PROCEEDINGS OF THE 26TH SYMPOSIUM ON RING THEORY

HELD AT TOKYO GAKUGEI UNIVERSITY, TOKYO

AUGUST 1 - 3, 1993

EDITED BY

Kanzo MASAIKE
Tokyo Gakugei University

Kunio YAMAGATA
University of Tsukuba

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OKAYAMA, JAPAN

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PREFACE

This volume contains the articles presented at the 26th Symposium on Ring Theory held at Tokyo Gakugei University, Tokyo, on August 1–3, 1993.

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October, 1993

Kanzo Masaike, Tokyo Gakugei University
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STRUCTURE OF RINGS SATISFYING CERTAIN CONDITIONS AND COMMUTATIVITY THEOREMS I

Tsunekazu Nishinaka

In his paper [9], Herstein introduced the concept of the hypercenter of a ring; the hypercenter, T_H, of the ring R is defined by $T_H(R) = \{a \in R \mid$ for each $x \in R$, there exists $n > 0$ such that $[a, x^n] = ax^n - x^n a = 0\}$. He showed that $T_H(R)$ coincides with the center of R if R has no non-zero nil ideal. Further, by making use of the result, in [10], he studied the rings R satisfying the following condition: For each $x, y \in R$, there exist $m, n > 0$ such that $[x^m, y^n] = 0$. Since then, several authors [3, 4, 5, 6, 8, 17] have studied various center-like subsets for rings and algebras in connection with the hypercenter, and also have observed rings satisfying certain conditions suggested naturally by those subsets. To generalize their results in [3, 5, 6, 8], we consider the following subsets of a ring R (as for the notations without mention see the bellow):

$S^* = S^*(R) = \{a \in R \mid$ for each $x \in R$, there exist $k > 0$ and a comonic $f(X) \in X\mathbb{Z}[X]$ such that $[a, f(x)]_k = 0\}$.

$T^* = T^*(R) = \{a \in R \mid$ there exist $k > 0$ and $n > 0$ such that for each $x \in R$, $[a, x^n - x^{n+1}f(x)]_k = 0$ for some $f(X) \in \mathbb{Z}[X]\}$.

$T^*_{(n,k)} = T^*_{(n,k)}(R) = \{a \in R \mid$ for each $x \in R$ there exists $f(X) \in \mathbb{Z}[X]$ such that $[a, x^n - x^{n+1}f(x)]_k = 0\}$, where n and k are positive integers.

We shall also study structure of semiprime rings satisfying the following conditions:

(H) For each $x, y \in R$, there exist comonic polynomials $f(X)$, $g(X) \in X\mathbb{Z}[X]$ and $k > 0$ such that $[f(x), g(y)]_k = 0$.

The detailed version of this paper has been submitted for publication elsewhere.

(H)$'_{(m)}$ For each $x \in R$, there exist $k > 0$ and $n > 0$ such that for each $y \in R$, $[x^m - x^{m+1}f(x), y^n - y^{n+1}g(y)]_k = 0$ for some polynomials $f(X), g(X) \in \mathbf{Z}[X]$, where m is a positive integer.

(H)$''_{(m,n,k)}$ For each $x, y \in R$, there exist polynomials $f(X), g(X) \in \mathbf{Z}[X]$ such that $[x^m - x^{m+1}f(x), y^n - y^{n+1}g(y)]_k = 0$, where m, n and k are positive integers.

Further, we consider the following condition which is studied coupling with the condition (H) in semiprime rings:

(S)$'$ For each $x, y \in R$, there exist integers α, β and γ such that $xy = \alpha yx + \beta x^2 + \gamma y^2 + f(x, y)$ for some $f(X, Y) \in \mathbf{Z}\langle X, Y\rangle$ each of whose monomial terms is of length ≥ 3.

In the present paper, we shall prove the following two theorems (§1 and §2):

Theorem 1.1 ([15, Theorem 1.1]). *If R is a reduced ring satisfying* (H), *then R is commutative.*

Theorem 2.1 ([15, Theorem 2.1]). *If R is a reduced ring, then $S^* = C$.*

By making use of the above two theorems, we have obtained some structure theorems for certain rings. In the preasent paper, we shall also exhibit several theorems which are especially interesting among those obtained in [15] and [16]. (§3, §4 and §5).

Throughout, R will represent a ring (not necessarily with 1) with center $C = C(R)$. Let $D = D(R)$ be the commutator ideal of R, and $J = J(R)$ the Jacobson radical of R. For $x, y \in R$, define extended commutators $[x, y]_k$ as follows: let $[x, y]_0 = x$, and proceed inductively $[x, y]_k = [[x, y]_{k-1}, y]$. Let $\mathbf{Z}$ denote the ring of integers, and $\mathbf{Z}\langle X, Y\rangle$ the polynomial ring over $\mathbf{Z}$ in the non-commuting indeterminates X and Y. We call a polynomial $f(X)$ in $X\mathbf{Z}[X]$ comonic if its lowest coefficient is 1 (i.e., $f(X) = X^m + X^{m+1}q(X)$ for some $m > 0$ and $q(X) \in \mathbf{Z}[X]$). Finally, for a subset U of R, we use the following notations: $\langle U\rangle$ (resp. (U)) is the subring (resp. ideal) of R generated by U. $C_R(U) = \{a \in R \mid [a, U] = 0\}$. $C^*_R(U) = \{a \in R \mid$ there exists $k > 0$ such that $[a, U]_k = 0\}$. $Ann(U) = \{a \in R \mid aU = Ua = 0\}$.

1. Proof of Theorem 1.1.

Lemma 1.1. *Let $\mathcal{E} = \{f(X) \in X\mathbf{Z}[X] \mid f(X)$ is a primitive polynomial*

(*i.e., the coefficients of* $f(X)$ *are relatively prime*) }. *Suppose that for each* $x, y \in R$ *there exist* $f(X), g(X) \in \mathcal{E}$ *and* $k_2 > k_1 > 0$ *such that* $[f(x), g(y)]_{k_1} = [f(x), g(y)]_{k_2}$. *If* R *is a torsion ring, then* D *is periodic.*

Proof. Let $R_p := \{x \in R \mid p^n x = 0 \text{ for some } n > 0\}$, p a prime. As is easily seen, R_p is an ideal of R and $R = \bigoplus_{p:\text{prime}} R_p$. By [2, Lemma 2], there exists a maximal periodic ideal $\mathcal{P}(R_p)$ of R_p such that $\overline{R_p} = R_p/\mathcal{P}(R_p)$ has no non-trivial periodic ideal. Since $\overline{R_p}$ has no non-zero nil ideal and $p\overline{R_p}$ is a nil ideal, we see that $\overline{R_p}$ is an algebra over GF(p), and also it has no non-zero algebraic ideal (see [2]). If $x, y \in \overline{R_p}$, then there exist $f(X), g(X) \in \mathcal{E}$ and positive integers k and d such that $[f(x), g(y)]_k = [f(x), g(y)]_{k+d}$. By [2, Lemma 4], we can easily see that there exist $\alpha > \beta > 0$ with $p^\alpha - p^\beta \equiv 0$ (mod d) such that $[f(x), g(y)^{p^\alpha}] = [f(x), g(y)]_{p^\alpha} = [f(x), g(y)]_{p^\beta} = [f(x), g(y)^{p^\beta}]$, and so we get $[f(x), g(y)^{p^\alpha} - g(y)^{p^\beta}] = 0$. Since it is easily seen that $g(X)^{p^\alpha} - g(X)^{p^\beta}$ is non-zero in $X\,\text{GF}(p)[X]$, we can see that $\overline{R_p}$ is commutative by [4, Theorem 3.6], and thus $D(R_p) \subseteq \mathcal{P}(R_p)$. Since $D(R) = \bigoplus D(R_p)$, we see that $D(R)$ is periodic.

Lemma 1.2. *Let* I *be an ideal of a ring* R *which is a division ring or a radical domain, and the characteristic* $Ch(I) = 0$. *Let* $a \in R$. *If for each* $x \in I$ *there exists* $k > 0$ *such that* $[x, a]_k = 0$, *then* $[I, a] = 0$.

Proof. Suppose that I is a radical domain, and further suppose, to the contrary, that there exists $x \in I$ such that $[x, a] \neq 0$. Then, by the hypothesis, $[x, a]_s = 0$ for some $s > 1$. Let k be the minimum integer in $\{t > 1 \mid [x, a]_t = 0\}$. Put $y = [x, a]_{k-2}$. Then $y \in I$. Since I is a radical domain, there exists $y^* \in I$ which is the quasi inverse of y. Embedding R into a ring with 1, we see that $1 + y^*$ is the inverse of $1 + y$. Put $u = 1 + y$. Since $[y, a]_2 = 0 \neq [y, a]$,

$$[u, a]_2 = 0 \neq [u, a]. \tag{1.1}$$

Noting that $0 = [uu^{-1}, a] = u[u^{-1}, a] + [u, a]u^{-1}$, we get

$$[u^{-1}, a] = -u^{-1}[u, a]u^{-1}. \tag{1.2}$$

By (1.1) and (1.2), we can easily see that $[u^{-1}, a]_2 = 2(u^{-1}[u, a])^2 u^{-1}$, and proceeding by induction, we get $[u^{-1}, a]_n = (-1)^n n!(u^{-1}[u, a])^n u^{-1}$ for all $n > 0$. Since $Ch(I) = 0$ and also $(u^{-1}[u,a])^n u^{-1} \in I$, we see that $[u^{-1},a]_n \neq 0$ and so $[y^*,a]_n \neq 0$ for all $n > 0$. On the other hand, since $y^* \in I$, we get that $[y^*,a]_{n'} = 0$ for some $n' > 0$, a contradiction.

In case that I is a division ring, we can get the conclusion in the same way.

Lemma 1.3. *Let R be a ring satisfying* (H). *If R is torsion free, then every primitive factorsubring of R is a division ring.*

Proof. By (H), for each prime p and each $x, y \in R$, there exist $f_0(X), g_0(X) \in \mathrm{Z}[X]$ and positive integers m, n and k such that

$$[(px)^m - (px)^{m+1} f_0(px), (py)^n - (py)^{n+1} g_0(py)]_k = 0.$$

It follows that

$$\begin{aligned} 0 &= [p^m x^m - p^{m+1} x^{m+1} f(x), p^n y^n - p^{n+1} y^{n+1} g(y)]_k \\ &= p^{m+nk} [x^m - px^{m+1} f(x), y^n - py^{n+1} g(y)]_k, \end{aligned}$$

where $f(X) = f_0(pX)$ and $g(X) = g_0(pX)$. Hence,

$$[x^m - px^{m+1} f(x), y^n - py^{n+1} g(y)]_k = 0. \tag{1.3}$$

Let R' be a primitive factorsubring of R. Suppose that R' is not a division ring. Then, we see that there exists a factorsubring of R' which is of type $(\mathrm{GF}(p))_2$ (p a prime) by the structure theorem for primitive rings. In (1.3), putting $x = e_{21} + e_{22}, y = e_{11} \in (\mathrm{GF}(p))_2$, we get that

$$0 = [(e_{21} + e_{22}), e_{11}]_k = [e_{21}, e_{11}]_{k-1} = e_{21} \neq 0,$$

a contradiction. Therefore, R' must be a division ring.

Let $\mathcal{E}^* = \{ f(X) \in X\mathrm{Z}[X] \mid f(X)$ is a comonic polynomial $\}$.

Lemma 1.4. *Let R be a domain. If for each $x, y \in R$, there exist $f(X), g(X) \in \mathcal{E}^*$ such that $[f(x), g(y)] = 0$, then R is commutative.*

Proof. If $Ch(R) \neq 0$, then D is periodic by Lemma 1.1, and so D is commutative by the well known Jacobson theorem. As is well known, a domain with a non-zero commutative ideal must itself be commutative, and thus $D = 0$. We may assume therefore that $Ch(R) = 0$. If R is semiprimitive, then it is a subdirect sum of primitive rings. Since $Ch(R) = 0$, R is a subdirect sum of division rings by Lemma 1.3. We may assume therefore that R is a division ring. Then, by [5, Remark 10], for each $x, y \in R$, there exists $f(X) \in \mathcal{E}^*$ such that $[x, f(y)] = 0$. For $x, y \in R$, consider the subring $\langle x, y \rangle$ of R generated by x and y. Then, we can easily see that for each $a \in \langle x, y \rangle$, there exists $f(X) \in \mathcal{E}^*$ such that $f(a) \in C(\langle x, y \rangle)$. Since $\langle x, y \rangle$ is domain, $\langle x, y \rangle$ is commutative by [7, Lemma 6], and so $[x, y] = 0$. We have thus seen that R is commutative in semiprimitive case. If R is not semiprimitive, then R has the non-zero radical J. Since J is a radical domain, we have that for each $x, y \in R$, there exists $f(X) \in \mathcal{E}^*$ such that $[x, f(y)] = 0$ again by [5, Remark 10]. Therefore, in the same way as above, we can see that J is commutative. Hence, R must be commutative.

Lemma 1.5. *Let R be a domain satisfying* (H), *and $a, b \in R$. If there exists $k > 0$ such that $[a, b]_k = 0$, then $[a, b] = 0$.*

Proof. If $Ch(R) = p \neq 0$, then D is a periodic domain by Lemma 1.1, and thus D is commutative. Hence, R must be commutative. We may assume therefore that $Ch(R) = 0$. Obviously, $C_R^*(b)$ is a subring of R and $b \in C_R^*(b)$. Also, $C_R^*(b)$ satisfies (H), and $C_{C_R^*(b)}(b) = C_R(b) \cap C_R^*(b) = C_R(b)$. Therefore, under the hypothesis $C_R^*(b) = R$, it is enough to show $b \in C(R)$.

First, we suppose that R is semiprimitive. Since R satisfies (H) and $Ch(R) = 0$, R is a subdirect sum of division rings R_i ($i \in I$) by Lemma 1.3. Let ϕ_i be the natural epimorphism of R onto R_i, and put $b_i = \phi_i(b)$. It suffices to show that $b_i \in C(R_i)$ ($i \in I$). Since R_i satisfies (H), as we saw at the first of the proof, we may assume that $Ch(R_i) = 0$. Then, for each $x \in R_i$, there exists $k > 0$ such that $[x, b_i]_k = 0$, and so $[x, b_i] = 0$ by Lemma 1.2.

Next, suppose that R has the non-zero radical J. Then, J is a radical domain, and for each $x \in J$, there exists $k > 0$ such that $[x, b]_k = 0$. Hence, by Lemma 1.2, we get that $[J, b] = 0$. Since R is a domain, as is well known, $C_R(J) \subseteq C(R)$, and so $b \in C(R)$.

Proof of Theorem 1.1. Since R is a subdirect sum of domains by [1, Theorem 2], we may assume that R is a domain. By (H), for each $x, y \in R$, there exist $f(X), g(X) \in \mathcal{E}^*$ and $k > 0$ such that $[f(x), g(y)]_k = 0$. Hence, we get that $[f(x), g(y)] = 0$ by Lemma 1.5. Therefore, R is commutative by Lemma 1.4.

2. Proof of Theorem 2.1.

Lemma 2.1. *S^* is a subring of R.*

Proof. If $a \in S^*$, then for each $x \in R$, there exist $k_1 > 0$ and $p(X) \in \mathcal{E}^*$ such that $[a, p(x)]_{k_1} = 0$, and if $b \in S^*$, then for $p(x)$, there exist $k_2 > 0$ and $q(X) \in \mathcal{E}^*$ such that $[b, q(p(x))]_{k_2} = 0$. Putting $k = max\{k_1, k_2\}$ and $h(X) = q(p(X))$,

$$[a, h(x)]_k = [a, q(p(x))]_k = \sum_i M_i(x)[a, p(x)]_k N_i(x) = 0,$$

where $M_i(X), N_i(X) \in X\mathbf{Z}[X]$. Hence,

$$[a, h(x)]_k = 0 = [b, h(x)]_k.$$

Furthermore, since $[a + b, h(x)]_k = [a, h(x)]_k + [b, h(x)]_k$ and $[ab, h(x)]_{2k} = \sum_{i=0}^{2k} \binom{2k}{i} [a, h(x)]_i [b, h(x)]_{2k-i} = 0$, we can easily see that $\langle a, b \rangle \subseteq S^*$, and so S^* is a subring of R.

Lemma 2.2. *Let p be a prime integer, R an algebra over* GF(p), *and $A(R)$ the algebraic hypercenter of R (see [4]). Then $S^*(R) = A(R)$.*

Proof. Obviously, $A(R) \subseteq S^*(R)$. On the other hand, since $[x, y]_{p^\alpha} = [x, y^{p^\alpha}]$ for all $x, y \in R$ and all $\alpha > 0$, we can easily see that $S^*(R) \subseteq A(R)$.

Lemma 2.3. *Let R be a prime ring with no non-zero nil ideal. If R has the non-zero radical J and $Ch(R) = p \neq 0$, then $S^* = C$.*

Proof. First, we claim that R has no non-zero periodic ideal. Suppose, to the contrary, that R has a periodic ideal $I \neq 0$. Since R is a prime ring, $I \cap J \neq 0$. For each $x \in I \cap J$, there exist positive integers n and d such that $x^n = x^{n+d}$, and so $x^n \in x^n J$. We see that $x^n = 0$. This implies a contradiction that $I \cap J$ is a non-zero nil ideal. Hence, R has no non-zero periodic ideal as claimed. R is an algebra over GF(p) and it has no non-zero periodic ideal; thus it has no non-zero algebraic ideal. Then $C =$ the algebraic hypercenter of R by [4, Theorem 1.6]. On the other hand, we see that $S^* =$ the algebraic hypercenter of R by Lemma 2.2. Therefore, we get that $S^* = C$.

Lemma 2.4. *If R is a division ring, then $S^* = C$.*

Proof. By Lemma 2.1, S^* is a subring of R. Since S^* satisfies (H), it is commutative by Theorem 1.1. Let K be the subfield of R generated by S^*. Then, it is clear that K is preserved re automorphisms of R, and so $S^* = C$ by Cartan, Brauer, Hua result.

Lemma 2.5. *Let R be a semiprimitive ring. If R is torsion free, then $S^* = C$.*

Proof. R is a subdirect sum of primitive rings R_i ($i \in I$). Let ϕ_i be the natural epimorphism of R onto R_i. It suffices to show that $\phi_i(S^*) \subseteq C(R_i)$ for all $i \in I$. In the same way of the proof of Lemma 1.3, if $a \in S^*$, then for each prime p and $y \in R$, there exist $m > 0$, $k > 0$ and $f(X) \in \mathrm{Z}[X]$ such that

$$[a, y^m - py^{m+1}f(y)]_k = 0. \tag{2.1}$$

If R_i is a division ring, then $S^*(R_i) = C(R_i)$ by Lemma 2.4, and so $\phi_i(S^*) \subseteq S^*(R_i) = C(R_i)$. We may assume therefore that R_i is not a division ring. Consider the case that $Ch(R_i) = p \neq 0$. If $a \in S^*$, then for each $y \in R_i$, there exists $b \in R$ such that $y = \phi_i(b)$ and $[\phi_i(a), y^m]_k = 0$ for some $m > 0$ by (2.1). Therefore, we see that $\phi_i(S^*) \subseteq S'(R_i)$. Hence, we get that $\phi_i(S^*) \subseteq C(R_i)$ by Lemma 2.5. Next, consider the case that $Ch(R_i) = 0$. Since R_i is a primitive ring which is not a division ring, by the density theorem, R_i acts densely on a vector space V_i over the division ring Δ_i with $dim V_i > 1$. Suppose that there exist $v \in V_i$ and $x \in \phi_i(S^*)$ such that v and vx are linearly independent. By the density action of R_i, there exists $y \in R_i$ such that $vy = 0$ and $vxy = vx$. Since $x \in \phi_i(S^*)$ and $\phi_i(R) = R_i$, there exist $m, k > 0$ and $f(X) \in \mathrm{Z}[X]$ such that $[x, y^m - 2y^{m+1}f(y)]_k = 0$ by (2.1). Put $g(X) = X^m - 2X^{m+1}f(X)$. Then, $g(1) = 1 - 2f(1) \neq 0$. On the other hand, we can easily see that $v[x, g(y)]_k = g(1)^k vx$, and thus $0 = v[x, g(y)]_k = g(1)^k vx \neq 0$, a contradiction. Hence, for each $v \in V_i$ and $x \in \phi_i(S^*)$, there exists $\lambda \in \Delta_i$ such that $vx = \lambda v$, and so $x \in C(R_i)$ by [5, Remark 13]. We have thus seen that $\phi_i(S^*) \subseteq C(R_i)$.

We are now in a position to prove Theorem 2.1.

Proof of Theorem 2.1. R is a subdirect sum of domains R_λ ($\lambda \in \Lambda$) by [1, Theorem 2]. Let ϕ_λ be the natural epimorphism of R onto R_λ. Since $\phi_\lambda(S^*(R)) \subseteq S^*(R_\lambda)$, it is enough to show that $S^*(R_\lambda) = C(R_\lambda)$ for all $\lambda \in \Lambda$. That is to say, we may assume that R is a domain.

First, we suppose that $Ch(R) = p \neq 0$. If there exists a non-zero periodic ideal I, then the domain I is commutative by the well-known Jacobson theorem. As is well known, a domain with a non-zero commutative ideal must itself be commutative, and thus $S^* = C = R$. Therefore, we may assume that R has no non-zero periodic ideal and thus it has no non-zero algebraic ideal over GF(p). Then, it follows that $S^* = C$ by [4, Theorem 1.6] and Lemma 2.2.

