a $\mathcal{H}$ homomorphism ϕ_A: $x_\mu \mathcal{H} \to x_\lambda \mathcal{H}$, defined by

$$\phi_A(x_\mu) = \sum_{B \sim A} x_\lambda T_{d_B}$$

the sum over all B in $\mathcal{J}(\mu,\lambda)$ which are row-equivalent to A. The homomorphisms ϕ_A for all row-standard A in (μ,λ) form a K basis of $\xi_\lambda S_\Lambda \xi_\mu$.

6. Weyl modules

If λ is a composition of n, the K space $x_\lambda \mathcal{H} y_{\lambda'}$ is one dimensional with basis $z_\lambda = x_\lambda T_{\pi_\lambda} y_{\lambda'}$, where the permutation π_λ is defined in a manner indicated by the following example:

$$\pi_\lambda: \begin{matrix} 1 & 5 & \\ 2 & & \\ 3 & 6 & 8 \\ 4 & 7 & \end{matrix} \quad \to \quad \begin{matrix} 1 & 2 & \\ 3 & & \\ 4 & 5 & 6 \\ 7 & 8 & \end{matrix} = t^\lambda, \quad \text{where } \lambda = (2,1,3,2).$$

If λ is in Λ, we may think z_λ is an element in M_Λ. The submodule $S_\Lambda z_\lambda = W_\lambda$ of $M_\Lambda y_{\lambda'}$ is called the Weyl module corresponding to λ. If two weights λ, μ in Λ are equivalent under $\sim$, then $W_\lambda = W_\mu$. Let Λ^+ be the set of partitions α such that $\alpha \sim \lambda$ for some λ in Λ. We can well-define W_α to be W_λ.

Let λ be a composition in Λ and μ a partition in Λ^+. The λ-weight space $(W_\mu)^\lambda = \xi_\lambda S_\Lambda \xi_\mu z_\mu$ has the following K basis: $\phi_A z_\mu$ for all A in $\mathcal{J}(\mu,\lambda)$ which are row-standard and strictly column standard. (Semi-standard basis theorem). This implies $\dim_K(W_\mu)^\lambda$ does not depend on K or q. In case $\Lambda = \Lambda(d,n)$, it follows that the Weyl module W_μ has the Schur function $\underline{\underline{S}}_\mu$ as its formal character.

The Weyl modules are highest weight modules in the sense that $(W_\mu)^\mu$ is one-dimensional and $(W_\mu)^\lambda \neq 0$ implies $\lambda \trianglelefteq \mu$ for λ, μ in Λ^+.

7. Irreducible (or simple) S_Λ modules

If $[n]! \neq 0$, the Weyl modules W_μ are irreducible for all μ in Λ^+. In general, there is a unique maximal submodule W_μ^{max} of W_μ. In fact, $W_\mu^{max} = W_\mu \cap W_\mu^\perp$ relative to some symmetric non-degenerate invariant inner product on $M_\Lambda y_{\mu'}$. The quotient modules $F_\mu = W_\mu / W_\mu^{max}$ are absolutely irreducible selfdual and mutually non-isomorphic for μ in Λ^+.

Let λ, μ be two partitions of n such that $\lambda \trianglerighteq \mu$. Assume that if μ is in Λ^+, then λ is also in Λ^+. For instance, $\Lambda = \Lambda(d,n)$ satisfies this condition. With the above assumption, the set of modules F_μ, μ in Λ^+ makes a complete set of representatives for the isomorphism classes of all irreducible S_Λ modules. For λ, μ in Λ^+, let $d_{\lambda\mu}$ be the multiplicity of F_μ as a composition factor of W_λ. The decomposition matrix $(d_{\lambda\mu})_{\lambda,\mu \text{ in } \Lambda^+}$ is lower triangular in the sense that $d_{\lambda\lambda} = 1$ and $d_{\lambda\mu} \neq 0$ implies $\lambda \trianglerighteq \mu$.

8. Schur functors

If e is an idempotent of a finite dimensional K algebra S we have a functor of Schur type [1,6.2]:

$$f\colon \underline{\text{mod}}\ S \to \underline{\text{mod}}\ eSe,\ V \mapsto eV.$$

As in the classical case, there are two applications.

Let Λ_1 and Λ_2 be two finite sets of compositions of n such that $\Lambda_1 \subset \Lambda_2$. There is an idempotent e in S_{Λ_2} such that $eM_{\Lambda_2} = M_{\Lambda_1}$, hence $eS_{\Lambda_2}e \simeq S_{\Lambda_1}$ so that we have a functor of Schur type:

$$f: \underline{\text{mod}}\ S_{\Lambda_2} \to \underline{\text{mod}}\ S_{\Lambda_1}.$$

Let $W_\lambda^{(i)}$ denote the Weyl module for S_{Λ_i} and $F_\lambda^{(i)}$ its irreducible quotient. Assume that both Λ_1^+ and Λ_2^+ satisfy the condition mentioned in 7. For any λ in Λ_2^+, we have

$$f(W_\lambda^{(2)}) = W_\lambda^{(1)} \text{ and } f(F_\lambda^{(2)}) = F_\lambda^{(1)} \quad \text{if } \lambda \text{ is in } \Lambda_1^+$$
$$f(W_\lambda^{(2)}) = 0 = f(F_\lambda^{(2)}) \quad \text{otherwise.}$$

It follows that for any partitions λ, μ the decomposition number $d_{\lambda\mu}$ does not depend on the choice of labellings Λ containing λ and μ, so that the matrix $\Delta_n = (d_{\lambda\mu})_{\lambda,\mu \text{ in } \mathcal{P}(n)}$ is well-defined. This matrix is obtained in [9] for $n \leq 10$ and q prime powers.

As a second application, assume Λ is full in the sense that $\Lambda^+ = \mathcal{P}(n)$. There is ω in Λ which has only 0 and 1 as its parts. Then $x_\omega = 1$ and if we put $e = \xi_\omega$ we have $eS_\Lambda e \simeq \mathcal{H}$ so that we have a functor of Schur type

$$f: \underline{\text{mod}}\ S_\Lambda \to \underline{\text{mod}}\ \mathcal{H}.$$

This is a q-analogue of the classical Schur functor. For any partition λ of n, $f(W_\lambda) = \mathcal{H}z_\lambda$, the Specht module, and the module $f(F_\lambda)$ is absolutely irreducible or zero. Dipper-James [4] has proved the following important criterion. Let e be the smallest integer $i > 1$ with $[i] = 0$. We put $e = \infty$ if such an integer does not exist. If $q \neq 1$, e equals the order of q. If $q = 1$, e equals char(K) (in positive characteristic) or ∞ (in characteristic zero).

<u>Irreducibility Criterion</u>. For a partition λ of n, $f(F_\lambda)$ is non-zero if and only if $\lambda_a - \lambda_{a+1} < e$ for all $a > 1$.

Such a partition is called column e-regular. The set of $f(F_\lambda)$ for all column e-regular partitions λ exhausts all irreducible $\mathcal{H}$ modules and mutually non-isomorphic. It follows that the part of the matrix Δ_n consisting of all columns μ which are column e-regular can be interpreted as the decomposition matrix of the Hecke algebra $\mathcal{H}$. If in particular $e > n$, then all partitions are column e-regular and we have $f(F_\lambda) = \mathcal{H}z_\lambda$. In this case $\mathcal{H}$ and S_Λ are semi-simple and the functor f is a category equivalence.

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PROCEEDINGS OF THE 25TH
SYMPOSIUM ON RING THEORY, 1992

LINEAR ACTIONS OF G_a ON POLYNOMIAL RINGS

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1. Introduction. Throughout this paper we fix an algebraically closed field k of characteristic $p \geq 0$. Let C be a k-algebra and let $S = C[x_1,\cdots,x_n]$ be a polynomial ring in n-variables over C. Then, for an element M of $GL_n(C)$, we denote by ψ_M the C-algebra automorphism of S defined by $\psi_M(f) = f((x_1,\cdots,x_n)M)$, where $f = f(x_1,\cdots,x_n)$ is an element of S.

Let G be an algebraic group over k and let $\rho : G \longrightarrow GL_n(k)$ be a rational representation of G. Then we have a linear action of G on a polynomial ring $A = k[x_1,\cdots,x_n]$ by letting $\sigma\cdot f = \psi_{\rho(\sigma)}(f)$ for $\sigma \in G$ and $f \in A$. We will call it the action of G on A associated with ρ. When G acts on A A^G stands for the ring of invariants, namely, $A^G = \{f \in A | \sigma\cdot f = f$ for every $\sigma \in G\}$.

In this paper we are interested in the case where G is the additive group G_a of k. The purpose is to study properties of $R = A^{G_a}$.

The paper has three sections. In the second section we show that R is Gorenstein if $ch(k) = 0$. In the third section we calculate the Poincaré series of R and we show that R is not a complete intersection in general.

The detailed version of this paper will be submitted for publication elsewhere.

2. Cohen-Macaulay property. We identify G_a with the subgroup $\{ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} | \lambda \in k\}$ of $SL_2(k)$. For a positive integer m, we denote by ρ_m the representation $SL_2(k) \longrightarrow GL_m(k)$ defined by $\rho_m\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = (a_{ij})$, where a_{ij} is the coefficient of X^{i-1} in the polynomial $(a + bX)^{m-j}(c + dX)^{j-1}$. The restriction of ρ_m to G_a is a representation $G_a \longrightarrow GL_m(k)$, which is also denoted by ρ_m.

Definition 2.1. A representation $\rho : G_a \longrightarrow GL_n(k)$ is said to be standard if ρ is equivalent to $\rho_{i_1} \oplus \cdots \oplus \rho_{i_m}$ for some positive integers $i_1, \cdots, i_m$ with $i_1 + \cdots + i_m = n$.

It is known that if $ch(k) = 0$, then every representation $\rho : G_a \longrightarrow GL_n(k)$ is standard (cf. [1], [6]).

Assume that $\rho : G_a \longrightarrow GL_n(k)$ is a standard representation. Then we may assume $\rho = \rho_{i_1} \oplus \cdots \oplus \rho_{i_m}$ for some $i_1, \cdots, i_m$. Let $B = A[y_0, y_1]$ be a polynomial ring in two variables over $A = k[x_1, \cdots, x_n]$ and consider the linear action of $SL_2(k)$ on B associated with the representation $\rho' = \rho \oplus \rho_2 : SL_2(k) \longrightarrow GL_{n+2}(k)$. Set $M = \begin{pmatrix} y_1 & 0 \\ -y_0 & 1/y_1 \end{pmatrix}$ and let $\rho'' : SL_2(k[y_0, y_1, \frac{1}{y_1}]) \longrightarrow GL_n(k[y_0, y_1, \frac{1}{y_1}])$ be the natural extension of $\rho : SL_2(k) \longrightarrow GL_n(k)$. Then $N = \rho''(M)$ belongs to $GL_n(k[y_0, y_1, \frac{1}{y_1}])$, and $\psi_N : B[1/y_1] \longrightarrow B[1/y_1]$ is a k-algebra automorphism of $B[1/y_1] = k[x_1, \cdots, x_n, y_0, y_1, 1/y_1]$. Hence if we set $R = A^{G_a}$ and $\tau = \psi_N|_R$, then τ is an injective k-algebra homomorphism from R to $B[1/y_1]$. Thus we have a k-algebra isomorphism $R \cong \tau(R)$.

Proposition 2.2. $\tau(R) = B^{SL_2(k)}$.

From this proposition we know that if $\rho : G_a \longrightarrow GL_n(k)$ is a standard representation, then the invariant subring R of A with respect to the action of G_a associated with ρ is isomorphic to $B^{SL_2(k)}$.

As is noted above, every representation of G_a is standard if $ch(k) = 0$. Furthermore, if $ch(k) = 0$, then $SL_2(k)$ is linearly reductive. Hence, by Proposition 2.2 and [3], we have the following

Theorem 2.3. Assume that $ch(k) = 0$. If G_a acts linearly on a polynomial ring $A = k[x_1, \cdots, x_n]$, then its ring of invariants A^{G_a} is Cohen-Macaulay.

Remark 2.4. It is easy to check that A^{G_a} is a unique factorization domain (cf. [5]). Since a Cohen-Macaulay UFD is Gorenstein, we know that, under the same assumption of Theorem 2.3, A^{G_a} is Gorenstein.

We conclude this section by remarking the following

Proposition 2.5. Let the notation and assumption be the same as above. Then we have $A^{SL_2(k)} = \{f \in A \mid \tau(f) = f\}$.

3. Poincaré series. In this section we consider a linear action of G_a on $A = k[x_0, x_1, \cdots, x_n]$ associated with the representation $\rho_{n+1} : G_a \longrightarrow GL_{n+1}(k)$ under the assumption that $ch(k) = 0$. In this case, we can determine $R = A^{G_a}$ when n is small.

Theorem 3.1. (1) If $n = 1$, then $R \cong k[X]$.

(2) If $n = 2$, then $R \cong k[X,Y]$.

(3) If $n = 3$, then $R \cong k[X,Y,Z,U]/(X^2U + Y^3 + Z^2)$.

(4) If $n = 4$, then $R \cong k[X,Y,Z,U,V]/(X^3V+Y^3+Z^2+X^2YU)$.

In view of this result it seems plausible that R has something better singularity than Gorensteinness. For example, R

is a complete intersection when $n \leq 4$, and it is natural to ask if this is the case in general. The purpose of this section is to show that R is not a complete intersection when $n = 5$.

For this purpose we will calculate the Poincaré series of R. Note that A is a graded ring with natural gradation and R is a graded subring of A. Let A_d (resp. R_d) be the homogeneous part of degree d of A (resp. R). Then we have $R = \oplus R_d$ and $P_R(t) = \Sigma_{d\geq 0} (\dim_k R_d)t^d$ is, by definition, the Poincaré series of R.

Let f be an element of A. Then there exist elements $f_0, f_1, \cdots, f_i, \cdots$ of A such that $\lambda \cdot f = \Sigma_{i\geq 0} f_i \lambda^i$ for every $\lambda \in G_a$. Define a map $D_i : A \longrightarrow A$ by $D_i(f) = f_i$ and let $\mathbb{D} = \{D_0, D_1, \cdots, D_i, \cdots\}$. Then $\mathbb{D}$ is a locally finite iterative higher derivation of A and R coincides with the ring of constants of $\mathbb{D}$ (cf. [4]). Moreover, since $ch(k) = 0$, we have $D_i = (1/i!)D^i$ for $i \geq 0$ where $D = D_1$. Note that D is a homogeneous k-derivation of A defined by $D(x_0) = 0$ and $D(x_i) = x_{i-1}$ for $i \geq 1$. Hence $D : A \longrightarrow A$ induces a k-linear map $D : A_d \longrightarrow A_d$ and we have $R_d = \{f \in A_d | D(f) = 0\}$.

For a monomial $m = x_0^{i_0} x_1^{i_1} \cdots x_n^{i_n}$ we set $w(m) = i_1 + 2i_2 + \cdots + ni_n$ and call it the weight of m. Let $A_{d,j}$ be the submodule of A_d spanned by all the monomials m of degree d and weight j. Then we have a direct sum decomposition $A_d = \bigoplus_{j=0}^{nd} A_{d,j}$. For simplicity we set $A_{d,-1} = (0)$. Then, as is easily seen, D induces a k-linear map $D : A_{d,j} \longrightarrow A_{d,j-1}$ for $j \geq 1$.

Proposition 3.2. The map $D : A_{d,j} \longrightarrow A_{d,j-1}$ is injective if $j > [nd/2]$ and is surjective if $j \leq [nd/2]$.

It follows from this proposition that $\dim_k R_d = \dim_k A_{d,[nd/2]} = \#\{(i_0, \cdots, i_n) | i_0 + \cdots + i_n = d,\ i_1 + 2i_2 + \cdots + ni_n = [nd/2]\}$.

Notation 3.3. For $f(x) = \Sigma_{i\geq 0}\, a_i x^i \in Q[[x]]$ and a positive integer j, we set

$$[f(x)]_{(j)} = a_0 + a_j x + \cdots + a_{nj} x^n + \cdots .$$

Now we can determine the Poincaré series of R. Let

$$F_i(x,y) = \frac{1-yx^i}{(1-y)(1-yx)\cdots(1-yx^n)}$$

$$G_i(x) = F(x, 1/x^i)$$

where $i = 0,1,\cdots,[n/2]$.

Theorem 3.4.

(1) If $n = 2m$, then $P_R(t) = \sum_{i=0}^{m-1} [G_i(t)]_{(m-i)}$

(2) If $n = 2m+1$, then $P_R(t) = \sum_{i=0}^{m} [(1+t)G_i(t^2)]_{(n-2i)}$.

If $n = 5$, then $P_R(t)$ is equal to

$$\frac{1+t^2+3t^3+3t^4+5t^5+4t^6+6t^7+6t^8+4t^9+5t^{10}+3t^{11}+3t^{12}+t^{13}+t^{15}}{(1-t)(1-t^2)(1-t^4)(1-t^6)(1-t^8)}$$

and we can show that $P_R(t)$ has a root which is not a root of unity. It is known that if R is a complete intersection, then every root of $P_R(t)$ must be a root of unity (cf. [7]). Thus we have the following

Corollary 3.5. If $n = 5$, then $R = A^{G_a}$ is not a complete intersection.

Remark 3.6. From Theorem 3.4, we have $a(R) = -(n+1)$, where $a(R)$ stands for the a-invariant of R (cf. [2]).

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PROCEEDINGS OF THE 25TH
SYMPOSIUM ON RING THEORY, 1992

IDEAL THEORY OF SKEW-POLYNOMIAL RINGS OF FROBENIUS TYPE

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§1. Introduction and Motivation:

Let R be a commutative Noetherian ring of characteristic $p > 0$ and let $\mathfrak{a}$ be an ideal of R. The tight closure $\mathfrak{a}^*$ of $\mathfrak{a}$ is defined by Hochster-Huneke, which is conditioned by the following condition:

DEFINITION (1.1). For an element $x \in R$, $x \in \mathfrak{a}^*$ if and only if there is an element $c \in R$ and an integer e such that c does not belong to any minimal prime of R and that

$$cx^{p^n} \in \mathfrak{a}^{[p^n]} \tag{1.1.1}$$

for all $n \geq e$. (Here $\mathfrak{a}^{[p^e]}$ denotes the ideal generated by p^e-th powers of all elements of $\mathfrak{a}$.)

For the easiest example, consider a subring $R = k[x^2, x^3]$ of a polynomial ring $k[x]$ and an ideal $\mathfrak{a} = x^2R$. Clearly $x^3 \notin \mathfrak{a}$, but it can be seen $x^3 \in \mathfrak{a}^*$. In fact, since $(x^3)^{p^n} = x^{3p^n} = (x^2)^{p^n}x^{p^n} \in \mathfrak{a}^{[p^n]}$ if $n \geq 1$, (1.1) is fulfilled by taking $c = 1$ and $e = 1$. Actually we can see that $\mathfrak{a}^* = (x^2, x^3)R$.

The notion of tight closure has certainly a great deal of applications in fields of commutative ring theory, related to the theory of rational singularities , homological conjectures and invariant theory etc. See Hochster-Huneke [1]. However it is not an easily accessible one, because of the complexity of its definition (1.1). In

The final version of this paper will be submitted for publication elsewhere.

this note we aim to simplify this notion using the skew-polynomial rings and to suggest some applications of this simplification. Moreover we shall propose some open questions in this context. Whole of the contents of this note is taken from the paper [2].

Now we begin with the definition of the skew-polynomial ring.

DEFINITION (1.2). The skew polynomial ring (of Frobenius type) $R[X;f]$ is defined by:

$$R[X;f] = R\langle X\rangle/(Xr - r^pX|\ r \in R)$$

That is, $R[X;f]$ is a ring which consists of all the polynomials of (noncommutative) variable X over R with the relation $Xr = r^pX$ $(r \in R)$. For simplicity we denote $R[X;f]$ by A.

Note that A is a direct sum of RX^n $(n \geq 0)$ as a left R-module. Note also that $Ax = \sum_{n\geq 0} Rx^{p^n}X^n$ for any $x \in R$. More generally, if $\mathfrak{a}$ is an ideal of R, then $A\mathfrak{a} = \sum_{n\geq 0} \mathfrak{a}^{[p^n]}X^n$. Therefore the condition (1.1.1) can be written in the form: $cX^eAx \subseteq A\mathfrak{a}$.

§2. Some easy results:

If the ring A is a right and/or left Noetherian, then various phenomena occurring in the theory of tight closure will be explained by this nature. But unfortunately we see that A is hardly Noetherian.

THEOREM (2.1).

(a) *A is left Noetherian if and only if R is a product of fields.*

(b) *A is right Noetherian if and only if R is an Artinian ring, all of whose residue fields are perfect.*

DEFINITION (2.2). R is said to have *sufficiently many units* if, for any $n \in \mathbb{N}$, there is $x \in R$ such that $x^{p^i} - x$ $(i = 1, 2, \dots, n)$ are units in R.

It is easily observed that this condition is satisfied either if R contains an infinite field or if R is a local ring whose residue field is infinite.

In the rest of this note we always assume that R has sufficiently many units.

We have mentioned in (2.1) that A never be a Noetherian ring unless R is an Artinian ring. But we see that A satisfies the ascending chain condition for two-sided ideals.

THEOREM (2.3). *For a subset $I \subset A$, the following two conditions are equivalent:*
(a) *I is a two-sided ideal of A,*

(b) *There is an ascending chain of ideals of R : $I_0 \subset I_1 \subset I_2 \subset \cdots \subset R$ such that $I = \sum_{i \geq 0} I_i X^i$.*

In particular, A satisfies the ascending chain condition for two-sided ideals.

Note that we can regard A as a graded ring by attaching degree 1 to X. Then it follows from the above theorem that every two-sided ideal of A is homogeneous.

Now recall the definition of A-modules: By definition, a set M is a left A-module if M has a left R-module structure with left action of X such that $Xrm = r^p Xm$ $(r \in R, m \in M)$. Likewise M is a right A-module if M is a right R-module with right X-action such that $mXr = mr^p X$ $(r \in R, m \in M)$.

EXAMPLE (2.4). We can define the left A-module structure on R by $Xr = 0$ for any $r \in R$. We denote this by R_0. Then it is easily proved that there is an isomorphism of left A-modules : $R_0 \cong A/AX$.

Similarly R_1 is defined as a left A-module R with $Xr = r^p$ for any $r \in R$. Then $R_1 \cong A/A(X-1)$.

THEOREM (2.5).
(a) *Assume that the Frobenius map $R \to R$ is a finite morphism. Then every injective R-module has a non-trivial structure of right A-module.*
(b) *Let R be a Gorenstein ring. Then every injective R-module has a non-trivial structure of left A-module.*

We shall give a brief sketch of the proof of (a).
We may assume that R is a local ring and that the injective module is the injective hull $E_R(k)$ of the residue field. Let S denote the ring R which we regard an R-algebra through the Frobenius map $R \to R$. Then it is easy to see that there is an isomorphism of S-modules $\rho : E_S(k) \cong \mathrm{Hom}_R(S, E_R(k))$. Since $S = R$ as a ring, we can identify $E_S(k)$ with $E_R(k)$. Then we define the right-action of X on $E_R(k)$ as the map $E_R(k) \to E_R(k)$ which sends x to $\rho(x)(1)$. It is straightforward to see that this action is well-defined. For (b) we need the local duality, and we omit the proof. See [2].

From the above observation we expect the following would be true.

CONJECTURE (2.6). Every injective R-module would have a non-trivial structure of right and left A-module.

Now we recall the localization of modules over A.

DEFINITION (2.7). Let $S = \{X^n |\ n \in \mathbb{N}\}$ and let N be a right A-module. Then we define the right localization as follows:

$$N_X = N \times S / \sim_r,$$

where $(a, X^n) \sim_r (b, X^m) \iff \exists t > n, m;\ aX^{t-n} = bX^{t-m}$. We denote the class of (a, X^n) by aX^{-n}.

Similarly we can define the left localization for a left A-module M:

$${}_XM = S \times M / \sim_\ell,$$

where $(X^n, a) \sim_\ell (X^m, b) \iff \exists t > n, m;\ X^{t-n}a = X^{t-m}b$. As above we denote the class of (X^n, a) by $X^{-n}a$.

Note that S satisfies the left Ore condition, hence ${}_XA$ is a ring, and ${}_XM$ is a left ${}_XA$-module. But A_X is not necessarily a ring.

We denote by R_{red} the ring R modulo nilpotent elements and by $Q(R_{\mathrm{red}})$ the total quotient ring of R_{red}. Furthermore $\overline{Q(R_{\mathrm{red}})}$ denotes the algebraic closure of $Q(R_{\mathrm{red}})$, in which the absolutely perfect closure of R defined as follows:

$$R^\infty = \left\{x \in \overline{Q(R_{\mathrm{red}})} |\ x^{p^n} \in R_{\mathrm{red}} \text{ for some } n \geq 0\right\}.$$

THEOREM (2.8). *There is an isomorphism of rings:*

$${}_XA \cong R^\infty[X, X^{-1}; f],$$

which sends $X^{-n}rX^m$ *to* $r^{p^{-n}}X^{-n+m}$.

As one of the applications of this theorem, we see that the left ${}_XA$-module ${}_X(R_1)$ is isomorphic to $(R^\infty)_1$ which is R^∞ having Frobenius endomorphism as the action of X.

Note that the localization is a functor from the category of left A-modules to the category of left ${}_XA$-modules, which is easily seen to be an exact functor. Also note that ${}_XA \cong R^\infty[X, X^{-1}; f]$ is a $\mathbb{Z}$-graded ring with natural grading : $\deg(X) = 1$. We can prove the following as a corollary of the above theorem:

THEOREM (2.9). *There is an exact equivalence of categories:*

$$(\textit{left graded } {}_XA\textit{-modules}) \longrightarrow (R^{\infty}\textit{-modules}).$$

§3. Tightly associated ideals:

For a given left A-module we may consider ideals of R which are, in a sense, associated with the module.

DEFINITION (3.1). Let M be a left A-module and let x be an element of M. Then we denote $\mathrm{ann}(Ax) = \{c \in A \mid cAx = 0\}$ that is obviously a two-sided ideal of A, hence, by (2.3), we may write it as $\mathrm{ann}(Ax) = \sum_{n \geq 0} I_n X^n$, where $\{I_n\}_{n \geq 0}$ is an ascending sequence of ideals of R. Now define an ideal $\mathfrak{a}(Ax)$ of R as follows:

$$\mathfrak{a}(Ax) = \bigcup_{n \geq 0} I_n = I_N \qquad (N \gg 0).$$

We call an R-ideal a *tightly associated ideal* of M if it is either a unit ideal or a prime ideal of R and if it is of the form $\mathfrak{a}(Ax)$ for a nonzero element $x \in M$. We denote the set of all tightly associated ideals of M by $\mathrm{Asst}(M)$, that is,

$$\mathrm{Asst}(M) = \{\mathfrak{p} \in \mathrm{Spec}\, R \cup \{R\} \mid \mathfrak{p} = \mathfrak{a}(Ax) \ \text{ for some } \ x \neq 0 \in M\}.$$

If we take a maximal element $\mathfrak{p}$ among R-ideals $\mathfrak{a}(Ax)$ ($x \neq 0 \in M$), it can be easily seen that $\mathfrak{p}$ is either a prime ideal or a unit ideal, hence $\mathfrak{p} \in \mathrm{Asst}(M)$. In particular, $M \neq 0$ if and only if $\mathrm{Asst}(M) \neq \emptyset$.

EXAMPLE (3.2). For the left A-module R_0 in (2.4), we can see that $\mathrm{Asst}(R_0) = \{R\}$. In fact, since we have $Xx = 0$ for any $x \in R_1$, $\mathrm{ann}(Ax)$ contains $\sum_{n \geq 1} RX^n$, therefore $\mathfrak{a}(Ax) = R$. On the other hand, if $R = R_{\mathrm{red}}$, then $\mathrm{Asst}(R_1) = \mathrm{Ass}_R(R)$ the set of ordinary associated prime ideals of R as an R-module.

The following is almost trivial:

LEMMA (3.3). *For a left A-module M, the following two conditions are equivalent:*
(a) $R \notin \mathrm{Asst}(M)$,

(b) the left action of X on M is injective.

Various other properties of tightly associated ideals are discussed in [2]. We expect the reader will consult with [2] for this, and we proceed to the next theme.

§4. Tight closures and primary decompositions:

Let $\mathfrak{a}$ be an ideal of R. If the tight closure $\mathfrak{a}^*$ is equal to $\mathfrak{a}$, we say that $\mathfrak{a}$ is tightly closed. The following theorem is straightforward from the definition of tightly associated ideals, but seems crucial in connecting the theory of tight closures and the ideal theory of skew-polynomial rings.

THEOREM (4.1). *The following two conditions are equivalent for an ideal* $\mathfrak{a}$ *of* R.

(a) $\mathfrak{a}^{[p^n]}$ *is tightly closed for any* $n \in \mathbb{N}$,

(b) $\mathrm{Asst}(A/A\mathfrak{a}) \subseteq \mathrm{Min}(R)$

Also this theorem suggests that the condition $\mathrm{Asst}(M) \subseteq \mathrm{Min}(R)$ for a left A-module M should be considerably important.

Now let I be a graded left ideal of A. Note and recall that a subset $I = \sum_{n\geq 0} I_n X^n \subseteq A$ is a graded left ideal of A if and only if each I_n is an R-ideal and $I_n^{[p]} \subseteq I_{n+1}$ for any $n \geq 0$. Note also that the ideal $A\mathfrak{a}$ in the above theorem is a graded one.

DEFINITION (4.2).

(a) Let $\mathfrak{p}$ be a prime ideal of R. A graded left ideal I of A is said to be a $\mathfrak{p}$-primary left ideal if each I_n is a $\mathfrak{p}$-primary ideal for any n.

(b) Let $I, I^{(1)}, I^{(2)}, \dots, I^{(\ell)}$ be graded left ideals of A and assume that

$$(*) \qquad I = \cap_{i=1}^{\ell} I^{(i)}.$$

We say that $(*)$ is a primary decomposition of I if each $I^{(i)}$ is a $\mathfrak{p}^{(i)}$-primary left ideal for any i.

It is a fundamental question to ask if every graded left ideal has a primary decomposition.

EXAMPLE (4.3). Let R be a regular local ring and let $\mathfrak{a}$ be an ideal of R. Then the graded left ideal $A\mathfrak{a}$ has a primary decomposition.

PROOF: Let $\mathfrak{a} = \mathfrak{a}^{(1)} \cap \mathfrak{a}^{(2)} \cap \cdots \cap \mathfrak{a}^{(\ell)}$ be a primary decomposition of $\mathfrak{a}$ in the Noetherian ring R. Note that A is a right flat R-module, because R is regular. Therefore we have that $A\mathfrak{a} = A\mathfrak{a}^{(1)} \cap A\mathfrak{a}^{(2)} \cap \cdots \cap A\mathfrak{a}^{(\ell)}$. Hence it is enough to show that if $\mathfrak{a}$ is a $\mathfrak{p}$-primary ideal of R, then $A\mathfrak{a}$ is a $\mathfrak{p}$-primary graded left ideal of A. For this, it suffices to prove that if $\mathfrak{a}$ is a $\mathfrak{p}$-primary ideal of R then so is $\mathfrak{a}^{[p]}$. To show this let $\mathfrak{P}$ be an associated prime ideal of $R/\mathfrak{a}^{[p]}$. Since the Frobenius map $R \to R$ is flat, we have a flat ring homomorphism $(R/\mathfrak{a})_{\mathfrak{P}} \to (R/\mathfrak{a}^{[p]})_{\mathfrak{P}}$. It follows from this that $\mathrm{depth}(R/\mathfrak{a})_{\mathfrak{P}} = 0$, hence that $\mathfrak{P}$ associates with $R/\mathfrak{a}$. Thus we have $\mathfrak{P} = \mathfrak{p}$ and $\mathfrak{a}^{[p]}$ must be a $\mathfrak{p}$-primary ideal. ∎

THEOREM (4.4). *Let $I = \sum_{n=0}^{\infty} I_n X^n$ be a graded left ideal of A and assume that all the prime ideals in $\cup_{n\geq 0} \mathrm{Ass}_R(R/I_n)$ have the same height. Then I has a primary decomposition.*

PROOF: Since $I_n^{[p]} \subseteq I_{n+1}$, we have a sequence of R-ideals $\sqrt{I_0} \subseteq \sqrt{I_1} \subseteq \sqrt{I_2} \subseteq \cdots$, which must terminates at some large N: $\sqrt{I_N} = \sqrt{I_{N+1}} = \sqrt{I_{N+2}} = \cdots$. Then by the assumption we see that

$$\cup_{n=0}^{\infty} \mathrm{Ass}(R/I_n) = \cup_{n=0}^{N} \mathrm{Min}(R/\sqrt{I_n}) = \mathrm{Min}(R/\sqrt{I_0}),$$

which we denote by $\{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_\ell\}$. Letting $I_n^{(i)} = (I_n)_{\mathfrak{p}_i} \cap R$, we have a primary decomposition $I^{(i)} = \cap_{i=1}^{\ell} I_n^{(i)}$ as an ideal of R. Then, since it is easily observed that $I_n^{(i)[p]} \subseteq I_{n+1}{}^{(i)}$ for each i and n, $I^{(i)} = \sum_{n=1}^{\infty} I_n^{(i)} X^n$ $(i = 1, 2, \dots, \ell)$ are graded left ideals of A which are primary and $I = \cap_{i=1}^{\ell} I^{(i)}$ as desired. ∎

COROLLARY (4.5). *Let R be a Cohen-Macaulay local ring and let $x_1, x_2, \dots, x_n$ be a system of parameters of R. Then the graded left ideal $A\{x_1, x_2, \dots, x_n\}$ has a primary decomposition.*

From these observations we expect that every graded left ideal has a primary decomposition.

CONJECTURE (4.6). Let R be a regular local ring. Then any graded left ideal of $A = R[X; f]$ would have a primary decomposition.

If this conjecture is true, then it will have some applications by combining with the following theorem, whose proof can be found in [2].

THEOREM (4.7). *Let $T \to R$ be a ring homomorphism of commutative Noetherian rings. We denote $A_R = R[X; f]$ and $A_T = T[X; f]$ to make the situation clear. Let M be a left A_R-module. We can regard M as a left A_T-module, which is denoted by M_T. Assume that, for any $\mathfrak{P} \in \mathrm{Spec}(R)$, $\mathfrak{P} \in \mathrm{Min}(R)$ if and only if $\mathfrak{P} \cap T \in \mathrm{Min}(T)$. Then the following two conditions are equivalent:*

(*a*) $\mathrm{Asst}(M) \subseteq \mathrm{Min}(R)$,

(b) $\mathrm{Asst}(M_T) \subseteq \mathrm{Min}(T)$.

We end this note by suggesting that the conjecture (4.6) has a good result as its application. See [1] for the definition of F-regularity and weak F-regularity.

COROLLARY (4.8). *Let S be a commutative Noetherian ring that is finite over a regular ring R. Suppose that the conjecture* (4.6) *is correct for R. Then if S is weakly F-regular then S is F-regular.*

Proof of Cor. (4.8) modulo (4.7):

Let $\mathfrak{a}$ be an ideal of S and let $\mathfrak{P}$ be a prime ideal of S with $\mathfrak{a} \subseteq \mathfrak{P}$. We want to show that $\mathfrak{a}S_{\mathfrak{P}}$ is tightly closed in $S_{\mathfrak{P}}$.

Let $M = S[X;f]/S[X;f]\mathfrak{a}$. Then, by (4.1), it is sufficient to show that $\mathrm{Asst}(M_{\mathfrak{P}}) \subseteq \mathrm{Min}(S_{\mathfrak{P}})$ for the left graded $S_{\mathfrak{P}}[X;f]$-module $M_{\mathfrak{P}}$. Because weak F-regular rings are CM, the assumption in (4.7) is satisfied for the ring extensions : $R \to S$ and $R_{\mathfrak{p}} \to S_{\mathfrak{P}}$, where $\mathfrak{p} = \mathfrak{P} \cap R$. Hence it suffices to show that $\mathrm{Asst}((M_{\mathfrak{P}})_{R_{\mathfrak{p}}}) \subseteq \mathrm{Min}(R_{\mathfrak{p}})$. On the other hand, the weak F-regularity of S gives that $\mathrm{Asst}(M) \subseteq \mathrm{Min}(S)$, hence that $\mathrm{Asst}(M_R) \subseteq \mathrm{Min}(R)$ by (4.7). Thus the following lemma will be enough to prove the corollary.

LEMMA (4.9). *Let R be a regular ring, for which Conjecture* (4.6) *is true, and let M be a graded left $A = R[X;f]$-module and $\mathfrak{p} \in \mathrm{Spec}(R)$. If $\mathrm{Asst}(M) \subseteq \mathrm{Min}(R)$, then $\mathrm{Asst}(M_{\mathfrak{p}}) \subseteq \mathrm{Min}(R_{\mathfrak{p}})$.*

Proof of (4.9): Suppose that there is $\mathfrak{P} \in \mathrm{Spec}(R) \cup \{R\}$ such that $\mathfrak{P}R_{\mathfrak{p}} \in \mathrm{Asst}(M_{\mathfrak{p}})$. By definition there is $x \in M$ whose natural image $\tilde{x}$ in $M_{\mathfrak{p}}$ is nonzero and $\mathfrak{P}R_{\mathfrak{p}} = \mathfrak{a}(A_{\mathfrak{p}}\tilde{x})$. Here x, hence $\tilde{x}$, can be taken as a homogeneous element. Then consider a graded left submodule $N = Ax$ of M, which is isomorphic, as a left A-module, to A/I for some graded left ideal I of A. Since $\mathrm{Asst}(N) \subseteq \mathrm{Asst}(M)$, we have $\mathrm{Asst}(A/I) \subseteq \mathrm{Min}(R)$, while it can be seen that $\mathfrak{P}A_{\mathfrak{p}} \subseteq IA_{\mathfrak{p}}$.

Now assume that (4.6) is OK for R. Then since I has a primary decomposition, we may write $I = J \cap K$ where J is the intersection of the primary components whose prime ideals are contained in $\mathfrak{p}$, and K is the intersection of other components. Note that $J = IA_{\mathfrak{p}} \cap A$. Since $\mathfrak{P}A_{\mathfrak{p}} \subseteq IA_{\mathfrak{p}}$, we have $\mathfrak{P}A \subseteq IA_{\mathfrak{p}} \cap A = J$. Therefore $\mathfrak{P}K \subseteq J \cap K = I$, because K is a left ideal. Now taking $z \in K - I$, we see $\mathfrak{P}Az \subseteq \mathfrak{P}K \subseteq I$, hence $\mathfrak{a}(A\bar{z})$ contains $\mathfrak{P}$ for the image $\bar{z} \in A/I$ of z. Hence there is $\mathfrak{Q} \in \mathrm{Asst}(A/I)$ with $\mathfrak{P} \subseteq \mathfrak{Q}$, but then $\mathfrak{Q}$ must be a minimal prime ideal of R. Thus $\mathfrak{P} = \mathfrak{Q}$ is also minimal. ∎

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PROCEEDINGS OF THE 25TH
SYMPOSIUM ON RING THEORY, 1992

ON LAMBEK TORSION THEORIES, II

Mitsuo HOSHINO and Shinsuke TAKASHIMA

Introduction

In this note, generalizing recent works of Masaike [5] and Hoshino [3], we will provide another approach to the theory of QF-3 rings. We will also provide an explanation to the symmetry established by Masaike [4, Theorem 2].

Recall that a ring R is said to be left (resp. right) QF-3 if there exists a minimal faithful left (resp. right) R-module, i.e. a faithful left (resp. right) R-module which appears as a direct summand in every faithful left (resp. right) R-module (see e.g. Tachikawa [12] for details). In his recent paper [5], K. Masaike showed that a left QF-3 ring R is right QF-3 if and only if it contains an idempotent f such that RfR is a minimal dense left ideal, and every finitely solvable system of congruences $\{x \equiv fx_\lambda \bmod I_\lambda\}$, with each I_λ a left ideal of R, is solvable. Generalizing this, we will provide a characterization of left and right QF-3 rings. To do so, we will introduce the notion of τ-absolutely pure rings in §1 and that of τ-semicompact modules in §2, where "τ-" means "relative to Lambek torsion theory". With those notions, we will show that a ring R is left and right QF-3 if and only if it is τ-absolutely pure, left and right τ-semicompact and contains idempotent e, f such that ReR and RfR are minimal dense right and left ideals, respectively.

The detailed version of this note will be submitted for publication elsewhere.

Throughout this note, R stands for an associative ring with identity and all modules are unitary modules. Sometimes, we use the notation ${}_R X$ (resp. X_R) to stress that the module X considered is a left (resp. right) module. We denote by Mod R (resp. Mod R^{op}) the category of left (resp. right) R-modules and by $(\ )^*$ both the R-dual functors. For a module X, we denote by E(X) its injective envelope and by $\varepsilon_X : X \to X^{**}$ the usual evaluation map. Recall that a module X is said to be torsionless if ε_X is a monomorphism, and to be reflexive if ε_X is an isomorphism. For an $X \in$ Mod R, we denote by $\tau(X)$ its Lambek torsion submodule. Namely, $\tau(X)$ denotes a submodule of X such that $\mathrm{Hom}_R(\tau(X), E({}_R R)) = 0$ and $X/\tau(X)$ is cogenerated by $E({}_R R)$. For also an $M \in$ Mod R^{op}, we denote by $\tau(M)$ its Lambek torsion submodule.

Let us recall several definitions. A module X is said to be torsion if $\tau(X) = X$, and to be torsionfree if $\tau(X) = 0$. A nonzero torsionfree module X is said to be cocritical if X/X' is torsion for every nonzero submodule X'. A submodule X' of a module X is said to be dense if X/X' is torsion, and to be closed if X/X' is torsionfree. A dense left (resp. right) ideal I is called a minimal dense left (resp. right) ideal if it is contained in every dense left (resp. right) ideal. Note that a minimal dense left ideal, if exists, has to be an idempotent two-sided ideal, that a minimal dense left ideal exists if and only if the class of all torsion $X \in$ Mod R is closed under taking direct products, and that, in case R is left and right perfect, there always exists an idempotent f with RfR a minimal dense left ideal.

The authors would like to express their gratitude to Prof. T. Sumioka for his valuable advice.

1. τ-absolute purity of rings

In this section, we introduce the notion of τ-absolutely pure rings. With that notion, we formulate the symmetry established by Masaike [4, Theorem 2].

To begin with, we recall several definitions. A module X is said to be τ-finitely generated if it contains a finitely generated dense submodule, and to be τ-finitely presented if there exists an exact sequence $0 \to Z \to Y \to X \to 0$ with Y finitely presented and Z torsion. A ring R is said to be left (resp. right) τ-artinian if it satisfies the descending chain condition on closed left (resp. right) ideals, to be left (resp. right) τ-noetherian if it satisfies the ascending chain condition on closed left (resp. right) ideals, and to be left (resp. right) τ-coherent if every finitely generated left (resp. right) ideal is τ-finitely presented.

Remarks. (1) A well known result of Miller and Teply [7, Theorem 1.4] says that a left τ-artinian ring R is left τ-noetherian.

(2) A ring R is left τ-noetherian if and only if every finitely generated left module is τ-finitely presented (see e.g. Sumioka [11]). Thus, a left τ-noetherian ring R is left τ-coherent.

(3) A module X is τ-finitely presented if and only if it is finitely generated and for every epimorphism $\pi : F \to X$, with F free of finite rank, $\mathrm{Ker}\,\pi$ is τ-finitely generated.

(4) It follows from a result of Chase [2, Appendix] that a semiprimary left τ-noetherian ring R is left τ-artinian.

(5) Assume that R is right τ-coherent and left perfect, and that $\mathrm{Ext}^1_R(M,R)$ is torsion for every finitely presented $M \in \mathrm{Mod}\,R^{op}$. Then it follows from a result of Masaike [4, Theorem 1] that R is left τ-artinian (cf. Stenström [10, Theorem 4.4]).

The next lemma will play a key role in our arguments.

Lemma 1.1 (Hoshino [3, Theorem A]). The following are equivalent.

(a) $\tau(X) = \mathrm{Ker}\,\varepsilon_X$ for every finitely presented $X \in \mathrm{Mod}\,R$.

(a)^{op} $\tau(M) = \mathrm{Ker}\,\varepsilon_M$ for every finitely presented $M \in \mathrm{Mod}\,R^{op}$.

(b) Every τ-finitely presented torsionfree $X \in \mathrm{Mod}\,R$ is torsionless.

(b)^{op} Every τ-finitely presented torsionfree $M \in \mathrm{Mod}\,R^{op}$ is torsionless.

(c) $\mathrm{Ext}_R^1(X,R)$ is torsion for every finitely presented $X \in \mathrm{Mod}\, R$.

(c)^{op} $\mathrm{Ext}_R^1(M,R)$ is torsion for every finitely presented $M \in \mathrm{Mod}\, R^{op}$.

In the following, a ring R will be said to be τ-absolutely pure if it satisfies the equivalent conditions of Lemma 1.1.

<u>Lemma 1.2</u>(Hoshino [3, Lemma 5]). The following are equivalent.

(a) $\tau(X) = \mathrm{Ker}\, \varepsilon_X$ for every finitely generated $X \in \mathrm{Mod}\, R$.

(b) Every finitely generated torsionfree $X \in \mathrm{Mod}\, R$ is torsionless.

(c) Every finitely generated submodule of $E({}_RR)$ is torsionless.

The next proposition generalizes results of Morita [8, Theorem 1] and Sumioka [11, Lemma 7].

<u>Proposition 1.3</u>(Cf. Hoshino [3, Proposition B]). Assume that R is right τ-coherent. The following are equivalent.

(a) R is τ-absolutely pure.

(b) $E({}_RR)$ is flat.

<u>Proposition 1.4</u>(Cf. Hoshino [3, Proposition C]). Assume that R is left τ-noetherian. The following are equivalent.

(a) R is τ-absolutely pure.

(b) Every finitely generated submodule of $E({}_RR)$ is torsionless.

(c) $E(R_R)$ is flat.

The next lemma is due essentially to Faith [2, Proposition 3.1].

<u>Lemma 1.5</u>. Assume that R is τ-absolutely pure. The following are equivalent.

(a) R is left τ-noetherian.

(b) R satisfies the ascending chain condition on annihilator left ideals.

Finally, we formulate the symmetry established by Masaike [4, Theorem 2] as follows.

Theorem 1.6. Assume that R is τ-absolutely pure. Then R is left τ-artinian if and only if it is right τ-artinian.

2. τ-semicompactness and reflexivity of modules

In this section, we introduce the notion of τ-semicompact modules, which is closely related to the notion of reflexive modules.

Recall that a homomorphism $\pi : X \to Y$ is called a τ-epimorphism if $\mathrm{Cok}\,\pi$ is torsion. In the following, a module X will be said to be τ-semicompact if for every inverse system of τ-epimorphisms $\{\pi_\lambda : X \to Y_\lambda\}$, with each $\mathrm{Im}\,\pi_\lambda$ torsionless, the induced homomorphism $\varprojlim \pi_\lambda$ is a τ-epimorphism. A ring R will be said to be left (resp. right) τ-semicompact if ${}_RR$ (resp. R_R) is τ-semicompact.

Remarks. (1) Given a module X, take a direct system of monomorphisms $\{j_\lambda : M_\lambda \to X^*\}$, with each M_λ finitely generated, such that $\varinjlim j_\lambda : \varinjlim M_\lambda \xrightarrow{\sim} X^*$. Then $\mathrm{Cok}\,\varepsilon_X \cong \mathrm{Cok}(\varprojlim(j_\lambda{}^* \circ \varepsilon_X))$.

(2) Even if R is commutative, our τ-semicompactness differs from the semicompactness, in the sense of Matlis [6], relative to Lambek torsion theory. However, for regular modules ${}_RR$ and R_R, our τ-semicompactness coincides with the semicompactness, in the sense of Stenström [10], relative to Lambek torsion theories.

(3) Assume that R satisfies the descending chain condition on annihilator left ideals, and that R has a minimal dense left ideal. Then R is left τ-semicompact.

Lemma 2.1. Assume that every finitely generated submodule of $E(_RR)$ is torsionless. Then ε_X is τ-epic for every τ-semicompact $X \in \text{Mod}\,R$.

Lemma 2.2. Assume that R contains an idempotent f such that fR is an injective right ideal and RfR is a minimal dense left ideal. Then every $X \in \text{Mod}\,R$, with ε_X τ-epic, is τ-semicompact.

Combining lemmas above together, we get the following.

Proposition 2.3(Masaike [5, Theorem 3]). Assume that every finitely generated submodule of $E(_RR)$ is torsionless, and that R contains an idempotent f such that fR is an injective right ideal and RfR is a minimal dense left ideal. Then for an $X \in \text{Mod}\,R$ the following are equivalent.

(a) X is reflexive.

(b) X is τ-semicompact and embeds, as a closed submodule, in a product of copies of $_RR$.

3. Idempotent generated minimal dense ideals

In this section, we collect several basic results on idempotent generated minimal dense ideals which we use in the next section.

Remark. For an idempotent f in R, RfR is a dense left ideal if and only if fR is a faithful right ideal.

Lemma 3.1(Rutter [9, Theorem 1.4]). For an idempotent f in R the following are equivalent.

(a) RfR is a minimal dense left ideal.

(b) fR_R is faithful and every simple homomorphic image of $_RRf$ is torsionless.

Corollary 3.2. Let f be an idempotent in R with RfR a minimal dense left ideal and fR an injective right ideal, and let f_1 be a local idempotent in R with $f_1 f = f_1 = ff_1$. Then $(Rf_1/Jf_1)^*$ is cocritical and embeds in f_1R_R, where J denotes the Jacobson radical of R.

Lemma 3.3(Rutter [9, Corollary 1.2]). Assume that R contains an idempotent f with fR a minimal faithful right module. Then RfR is a minimal dense left ideal.

Lemma 3.4. Let f be an idempotent in R with RfR a minimal dense left ideal. Then ${}_RRfX$ is simple for every $X \in$ Mod R with $X/\tau(X)$ cocritical.