Next, suppose that $Ch(R) = 0$. Since S^* is a domain satisfying (H), it is a commutative subring of R by Theorem 1.1. Therefore, if R has the non-zero radical J, then $S^* = C$ by [5, Remark 9]. On the other hand, if R is semiprimitive, then $S^* = C$ by Lemma 2.5.

3. The subsets S^*, T^* and $T^*_{(n,k)}$ in semiprime rings. In section 2, we saw that $S^*(R) = C(R)$ if R is a reduced ring. However, if R is a semiprime ring, (or a ring with no non-zero nil ideal) then $S^*(R)$, even if $T^*_{(n,k)}(R)$ ($nk > 1$), does not coincide with $C(R)$ in general. In this section, we shall study structure of semiprime rings R in which $S^*(R)$, $T^*(R)$ and $T^*_{(n,k)}$ ($nk > 1$) do not coincide with $C(R)$, respectively.

Throughout this section, for $n > 0$, R_n will denote the ring of $n \times n$-matrices over a ring R. We call a field F periodic if it is an algebraic extension field over a finite field.

we have obtained the followings:

Theorem 3.1([15, Theorem 3.1]). *If R is a semiprime ring satisfying* (S)$'$, *then R is a subdirect sum of rings each of which has one of the following types.*

(i) *a prime ring R' with $S^*(R') = C(R')$.*

(ii) *a dense ring of linear transformations on a vector space V over F, where F is a periodic field and $dim_F\ V > 1$.*

Theorem 3.2([15, Theorem 3.2]). *If R is a prime ring, then one of the following properties hold*:

(i) $T^*(R) = C(R)$.

(ii) *R is isomorphic to F_t, where F is a periodic field and $t > 1$ an integer*

Theorem 3.3([15, Theorem 3.3]). *Let n and k be positive integers. If R is a prime ring, then one of the following properties hold*:

(i) $T^*_{(n,k)}(R) = C(R)$.

(ii) *R is isomorphic to F_t, where F is a periodic field and $1 < t \leq kn$.*

If R is n-algebraic over a subring A (see [6]), then $C_R(A) \subseteq T_{(n,1)}(R)$. On account of this, we can say that Theorem 3.3 improves [6, Theorem 3]. Needless to say that Theorem 3.2 and Theorem 3.3 can be extended to the results for semiprime rings, respectively: If R is a semiprime ring, then R is a subdirect sum of rings each of which is of type (i) or (ii).

4. Structure of semiprime rings satisfying (H). In section 1, we saw that a reduced ring satisfying (H) is commutative. However, a ring satisfying (H) with no non-zero nil ideal need not be commutative. We begin this section by stating the following conjecture:

Conjecture 4.1. *Let R be a ring with no non-zero nil ideal. If R satisfies* (H), *then R is a subdirect sum of rings each of which has one of the following types.*

(i) *a commutative domain.*

(ii) *a dense ring of linear transformations on a vector space V over F, where F is a periodic field and $dim_F\ V > 1$.*

We claim that if the answer of Köthe conjecture (i.e., a ring which has a non-zero one-sided nil ideal contains a non-zero two-sided nil ideal) is positive, then the answer of our conjecture is also positive (see Theorem 4.2). We cannot answer the above conjecture in the present paper, however, by making use of Theorem 1.1, we have proved the following theorems with respect to rings satisfying (H), which includes a generalization of [8, Theorem 3]:

Theorem 4.1([15, Theorem 4.1]). *Let R be a ring with no non-zero nil ideal. If R is a torsion ring satisfying* (H), *then R is a subdirect sum of rings each of which has one of the following types.*

(i) *a commutative domain.*

(ii) *a dense ring of linear transformations on a vector space V over F, where F is a periodic field and $dim_F\ V > 1$.*

Theorem 4.2([15, Theorem 4.2]). *Let R be a ring with no non-zero nil right ideal. If R satisfies* (H), *then R is a subdirect sum of rings each of which has one of the following types.*

(i) *a commutative domain.*

(ii) *a dense ring of linear transformations on a vector space V over F, where F is a periodic field and $dim_F\ V > 1$.*

Theorem 4.3([15, Theorem 4.3]). *Suppose that R satisfies* (S)$'$ and (H). *If R is a semiprime ring, then R is a subdirect sum of rings each of which has one of the following types.*

(i) *a commutative domain.*

(ii) *a dense ring of linear transformations on a vector space V over F, where F is a periodic field and $dim_F\ V > 1$.*

Note that Theorem 4.2 generalizes [8, Theorem 3].

Next, we shall introduce the following two theorems with respect to rings satisfying $(\mathrm{H})'_{(m)}$ and $(\mathrm{H})''_{(m,n,k)}$, respectively.

Theorem 4.4([15, Theorem 4.4]). *Let R be a semiprime ring, and m a positive integer. If R satisfies $(\mathrm{H})'_{(m)}$, then R is a subdirect sum of rings each of which has one of the following types.*

(i) *a commutative domain.*

(ii) *F_t, where F is a periodic field and $t > 1$ an integer*

Theorem 4.5([15, Theorem 4.5]). *Let R be a prime ring, and m, n and k positive integers. If R satisfies $(\mathrm{H})''_{(m,n,k)}$, then R is of one of the following forms:*

(i) *a commutative domain.*

(ii) F_t, *where* F *is a periodic field and* $1 < t \leq max\{kn, m\}$.

In Theorem 4.5, as $k = 1$ and $m = n$, it means [6, Corollary 5.3].

5. Structure of rings satisfying the conditions $(H)_{(f)}$ and (S). Given a finite field K with a non-trivial automorphism σ, we put $M_\sigma(K) = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \sigma(\alpha) \end{pmatrix} \middle| \alpha, \beta \in K \right\}$.

Let $f(X)$ be in $XZ[X]$, and n a positive integer. We consider the following conditions:

$(H)_{(f)}$ For each $x, y \in R$, there exists $k > 0$ such that $[f(x), f(y)]_k = 0$.

(S) For each $x, y \in R$, there exists $f(X,Y) \in Z\langle X,Y\rangle[X,Y]Z\langle X,Y\rangle$ each of whose monomial terms is of length ≥ 3 such that $[x,y] = f(x,y)$.

$Q(n)$ If $x, y \in R$, and $n[x,y] = 0$, then $[x,y] = 0$.

In his paper [11], Kobayashi determined the structure of unital rings (rings with unit element 1) satisfying the identity $[X^n, Y^n] = 0$ and $Q(n)$. The condition (S) have been introduced by Streb [18]. We should note that every unital ring satisfying $Q(n)$ and the identity $[X^n, Y^n] = 0$ satisfies (S). Recently, Komatsu and Tominaga [14] investigated the rings satisfying (S) with nil commutator ideals, in which they improved the main result in [11] as follows:

Theorem A [14, Theorem 3.6]. *Let* R *be a ring, and* $0 < n \in Z$. *Then the following conditions are equivalent:*

(1) R *satisfies the identity* $[X - X^m, Y - Y^m] = 0$ *for some* $m > 0$, *and satisfies the identity* $[X^n, Y^n] = 0$.

(2) R *satisfies* (S) *and the identity* $[X^n, Y^n] = 0$

(3) R *is a subdirect sum of rings each of which has one of the following types:*

(i) *a commutative ring.*

(ii) $M_\sigma(K)$, *where* $(|K| - 1)/(|K^\sigma| - 1) \mid n$.

(In fact, Theorem A is stated for certain algebras more general than rings.)

In this section, we shall introduce two structure theorems which extend Theorem A to rings satisfying the conditions $(H)_{(f)}$ and (S). If R contains the unit element 1, then we can state the next:

Theorem 5.1([16, Theorem 2.1]). *Let $f(X)$ be a comonic polynomial in $X\mathbf{Z}[X]$ with $f(1) \neq 0$, and R a ring with 1. If R satisfies the conditions* $(\mathrm{H})_{(f)}$, (S) *and* $\mathrm{Q}(f(1))$, *then R is a subdirect sum of rings each of which has one of the following types:*

(i) *a commutative ring.*

(ii) $\mathrm{M}_\sigma(K)$, *where* $(Ch(K), f(1)) = 1$, *and* $|K|/|K^\sigma| \leq deg\ f(X)$.

In case that R does not always contain 1, we don't know whether the above statement is true or not. But we can state the following:

Theorem 5.2([16, Theorem 2.2]). *Let $f(X)$ be a comonic polynomial in $X\mathbf{Z}[X]$ such that all the coefficients of $f(X)$ are non-negative. If R satisfies the conditions* $(\mathrm{H})_{(f)}$, (S) *and* $\mathrm{Q}(f(1)!)$, *then R is a subdirect sum of rings each of which has one of the following types:*

(i) *a commutative ring.*

(ii) $\mathrm{M}_\sigma(K)$, *where* $(Ch(K), f(1)!) = 1$, $(|K| - 1)/(|K^\sigma| - 1) \mid f'(1)$ *and* $|K|/|K^\sigma| \leq deg\ f(X)$.

Theorem 5.1 and Theorem 5.2 generalize [11, Theorem] and Theorem A, respectively.

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STRUCTURE OF RINGS SATISFYING CERTAIN CONDITIONS AND COMMUTATIVITIY THEOREMS. II

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0. Introduction . In [2], Y. Kobayashi defined the additive map Φ of $\mathbf{Z}\langle X,Y\rangle$ to $\mathbf{Z}$, and indicated that for $f(X,Y) \in \mathbf{Z}\langle X,Y\rangle, \Phi(f(X,Y))$ is closely related with the commutativity of rings with 1 satisfying the polynomial identity $f(X,Y) = 0$. In [3], he turned his attention to the fact that $\Phi((XY)^n - X^nY^n) = -\dfrac{n(n-1)}{2}$ for $n > 1$, and investigated the structure of $\dfrac{n(n-1)}{2}$-torsion free rings with 1 satisfying the polynomial identity $(XY)^n - X^nY^n = 0$. Coincidentally, he proved the following ([3, Theorem]): Let R be a ring with 1. If $E(R) = \{ n \in \mathbf{Z} \mid n > 0 \text{ and } (xy)^n = x^ny^n \text{ for all } x, y \in R \}$ contains integers $n_1, \cdots, n_r \geq 2$ such that $(\dfrac{n_1(n_1-1)}{2}, \cdots, \dfrac{n_r(n_r-1)}{2}) = 1$ and some of n_i's is even, then R is commutative. In connection with the above theorem, Y. Kobayashi and the present author respectively raised the following conjectures:

Conjecture 0.1 ([4, Conjecture 1]). Let R be a ring with 1. If $E(R)$ contains integers $n_1, \cdots, n_r \geq 2$ such that R is $(\dfrac{n_1(n_1-1)}{2}, \cdots, \dfrac{n_r(n_r-1)}{2})$-torsion free and some of n_i's is even, then R is commutative.

This note is derived from [14] .

Conjecture 0.2 ([13, Conjecture (I)]). Let R be a ring with 1. If for each $x, y \in R$, there exist integers $n_i \geq 2$ $(i = 1, \cdots, r)$ such that $(\frac{n_1(n_1-1)}{2}, \cdots, \frac{n_r(n_r-1)}{2}) = 1$ and some of n_i's is even and such that $(xy)^{n_i} = x^{n_i}y^{n_i}$ $(i = 1, \cdots, r)$, then R is commutative.

In [5] and [6], Y. Kobayashi gave partially affirmative answers to the above conjectures. In §§2 and 4 of the present note, those results will be improved more precisely and satisfactorily.

Meanwhile, J. Grosen [1] generalized some known commutativity theorems for a ring with 1 and satisfying certain polynomial identities by assuming that the identities hold merely for the elements of a certain subset of the ring rather than for all elements of the ring. Almost all the results obtained in [1] have been improved and sharpened in [10]. In §3 of the present note, we shall give some commutativity theorems for a ring with 1 and satisfying polynomial identities of the form $(XY)^n - X^nY^n = 0$ merely for the elements of a certain subset of the ring.

Recently, W. Streb [15] gave a classification of non-commutative rings. H. Komatsu and H. Tominaga applied the classification to the proof of some commutativity theorems, in [8], [9], [10] and [11]. Several results obtaind in [9], [11] will play an essential role in our subsequent study.

1. Preliminaries . Throughout the present note, R will represent a ring with 1. Let $n_1, n_2, \cdots, n_r$ and k be positive integers, and M a non-empty subset of R. We use the following notations:

$(n_1, n_2, \cdots, n_r) =$ the greatest common divisor of $n_1, n_2, \cdots,$ and n_r.

$|E| =$ the cardinal number of a set E.

$C = C(R) =$ the center of R .

$D = D(R) =$ the commutator ideal of R.

$N = N(R) =$ the set of all nilpotent elements in R.

$J = J(R) =$ the Jacobson radical of R.

$U = U(R) =$ the set of units in R.

$Q =$ the intersection of the set of non-units in R with the set of quasi-regular elements in $R = (1+U)\setminus U$ ($\supseteq N \cup J$).

As usual, for $x, y \in R$, let $[x,y] = xy - yx$.

$\mathbf{Z} =$ the set of integers.

$\mathbf{Z}\langle X,Y\rangle =$ the polynomial ring over $\mathbf{Z}$ in the non-commuting indeterminates X and Y.

$\mathbf{K} = \mathbf{Z}\langle X,Y\rangle[X,Y]\mathbf{Z}\langle X,Y\rangle$.

$\mathbf{K}_k =$ the set of all $f(X,Y) \in \mathbf{K}$ each of whose monomial terms is of length $\geq k$ (together with 0).

$\mathbf{W} =$ the set of all words in X and Y, namely products of factors each of which is X or Y (together with 1).

As is well-known, $\mathbf{K} = \mathbf{K}_2$ coincides with the kernel of the natural homomorphism of $\mathbf{Z}\langle X,Y\rangle$ onto $\mathbf{Z}[X,Y]$. Let $f(X,Y) = \sum f_{ij}(X,Y)$ be a polynomial in $\mathbf{Z}\langle X,Y\rangle$, where $f_{ij}(X,Y)$ is a homogeneous polynomial with degree i in X and degree j in Y. Then we can easily see that $f(X,Y)$ is in $\mathbf{K}$ if and only if for each i,j, the sum of the coefficients of $f_{ij}(X,Y)$ equals zero.

Following [2], we denote by Φ the additive map of $\mathbf{Z}\langle X,Y\rangle$ to $\mathbf{Z}$ defined as follows: For each monic monomial $X_1\cdots X_r$ (X_i is either X or Y), $\Phi(X_1\cdots X_r)$ is the number of pairs (i,j) such that $1 \leq i < j \leq r$ and $X_i = X$, $X_j = Y$. We can easily see that, for any $f(X,Y) \in \mathbf{Z}\langle X,Y\rangle$, $\Phi(f(X,Y))$ equals the coefficient of XY occurring in $f(1+X,1+Y)$. Now, let $f(X,Y) \in \mathbf{K}$. Then $f(1+X,1+Y) \in \mathbf{K}$, and so there exists $g(X,Y) \in \mathbf{K}_3$ such that $f(1+X,1+Y) = \Phi(f(X,Y))[X,Y] + g(X,Y)$.

Further, we put

$$e(k) = \begin{cases} k & \text{if } k \text{ is even,} \\ k-1 & \text{if } k \text{ is odd.} \end{cases}$$

We consider the following conditions:

(S) For each $x, y \in R$, there exists $f(X,Y) \in \mathbf{K}_3$ such that $[x,y] = f(x,y)$.

$Q(k)$ If $x, y \in R$ and $k[x,y] = 0$ then $[x,y] = 0$.

Definition 1.0. A ring is called a *factorsubring* of R if it is a factor ring (homomorphic image) of a subring of R.

By [9, Theorem 1.2, Proposition 1.6 and Proposition 1.7], we obtain the next theorem which plays an important role in our study.

Theorem 1.1. *Let R be a non-commutative ring with 1. Then there exists a factorsubring of R which is of type* a)[1], b), c), d)[1] *or* e)[1]*:*

a)[1] $\begin{pmatrix} \mathrm{GF}(p) & \mathrm{GF}(p) \\ 0 & \mathrm{GF}(p) \end{pmatrix}$, *where p a prime number.*

b) $\mathrm{M}_\sigma(K) = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \sigma(\alpha) \end{pmatrix} \,\middle|\, \alpha, \beta \in K \right\}$, *where K is a finite field with a non-trivial automorphism σ.*

c) *A non-commutative division ring.*

d)[1] *A domain which is generated by 1 and a simple radical subring.*

e)[1] *A ring $B = \langle 1, x, y \rangle$ with 1 such that $D(B)$ is the heart of B and $x, y \in \mathrm{Ann}_B(D(B))$.*

By [9, Proposition 1.3 (2), Lemma 1.4 (1) and (4), and Proposition 1.7], we obtain

Lemma 1.2. *Let R be a ring with 1. If $xy \neq 0 = yx$ for some $x, y \in R$, then there exists a factorsubring of R which is of type* a)[1] *or* e)[1].

2. On Conjecture 0.1 . Given $x, y \in R$, we denote by $E(x, y)$ the set of integers $n > 1$ such that $(xy)^n = x^n y^n$; and $\widetilde{E}(x, y) = E(x, y) \cap E(y, x)$. For a positive integer n, an element x of a module G is said to be *n-torsion free* if the order of x is infinite or relatively prime to n. Obviously, all the elements of G are n-torsion free if and only if $nx = 0$ implies $x = 0$ for any $x \in G$.

The purpose of this section is to give a complete answer to Conjecture 0.1. In [5], Y. Kobayashi proved the following theorem which is a partial answer to Conjecture 0.1.

Theorem A. *Let R be a ring with 1. If for any $x, y \in R$, $\widetilde{E}(x,y)$ contains (at least one) even integers $n_1, \cdots, n_s$ and odd integers $n_{s+1}, \cdots, n_r$ $(r \geq s \geq 1)$ such that $(n_1, \cdots, n_s\ ,\ n_{s+1} - 1, \cdots, n_r - 1)$ is 2 (or a multiple of 4) and $[x, y]$ is $(\frac{n_1(n_1-1)}{2}, \cdots, \frac{n_r(n_r-1)}{2})$-torsion free, then R is commutative.*

In connection with the above theorem, in [6], he determined the structure of $\frac{n(n-1)}{2}$-torsion free rings with 1 satisfying the identity $(xy)^n = x^n y^n$, when n is a positive even integer. Recently, this result has been generalized by H. Komatsu and H. Tominaga (see [11, Theorem 2.12]). The main theorems of this section can be stated as follows:

Theorem 2.1. *Let R be a ring with 1. Suppose that, for each $x, y \in R$, $\widetilde{E}(x,y)$ contains $n_1, \cdots, n_s$ such that $(e(n_1), \cdots, e(n_s)) \equiv 0 \pmod 4$ and $[x,y]$ is $(\frac{n_1(n_1-1)}{2}, \cdots, \frac{n_s(n_s-1)}{2})$-torsion free. Then R is commutative.*

Theorem 2.2. *Let R be a ring with 1, and n a positive integer such that $n \equiv 2 \pmod 4$. Then the following conditions are equivalent:*

1) *R satisfies $Q(\frac{n(n-1)}{2})$ and the identity $(XY)^n - X^n Y^n = 0$.*

2) *R satisfies $Q(\frac{n(n+1)}{2})$ and the identity $(XY)^n - Y^n X^n = 0$.*

3) *R satisfies $Q(\frac{n(n+1)}{2})$ and the identity $(XY)^{n+1} - X^{n+1} Y^{n+1} = 0$.*

4) *For each $x, y \in R$, $\widetilde{E}(x,y)$ contains $n_1, \cdots, n_s$ and $m_1, \cdots, m_r$ such that $(e(n_1), \cdots, e(n_s)) = n$ and $[x,y]$ is $(\frac{m_1(m_1-1)}{2}, \cdots, \frac{m_r(m_r-1)}{2})$-torsion free.*

5) *R is a subdirect sum of rings each of which has one of the following types:*

i) *A commutative ring.*

ii) *$\mathrm{M}_\sigma(K)$, where K is a finite field of characteristic 2 with a non-trivial automorphism σ such that $(|K|-1)/(|K^\sigma|-1)$ divides $\frac{n}{2}$. Where, $K^\sigma = \{\alpha \in K | \sigma(\alpha) = \alpha\}$.*

Our theorems are derived by the use of next lemma together with Lemmas 1.2, [9, Lemma 2.1, Theorem 3.6] and [11, Theorem 2.12(II)].

Lemma 2.3. *Let R be a ring with 1. Suppose that, for each $x, y \in R$, $\widetilde{E}(x,y)$ contains $m_1, \cdots, m_r$ such that $[x,y]$ is $(\frac{m_1(m_1-1)}{2}, \cdots, \frac{m_r(m_r-1)}{2})$-torsion free. Then there hold the following:*

(1) $D \subseteq N$.

(2) $2[N,R] = 0$, *namely* $2N \subseteq C$.

(3) *R satisfies* (S).

(4) *R is completely reflexive, namely $xy = 0$ implies $yx = 0$ for any $x, y \in R$.*

(5) *Let $a \in N$, and $x \in R$. If $n \in \widetilde{E}(1+a, x)$ then $[a, x^{e(n)}] = 0$.*

3. Commutativity theorems for rings satisfying the polynomial identities of the form $(XY)^n - X^nY^n = 0$ on certain subsets. In this section, we shall generalize some known commutativity theorems for a ring R satisfying the polynomial identities of the form $(XY)^n - X^nY^n = 0$ by assuming that the identities hold merely for the elements of a certain subset of R rather than for all elements of R.