As pointed out by Stenström [10, Proposition 2.5], the argument of Matlis [6, Propositions 2 and 3] yields the following

Proposition 3.5. Let f be an idempotent in R with RfR a minimal dense left ideal. The following are equivalent.

(a) fR is an injective right ideal.

(b) R is τ-absolutely pure and left τ-semicompact.

Proposition 3.6. Let f be an idempotent in R with RfR a minimal dense left ideal. Assume that every finitely generated submodule of $E({}_RR)$ is torsionless, and that R is left τ-semicompact. Then fRf is a semiperfect ring.

4. QF-3 rings

In this section, generalizing a result of Masaike [5, Theorem 5], we provide a characterization of left and right QF-3 rings.

To point out the difference between "one-sided QF-3" and "two-sided QF-3", we first provide a characterization of right QF-3 rings.

Proposition 4.1. The following are equivalent.

(1) R is right QF-3.

(2)(a) R is τ-absolutely pure.

(b) R is left τ-semicompact.

(c) R contains an idempotent f such that RfR is a minimal dense left ideal and fRf is a semiperfect ring.

(d) Every cocritical right module has a nonzero socle.

Theorem 4.2. The following are equivalent.

(1) R is left and right QF-3.

(2)(a) R is τ-absolutely pure.

(b) R is left and right τ-semicompact.

(c) R contains idempotents e, f such that ReR and RfR are minimal dense right and left ideals, respectively.

5. Maximal two-sided quotient rings

In this section, we deal with the case where R has a maximal two-sided quotient ring. Recall that a maximal left (resp. right) quotient ring Q_ℓ (resp. Q_r) is defined as a biendomorphism ring of $E({}_RR)$ (resp. $E(R_R)$), and that R is said to have a maximal two-sided quotient ring if $Q_\ell \simeq Q_r$ as ring extensions of R.

In the following, we denote by $\mathrm{Mod}\, R/\tau$ the quotient category of $\mathrm{Mod}\, R$ over the full subcategory consisting of all torsion $X \in \mathrm{Mod}\, R$. Also, $\mathrm{Mod}\, R^{op}/\tau$ denotes the quotient category of $\mathrm{Mod}\, R^{op}$ over the full subcategory consisting of all torsion $M \in \mathrm{Mod}\, R^{op}$.

The next lemma seems to be well known.

Lemma 5.1. Assume that R is left τ-artinian. Then

(1) R has a maximal left quotient ring Q which is semiprimary.

(2) There exists a semiprimary ring A such that $\mathrm{Mod}\, R/\tau \xrightarrow{\sim} \mathrm{Mod}\, A$.

According to Proposition 1.4, a result of Masaike [4, Theorem 2] implies the following

Proposition 5.2. Assume that R is τ-absolutely pure and left τ-artinian. Then

(1) R has a maximal two-sided quotient ring Q which is semi-primary left and right QF-3.

(2) There exist a left artinian ring A and a right artinian ring B such that $\mathrm{Mod}\, R/\tau \cong \mathrm{Mod}\, A$, $\mathrm{Mod}\, R^{op}/\tau \cong \mathrm{Mod}\, B^{op}$ and A is Morita dual to B.

In case R is commutative, the next proposition is well known (see Bass [1, Proposition 6.1]).

Proposition 5.3(Hoshino [3, Proposition F]). Assume that R is left and right noetherian. The following are equivalent.

(1) $E({}_R R)$ is flat.

(2)(a) R has a maximal two-sided quotient ring.

(b) X^* is reflexive for every finitely generated $X \in \mathrm{Mod}\, R$.

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PROCEEDINGS OF THE 25TH
SYMPOSIUM ON RING THEORY, 1992

ON QF–RINGS WITH CYCLIC NAKAYAMA PERMUTATIONS

KIYOICHI OSHIRO AND SEOG HOON RIM

The concept of skew matrix ring was introduced by H.Kuppish [3] and K.Oshiro [4] independently.

In 1987, Oshiro provided a structure theorem of certain skew matrix ring over a local ring (Theorem 1).

In this paper we want to show the following :

Theorem. *If R is a basic QF-ring such that for any idempotent in R, eRe is QF-ring with cyclic Nakayama permutation, then there exists a local QF-ring Q, an element c in the Jacobson radical of Q and a ring automorphism σ of Q for which R is represented as a skew-matrix ring.*

Throughout this paper R will denote always associative ring with identity and all R-modules are unitary. The notation M_R (resp. ${}_RM$) is used to denote that M is a right (resp. left) R-module. For a given R-module $M, J(M)$ and $S(M)$ denote its Jacobson radical and socle, respectively. For R-modules M and N, $M \underset{\sim}{\subset} N$ means that M is isomorphic to a submodule of N. And, for R-modules M and N, we put $(M,N) = \mathrm{Hom}_R(M,N)$ and in particular, we put $(e,f) = (eR,fR) = \mathrm{Hom}_R(eR,fR)$ for idempotents e,f in R.

Let R be a ring which is represented as a matrix form:

The final version of this paper will be submitted for publication elsewhere

$$R = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ & \cdots & \\ A_{n1} & \cdots & A_{nn} \end{pmatrix}$$

Then we use $< a >_{ij}$ to denote the matrix of R whose (i,j)-position is a but other positions are zero. Consider another ring which is also represented as a matrix form :

$$T = \begin{pmatrix} B_{11} & \cdots & B_{1n} \\ & \cdots & \\ B_{n1} & \cdots & B_{nn} \end{pmatrix}$$

When we say is a map from R to T, this word means that τ_{ij} is a map from A_{ij} to B_{ij} and $\tau(< a >_{ij}) =< \tau_{ij}(a) >_{ij}$. In the above ring R. We put $Q_i = A_{ii}$ for $i = 1, \cdots, n$. Consider a ring U which is a isomorphic to Q_k; $\xi : U \simeq Q_k$. Then we can exchange Q_k by U and make a new ring $R(Q_k, U, \xi)$ which is canonically isomorphic to R. We often identify R with $R(Q_k, U, \xi)$.

Let R be an artinian ring and $E = \{e_1, \cdots, e_n\}$ be a complete set of orthogonal primitive idempotents of R. The following result due to Fuller ([2]) which is a very basic result: Let f be in E. ${}_RRf$ is injective iff there exists e in E such that $(eR; Rf)$ is an i-pair, that is ${}_RRe/J({}_RRe) \simeq {}_RS({}_RRf)$ and $fR_R/J(fR_R)_R \simeq S(eR_R)_R$. In this case, eR_R is also injective. We note that if R is a basic artinian ring and $(eR; Rf)$ is an i-pair, then $S({}_{eRe}eRf) = S(eRf_{fRf})$ and

$$S(eR_R) = \begin{pmatrix} & 0 & \\ 0 & S(eRf) & 0 \\ & 0 & \end{pmatrix} = S({}_RRf)$$

Let R be a basic QF-ring and $E = \{e_1, \cdots, e_n\}$ be a complete set of orthogonal primitive idempotents. For each $e_i \in E$, there exists a unique $f_i \in E$ such that (e_iR, Rf_i) is an i-pair. Then $\begin{pmatrix} e_1 & e_2 & \cdots & e_n \\ f_1 & f_2 & \cdots & f_n \end{pmatrix}$ is a permutation of R. This permutation is called a Nakayama permutation. If there exists a ring automorphism ϕ of R satisfying $\phi(e_i) = f_i$, $i = 1, \cdots, n$, then ϕ is called a Nakayama automorphism of R.

For a ring R, End(R) and Aut(R) stand for the set of all ring endomorphisms of R and that of all automorphisms of R, respectively.

1. SKEW MATRIX RING

In this section we consider some structure theorem on skew matrix ring. After the first named author published the paper [4] in which these rings are introduced, Kuppish pointed out that he already introduced these rings in [3]. We note that almost results in this section were reported in [4].

Let Q be a ring and let $c \in Q$ and $\sigma \in$ End (Q) such that

$$\sigma(c) = c, \ \sigma(q)c = cq \text{ for all } q \in Q.$$

By R we denote the set of all $n \times n$ matrices over Q;

$$R = \begin{pmatrix} Q & \cdots & Q \\ & \cdots & \\ Q & \cdots & Q \end{pmatrix}$$

We define a multiplication in R which depends on (σ, c, n) as follows: For $(x_{ik}), (y_{ik})$ in R,

$$(z_{ik}) = (x_{ik})(y_{ik})$$

where z_{ik} is defined as follows:

(1) If $i \leq k$, $z_{ik} = \sum_{j<i} x_{ij}\sigma(y_{jk})c + \sum_{i\leq j\leq k} x_{ij}y_{jk} + \sum_{k<j} x_{ij}y_{jk}c$

(2) If $k < i$, $z_{ik} = \sum_{j\leq k} x_{ij}\sigma(y_{jk}) + \sum_{k<j<i} x_{ij}\sigma(y_{jk})c + \sum_{i\leq j} x_{ij}y_{jk}$

We may understand this operation as follows:

$$< a >_{ij} < b >_{jk} = \begin{cases} < a\sigma(b) >_{ik} & (j \leq k < i) \\ < a\sigma(b)c >_{ik} & (k < j < i \text{ or } j < i \leq k) \\ < ab >_{ik} & (i = j) \\ < abc >_{ik} & (i \leq k < j) \\ < ab >_{ik} & (k < i < j \text{ or } i < j \leq k) \end{cases}$$

Note that this operation satisfies associative law, i.e.,

$$(< x >_{ij} < y >_{jk}) < z >_{kl} = < x >_{ij} (< y >_{jk} < z >_{kl})$$

Therefore R becomes a ring by this multiplication together with the usual sum of matrices. We call R the skew matrix ring over Q with respect to (σ, c, n), and denote it by

$$R = \begin{pmatrix} Q & \cdots & Q \\ & \cdots & \\ Q & \cdots & Q \end{pmatrix}_{\sigma,c,n}$$

$$R = \begin{pmatrix} Q & \cdots & Q \\ & \cdots & \\ Q & \cdots & Q \end{pmatrix}_{\sigma,c}$$

if there are no confusions.

When $n = 2$, the multiplication is:

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} = \begin{pmatrix} x_1y_1 + x_2y_3c & x_1y_2 + x_2y_4 \\ x_3\sigma(y_1) + x_4y_3 & x_3\sigma(y_2)c + x_4y_4 \end{pmatrix}$$

Now, in the skew-matrix ring R above, we put $e_i = < 1 >_{ii}, i = 1, \cdots, n$. Then $\{e_1, \cdots, e_n\}$ is a set of orthogonal idempotents with $1 = e_1 + \cdots + e_n$, and

$$e_iR = \begin{pmatrix} & 0 & \\ Q & \cdots & Q \\ & 0 & \end{pmatrix} < i$$

$$Re_j = \begin{pmatrix} & Q & \\ 0 & \vdots & 0 \\ & Q & \end{pmatrix} < j$$

If Q is a local ring, then each e_i is a primitive idempotent.

Proposition 1. *The mapping $\tau : R \longrightarrow R$ given by*

$$\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ & \cdots & & \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix} \longrightarrow \begin{pmatrix} x_{nn} & x_{n1} & \cdots & x_{n,n-1} \\ \sigma(x_{1n}) & \sigma x_{11} & \cdots & \sigma(x_{1,n-1}) \\ & \cdots & & \\ \sigma(x_{n-1,n}) & \sigma(x_{n-1,1}) & \cdots & \sigma(x_{n-1,n-1}) \end{pmatrix}$$

is a ring homomorphism; in particular if $\sigma \in$ Aut (Q), then $\tau \in$ Aut(R).

Proof. Straightforward.

We put

$$W_i = \begin{pmatrix} & & & 0 & & & \\ Q & \cdots & Q & Qc & Q & \cdots & Q \\ & & & 0 & & & \end{pmatrix} < i$$

Then W_i is a submodule of e_iR_R. For $i = 2, \cdots, n$, let $\phi_i : e_iR \longrightarrow W_{i-1}$ be a map given by

$$\begin{pmatrix} & & 0 & & & \\ x_1 & \cdots & x_{i-1} & x_i & \cdots & x_n \\ & & 0 & & & \end{pmatrix} < i \longrightarrow \begin{pmatrix} & & 0 & & & \\ x_1 & \cdots & x_{i-1}c & x_i & \cdots & x_n \\ & & 0 & & & \end{pmatrix} < i-1$$

and let $\phi_1 : e_1R \longrightarrow W_n$ be a map given by

$$\begin{pmatrix} x_1 & \cdots & x_n \\ 0 & \cdots & 0 \\ & \cdots & \\ 0 & \cdots & 0 \end{pmatrix} < 1 \longrightarrow \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 \\ \sigma(x_1) & \cdots & \sigma(x_{n-1}) & \sigma(x_n)c \end{pmatrix} < n$$

Then it is easy to check the following

Proposition 2. *For each $\phi_i, i = 1, 2, \cdots, n$, is a homomorphism. In particular, if $\sigma \in Aut\ (Q)$, then each ϕ_i is an onto homomorphism and*

$$\mathrm{Ker}\phi_1 = \begin{pmatrix} 0 & \cdots & 0 & (0:c) \\ 0 & \cdots & 0 & 0 \\ & \cdots & & \\ 0 & \cdots & 0 & 0 \end{pmatrix} < 1$$

$$\mathrm{Ker}\phi_i = \begin{pmatrix} & 0 & \\ 0 & (0:c) & 0 \\ & 0 & \end{pmatrix} < i \text{ for} i = 2, \cdots, n.$$

where $(0 : c)$ is a right (or left) annihilator ideal of c.

Theorem 1. *If Q is a local ring, $\sigma \in Aut\ (Q)$ and $c \in J(Q)$, then the skew matrix ring R over Q w.r.to (σ, c, n) is a basic indecomposable QF-ring and $\begin{pmatrix} e_1 & e_2 & \cdots & e_n \\ e_n & e_1 & \cdots & e_{n-1} \end{pmatrix}$ is a Nakayama permutation where $\{e_1, e_2, \cdots, e_n\}$ is a set of orthogonal idempotents with $e_1 + e_2 + \cdots + e_n = 1$; whence R has a Nakayama automorphism by Proposition 1. Futhermore, for any idempotent e in R, eRe is represented as a skew-matrix ring over Q with respect to $(\sigma, c, k \leq n)$; so eRe is a QF-ring with cyclic Nakayama permutation.*

Proof. Put $X = S(Q_Q)$ $(= S(\,_QQ))$. Nothing $cX = Xc = 0$, we can easily see that

$$S(e_i R) = \begin{pmatrix} 0 & \cdots & 0 & X \\ & 0 & & \end{pmatrix} = S(Re_n)$$

for $i = 1, 2, \cdots$. Hence it follows that $(e_1 R : Re_n), (e_2 R : Re_1), \cdots (e_n R : Re_{n-1})$ are i-pairs. Therefore R is a QF-ring with a Nakayama authomorphism (cf. Proposition 1). For any subset $\{f_1, \cdots, f_k\} \subseteq E$, clealy, fRf is represented as a skew matrix ring over Q with respect to (σ, c, k), where $f = f_1 + \cdots + f_k$; whence so is represented eRe for any idempotent e in R.

By Theorem 1 and Proposition 2, we obtain

Corollary 1.(cf.[3]). *If Q is a Nakayama ring, $\sigma \in Aut\ (Q)$ and $cQ = J(Q)$, then the skew matrix ring R over Q w.r.to (σ, c, n) is a basic indecomposable QF-serial ring such that $\{e_n R, e_{n-1} R, \cdots, e_1 R\}$ is a Kuppisch series and $\begin{pmatrix} e_1 & e_2 & \cdots & e_n \\ e_n & e_1 & \cdots & e_{n-1} \end{pmatrix}$ is a Nakayama permutation. Futhermore, R has a Nakayama automorphism.*

2. Main Theorem

In this section we prove the following main theorem which is the converse of the above Theorem 1.

Theorem 2. *If R is a basic QF-ring such that for any idempotent e in R, eRe is a QF-ring with cyclic Nakayama permutation, then there exists a local QF-ring Q, an element c in the Jacobson radical of Q and a ring automorphism σ of Q for which R is represented as a skew-matrix ring:*

$$R \simeq \begin{pmatrix} Q & \cdots & Q \\ & \cdots & \\ Q & \cdots & Q \end{pmatrix}_{\sigma, c, n}$$

Let R be a basic QF-ring such that, for any idempotent e in R, eRe is a QF-ring with cyclic Nakayama permutation, and let E be a complete set of orthogonal primitive idempotents of R with $1 = \sum\{e|e \in E\}$. First we consider the cardinal $|E|$ of E is 2; let $E = \{e, f\}$. We represent R as

$$R = \begin{pmatrix} Q & A \\ B & T \end{pmatrix}$$

where $Q = (e, e)$, $A = (f, e)$, $B = (e, f)$, $T = (f, f)$. Since $\begin{pmatrix} e & f \\ f & e \end{pmatrix}$ is a Nakayama permutation, we see

$$S(eR) = S(Rf) = \begin{pmatrix} 0 & S(A) \\ 0 & 0 \end{pmatrix}, S(fR) = S(Re) = \begin{pmatrix} 0 & 0 \\ S(B) & 0 \end{pmatrix}$$

Nothing these facts, we can easily prove the following:

Lemma 1.

1) $\{a \in A|aB = 0\} = \{a \in A|Ba = 0\}$.
2) $\{b \in B|bA = 0\} = \{b \in B|Ab = 0\}$.

We denote the sets in 1) and 2) by A^* and B^*, respectively. Note that A^*, B^* are submodules of ${}_QA_T$, ${}_TB_Q$, respectively.

3)

$$\begin{pmatrix} 0 & A^* \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ B^* & 0 \end{pmatrix}$$

are ideals of R.

Now, we denote the factor ring $\bar{R} = \begin{pmatrix} Q & A \\ B & T \end{pmatrix} / \begin{pmatrix} 0 & 0 \\ B^* & 0 \end{pmatrix}$ by $\begin{pmatrix} Q & A \\ \bar{B} & T \end{pmatrix}$, and $r + \begin{pmatrix} 0 & 0 \\ B^* & 0 \end{pmatrix}$ by $\bar{r}$ for $r \in R$. Then $\{\bar{e}, \bar{f}\}$ is a complete set of orthogonal primitive idempotents of $\bar{R}$ and

$$S(\bar{f}\bar{R}) = \begin{pmatrix} 0 & 0 \\ 0 & S(T) \end{pmatrix}$$

Since eR_R is injective and $S(\bar{f}\bar{R}_R)_R$ is simple, we see $\begin{pmatrix} Q & A \\ 0 & 0 \end{pmatrix} \supseteq \begin{pmatrix} 0 & 0 \\ \bar{B} & T \end{pmatrix}$ as R (and as $\bar{R}$)-module. Since $S(A_T)_T$ is simple, it follows

$$A_T \simeq T_T$$

whence $\alpha T = A$ for some $\alpha \in A$. If $Q\alpha \not\subseteq {}_QA \simeq {}_QQ$, then $S(Q)\alpha = S(Q)Q\alpha = 0$; whence $S(Q)A = 0$, a contradiction. Hence

$$Q\alpha = \alpha T = A.$$

If $q \in Q$, then there exists $t \in T$ such that $q\alpha = \alpha t$. Then the mapping $\psi : Q \longrightarrow T$ given by $\psi(q) = t$ is a ring isomorphism. We exchange T by Q with respect to the isomorphism ψ;

$$R = \begin{pmatrix} Q & A \\ B & Q \end{pmatrix}.$$

Then

$$q\alpha = \alpha q \text{ for all } q \in Q.$$

Next, considering the factor ring $\begin{pmatrix} Q & A \\ B & Q \end{pmatrix} / \begin{pmatrix} 0 & A^* \\ 0 & 0 \end{pmatrix}$, we can obtain $\beta \in B$, $\sigma \in \mathrm{Aut}(Q)$ such that $B = Q\beta = \beta Q$ and

$$\beta q = \sigma(q)\beta \text{ for all } q \in Q.$$

We put $c = \alpha\beta$. Nothing , we see

$$\beta(\alpha\beta) = (\beta\alpha)\beta,$$

Further $\alpha\beta = \beta\alpha$. For, if $\alpha\beta - \beta\alpha \neq 0$, then $(\alpha\beta - \beta\alpha)A \neq 0$; so $0 \neq (\alpha\beta - \beta\alpha)\alpha = \alpha\beta\alpha - \beta\alpha\alpha = \alpha\beta\alpha - \alpha\beta\alpha$, contradiction. Thus $\alpha\beta = \beta\alpha$ and hence

$$\sigma(c) = c.$$

And we can see easily that $c \in J(Q)$ and $\sigma(q)c = cq$ for any $q \in Q$. Now, for

$$X = \begin{pmatrix} x_1 & x_2\alpha \\ x_3\beta & x_4 \end{pmatrix}, Y = \begin{pmatrix} y_1 & y_2\alpha \\ y_3\beta & y_4 \end{pmatrix} \in R = \begin{pmatrix} Q_1 & Q\alpha \\ Q\beta & Q \end{pmatrix},$$

we calculate XY, and see

$$XY = \begin{pmatrix} x_1y_1 + x_1y_3c & (x_1y_2 + x_2y_4)\alpha \\ (x_3\sigma(y_1) + x_4y_3)\beta & x_3\sigma(y_2)c + x_4y_4 \end{pmatrix}.$$

Thus we see that R is isomorphic to the skew matrix ring $\begin{pmatrix} Q & Q \\ Q & Q \end{pmatrix}_{\sigma,c}$ by the mapping

$$\begin{pmatrix} x_1 & x_2\alpha \\ x_3\beta & x_4 \end{pmatrix} \longrightarrow \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$$

Next, consider the case of $|E| = 3$; put $E = \{e_1, e_2, e_3\}$. We may assume that $\begin{pmatrix} e_1 & e_2 & \cdots & e_n \\ e_n & e_1 & \cdots & e_{n-1} \end{pmatrix}$ is a Nakayama permutation. We represent R as

$$R = \begin{pmatrix} (e_1, e_1) & (e_2, e_1) & (e_3, e_1) \\ (e_1, e_2) & (e_2, e_2) & (e_3, e_2) \\ (e_1, e_3) & (e_2, e_3) & (e_3, e_3) \end{pmatrix} = \begin{pmatrix} Q_1 & A_{12} & A_{13} \\ A_{21} & Q_2 & A_{23} \\ A_{31} & A_{32} & Q_3 \end{pmatrix}$$

We Put $Q = Q_1$. Considering $\begin{pmatrix} Q_1 & A_{12} \\ A_{21} & Q_2 \end{pmatrix}$, $\begin{pmatrix} Q_1 & A_{13} \\ A_{31} & Q_3 \end{pmatrix}$ and $\begin{pmatrix} Q_2 & A_{23} \\ A_{32} & Q_3 \end{pmatrix}$, we can assume that $Q = Q_2 = Q_3$ by the argument above;

$$R = \begin{pmatrix} Q & A_{12} & A_{13} \\ A_{21} & Q & A_{23} \\ A_{31} & A_{32} & Q \end{pmatrix}.$$

and then note that ${}_Q(A_{ij})_Q \simeq {}_QQ_Q$ for each ij.

Nothing that

$$S(e_1R) = S(Re_3) = \begin{pmatrix} 0 & 0 & S(A_{13}) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$S(e_2R) = S(Re_1) = \begin{pmatrix} 0 & 0 & 0 \\ S(A_{21}) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$S(e_3R) = S(Re_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & S(A_{32}) & 0 \end{pmatrix},$$

we prove the following

Lemma 2.

(1)
$$\begin{aligned} \{x \in A_{32} | xA_{23} = 0\} &= \{x \in A_{32} | A_{23}x = 0\} \\ &= \{x \in A_{32} | xA_{21} = 0\} \\ &= \{x \in A_{32} | A_{13}x = 0\}. \end{aligned}$$

(2)
$$\begin{aligned} \{x \in A_{21} | xA_{12} = 0\} &= \{x \in A_{21} | xA_{13} = 0\} \\ &= \{x \in A_{21} | A_{12}x = 0\} \\ &= \{x \in A_{21} | A_{32}x = 0\}. \end{aligned}$$

(3) $$\{x \in A_{13} | xA_{31} = 0\} = \{x \in A_{13} | A_{31}x = 0\} = \{x \in A_{13} | xA_{32} = 0\} = \{x \in A_{13} | A_{21}x = 0\}$$

Proof. 1) By Lemma 1, $\{x \in A_{32} | xA_{23} = 0\} = \{x \in A_{32} | A_{23}x = 0\}$. Let $x \in A_{32}$ such that $xA_{23} = 0$. If $xA_{21} \neq 0$, then $A_{23}xA_{21} \neq 0$; whence $A_{23}x \neq 0$, a contradiction. If $A_{13}x \neq 0$, then $A_{13}xA_{23} \neq 0$; whence $xA_{23} \neq 0$, a contradiction. Thus $\{x \in A_{32} | xA_{23} = 0\} \subseteq \{x \in A_{32} | xA_{21} = 0\}$ and $\{x \in A_{32} | xA_{23} = 0\} \subseteq \{x \in A_{32} | A_{13}x = 0\}$.