Let k be a positive integer, and A a subset of R. We consider the following conditions:

$P_0(k,A)$ $\quad (xy)^k = x^ky^k \quad$ for all $x, y \in A$.

$P_0^*(k,A)$ $\quad (xy)^k = y^kx^k \quad$ for all $x, y \in A$.

The statements in the following theorem are included in [12, Theorem 2] and [16, Theorem 4], respectively.

Theorem B. *Suppose that a ring R with 1 satisfies $P_0(k,R)$ ($k = n, n+2, n+4$).*

(1). *If n is even, then R is commutative.*

(2) *If $x^4 \in C$ for all x in R, then R is commutative.*

More recently, H. Komatsu and H. Tominaga proved [10, Theorems 2.4 and 2.7] which encompass several results of J. Grosen [1]. From [10, Theorems 2.4 and 2.7], we readily obtain

Theorem C. (1) *Suppose that a ring R with 1 satisfies $P_0(k, R \setminus Q)$ $(k = m, m+1, n, n+1)$. If R satisfies $Q((m,n))$, then R is commutative.*

(2) *Suppose that a ring R with 1 satisfies $P_0(n+1, R \setminus Q)$ (or $P_0^*(n, R \setminus Q)$). If R satisfies $Q(n(n+1))$, then R is commutative.*

Obviously, Theorem C (2) includes [7, Theorem 1 (b) and Theorem 2 (b)]. The first main theorem of this section is stated as follows:

Theorem 3.1. *Let R be a ring with 1. Let $n_1, \cdots, n_r$ be positive integers such that $(\frac{n_1(n_1-1)}{2}, \cdots, \frac{n_r(n_r-1)}{2}) = 1$. If R satisfies $P_0(n_i, R \backslash J)$ $(i = 1, \cdots, r)$, then R is commutative.*

Theorem 3.1 is deduced by the use of following two lemmas together with [9, Lemma 2.1] and [11, Proposition 2.9(2), Lemma 2.10(2)].

Lemma 3.2. *Let R be a ring with 1. Let k, m, n be non-negative integers, and $f: R \to R$ a function such that $f(x) = f(x+1)$ for all $x \in R$. If $f(x)(x+k)^m x^n = 0$ (or $x^n(x+k)^m f(x) = 0$) for all $x \in R$, then $(k+1)^{mn} f(x) = 0$. In particular, if $f(x)x^n = 0$ (or $x^n f(x) = 0$) for all $x \in R$, then $f(x) = 0$.*

Lemma 3.3. *Let R be a ring with 1. Suppose that R satisfies $P_0(n, R \backslash Q)$ $(n > 1)$. Then, for each $u \in U$, $u^{n(n-1)} \in C$, and $D \subseteq N$. In particular, if R satisfies $P_0(k, R \backslash Q)$ $(k = n\ (\geq 1), n+2, n+4)$, then $u^2 \in C$ for each $u \in U$.*

Corollary 3.4. *Let R be a ring with 1, and n a positive integer. If R satisfies $P_0(k, R \backslash J)$ $(k = n, n+2, n+4)$, then R is commutative.*

Theorem 3.1′. *Let R be a ring with 1. Let $n_1, \cdots, n_r$ be positive integers such that $(\frac{n_1(n_1-1)}{2}, \cdots, \frac{n_r(n_r-1)}{2}) = 1$. If R satisfies $P_0(n_i, R \backslash N)$ $(i = 1, \cdots, r)$, then R is commutative.*

Corollary 3.4′. *Let R be a ring with 1, and n a positive integer. If R satisfies $P_0(k, R\backslash N)$ $(k = n, n+2, n+4)$, then R is commutative.*

Corollary 3.5. *Let R be a ring with 1.*

(1) *If there exist positive integers $n_1, \cdots, n_r$ with $(\frac{n_1(n_1-1)}{2}, \cdots, \frac{n_r(n_r-1)}{2}) = 1$ such that R satisfies $P_0(n_i, R)$ $(i = 1, \cdots, r)$, then R is commutative.*

(2) *If there exist positive integers m, n with $(m, n) = 1$ or 2 such that R satisfies $P_0(k, R)$ $(k = m, m+1, n, n+1)$, then R is commutative.*

(3) *If there exists a positive integer n such that R satisfies $P_0(k, R)$ $(k = n, n+2, n+4)$, then R is commutative.*

Needless to say, Theorem B is included in Corollary 3.5 (3).

Now, by making use of Corollary 3.5, we obtain the following two theorems, which are related with Theorem C.

Theorem 3.6. *Let R be a ring with 1. Suppose that R satisfies $P_0(k, R\backslash Q)$ $(k = n, n+2, n+4)$.*

(1) *If n is even, then R is commutative.*

(2) *If $2[x, a] = 0$ implies $[x, a] = 0$ for each $a \in Q$ and $x \in R$, then R is commutative.*

Theorem 3.7. *Let R be a ring with 1. Suppose that R satisfies $P_0^*(k, R\backslash Q)$ $(k = n, n+2, n+4)$. Then R is commutative.*

4. On Conjecture 0.2.

In [5, Theorem 2], Y. Kobayashi proved the following theorem which gives an affirmative answer to Conjecture 0.2 in a somewhat weak form.

Theorem D. *Let R be a ring with 1. If for each $x, y \in R$, $\widetilde{E}(x, y)$ contains integers $n_1, \cdots, n_r$ such that $(\frac{n_1(n_1-1)}{2}, \cdots, \frac{n_r(n_r-1)}{2}) = 1$ and some of n_i's is even, then R is commutative.*

In this section, we give a generaliation of Theorem D. We consider the following conditions:

($*$) For each $x, y \in R$, there exist integers $k \geq 0$, $n > 1$ and words $w(x, y)$, $w'(x, y) \in \mathbf{W}$ such that

$$w(x, y)\{(xy)^n - x^n y^n\}w'(x, y) = 0 = y^k\{(yx)^n - y^n x^n\}x^k.$$

($\sharp$) For each $x, y \in R$, there exist non-negative integers $r_1 \leq r_2 \leq r_3 \leq r_4 \leq r_5 \leq r_6 \leq r_7 \leq r_8$ with $1 < r_8$, positive integers n_i $(1 \leq i \leq r_8)$, m_i $(r_2 + 1 \leq i \leq r_8)$, l_i $(r_4 + 1 \leq i \leq r_8)$, and words $w_i(x, y), w_i'(x, y) \in \mathbf{W}$ $(1 \leq i \leq r_8)$ such that

$$(\sharp)_0 \quad \left(\frac{n_1(n_1+1)}{2}, \cdots, \frac{n_{r_2}(n_{r_2}+1)}{2}, m_{r_2+1}n_{r_2+1}, \cdots, m_{r_4}n_{r_4}, l_{r_4+1}m_{r_4+1}n_{r_4+1}, \cdots, l_{r_8}m_{r_8}n_{r_8}\right) = 1,$$

$$(\sharp)_1 \quad w_i(x, y)\{(xy)^{n_i} - y^{n_i}x^{n_i}\}w_i'(x, y) = 0 \qquad (1 \leq i \leq r_1),$$

$$(\sharp)_2 \quad w_i(x, y)\{(yx)^{n_i} - x^{n_i}y^{n_i}\}w_i'(x, y) = 0 \qquad (r_1 + 1 \leq i \leq r_2),$$

$$(\sharp)_3 \quad w_i(x, y)\{(x^{m_i}y^{m_i})^{n_i} - (y^{m_i}x^{m_i})^{n_i}\}w_i'(x, y) = 0 \qquad (r_2 + 1 \leq i \leq r_3),$$

$$(\sharp)_4 \quad w_i(x, y)[x^{m_i}, y^{n_i}]w_i'(x, y) = 0 \qquad (r_3 + 1 \leq i \leq r_4),$$

$$(\sharp)_5 \quad w_i(x, y)[x^{l_i}, (x^{m_i}y^{m_i})^{n_i}]w_i'(x, y) = 0 \qquad (r_4 + 1 \leq i \leq r_5),$$

$$(\sharp)_6 \quad w_i(x, y)[x^{l_i}, (y^{m_i}x^{m_i})^{n_i}]w_i'(x, y) = 0 \qquad (r_5 + 1 \leq i \leq r_6),$$

$$(\sharp)_7 \quad w_i(x, y)[y^{l_i}, (x^{m_i}y^{m_i})^{n_i}]w_i'(x, y) = 0 \qquad (r_6 + 1 \leq i \leq r_7),$$

$$(\sharp)_8 \quad w_i(x, y)[y^{l_i}, (y^{m_i}x^{m_i})^{n_i}]w_i'(x, y) = 0 \qquad (r_7 + 1 \leq i \leq r_8).$$

Now, the main theorem of this section is stated as follows:

Theorem 4.1. *Let R be a ring with 1. If R satisfies ($*$) and ($\sharp$), then R is commutative.*

According to Theorem 1.1, in order to complete the proof of Theorem 4.1, it suffices to prove the following two lemmas.

Lemma 4.2. *If R is of type* a)^1, c) *or* d)^1, *then R dose not satisfy ($*$).*

Lemma 4.3. *If R is of type* b) *or* e)[1], *then R dose not satisfy* ($\sharp$).

Noting that $(xy)^n - x^n y^n = x\{(yx)^{n-1} - x^{n-1}y^{n-1}\}y$ for any positive integer n, we obtain the next as an important special case of Theorem 4.1.

Corollary 4.4. *Let R be a ring with 1. Suppose that for each $x, y \in R$, there exist positive integers $r \leq s$ and $n_i > 1$ $(i = 1, \cdots, s)$ such that*

1) $(\frac{n_1(n_1-1)}{2}, \cdots, \frac{n_s(n_s-1)}{2}) = 1$,

2) $(xy)^{n_i} = x^{n_i}y^{n_i}$ $(i = 1, \cdots, r)$,

3) $(yx)^{n_i} = y^{n_i}x^{n_i}$ $(i = r, \cdots, s)$.

Then R is commutative.

Needless to say, Corollary 3.5 is a direct consequence of Corollary 4.4.

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REMARKS ON MODULAR GROUP ALGEBRAS OF FINITE GROUPS IN CHARACTERISTIC 3 *†

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1. Introduction. Here we discuss on modules over the group algebra FG of a finite group G over a field F of prime characteristic p. We mean finitely generated right FG-modules just by FG-modules. In modular representation theory of finite groups, it is of course important to study structure of indecomposable FG-modules, in particular, *projective* indecomposable modules (p.i.m.'s) over FG since any indecomposable FG-module is a factor moudule of a direct sum of finite number of p.i.m.'s over FG. So, let's begin to consider the most important p.i.m., first of all. That is, the projective cover $P = P(F_G)$ of the trivial FG-module F_G, where $F_G = F(\sum_{g\in G} g) \subseteq FG$ and this becomes a one-dimensional FG-module on which all elements in G act trivially. In this short note we discuss mainly on this particular p.i.m. P. We need a little bit more notation, say, J and $j(M)$ for an FG-module M. Namely, $J = J(FG)$ is the Jacobson radical of FG, and the Loewy length $j(M)$ of M is the least positive integer j such that $MJ^j = 0$.

2. The Loewy length $j(P)$ for $P = P(F_G)$. In this section we discuss on the Loewy length $j(P)$ of our special p.i.m. $P = P(F_G)$.

(2.1) Fact. *If $j(P) \leqq 2$, then the structure of G is completely determined by theorems of Maschke and D.A.R. Wallace.*

The next step is of course the case $j(P) = 3$. Namely,

*The final and detailed version of this note will presumably be submitted for publication elsewhere.

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(2.2) Fact. *There are several nice results on the structure of G under the condition* $j(P) = 3$. *For details, see original beautiful papers of Erdmann [1] and Okuyama [4]. In particular, Okuyama proved that the condition* $j(P) = 3$ *implies that Sylow 2-subgroups of G are dihedral (can be elementary abelian of order 4) in the case* $p = 2$ *(see Okuyama [4, Theorem 2]).*

Because of (2.2) it seems natural to give the following question.

(2.3) Question. *For p odd, what can we say on the structure of G under the condition* $j(P) = 3$?

Now, let's look at a higher step, that is to say, the case $j(P) = 4$. In author's point of view the condition $j(P) = 4$ seems stronger (or more mysterious) than the condition $j(P) = 3$. The reason for it comes from the next elementary observations.

(2.4) Observation (Dade-Janusz-Kupisch, see [1, VII]). *If G has cyclic Sylow p-subgroups and if* $j(P) = 4$, *then* $p = 2$ *(and G is a 2-nilpotent group with cyclic Sylow 2-subgroups of order 4).*

(2.5) Observation (Ninomiya [3] and Willems [5]). *If G is p-solvable and if* $j(P) = 4$, *then* $p = 2$ *(hence the structure of G is* almost *determined).*

The author is afraid of taking readers to a wrong way. It seems, however, natural (or reasonable) to give the next question (he has no confidence to call it just a *conjecture*).

(2.6) Question (Koshitani). *Is it true that the condition* $j(P) = 4$ *would imply* $p = 2$?

As a matter of fact we have another partial afirmative answer to the question (2.6). Namely,

(2.7) Observation (Koshitani) . *If* $p = 3$ *and if* G *has elementary abelian Sylow 3-subgroups of order 9, then* $j(P) \neq 4$ *(actually, it holds even that* $j(P) \geqq 5$ *and this is the best possible).*

The author should confess that the above observation (2.7) deeply is due to, so-called, the classification of finite simple groups which almost nobody has been able to understand its proof perfectly yet, though.

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Artinian rings related to almost relative projectivity

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This is a summary of the author's paper [4] and [7]. In this paper we always assume that R is a (two-sided) artinian ring and every module is a finitely generated and unitary right R-module.

1. Preliminary We shall recall the definition of almost relative projectivity.

Let M and N be R-modules. Consider a diagram

$$\begin{array}{ccccc} & & M & & \\ & & \downarrow h & & \\ N & \xrightarrow{\nu} & N/K & \to & 0, \end{array}$$

where h is a homomorphism, ν is the natural epimorphism and K is a submodule of N.

We consider the two conditions: 1) there exists $\tilde{h}$: M → N such that $\nu\tilde{h} = h$, 2) there exist a non-zero direct summand N' of N and $\breve{h}$:N' → M such that $h\breve{h} = \nu|N'$.

If 1) holds for any h and any K in the above diagram, then we say that M is N-projective [1]. If 1) or 2) holds, we say that M is almost N-projective [3].

The final detailed version of the last part of this paper has been submitted to Osaka Journal of Mathematics for publication.

If a module P is always Q-projective for any R-module Q, then P is projective. Similarly we call a module P' is almost projective if P' is always almost Q'-projective for any Q' [6]. Further if the Jacobson radical J of R is projective, we call R is hereditary. Similarly we call R is almost right hereditary if J is almost projective as a right R-module [5].

2. Hereditary rings with $J^2 = 0$

We have studied the following several conditions in [4]. Let M and N be R-mdoules.

(1) If M is N-projective then M' is N-projective for any submodule M' of M.

In (1) we put

(1') $M = eR/A$ and $N = fR/B$, where e, f are primitive idempotents in R and $A \subset eR$ and $B \subset fR$.

In (1') we assume $A = 0$ and $B = 0$, then

(2) if every submodule C in eR is fR-projective.

We can easily see that (2) is equivalent to R being hereditary. Relating this fact we have

Theorem 1 ([2] and [4]). *Let R be artinian. Then the following are equivalent:*

1) R is a hereditary ring with $J^2 = 0$.

2) (1) holds for any M and N.

3) (1') holds for any eR/A and fR/B.

(4) Any submodule of quasi-projective module is again quasi-projective.

In (1') we assume e = f, i.e.,

(3) If eR/A is eR/B-projective, then C/A is always eR/B-projective.

If (3) holds, then eJe = 0 for any primitive idempotent e [4]. However the converse is not true. Only in hereditary algebras over an algebraically closed field we have given the structure of R with (3) [4]. We do not know a characterization of R with (3).

Next we study the conditions which are given by replacing projectivity by almost projectivity.

(4) If M is almost N-projective, then M' is always almost N-projective.

We note that if (1') holds, i.e., M and N are local in (1), then (1) holds for any modules M' and N'. However this fact is not true in the case (4). We obtain the following results for (4).

Theorem 2 ([4]). *(4) holds when M and N are local if and only if* $J^2 = 0$.

Theorem 3 ([7]). *The following are equivalent:*

1) (4) holds when M is local.

2) (4) holds when M is a direct sum of local modules.

3) R is a right almost hereditary ring with $J^2 = 0$.

We do not know whether 3) in the above implies that (4) holds for any M and N or not.

3. Almost hereditary rings with $J^3 = 0$

We shall study a little stronger condition than (4).

(5) If M is N-projective, then M' is almost N-projective for any submodule M' of M.

We can easily see that if (5) holds for local modules M and N, then every submodule of indecomposable quasi-projective module is again quasi-projective, however the converse is not true, cf. Theorem 1.

We can show that if (5) holds for local modules M and N, then $J^3 = 0$ and if $eJ^2 \neq 0$, then

(#) $eJ \approx f_1R \oplus f_2R \oplus \cdots \oplus f_sR \oplus S_1 \oplus \cdots \oplus S_t$,

where the f_iR are uniserial and projective and the S_j are simple.

We put $eJ = P_1 \oplus P_2 \oplus \cdots \oplus P_k \oplus S_1 \oplus \cdots$, where $P_1 \approx f_{11}R \oplus \cdots$, $P_2 \approx f_{12}R \oplus \cdots$, $P_k \approx f_{k1}R \oplus \cdots$, and $f_{ij}R \not\approx f_{hm}R$ for $i \neq h$. Then we obtain

Theorem 4 ([7]). *Let R be artinian. Then (5) holds for local modules M and N if and only if i) $J^3 = 0$ and eJ has the* *decomposition* (#), *ii) fR/fJ is never isomorphic to any simple component of Soc(R), iii) if* $eR \supset fR$ *and* $e'R \supset f'R$ *in* (#) *and* $eR \not\approx e'R$, *then* $fR \not\approx f'R$ *and iv) for simple submodule S in* $P_1 \oplus P_2 \oplus \cdots \oplus P_k$, $eReS = \Sigma \oplus Soc(P_{i_j})$, *where* $\{i_j\} \subset \{1,2,\cdots,k\}$.

Theorem 5 ([7]). *Let R be as above. Then* (5) *holds for local modules M and direct sums of local modules N if and only if i)* ~ *iv) hold and v) the projective cover of Soc(R) is a direct sum of uniserial modules.*

Finally we obtain

Theorem 6 ([7]). (5) *holds for local modules M and any R-modules N if and only if i)* ~ *v) hold and vi) for any simple submodule S in* $fR \subset eR$ *in* (#) *and S' in gR, if* $\theta: S \to S'$, *then* θ *is extensible to an element in* $Hom_R(fR,gR)$ *or* θ^{-1} *is extensible to an element in* $Hom_R(gR,fR)$ *and R is right almost hereditary, where g is any primitive idempotent.*

We note that the last condition vi) is very closed to a fact that R is left QF-2. Hence we obtain

Corollary. *Assume that R is left QF-2. Then* (5) *holds for local modules M and N if and only if* $J^3 = 0$ *and eJ has the decomposition* (#) *if* $eJ^2 \neq 0$. (5) *holds for local modules M and any N if and only if eJ has the above property and R is right* *almost hereditary.*

We believe that if the condition in Theorem 6 holds then (5) holds for any modules M and N.

Let $L \supsetneq K$ be fields and put

$$R = \begin{pmatrix} K & K & K & K \\ 0 & K & K & 0 \\ 0 & 0 & K & P \\ 0 & 0 & 0 & K \end{pmatrix},$$

where $e_{13}e_{34} = e_{23}e_{34} = o$ and $P = L$, K or 0.
If $P = L$, Then R satisfies (5) for local modules M and N, but not when N is a direct sum of local modules. If $P = K$, then (5) holds for local modules M and direct sums of local modules N, but not for for any R-module N. If $P = 0$, (5) holds for local modules M and any modules N.

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ON A GENERALIZATION OF FREE CALCULUS

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In [1], R.H.Fox has introduced the derivation over free groups and he caluculated the derivative over free groups. In this paper, we generalize the notion of derivation and expand the free caluculus by using our derivation . Furthermore we examine a group invariant by using free calculus. The detailed version of this paper will appear in [2].

1 Generalization of derivation

Let Z be a ring of integer and G be a finitely presented group. We denote $End(G)$ to the set of endomorphism of G. We can assume $End(G)$ is contained in $End_Z(G)$.

Definition 1.1
For any $\alpha, \beta \in End(G)$, we define (α, β) - derivation D as follows.
(1) D is a Z -linear endomorphism of $Z(G)$.
(2) $D(gh) = \alpha(g)D(h) + D(g)\beta(h)$ for any $g, h \in G$.

Remark 1.2
(1) If α = identity and $\beta \equiv 1$, then we have the Fox's derivation.
(2) For any $\alpha, \beta \in End(G)$, there exists (α, β) - derivation D. For example, we can put $D(x) = \alpha(x) - \beta(x)$, for any $x \in Z(G)$.

For free groups of finite rank, we have the following proposition.

Proposition 1.3
Let F be a free group with free basis $x_1, x_2, \cdots, x_n$. Then for any $\alpha, \beta \in End(G)$, and for any $d_1, d_2, \cdots, d_n \in F$, there exists unique (α, β) - derivation D such that $D(x_i) = d_i$.

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We will use the notation $\frac{\partial^{(\alpha,\beta)}}{\partial x_j}$ to denote the derivation which satisfy $\frac{\partial^{(\alpha,\beta)}}{\partial x_j}(x_i) = \delta_{ij}$ (Kronecker' delta). Let G be a finitely presented group and H be a group such that there exist onto homo η from G to H.

Definition 1.4
Let (G, H, η) be as above. If for any $\alpha \in End(H)$, there exist $\tilde{\alpha} \in End(G)$ such that $\alpha \circ \eta = \eta \circ \tilde{\alpha}$, we say $\tilde{\alpha}$ is a pull back of α.

In general, $\tilde{\alpha}$ may not exist. However if G is a free group, there exists a pull back.

Proposition 1.5
Let F be a free group of finite rank and H be a group with onto homo φ from F to H. Then for any $\alpha \in End(H)$, there exists a pull back $\tilde{\alpha} \in End(F)$.

Since G is finitely presented, we can write $G = \langle x_1, x_2, \cdots, x_n \mid r_1, r_2, \cdots, r_m \rangle$, where $x = \{x_1, x_2, \cdots, x_n\}$ is a set of generator of G and $r = \{r_1, r_2, \cdots, r_m\}$ is a set of relation of G. Let F be a free group with free basis x and π be the represention map from F to G. Then $\eta \circ \pi$ is an onto homo from F to H. We fix $\alpha, \beta \in End(H)$. By proposition 1.5, we have a pull back $\tilde{\alpha}, \tilde{\beta} \in End(F)$, and by proposition 1.3, we get $(\tilde{\alpha}, \tilde{\beta})$ - derivation $\frac{\partial^{(\tilde{\alpha},\tilde{\beta})}}{\partial x_j}$) on $Z[F]$.

Definition 1.6
$J_{\alpha,\beta}(G, H, \eta, ; x \mid r) \stackrel{\text{def}}{=} \left(\bar{\eta} \circ \bar{\pi}\left(\frac{\partial^{(\tilde{\alpha},\tilde{\beta})}}{\partial x_j}\right)\right)_{i=1,\cdots,m; j=1,\cdots,n} \in M_{n,m}(Z[H])$, where $\bar{\eta}, \bar{\pi}$ is a ring homo extended by η, π respectively.

Remark 1.7
$J_{\alpha,\beta}(\cdots)$ does not depend the choice of $\tilde{\alpha}, \tilde{\beta}$.

Let $G, H, \eta, and \alpha, \beta$ as above. If G has two differnt representation $G = \langle x_1, \cdots, x_n \mid r_1, \cdots, r_m \rangle = \langle y_1, \cdots, y_p \mid s_1, \cdots, s_q \rangle$, then by an elementary matrix transformation (Tietze' transformation), $J_{\alpha,\beta}(G, H, \eta; x \mid r)$ is equivalent to $J_{\alpha,\beta}(G, H, \eta; y \mid s)$. We denote the equivalence class of $J_{\alpha,\beta}(G, H, \eta; x \mid r)$ to $A_{\alpha,\beta}(G, H, \eta)$.

Definition 1.8
$A_{\alpha,\beta}(G, H) \stackrel{\text{def}}{=} \{A_{\alpha,\beta}(G, H, \eta) \mid \eta \in Hom(G, H)\}$

Theorem 1.9
If G_1 is isomorphic to G_2, then $A_{\alpha,\beta}(G_1, H) = A_{\alpha,\beta}(G_2, H)$.

2 Group Invariant

In this section, by using free calculus, we examine a group invariant. Let $M_{s,t}(Z)$ be the set of $s \times t$ matrices over Z. Then we define $M(Z) = \cup_{s,t} M_{s,t}(Z)$. Note that $M(Z)$ have a ring structure. Let H be a group and P_N be the permutation group. Note that P_N is a subset of $M_N(Z)$ and as a group, P_N is isomorphic to N th symmetric group. We fix a homomorphism φ from H to P_N. Note that if $R = M_N(Z)$, then $M_{s,t}(R) \cong M_{sN,tN}(Z)$ as a set. Thus $Z[H] \cong Z[P_N] \subseteq Z[M_N(Z)]$. Now let G be a finitely presented group and H be a group, and assume that there exists an onto homo η from G to H. In section 1, we have defined $J_{\alpha,\beta}(G, H, \eta; x \mid r)$. Now $\tilde{\varphi}(J_{\alpha,\beta}(G, H, \eta; x \mid r)$ is in $M(Z)$, where $\tilde{\varphi}$ ia the ring homo extended by φ. If G has another representation, then clearly, $\tilde{\varphi}(J_{\alpha,\beta}(G, H, \eta; \cdots)$ are equivalent. We denote $A_{\alpha,\beta}(G, H, \eta)$ to the equivalence class of the matrix. Let A be a $s \times t(t > s)$ matrix over Z. Then we define $E_k(A) =$ the ideal of Z generated by $(t-k)$ minors of A, where $k = 1, 2 \cdots$. Note that $E_k(A) = 0$ if $t - k > s$, and $E_k(A) = Z$ if $t - k \leq 0$. Since Z is PID, $E_k(A) = (e_k)$ for some $e_k \in Z$.

Proposition 2.1

Let A be a $s \times t$ matrix over Z. Then
(1) $e_k \geq 0$.
(2) $e_{k+1} \mid e_k$.
(3) $e_k = 1$ for sufficiently large k.

We can apply this to the case $A = A_{\alpha,\beta}(G, H, \eta)$.
Let $e_{\alpha,\beta}\varphi(G, H, \eta) = e(A_{\alpha,\beta}(G, H, \eta))$.

Definition 2.2

$\Delta_{\alpha,\beta}\varphi(G, H) \stackrel{\text{def}}{=} \{e_{\alpha,\beta}\varphi(G, H) \mid \eta \in Hom(G, H)\}$.

Now we have the folowing.

Theorem 2.3

If $G_1 \cong G_2$, then $\Delta_{\alpha,\beta}\varphi(G_1, H) = \Delta_{\alpha,\beta}\varphi(G_2, H)$.

We shall give an examlpe to illustrate our invariant.

Example

Let q a be positive integer and p be a prime number. Consider that $G = G_{p,q} = \langle x, y \mid x^p = y^q \rangle$. We set $H = Z/pZ = < t >$, the cyclic group of order p. Let $T = (\delta_{i+1,j})_{i=1,\cdots,p, j=1,\cdots,p}$ be an element of p th permutation group P_p, where we assume that $p + 1 = 1$. We define a group homo φ from Z/pZ to P_p as $\varphi(t) = T$. Clearly $Hom(G_{p,q}, Z/pZ) = \eta_k \mid k = 0, \cdots, p-1$, where $\eta_k(x) = t^k, \eta_k(y) = 1$. Assume that $\alpha =$ identity and $\beta \equiv 1$. Now $\left(\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}\right) = \left(1 + x + \cdots + x^{p-1}, -\left(1 + y + \cdots + y^{q-1}\right)\right)$. So $J_{\alpha,\beta}(G_{p,q}, Z/pZ, \eta_k; x \mid r) = (1 + t + \cdots + t^{p-1}, -q)$. Hence $\varphi(J_{\alpha,\beta}(G_{p,q}, Z/pZ, \eta_k; x \mid r) = (E_p + T + \cdots +$

$T^{p-1}, -qE_p)$. Now $e_{\alpha,\beta}\varphi(G,H) = (\underbrace{0+\cdots+0}_{p}, q^{p-1}, q^{p-2}, \cdots, q, 1, 1, \cdots) \overset{\text{put}}{=}$ $e[p,q]$, so we have $\Delta_{\alpha,\beta}\varphi(G,H) = e[p,q]$.

Inparticular, if q, q' are positive integer, and p is a prime number such that $(p,q) = (p,q') = 1$, then $q \neq q'$ implies $G_{p,q} \not\cong G_{p,q'}$.

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Resolutions of determinantal ideals— A Counterexample on Symmetric Matrices*

MITSUYASU HASHIMOTO

Abstract

This note is a survey on the problem of minimal free resolutions of determinantal ideals of generic symmetric matrices, with emphasis on new counterexamples obtained by J. Andersen and the author.

1 The problem

Let A be a noetherian ring, I an ideal of A, and M a finitely generated A-module. We define the *I-depth* of M to be $\min\{i \mid \mathrm{Ext}^i_A(A/I, M) \neq 0\}$ and denote it by $\mathrm{depth}(I, M)$.

If A is a local ring with the maximal ideal $\mathfrak{m}$, then $\mathrm{depth}(\mathfrak{m}, M)$ is sometimes denoted by $\mathrm{depth}\, M$. In this case, we have $\mathrm{depth}\, M \leq \dim M$ for $M \neq 0$, where $\dim M$ is the Kurll dimension of $A/\mathrm{ann}_A M$. We say that M is Cohen-Macaulay when the equality holds, or $M = 0$. We say that the local ring A is Cohen-Macaulay when so is A as an A-module. A noetherian ring (which may not be local) A is said to be Cohen-Macaulay when its localization at any maximal ideal is Cohen-Macaulay local.

Cohen-Macaulay property is one of the most important notion in the modern commutative ring theory.

Lemma 1.1 *Let A be a d-dimensional graded K-algebra (K a field) generated by finite degree one elements. Then, the following hold.*

1 *A is Cohen-Macaulay if and only if* $\mathrm{depth}(A_+, A) = d$, *where A_+ is the ideal of A consisting of all degree positive elements.*

2 *(K is assumed to be infinite) Let $\theta_1, \dots, \theta_d$ be degree one elements such that A is a finite module over $K[\underline{\theta}] = K[\theta_1, \dots, \theta_d] \subset A$ (such $\theta_1, \dots, \theta_d$ do exist). Then, A is Cohen-Macaulay if and only if A is a* free *$K[\underline{\theta}]$-module (hence, this condition does not depend on the choice of $\theta_1, \dots, \theta_d$).*

3 *Let $a_1, \dots, a_r$ be the degree one generator of A as a K-algebra so that the map*

$$S = K[x_1, \dots, x_r] \to K[a_1, \dots, a_r] = A \quad (x_i \mapsto a_i)$$

is a surjective map of graded K-algebras. Then, A is Cohen-Macaulay if and only if $\mathrm{pd}_S A = r - d$, *where* pd *denotes the projective dimension.*

*A detailed version of this paper will be submitted for publication elsewhere.

Gorenstein property is also important homological property. A noetherian local ring A is said to be Gorenstein when its self-injective dimension is finite. A noetherian ring is said to be Gorenstein when its localization at any maximal ideal is Gorenstein. Any Gorenstein ring is Cohen-Macaulay, but the converse is not true in general.

Lemma 1.2 *Let A be a d-dimensional Cohen-Macaulay graded K-algebra (K a field) generated by finite degree one elements. Then, the following hold.*

1 *A is Gorenstein if and only if* $\operatorname{Ext}_A^d(A/A_+, A) \cong K$.

2 *Let $F_A(t) = \sum_{i\geq 0}(\dim_K A_i)t^i$. Then, $(1-t)^d F_A(t)$ is a polynomial in t, say, $h_0 + h_1 t + \cdots + h_s t^s$ ($h_s \neq 0$). If A is Gorenstein, then $h_s = 1$. The converse is true when A is an integral domain.*

3 *Let $a_1, \ldots, a_r$ be degree-one generators of A, and consider A as a module over $S = K[x_1, \ldots, x_r]$. Then, the following are equivalent.*

- **a** *A is Gorenstein.*
- **b** $\operatorname{Ext}_S^{r-d}(A, S)$ *is cyclic as an S-module.*
- **b'** $\operatorname{Ext}_S^{r-d}(A, S) \cong A$ *as an S-module.*

For a graded K-algebra A, a graded A-module M is said to be free when M is a direct sum of modules of the form $A(i)$, where $A(i)$ is simply A as an A-module, and the grading is given by $A(i)_j = A_{i+j}$. Clearly, a free module is projective in the category of graded A-modules. Assume that A is generated by finite elements of positive degree. For a finitely generated graded A-module M and its subset $S = \{m_1, \ldots, m_r\}$, S generates M if and only if the image of S generates M/A_+M (an analogue of Nakayama's lemma). So, S is a set of minimal generators if and only if its image in M/A_+M is a K-basis.

Let R be a commutative ring with unity. For a matrix $(a_{i,j}) \in \operatorname{Mat}_{m,n}(R)$ with coefficients in R and a positive integer t, we define the *determinantal ideal* $I_t((a_{i,j}))$ of the matrix $(a_{i,j})$ to be the ideal of R generated by all t-minors of $(a_{i,j})$.

There has been much interest in determinantal ideals and the varieties defined by them in commutative ring theory and algebraic geometry.

Especially, the determinantal ideal of the *generic matrix* $(y_{i,j})$ ($y_{i,j}$ are independent variables) and the determinantal ideal of the *generic symmetric matrix* (explained below), which is the central object in this note, have been studied by many authors.

Consider the polynomial ring $S = R[x_{i,j}]_{1\leq i\leq j\leq n}$ over R, where n is a positive integer. With letting $x_{j,i} = x_{i,j}$, we can form a 'generic symmetric matrix' $(x_{i,j})$. We set $I_t = I_t(x_{i,j})(\subset S)$. With letting each $x_{i,j}$ of degree one, S is a graded R-algebra, and I_t is a homogeneous ideal generated by its degree t-component.

Note that S is the coordinate ring of the affine space $X = \operatorname{Sym}_n(R)$, the space of all $n \times n$ symmetric matrices with coefficients in R. The quotient S/I_t corresponds to the set Y_t of all symmetric matrices whose rank is smaller than t (because the rank of a matrix is smaller than t if and only if all of its t-minors vanish).

When R is a field, it is known that S/I_t is a Cohen-Macaulay normal domain of dimension $n(n+1)/2-(n-t+1)(n-t+2)/2$ [12]. It is Gorenstein if and only if $n-t$ is even (see e.g., [10, Remark 6.3]).

Our main problem is the following.

Problem 1.3 **1** Let R be an arbitrary ring. Construct a graded minimal free resolution of S/I_t as an S-module.

2 Let $R=K$ be a field. Calculate the i^{th} Betti number $\beta_i^R = \dim_K \operatorname{Tor}_i^S(S/I_t, S/S_+)$ for each i, where $S_+ = I_1 = (x_{i,j})\cdot S$.

Here, a graded S-complex (i.e., a chain complex in the category of graded S-modules)

$$\mathsf{F} : \cdots \to F_i \xrightarrow{\partial_i} F_{i-1} \to \cdots \to F_0 \to 0$$

is said to be a free resolution of a graded S-module M when each F_i is free, $H_i(\mathsf{F})=0$ $(i>0)$ and $H_0(\mathsf{F})=M$. It is called minimal when the boundary maps of $S/S_+\otimes \mathsf{F}$ are all zero. A graded minimal free resolution is unique up to isomorphism. It exists when the base ring R is a field.

It is known that S/I_t is free as an R-module (e.g., [10]) so that $\operatorname{Tor}_i^R(M, S/I_t)=0$ for $i>0$ and any R-module M. Hence, if F is a projective resolution of S/I_t over the base ring R, and if R' is an R-algebra, then $R'\otimes_R \mathsf{F}$ is a projective resolution of $R'\otimes_R S/I_t$. If F is graded minimal free, then so is $R'\otimes_R S/I_t$. So, if **1** of the problem is solved for the ring of integers $\mathbf{Z}$, then **1** is solved for any R, because we can get the resolution by base change $R\otimes_{\mathbf{Z}}?$.

Let F be a graded minimal free resolution of S/I_t. Then, $H_i(S/S_+\otimes_S \mathsf{F}) = S/S_+\otimes F_i$ is an R-free module, and we have

$$\infty > \operatorname{rank}_R \operatorname{Tor}_i^S(S/S_+, S/I_t) = \operatorname{rank}_S F_i.$$

Note that the right hand side is invariant under the base change. In particular, for any R-algebra K which is a field, we have $\beta_i^K = \operatorname{rank}_S F_i$. Thus, the problem **2** is easier than **1** (for example, if **1** is solved for any field, then **2** is completely solved).

Assume that R is a field. Since S/I_t is Cohen-Macaulay of dimension $\dim S-(n-t+2)(n-t+1)/2$, we have $\operatorname{pd}_S S/I_t = (n-t+2)(n-t+1)/2$. We set $h=(n-t+2)(n-t+1)/2$. Then, we have $\beta_h^p \neq 0$ and $\beta_i^p = 0$ for $i>h$. The ring S/I_t is Gorenstein if and only if $\beta_h = 1$ by Lemma 1.2. Let F be a graded minimal free resolution of S/I_t. Then, we have

$$H_i(\operatorname{Hom}_S(\mathsf{F}, S)) = \operatorname{Ext}_S^{-i}(S/I_t, S) = 0$$

unless $i=-h$ by Lemma 1.1, since S/I_t is Cohen-Macaulay of codimension h. So the complex $\operatorname{Hom}_S(\mathsf{F}, S)[-h]$ ([ ] denotes the shift of the degree as a chain complex) is a minimal free resolution of the S-module $\operatorname{Ext}_S^h(S/I_t, S)$. When S/I_t is Gorenstein, we have $\operatorname{Ext}_S^h(S/I_t, S)\cong S/I_t$. This shows that $\operatorname{Hom}_S(\mathsf{F}, S)[-h]$ is a graded minimal free resolution of S/I_t (the grading as a graded S-module may be different, so we should say

$\mathrm{Hom}_S(\mathsf{F}, S)(a)[h]$ is a graded minimal free resolution of S/I_t for some $a \in \mathbf{Z}$). This shows that

$$F_i \cong \mathrm{Hom}_S(\mathsf{F}, S)[h]_i(a) = \mathrm{Hom}_S(F_{h-i}, S)(a),$$

and we have $\beta_i = \beta_{h-i}$.

Why is the problem a problem? First, constructing a graded minimal free resolution of S/I as an S-module (for a homogeneous polynomial ring $S = K[x_1, \ldots, x_r]$ over a field K and its homogeneous ideal I) has been considered as an ultimate aim of homological study of the algebra S/I—knowing a minimal free resolution yields ample information on the ring in question. For example, S/I is Cohen-Macaulay if and only if $\beta_i(S/I) = 0$ for $i > \dim S - \dim S/I$. It is Gorenstein if and only if it is Cohen-Macaulay and $\beta_{\dim S - \dim S/I}(S/I) = 1$. So the Betti numbers β_i of an algebra contain a lot of information of the algebra (however, nowadays the progress of the theory of commutative algebra provides us a lot of tools for studying important homological properties (such as Cohen-Macaulay property) of commutative algebras without constructing a resolution).

Secondly, the theory of the resolution of determinantal ideals is an interaction between the theory of commutative algebra and the representation theory of algebraic groups, and is interesting itself.

The number β_i^K depends only on the characteristic p of K, so we also write β_i^p.

When there exists a graded minimal free resolution F of S/I_t over the ring of integers so that the resolution is obtained by base change for an arbitrary ring? Clearly, if such a resolution exists over $\mathbf{Z}$, then β_i^p is independent of p. The converse is true.

Lemma 1.4 ([16, Chapter 4, Proposition 2], [7, Proposition II.3.4]) *Assume that R is a noetherian reduced ring such that any finitely generated projective R module is free. Let $A = R[x_1, \ldots, x_n]$ be a homogeneous polynomial ring over R, and M a finitely generated graded A-module which is flat as an R-module. Then, the following are equivalent for any $i \geq 0$.*

1 *There exists a graded minimal free complex*

$$0 \to \mathsf{F}_{i+1} \xrightarrow{\partial_{i+1}} \mathsf{F}_i \xrightarrow{\partial_i} \cdots \to \mathsf{F}_1 \xrightarrow{\partial_1} \mathsf{F}_0 \to 0$$

such that $H_0\mathsf{F} = M$ and $H_k\mathsf{F} = 0$ for $1 \leq k \leq i$.

2 *For any $0 \leq k \leq i$ and $j \in \mathbf{N}_0$, the numbers*

$$\beta_{kj}^K(M) \stackrel{def}{=} \dim_{R/\mathfrak{M}}[\mathrm{Tor}_k^{R/\mathfrak{M} \otimes_R A}(R/\mathfrak{M} \otimes_R A/A_+, R/\mathfrak{M} \otimes_R M)]_j$$

is independent of the maximal ideal $\mathfrak{M}$ of R, where $[\ ]_j$ denotes the degree j component of a graded A-module.

3 *For any $0 \leq k \leq i$, the Betti numbers $\beta_k^K(M) = \beta_k(R/\mathfrak{M} \otimes_R M)$ (over the field $K = R/\mathfrak{M}$) is independent of the maximal ideal $\mathfrak{M}$ of R.*

4 *For any $0 \leq k \leq i$, $\mathrm{Tor}_k^A(A/A_+, M)$ is a free R-module.*

Thus, there exists a graded minimal free resolution of S/I_t over $\mathbf{Z}$ if and only if $\beta_i^p(S/I_t)$ is independent of p for any i.

Problem 1.5 Are the Betti numbers $\beta_i^p(S/I_t)$ independent of the characteristic?

2 A history

Clearly, the Koszul complex is a graded minimal free resolution of $(x_{i,j}) = I_1$. When $n = t$, we have $I_t = (\det(x_{i,j}))$ is principal, and the sequence

$$0 \to S(-n) \xrightarrow{\det(x_{i,j})} S \to S/I_n \to 0$$

is a graded minimal free resolution.