Let $x \in A_{32}$ such that $xA_{21} = 0$. If $xA_{23} \neq 0$, then we see from ${}_QQ \simeq {}_QA_{31}$ that $xA_{23}A_{31} \neq 0$; so $xA_{21} = 0$. If $xA_{23} \neq 0$, a contradiction. Hence $\{x \in A_{32} | xA_{23} = 0\} = \{x \in A_{32} | xA_2 = 0\}$. Let $x \in A_{32}$ such that $A_{13}x = 0$. If $xA_{23} \neq 0$, then $A_{13}xA_{23} \neq 0$; so $A_{13}x \neq 0$, a contradiction. Hence $\{x \in A_{32} | xA_{23} = 0\} = \{x \in A_{32} | A_{13}x = 0\}$. Similarly we can prove 2) and 3).

We put the sets in 1), 2) and 3) above by $A^*_{32}, A^*_{21}, A^*_{13}$, respectively. we see ${}_Q(A^*_{32})_Q, {}_Q(A^*_{21})_Q, {}_Q(A^*_{31})_Q$ are submodules of ${}_Q(A_{32})_Q, {}_Q(A_{21})_Q, {}_Q(A_{13})_Q$, respectively. Futher we put

$$X_{13} = \begin{pmatrix} 0 & 0 & A^*_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X_{21} = \begin{pmatrix} 0 & 0 & 0 \\ A^*_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X_{32} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & A^*_{32} & 0 \end{pmatrix}$$

These are ideals of R. Consider the factor rings $\bar{R} = R/X_{ij}$ where $X_{ij} = X_{13} + X_{21} + X_{32}$,and put $\bar{r} = r + X_{ij}$ for $r \in R$. We can easily see that

$$S(\bar{e}_1\bar{R})_{\bar{R}} = S(\bar{e}_1\bar{R})_R = \begin{pmatrix} 0 & S(A_{12}) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$S(\bar{e}_2\bar{R})_{\bar{R}} = S(\bar{e}_2\bar{R})_R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & S(A_{23}) \\ 0 & 0 & 0 \end{pmatrix}$$

$$S(\bar{e}_3\bar{R})_{\bar{R}} = S(\bar{e}_3\bar{R})_R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ S(A_{31}) & 0 & 0 \end{pmatrix}$$

Therefore there are monomorphism $\theta_3 : \bar{e}_3\bar{R}_R \longrightarrow e_2R_R$, $\theta_2 : \bar{e}_2\bar{R}_R \longrightarrow e_1R_R$ and $\theta_1 : \bar{e}_1R_R \longrightarrow e_3R_R$. We put $\gamma_i = \theta_i\eta_i$ for $i = 1, 2, 3$, where η_i is a canonical homomorphism: $e_iR_R \longrightarrow \bar{e}_i\bar{R}_R$.

Nothing

$$\gamma_1 = (\begin{pmatrix} 0 & A_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & A_{32} & 0 \end{pmatrix},$$

$$\gamma_2 = (\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & A_{23} \\ 0 & 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & 0 & A_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\gamma_3 = (\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A_{31} & 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & 0 & 0 \\ A_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and using Lemma 1, we can easily prove the following

Lemma 3.

(1)
$$\begin{aligned}\{x \in A_{31} | xA_{12} = 0\} &= \{x \in A_{31} | xA_{13} = 0\} \\ &= \{x \in A_{31} | A_{13}x = 0\} \\ &= \{x \in A_{31} | A_{23}x = 0\}.\end{aligned}$$

(2)
$$\begin{aligned}\{x \in A_{23} | xA_{31} = 0\} &= \{x \in A_{23} | xA_{32} = 0\} \\ &= \{x \in A_{23} | A_{32}x = 0\} \\ &= \{x \in A_{23} | A_{12}x = 0\}.\end{aligned}$$

$$(3) \qquad \{x \in A_{12} | xA_{21} = 0\} = \{x \in A_{12} | xA_{23} = 0\}$$
$$= \{x \in A_{12} | A_{31}x = 0\}$$
$$= \{x \in A_{12} | A_{21}x = 0\}$$

We denote the sets in 1), 2) and 3) by A^*_{31}, A^*_{23} and A^*_{12}, respectively, and put

$$X_{31} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A^*_{31} & 0 & 0 \end{pmatrix}, X_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & A^*_{23} \\ 0 & 0 & 0 \end{pmatrix}, X_{12} = \begin{pmatrix} 0 & A^*_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then

$$(4) \qquad \gamma_3(X_{31}) = X_{21}, \gamma_2(X_{23}) = X_{13} \text{and} \gamma_1(X_{12}) = X_{32}.$$

Lemma 4. *There exists* $\alpha_{12} \in A_{12}, \alpha_{21} \in A_{21}$, $c \in J(Q)$ *and* $\sigma \in Aut(Q)$ *such that*

$$(1) \qquad c = \alpha_{12}\alpha_{21} = \alpha_{21}\alpha_{12}$$
$$\alpha_{12}q = q\alpha_{12} \text{ for all } q \in Q$$
$$\sigma(q)\alpha_{21} = \alpha_{21}q \text{ for all } q \in Q$$

$$(2) \qquad \begin{pmatrix} Q & A_{12} \\ A_{21} & Q \end{pmatrix} \simeq \begin{pmatrix} Q & Q \\ Q & Q \end{pmatrix}_{\sigma,c}$$

by the mapping:

$$\begin{pmatrix} q_{11} & q_{12}\alpha_{12} \\ q_{21}\alpha_{21} & q_{22} \end{pmatrix} \longrightarrow \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$$

$$\text{(3)} \qquad \mathrm{Im}\theta_3 = \begin{pmatrix} 0 & 0 & 0 \\ A_{21} & cQ & A_{23} \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathrm{Im}\theta_2 = \begin{pmatrix} cQ & A_{12} & A_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathrm{Im}\theta_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A_{31} & A_{32} & Qc \end{pmatrix}$$

4) Im θ_1, Im θ_2, Im θ_3, Im $\theta_2\theta_3$, Im $\theta_1\theta_2$ and Im $\theta_3\theta_1$ are quasi-injective (or equivalently, fully invariant) submodules of e_3R, e_1R_R, e_2R_R, e_1R_R, e_3R_R and e_2R_R, respectively.

Proof. Considering $\begin{pmatrix} Q & A_{12} \\ A_{21} & Q \end{pmatrix}$, we get $\alpha_{12} \in A_{12}$, $\alpha_{21} \in A_{21}$, $c \in J(Q)$ and $\sigma \in \mathrm{Aut(Q)}$ for which 1) and 2) hold. Furthermore, considering $\begin{pmatrix} Q & A_{23} \\ A_{32} & Q \end{pmatrix}$ and $\begin{pmatrix} Q & A_{13} \\ A_{31} & Q \end{pmatrix}$, we get $c_2, c_3 \in J(Q)$ and $\sigma_2, \sigma_3 \in \mathrm{Aut(Q)}$ for which

$$\begin{pmatrix} Q & A_{23} \\ A_{32} & Q \end{pmatrix} \simeq \begin{pmatrix} Q & Q \\ Q & Q \end{pmatrix}_{\sigma_2, c_2}, \begin{pmatrix} Q & A_{13} \\ A_{31} & Q \end{pmatrix} \simeq \begin{pmatrix} Q & Q \\ Q & Q \end{pmatrix}_{\sigma_3, c_3},$$

we see that

$${}_Q(\bar{A}_{12})_Q \simeq_Q cQ_Q,\ {}_Q(\bar{A}_{21})_Q \simeq {}_QcQ_Q,\ {}_Q(\bar{A}_{13})_Q \simeq {}_Qc_3Q_Q,$$

$${}_Q(\bar{A}_{31})_Q \simeq_Q c_3Q_Q,\ {}_Q(\bar{A}_{32})_Q \simeq {}_Qc_2Q_Q,\ {}_Q(\bar{A}_{23})_Q \simeq {}_Qc_2Q_Q,$$

where $\bar{A}_{ij} = A_{ij}/A^*_{ij}$.

Further, as

$$e_1R/X_{12}+X_{13} = \begin{pmatrix} Q & \bar{A}_{12} & \bar{A}_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \simeq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A_{31} & \bar{A}_{32} & c_3Q \end{pmatrix} \subseteq e_3R/X_{32}$$

$$e_2R/X_{21}+X_{23} = \begin{pmatrix} 0 & 0 & 0 \\ \bar{A}_{21} & Q & \bar{A}_{23} \\ 0 & 0 & 0 \end{pmatrix} \simeq \begin{pmatrix} Qc & A_{12} & \bar{A}_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \subseteq e_1R/X_{32}$$

$$e_3R/X_{31}+X_{32} = \begin{pmatrix} Q & Q & Q \\ 0 & 0 & 0 \\ \bar{A}_{31} & \bar{A}_{32} & Q \end{pmatrix} \simeq \begin{pmatrix} 0 & 0 & 0 \\ \bar{A}_{21} & Qc_2 & A_{23} \\ 0 & 0 & 0 \end{pmatrix} \subseteq e_2R/X_{21}$$

we see that $(\bar{A}_{ij})_Q \simeq (\bar{A}_{kj})_Q$ for $i \neq k$ and $cQ_Q \simeq c_2Q_Q \simeq c_3Q_Q$. Since cQ_Q, c_2Q_Q and c_3Q are fully invariant submodules of Q, it follows that $cQ = c_2Q = c_3Q$. Hence 3) is proved. 4) is clear.

Lemma 5. *1) For any $\psi \in (e_3, e_2)$, Im $\psi \subseteq$ Im θ_3. For any $\psi \in (e_2, e_1)$, Im $\psi \subset$ Im θ_2. For any $\psi \in (e_1, e_3)$, Im $\psi \subset$ Im θ_1.*

2) For any $\psi \in (e_3, e_2)$, Im $\psi \subseteq$ Im $\theta_2\theta_3$. For any $\psi \in (e_2, e_3)$, Im $\psi \subseteq$ Im $\theta_1\theta_2$. For any $\psi \in (e_1, e_2)$, Im $\psi \subseteq$ Im $\theta_3\theta_1$.

Proof. let $\psi \in (e_3, e_2)$. If $x \in A^*_{32}$ and $\begin{pmatrix} 0 & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & 0 \end{pmatrix} \psi(< x >_{32}) \neq 0$, then $\psi(< x >_{32}) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & A_{23} \\ 0 & 0 & 0 \end{pmatrix} \neq 0$, but $< x >_{23} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & A_{23} \\ 0 & 0 & 0 \end{pmatrix} = 0$, which is impossible. Hence $\psi(\{< x >_{32} | x \in$

$A_{32}\}) = 0$ and there exists an epimorphism from $\operatorname{Im}\theta_3 = \begin{pmatrix} 0 & 0 & 0 \\ A_{21} & cQ & A_{23} \\ 0 & 0 & 0 \end{pmatrix}$ to $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A_{31} & A_{32} & Q \end{pmatrix} / \operatorname{Ker}\psi \simeq \operatorname{Im}\psi$. Since $\operatorname{Im}\theta_3$ is a fully invariant submodule of e_2R, we see $\operatorname{Im}\psi \subset \operatorname{Im}\theta_3$.

Similarly we can see the rest parts.

Next for $\psi \in (e_3, e_1)$, we see $\psi(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A^*_{31} & A^*_{32} & 0 \end{pmatrix}) = 0$. Hence it follows that $\operatorname{Im}\psi \subseteq \operatorname{Im}\theta_2\theta_3$.

Now consider the factor ring $\bar{R} = R/X_{32}$ and denote $r + X_{32}$ by $\bar{r}$ for $r \in R$. We represent $\bar{R}$ as

$$\begin{aligned} \bar{R} &= \bar{e}_1\bar{R} \oplus \bar{e}_2\bar{R} \oplus \bar{e}_3\bar{R} \\ &= \begin{pmatrix} (e_1, e_1) & (e_2, e_1) & (\bar{e}_3, e_1) \\ (e_1, e_2) & (e_2, e_2) & (\bar{e}_3, e_2) \\ (e_1, \bar{e}_3) & (e_2, \bar{e}_3) & (\bar{e}_3, \bar{e}_3) \end{pmatrix} \\ &= \begin{pmatrix} Q & A_{12} & A_{13} \\ A_{21} & Q & A_{23} \\ A_{31} & \bar{A}_{32} & Q \end{pmatrix} \end{aligned}$$

where $\bar{A}_{32} = A_{32}/A^*_{32}$.

Lemma 6. *The mapping*

$$\tau = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix} : \begin{pmatrix} Q & A_{12} & A_{12} \\ A_{21} & Q & Q \\ A_{21} & I & Q \end{pmatrix} \longrightarrow \bar{R} = \begin{pmatrix} Q & A_{12} & A_{13} \\ A_{21} & Q & A_{23} \\ A_{31} & A_{32} & Q \end{pmatrix}$$

where $I = \theta_3\bar{A}_{32}$, given by

$$\begin{pmatrix} q_{11} & q_{12} & p_{12} \\ q_{21} & q_{22} & p_{22} \\ t_{21} & t_{22} & y_{22} \end{pmatrix} \longrightarrow \begin{pmatrix} q_{11} & q_{12} & p_{12}\theta_3 \\ q_{21} & q_{22} & p_{22}\theta_3 \\ \theta_3^{-1}t_{21} & \theta_3^{-1}t_{22} & \theta_3^{-1}y_{22}\theta_{33} \end{pmatrix}$$

is a ring isomorphism.

Proof. By Lemma 5, τ is well-defined and furthermore it is a ring monomorphism. Nothing e_1R_R is injective, we can see that α_{13} is an onto mappping. And nothing e_2R_R is injective, we see that τ_{23} and τ_{33} are onto mapping. It is easy to see that τ_{31} is an onto mapping. τ_{32} is a clearly onto mapping. Hence τ is a ring isomorphism.

By the lemma above, we see $(\bar{A}_{32})_R \simeq I_R$ and hence we see that $I = cQ$. Hence

$$\tau : \begin{pmatrix} Q & A_{12} & A_{12} \\ A_{21} & Q & Q \\ A_{21} & I & Q \end{pmatrix} \simeq \begin{pmatrix} Q & A_{12} & A_{13} \\ A_{21} & Q & A_{23} \\ A_{31} & A_{32} & Q \end{pmatrix}$$

We put $< \alpha_{31} >_{31} = \tau(< \alpha_{21} >_{31}), < \alpha_{13} >_{13} = \tau(< \alpha_{12} >_{13})$ and $\alpha_{32} = \alpha_{31}\alpha_{12}$. Since A_{32}^* is a small submodule of A_{32}, we see that $\alpha_{32}Q = A_{32}$. Hence R is represented as $R \simeq \begin{pmatrix} Q & \alpha_{12}Q & \alpha_{12}Q \\ \alpha_{21}Q & Q & \alpha_{23}Q \\ \alpha_{31}Q & \alpha_{32}Q & Q \end{pmatrix}$ with relations:

$$c = \alpha_{21}\alpha_{12} = \alpha_{12}\alpha_{21}$$
$$\sigma(c) = c$$
$$\alpha_{12}q = q\alpha_{12} \text{ for all } q \in Q$$
$$\sigma(q)\alpha_{21} = \alpha_{21}q \text{ for all } q \in Q.$$

Putting $\alpha_{ii} = 1$ for $i = 1, 2, 3$, we further obtain the following relations $(*)$:

If $i > j$,

$$\sigma(q)\alpha_{ij} = \alpha_{ij}q \text{ for all } q \in Q$$

$$\alpha_{ij}\alpha_{jk} = \begin{cases} \alpha_{ik} & (i > k \geq j) \\ \alpha_{ik}c & (k \geq i \text{ or } j > k) \end{cases}$$

If $i = j$,

$$q\alpha_{ij} = \alpha_{ij}q \text{ for all } q \in Q$$

$$\alpha_{ij}\alpha_{jk}$$

If $i < j$,

$$q\alpha_{ij} = \alpha_{ij}q \text{ for all } q \in Q$$

$$\alpha_{ij}\alpha_{jk} = \begin{cases} \alpha_{ik}c & (i \leq k < j) \\ \alpha_{ik} & (k < i \text{ or } j \leq k) \end{cases}$$

for $1 \leq i, j \leq 3$.

By these relations, we see that R is isomorphic to the skew-matrix ring $\begin{pmatrix} Q & Q & Q \\ Q & Q & Q \\ Q & Q & Q \end{pmatrix}_{\sigma,c}$ by the mapping

$$\begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix} \longrightarrow \begin{pmatrix} q_{11}\alpha_{11} & q_{12}\alpha_{12} & q_{13}\alpha_{13} \\ q_{21}\alpha_{21} & q_{22}\alpha_{22} & q_{23}\alpha_{23} \\ q_{31}\alpha_{21} & q_{32}\alpha_{32} & q_{33}\alpha_{33} \end{pmatrix}$$

For the mathematical induction, we assume that for any basic QF-ring R with $n-1$ orthogonal primitive idempotents such that for any idempotent e in R, eRe is a QF-ring with cyclic Nakayama permutation, then R is represented as skew matrix ring over a local QF-ring Q w.r.to $(\sigma, c, n-1)$, where $\sigma \in \mathrm{Aut}\ (Q)$ and $c \in J(Q)$.

Now, consider the case $|E| = n$; let $E = \{e_1, e_2, \cdots, e_n\}$. We may assume that $\begin{pmatrix} e_1, & e_2 & \cdots & e_n \\ e_n & e_1 & \cdots & e_{n-1} \end{pmatrix}$ is a Nakayama permutation. We represent R as

$$R = \begin{bmatrix} Q_1 & A_{12} & A_{13} & \cdots & A_{1n} \\ A_{21} & Q_2 & A_{23} & \cdots & A_{2n} \\ & \cdots & & \cdots & \\ A_{n1} & A_{n2} & A_{n3} & \cdots & Q_n \end{bmatrix}$$

where $Q_i = (e_i, e_i)$ and $A_{ij} = (e_j, e_i)$. Considering $(n-1) \times (n-1)$ minor matrices in R including diagonal line, and applying induction hypothesis, we can assume that $Q_1 = Q_2 = \cdots = Q_n$ and ${}_Q(A_{ij})_Q \cong_Q Q_Q$ for each ij.

Specially we look at the first minor matrix

$$R_0 = \begin{bmatrix} Q & A_{12} & \cdots & A_{1\ n-1} \\ A_{21} & Q & \cdots & A_{2\ n-1} \\ & \cdots & & \cdots \\ A_{n-1\ 1} & A_{n-1\ 2} & \cdots & Q \end{bmatrix}$$

which is isomorphic to skew matrix ring over Q w.r.to $(\sigma, c, n-1)$ where $\sigma \in \mathrm{Aut}\ (Q)$ and $c \in J(Q)$.

Now we consider an extension ring R_1 of R_0,

$$R_1 = \left(\begin{array}{ccc|c} & & & A_{1n-1} \\ & R_0 & & \vdots \\ & & & A_{n-2n-1} \\ & & & Q \\ \hline A_{n-11} & \cdots & A_{n-1n-2}\quad cQ & Q \end{array}\right)$$

By the similar argument which is used $n = 3$ case.

R_1 is isomorphic to

$$R_2 = \left(\begin{array}{ccc|c} & R_0 & & \begin{array}{c} A_{1n} \\ \vdots \\ A_{n-2n} \\ A_{n-1n} \end{array} \\ \hline A_{n1} & \cdots & \bar{A}_{nn-1} & Q \end{array}\right)$$

where $\bar{A}_{nn-1} = A_{nn-1}/A^{*}_{nn-1}$.

$$\begin{aligned} A^{*}_{nn-1} &= \{x \in A_{nn-1} | xA_{n-1j} = 0 \; j = 1, 2, \cdots n-2, n\} \\ &= \{x \in A_{nn-1} | A_{in}x = 0 \; i = 1, 2, \cdots, n-1\} \end{aligned}$$

,which is (Q, Q)-submodule of $A_{n,n-1}$.

Note that $\bar{A}_{nn-1}$ is isomorphic to cQ.

Let $\tau = (\tau_{ij})$ be the ring isomorphism between R_1 and R_2. We put $< \alpha_{in} >_{in} = \tau(< \alpha_{in-1} >_{in})$ and $< \alpha_{nj} >_{nj} = \tau(< \alpha_{n-1j} >_{nj})$ where $i = 1, 2, \cdots, n-2$ and $j = 1, 2, \cdots, n-2$. We take $\alpha_{nn-1} = \alpha_{nn-2}\alpha_{n-2n-1} \in A_{nn-1}$ and $\tau(< 1 >_{n-1n}) = \alpha_{n-1n}$. Since A^{*}_{nn-1} is a small submodule of A_{nn-1}, we see that $\alpha_{nn-1}Q = A_{nn-1}$. Hence

$$R \simeq \left(\begin{array}{ccc|c} & R_0 & & \begin{array}{c} \alpha_{1n}Q \\ \vdots \\ \alpha_{n-1n}Q \end{array} \\ \hline \alpha_{n1}Q & \cdots & \alpha_{nn-1}Q & Q \end{array}\right)$$

$$\simeq \left(\begin{array}{ccc|c} & (\alpha_{ij}Q)_{n-1n-1} & & \begin{array}{c} \alpha_{1n}Q \\ \vdots \\ \alpha_{n-1n}Q \end{array} \\ \hline \alpha_{n1}Q & \cdots & \alpha_{nn-1}Q & Q \end{array}\right) \text{(induction hypothesis)}$$

with the relations $(*)$ for $1 \leq i, j \leq n$.

We see that R is isomorphic to the skew matrix ring

$$\begin{pmatrix} Q & Q & \cdots & Q \\ & \cdots & & \\ Q & Q & \cdots & Q \end{pmatrix}_{\sigma,c,n} \quad \text{by the mapping}$$

$$(q_{ij}) \longrightarrow (q_{ij}\alpha_{ij}).$$

Our argument inductively works. So, we can prove our theorem for any case $n = |E|$.

Combining Theorem 1 and 2 we have the following;

Corollary 2. *If R is a basic QF-ring such that for any idempotent e in R, eRe is a QF-ring with cyclic Nakayama permutation, then R has a Nakayama automorphism.*

Acknowledgements

The second author wishes to express his appreciation to Professor Oshiro who introduced this topic to him. He also thanks to KOSEF-TGRC for the grant 1992 and the Organiging Committee of the 27th Ring Theory Symposium for their financial supports.

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PROCEEDINGS OF THE 25TH
SYMPOSIUM ON RING THEORY, 1992

ON A CLASSIFICATION OF CANONICAL ALGEBRAS

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0. Introduction

Let K be an algebraically closed field. In his paper [6], C.M.Ringel has introduced and studied the canonical K-algebras. In this note, we will introduce canonical algebras and remark that, by [4] and [6], they are classifid by checking Coxeter matrix-orbit of some C-module W_0. And we will consider the component containing W_0.

1. Definitions

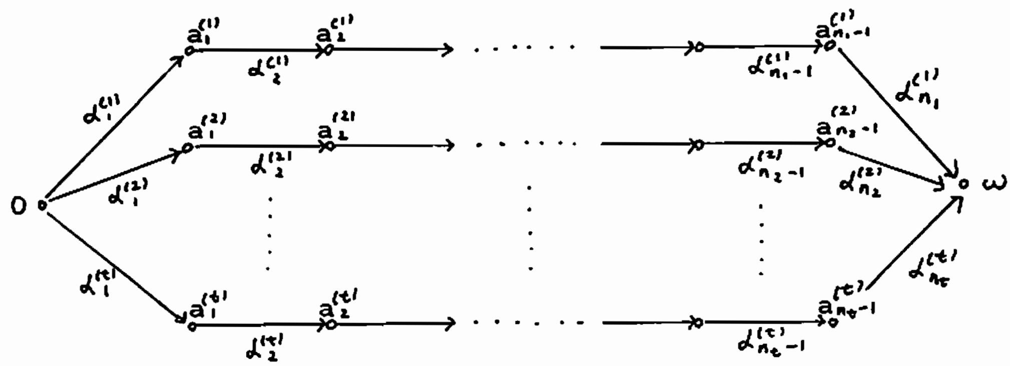
Let K be an algebraically closed field, and let A be a finite dimensional K-algebra. Let A-mod be the category of all finite dimensional left A-modules. In this note, objects of A-mod is said to be A-modules. Let {P(1),⋯⋯, P(ℓ) } be the isomorphism classes of indecomposable projective A-modules. Let $K_0(A)$ be the Grothendieck group of A. We know that $K_0(A)$ can identified with $\mathbb{Z}^{\ell}$. Let M be an A-module, and let $\underline{\dim}M$ be the row vector with it's i-component being $(\underline{\dim}M)_i = \dim_K \mathrm{Hom}_A(P(i),M)$. We call $\underline{\dim}M$ the dimension vector of M. Clearly, $\underline{\dim}M \in K_0(A)$. Let B be the following matrix;

$$B = (\underline{\dim}P(1)^T, \underline{\dim}P(2)^T, \cdots\cdots\cdots, \underline{\dim}P(\ell)^T)$$

The detailed version of section4 will be submitted for publication elsewhere.

B is called the Cartan matrix of A. If B^{-T} is defined, then let Φ be $-B^{-T}B$. Φ is called the Coxeter matrix of A.