When $t = n - 1$, a graded minimal free resolution of S/I_t was constructed explicitly [5], [8]. The free resolution is length three (this is because $\operatorname{pd}_S S/I_t = (n-t+2)(n-t+1)/2 = 3$).

After a progress in the characteristic-free representation theory [2] and its application to the theory of resolution of determinantal ideals [1], [11], Kurano proved the following [10, Theorem 5.1].

Theorem 2.1 *The second Betti number $\beta_2^p(S/I_t)$ is independent of the characteristic p of the base field.*

Clearly, we have $\beta_0^p = 1$. It is not so difficult to show that the set

$$\{\det(x_{\alpha(i),\beta(j)})_{1\le i,j\le n} \mid 1 \le \alpha(1) < \cdots < \alpha(n) \le n,\ 1 \le \beta(1) < \cdots < \beta(n),\ \beta(i) \ge \alpha(i)\},$$

whose cardinality is $\binom{n}{t}^2 - \binom{n}{t+1}\binom{n}{t-1}$, minimally generates I_t. It follows that $\beta_1^p = \binom{n}{t}^2 - \binom{n}{t+1}\binom{n}{t-1}$ is independent of p.

Consider the case $n - t = 2$. In this case, we have $h = \operatorname{pd}_S S/I_t = 6$. As $n - t$ is even, $\beta_{6-i}^p = \beta_i^p$ for any i. Since β_0^p, β_1^p and β_2^p are independent of p, so are β_6^p, β_5^p and β_4^p. Let K be a field of characteristic p, and F a graded minimal free resolution of S/I_t over the base field K. When we denote the quotient field of S (it is a rational function field) by L, the sequence $L \otimes_S \mathsf{F}$ is a resolution of $L \otimes_S S/I_t = 0$. That is, $L \otimes_S \mathsf{F}$ is an exact sequence of finite dimensional L-vector spaces. As we have $\dim_L(L \otimes_S \mathsf{F})_i = \operatorname{rank}_S F_i = \beta_i$, the alternating sum $\sum_i (-1)^i \beta_i^p$ is zero. Combining this with the fact β_i^p is independent of p for $i \ne 3$, we have β_3^p is also independent of p. This proves

Theorem 2.2 ([10, Theorem 6.4]) *There exists a graded minimal free resolution of S/I_t when $n - t = 2$.*

However, there is no known description of the explicit form of the resolution for this case. There is no affirmative result on the existence of the graded minimal free resolution of S/I_t for the case $2 \le t \le n - 3$.

When R is a field of characteristic zero, there is a striking result. Let $V = R^n$. The polynomial ring $S = R[x_{i,j}]$ is identified with the symmetric algebra $S(S_2V)$ by the correspondence $x_{i,j} \mapsto x_i x_j \in S_2V$, where $\{x_1, \ldots, x_n\}$ is a basis of V. The group $\mathrm{GL}(V)$ acts on S_2V by $g \cdot (vw) = (gv)(gw)$ for $v, w \in V$, hence it acts on $S = S(S_2V)$. It is easy to check that I_t is invariant under this action. Since a polynomial representation of $\mathrm{GL}(V)$ is completely reducible, the minimal free resolution F of S/I_t has a natural structure of $\mathrm{GL}(V)$-complex. Thus, $\operatorname{Tor}_i^S(S/I_t, S/S_+)$ has a natural structure of a finite-dimensional polynomial representation of $GL(V)$. Józefiak-Pragacz-Weyman determined the irreducible decomposition of $\mathrm{Tor}_i^S(S/I_t, S/S_+)$ completely (the decomposition is multiplicity-free, as a result). This is more than complete determination of β_i^0, because dimensions of irreducible polynomial representations are completely known. But it is not as much as the explicit construction of the resolution— to describe the boundary map is another problem (and is still open). For the statement and the proof, see [9]. It should be noted that the condition $n-t$ even for Gorensteinness of S/I_t follows directly from their resolution [9, Corollary 3.27]. We know that Gorenstein property of S/I_t depends only on its Poincaré series $\sum_i \dim_K(S/I_t)_i t^i$ (Lemma 1.2) which is independent of K, so this criterion is valid for any field. Their proof of the characteristic-zero resolution depends on Bott's vanishing theorem on line bundles over flag varieties and the decomposition formula of the plethysm $\bigwedge S_2 E$, which are valid only over the field of characteristic zero. And, polynomial representations of GL in positive characteristic are not completely reducible in general. So their approach is not translated into the context of positive characteristic directly.

However, after the success of the use of the representation theory in characteristic zero, it has been tried to develop the characteristic-free representation theory of GL and to apply to the problem of resolutions of determinantal ideals as we mentioned above. Kurano generalized the plethysm formula [10, Lemma 3.5] to the characteristic-free case (it is not a direct-sum decomposition formula any more).

3 Andersen's counterexample

Recently, a counterexample for Problem 1.5 was found by J. Andersen.

Theorem 3.1 ([3, Theorem 5.4.1]) *When $n = 7$ and $t = 2$, $\beta_5^5 > \beta_5^0$. In particular, there is no minimal free resolution of S/I_t over the ring of integers $\mathbb{Z}$ in this case.*

She used the combinatorial method developed by J. Eagon and J. Roberts [4] for the application to the resolution of determinantal ideals of generic matrices. Their method used on determinantal ideals can be formulated in terms of semigroup rings as follows.

Let $A = K[M_1, \ldots, M_n] \subset K[t_1, \ldots, t_r]$ be a semigroup ring over a field K, where $M_i = t_1^{a(i)_1} \cdots t_r^{a(i)_r}$ is a monomial. Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring. With letting x_i of degree $(a(i)_1, \ldots, a(i)_r)$, S is an $\mathbb{N}_0^r$-graded K-algebra, and the surjective map $S \to A$ given by $x_i \mapsto M_i$ is a homomorphism of $\mathbb{N}_0^r$-graded K-algebras, where $\mathbb{N}_0$ is the monoid of non-negative integers. Then, there is a unique $\mathbb{N}_0^r$-graded minimal free resolution F of A as as S-module, and $\mathrm{Tor}_i^S(S/S_+, A)$ is also $\mathbb{N}_0^r$-graded for any i. This $\mathbb{N}_0^r$-grading is also determined by the $\mathbb{N}_0^r$-graded resolution of S/S_+, the Koszul complex. Let $K_.(\underline{x})$ be the Koszul complex so that we have

$$\mathrm{Tor}_i^S(S/S_+, A) = H_i(A \otimes_S K_.(\underline{x})).$$

We have a direct decomposition

$$\mathrm{Tor}_i^S(S/S_+, A) = \bigoplus_{\lambda \in \mathbb{N}_0^r} H_i((A \otimes_S K_.(\underline{x}))_\lambda).$$

What is interesting is that the summand in the right-hand side is expressed in terms of (reduced) homology of certain simplicial complex.

Let $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{N}_0^r$. We define an abstract simplicial complex Σ_λ as follows. The vertex set is (a subset of) $\{x_1, \ldots, x_n\}$. A subset $\underline{m} = \{x_{m(1)}, \ldots, x_{m(s)}\}$ $(1 \leq m(1) < \cdots < m(s) \leq n)$ is an $(s-1)$-face of Σ_λ if and only if the monomial $M_{m(1)} \cdots M_{m(s)}$ divides $t^\lambda = t_1^{\lambda_1} \cdots t_r^{\lambda_r}$.

Proposition 3.2 (see [4, Corollary 3.3], [3, p.6]) *We have an isomorphism*

$$\mathrm{Tor}_i^S(S/S_+, A) = \bigoplus_{\lambda \in \mathbb{N}_0^r} \tilde{H}_{i-1}(\Sigma_\lambda, K),$$

where $\tilde{H}_*(?, K)$ denotes the reduced homology with coefficients in K.

Proof. It suffices to show that the degree λ-component

$$M(\lambda) \stackrel{def}{=} (A \otimes_S K_.(\underline{x}))_\lambda$$

of the Koszul complex, whose i^{th} homology is $\mathrm{Tor}_i^S(S/S_+, A)_\lambda$, is isomorphic to the complex $\tilde{C}_.(\Sigma_\lambda, K)[-1]$. By definition, the complex degree s-component $M(\lambda)_s$ of $M(\lambda)$ has a basis

$$X(\lambda)_s = \{N \otimes x_{m(1)} \wedge \cdots \wedge x_{m(s)} \mid N \cdot M_{m(1)} \cdots M_{m(s)} = t^\lambda\}.$$

So there is a map $X(\lambda)_s \to (\tilde{\Sigma}_\lambda)_{s-1}$ given by

$$N \otimes x_{m(1)} \wedge \cdots \wedge x_{m(s)} \mapsto \{x_{m(1)}, \ldots, x_{m(s)}\},$$

where $\tilde{\Sigma}_\lambda = \Sigma_\lambda \cup \{\emptyset\}$. Since $N = t^\lambda / M_{m(1)} \cdots M_{m(s)}$ is determined by λ and $x_{m(1)}, \ldots, x_{m(s)}$, it is clear that this correspondence is bijective. It is also clear that the linear extension $M(\lambda) \to \tilde{C}_.(\Sigma_\lambda, K)[-1]$ is compatible with the boundary map. □

J. Andersen applied this proposition to our problem. Consider our $S = K[x_{i,j}]_{1 \leq i \leq j \leq n}$, and the homomorphism $\Phi : S \to K[t_1, \ldots, t_n]$ given by $x_{i,j} \mapsto t_i t_j$ so that the image of Φ is the semigroup ring generated by all degree two monomials. It is well-known that $\mathrm{Ker}\,\Phi = I_2$. So the proposition is applicable (only) for the case $t = 2$. Using a computer-aided calculation, Andersen found that $\dim_K H_4(\Sigma_{(2,2,2,2,2,2,2)}, K)$ depends on the characteristic, and the theorem was proved.

She also proved some combinatorial statement on the simplicial complex Σ_λ. For example, she proved

Theorem 3.3 ([3, Theorem 3.4.1]) *β_3^p is independent of the characiteristic p when $t =$ 2.*

4 Another example

The third Betti number β_3^p is independent of p when $t = 2$. One might ask, what about the case $t \geq 3$, then?

Theorem 4.1 *If $n = 11$ and $t = 3$, then $\beta_3^3 > \beta_3^0$.*

The proof depends heavily on characteristic-free representation theory. As we mentioned, Kurano generalized the plethysim formula, which is a decomposition formula of $S_r S_2 E$, to the characteristic-free form.

We can generalize it again to the case of maps. When the characteristic is zero, for a map of finite free modules $\varphi : G \to F$, the complex $S_r S_2 \varphi$ is defined [14], and is decomposed into a direct sum of Schur complexes. Schur complexes in characteristic zero case is defined by Nielsen [14], and is a generalization of irreducible representations of GL to the complex version in some sense. We can generalize the decomposition formula of $S_r S_2 \varphi$ to the characteristic-free version, as the decomposition formula of $S_r \bigwedge^2 \varphi$ is generalized in [6]. The generalization of Schur complexes (to the characteristic free case) is due to Akin-Buchsbaum-Weyman [2], and we use it.

Next, we prove that

$$I_t \otimes_S S(S_2 \mathrm{id}_V) \subset S(S_2 \mathrm{id}_V)$$

is quasi-isomorphic to the complex $I_t \otimes_S K_.(x_{i,j})$ (in the category of $\mathrm{GL}(V)$-complexes), whose i^{th} homology is isomorphic to $\mathrm{Tor}_{i+1}^S(S/S_+, S/I_t)$.

Roughly speaking, the generalized decomposition formula applied to $S(S_2 \mathrm{id}_V)$ tells us that there is a filtration of $I_t \otimes_S S(S_2 \mathrm{id}_V)$, and its factor complexes are reasonably computable.

This is an out-line of the proof of the theorem. Thus, β_3^p depends on p in general.

Let us end with some open problems on the resolution of determinantal ideals of symmetric matrices.

Problem 4.2 1 Find an explicit form of the graded minimal free resolution of S/I_t for the case $n - t = 2$.

2 Is there any graded minimal free resolution of S/I_t over $\mathbb{Z}$ when $n - t = 3$?

3 Construct a graded minimal free resolution (if any) S/I_t^r for $r \geq 2$ when $n - t = 1$.

4 Describe the boundary maps of the Józefiak-Pragacz-Weyman resolution explicitly.

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A NOTE ON AUSLANDER-REITEN QUIVERS FOR INTEGRAL GROUP RINGS.

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0.Introduction

Let G be a finite group and $\mathcal{O}$ be a complete discrete valuation ring, with the maximal ideal (π) and residue field $k=\mathcal{O}/(\pi)$ of characteristic $p>0$. R will be used to denote either $\mathcal{O}$ or k. Let Θ be a connected component of the stable Auslander-Reiten quiver $\Gamma_s(RG)$ of group algebra RG and set $V(\Theta) = \{ vx(M) \mid M$ is an indecomposable RG-module in $\Theta\}$, where vx(M) denotes the vertex of M. Due to Kawata ([4, Proposition 3.2]), we know that there is a minimal element Q in $V(\Theta)$ with respect to the partial order $\leq_G$ which is uniquely determined up to G-conjugation. We call Q a vertex of Θ.

Let $N=N_G(Q)$ and f be the Green correspondence with respect to (G,Q,N). Choose an indecomposable RG-module M_0 in Θ with Q as its vertex. Let Δ be the connected component of $\Gamma_s(RN)$ containing $fM_0=L_0$. In the case R=k, Kawata has shown the following theorem which extends the Green correspondence in his paper [4]:

There is a graph monomorphism from Θ to Δ which preserves edge-multiplicity and direction.

The purpose of this note is to ensure that the above result also holds for $\mathcal{O}$G-lattices (i.e., finitely generated $\mathcal{O}$-free $\mathcal{O}$G-modules). The important tools used here can be

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The detailed version of this paper will be submitted for publication elsewhere.

found in [4], indeed the whole argument in [4] is also valid for $\mathcal{O}G$-lattices with some modifications. In this note, we shall provide a slightly simple proof by examining the middle terms of Auslander-Reiten sequences (see Theorem 1.5 and Corollary 1.6 below). The graph correspondence stated above is not always isomorphic. We shall give an example of this type for $\mathcal{O}G$-lattices in section 2.

The notation is almost standard. We shall work over the group ring RG. All the modules considered here are finitely generated free over R. We write $W|W'$ for RG-modules W and W', if W is a direct summand of W'. For an indecomposable non-projective RG-module M, we denote by $\mathcal{A}(M)$ the Auslander-Reiten (abbreviated AR-) sequences terminating at M. Concerning some basic facts and terminologies used here, we refer [1], [6] and [7], for example.

1.The middle terms of AR-sequences

To begin with, we shall exhibit some results on the AR-sequences for RG-modules, which are well-known or proved in [4] for kG-modules. We can easily see that they are also valid for $\mathcal{O}G$-lattices.

Lemma1.1 ([4, Lemma 2.3]). Let M be an indecomposable non-projective RG-module and H be a subgroup of G. Then the restricted exact sequence $\mathcal{A}(M)_H$ does not split if and only if $vx(M) \leq_G H$.

Lemma1.2 ([4, Lemma 2.4]). Let H be a subgroup of G. Let M and L be indecomposable non-projective modules for G and H respectively. Assume that L is a direct summand of $L^G{}_H$ with multiplicity one, and that M is a direct summand of L^G such that $L|M_H$. Then $\mathcal{A}(L)^G \cong \mathcal{A}(M)\oplus\mathcal{E}$, where $\mathcal{E}$ is a split sequence.

Lemma 1.3 (see [4, Lemma 2.5]). Let P be a non-trivial p-subgroup of G. Let M and L be indecomposable non-projective modules for G and $N_G(P)$ respectively. Assume that $\mathcal{A}(L)^G \cong \mathcal{A}(M) \oplus \mathcal{E}$, where $\mathcal{E}$ is a split sequence and that $P \leq_{N_G(P)} vx(L)$. Then $\mathcal{A}(L)^G{}_{N_G(P)} \cong \mathcal{A}(L) \oplus \mathcal{E}'$, where $\mathcal{E}'$ is a P-split sequence.

Remark. In the above lemma, by making use of Krull-Schmidt theorem for the category of morphisms, we have that $\mathcal{A}(M)_{N_G(P)} \cong \mathcal{A}(L) \oplus \mathcal{E}''$, where $\mathcal{E}''$ is a P-split sequence.

As we have mentioned in the introduction, Θ has a vertex. More precisely, the following holds.

Lemma 1.4 ([4, Lemma 3.1]). Let Ξ be a connected subgraph of $\Gamma_s(RG)$. Take any $Q \in V(\Xi)$ with the smallest order among those p-subgroups in $V(\Xi)$. Then for any indecomposable RG-module $M \in \Xi$, M_Q has an indecomposable direct summand whose vertex is Q.

Now we return to the situation in the introduction. Let Q be a vertex of Θ, put $N = N_G(Q)$. Let Λ be a subquiver of Δ consisting of $L_0 = fM_0$ and all the RN-modules L in Δ with the property : There exist RN-modules $L_0, L_1, L_2, \cdots, L_m = L$ such that L_n and L_{n+1} are connected by an irreducible map for all n with $0 \leq n \leq m-1$ and $Q \leq_G vx(L_n)$ for all n.

Remark ([4, Lemma 4.1]). For any indecomposable RN-module L in Λ, $Q \leq vx(L)$ holds by Lemma 1.4.

We shall show that $\Theta \cong \Lambda$ as graphs. Theorem 1.5 below is essential.

Let $\mathfrak{X}$ be the set of all p-subgroups of N whose orders are smaller than $|Q|$. Let L be an indecomposable RN-module in Λ, and M be an indecomposable RG-module in Θ. Assume that L and M satisfy the following two conditions:

(1) $L^G \cong M \oplus W$, where W is a $\mathfrak{X}$-projective RG-module.

(2) $M_N \cong L \oplus Z$, where Z_Q is a $\mathfrak{X}$-projective RQ-module.

Now we examine the relation of the middle terms of $\mathscr{A}(L)$ and $\mathscr{A}(M)$. Let $\mathbb{Y}$ be the set of all indecomposable direct summands of the middle term of $\mathscr{A}(L)$ whose vertices contain (a G-conjugation of) Q. Let $\mathbb{X}$ be the set of all indecomposable direct summands of the middle term of $\mathscr{A}(M)$. Then the modules of $\mathbb{Y}$ and $\mathbb{X}$ inherit the above conditions (1) and (2). More precisely, the following holds:

Theorem1.5. With the above notations. For each $Y_i \in \mathbb{Y}$, $(Y_i)^G$ has a unique indecomposable direct summand, say X_i, such that $Q \leq_G vx(X_i)$. The correspondence Ψ: $Y_i \to X_i$ is a bijective mapping from $\mathbb{Y}$ to $\mathbb{X}$ satisfying the following two conditions:

(1') $(Y_i)^G \cong X_i \oplus U_i$, where U_i is a $\mathfrak{X}$-projective RG-module.

(2') $(X_i)_N \cong Y_i \oplus V_i$, where $(V_i)_Q$ is a $\mathfrak{X}$-projective RQ-module.

Moreover, $X_i \cong X_j$ holds if and only if $Y_i \cong Y_j$ holds, when $\Psi(Y_i) = X_i$ and $\Psi(Y_j) = X_j$.

Proof. Let Y_i be a element of $\mathbb{Y}$. First, we prove that $(Y_i)^G{}_N \cong Y_i \oplus Y'_i$, where $(Y'_i)_Q$ is $\mathfrak{X}$-projective. In particular, $Y_i \,|\, (Y_i)^G{}_N$ with multiplicity one by Lemma 2.4. By the conditions (1) and (2), $L^G{}_N \cong L \oplus L'$, where $(L')_Q$ is $\mathfrak{X}$-projective. Let Y be the middle term of $\mathscr{A}(L)$. By Lemma 1.3, $Y^G{}_N \cong Y \oplus Y'$, where Y' is the middle term of a Q-split sequence terminating at L'. Thus, $(Y'_i)_Q \,|\, (L' \oplus \tau(L'))_Q$ and $(Y'_i)_Q$ is $\mathfrak{X}$-projective, where τ denotes Auslander-Reiten translation.

Next we prove that $(Y_i)^G$ has a unique indecomposable direct summand whose vertex contains Q. By Lemma 1.2, $\mathscr{A}(L)^G \cong \mathscr{A}(M) \oplus \mathcal{E}$, where $\mathcal{E}$ is a split sequence terminating at W. So, $Y^G \cong X \oplus$ ($\mathfrak{X}$-projective RG-modules), where X is the middle term of $\mathscr{A}(M)$. Let Y_t be an indecomposable direct summand of Y. If $(Y_t)^G$ have an indecomposable direct summand of X, then $Q \leq_G vx(Y_t)$. So, if $Y_t \notin \mathbb{Y}$, $(Y_t)^G$ is $\mathfrak{X}$-projective. On the other hand, for

$Y_i \in \mathbb{Y}$, $(Y_i)^G$ has a unique indecomposable direct summand, say X_i, satisfying $Y_i | (X_i)_N$, because $Y_i | (Y_i)^G{}_N$ with multiplicity one. Moreover, the condition $Y_i | (X_i)_N$ implies that $Q \leq_G vx(X_i)$ and $X_i \in \mathbb{X}$. Now we have to show the uniqueness of X_i. Let X'_i be an indecomposable summand of $(Y_i)^G$ such that $Q \leq_G vx(X'_i)$. Because $X'_i | (Y_i)^G$, we have that $(X'_i)_N | Y_i \oplus Y'_i$ and $(X'_i)_Q | (Y_i \oplus Y'_i)_Q$. We know that $(Y'_i)_Q$ is $\mathfrak{X}$-projective, and that $(X'_i)_Q$ and $(Y_i)_Q$ have indecomposable direct summands whose vertices are Q by Lemma 1.4. This implies that $Y_i | (X'_i)_N$ and $X'_i \cong X_i$.

Thus, for any $Y_i \in \mathbb{Y}$, we have that $(Y_i)^G \cong X_i \oplus$ ($\mathfrak{X}$-projective RG-modules), where $X_i \in \mathbb{X}$, and that $\oplus\sum(Y_i)^G \cong \mathbb{X} \oplus$ ($\mathfrak{X}$-projective RG-modules), where the left-side sum runs over all $Y_i \in \mathbb{Y}$. Moreover, $(X_i)_N \cong Y_i \oplus$(some direct summands of Y'_i). Hence, the correspondence $\Psi: Y_i \to X_i$ gives a bijective mapping from $\mathbb{Y}$ to $\mathbb{X}$, and we see that (1') and (2') hold. The last statement of the theorem is straightforward by (1') and (2').

Remark for Theorem1.5. Assume that $L \in \Lambda$ and $M \in \Theta$ satisfy the conditions (1) and (2). Then the middle terms of $\mathcal{A}(\tau^{-1}(L))$ and $\mathcal{A}(\tau^{-1}(M))$ have also the same properties as those of $\mathcal{A}(L)$ and $\mathcal{A}(M)$ do in the above theorem.

Corollary1.6 ([4, Theorem 4.6]). For any RN-module $L \in \Lambda$, L^G have a unique indecomposable direct summand M such that $Q \leq_G vx(M)$. The correspondence $L \to M$ gives rise to graph isomorphism from Λ to Θ, which preserves edge-multiplicity and direction. And the correspondents satisfy the conditions (1) and (2).

Proof. First, we recall that $(L_0)^G$ have a unique indecomposable direct summand M_0 such that $Q \leq_G vx(M_0)$, and that L_0 and M_0 satisfy (1) and (2). By successive use of Theorem 1.5 and its remark, the proof will be done.

2. An Example

As we have seen in Corollary 1.6, there is a graph monomorphism from Θ to Δ. But this correspondence is not always isomorphic (i.e., The case $\Lambda \subsetneq \Delta$ may occur). In this section, we shall provide an example of this type.

Throughout this section, we assume that p=2, $\mathcal{O}$ is of rank one (i.e., $(\pi)=(p)$) and has all the 3th root of unity, and that P denotes the cyclic group of order 2. Set $G=\mathfrak{A}_5\times P$, $N=\mathfrak{A}_4\times P$ and $Q=V\times P$, where $\mathfrak{A}_n$ and V denote the alternating group of degree n and a Sylow 2-subgroup of $\mathfrak{A}_4$, respectively. Then G and N have the common Sylow 2-subgroup Q and $N=N_G(Q)$. The desired correspondence happens between the connected components, Θ (2.2) in $\Gamma_s(\mathcal{O}G)$ and Δ (2.1) in $\Gamma_s(\mathcal{O}N)$.

$\mathcal{O}_P{}^N \cong \hat{P}\mathcal{O}N \cong \mathcal{O}\mathfrak{A}_4$ has just three isomorphism classes of primitive idempotents, say e_t, e_1 and e_2, where e_t corresponds to the trivial $\mathcal{O}N$-lattice. Put $Q_1=e_1\hat{P}\mathcal{O}N$. Q_1 has period 2. The connected component which contains Q_1 in $\Gamma_s(\mathcal{O}N)$, say Δ, is isomorphic to $\mathbb{Z}A_\infty/(2)$ and has the following form ;

$$(2.1) \qquad \Delta : \quad \begin{array}{ccccccc} Q_1 & — & L & — & L_1 & — & L_2 — \cdots \\ & \backslash & & \backslash & & \backslash & \\ & & \Omega(Q_1) & - & \Omega(L) & - & \Omega(L_1) - \cdots , \end{array}$$

where $vx(Q_1)=P$, $vx(L)=Q$ and $vx(L_i)=Q$ for i=1, 2, $\cdots$.

Let M be the Green correspondent of L with respect to (G,Q,N) and Θ be the connected component which contains M in $\Gamma_s(\mathcal{O}G)$. M has period 2, so $\Theta \cong \mathbb{Z}A_\infty/(2)$. And Θ has the following form ;

$$(2.2) \qquad \Theta : \quad \begin{array}{ccccc} M & — & M_1 & — & M_2 - \cdots \\ & \backslash & & \backslash & \\ & & \Omega(M) & - & \Omega(M_1) - \cdots , \end{array}$$

where M_i is the Green correspondent of L_i for i=1, 2, $\cdots$, and all the indecomposable $\mathcal{O}G$-lattices in Θ have Q as their vertices.

Example. With the above notations. Put $M=M_0$ and $L=L_0$. Let Λ be the subquiver of Δ removed Q_1 and $\Omega(Q_1)$ from Δ. Then $\Lambda \cong \Theta$ holds by Corollary 1.6. That is, Kawata's correspondence between Θ (2.2) and Δ (2.1) is not isomorphic.

Remark. In the case R=k, a similar example to our one has already given by Okuyama in [8].

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On Auslander-Reiten components for certain group modules and an additive function

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Throughout this note G is a finite group and R denotes either a field k of characteristic $p>0$, or a complete discrete valuation ring $\mathcal{O}$ of characteristic zero with residue class field of characteristic p. Let $\Gamma_s(RG)$ be the stable Auslander-Reiten quiver of the group ring RG and Θ a connected component of $\Gamma_s(RG)$ which contains no loops. In [W] Webb constructs a subadditive function on Θ using the cohomology theory, and in the case when $R=k$ is a field Okuyama [O] also constructs a subadditive function using an inner product on the Green ring $a_k(G)$ (see also[Bs] and [E-S]). Then by the result of Happel, Preiser and Ringel [H-P-R] the tree class of Θ is either a Euclidean diagram, a Dynkin diagram or one of five infinite trees A_∞, B_∞, C_∞, D_∞ or A_∞^∞. Our aim in this note is to give a simple construction of an additive function on Θ, which is a modification of Okuyama's one, when the RG-modules in Θ are not periodic. This is shown in Section 2. In Section 3 we consider the case where $R=k$ is algebraically closed and give some condition which implies that Θ is isomorphic to $\mathbb{Z}A_\infty$.

The notation is almost standard. All RG-modules considered here are lattices, that is modules which are finitely generated free R-modules. An RG-module W is periodic if $\tilde{W} \cong \Omega^n \tilde{W}$ for some positive integer n, where Ω is the Heller operator and $\tilde{W}$ is the projective-free part of W. For a non-projective indecomposable RG-module W, we write $\mathcal{A}(W)$ to denote the Auslander-Reiten sequence (AR-sequence for short) $0 \to \tau W \to m(W) \to W \to 0$ terminating at W, and we write $m(W)$ to denote the middle term of $\mathcal{A}(W)$. If $R=k$ is a field then $\tau = \Omega^2$, and if $R=\mathcal{O}$ then $\tau=\Omega$. Concerning some basic facts and terminologies used here, we refer to [Bn], [E1] and [R2].

1. Preliminaries

In this section we collect some results which will be used to construct an additive function.

Lemma 1.1([C, Lemma 2.5]). Suppose that H is a normal subgroup of G with G/H cyclic. If W is an RG-module such that $W{\downarrow}_H$ is a projective RH-module, then W is periodic.

The symbol $\otimes$ denotes the tensor product over the coefficient ring R. For an RG-module W, we write W^* to denote the dual $\mathrm{Hom}_R(W, R)$ of W.

Lemma 1.2([A-C, Proposition 4.8]). Let W be an RG-module. Then $W \mid W\otimes W^*\otimes W$. In particular if W is not projective, then $W\otimes W$ is not projective.

Lemma 1.3. Let W be a non-projective indecomposable RG-module, and $\mathcal{A}(W): 0 \to \tau W \to m(W) \to W \to 0$ be the AR-sequence terminating at W.

(1)([A-C, Proposition 2.3]) Let X be an indecomposable RG-module. Then the tensor sequence $0 \to \tau W\otimes X \to m(W)\otimes X \to W\otimes X \to 0$ splits if and only if $W \nmid X\otimes X^*\otimes W$.

(2)([R1, 2.10 Proposition]) Let H be a subgroup of G. Then $\mathcal{A}(W)$ splits on restriction to H if and only if W is not H-projective.

Lemma 1.4. Let Θ be a connected component of $\Gamma_s(RG)$ and suppose that V and W are indecomposable RG-modules in Θ. Let X be an RG-module and H a subgroup of G. Then the following hold.

(1)([O, Lemma 3]) $V\otimes X$ is projective if and only if $W\otimes X$ is projective.

(2) $V\otimes X$ is periodic if and only if $W\otimes X$ is periodic.

(3) $V{\downarrow}_H$ is projective (resp. periodic) if and only if $W{\downarrow}_H$ is projective (resp. periodic).

Proof. (1) If $R = \mathcal{O}$, using the same argument in the proof of [O, Lemma 3], we get the result.

(2) We have only to consider the case where all RG-modules in Θ are not periodic, and it suffices to show the result in the case when there exists an irreducible

map from V to W. Assume that $V\otimes X$ is periodic. Then $\tau^{-1}V\otimes X$ is also periodic. Since $\tau^{-1}V$ is not periodic, $\tau^{-1}V \nmid X\otimes X^*\otimes V$. Hence the tensor sequence $0 \to V\otimes X \to m(V)\otimes X \to \tau^{-1}V\otimes X \to 0$ splits by Lemma 1.3(1). This implies that $W\otimes X$ is periodic since $W\otimes X$ is a direct summand of $m(\tau^{-1}V)\otimes X$. The same argument as above shows that if $W\otimes X$ is periodic, then $V\otimes X$ is periodic.

(3) Set $X = R_H\uparrow^G$, where R_H is the trivial RH-module. Then the result follows from (1) and (2).

For a connected component Θ of $\Gamma_s(RG)$, set $V(\Theta) = \{ vx(W) \mid W \in \Theta \}$, where $vx(W)$ denotes a vertex of W.

Lemma 1.5([K1, Proposition 3.2]). Let Q_0 be an element of $V(\Theta)$ which is minimal with respect to the partial order $\leq_G$. Then for any $H \in V(\Theta)$ we have $Q_0 \leq_G H$. In particular Q_0 is uniquely determined up to conjugation.

Proof. If $R = \mathcal{O}$, using the same argument in [K1, Section 3], we get the result.

2. Additive functions

Let Θ be a connected component of $\Gamma_s(RG)$. In this section we assume that the RG-modules in Θ are not periodic.

Lemma 2.1. There exists a subgroup Q of G which satisfies the following two conditions for any indecomposable RG-module W in Θ:

(A1) W is not Q-projective;

(A2) $W\downarrow_Q$ is not projective.

Proof. Let Q_0 be a minimal element of $V(\Theta)$ (see Lemma 1.5). Choose an indecomposable RG-module W_0 in Θ with Q_0 its vertex. Let S_0 be a Q_0-source of W_0. Then S_0 is not periodic. Let Q be a normal subgroup of Q_0 with $|Q_0 : Q| = p$. We show that Q satisfies the conditions (A1) and (A2). Since $Q < Q_0 \leq_G vx(W)$ for any indecomposable RG-module W in Θ, Q satisfies (A1). By Lemma 1.1, $S_0\downarrow_Q$ is not projective and hence $W_0\downarrow_Q$ is not projective. Thus $W\downarrow_Q$ is not projective for any W in Θ by Lemma 1.4(3).

Now we construct an additive function on Θ. Let Q be a subgroup of G satisfying the conditions (A1) and (A2) in Lemma 2.1. For an indecomposable RG-module W in Θ, let $d_Q(W)$ be the number of non-projective indecomposable direct summands of $W\downarrow_Q$.

<u>Theorem 2.2.</u> d_Q is an additive function on Θ with $d_Q(W) = d_Q(\tau W)$.

<u>Proof.</u> By the condition (A2) $W\downarrow_Q$ is not projective for any W in Θ. Thus $d_Q(W)$ is positive and $d_Q(W) = d_Q(\tau W)$. Let $0 \to \tau W \to m(W) \to W \to 0$ be the AR-sequence terminating at W. Then $m(W) = \oplus\, a'_{VW} V$, where a'_{VW} is the length of $\mathrm{Irr}(V,W) = \mathrm{Rad}(V,W)/\mathrm{Rad}^2(V,W)$ as an $\mathrm{End}_{RG}(V)$-module. By the condition (A1) and Lemma 1.3(2), $\mathcal{A}(W)$ splits on restriction to Q. Thus we have $d_Q(W) + d_Q(\tau W) = \Sigma a'_{VW}\, d_Q(V)$ and therefore d_Q is additive.

We shall also construct an additive function on Θ using the tensor product with a certain periodic RG-module.

<u>Lemma 2.3.</u> Let Θ be a connected component of $\Gamma_s(RG)$. Then there is a periodic RG-module L such that $W \otimes L$ is not projective for any indecomposable RG-module W in Θ.

<u>Proof.</u> Let W_0 be an indecomposable RG-module in Θ. Then for some subgroup H of G, $W_0\downarrow_H$ has a periodic (non-projective) direct summand, say L_0, by virtue of Lemma 1.1. Let L be a periodic RG-module such that $L_0 \mid L\downarrow_H$. We claim that $W_0 \otimes L$ is not projective: since $L_0 \otimes L_0 \mid (W_0 \otimes L)\downarrow_H$ and $L_0 \otimes L_0$ is not projective by Lemma 1.2. Hence $W \otimes L$ is not projective for any W in Θ by Lemma 1.4(1).

Let L be a periodic RG-module such that $W \otimes L$ is not projective for any indecomposable RG-module W in Θ. For W in Θ, let $d_L(W)$ be the number of non-projective indecomposable direct summands of $W \otimes L$.

<u>Theorem 2.4.</u> d_L is an additive function on Θ with $d_L(W) = d_L(\tau W)$.

<u>Proof.</u> For any W in Θ, $W \otimes L$ is not projective, and thus $d_L(W)$ is positive and $d_L(W) = d_L(\tau W)$. Since $L \otimes L^* \otimes W$ is periodic and W is not periodic, it follows that $W \nmid L \otimes L^* \otimes W$. Thus the tensor sequence $\mathcal{A}(W) \otimes L$ splits by Lemma 1.3(1). Then using an argument similar to the one in the proof of Theorem 2.2, we see that d_L is additive.

3. Auslander-Reiten components and certain group modules

In this section we consider a connected component of $\Gamma_s(kG)$ containing an indecomposable kG-module whose k-dimension is not divisible by p under the following hypothesis:

(#) k is an algebraically closed field of characteristic $p>0$ and a Sylow p-subgroup P of G is not cyclic, dihedral, semidihedral or generalized quaternion.

The purpose of this section is to show the following theorem.

Theorem 3.1([K2, Theorem 2.1]). Assume (#). Suppose that Θ is a connected component of $\Gamma_s(kG)$ and Θ contains an indecomposable kG-module whose k-dimension is not divisible by p. Then Θ is isomorphic to $\mathbf{Z}A_\infty$.

The following lemma is useful.

Lemma 3.2([K2, Lemmas 1.8 and 1.9]). Let Θ be a connected component of $\Gamma_s(kG)$.

(1) Suppose that the tree class of Θ is A_∞. Let $T: M_1 \leftarrow M_2 \leftarrow \cdots \leftarrow M_n \leftarrow \cdots$ be a tree in Θ such that $\Theta \cong \mathbf{Z}T/\Pi$ for some admissible group of automorphisms $\Pi \subseteq \operatorname{Aut} \mathbf{Z}T$. Then $\dim_k M_n \equiv n(\dim_k M_1) \pmod p$ for all $n \geq 1$.

(2) If the tree class of Θ is A_∞^∞, then $\dim_k M \equiv \dim_k M' \pmod p$ for all indecomposable kG-modules M and M' in Θ.

(3) Suppose that the tree class of Θ is D_∞. Let

$$T:\ \begin{array}{l} M \leftarrow M_2 \leftarrow M_3 \leftarrow \cdots \leftarrow M_n \leftarrow \cdots \\ \downarrow \\ M' \end{array}$$

be a tree in Θ with $\Theta \cong \mathbf{Z}T$.

Then $\dim_k M \equiv \dim_k M' \pmod p$ and $\dim_k M_n \equiv 2(\dim_k M) \pmod p$ for all $n \geq 2$.

Proof. Let H be a subgroup of G of order p. Then the group algebra kH has only p non-isomorphic indecomposable modules, say $V_1, V_2, \cdots, V_{p-1}$ and V_p, where $\dim_k V_t = t$ $(1 \leq t \leq p)$ and V_p is projective. For a kG-module M, let $d_H^t(M)$ be the multiplicity of V_t in $M{\downarrow}_H$. Then each d_H^t $(1 \leq t \leq p-1)$ is either the function with constant value 0 or an additive function on Θ. Hence the result follows from [Bn, Lemma 2.30.5].

Outline of Proof of Theorem 3.1. The tree class of Θ is A_∞, D_∞ or A_∞^∞.

Step 1. The tree class of Θ is not A_∞^∞.

Proof. We shall derive a contradiction assuming that the tree class of Θ is A_∞^∞. Let $T: \cdots \to W_n \to \cdots \to W_2 \to M \leftarrow M_2 \leftarrow M_3 \leftarrow \cdots \leftarrow M_n \leftarrow \cdots$ be a tree in Θ with $\Theta \cong \mathbf{Z}T$. Note that $p \nmid \dim_k M$, $p \nmid \dim_k M_n$ and $p \nmid \dim_k W_n$ for all $n \geq 2$ from Lemma 3.2(2). On the other hand the connected component Δ_0 containing the trivial kG-module k is isomorphic to $\mathbf{Z}A_\infty$ by [W]. Let $T_0: k = L_1 \leftarrow L_2 \leftarrow \cdots \leftarrow L_n \leftarrow \cdots$ be a tree in Δ_0 with $\Delta_0 \cong \mathbf{Z}T_0$. Tensoring modules $L_1, L_2, \cdots, L_p$ with M, we have $M_n \oplus W_n \mid L_n \otimes M$ for $n \leq p$. In particular we obtain $M_p \oplus W_p \mid L_p \otimes M$. But this is a contradiction, since $p \mid \dim_k L_p$ from Lemma 3.2(1) and thus $L_p \otimes M$ has no indecomposable direct summand whose k-dimension is not divisible by p from [B-C].

Step 2. The tree class of Θ is not D_∞.

Proof. Assume contrary that the tree class of Θ is D_∞. Let

$$
\begin{array}{l}
T: M \leftarrow M_2 \leftarrow M_3 \leftarrow \cdots \leftarrow M_n \leftarrow \cdots \\
\qquad\quad \downarrow \\
\qquad\quad W
\end{array}
$$

be a tree in Θ with $\Theta \cong \mathbf{Z}T$.

Note that $p \nmid \dim_k M$ and $p \nmid \dim_k W$ from Lemma 3.2(3). Let $\mathcal{A}(k): 0 \to \Omega^2 k \to m(k) \to k \to 0$ be the AR-sequence terminating at k. By [A-C] the tensor sequences $\mathcal{A}(k)\otimes M$ and $\mathcal{A}(k)\otimes W$ are the AR-sequences $\mathcal{A}(M)$ modulo projectives and $\mathcal{A}(W)$ modulo projectives respectively. Hence we have $M_2 \cong m(k)\otimes M \cong m(k)\otimes W$ (mod projectives). Thus $m(k)\otimes M\otimes M^* \cong m(k)\otimes W\otimes M^*$ (mod projectives). Note that $m(k)\otimes M\otimes M^*$ and $m(k)\otimes W\otimes M^*$ are the middle terms of the tensor sequences $\mathcal{A}(k)\otimes M\otimes M^*$ and $\mathcal{A}(k)\otimes W\otimes M^*$ respectively.

Since the multiplicity of k in $M\otimes M^*$ is one, it follows that the number of indecomposable direct summands of $m(k)\otimes M\otimes M^*$ lying in Δ_0 is odd. However k is not a direct summand of $W\otimes M^*$. Hence the number of indecomposable direct summands of $m(k)\otimes W\otimes M^*$ lying in Δ_0 is even, a contradiction.

By Steps 1 and 2, the tree class of Θ is A_∞. Since a Sylow p-subgroup P of G is not generalized quaternion, indecomposable kG-modules whose k-dimension is not divisible by p are not periodic. Hence Θ is isomorphic to $\mathbf{Z}A_\infty$.

Corollary 3.3([K2, Corollary 2.7]). Assume (#). Let M be an indecomposable kP-madule and Θ a connected component of $\Gamma_s(kP)$ containing M.

(1) Suppose that p is odd and $\dim_k M = 2$. Then Θ is isomorphic to $\mathbf{Z}A_\infty$ and M lies at the end of Θ.

(2) Suppose that $p \neq 3$ and $\dim_k M = 3$. Then Θ is isomorphic to $\mathbb{Z}A_\infty$ and M lies at the end of Θ.