For $n_1, \cdots, n_t \in \mathbb{N}$ with $t \geq 2$, we denote by $\Delta(n_1, \cdots, n_t)$ the following quiver;



Let $\alpha^{(s)}$ be the pass $\alpha^{(s)}_{n_s-1} \cdots \alpha^{(s)}_2 \alpha^{(s)}_1$ and let I be the vector space with basis $\alpha^{(1)}, \cdots, \alpha^{(t)}$. A subspace J of I is said to be generic provided $\dim J = t-2$ and J intersects any 2-dimensional coordinate subspace $\langle \alpha^{(s)}, \alpha^{(s')} \rangle$ (where $s \neq s'$) in zero. The algebras given by the quiver $\Delta(n_1, \cdots, n_t)$ with generic relations J is said to be the canonical algebras of type $(n_1, \cdots, n_t)$.

Let C be a canonical algebra of type $(n_1, \cdots, n_t)$. Then we can define the linear form $\eta : K_0(C) \rightarrow \mathbb{Z}$ given by $-x_0 + x_\omega$ (where x_a is a-component of x). Let $\mathcal{P}$, $\mathcal{T}$ or $\mathcal{Q}$ be the module class of all indecomposable C-modules M with $\eta(\underline{\dim} M) < 0$, $= 0$ or > 0, respectively. Aiso, let $T_{n_1 \cdots n_t}$ be the quiver obtained from $\Delta(n_1, \cdots, n_t)$ by deleting the vertex ω, and let $R = \text{rad } P(\omega)$, where $P(\omega)$ is the indecomposable projective C-module corresponding to ω. Let C_0 be the hereditary algebra given by $T_{n_1 \cdots n_t}$, and let W_0 be the indecomposable injective C_0-module corresponding to the vertex 0. C.M.Ringel has proved in [6] the following;

Theorem 1.1 ([6] 3.7) $\mathcal{T}$ is a sincere stable tubular $\mathbb{P}_1K$-family of type $(n_1,\cdots,n_t)$, separating $\mathcal{P}$ from $\mathcal{Q}$ (where $\mathbb{P}_1K$ is the projective line over K).

In the proof, W_0 and R played important roles.

Theorem 1.2 ([6] 3.7) For an indecomposable C-module M, M is given in the form $M = (M_a, \alpha_i^{(s)})$, where a runs through the vertices and $\alpha_i^{(s)}$ through the arrows of $\Delta(n_1,\cdots,n_t)$, thus $1 \leq i \leq n_s$, $1 \leq s \leq t$. M belongs to $\mathcal{P}$ if and only if all $\alpha_i^{(s)}$ are mono, and not all $\alpha_i^{(s)}$ are isomorphisms. Dually M belongs to $\mathcal{Q}$ if and only if all $\alpha_i^{(s)}$ are epi and not all $\alpha_i^{(s)}$ are isomorphisms. Finally, M belongs to $\mathcal{T}$ if and only if not all $\alpha_i^{(s)}$ are mono and also not all $\alpha_i^{(s)}$ are epi, or all $\alpha_i^{(s)}$ are isomorphisms.

Remark C_0-module W_0 is also C-module. And <u>dim</u>M is

$$\underline{\dim}M = (1,\underbrace{1,\cdots,1}_{n_1-1},1,\cdots\cdots,\underbrace{1,\cdots,1}_{n_t-1},0)$$

<u>dim</u>R is the following form;

$$\underline{\dim}R = (2,\underbrace{1,\cdots,1}_{n_1-1},1,\cdots\cdots\underbrace{1,\cdots,1}_{n_t-1},0)$$

2. Auslander-Riten translation and Coxeter matrix

Given a canonical algebra C of type $(n_1,\cdots,n_t)$, we can assume $n_1 \geq n_2 \geq \cdots \geq n_t$. If $t \geq 3$, we can assume, in addition, that $n_t \geq 2$ ([6]3.7). Note that we may consider C as a one-point extension $C=C_0[R]$, where R = rad $P(\omega)$, and where C_0 is the algebra given by the quiver $T_{n_1\cdots n_t}$.

In this section, we are going to study the relation between Coxeter matrix Φ of C and the Auslander-Reiten translation τ. Clearly, Φ is of the following form;

$$
\Phi = \left(\begin{array}{c|c|c|c|c|c}
-1 & -1 \cdots -1 & -1 \cdots -1 & \cdots\cdots & -1 \cdots -1 & -2 \\
\hline
\begin{matrix}1\\0\\ \vdots\\0\end{matrix} & K(1,1) & K(1,2) & \cdots\cdots & K(1,t) & \begin{matrix}1\\0\\ \vdots\\0\end{matrix} \\
\hline
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\hline
\begin{matrix}1\\0\\ \vdots\\0\end{matrix} & K(t,1) & K(t,2) & \cdots\cdots & K(t,t) & \begin{matrix}1\\0\\ \vdots\\0\end{matrix} \\
\hline
2-t & L(1) & L(2) & \cdots\cdots & L(t) & 3-t
\end{array}\right)
$$

where

$$
K(i,i) = \begin{cases} \begin{pmatrix} 0 & & O \\ 1 & \ddots & \\ O & 1 & 0 \end{pmatrix} & (n_i-1)\times(n_i-1)\,\text{matrix} \quad (n_i>2) \\ (0) & (n_i=2) \end{cases}
$$

$$
K(i,j) = \begin{cases} \begin{pmatrix} 1 \cdots 1 \\ \hline O \end{pmatrix} & (n_i-1)\times(n_j-1)\,\text{matrix} \quad (n_i>2) \\ (1 \cdots 1) & (n_i=2) \end{cases}
$$

(where $i \neq j$)

$$
L(i) = (\underbrace{2-t, \cdots, 2-t}_{n_i-2}, 3-t)
$$

Theorem 2.1 Suppose that, for an indecomposable C-module X in $\mathcal{P}$, $\tau^m X$ is defined for a non-negative integer m. If $\tau^m X$ is non-projective and if $\eta(\underline{\dim}X)=-1$, then $\mathrm{Hom}_C(\tau^m X, C)=0$, and $\underline{\dim}\,\tau^{m+1}X=(\underline{\dim}\,x)\Phi^{m+1}$.

3. A classification of canonical algebras

On the proofs of this section, we were advised by H.Lenzing.

We recall that W_0 is the indecomposable injective C_0-module corresponding to the vertex 0.

By results of [4] and [6], we obtain the following results immediately.

Theorem 3.1 (1) The following propertis are equivalent;

(a) $T_{n_1,\dots,n_t}$ is of Dynkin type.

(a') There exists a positive integer m such that the ω-component of $(\underline{\dim} W_0)\Phi^m$ is negative.

(2) The following properties are equivalent;

(b) $T_{n_1,\dots,n_t}$ is of Euclidean type.

(b') For any positive integer m, an ω-component of $(\underline{\dim} W_0)\Phi^m$ is non-negative, and there exists an positive integer m such that the ω-component of $(\underline{\dim} W_0)\Phi^m$ is zero.

(3) The following properties are equivalent;

(c) $T_{n_1,\dots,n_t}$ is of neither Dynkin type nor Euclidean type.

(c') For any positive integer m, an ω-component of $(\underline{\dim} W_0)\Phi^m$ is positive.

Corollary 3.2 (1) Theorem3.1(1)(a) is equivalent to the following;

(a″) There exists a positive integer m′ and an indecomposable projective C-module P such that $\underline{\dim} P = (\underline{\dim} W_0)\Phi^{m'}$.

(2) Theorem3.1(2)(b) is equivalent to the following;

(b″) There exists a positive integer m′ such that $\underline{\dim} W_0 = (\underline{\dim} W_0)\Phi^{m'}$.

(3) Theorem3.1(3)(c) is equivalent to following;

(c″) For any positive integer m′ and any indecomposable projective C-module P, $\underline{\dim} P \neq (\underline{\dim} W_0)\Phi^{m'}$, $\underline{\dim} W_0 \neq (\underline{\dim} W_0)\Phi^{m'}$.

Corollary 3.3 (1) Corollary3.2(1)(a″) is equivalent to the following;

(a‴) There exist a positive integer m' and an indecomposable C-module P such that $P \simeq \tau^{m'} W_0$.

(2) Corollary3.2(2)(b″) is equivalent to the following;

(b‴) There exists a positive integer m such that $W_0 \simeq \tau^{m'} W_0$.

(3) Corollary3.2(3)(c″) is equivalent to the following;

(c‴) For any positive integer m and any indecomposable projective C-module P, $P \not\simeq \tau^{m'} W_0$, $W_0 \not\simeq \tau^{m'} W_0$.

4.

We will consider the component containing the module W_0, in Auslander-Reiten quiver of C-mod. Combining corollary 3.2 with results in [6], we obtain the following;

Proposition 4.1 The following properties are equivalent. Let $\mathcal{P}_0$ be a preprojective component of A.-R. quiver of C-mod.

(1) $T_{n_1 \cdots n_t}$ is of Dynkin type.

(2) $\mathcal{P} = \mathcal{P}_0$

(3) W_0 is contained in $\mathcal{P}_0$.

Suppose that $T_{n_1 \cdots n_t}$ is of Euclidean type. $(\underline{\dim} W_0)\Phi^m$ is periodic with minimal period n_1 ([6]3.5). Therefore $\tau^{n_1} W_0 \simeq W_0$ by corollary 3.3. C.M.Ringel has proved in [6] 5 that there exists a tube containing W_0, τW_0, ⋯, $\tau^{n_1 - 1} W_0$. We obtain the following;

Proposition 4.2 If $T_{n_1 \cdots n_t}$ is of Euclidean type, then W_0, τW_0, ⋯, $\tau^{n_1 - 1} W_0$ form a mouth of some stable tube.

If $T_{n_1 \cdots n_t}$ is of neither Dynkin type nor Euclidean type, $\mathcal{P}$ is strictly wild([6] 3).

Proposition 4.3 Suppose that $T_{n_1 \cdots n_t}$ is of neither Dynkin type nor Euclidean type. Let d be the lowest common multiple of $n_1, \cdots, n_t$. Then the following conditions are equivalent.

(1) $(2-\sum_{s=1}^{t}(1-\frac{1}{n_s}))\times d = -1$.

(2) W_0 and R belong to the same τ-orbit.

In this case, $W_0 \simeq \tau^d R$.

Remark $(n_1, \cdots, n_t)$ satisfies the condition (1) of this proposition if and only if it is one of the following types;

(2,2,2,2,2)	(3,2,2,2)	(4,2,2,2)	(4,4,4)
(4,3,3)	(6,3,3)	(5,5,2)	(6,6,2)
(5,4,2)	(6,4,2)	(8,4,2)	(7,3,2)
(8,3,2)	(9,3,2)	(12,3,2)	

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PROCEEDINGS OF THE 25TH
SYMPOSIUM ON RING THEORY, 1992

THE MOD 2 COHOMOLOGY ALGEBRAS OF FINITE GROUPS WITH SEMIDIHEDRAL SYLOW 2-SUBGROUPS

Hiroki SASAKI

1. Introduction. The following 2-groups are known as noncommutative 2-groups that have cyclic maximal subgroups:

(1) dihedral 2-group

$$D_n = \langle\, x, y \mid x^{2^{n-1}} = y^2 = 1, y^{-1}xy = x^{-1} \,\rangle,\ n \geq 3;$$

(2) generalized quaternion 2-group

$$Q_n = \langle\, x, y \mid x^{2^{n-2}} = y^2 = z, z^2 = 1, y^{-1}xy = x^{-1} \,\rangle,\ n \geq 3;$$

(3) semidihedral 2-group

$$S_n = \langle\, x, y \mid x^{2^{n-1}} = y^2 = 1, y^{-1}xy = x^{-1+2^{n-2}} \,\rangle,\ n \geq 4;$$

(4)

$$M_n(2) = \langle\, x, y \mid x^{2^{n-1}} = y^2 = 1, y^{-1}xy = x^{1+2^{n-2}} \,\rangle,\ n \geq 4.$$

The structure theories of finite groups with Sylow 2-subgroups isomorphic with these 2-subgroups have been deeply studied by many authors. Among them we mention the works [25] by W. J. Wong and [1] by J. Alperin, R. Brauer and D. Gorenstein. In [25] investigated were finite groups that have S_n and $M_n(2)$ as Sylow 2-subgroups. A result showed that a finite group containing $M_n(2)$ as Sylow 2-subgroup has a normal 2-complement. Therefore the mod 2 cohomology algebra of such a finite group is isomorphic with that of $M_n(2)$. In the paper [1] the finite simple groups with semidihedral Sylow 2-subgroups were classified.

The detailed version of this paper will be submitted for publication elsewhere.

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Recently Martino and Priddy [18] and Martino [17] determined the stable decompositions of the classification spaces of finite groups with dihedral, generalized quaternion and semidihedral Sylow 2-subgroups. As a consequence the mod 2 cohomology algebras of such finite groups were obtained.

On the other hand Asai and Sasaki [3] has determined the cohomology algebras of finite groups with dihedral Sylow 2-subgroups from a module theoretic point of view. Main tools there were the theory of cohomology varieties of modules, which was originated by Quillen and has been developed by Alperin, Benson, Carlson, Evens and others, and the theory of relatively projective covers of modules with respect to subgroups, which was introduced by Knörr [16]. Recently Okuyama [21] has introduced the notion of relative projectivity of modules with respect to "modules". This theory generalizes the theory of relative projectivity with respect to subgroups.

Now the mod 2 cohomology algebras of finite groups with semidihedral Sylow 2-subgroups are as follows. Let k be a field of characteristic 2.

Theorem 1. (H. Munkholm [19], L. Evens and S. Priddy [13]) *It holds that*

$$H^*(S_n, k) = k[\xi, \eta, \chi, \tau]/(\xi^2 - \xi\eta, \xi^3, \xi\chi, (\xi^2 + \eta^2)\tau - \eta^6 - \chi^2)$$

where $\deg \xi = \deg \eta = 1, \deg \chi = 3$ *and* $\deg \tau = 4$.

Let

$$v = x^{2^{n-3}}, \quad z = x^{2^{n-2}}.$$

Then the finite groups with Sylow 2-subgroup S_n are classified into the following four types:

I $\langle y\rangle \sim \langle z\rangle$, $\langle yx\rangle \sim \langle v\rangle$;
II $\langle y\rangle \sim \langle z\rangle$, $\langle yx\rangle \not\sim \langle v\rangle$;
III $\langle y\rangle \not\sim \langle z\rangle$, $\langle yx\rangle \sim \langle v\rangle$;
IV $\langle y\rangle \not\sim \langle z\rangle$, $\langle yx\rangle \not\sim \langle v\rangle$.

Theorem 2. (J. Martino [17]) *Let G be a finite group with semidihedral Sylow 2-subgroup S_n.*

(1) *If the group G is of type I, then*

$$H^*(G, k) = k[\beta, \gamma, \delta]/(\beta^2\gamma - \delta^2)$$

where $\deg \beta = 3, \deg \gamma = 4$ *and* $\deg \delta = 5$.

(2) *If the group G is of type II, then*

$$H^*(G, k) = k[\alpha, \beta, \gamma, \delta]/(\alpha^3, \alpha\beta, \alpha\delta, \beta^2\gamma - \delta^2)$$

where $\deg \alpha = 1, \deg \beta = 3, \deg \gamma = 4$ *and* $\deg \delta = 5$.

(3) *If the group G is of type III, then*

$$H^*(G,k) = k[\alpha,\beta,\gamma]/(\alpha^6 - \alpha^2\gamma - \beta^2)$$

where $\deg\alpha = 1, \deg\beta = 3$ *and* $\deg\gamma = 4$.

(4) *If the group G is of type IV, then*

$$H^*(G,k) \simeq H^*(S,k).$$

The purpose of this report is to show that the theorems can be established by using the theory of relative projectivity with respect to modules and the theory of cohomology varieties.

As we have mentioned, in Wong [25] and Alperin, Brauer and Gorenstein [1], the structure of finite groups in question was investigated. But we do not depend on such structure theory.

In Section 2 we shall recall some facts from cohomology theory of finite groups. In Section 3, following Okuyama [21], we shall introduce the theory of relative projectivity. In Section 4 we shall introduce a nice extension

$$0 \longrightarrow k \longrightarrow V \longrightarrow \Omega^3(k) \longrightarrow 0.$$

As a matter of fact this extension is a relatively injective hull of k with respect to "the kG-module V". The important is that the kG-module V is selfdual and the tensor product of V with the extension does split. Let us denote by γ the element in $H^4(G,k)$ corresponding to the extension above. The element γ is not a zero-divisor in the cohomology algebra $H^*(G,k)$. The extension above will give us much information on the cohomology algebra. First a dimension formula for the cohomology groups $H^n(G,k)$ will be stated in Section 5. Second a homogeneous element β of degree 3 that together with the element γ forms a system of parameters for the cohomology algebra will be detected from the extension above in Section 6. Hence the cohomology algebra $H^*(G,k)$ is generated by homogeneous elements degree up to 5 over the subalgebra $k[\beta,\gamma]$ generated by the elements β and γ. In the final section we shall sketch our arguments that determine generators and relations.

2. Cohomology algebra of finite groups. In this and the next sections let G be an arbitrary finite group and let k be a field of characteristic p dividing the order of G.

By a kG-module we shall always mean a finitely generated right kG-module. For U and V kG-modules we denote by $(U,V)_G$ the set of kG-homomorphisms of U to V.

The nth cohomology group $H^n(G,k)$ is isomorphic to the k-space $(\Omega^n(k),k)_G$. For an element ρ in $H^n(G,k)$ we denote by $\widehat{\rho}$ the kG-homomorphism of $\Omega^n(k)$ to k that corresponds to ρ. If the element ρ is not the zero element, then we denote by L_ρ the kernel of $\widehat{\rho} : \Omega^n(k) \to k$. While if $\rho = 0$, then we define $L_\rho = \Omega^n(k) \oplus \Omega(k)$. The module L_ρ is called a Carlson module.

For $H \leq G$, M a kG-module and α in $H^n(G, M)$ we denote by α_H the restriction $\mathrm{Res}^G_H(\alpha)$ of α to H.

A direct product of r copies of cyclic group of order p is called an elementary abelian p-group of rank r. The p-rank of a finite group G is defined to be the maximal rank of elementary abelian p-subgroups of G.

Our aim is to determine generators of $H^\bullet(G, k)$ and relations. The following is of fundamental importance.

Theorem 2.1. (Quillen [23,24]) *The p-rank of a finite group G equals to the Krull dimension of the cohomology algebra $H^\bullet(G, k)$.*

Hence if the p-rank of a finite group G is r, then there exist r homogeneous elements $\zeta_1, \ldots, \zeta_r$ for which the cohomology algebra $H^\bullet(G, k)$ is finitely generated over the subalgebra $k[\zeta_1, \ldots, \zeta_n]$ generated by $\zeta_1, \ldots, \zeta_r$. By works of Carlson this condition is equivalent to the condition that $L_{\zeta_1} \otimes \cdots \otimes L_{\zeta_r}$ is projective. When the p-rank of G is 2, we can say about bases over $k[\zeta_1, \zeta_2]$:

Lemma 2.1. (Okuyama and Sasaki [22]) *For ζ_1 in $H^{d_1}(G, k)$ and ζ_2 in $H^{d_2}(G, k)$, if the tensor product $L_{\zeta_1} \otimes L_{\zeta_2}$ is a projective module, then it holds that for $n \geq d_1 + d_2 - 1$*

$$H^n(G, k) = H^{n-d_1}(G, k)\zeta_1 + H^{n-d_2}(G, k)\zeta_2.$$

Namely

$$H^\bullet(G, k) = \Big[\bigoplus_{n=1}^{d_1+d_2-2} H^n(G, k) \Big] k[\zeta_1, \zeta_2].$$

The following two facts also show importance of Carlson modules. A kG-module V with no projective direct summands is said to be periodic if there exists a number n such that $\Omega^n(V) \simeq V$. Again by works of Carlson a kG-module V is periodic if and only if there exists a homogeneous element ρ such that $V \otimes L_\rho$ is projective.

Since the nth cohomology group $H^n(G, k)$ is isomorphic to $\mathrm{Ext}^1_{kG}(\Omega^{n-1}(k), k)$ by dimension shifting, an element ρ in $H^n(G, k)$ represents an extension of $\Omega^{n-1}(k)$ by k. Such an extension is of the form

$$0 \longrightarrow k \longrightarrow \Omega^{-1}(L_\rho) \longrightarrow \Omega^{n-1}(k) \longrightarrow 0.$$

3. Relative projectivity with respect to modules. In this section, following Okuyama [21], we explain the theory of relatively projective modules with respect to modules briefly. We quote some results we need from [21]. Refer to [21] for the proofs and more detailed description of the theory.

Definition 3.1. For a kG-module V let

$$\mathcal{P}(V) = \{\ R \mid R \text{ is a direct summand of } V \otimes A \text{ for a } kG\text{-module } A\ \}.$$

A module in this set is said to be $\mathcal{P}(V)$-*projective.*

A direct sum of kG-modules is $\mathcal{P}(V)$-projective if and only if every direct summand is $\mathcal{P}(V)$-projective. A tensor product of a $\mathcal{P}(V)$-projective module and an arbitrary kG-module is $\mathcal{P}(V)$-projective. A kG-module is $\mathcal{P}(V)$-projective if and only if its syzygy is $\mathcal{P}(V)$-projective. Moreover the set $\mathcal{P}(V)$ is closed under dual from the following lemma of Auslander and Carlson.

Lemma 3.1. (Auslander and Carlson [4]) *For V a kG-module, define a k-linear map*

$$t_V : V^* \otimes V \longrightarrow k\ ;\ \lambda \otimes v \longmapsto \lambda(v) \quad (\lambda \in V^*, v \in V).$$

Then the map t_V is a kG-homomorphism and the induced homomorphism

$$1_V \otimes t_V : V \otimes (V^* \otimes V) \longrightarrow V$$

is a splitting epimorphism.

Due to this lemma a kG-module is $\mathcal{P}(V)$-projective if and only if its dual is $\mathcal{P}(V)$-projective. Owing to this fact a $\mathcal{P}(V)$-projective module is also said to be $\mathcal{P}(V)$-*injective.*

Definition 3.2. An exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is said to be $\mathcal{P}(V)$-*split* if the tensor product

$$0 \longrightarrow A \otimes R \longrightarrow B \otimes R \longrightarrow C \otimes R \longrightarrow 0$$

splits for all $\mathcal{P}(V)$-projective kG-modules R. Note that if the tensor product

$$0 \longrightarrow A \otimes V \longrightarrow B \otimes V \longrightarrow C \otimes V \longrightarrow 0$$

splits, then the sequence is $\mathcal{P}(V)$-split.

Definition 3.3. For M a kG-module an exact sequence

$$0 \longrightarrow X \longrightarrow R \longrightarrow M \longrightarrow 0$$

is called a $\mathcal{P}(V)$-*projective resolution* of M if

(1) R is $\mathcal{P}(V)$-projective;
(2) the sequence is $\mathcal{P}(V)$-split.

If furthermore

(3) the kernel X has no $\mathcal{P}(V)$-projective direct summands

then we call this resolution a $\mathcal{P}(V)$-*projective cover* of the module M and we denote by $\Omega_{\mathcal{P}(V)}(M)$ the kernel X.

Because of Lemma 3.1 an arbitrary kG-module has a $\mathcal{P}(V)$-projective resolution. Okuyama showed that $\mathcal{P}(V)$-projective cover does exist for every kG-module.

Theorem 3.1. (Okuyama [21]) *Let V be a kG-module. An arbitrary kG-module has a $\mathcal{P}(V)$-projective cover, which is uniquely determined up to isomorphism of sequences.*

Dually we can define a $\mathcal{P}(V)$-injective hull of a kG-module.

A kG-module M is $\mathcal{P}(V)$-projective if and only if $\Omega_{\mathcal{P}(V)}(M) = 0$. If a kG-module M is indecomposable and not $\mathcal{P}(V)$-projective, then $\Omega_{\mathcal{P}(V)}(M)$ is also indecomposable with the same vertices as M.

Example 3.1. (1) If V is projective, then a $\mathcal{P}(V)$-projective cover is a projective cover in the usual sense.

(2) Let $\mathcal{H}$ be a set of subgroups of G. For $V = \bigoplus_{H\in\mathcal{H}} k_H^G$, a $\mathcal{P}(V)$-projective cover coincides with a relatively $\mathcal{H}$-projective cover, which was introduced by Knörr [16].

The following two propositions are concerned with relations of relative projectivity and restrictions and inductions of modules.

Proposition 3.1. *Let V be a kG-module. The restriction of a $\mathcal{P}(V)$-projective cover of a kG-module M to a subgroup H of G is a $\mathcal{P}(V_H)$-projective resolution of the restriction M_H. In particular*

$$\Omega_{\mathcal{P}(V)}(M)_H \simeq \Omega_{\mathcal{P}(V_H)}(M_H) \oplus R$$

for some $\mathcal{P}(V_H)$-projective module R.

Proposition 3.2. *Let H be a subgroup of G and let W be a kH-module. Let M be a kG-module.*

(1) *For a $\mathcal{P}(W)$-projective cover*

$$0 \longrightarrow X \longrightarrow S \xrightarrow{g} M_H \longrightarrow 0$$

of the restriction M_H, we define a kG-homomorphism $f : S^G \rightarrow M$ by

$$s \otimes x \longmapsto g(s)x \quad (s \in S,\ x \in G).$$

Then the kG-homomorphism f is an epimorphism and

$$0 \longrightarrow \ker f \longrightarrow S^G \xrightarrow{f} M \longrightarrow 0$$

is a $\mathcal{P}(W^G)$-projective resolution of M.

(2) *Similarly if*

$$0 \longrightarrow M_H \xrightarrow{g} T \longrightarrow Y \longrightarrow 0$$

is a $\mathcal{P}(W)$-injective hull of M_H, then the sequence

$$0 \longrightarrow M \xrightarrow{f} T^G \longrightarrow \operatorname{coker} f \longrightarrow 0$$

where $f : M \to T^G$ is defined by

$$a \longmapsto \sum_{x \in H \backslash G} g(ax^{-1}) \otimes x \quad (a \in M)$$

is a $\mathcal{P}(W^G)$-injective resolution of M.

Next we state some results that are concerned with relatively injective hull of the trivial module.

Proposition 3.3. *Let V be an indecomposable kG-module. Then there exists an indecomposable kG-direct summand R of $V^* \otimes V$ such that*

$$0 \longrightarrow \ker t_V \cap R \longrightarrow R \xrightarrow{t_V|R} k \longrightarrow 0$$

is a $\mathcal{P}(V)$-projective cover of k. Dually a $\mathcal{P}(V)$-injective hull of k is of the form

$$0 \longrightarrow k \longrightarrow S \longrightarrow Y \longrightarrow 0$$

where S is an indecomposable kG-direct summand of $V^* \otimes V$.

An exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

represents an element in $\mathrm{Ext}_{kG}(C, A)$. We denote by ε the element that corresponds to the extension and by $[\varepsilon]$ the extension. The group $\mathrm{Ext}_{kG}(C, A)$ is isomorphic to $(\Omega(C), A)_G/(\text{projectives})$. Let $\widehat{\varepsilon}$ denote a kG-homomorphism of $\Omega(C)$ to A that represents the class corresponding to ε. Then, as is easily seen, the tensor product $[\varepsilon] \otimes V$ of the extension $[\varepsilon]$ and a kG-module V splits if and only if the element $\varepsilon \otimes V$ in $\mathrm{Ext}_{kG}(C \otimes V, A \otimes V)$ vanishes. Clearly this is equivalent to that the tensor product homomorphism $\widehat{\varepsilon} \otimes 1_V : \Omega(C) \otimes V \to A \otimes V$ is a projective kG-homomorphism. From this observation we have

Lemma 3.2. *Let ρ be an element in $H^n(G,k)$. Then the extension*

$$0 \longrightarrow k \longrightarrow \Omega^{-1}(L_\rho) \longrightarrow \Omega^{n-1}(k) \longrightarrow 0$$

of $\Omega^{n-1}(k)$ by k corresponding to the element ρ is a $\mathcal{P}(L_\rho)$-injective hull of the trivial module k if and only if the tensor product $\widehat{\rho} \otimes 1_{L_\rho}$ is a projective kG-homomorphism.

Example 3.2. Suppose that the field k has an odd characteristic. Let n be an even number. Then by Lemma 3.2 and Benson [6] Proposition 5.9.6 for every element ρ in $H^n(G,k)$ the extension

$$0 \longrightarrow k \longrightarrow \Omega^{-1}(L_\rho) \longrightarrow \Omega^{n-1}(k) \longrightarrow 0$$

corresponding to ρ is a $\mathcal{P}(L_\rho)$-injective hull of the trivial module k.

The following two examples are also due to Okuyama.

Example 3.3. Suppose that the field k has characteristic 2. We consider a four-group $E = \langle y, z \rangle$. Let L be the kE-submodule in $kE \oplus kE$ generated by the two elements

$$a = (y-1, \alpha(z-1)), \quad b = (z-1, y-1)$$

where α is a nonzero scalar. The module L is periodic of period 1.

(1) We have the extension

$$0 \longrightarrow k \xrightarrow{s} L \longrightarrow \Omega(k) \longrightarrow 0$$

where

$$s : k \to L\ ;\ 1 \longmapsto a(z-1)\ (= b(y-1)).$$

This extension is in fact a $\mathcal{P}(L)$-injective hull of k_E.

In Conlon [10] and Benson [5] described are the indecomposable modules and the representation algebra for the four-group. The results follows from their descriptions. However this is also verified from the point of view of Lemma 3.2.

Regarding $H^1(E,k)$ as $\mathrm{Hom}(E,k)$, let λ and μ in $H^1(E,k)$ be the duals of the elements y and z, respectively. Take the element $\rho = \alpha\lambda^2 + \mu^2$ in $H^2(E,k)$. It is easily checked that the Carlson module L_ρ is in fact our kE-module L and our extension corresponds to the element ρ.

Since $\Omega^2(k) \otimes L \simeq L \oplus (kE)^4$, to verify that the tensor product $\widehat{\rho} \otimes 1_{L_\rho}$ is projective it is enough to show that $(\widehat{\rho} \otimes 1_L)(L) = 0$ under the isomorphism. This can be carried out by calculation.

(2) The dual L^* of L is isomorphic with L. Hence making dual of the $\mathcal{P}(L)$-injective hull of k above, we obtain a $\mathcal{P}(L)$-projective cover of k :

$$0 \longrightarrow \Omega^{-1}(k) \longrightarrow L \xrightarrow{q} k \longrightarrow 0$$

The kE-homomorphism $q : L \to k$ is defined by

$$q : \begin{cases} a \longmapsto 1 \\ b \longmapsto 0. \end{cases}$$

Example 3.4. Let

$$S = \langle\, x, y \mid x^{2^{n-1}} = y^2 = 1, y^{-1}xy = x^{-1+2^{n-2}} \,\rangle,\ n \geq 4$$

be a semidihedral 2-group of order 2^n. Let $z = x^{2^{n-2}}$ and let $E = \langle y, z\rangle$ be a four-subgroup in S. Take an element ρ in $H^2(E, k)$ as in the preceding example. We write L for L_ρ and take the same generators a and b of L as the preceding example. The extension

$$0 \longrightarrow k_S \xrightarrow{g} L^S \longrightarrow X \longrightarrow 0$$

where the kS-homomorphism $g : k_S \to L^S$ is defined by

$$1 \longmapsto a(x-1)^{2^{n-1}-1}$$

is a $\mathcal{P}(L^S)$-injective hull of k_S; and the extension

$$0 \longrightarrow Y \longrightarrow L^S \xrightarrow{f} k_S \longrightarrow 0$$

where the kS-homomorphism $f : L^S \to k_S$ is defined by

$$f : \begin{cases} a \longmapsto 1 \\ b \longmapsto 0 \end{cases}$$

is a $\mathcal{P}(L^S)$-projective cover of k_S by Proposition 3.2. Moreover the kernel Y is isomorphic with $\Omega^{-3}(k_S)$ and the cokernel X is isomorphic with $\Omega^3(k_S)$. Summarizing, the extension

$$0 \longrightarrow k_S \xrightarrow{g} L^S \longrightarrow \Omega^3(k_S) \longrightarrow 0$$

is a $\mathcal{P}(L^S)$-injective hull of k_S; and the extension

$$0 \longrightarrow \Omega^{-3}(k_S) \longrightarrow L^S \xrightarrow{f} k_S \longrightarrow 0$$

is a $\mathcal{P}(L^S)$-projective cover of k_S.

If the scalar α is a primitive cubic root ω of unity, then the module L^S is defined over the prime field $\mathbf{F}_2$. Let $v = x^{2^{n-3}}$ and $D = \langle v, y\rangle$. The induced module L^D is in fact defined over the prime field $\mathbf{F}_2$.

4. A relatively injective hull of the trivial module. Henceforth we let k be a field of chracteristic 2 with a primitive cubic root of unity, and let G denote a finite group with semidihedral Sylow 2-subgroup

$$S = \langle\, x, y \mid x^{2^{n-1}} = y^2 = 1, y^{-1}xy = x^{-1+2^{n-2}} \,\rangle,\ n \geq 4$$

unless otherwise stated. In this section for such a group G we shall introduce an extension

$$0 \longrightarrow k_G \longrightarrow V \longrightarrow \Omega^3(k_G) \longrightarrow 0$$

of $\Omega^3(k_G)$ by k_G that is a $\mathcal{P}(V)$-injective hull of k_G. First we define a kG-module V and then show that a $\mathcal{P}(V)$-injective hull of k_G is of the form above.

We set

$$z = x^{2^{n-2}}, \quad E = \langle y, z\rangle.$$

Take an element $\rho = \omega\lambda^2 + \mu^2$ in $H^2(E, k)$, where ω is a primitive cubic root of unity, and let

$$L = L_\rho, \quad U = L^S$$

as in Example 3.4. We use the same notations as Example 3.4 for the generators of the modules and the $\mathcal{P}(U)$-injective hull and the $\mathcal{P}(U)$-projective cover of the trivial module. We list some properties of the kS-module U.

Lemma 4.1. (1) *The module U is an indecomposable module with vertex E and source L.*

(2) *For a subgroup T of S that contains no conjugate subgroup to the four-group E the restriction U_T is projective.*

(3) *The tensor product $U \otimes U$ decomposes into a direct sum of two copies of the kS-module U and a projective kS-module.*

(4) *The module U is periodic of period 1.*

(5) *The module U is defined over the prime field $\mathbb{F}_2$.*

Lemma 4.2. *It holds that $C_G(E) = E \times O_{2'}(C_G(E))$.*

We regard the kE-module L as a $kC_G(E)$-module by making $O_{2'}(C_G(E))$ act trivially on L.

We set

$$v = x^{2^{n-3}}.$$

Then one has $N_S(E) = \langle E, v\rangle$ so that the index $|N_G(E) : C_G(E)|$ is 6 or 2. When $|N_G(E) : C_G(E)| = 6$, there exists an element u in $N_G(E)$ such that

$$y^u = z, \; z^u = yz, \; (yz)^u = y.$$

Let

$$H = \begin{cases} \langle C_G(E), u\rangle & \text{when } |N_G(E) : C_G(E)| = 6, \\ C_G(E) & \text{when } |N_G(E) : C_G(E)| = 2. \end{cases}$$

Then it holds that

$$|N_G(E) : H| = 2, \quad O_{2'}(C_G(E)) = O_{2'}(H).$$

We shall often write N for $N_G(E)$ and C for $C_G(E)$.

When $|N_G(E) : C_G(E)| = 6$, for i, $0 \le i \le 2$, we let k_i be the one-dimensional kH-module on which C acts trivially and u acts as multiplication by ω^i. Since the subgroup $O_{2'}(H)$ acts on L_C^H trivially and the factor group $H/O_{2'}(H)$ is isomorphic with the alternating group A_4 in this case, we have

Proposition 4.1. *Assume that* $|N_G(E) : C_G(E)| = 6$.

(1) *The induced module* L_C^H *is the direct sum of three indecomposable* kH*-modules* M_0, M_1 *and* M_2 *that satisfy the following conditions:*

(i) M_i *has vertex* E *and source* L;

(ii) $M_i/\operatorname{rad} M_i \simeq k_i \oplus k_{i-1}$ *and* $\operatorname{soc} M_i \simeq k_i \oplus k_{i+1}$;

(iii) $\Omega(M_i) \simeq M_{i-1}$ *and* M_i *is periodic of period 3;*

(iv) $M_0 \otimes M_0 \simeq M_0 \oplus M_0 \oplus$ *(projective),* $M_0 \otimes M_0^{\tau}$ *is projective, and* $M_0 \simeq M_0^{*}$.

where the subscripts are considered modulo 3.

(2) *The induced module* M_i^N *is an indecomposable* kN*-module with vertex* E *and source* L*, and periodic of period 3.*

(3) *The induced module* M_0^N *is defined over the prime field* $\mathbf{F}_2$.

While when $|N_G(E) : C_G(E)| = 2$, we have

Proposition 4.2. *Assume that* $|N_G(E) : C_G(E)| = 2$.

(1) *The induced module* L_C^N *is an indecomposable* kN*-module with vertex* E *and source* L*, and periodic of period 1.*

(2) *The induced module* L_C^N *is defined over the prime field* $\mathbf{F}_2$.

We let

$$W = \begin{cases} M_0^N & \text{when } |N_G(E) : C_G(E)| = 6 \\ L_C^N & \text{when } |N_G(E) : C_G(E)| = 2 \end{cases}$$

and let

V be the Green correspondent of W with respect to $(G, E, N_G(E))$.

The kG-module V has the following properties.

Proposition 4.3. (1) *The* kG*-module* V *is an indecomposable* kG*-module with vertex* E *and source* L *lying in the principal block.*

(2) *The restriction* V_S *of* V *to* S *decomposes into a direct sum of the* kS*-module* U *and a projective* kS*-module.*

(3) *The induced module* W^G *of* W *to* G *decomposes into a direct sum of the* kG*-module* V *and a projective* kG*-module.*

(4) *The tensor product* $V \otimes V$ *decomposes into a direct sum of two copies of the* kG*-module* V *and a projective* kG*-module.*

(5) *The* kG*-module* V *is isomorphic with its dual* V^*.

(6) *The* kG*-module* V *is periodic of period 3 when* $|N_G(E) : C_G(E)| = 6$, *and 1 when* $|N_G(E) : C_G(E)| = 2$.

(7) *The* kG*-module* V *is defined over the prime field* $\mathbf{F}_2$.

The kG-module V is the very module we want.

Theorem 4.1. (1) *A* $\mathcal{P}(V)$*-injective hull of the trivial module* k_G *is of the following form:*

$$0 \longrightarrow k_G \xrightarrow{\tilde{g}} V \longrightarrow \Omega^3(k_G) \longrightarrow 0.$$

(2) *A* $\mathcal{P}(V)$*-projective cover of the trivial module* k_G *is of the following form:*

$$0 \longrightarrow \Omega^{-3}(k_G) \longrightarrow V \xrightarrow{\tilde{f}} k_G \longrightarrow 0.$$

(3) *The restrictions of the extensions above to* S *are*

$$0 \longrightarrow k_S \xrightarrow{g} U \oplus P \xrightarrow{h} \Omega^3(k_S) \oplus P \longrightarrow 0$$

$$0 \longrightarrow \Omega^{-3}(k_S) \oplus P \xrightarrow{e} U \oplus P \xrightarrow{f} k_S \longrightarrow 0$$

where P *is a projective* kS*-module.*

Proof. We prove only (1). Let $A = N_G(S)$. Then it holds that $A = O_{2'}(A) \times S$. We regard the kS-module U as a kA-module by making $O_{2'}(A)$ act on U trivially. Since the subgroup $O_{2'}(A)$ acts on $\Omega^3(k_A)$ trivially, it follows that

$$\Omega^3(k_A)_S = \Omega^3(k_S).$$

Thus we see that the extension

$$0 \longrightarrow k_A \longrightarrow U \longrightarrow \Omega^3(k_A) \longrightarrow 0$$

is a $\mathcal{P}(U)$-injective hull of k_A. By Proposition 3.3 (2) the middle term of a $\mathcal{P}(V)$-injective hull of k_G is an indecomposable direct summand of $V^* \otimes V$. We have observed that

$$V^* \otimes V \simeq V \otimes V \simeq V \oplus V \oplus \text{ (projective)}$$

in the preceding proposition. Hence a $\mathcal{P}(V)$-injective hull of k_G is of the form

$$0 \longrightarrow k_G \longrightarrow V \longrightarrow X \longrightarrow 0.$$

The cokernel X is indecomposable with vertex S. We show that the module X is isomorphic with $\Omega^3(k_G)$ by using Green correspondence with respect to (G, S, A). The restriction of the $\mathcal{P}(V)$-injective hull of k_G above to A is of the form

$$0 \longrightarrow k_A \longrightarrow U \oplus P \longrightarrow \Omega^3(k_A) \oplus P \longrightarrow 0$$

where P is a projective kA-module by Propositions 3.1 and 4.3 (2). Namely we have

$$X_A \simeq \Omega^3(k_A) \oplus P.$$

Thus the module X is the Green correspondent of $\Omega^3(k_G)$. On the other hand the module $\Omega^3(k_G)$ is also a Green correspondent of $\Omega^3(k_A)$, since $\Omega^3(k_G)_A \simeq \Omega^3(k_A) \oplus$ (projective). Consequently the module X is isomorphic with $\Omega^3(k_G)$, as desired. □

5. Dimensions of cohomology groups. The $\mathcal{P}(V)$-injective hull

$$0 \longrightarrow k_G \xrightarrow{\tilde{g}} V \xrightarrow{\tilde{h}} \Omega^3(k_G) \longrightarrow 0$$

gives us much information about the cohomology algebra. In this section we shall deduce a formula for the dimensions of cohomology groups $H^n(G, k)$.

We denote by γ the element in $H^4(G, k)$ that corresponds to the extension above.

Lemma 5.1. *The induced homomorphism*

$$\tilde{g}^* : (V, k)_G \longrightarrow (k, k)_G$$

is the zero homomorphism so that

$$\tilde{h}^* : (\Omega^3(k), k)_G \simeq (V, k)_G.$$

In particular

$$\dim H^3(G, k) = \begin{cases} 1 & \text{when } |N_G(E) : C_G(E)| = 6 \\ 2 & \text{when } |N_G(E) : C_G(E)| = 2. \end{cases}$$

Proof. We note that the socle of the module V is contained in the radical of V, since the module V is indecomposable but not simple. Hence we have $\tilde{g}^* = 0$. Since $W^G \simeq V \oplus$ (projective) by Proposition 4.3 (3), we can calculate the dimension as follows:

$$\dim (V, k)_G = \begin{cases} 1 & \text{when } |N : C| = 6 \\ 2 & \text{when } |N : C| = 2. \end{cases}$$

□

Applying the functor $\mathrm{Ext}^*_{kG}(-, k)$ to the extension we can deduce the following theorem.

Theorem 5.1. (1) *The element γ is not a zero-divisor in $H^*(G,k)$.*

(2) *When $|N_G(E):C_G(E)| = 6$, we have for $n \geq 0$*

$$\dim H^{n+4}(G,k) = \dim H^n(G,k) + \begin{cases} 0 & \text{if } n \equiv 0 \pmod 3 \\ 1 & \text{if } n \equiv 1 \pmod 3 \\ 1 & \text{if } n \equiv 2 \pmod 3). \end{cases}$$

(3) *When $|N_G(E):C_G(E)| = 2$, we have*

$$\dim H^{n+4}(G,k) = \dim H^n(G,k) + 2 \quad \text{for } n \geq 0.$$

Proof. Let us consider the long exact Ext-sequence

$$\begin{aligned} 0 \longrightarrow (\Omega^3(k),k)_G &\xrightarrow{\tilde{h}_0^*} (V,k)_G \\ &\xrightarrow{\tilde{g}_0^*} (k,k)_G \xrightarrow{\Delta} \mathrm{Ext}^1_{kG}(\Omega^3(k),k) \xrightarrow{\tilde{h}_1^*} \mathrm{Ext}^1_{kG}(V,k) \\ &\xrightarrow{\tilde{g}_1^*} \mathrm{Ext}^1_{kG}(k,k) \xrightarrow{\Delta} \mathrm{Ext}^2_{kG}(\Omega^3(k),k) \xrightarrow{\tilde{h}_2^*} \mathrm{Ext}^2_{kG}(V,k) \longrightarrow \cdots \\ &\xrightarrow{\tilde{g}_n^*} \mathrm{Ext}^n_{kG}(k,k) \xrightarrow{\Delta} \mathrm{Ext}^{n+1}_{kG}(\Omega^3(k),k) \xrightarrow{\tilde{h}_{n+1}^*} \mathrm{Ext}^{n+1}_{kG}(V,k) \longrightarrow \cdots. \end{aligned}$$

If the induced homomorphism $\tilde{g}_n^* : \mathrm{Ext}^n_{kG}(V,k) \to \mathrm{Ext}^n_{kG}(k,k)$ is the zero homomorphism for each $n \geq 0$, then we see that the element γ in $H^4(G,k)$ is not a zero-divisor, since the connecting homomorphism Δ can be interpreted as the multiplication by the element γ through the isomorphism of $\mathrm{Ext}^{n+1}_{kG}(\Omega^3(k),k)$ with $\mathrm{Ext}^{n+4}_{kG}(k,k)$; and

$$\dim H^{n+4}(G,k)/\dim H^n(G,k)\gamma = \dim \mathrm{Ext}^{n+1}_{kG}(V,k) \quad \text{for } n \geq 0.$$

We have already observed in Lemma 5.1 that the induced homomorphism $\tilde{g}_0^*$ is the zero homomorphism.

We first treat the case when $|N:C| = 2$. Since the syzygy $\Omega(V)$ is isomorphic with V, we have by Asai [2] Theorem 3.3 for each $n, n \geq 0$

$$[\, \tilde{g}_n^* : \mathrm{Ext}^n_{kG}(V,k) \longrightarrow \mathrm{Ext}^n_{kG}(k,k)\,] = 0.$$

It follows also from $\Omega(V) \simeq V$ that

$$\dim \mathrm{Ext}^{n+1}_{kG}(V,k) = 2.$$

Hence we obtain the formula for this case.

Next let us assume that $|N:C| = 6$. Note that the argument above is valid for the Sylow 2-subgroup S and the kS-module U. Because $\Omega^3(V)$ is isomorphic with V, if the induced homomorphisms $\widetilde{g}_1^*$ and $\widetilde{g}_2^*$ are the zero homomorphisms, then again by Asai [2] Theorem 3.3 every $\widetilde{g}_n^*$ is the zero homomorphism for each $n \geq 0$. To see that $\widetilde{g}_1^*$ and $\widetilde{g}_2^*$ are the zero homomorphisms it is enough to note by Theorem 4.1 that the induced homomorphisms $g_n^* : \mathrm{Ext}_{kS}^n(U,k) \rightarrow \mathrm{Ext}_{kS}^n(k,k)$ are the zero homomorphisms. As in the proof of Lemma 5.1 we see that

$$\mathrm{Ext}_{kG}^n(V,k) \simeq \mathrm{Ext}_{kH}^n(M_0,k) = \begin{cases} k & n=0 \\ 0 & n=1 \\ k & n=2 \end{cases}$$

by Proposition 4.1, whence we have

$$\dim \mathrm{Ext}_{kG}^{n+1}(V,k) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 3 \\ 1 & \text{if } n \equiv 1 \pmod 3 \\ 1 & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

This completes the proof of the theorem. □

The following lemma is also deduced from our $\mathcal{P}(V)$-injective hull.

Lemma 5.2. *The second cohomology group $H^2(G,k)$ has the same dimension as the first cohomology group $H^1(G,k)$.*

6. Homogeneous system of parameters. Since the element γ in $H^4(G,k)$ corresponds to the extension

$$0 \longrightarrow k \longrightarrow V \longrightarrow \Omega^3(k) \longrightarrow 0$$

the kG-module V is isomorphic with $\Omega^{-1}(L_\gamma)$ so that the Carlson module L_γ is periodic. Therefore there must be a homogeneous element β such that the tensor product $L_\beta \otimes L_\gamma$ is projective. Such an element β and the element γ form a system of parameters for the cohomology algebra $H^*(G,k)$.

By Lemma 5.1 we can take a kG-homomorphism

$$\widehat{\beta} : \Omega^3(k) \longrightarrow k$$

such that

$$\widetilde{f} = \widehat{\beta}\widetilde{h}.$$

Theorem 6.1. *The tensor product $L_\beta \otimes L_\gamma$ is projective. In particular it holds that*

$$H^n(G,k) = H^{n-3}(G,k)\beta + H^{n-4}(G,k)\gamma \qquad \textit{for } n \geq 6$$

whence one has

$$H^*(G,k) = \left[\bigoplus_{i=1}^{5} H^i(G,k)\right] k[\beta,\gamma].$$

Proof. Form a pull-back of $L_\beta \hookrightarrow \Omega^3(k)$ and $\tilde{h} : V \to \Omega^3(k)$ to obtain the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & k & \longrightarrow & \Omega^{-3}(k) & \longrightarrow & L_\beta & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & k & \xrightarrow{\tilde{g}} & V & \xrightarrow{\tilde{h}} & \Omega^3(k) & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle \tilde{\beta}} & & \\
 & & & & k & = & k & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Tensoring the kG-module V to the commutative diagram above we obtain

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V & \longrightarrow & \Omega^{-3}(k)\otimes V & \longrightarrow & L_\beta \otimes V & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V & \xrightarrow{\tilde{g}\otimes 1_V} & V \otimes V & \xrightarrow{\tilde{h}\otimes 1_V} & \Omega^3(k)\otimes V & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \tilde{f}\otimes 1_V} & & \downarrow{\scriptstyle \tilde{\beta}\otimes 1_V} & & \\
 & & & & V & = & V & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

The middle horizontal sequence splits, because the sequence

$$0 \longrightarrow k \longrightarrow V \longrightarrow \Omega^3(k) \longrightarrow 0$$

is $\mathcal{P}(V)$-split. Therefore the upper horizontal sequence also splits, whence we have

$$\Omega^{-3}(k)\otimes V \simeq V \oplus L_\beta \otimes V.$$

This together with the fact that $\Omega^{-3}(k)\otimes V \simeq V\oplus$ (projective), which follows from Lemma 4.3 (6), implies that the tensor product $L_\beta \otimes L_\gamma$ is projective, since the module V is isomorphic with $\Omega^{-1}(L_\gamma)$. The latter assertion follows from Lemma 2.1. □

We have obtained a dimension formula for the cohomology groups and a homogeneous system of parameters for the cohomology algebra. Therefore we must be able to determine generators and relations by investigating cohomology groups of low homogeneous degree closely. Since the kG-module V is defined over the prime field $\mathbf{F}_2$, the elements β and γ are defined over the prime field $\mathbf{F}_2$.