Proof. There exists an elements x of P such that x does not act on M trivially. Let $H = <x>$ and L a non-projective indecomposable direct summand of $M\downarrow_H$. Note that $L \cong \Omega^2 L$. Since Θ is isomorphic to $\mathbb{Z}A_\infty$ by Theorem 3.1, for any indecomposable module W in Θ, W is not H-projective and L is a direct summand of $W\downarrow_H$.

Let $d(W)$ be the multiplicity of L in $W\downarrow_H$. Then from the same argument in the proof of Theorem 2.2, d is an additive function on Θ. Now $d(M) = 1$. This implies that M lies at the end of Θ.

Remark. In [E2], Erdmann proved that there are infinitely many kP-modules of dimension 2 or 3 lying at the ends of $\mathbb{Z}A_\infty$-components under the hypothesis (#) ([E2, Propositions 4.2 and 4.4]). Consequently she showed that for a block B over an algebraically closed field, the stable Auslander-Reiten quiver $\Gamma_s(B)$ has infinitely many components isomorphic to $\mathbb{Z}A_\infty$ if a defect group of B is not cyclic, dihedral, semidihedral or generalized quaternion ([E2, Theorem 5.1]).

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RATIONAL REPRESENTATIONS, TYPES AND EXTENSIONS OF 2-GROUPS

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Let G be a finite group and θ a character of G. We do not know, in general, how to construct a representation Θ with character θ. The purpose of the paper is to discuss some recent progress in constructing a representation with character θ of G. In particular, we deal with representation theory of 2-groups, emphasizing the role of generalized quaternion, dihedral and semidihedral groups.

Notation. A group G means a finite group. The set of irreducible complex characters of G is denoted by $\mathrm{Irr}(G)$. $\mathbf{Q}$ is the rational numbers. For a natural number n, ζ_n is a primitive n-th root of unity. The generalized quaternion, dihedral and semidihedral groups of order 2^{n+1} are denoted by Q_n, D_n and SD_n, respectively:

$$Q_n = \langle a,\ b \mid a^{2^n} = 1,\ b^2 = a^{2^{n-1}},\ bab^{-1} = a^{-1}\rangle,\ (n \geq 2),$$
$$D_n = \langle a,\ b \mid a^{2^n} = b^2 = 1,\ bab^{-1} = a^{-1}\rangle,\ (n \geq 2),$$
$$SD_n = \langle a,\ b \mid a^{2^n} = b^2 = 1,\ bab^{-1} = a^{-1+2^{n-1}}\rangle,\ (n \geq 3).$$

The cyclic group of order 2^n is denoted by C_n:

$$C_n = \langle a \mid a^{2^n} = 1\rangle,\ (n \geq 0).$$

1. Irreducible complex representations

Let G be a finite group of exponent n and $\chi \in \mathrm{Irr}(G)$. Brauer's theorem [2] asserts that there exists a representation X with character χ,

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This paper is in final form and no version of it will be submitted for publication elsewhere .

which is realized in $\mathbf{Q}(\zeta_n)$. Of course, Brauer's proof gives no means of constructing a representation X with character χ. It seems that for few groups G (except symmetric groups, monomial groups, etc.), irreducible complex representations of G have been constructed in $\mathbf{Q}(\zeta_n)$.

Let $m_{\mathbf{Q}}(\chi)$ denote the Schur index of χ over $\mathbf{Q}$. Then we have the following:

Problem 1.1. Find a field K with $[K : \mathbf{Q}(\chi)] = m_{\mathbf{Q}}(\chi)$ and find a representation X with character χ, which is realized in K.

The following is an easier version of this problem:

Problem 1.2. Let $\chi \in \mathrm{Irr}(G)$ with $m_{\mathbf{Q}}(\chi) = 1$. Find a representation X with character χ, which is realized in $\mathbf{Q}(\chi)$.

Let p be a prime and $q = p^a$, where a is a natural number. All the irreducible representations of the special linear group $SL(2,\ q)$ was constructed by Kloosterman [13] for the case $q = p$, and later by Gel'fand–Graev [5] and Tanaka [17] for $q = p^a$. They constructed representations of $SL(2,\ q)$ in sufficiently large cyclotomic fields.

Meanwhile, Janusz [12] showed that $m_{\mathbf{Q}}(\chi) = 1$ or 2 for each $\chi \in \mathrm{Irr}(SL(2,\ q))$. In fact, Janusz determined the local indices of each simple component of the group algebra $\mathbf{Q}[SL(2,\ q)]$. In particular, $m_{\mathbf{Q}}(\chi) = 1$ for each $\chi \in \mathrm{Irr}(PSL(2,\ q))$. Recently, Böge [1] settled the Problem 1.2 for some $\chi \in \mathrm{Irr}(PSL(2,\ p))$, $(p \neq 2)$. Namely, for every $\chi \in \mathrm{Irr}(PSL(2,\ p))$ with $\chi(1) = p - 1$, Böge constructed a representation X with character χ, which is realized in $\mathbf{Q}(\chi)$.

It is interesting to consider Problems 1.1 and 1.2 for monomial characters χ of G. Namely, suppose that there is a subgroup H of G and a linear character ψ of H with $\chi = \psi^G$. Let $\mathbf{Q}(\psi) = \mathbf{Q}(\zeta_t)$ for some integer t. Then the induced representation X with character χ is realized in $\mathbf{Q}(\zeta_t)$. The problem is : Find an extension K of $\mathbf{Q}(\chi)$ with $[K : \mathbf{Q}(\chi)] = m_{\mathbf{Q}}(\chi)$ and a representation X' of G such that X' is realized in K and that X' is equivalent to X. In particular, if $m_{\mathbf{Q}}(\chi) = 1$, find $X' \sim X$, which is realized in $\mathbf{Q}(\chi)$. Of course, if $[\mathbf{Q}(\zeta_t) : \mathbf{Q}(\chi)] = m_{\mathbf{Q}}(\chi)$, then the induced representation X with character χ is a required representation. Such examples are faithful representations of the generalized quaternion groups of order 2^{n+1}, $(n \geq 2)$.

2. Irreducible rational representations

Let $\chi \in \mathrm{Irr}(G)$ and put

$$\Omega(\chi) = m_{\mathbf{Q}}(\chi) \sum_{\sigma} \chi^{\sigma}, \quad \sigma \in \mathrm{Gal}(\mathbf{Q}(\chi)/\mathbf{Q}).$$

Then $\Omega(\chi)$ is the character of an irreducible rational representation $\tilde{X}$ of G, which corresponds to χ. Conversely, if $\tilde{X}$ is an irreducible rational representation of G, then there exists $\chi \in \mathrm{Irr}(G)$ such that $\Omega(\chi)$ is the character of $\tilde{X}$.

It seems that constructing an irreducible rational representation $\tilde{X}$ with character $\Omega(\chi)$ is more difficult than constructing an irreducible complex representation X with character χ. Even if X is known, it is not so easy to get $\tilde{X}$. For instance, let the exponent of G be n and suppose that one has obtained an irreducible complex representation X with character χ, which is realized in $\mathbf{Q}(\zeta_n)$. Let $\mathrm{Gal}(\mathbf{Q}(\chi)/\mathbf{Q}) = \{\sigma_1, \sigma_2, \ldots, \sigma_s\}$, $\sigma_1 = 1$ and put $m = m_{\mathbf{Q}}(\chi)$. Each σ_i is extended to an automorphism of $\mathbf{Q}(\zeta_n)$, which is also denoted by σ_i. For $x \in G$, put

$$X'(x) =$$
$$[\overbrace{X(x), \ldots, X(x)}^{m}, \overbrace{X(x)^{\sigma_2}, \ldots, X(x)^{\sigma_2}}^{m}, \ldots, \overbrace{X(x)^{\sigma_s}, \ldots, X(x)^{\sigma_s}}^{m}],$$

where the right side denotes the matrix of degree $ms\chi(1)$ whose entries are all 0 except the above matrices $X(x), \ldots, X(x), X(x)^{\sigma_2}, \ldots, X(x)^{\sigma_2}, \ldots, X(x)^{\sigma_s}, \ldots, X(x)^{\sigma_s}$ on the diagonal. It is clear that X' is a representation of G with character $\Omega(\chi)$. Hence there exists a non-singular matrix P of degree $ms\chi(1)$ such that the entries of P are in $\mathbf{Q}(\zeta_n)$ and that

$$\tilde{X}(x) = P^{-1}X'(x)P \quad (x \in G)$$

is a rational representation of G with character $\Omega(\chi)$. But it seems difficult to find such a matrix P.

So we ask to what extent we can construct rational representations of a finite group G. For a linear character ψ of G, we know well an irreducible rational representation $\tilde{\Psi}$, whose character is $\Omega(\psi)$:

Proposition 2.1 (cf. [24, Proposition 1]). *Let ψ be a linear character of G. Let N be the kernel of ψ with $t = [G : N]$. Let $G = \cup_{i=0}^{t-1} Ny^i$. Then*

$$\psi(xy^i) = \zeta^i, \quad (0 \le i < t;\ x \in N),$$

where ζ is a primitive t-th root of unity. Let

$$f(X) = X^s - a_{s-1}X^{s-1} - \cdots - a_1X - a_0$$

be the irreducible polynomial over $\mathbf{Q}$ *such that* $f(\zeta) = 0$, *where* $s = \varphi(t)$, φ *being the Euler's function. Put*

$$\bar{\Psi}(xy^i) = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ a_0 & a_1 & \cdots & \cdots & a_{s-1} \end{pmatrix}^i, \quad (0 \le i < t;\ x \in N).$$

Then $\bar{\Psi}$ *is an irreducible rational representation of* G, *whose character is* $\Omega(\psi)$.

Next everyone will think of induced representations. Let H be a subgroup of G and $\varphi \in \mathrm{Irr}(H)$ with $\varphi^G \in \mathrm{Irr}(G)$. Suppose that we have constructed an irreducible rational representation $\bar{\Phi}$ with character $\Omega(\varphi)$. If $\Omega(\varphi^G) = \Omega(\varphi)^G$, then $\bar{\Phi}^G$ is an irreducible rational representaion of G with character $\Omega(\varphi^G)$. So we ask when $\Omega(\varphi^G) = \Omega(\varphi)^G$ happens. For this problem we have:

Proposition 2.2 ([24, Proposition 3]). *Let* H *be a subgroup of* G *and* $\varphi \in \mathrm{Irr}(H)$ *such that* $\varphi^G \in \mathrm{Irr}(G)$. *Then* $m_{\mathbf{Q}}(\varphi^G)$ *divides* $m_{\mathbf{Q}}(\varphi)[\mathbf{Q}(\varphi) : \mathbf{Q}(\varphi^G)]$. *Furthermore, the induced character* $\Omega(\varphi)^G$ *of* G *is a character of an irreducible rational representation of* G, *if and only if*

$$m_{\mathbf{Q}}(\varphi^G) = m_{\mathbf{Q}}(\varphi)[\mathbf{Q}(\varphi) : \mathbf{Q}(\varphi^G)]. \tag{2.1}$$

In this case, $\Omega(\varphi)^G = \Omega(\varphi^G)$.

Corollary 2.3 ([24, Corollary 4]). *If* $\mathbf{Q}(\varphi) = \mathbf{Q}(\varphi^G)$, *then* $\Omega(\varphi)^G = \Omega(\varphi^G)$.

In the formula (2.1), the computation of Schur index $m_{\mathbf{Q}}(\varphi^G)$ is not so easy. For the case that $H \triangleleft G$ and φ is linear, we have the following theorem:

Theorem 2.4 (Yamada [24, Theorem 13]). *Let H be a normal subgroup of a finite group G. Let χ be a complex irreducible character of G which is induced from a linear character ψ of H. Set $F = \{f \in G \; ; \; \psi^f = \psi^{\tau(f)}$ for some $\tau(f) \in \mathrm{Gal}(\mathbf{Q}(\psi)/\mathbf{Q}(\chi))\}$. Then F/H is isomorphic to $\mathrm{Gal}(\mathbf{Q}(\psi)/\mathbf{Q}(\chi))$ and $\mathbf{Q}(\chi) = \mathbf{Q}(\psi^F)$. Let Hf_i ($i = 1, 2, \cdots, t$; $f_1 = 1$) be all the distinct cosets of H in F. Set $\tau(f_i) = \tau_i$. Let $f_i f_j = n_{ij} f_{\nu(i,\, j)}$, $n_{ij} \in H$, $\nu(i,\, j) \in \{1, 2, \cdots, t\}$. Put $\beta(\tau_i, \tau_j) = \psi(n_{ij})$. Then $\Omega(\chi) = \Omega(\psi^G) = \Omega(\psi)^G$ if and only if the cyclotomic algebra $(\beta, \mathbf{Q}(\psi)/\mathbf{Q}(\psi^F))$ is a division algebra.*

Remark 2.5. The index of a cyclotomic algebra is computed by the formulas as are given by Yamada [20] and [23].

Remark 2.6. If G is a metabelian group, then every irreducible character χ of G is induced from a linear character ψ of a normal subgroup of G. (See [19, Theorem 1].)

3. Imprimitive Q[G]-modules

We recall the definition of imprimitive irreducible $\mathbf{Q}[G]$-modules (see Roquette [16]). As is remarked in [16], the concept of imprimitivity originates in Witt [18].

Let G be a finite group. Let $\mathfrak{M}$ be an irreducible $\mathbf{Q}[G]$-module. Put $\mathbf{S}=\mathrm{End}_{\mathbf{Q}[G]}(\mathfrak{M})$, the skew-field of $\mathbf{Q}[G]$-endomorphisms of $\mathfrak{M}$. We regard $\mathfrak{M}$ as the right $\mathbf{Q}[G]$- and left $\mathbf{S}$-module. $\mathfrak{M}$ is called *imprimitive*, if there exist left $\mathbf{S}$-modules $\mathfrak{N}_1, \mathfrak{N}_2, \ldots, \mathfrak{N}_r$ ($r \geq 2$) such that $\mathfrak{M} = \mathfrak{N}_1 \bigoplus \mathfrak{N}_2 \bigoplus \cdots \bigoplus \mathfrak{N}_r$ and that $\mathfrak{N}_1, \mathfrak{N}_2, \ldots, \mathfrak{N}_r$ are transitively permuted by G. Put $\mathfrak{N} = \mathfrak{N}_1$. Let H be the group of elements $h \in G$ such that $\mathfrak{N}h = \mathfrak{N}$. Then $|G : H| = r$. If $G = Hg_1 \cup Hg_2 \cup \cdots \cup Hg_r$, then

$$\mathfrak{M} = \mathfrak{N}g_1 \oplus \mathfrak{N}g_2 \oplus \cdots \oplus \mathfrak{N}g_r,$$
$$\mathfrak{M} \cong \mathfrak{N} \otimes_{\mathbf{Q}[H]} \mathbf{Q}[G].$$

If $\mathfrak{M}$ is not imprimitive, then $\mathfrak{M}$ is called *primitive*.

Lemma 3.1 (Roquette [16, p.243]). *Notation being the same as above,*

$$\mathbf{S} = \mathrm{End}_{\mathbf{Q}[G]}(\mathfrak{M}) \cong \mathrm{End}_{\mathbf{Q}[H]}(\mathfrak{N}).$$

Lemma 3.2 (Roquette [16, Lemma 1]). *If G has a faithful, primitive, irreducible $\mathbf{Q}[G]$-module $\mathfrak{M}$, then every abelian normal subgroup of G is cyclic.*

Theorem 3.3 (Roquette [16, Lemma 3]). *Let p be a prime. Let G be a p-group such that every abelian normal subgroup of G is cyclic. Then G itself is cyclic, except $G = Q_n$ $(n \geq 2)$ or $G = D_n$ $(n \geq 3)$ or $G = SD_n$ $(n \geq 3)$.*

From Lemma 3.1, Lemma 3.2 and Theorem 3.3 we have

Theorem 3.4 (Roquette [16]). *Let G be a finite group. Let Θ be an irreducible rational representation of G. Then there exist subgroups $H \triangleright N$ of G and an irreducible rational representation Ξ of H with the following properties (possibly $H = G$ and $\Xi = \Theta$) :*
(i) *Every abelian normal subgroup of H/N is cyclic.*
(ii) $\Theta = \Xi^G$, $\ker \Xi = N$ *and* $\mathrm{End}_{\mathbf{Q}[G]}(\mathfrak{M}_\Theta) \cong \mathrm{End}_{\mathbf{Q}[H]}(\mathfrak{M}_\Xi)$,
where $\mathfrak{M}_\Theta$ (resp. $\mathfrak{M}_\Xi$) is the irreducible $\mathbf{Q}[G]$-module (resp. $\mathbf{Q}[H]$-module) affording the representation Θ (resp. Ξ).
(iii) *If G is a p-group, then*
H/N is isomorphic to a cyclic group for $p \neq 2$;
$H/N \cong C_n$ $(n \geq 0)$, Q_n $(n \geq 2)$, D_n $(n \geq 3)$ or SD_n $(n \geq 3)$ for $p = 2$.

Next we will refine Corollary 2.3. For $\chi \in \mathrm{Irr}(G)$, $A(\chi, \mathbf{Q})$ denotes the simple component of the group algebra $\mathbf{Q}[G]$, which corresponds to χ.

Theorem 3.5 (cf. [8, Theorem 2.4]). *Let G be a finite group and H a subgroup of G. Let $\varphi \in \mathrm{Irr}(H)$ such that $\varphi^G \in \mathrm{Irr}(G)$ and $\mathbf{Q}(\varphi^G)=\mathbf{Q}(\varphi)$. Put $\chi = \varphi^G$ and $K = \mathbf{Q}(\chi)=\mathbf{Q}(\varphi)$. Let $\tilde{X}$ and $\tilde{\Phi}$ be the irreducible rational representations of G and H with characters $\Omega(\chi)$ and $\Omega(\varphi)$, respectively. Let $\mathfrak{M}$ and $\mathfrak{N}$ be the irreducible $\mathbf{Q}[G]$- and $\mathbf{Q}[H]$-modules which afford the representations $\tilde{X}$ and $\tilde{\Phi}$, respectively. Then*

$$\Omega(\chi) = \Omega(\varphi)^G, \ \tilde{X} = \tilde{\Phi}^G, \ \mathfrak{M} \cong \mathfrak{N} \otimes_{\mathbf{Q}[H]} \mathbf{Q}[G],$$
$$\mathbf{S} = \mathrm{End}_{\mathbf{Q}[G]}(\mathfrak{M}) = \mathrm{End}_{\mathbf{Q}[H]}(\mathfrak{N}),$$
$$m = m_{\mathbf{Q}}(\chi) = m_{\mathbf{Q}}(\varphi) = \sqrt{|\mathbf{S} : K|},$$
$$A(\chi, \mathbf{Q}) \cong M_{\chi(1)/m}(\mathbf{S}), \ \ A(\varphi, \mathbf{Q}) \cong M_{\varphi(1)/m}(\mathbf{S}), \text{ and}$$

$\mathfrak{M}$ is imprimitively induced from $\mathfrak{N}$.

The proof of Theorem 3.5 uses the Brauer–Witt theorem (cf. [21, p.31]).

We now define the concept of imprimitivity for complex irreducible characters.

Definition 3.6. Let G be a finite group and $\chi \in \mathrm{Irr}(G)$. χ is called *imprimitive*, if there exists a proper subgroup H of G with $\varphi \in \mathrm{Irr}(H)$ such that $\chi = \varphi^G$ and $\mathbf{Q}(\chi)=\mathbf{Q}(\varphi)$. If χ is not imprimitive, then χ is called *primitive.*

Using again the Brauer–Witt theorem, we have

Theorem 3.7 ([8, Theorem 2.6]). *Let G be a finite group with $\chi \in \mathrm{Irr}(G)$. Let $\mathfrak{M}$ be an irreducible $\mathbf{Q}[G]$-module which affords the character $\Omega(\chi)$. Then $\mathfrak{M}$ is imprimitive if and only if χ is imprimitive.*

Corollary 3.8 ([8, Corollary 2.8]). *If a finite group G has a faithful, primitive, irreducible character χ, then every abelian normal subgroup of G is cyclic.*

Corollary 3.9 ([8, Corollary 2.9]). *Let G be a finite group and $\chi \in \mathrm{Irr}(G)$. Then there exist subgroups $H \triangleright N$ of G with the following properties:*

(i) *Every abelian normal subgroup of H/N is cyclic.*

(ii) *There exists $\psi \in \mathrm{Irr}(H)$ such that $\ker\psi = N$, $\chi = \psi^G$ and $\mathbf{Q}(\chi) = \mathbf{Q}(\psi)$.*

(Possibly, $H = G$ and $\psi = \chi$.)

Finally, we have

Theorem 3.10. *Let p be an odd prime and G a p-group. Let $\chi \in \mathrm{Irr}(G)$. Then there exist a subgroup H of G and a linear character ψ of H such that $\chi = \psi^G$, $\mathbf{Q}(\chi) = \mathbf{Q}(\psi)$, and so $\Omega(\chi) = \Omega(\psi^G) = \Omega(\psi)^G$.*

Theorem 3.11. *Let G be a 2-group and $\chi \in \mathrm{Irr}(G)$. Then there exist subgroups $H \triangleright N$ of G and $\varphi \in \mathrm{Irr}(H)$ such that $\chi = \varphi^G$, $\mathbf{Q}(\chi) = \mathbf{Q}(\varphi)$, $\ker\varphi = N$ and*

$$H/N \cong Q_n\ (n \geq 2),\ D_n\ (n \geq 3),\ SD_n\ (n \geq 3) \text{ or } C_n\ (n \geq 0).$$

4. Rational representations of p-groups

Recently, Ford [4] obtained a remarkable result about rational representations of p-groups:

Theorem 4.1 (Ford [4]). *Each irreducible rational representation of a finite p-group is induced from the faithful irreducible rational representation of degree $p-1$ on a section of order p.*

Let G be a p-group. Using Ford's theorem, let us attempt to determine all the irreducible rational representations of G. Then we must settle the following two problems:

1. Determine all the sections of order p of G.

2. For the faithful irreducible rational representation Ω on a section of order p, determine whether the induced representation Ω^G is irreducible or not.