7. The cohomology algebra. In this section we shall establish the main theorems. In order to determine the first cohomology group $H^1(G,k)$ we prepare a lemma that concerns with relatively injective hulls of the trivial module with respect to subgroups. Before stating the lemma we introduce some more notations. For sets $\mathcal{H}$ and $\mathcal{K}$ of subgroups of a finite group X we write $\mathcal{H} \leq_X \mathcal{K}$ to mean that a conjugate of every subgroup in $\mathcal{H}$ is contained in a subgroup that belongs to $\mathcal{K}$. If $\mathcal{H} \leq_X \mathcal{K}$ and $\mathcal{K} \leq_X \mathcal{H}$, then we write $\mathcal{H} =_X \mathcal{K}$. By a similar argument to Asai and Sasaki [3] Proposition 5.1 we have the following.

Lemma 7.1. *Let G be a finite group. If a maximal subgroup T of a Sylow 2-subgroup S of G satisfies*

$$\{\, T^g \cap S \mid g \in G \,\} =_S \{T\}$$

then for a field k of characteristic 2 a relatively T-injective hull of the trivial kG-module k is of the form

$$0 \longrightarrow k_G \longrightarrow \mathrm{Sc}\,T \longrightarrow k_G \longrightarrow 0$$

where $\mathrm{Sc}\,T$ is a Scott module with vertex T.

Recall that $v = x^{2^{n-3}}$ and $z = x^{2^{n-2}}$ and the finite groups with Sylow 2-subgroup S are classified into the following four types:

I $\langle y\rangle \sim \langle z\rangle$, $\langle yx\rangle \sim \langle v\rangle$;
II $\langle y\rangle \sim \langle z\rangle$, $\langle yx\rangle \not\sim \langle v\rangle$;
III $\langle y\rangle \not\sim \langle z\rangle$, $\langle yx\rangle \sim \langle v\rangle$;
IV $\langle y\rangle \not\sim \langle z\rangle$, $\langle yx\rangle \not\sim \langle v\rangle$.

Let us define two elements ξ and η in $H^1(S,k)$ by

$$\xi = x^*, \quad \eta = y^*$$

regarding $H^1(S,k)$ as $\mathrm{Hom}(S,k)$. Obviously we have $H^1(S,k) = \langle \xi, \eta\rangle$.

Using Lemma 7.1 with the maximal subgroups $D = \langle x^2, y\rangle$ and $Q = \langle x^2, yx\rangle$ we see the following.

Lemma 7.2. (1) *If the group G is of type I, then $H^1(G,k) = 0$; and $|N_G(E) : C_G(E)| =$ 6.*

(2) *If the group G is of type II, then there exists an element α in $H^1(G,k)$ such that $H^1(G,k) = \langle\alpha\rangle$ and $\alpha_S = \xi$; and $|N_G(E) : C_G(E)| = 6$.*

(3) *If the group G is of type III, then there exists an element α in $H^1(G,k)$ such that $H^1(G,k) = \langle\alpha\rangle$ and $\alpha_S = \xi + \eta$; and $|N_G(E) : C_G(E)| = 2$.*

(4) *If the group G is of type IV, then the group G has a normal 2-complement. In particular $H^*(G,k) \simeq H^*(S,k)$; and $|N_G(E) : C_G(E)| = 2$.*

Remark. If a finite group G is of type II, then the kernel of the element α in $H^1(G,k)$ is a normal subgroup of index 2 with dihedral Sylow 2-subgroup D. If a finite group G is of type III, then the kernel of the element α in $H^1(G,k)$ is a normal subgroup of index 2 with Sylow 2-subgroup Q. The subgroup Q is a quaternion (when $n = 4$) or generalized quaternion (when $n \geq 5$) group. Clearly a finite group of type I has no normal subgroups of index 4. Of course these have been known as a first step of structure theory. We also note that the principal block algebra of a finite group of type IV is Morita equivalent to the group algebra of its Sylow 2-subgroup.

By the results in Section 5 and this lemma we can determine the dimensions of the cohomology groups $H^n(G,k), n \geq 0$.

Before making further steps we state a lemma, which is deduced from our $\mathcal{P}(V)$-injective hull of k_G and the fact that the kG-module V is E-projective.

Lemma 7.3. *If an element ζ in $H^n(G,k)$, $n \geq 4$, restricts to the zero element in $H^n(E,k)$, then there exists an element ρ in $H^{n-4}(G,k)$ such that*

$$\zeta = \gamma\rho.$$

Every four-subgroup of the group G is conjugate to the four-group E. Hence by Quillen's theorem an element ζ in $H^n(G,k)$ restricts to the zero element in $H^n(E,k)$ if and only if ζ is nilpotent. Thus Lemma 7.3 can be restated as follows.

Lemma 7.4. *Let $n \geq 4$ and let r be the residue class of n modulo 4. A nilpotent element of degree n is a product of a nilpotent element of degree r and a power of the element γ.*

We first treat the semidihedral group S. The arguments we have done are valid for S with $G = S$ and $V = U$. Let τ in $H^4(S,k)$ be the element corresponding to the $\mathcal{P}(U)$-injective hull

$$0 \longrightarrow k_S \longrightarrow U \longrightarrow \Omega^3(k_S) \longrightarrow 0$$

of the trivial module k_S and let χ in $H^3(S,k)$ be the element satisfying

$$f = \widehat{\chi}h.$$

Then the element τ is not a zero-divisor and the elements χ and τ form a system of parameters for the cohomology algebra $H^*(S,k)$.

Using Lemma 7.4, restriction maps and some other results, we obtain

Lemma 7.5. *The following hold.*
(1) $H^2(S,k) = \langle \xi^2, \eta^2 \rangle$; $\xi^2 = \xi\eta$.
(2) $H^3(S,k) = \langle \eta^3, \chi \rangle$; $\chi_{(y)} = 0, \xi^3 = 0$.
(3) $H^4(S,k) = \langle \eta^4, \eta\chi, \tau \rangle$; $\xi\chi = 0$.
(4) $H^5(S,k) = \langle \eta^5, \eta^2\chi, \eta\tau, \xi\tau \rangle$.

Lemma 7.6. *It holds that*

$$\chi^2 = (\xi^2 + \eta^2)\tau + \eta^6.$$

Proof. Step 1. *One has* $\tau_E = \lambda^4 + \lambda^2\mu^2 + \mu^4$.

Sketch of the proof of Step 1. This can be shown by considering the restriction of the $\mathcal{P}(U)$-injective hull

$$0 \longrightarrow k_S \longrightarrow U \longrightarrow \Omega^3(k_S) \longrightarrow 0$$

to the four-group E.

Step 2. *One has* $\chi_E = \lambda\mu(\lambda + \mu)$.

Sketch of the proof of Step 2. We can verify the equation by considering the restriction maps to subgroups of order 2.

Step 3. *Conclusion.*

Proof of Step 3. By Steps 1 and 2 the element $\chi^2 + \eta^6 + \eta^2\tau$ restricts to the zero element in $H^6(E,k)$. Then by Lemma 7.4 this element is a scalar multiple of $\xi^2\tau$. Since the elements χ, τ, η and ξ are all defined over the prime field $\mathbf{F}_2$, this element is $\xi^2\tau$ or 0. To see that this is $\xi^2\tau$ we consider the cohomology algebra of the maximal subgroup $Q = \langle x^2, yx \rangle$. The subgroup Q is a quaternion (when $n = 4$) or generalized quaternion (when $n \geq 5$) group. Both of the elements ξ and η restrict to the dual $(yx)^*$ in $H^1(Q,k)$ and the restriction τ_Q is not a zero-divisor in $H^*(Q,k)$. Let θ in $H^1(Q,k)$ be the dual $(x^2)^*$ of the element x^2. Then the cohomology algebra $H^*(Q,k)$ is generated by the elements ξ_Q, θ and τ_Q. The relations are as follows:

$$\theta^2 + \theta\xi_Q + \xi_Q^2 = 0,\ \xi_Q^3 = 0,\ \theta^3 = 0 \qquad \text{when } n = 4;$$
$$\theta^2 + \theta\xi_Q = 0,\ \xi_Q^3 = 0 \qquad \text{when } n \geq 5.$$

(See for example Martino and Priddy [18] or Asai and Sasaki [3].) Hence the square χ^2 restricts to the zero element in $H^6(Q,k)$. Suppose that $\chi^2 + \eta^6 + \eta^2\tau = 0$. Then restricting to Q, we obtain $\eta_Q^2\tau_Q = 0$, a contradiction. Consequently we have $\chi^2 = \eta^6 + \eta^2\tau + \xi^2\tau$, as desired. □

We have established that the cohomology algebra $H^*(S,k)$ is generated by the elements ξ, η, χ and τ; these generators satisfy the following :

$$\xi^2 = \xi\eta,\ \xi^3 = 0,\ \xi\chi = 0,\ \chi^2 = \eta^6 + \eta^2\tau + \xi^2\tau.$$

Considering the dimensions of the cohomology groups $H^n(S,k)$, $n \geq 0$, we see that the relations above are enough. Namely we have established Theorem 1. We note that the same result holds for the cohomology algebra $H^*(S,\mathbf{F}_2)$ of S with coefficients in the prime field $\mathbf{F}_2$.

Let us proceed to the general cases. First of all we note that

$$\beta_S = \chi, \quad \gamma_S = \tau$$

by Theorem 4.1. The elements β, γ and α (for the groups of types II and III) are defined over the prime field $\mathbf{F}_2$.

Proposition 7.1. *Assume that the group G is of type I. Then the following hold:*

(1) $H^2(G,k) = 0, H^3(G,k) = \langle\beta\rangle, H^4(G,k) = \langle\gamma\rangle$.

(2) *There exists an element δ in $H^5(G,k)$ such that $H^5(G,k) = \langle\delta\rangle$ and*

$$\delta^2 = \beta^2\gamma.$$

Proof. By Lemmas 5.3, 7.2 (1) and Theorem 5.1 (1) we have

$$\dim H^1(G,k) = \dim H^2(G,k) = 0,$$
$$\dim H^3(G,k) = \dim H^4(G,k) = \dim H^5(G,k) = 1.$$

Hence the assertion (1) holds. We can take a nonzero element δ in $H^5(G,k)$ that is defined over the prime field $\mathbf{F}_2$. Using Theorems 5.1 (1) and 6.1 we have $H^{10}(G,k) = \langle\beta^2\gamma\rangle$. Thus the square δ^2 is $\beta^2\gamma$ or 0. By virtue of Lemma 7.4 the square δ^2 must not vanish. Therefore it follows that $\delta^2 = \beta^2\gamma$. □

The remaining cases are also studied similarly.

Proposition 7.2. *Assume that the group G is of type II. Then the following hold:*

(1) $H^2(G,k) = \langle\alpha^2\rangle, H^3(G,k) = \langle\beta\rangle, H^4(G,k) = \langle\gamma\rangle$;

$$\alpha^3 = 0,\ \alpha\beta = 0.$$

(2) *There exists an element δ in $H^5(G,k)$ such that $H^5(G,k) = \langle\alpha\gamma, \delta\rangle$ and*

$$\alpha\delta = 0,\ \delta^2 = \beta^2\gamma.$$

Proposition 7.3. *Assume that the group G is of type III. Then the following hold:*

(1) $H^2(G,k) = \langle\alpha^2\rangle, H^3(G,k) = \langle\alpha^3,\beta\rangle, H^4(G,k) = \langle\alpha^4,\alpha\beta,\gamma\rangle$.

(2) $H^5(G,k) = \langle\alpha^5,\alpha^2\beta,\alpha\gamma\rangle$ and

$$\beta^2 = \alpha^6 + \alpha^2\gamma.$$

Again considering the dimensions we obtain Theorem 2 as in the case of the semidihedral group S. We note that the same result holds for the cohomology algebra $H^*(G,\mathbf{F}_2)$ of G with coefficients in the prime field $\mathbf{F}_2$.

Acknowledgements

Thanks are due to Professor T. Okuyama for permission to quote from the work [21].

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PROCEEDINGS OF THE 25TH
SYMPOSIUM ON RING THEORY, 1992

NON-COMMUTATIVE VALUATION RINGS

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0. Introduction. In [D1], Dubrovin introduced a notion of non-commutative valuation rings in simple Artinian rings, and proved some elementary properties of it. He obtained in [D2] more detailed results concerning orders in finite dimensional central simple algebras over fields.

In this note, we introduce the non-commutative valuation ring in the sense of [D1], and investigate prime ideals of it. The key result is Proposition 2.8 which states that, for any ideal A of a valuation ring R, $\bigcap A^n$ is a prime ideal of R. Using this result, we characterize branched and unbranched prime ideals.

1. Definition and some properties of non-commutative valuation rings. First, we shall define the non-commutative valuation ring. Let Q be a simple Artinian ring, and adjoin a new symbol ∞ to Q. We define $a + \infty = \infty + a = \infty$ for any $a \in Q$ and $c \cdot \infty = \infty \cdot c = \infty$ for any unit $c \in Q$. The operations $\infty + \infty$ and $\infty \cdot a$, where a is not a unit, remains undefined.

Definition 1.1 (Definition 5 of [D1, §1]). A *right place* of a simple Artinian ring Q into a simple Artinian ring D is a mapping f of the set (Q, ∞) onto (D, ∞) such that

$$f(ab) = f(a)f(b),\ f(1) = 1,\ f(a+b) = f(a) + f(b)$$

for any a, $b \in (Q, \infty)$ whenever the right hand sides are defined, and for any $q \in Q$ such that $f(q) = \infty$, there is an element $r \in Q$ such that $f(r) \neq \infty$ and $f(qr) \neq \infty$, 0. The

The detailed version of this paper will be submitted for publication elsewhere.

left place is defined similarly. Right and left place is called *place*.

Then we have the following.

Proposition 1.2 (Proposition 3 of [D1, §1]). Let Q and D be simple Artinian rings and let $f: Q \longrightarrow D$ be a place. Put $R = \{q \in Q \mid f(q) \neq \infty\}$ and $m(R) = \{q \in Q \mid f(q) = 0\}$. Then R is a subring of Q and $m(R)$ is a maximal ideal of R. Further, R is a local ring (i.e. $R/J(R)$ is a simple Artinian ring, where $J(R)$ denotes the Jacobson radical of R), and for any $q \in Q - R$, there exist $r_1, r_2 \in R$ such that $r_1 q, q r_2 \in R - m(R)$. Conversely, if R is a subring of Q and $m(R)$ is an ideal of R such that $R/m(R)$ is a simple Artinian ring, and if, for any $q \in Q - R$, there exist $r_1, r_2 \in R$ such that $r_1 q, q r_2 \in R - m(R)$, then the mapping

$$f(a) = a + m(R) \text{ if } a \in R,$$

$$f(a) = \infty \qquad \text{ if } a \notin R,$$

defines a place of Q into $R/m(R)$.

Theorem 1.3 (Theorem 4 of [D1, §1]). *The following are equivalent:*
(1) R is a rihgt semi-hereditary, right and left Goldie finite dimensional, prime local ring.
(2) R is a local Bezout ring (i.e. every finitely generated right or left ideal of R is principal) and is a prime Goldie ring.
(3) R is the set of elements of a simple Artinian ring Q at which a place of Q takes finite values.
(4) R is a right and left n-chain ring in a primitive ring Q, and there is a primitive ideal M of R such that the Goldie dimension of $R/M \geq n$. (R is called a right n-chain ring if, for any elements $a_0, a_1, \cdots, a_n \in Q$, there is an $i (0 \leq i \leq n)$ such that $a_i \in \Sigma_{j \neq i} a_j R$.)

Definition 1.4 (Definition 6 of [D1, §1]). A ring R is called a *non-commutative valuation ring* if R satisfies the conditions of Theorem 1.3.

Next we shall give some elementary properties of a non-commutative valuation ring.

Theorem 1.5 (Theorem 4 of [D1, §2]).*Let R be a valuation ring in a simple Artinian ring Q. Then*
(1) Two-sided R-ideals are linearly ordered by inclusion.
(2) Any overring S of R is also a valuation ring in Q, and the Jacobson radical $J(S)$ is a prime ideal of R.

(3) $R/J(S)$ is a valuation ring in the simple Artinian ring $A/J(S)$.
(4) $C(J(S)) = \{c \in R \mid [c+J(S)]$ is regular mod $J(S)\}$ is a regular Ore set of R and $S = R_{J(S)}$, the localization of R with respect to $C(J(S))$.

In the case of algebras, we have more results.

Theorem 1.6 (Theorem 1 of [D2, §2]).*Let V be a commutative valuation ring with quotient field K, and let R be a valuation ring in a finite dimensional central simple K-algebra Σ with its center V and $KR = \Sigma$. Then*
(1) For any prime ideal P of R, $C(P) = \{c \in R \mid [c+P]$ is regular mod $P\}$ is a regular Ore set of R, and so there exists the localization of R at P. Then we have $R_P = R_p$, where $p = P \cap V$ and R_p denotes the localization of R with respect to $V - p$.
(2) The mapping $P \longrightarrow R_P$ is an inclusion reversing bijection between the set of prime ideals of R and the set of overrings of R. The inverse mapping is $S \longrightarrow J(S)$.

2. Prime ideals in non-commutative valuation rings. Throughout this section, let V be a valuation domain with the quotient field K, and let R be a valuation ring in the sense of [D1] in a finite dimensional central simple K-algebra Σ.

For any ideal A of R, we define $\sqrt{A} = \{P :$ prime ideal of $R \mid P \supseteq A\}$, the *radical* of A. The following is trivial from Theorem 1.5 (1).

Lemma 2.1. *If A is an ideal of R, then $\sqrt{A}$ is a prime ideal of R.*

For any R-ideal A, let $O_r(A) = \{q \in Q \mid Aq \subseteq A\}$, the *right order* of A. Similarly the *left order* of A is defined. Concerning the right and left orders of a prime ideal of R, we have following.

Lemma 2.2. *Let P be a prime ideal of R. Then $R_P = O_r(P) = O_l(P)$.*

From Lemma 2.2 and Theorem 1.6, we have

Lemma 2.3. *Let P_1 and P_2 be prime ideals of R such that $P_1 \subseteq P_2$. Then we have $O_r(P_1) \supseteq O_r(P_2)$.*

An ideal Q of R ia called a *primary ideal* if $xRy \subseteq Q$ and $x \notin Q$, then $y \in \sqrt{Q}$, and if $xRy \subseteq Q$ and $x \notin \sqrt{Q}$, then $y \in Q$. If $\sqrt{Q} = P$, then we say that Q is a *P-primary ideal.*

Lemma 2.4. *For any ideal A of R, we have $O_r(A) \subseteq O_r(\sqrt{A})$. If A is a primary ideal, then the equality holds.*

We note that the equality in Lemma 2.4 does not hold in general. Next we give a condition for an ideal of R to be a primary ideal.

Lemma 2.5. *Let $Q \subseteq P$ be ideals of R and assume that P is prime. Then the following are equivalent.*
(i) Q is a P-primary ideal.
(ii) $\sqrt{Q} = P$ and Q is an ideal of R_P.

Corollary 2.6. *If Q_1 and Q_2 are P-primary ideals of R, then Q_1Q_2 is also a P-primary ideal* (See [G]).

Let P be a prime ideal of R. If T is a ring such that $R \subseteq T \subseteq R_P$, then, by Lemma 2.5, a P-primary ideal of R is an ideal of T. Then we have

Lemma 2.7. *Let Q be a P-primary ideal of R. Then for any ring T such that $R \subseteq T \subseteq R_P$, Q is P-primary ideal as an ideal of T.*

Now we give the key result in this note.

Proposition 2.8. *For any ideal A of R, $\bigcap A^n$ is a prime ideal.*

Next lemma follows from Proposition 2.8.

Lemma 2.9. *Let A be an ideal of R.*
(1) If $A^k = A^{k+1}$ for some $k > 0$, then A is an idempotent prime ideal.
(2) Let P be a prime ideal of R such that $P \subset A$. Then $P \subseteq \bigcap A^n$.
(3) If B is an ideal of R and $A \subset \sqrt{B}$, then $A^n \subseteq B$ for some $n > 0$.

Lemma 2.10. *Let S be a set of prime ideals of R and let $P = \bigcup_{P' \in S} P'$. Then*
(1) $O_r(P) = \bigcap O_r(P')$.
(2) P is a prime ideal.

A prime ideal P is said to be *branched* if there exists a P-primary ideal Q of R such that $Q \neq P$. In other case, P is called an *unbranched* prime ideal. Now we have the following, concerning branched and unbranched prime ideals of non-commutative valuation rings.

Theorem 2.11. *Let P be a prime ideal of R, and let $P_0 = \bigcap P^n$.*
(1) If P is branched and $P \neq P^2$, then
(i) $\{P^k \mid k > 0\}$ is the full set of P-primary ideals of R,
(ii) $P = zT = Tz$ for some $z \in P$, where $T = O_r(P)$,
(iii) there is no prime ideal P' such that $P \supset P' \supset P_0$ and P_0 is a prime ideal.
(2) If P is branched and $P = P^2$, then
(i) for any P-primary ideal $Q(\neq P)$, $\bigcap Q^n = \bigcap\{Q_\lambda \mid Q_\lambda :$ P-primary ideal $\}$,
(ii) $Q_0 := \bigcap\{Q_\lambda \mid Q_\lambda :$ P-primary ideal $\}$ is a prime ideal,
(iii) there is no prime ideal P' such that $P \supset P' \supset Q_0$.
(iv) $P = \bigcup\{Q_\lambda \mid Q_\lambda : P$-primary with $Q_\lambda \neq P\}$.
(3) The following are equivalent:
(i) P is branched.
(ii) $P = \sqrt{A}$ for some ideal $A(\neq P)$.
(iii) $P = \sqrt{RaR}$ for some $a \in R$.
(iv) P is not the union of prime ideals P' such that $P' \subset P$.
(v) There is a prime ideal M such that $M \subset P$ and there are no prime ideals P' such as $M \subset P' \subset P$.
(4) If P is unbranched, then $P = \bigcup\{P_\lambda \mid P_\lambda(\subset P)$: prime ideal $\}$.

Corollary 2.12. *Let P be a prime ideal of R and let $p = P \cap V$. Then*
(1) p is branched if and only if P is branched.
(2) p is idempotent if and only if P is idempotent. In this case, we have $P = pR$.

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PROCEEDINGS OF THE 25TH
SYMPOSIUM ON RING THEORY, 1992

THE INJECTIVITY OF QUASI-PROJECTIVE MODULES AND THE PROJECTIVITY OF QUASI-INJECTIVE MODULES

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Let R be a semiprimary ring with identity 1 and R-modules unitary. For a left R-module M we denote the injective hull of M by $E(M)$, the socle of M by $S(M)$, the Jacobson radical of M by $\mathrm{rad}(M)$ and the top $M/\mathrm{rad}(M)$ by $T(M)$. And for subsets S and T of R we write $l_S(T) = \{x \in S \mid xT = 0\}$ and $r_S(T) = \{x \in S \mid Tx = 0\}$. We put $J := \mathrm{rad}(R_R)$.

§1. i-pair.

In the following theorem, K. R. Fuller gave necessary and sufficient conditions for projective left modules to be injective over a left artinian ring.

Theorem A [4, Theorem 3.1]. *If e is an idempotent element of a left artinian ring R, then the following statements are equivalent:*

(a) *${}_RRe$ is injective.*

(b) *For each e_i in a basic set of idempotents for e, there is a primitive idempotent f_i of R such that*

$$S(Re_i) \approx T(Rf_i) \quad \text{and} \quad S(f_iR) \approx T(e_iR).$$

(c) *There exists an idempotent f of R such that*

(i) $l_{fR}(Re) = 0 = r_{Re}(fR)$;

(ii) *The functors*

$$\mathrm{Hom}_{fRf}(-, fRe) \quad \text{and} \quad \mathrm{Hom}_{eRe}(-, fRe)$$

define a duality between the category of finitely generated left fRf-modules and the category of finitely generated right eRe-modules.

Moreover, if ${}_RRe$ is injective, then the f_iR_R of (b) *and the fR_R of* (c) *are also injective.*

The detailed version of this paper will be submitted for publication elsewhere.

With respect to this theorem, we give two theorems in [1] which is a joint work with Prof. Oshiro.

For primitive idempotents e and f we call that the pair (fR, Re) is an *i-pair* if $S(fR_R) \approx T(eR_R)$ and $S({}_RRe) \approx T({}_RRf)$.

Theorem 1. *Let e be a primitive idempotent of R. Then the following two conditions are equivalent.*

(1) *${}_RRe$ is injective.*

(2) *There exists a primitive idempotent f of R such that*

 (i) *(fR, Re) is an i-pair,*

 (ii) *$l_{fR}r_{Re}(X) = X$ for any left fRf-submodule X of fR.*

Theorem 2. *Let e and f be primitive idempotents of R such that (fR, Re) is an i-pair. Then the following three conditions are equivalent.*

(1) *Re_{eRe} is artinian.*

(2) *${}_{fRf}fR$ is artinian.*

(3) *Both ${}_RRe$ and fR_R are injective.*

Example. Let D and F be division rings with $D \subsetneqq F$. We denote the factor ring S/T by R, where

$$S = \begin{pmatrix} D & 0 & 0 \\ D & D & 0 \\ F & F & D \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ F & 0 & 0 \end{pmatrix}.$$

Then R is a semiprimary ring. In R we put

$$e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then we see that ${}_{eRe}eR$ and Rf_{fRf} are artinian and (eR, Rf) is an i-pair. Therefore by Theorem 2 both eR_R and ${}_RRf$ are injective. But if F is infinite dimensional over D, R is neither left nor right artinian.

§2. The injectivity of quasi-projective modules and the projectivity of quasi-injective modules.

In §§2 and 3, we give the results in [2]. In this section, we generalize Theorem A by giving necessary and sufficient conditions for quasi-projective modules to be injective and ones for quasi-injective modules to be projective.

For idempotents e, f and g of R we call that R satisfies the condition $D_r[f, g, e]$ (resp. $D_l[f, g, e]$) if the descending chain condition (abbreviated DCC) holds on $\{ r_{gRe}(I) \mid I$ is a left fRf-submodule of $fRg \}$ (resp. $\{ l_{fRg}(I') \mid I'$ is a right eRe-submodule of $gRe \}$) (equivalently the ascending chain condition (abbreviated ACC) holds on $\{ l_{fRg}(I') \mid I'$ is a right eRe-submodule of $gRe \}$ (resp. $\{ r_{gRe}(I) \mid I$ is a left fRf-submodule of $fRg \}$)). And if both conditions $D_r[f, g, e]$ and $D_l[f, g, e]$ are satisfied, we call that R satisfies the condition $D[f, g, e]$.

Theorem 3. *Let e and f be primitive idempotents of R. Suppose that R satisfies* $D_r[f,1,e]$ *and* $D_l[f,f,e]$. *Then the following four conditions are equivalent.*

(1) *$E(T({}_RRf))$ is quasi-projective with the projective cover $\phi: Re \to E(T({}_RRf))$.*

(2) *fR_R is quasi-injective with $S(fR_R) \approx T(eR_R)$.*

(3) (i) *$S(fR_R) \approx T(eR_R)$,*

(ii) *$S({}_{fRf}fRe)$ is simple.*

(4) (i) *$l_{fR}(Re) = 0$,*

(ii) *fRe defines the Morita duality between the category of finitely generated left fRf-modules and the category of finitely generated right $eRe/e\text{-}r_{Re}(fR)$-modules.*

Moreover, if (1) is satisfied, $\mathrm{Ker}\phi = r_{Re}(fR)$.

The equivalence between (1) and (3) of Theorem 3 are easily generalized to the case that e and f are not primitive. Concretely, let $e = e_1 + \cdots + e_n$ and $f = f_1 + \cdots + f_n$ be decompositions to pairwise orthogonal primitive idempotents of R. And suppose that R satisfies $D_r[f_i,1,e_i]$ for any $i \in \{1,\cdots,n\}$. Then (1) is equivalent to the following (3') which is naturally generalized from (3).

(3') *For each $i \in \{1,\cdots,n\}$ the following two conditions are satisfied.*

(i) *$S(f_iR_R) \approx T(e_iR_R)$,*

(ii) *$S({}_{f_iRf_i}f_iRe_i)$ is simple.*

Moreover, if the conditions are satisfied, $\mathrm{Ker}\phi = \sum_{i=1}^{n}\oplus r_{Re_i}(f_iR)$.

But the equivalence between (2), (3) and (4) of Theorem 3 are not generalized naturally.

Example. Put

$$R = \begin{pmatrix} F & 0 & F \\ 0 & F & F \\ 0 & 0 & F \end{pmatrix},$$

where F is a field. And let e_i be the primitive idempotent of R which corresponds to the vertex i for each $i \in \{1,2,3\}$. Then $S(e_iR_R) \approx T(e_3R_R)$ and $S({}_{e_iRe_i}e_iRe_3)$ is simple for each $i = 1,2$. But $(e_1+e_2)R_R$ is not quasi-injective. In fact, since ($I :=$) $E(e_1R_R) = E(e_2R_R)$, we can represent $\mathrm{End}_R(\ E(e_1R_R) \oplus E(e_2R_R)\ )$ as

$$\begin{pmatrix} End_R(I) & End_R(I) \\ End_R(I) & End_R(I) \end{pmatrix}.$$

Then

$$\begin{pmatrix} 0 & 1_I \\ 0 & 0 \end{pmatrix}\begin{pmatrix} e_1R \\ e_2R \end{pmatrix} = \begin{pmatrix} e_2R \\ 0 \end{pmatrix} \nsubseteq \begin{pmatrix} e_1R \\ e_2R \end{pmatrix},$$

where 1_I denotes the identity map. So we see that (2) of Theorem 3 is not satisfied. Further ${}_{e_3Re_3}e_3Re_3$ is simple, but ${}_{eRe}\mathrm{Hom}_{e_3Re_3}(\ e_3Re_3,\ eRe_3\ ) \approx {}_{eRe}eRe_3$ is not simple, where $e := e_1 + e_2$. Therefore (4) of Theorem 3 is not satisfied.

§3. The projective cover of injective modules.

In the following theorem, Fuller gave a natural one to one correspondence between the homogeneous components of the k-th upper (resp. lower) Loewy factor of an injective right R-module E and the k-th lower (resp. upper) Loewy factor of ${}_RRf$ whenever fR is the projective cover of the socle of E, where f is a primitive idempotent in a right artinian ring R.

Theorem B [4, Theorem 2.4]. *Let E be an indecomposable injective left module over a left artinian ring R. Let f be a primitive idempotent of R with $T(Rf) \approx S(E)$. If e is any primitive idempotent of R, then $T(Re)$ appears in the k-th upper (lower) Leowy factor of E if and only if $T(eR)$ appears in the k-th lower (upper) Loewy factor of fR.*

Now we shall give a complete correspondence between simple submodules of the 1st upper Loewy factor of E and the 1st lower Loewy factor of ${}_RRf$.

Theorem 4. *Let $e, f_1, \cdots, f_n$ be primitive idempotents of R. Suppose that R satisfies $\mathrm{D}_l[f_i, 1, e]$ and both $S(f_iRe_{eRe})$ and $S({}_{f_iRf_i}f_iRe)$ are simple for any $i \in \{1, \cdots n\}$. Then $S({}_RRe) \approx \bigoplus_{i=1}^{n} T({}_RRf_i)$ if and only if $\bigoplus_{i=1}^{n} f_iR$ is the projective cover of $E(T(eR_R))$.*

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PROCEEDINGS OF THE 25TH
SYMPOSIUM ON RING THEORY, 1992

The calculation of direct summands of kG-modules for the modular group algebra kG of a finite group G

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Recently we can see a lot of softwares which calculate the finite field, the ring, the group, the vector space ... And we know computational methods are useful for the representation theory. In this paper, we will show a new procedure which investigate the socle series and some information about direct summands of a module over a group algebra.

1. Notation

Let G be a finite group with a set of generators $\{g_1, ..., g_l\}$ and k a splitting field for G such that the characteristic of k divides the order of the group G. And kG is a group algebra.

Let M be a kG-module and g an element of G, then a matrix $M(g)$ represents the action of g on a k-basis of M from right. So we can consider the vector space M as a right kG-module. We identify a set $\{M(g_1), ..., M(g_l)\}$ and a right kG-module M.

Let A_1, A_2, B, and C be matrices, and M, N, and S denote maps from G to a matrix ring over k. Then an equation

$$A_1 \cdot (M) \cdot A_2 = (N) \cdot B + C \cdot (S)$$

means $A_1 \cdot M(g) \cdot A_2 = N(g) \cdot B + C \cdot S(g)$ for all g in G. A matrix P represents homomorphism from a kG-module M to a kG-module N, if and only if, $(M) \cdot P = P \cdot (N)$. So M and N are isomorphic, if and only if, there is a regular matrix P such that $(M) \cdot P = P \cdot (N)$. Assume a kG-module M has a submodule N and M/N isomorphic to S. Then we can represent

$$M = \begin{pmatrix} N & 0 \\ D & S \end{pmatrix}$$

by suitable basis of M. A series of submodule of M

$$0 = M_0 \subset M_1 \subset ... \subset M_n = M, \qquad \text{where } S_i = M_i/M_{i-1} \text{ is simple}$$

The final version of this paper will be submitted for publication elsewhere.

is called a composition series of M and each S_i is a composition factor of M. Let Soc(M) denote the socle of M, namely the sum of all simple submodule of M. So we can define $\mathrm{Soc}^i(M)$ as following,

$$\mathrm{Soc}^0(M) = 0, \qquad \frac{\mathrm{Soc}^i(M)}{\mathrm{Soc}^{i-1}(M)} = \mathrm{Soc}\left(\frac{M}{\mathrm{Soc}^{i-1}(M)}\right).$$

Then there is a number m such that $\mathrm{Soc}^m(M) = M$ and $\mathrm{Soc}^{m-1}(M)$ is a proper submodule of M, and we call this number m the Socle length of M. Moreover a series of semi-simple modules $\{\mathrm{Soc}^1(M), \mathrm{Soc}^2(M)/\mathrm{Soc}^1(M), ..., \mathrm{Soc}^m(M)/\mathrm{Soc}^{m-1}(M)\}$ is called the Socle series of M and $\mathrm{Soc}^i(M)/\mathrm{Soc}^{i-1}(M)$ is the i^{th} Socle layer of M.

2. A simple way to get the Socle series and direct summands

(1) Socle series

Let $\{S_i\}$ be a set of representative of isomorphic classes of composition factors of M. Solve a matrix equation $(S_i)\cdot X = X\cdot(M)$ then the low vectors of the solution matrix P_i are basis of submodule which is isomorphic to S_i. So we can get Soc(M) by all solutions of above equations of all i.

(2) Direct summands

First we calculate the endomorphism ring of M. It is a ring which is generate from solutions of a matrix equation $(M)\cdot X = X\cdot(M)$. Next find a fitting element of this ring. Then the fitting element makes a projection map from M to the direct summand of M. We can see in detail in Schneider[1].

In both ways, a number of variable to solve the equation become huge if the dimension of M is big. So we will show another way in the section 4.

3. Preparation

We will show 4 lemmas for the section 4. In this section, we fix M and N as a kG-module and a kG-submodule of M such that $S = M/N$ is simple.

LEMMA 1. *A module M is isomorphic to $N \oplus S$, if and only if, there is a kG-submodule S' of M such that $M = N \oplus S'$.*

Proof) Easy.

LEMMA 2. *Suppose*

$$M = \begin{pmatrix} N & 0 \\ D & S \end{pmatrix}$$

then M is isomorphic to $N \oplus S$, if and only if, there is a matrix Q such that $(D) = (S)\cdot Q - Q\cdot(N)$.

Proof) From Lemma 1, M is isomorphic to $N \oplus S$, if and only if, there is a regular matrix P such that

$$P = \begin{pmatrix} 1 & & 0 & \\ & \ddots & & \\ & & 1 & \\ & A & & B \end{pmatrix} \quad \text{and} \quad P\cdot(M) = \begin{pmatrix} N & 0 \\ 0 & S \end{pmatrix}\cdot P.$$

So we can get two equations $A\cdot(N) + B\cdot(D) = (S)\cdot A$ and $(S)\cdot B = B\cdot(S)$. Since the matrix P is regular, the matrix B also is regular. Thus $B^{-1}\cdot A\cdot(N) + (D) = (S)\cdot B^{-1}\cdot A$. From above argument,if M is isomorphic to $N \oplus S$, $B^{-1}\cdot A$ satisfy the condition of Q. On the other side, if there is a matrix Q then a matrix P where A is Q and B is the unit matrix, satisfy the condition of P.

REMARK: Usually, if we want to know $M \cong N \oplus S$, we must find a matrix P such that

$$P\cdot(M) = \begin{pmatrix} N & 0 \\ 0 & S \end{pmatrix}\cdot P.$$

But lemma 2 shows it is enough to find the matrix Q instead of the matrix P if S is simple. For example, if the dimension of M and N are 100 and 90, then we need 10000 variables for P, but 900 for Q.

LEMMA 3. *Suppose N has kG-submodule N_1 and $N_2 = N/N_1$, thus*

$$M = \begin{pmatrix} N_1 & & 0 \\ E & N_2 & \\ D_1 & D_2 & S \end{pmatrix}.$$

If M is isomorphic to $N \oplus S$, there is a matrix Q such that $(D_2) = (S)\cdot Q - Q\cdot(N_2)$. In particular, if $N \cong N_1 \oplus N_2$, M is isomorphic to $N \oplus S$, if and only if, there are Q_1 and Q_2 where $(D_i) = (S)\cdot Q_i - Q_i\cdot(N_i)$ $(i = 1, 2)$.

Proof) From lemma 2, there is a matrix Q' where

$$(D_1 \quad D_2) = (S)\cdot Q' - Q'\cdot\begin{pmatrix} N_1 & 0 \\ E & N_2 \end{pmatrix}.$$

Let the matrix Q' be separated Q_1 and Q_2 where Q_i has the same size of the matrix D_i. Then the matrix Q_2 satisfy the first equation of this lemma. And if E is a zero matrix, Q_i satisfy the second equation.

Let $N_1 = \mathrm{Soc}(N)$, $N_2 = \mathrm{Soc}^2(N)/\mathrm{Soc}^1(N)$ and $N' = N/\mathrm{Soc}^2(N)$. Since N_2 is semisimple, we can consider $N_2 = T_1 \oplus T_2$ where T_1 is isomorphic to a direct sum of S and T_2 does not have any composition factors which is isomorphic to S. Then we can get the following lemma.

LEMMA 4. *Suppose $M/Soc(N) \cong N/Soc(N) \oplus S$. So*

$$M = \begin{pmatrix} N_1 & & & & \\ E_1 & T_1 & & & \\ E_2 & 0 & T_2 & & \\ F_1 & F_2 & F_3 & N' & \\ D & 0 & 0 & 0 & S \end{pmatrix}.$$

Then M is isomorphic to $N \oplus S$, if and only if, there is a matrix Q such that

$$(D \quad 0) = (S)\cdot Q - Q\begin{pmatrix} N_1 & \\ E_1 & T_1 \end{pmatrix}.$$

Moreover if N_2 does not have S as a composition factor, we can use the equation $(D) = (S)\cdot Q - Q\cdot(N_1)$ instead of above one.

Proof) Let assume M is isomorphic to $N \oplus S$. From Lemma 2, there is a matrix Q' such that

$$(D \quad 0 \quad 0 \quad 0) = (S)\cdot Q' - Q'\cdot\begin{pmatrix} N_1 & & & \\ E_1 & T_1 & & \\ E_2 & 0 & T_2 & \\ F_1 & F_2 & F_3 & N' \end{pmatrix}$$

So $(D) = (S)\cdot Q_1 - Q_1\cdot(N_1) - Q_2\cdot(E_1) - Q_3\cdot(E_2) - Q_4\cdot(F_1)$ and

$$(S)\cdot(Q_2\ Q_3\ Q_4) = (Q_2\ Q_3\ Q_4)\cdot\begin{pmatrix} T_1 & & \\ 0 & T_2 & \\ F_2 & F_3 & N' \end{pmatrix} \quad \text{where } Q' = (Q_1\ Q_2\ Q_3\ Q_4).$$

The second equation shows the matrix $(Q_2 \quad Q_3 \quad Q_4)$ represents a homomorphism from S to N/N_1. Since S is simple, An image of this map must be in $\mathrm{Soc}(N/N_1) = N_2 = T_1 \oplus T_2$. This means that the matrix Q_4 is a zero matrix. And the matrix Q_3 also is a zero matrix because of the definition of T_2. From first equation, $Q = (Q_1 \quad Q_2)$ satisfy the condition of this lemma. If N_2 does not have S as a composition factor, the matrices Q_2 and T_1 are disappear. The rest of the proof is just a calculation of matrices.

4. The outline of the procedure

In this section, we will show a procedure which calculate the socle series and some direct summands of a kG-module M.

For convenience, we call this procedure *AUSBAU*. The input of *AUSBAU* is a sequence of matrices which are corresponding to the representation of the generators of the group G. Then *AUSBAU* calculates the structure of the socle series and some direct summands.

EXAMPLE: A group G is the alternative group with degree 6. A field k has characteristic 3. Let M' be a permutation module of G and $M = M' \otimes M'$. So the dimension of M is 36. By this procedure *AUSBAU*, we can get following information.

$$M = \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} \oplus \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} \oplus \begin{pmatrix} & 4 & \\ 1 & 3 & 3' \\ & 4 & \end{pmatrix} \oplus 9.$$

Each number is the dimension of the composition factors of M. So *AUSBAU* decompose M to 4 direct summands. And each layer of a direct summand shows the socle series of M.

Let see how *AUSBAU* calculate the Socle series of kG-module X.

Step 1 Calculate the the composition series $\{X_i\}$ of X by the procedure *Meat Axe* which is made by Parker[2]. Let the number i be 2.

Step 2 Let $M = X_i$, $N = X_{i-1}$ and $S = X_i/X_{i-1}$. We can say that we have already known the direct summands $\{N_j\}$of N, the Socle series of N_j and the Socle length l_j of N_j for each number j. Let $\widetilde{N_j} = N/\bigoplus_{s\neq j} N_s$, then N_j can be embedded in $\widetilde{N_j}$ and $\widetilde{N_j}/N_j \cong S$ for each number j.

Step 3 Let the number l be $l_j - 1$.

Step 4 Check whether $\widetilde{N_j}/\mathrm{Soc}^l(N_j) \cong (N_j/\mathrm{Soc}^l(N_j)) \oplus S$

Step 5 If the answer of step 4 is true, then the number l be $l-1$ and do step 4 again. And if the answer is false, S is in the $(l+1)^{th}$ Socle layer of $\widetilde{N_j}$ and we can get the Socle series of $\widetilde{N_j}$. So let the number s_j be $l+1$. Moreover, if S is in the first Socle layer, $\widetilde{N_j} \cong N_j \oplus S$ and N_j is a direct summand of M.

Step 6 Repeat from step 3 to step 5 for all N_j.

Step 7 Let the number s_0 be maximum of s_j, then S is in the s_0^{th} Socle layer of M so we can get the socle series of M. And step 5 gives some direct summands of M.

Step 8 Let the number i be $i+1$ and repeat from step 2 to step 7 until the module X_i become X.

Let see step 4 in detail. To avoid complexity, $M = \widetilde{N_j}/\mathrm{Soc}^l(N_j)$ and $N = N_j/\mathrm{Soc}^l(N_j)$. So $S \cong M/N$ and $\mathrm{Soc}(N) = \mathrm{Soc}^{l+1}(N_j)/\mathrm{Soc}^l(N_j)$. Thus $M/\mathrm{Soc}(N) \cong \widetilde{N_j}/\mathrm{Soc}^{l+1}(N_j)$ and $N/\mathrm{Soc}(N) \cong N_j/\mathrm{Soc}^{l+1}(N_j)$. From step 5,

$$M/\mathrm{Soc}(N) \cong (N/\mathrm{Soc}(N)) \oplus S$$

So we can use Lemma 4 and step 4 is true, if and only if, the following matrix equation has a solution.

$$(D \quad 0) = (S)\cdot Q - Q\cdot (V).$$

The module V is a minimal submodule of $\mathrm{Soc}^2(N)$ such that $\mathrm{Soc}(N) \subset V$ and $\mathrm{Soc}^2(N)/V$ has no composition factor which is isomorphic to S.

In particular, if $\mathrm{Soc}^2(N)/\mathrm{Soc}(N)$ has no composition factors which is isomorphic to S, it is enough to solve a equation $(D) = (S)\cdot Q - Q\cdot(\mathrm{Soc}(N))$. Moreover we can use Lemma 2 because $\mathrm{Soc}(N)$ is semi-simple. So this equation decompose to $(D_i) = (S)\cdot Q_i - Q_i\cdot(W_i)$ where W_i is the composition factor of $\mathrm{Soc}(N)$.

Even there is a composition factors which are isomorphic to S in $\mathrm{Soc}^2(N)/\mathrm{Soc}(N)$, we can expect that V often decompose to some direct summands and the size of the matrix equation become small by lemma 2.

Now we know that the main calculation of this procedure *AUSBAU* is to solve the equation $(D) = (S)\cdot Q - Q\cdot(W)$ and almost W is simple. Thus even the dimension of the module is big. If simple modules in the composition factors are enough small then *AUSBAU* can work very well.

5. The decomposition of the module to direct summands

Unfortunately, we can not say the procedure *AUSBAU* decompose the module to the direct sum of indecomposable summands. But we can guarantee the following low boundary.

THEOREM. *Let X be a kG-module and $X = X_1 \oplus X_2$ such that the set of isomorphism classes of the composition factors of X_1 and X_2 are disjoint. Then the procedure AUSBAU decompose X to X_1 and X_2.*

The proof of theorem) Let $\{M_i\}$ be composition series of X. Because of assumption of X, $M_i = (M_i \cap X_1) \oplus (M_i \cap X_2)$ for all i. So what we need to do is to prove that if M_i is decompose to $(M_i \cap X_1)$ and $(M_i \cap X_2)$, then *AUSBAU* decompose M_{i+1} to $(M_{i+1} \cap X_1)$ and $(M_{i+1} \cap X_2)$. But

$$\frac{M_{i+1} \cap X_1}{M_i \cap X_1} \oplus \frac{M_{i+1} \cap X_2}{M_i \cap X_2} \cong \frac{M_{i+1}}{M_i} = S_i \quad : \text{simple.}$$

So we can assume $M_{i+1} \cap X_2 = M_i \cap X_2$. Then $M_{i+1}/(M_i \cap X_1) \cong S_i \oplus (M_i \cap X_2)$. And we can get $(M_i \cap X_2)$ as a direct summand of M_{i+1} in step 5 of *AUSBAU* .

COROLLARY. *Let X be a kG-module and $X = X_1 \oplus X_2$ such that X_1 and X_2 are in the different blocks. Then the procedure AUSBAU decompose X to X_1 and X_2.*

As concrete examples, there is a log-file of the computer system *CAYLEY* in the appendix. In this log-file, we can see the calculation about the representations over the group algebra where the group is the Mathieu group of the degree 22 and the field has characteristic 3.

REFERENCES

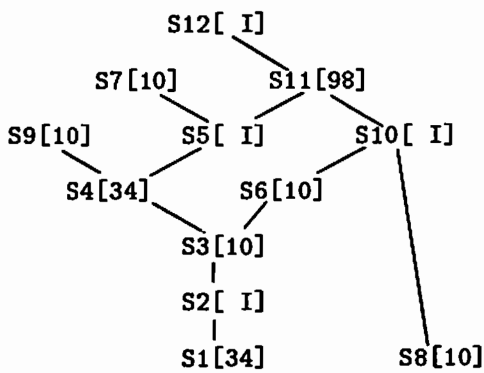
1. G. Schneider (1990), *Computing with Endomorphism Rings of Modular Representations*, Journal of Symbolic Computation vol 9, pp 607 – 636.
2. R. A. Parker (1984), *The computer calculations of modular characters (The meat-axe)*, In "Computational Group Theory" (ed. M.D. Atkinson). Academic Press, London.

Appendix

```
CAYLEY  V3.8.3 (IBM RS6000) Tue Jun 16 1992 16:31:22  STORAGE 8000000

library m22;
Library module found as /lokal3/cayley/caylibs/pergps/m22

 M22 - Mathieu group on 22 letters - degree 22
 Order 443 520 = 2^7 * 3^2 * 5 * 7 * 11;  Base 1,2,3,4,5
 Group : G; Generators: A, B, C.

gp=<c,a>;                                <===== Make a group gp
print order(gp);
    443520
f=field(4,w);                            <===== f=GF(4)
vs=vector space(22,f);
```

```
r=permutation module(vs,gp);
library ausbau2;
Library module found as /u/snoopy/waki/caylibs/ausbau2


library dec2;
Library module found as /u/snoopy/waki/caylibs/dec2


cf=composition factors(r);


rr=tensor(r,cf[2]);

socser(rr;rsq,soc);
     +++ Construct composition series of rep  +++
     +++        Change basis of rep           +++
     +++          Make a indsq                +++
    The dimension of composition factors are
    SEQ( 34, 1, 10, 34, 1, 10, 10, 10, 10, 1, 98, 1 )


    This is some information about relations of composition factors
    1   SEQ( 2 )
    2   SEQ( 3 )
    3   SEQ( 4, 6 )
    4   SEQ( 5, 9 )
    5   SEQ( 7, 11 )
    6   SEQ( 10 )
    7   SEQ(  )
    8   SEQ( 10 )
    9   SEQ(  )
    10   SEQ( 11 )
    11   SEQ( 12 )
    12   SEQ(  )
print soc;
    SEQ( SEQ(
     SEQ( 8, 1 ),
     SEQ( 2 ),
     SEQ( 3 ),
     SEQ( 4, 6 ),
     SEQ( 5, 9, 10 ),
     SEQ( 7, 11 ),
     SEQ( 12 ) ) )
quit;
```

rr =



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