In order to deal with the above second problem, we probably need to know all the irreducible complex characters χ and the irreducible rational characters $\Omega(\chi)$ of G. Since G is monomial, every irreducible complex representation X of G is obtained, when its character χ is obtained as an induced character. Hence we may say that determining the irreducible rational representations of G is more difficult than determining the irreducible complex representations of G.

Here we remark that Ford's theorem is essentially contained in the works of Roquette [16] and Rasmussen [15]. We define the group M_n of order 2^{n+1} as follows:

$$M_n = \langle a,\ b \mid a^{2^n} = b^2 = 1,\ bab^{-1} = a^{1+2^{n-1}}\rangle,\ n \geq 3.$$

Proposition 4.2 (Rasmussen [15]). *Let p be a prime. Let C'_{p^n} denote the cyclic group of order p^n. Let ξ denote the unique faithful irreducible rational character of $G = C'_{p^n},\ Q_n,\ D_n,\ SD_n$ or M_n. Then,*
(i) $\xi = 1^G_{\langle 1\rangle} - 1^G_{\langle a^{p^{n-1}}\rangle}$ *for* $G = C'_{p^n} = \langle a\rangle,\ a^{p^n} = 1,\ (n \geq 1)$.
(ii) $\xi = 1^G_{\langle 1\rangle} - 1^G_{\langle a^{2^{n-1}}\rangle}$ *for* $G = Q_n = \langle a,\ b \mid a^{2^n} = 1,\ b^2 = a^{2^{n-1}},\ bab^{-1} = a^{-1}\rangle,\ (n \geq 2)$.
(iii) $\xi = 1^G_{\langle b\rangle} - 1^G_{\langle a^{2^{n-1}},\ b\rangle}$ *for* $G = D_n\ (n \geq 2),\ SD_n\ (n \geq 3)$ *or* $M_n\ (n \geq 3)$, *where* $D_n = \langle a,\ b \mid a^{2^n} = b^2 = 1,\ bab^{-1} = a^{-1}\rangle$ *and* $SD_n = \langle a,\ b \mid a^{2^n} = b^2 = 1,\ bab^{-1} = a^{-1+2^{n-1}}\rangle$.

Observe that for $G = C'_{p^n}$ $(n \geq 1)$,

$$\xi = 1_{\langle 1\rangle}^G - 1_{\langle a^{p^{n-1}}\rangle}^G = \omega^G, \qquad \omega = 1_{\langle 1\rangle}^{\langle a^{p^{n-1}}\rangle} - 1_{\langle a^{p^{n-1}}\rangle},$$

where ω is the faithful irreducible rational character of the subgroup $\langle a^{p^{n-1}}\rangle$ of order p. The faithful irreducible rationl representation Ω with character ω is explicitly given in Proposition 2.1.

Similarly, for $G = Q_n$, we have

$$\xi = 1_{\langle 1\rangle}^G - 1_{\langle a^{2^{n-1}}\rangle}^G = \omega^G, \qquad \omega = 1_{\langle 1\rangle}^{\langle a^{2^{n-1}}\rangle} - 1_{\langle a^{2^{n-1}}\rangle},$$

where ω is the faithful irreducible character of the subgroup $\langle a^{2^{n-1}}\rangle$ of order 2, and the degree of ω is 1. So ω can be regarded as the faithful irreducible rational representation Ω of the subgroup $\langle a^{2^{n-1}}\rangle$.

For $G = D_n$, SD_n or M_n, observe that

$$\xi = 1_{\langle b\rangle}^G - 1_{\langle a^{2^{n-1}},\, b\rangle}^G = \omega^G, \quad \omega = 1_{\langle b\rangle}^{\langle a^{2^{n-1}},\, b\rangle} - 1_{\langle a^{2^{n-1}},\, b\rangle},$$

where ω is regarded as the faithful irreducible rational character on the section $\langle a^{2^{n-1}},\ b\rangle/\langle b\rangle$ of order 2. Consequently, ω can be regarded as the faithful irreducible rational representation Ω on the above section.

These excellent observation has been made by Ford [4]:

Proposition 4.3. *Let $G = C'_{p^n}$, Q_n, D_n, SD_n or M_n. Let Ξ denote the unique faithful irreducible rational representation of G. Then Ξ is, as is shown above, induced from the faithful irreducible rational representation Ω on a certain section of G of order p.*

Finally we remark that Theorem 4.1 follows immediately from Theorem 3.4 and Proposition 4.3.

For the rest of this section, we consider determining explicitly all irreducible rational representations of a finite p-group G, where p is an odd prime. Let $\chi \in \mathrm{Irr}(G)$. By Theorem 3.10, there exist a subgroup H and a linear character ψ of H such that

$$\chi = \psi^G \quad \text{and} \quad \mathbf{Q}(\chi) = \mathbf{Q}(\psi).$$

We call the above H and ψ, a *required pair* $\{H,\ \psi\}$ for $\chi \in \mathrm{Irr}(G)$. By Theorem 3.5, $\Omega(\chi) = \Omega(\psi^G) = \Omega(\psi)^G$. And the irreducible rational representation $\tilde{\Psi}$ with character $\Omega(\psi)$ is explicitly given by Proposition 2.1. Consequently, $\tilde{\Psi}^G$ is the irreducible rational representation of G with character $\Omega(\chi)$.

Thus, in order to obtain the irreducible rational representation $\tilde{X}$ with character $\Omega(\chi)$, it is enough to obtain a required pair $\{H, \psi\}$ for $\chi \in \mathrm{Irr}(G)$. It dose not seem that obtaining a required pair $\{H, \psi\}$ for $\chi \in \mathrm{Irr}(G)$ is an easy task.

In Iida–Yamada [9], we have carried out this for a metacyclic p-group G $(p \neq 2)$. Thus, irreducible rational representations of a metacyclic p-group have been explicitly obtained. For the details, see Theorems 4.5 and 4.6 of [9].

5. Types of 2-groups

Let G be a finite 2-group and $\chi \in \mathrm{Irr}(G)$. By Theorem 3.11 there exist subgroups $H \triangleright N$ of G and $\varphi \in \mathrm{Irr}(H)$ such that $\chi = \varphi^G$, $\mathbf{Q}(\chi) = \mathbf{Q}(\varphi)$, $\ker\varphi = N$ and

(5.1) $\quad H/N \cong Q_n\ (n \geq 2),\ D_n\ (n \geq 3),\ SD_n\ (n \geq 3)$ or $C_n\ (n \geq 0)$.

According to (5.1), χ is defined to be *of Q-type*, *D-type*, *SD-type* or *C-type*. We will call the above H and φ, a *required pair* $\{H, \varphi\}$ for $\chi \in \mathrm{Irr}(G)$. Note that the definition of a required pair for 2-groups is different from that for p-groups $(p \neq 2)$. By the results of the previous section, the irreducible rational representation with character $\Omega(\chi)$ is obtained if a required pair $\{H, \varphi\}$ for χ is explicitly obtained.

We now state the relation between the *type* of χ and the Frobenius–Schur indicator $\nu(\chi)$. Let G be a finite group and $\chi \in \mathrm{Irr}(G)$. Then,

$$\nu(\chi) = \frac{1}{|G|}\sum_{g\in G}\chi(g^2) = 0,\ \pm 1.$$

It is well known that the number of characters χ with $\nu(\chi) = \pm 1$ is equal to the number of real classes of G. Consequently, the number of characters χ with $\nu(\chi) = 0$ is equal to the number of non-real classes of G. The following problem is an old one (cf. [3, Problem 14]):

Problem 5.1. Describe the number of characters χ with $\nu(\chi) = 1$ purely in group theoretic terms.

For 2-groups G, we have:

Proposition 5.2. *Let G be a 2-group and $\chi \in \mathrm{Irr}(G)$.*
(i) *If χ is of Q-type, then $\nu(\chi) = -1$.*
(ii) *If χ is of D-type or χ is of C-type with $\mathbf{Q}(\chi) = \mathbf{Q}$, then $\nu(\chi) = 1$.*
(iii) *If χ is of SD-type, then $\nu(\chi) = 0$.*
(iv) *If χ is of C-type with $\mathbf{Q}(\chi) \neq \mathbf{Q}$, then $\nu(\chi) = 0$.*

Thus we have the following problem, which is a refinement of Problem 5.1 for 2-groups G:

Problem 5.3. Let G be a 2-group. Denote by n_Q, n_D, n_{SD} and n_C the numbers of irreducible complex characters χ such that χ is of Q-type, D-type, SD-type and C-type, respectively. Describe the numbers n_Q, n_D, n_{SD} and n_C purely in group theoretic terms.

The following problem is an easier version of Problems 5.1 and 5.3:

Problem 5.4. Let G be a 2-group with $\mathrm{FIrr}(G) \neq \varnothing$, where $\mathrm{FIrr}(G)$ is the set of faithful irreducible complex characters of G. Denote by n'_Q, n'_D, n'_{SD} and n'_C the numbers of faithful irreducible complex characters χ such that χ is of Q-type, D-type, SD-type and C-type, respectively. Describe the numbers n'_Q, n'_D, n'_{SD} and n'_C purely in group theoretic terms. In particular, for what 2-groups G, are all $\chi \in \mathrm{FIrr}(G)$ of the same type?

Let G be a metacyclic 2-group and $\chi \in \mathrm{Irr}(G)$. We ask of what type is χ. We may assume that χ is faithful. In Iida–Yamada [8], we have completely determined the type of any $\chi \in \mathrm{FIrr}(G)$ for a metacyclic 2-group G:

Theorem 5.5 (Iida–Yamada [8]).
(i) *For $\chi \in \mathrm{FIrr}(Q_n)$, $(n \geq 2)$, χ is of Q-type.*
(ii) *For $\chi \in \mathrm{FIrr}(D_n)$, $(n \geq 3)$, χ is of D-type.*
(ii)′ *For $\chi \in \mathrm{FIrr}(D_2)$, χ is of C-type.*
(iii) *For $\chi \in \mathrm{FIrr}(SD_n)$, $(n \geq 3)$, χ is of SD-type.*
(iv) *For $\chi \in \mathrm{FIrr}(M_n)$, $(n \geq 3)$, χ is of C-type.*

Theorem 5.6 (Iida–Yamada [8]). *Let G be a metacyclic 2-group with $\mathrm{FIrr}(G) \neq \varnothing$. Suppose that $G \neq Q_n$ $(n \geq 2)$, D_n $(n \geq 3)$, SD_n $(n \geq 3)$. Then every $\chi \in \mathrm{FIrr}(G)$ is of C-type except*

$$G = \langle a,\ b \mid a^{2^n} = b^{2^t} = 1,\ bab^{-1} = a^{-1+2^{n-t}} \rangle, \quad (2 \leq t \leq n-2).$$

For the above groups G, every $\chi \in \mathrm{FIrr}(G)$ is of SD-type.

For Problem 5.4, we have

Theorem 5.7 (Iida–Yamada [8]). *Let G be a metacyclic 2-group with $\mathrm{FIrr}(G) \neq \varnothing$. Then all $\chi \in \mathrm{FIrr}(G)$ are of the same type.*

As for irreducible rational representations of a metacyclic 2-group G, we need to obtain a required pair $\{H, \varphi\}$ for each $\chi \in \mathrm{Irr}(G)$. Again we may assume χ is faithful. In Iida–Yamada [10] we have explicitly given a required pair $\{H, \varphi\}$ for each $\chi \in \mathrm{FIrr}(G)$ of a metacyclic 2-group G. Thus we have determined all irreducible rational representations of a metacyclic 2-group (for the details, see Theorems 16–20 of Iida–Yamada [10]).

6. Extensions of 2-groups

We have seen that the groups Q_n ($n \geq 2$), D_n ($n \geq 3$) and SD_n ($n \geq 3$) are most fundamental in the theory of representations of 2-groups. Besides, they have the following remarkable properties, which characterize them.

Theorem 6.1. *Let G be a nonabelian 2-group of order 2^{n+1}. Then the following conditions are equivalent.*

(1) $G \cong Q_n$ *or* D_n *or* SD_n.

(2) *The number of involutions of G is* $\equiv 1 \pmod 4$. (Thompson) (*See* [11, (4.9)] *or* [14, Theorem 6.2].)

(3) $[G : G'] = 4$. (Taussky) (*See* [6, Satz III, 11.9(a)].)

(4) *G has class n.* (*See* [6, Satz III, 11.9(b)].)

(5) *G has no noncyclic abelian normal subgroup.* (Roquette) (*The dihedral group D_2 of order 8 is the only exception among Q_n, D_n, SD_n.*)

(6) *The Artin exponent of G is not equal to 2^n.* (Lam)

We would like to find properties, which characterize the groups Q_n, D_n and SD_n in terms of irreducible complex characters. In Iida–Yamada [7], we have proved:

Theorem 6.2 (Iida–Yamada [7]). *Let $n \geq 3$. Then there exists one and only one group $G \supset Q_n$ (resp. $G \supset D_n$) such that $[G : Q_n] = 2$ (resp. $[G : D_n] = 2$) and $\varphi^G \in \mathrm{Irr}(G)$, where $\varphi \in \mathrm{FIrr}(Q_n)$ (resp. $\varphi \in \mathrm{FIrr}(D_n)$). If $\varphi \in \mathrm{FIrr}(SD_n)$, then there exist only two groups G_1 and G_2 ($G_1 \not\cong G_2$) such that $G_i \supset SD_n$, $[G_i : SD_n] = 2$ and $\varphi^{G_i} \in \mathrm{Irr}(G_i)$, ($i = 1, 2$).*

This result originates from the work of Yamada [22], and might contribute to characterize Q_n, D_n, SD_n in terms of irreducible complex characters. In fact, let G_0 be an arbitary 2-group with $\mathrm{FIrr}(G_0) \neq \emptyset$ and $G_0 \neq Q_n$, D_n, SD_n. It seems that usually, there are many groups G such that $[G : G_0] = 2$ and $\varphi^G \in \mathrm{Irr}(G)$, where $\varphi \in \mathrm{FIrr}(G_0)$. But this problem relies very much on future investigation.

Finally we mention that the groups G which satisfy the following conditions are completely determined in Iida–Yamada [7]:
(i) G is an extension of $G_0 = Q_n$, D_n or SD_n;
(ii) $[G : G_0] = 2$ or 4;
(iii) $\varphi^G \in \mathrm{Irr}(G)$ for $\varphi \in \mathrm{FIrr}(G_0)$.
For the details, see Theorems 3–6 of [7].

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ON FINITELY GENERATED P.I. ALGEBRAS WITHOUT INFINITE SET OF CENTRAL ORTHOGONAL IDEMPOTENTS

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As was shown by Posner [9], every prime P.I. algebra has a simple Artinian classical algebra of two-sided quotients, which turned out to be the algebra of central quotients (e.g. [11, Theorem 3.3]). On the other hand, semiprime P.I. algebras do not necessarily have classical algebras of right (or left) quotients ([3, Theorem 8], [11, Example 5.7]). A theorem of Rowen [12, Theorem 1.7.34, p. 58] runs as follows: A semiprime P.I. algebra R satisfies ACC on annihilator ideals if and only if the algebra of central quotients of R is semisimple Artinian. When this is the case, one can easily see that the algebra of central quotients of R is a classical algebra of two-sided quotients of R. In this paper we study algebras of central quotients of finitely generated P.I. algebras without infinite set of central orthogonal idempotents.

Throughout this paper "ring" means "associative ring with identity element" and "algebra" means "associative algebra over a commutative ring." "Ideal" means "two-sided ideal." For a ring R, $J(R)$ and $P(R)$ denote the Jacobson radical and the prime radical of R respectively. $E(R)$ denotes the set of all idempotents in R. For any non-empty subset X of a ring R, we denote the right (resp. left) annihilator of X in R by

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$r(X)$ (resp. $\ell(X)$).

Let K be a commutative ring and let R be a K-algebra with center C. Let S be a multiplicatively closed subset of C. The localization of R by S is denoted by R_S, which is a K-algebra. If S consists of all non-zero-divisors of C, R_S is called the algebra of central quotients of R and is written $Q_C(R)$. For details of localization, see [11, §3] or [12, §1.7]. Let R be a semiprime P.I. algebra with center C. If A is a nonzero ideal of R, then $A \cap C \neq 0$ ([11, Theorem 2.10] or [12, Theorem 1.6.27, p. 47]). Hence if c is a non-zero-divisor of C, then c is a non-zero-divisor in R. For any ideal A of a semiprime ring R, we have $\ell(A) = r(A)$, which is called an annihilator ideal in R. As is easily seen, every annihilator ideal in a semiprime ring is a semiprime ideal.

Let R be a finitely generated semiprime P.I. algebra over a commutative Noetherian ring K. Then R has a classical algebra of two-sided quotients Q of R, which is semisimple Artinian [10, Theorem 2.5, p. 108]. Noting that a finitely generated P.I. algebra over a commutative Noetherian ring satisfies ACC on semiprime ideals [10, Corollary 2.2, p. 106], we see that Q is the algebra of central quotients of R.

A ring R is called a right (resp. left) p.p. ring if every principal right (resp. left) ideal of R is projective. R is called a generalized right (resp. left) p.p. ring if for any element a of R, there is a positive integer n such that a^nR (resp. Ra^n) is projective.

We begin with the following

Proposition 1. Let R be a semiprime P.I. algebra with center C and let $Q = Q_C(R)$, the algebra of central quotients of R. Suppose that C is a p.p. ring without infinite set of orthogonal idempotents. Then Q is semisimple Artinian and hence Q is a classical algebra of two-sided quotients of R. If moreover R is right hereditary, then R is right Noetherian.

Proof. By Small [4, Lemma 8.4, p. 112], C satisfies ACC on annihilator ideals. Hence by [12, Theorem 1.7.34, p. 58], Q is semisimple Artinian. The last assertion follows from Sandomierski ([4, Corollary 8.25, p. 124] or [13, Corollary 2]).

Next we state and prove our main result.

Theorem 1. Let R be a finitely generated P.I. algebra over its center C and let $Q = Q_C(R)$. Suppose that C is a generalized p.p. ring without infinite set of orthogonal idempotents. Then Q/P(Q) has a classical ring of two-sided quotients which is semisimple Artinian. In particular, Q has no infinite sets of orthogonal idempotents.

Proof. Let K be a classical ring of quotients of C. Since C is a generalized p.p. ring, K is a π-regular ring and E(K) = E(C) [6, Theorem 2]. Thus K has no infinite sets of orthogonal idempotents. Being a reduced π-regular ring, K/P(K) is a von Neumann regular ring without infinite set of orthogonal idempotents. Hence by [8, Theorem 2.1], K/P(K) is Artinian. (In fact, K/P(K) is a finite direct sum of fields.) The center of Q is K [7, Lemma 1.1] and Q is a finitely generated P.I. algebra over K. Hence Q/P(Q) is a finitely generated semiprime P.I. algebra over K + P(Q)/P(Q), which is isomorphic to K/P(K). The conclusion now follows from [10, Theorem 2.5, p. 108].

Remark. As was noted previously, the above classical ring of two-sided quotients of Q/P(Q) is the algebra of central quotients of Q/P(Q).

The following corollary contains [5, Lemma 2].

Corollary 1. Let R be a finitely generated P.I. algebra over its center C and let $Q = Q_C(R)$. Suppose that R is a right (or left) p.p. ring such that C has no infinite sets of orthogonal idempotents. Then Q/P(Q) has a classical ring of two-sided quotients which is semisimple Artinian. In particular, R has no infinite sets of orthogonal idempotents.

Proof. If R is a right p.p. ring, then C is a p.p. ring and every non-zero-divisor of C is a non-zero-divisor in R [2, Corollary 8.2]. Hence R can be embedded in Q and our assertions are direct consequences of Theorem 1.

As an application we obtain the following

Corollary 2. Let R be a finitely generated P.I. algebra over its center C and suppose that C has no infinite sets of orthogonal idempotents. If R is right semihereditary, then R is left semihereditary.

Proof. For every positive integer n, M(n,R), the ring of $n \times n$ matrices over R, is a right p.p. ring ([4, Theorem 8.17, p. 120] or [14, Proposition]) which is a finitely generated algebra over its center C. M(n,R) is a P.I. algebra [10, Chap. III, Theorem 3.2, p. 71] and hence it has no infinite sets of orthogonal idempotents (Corollary 1). By Small ([4, Corollary 8.19, p. 121] or [14, Theorem 3]), R is left semihereditary.

Now we consider the special case in which R is a finitely generated module over its center.

Theorem 2. Let R be a ring which is a finitely generated module over its center C. Let $Q = Q_C(R)$. If C is a generalized p.p. ring, then $J(Q) = P(Q)$ and $Q/J(Q)$ is a von Neumann regular ring. In particular, if C is a p.p. ring, then $J(Q)$ is nilpotent.

Proof. Let K be a classical ring of quotients of C. By hypothesis K is a π-regular ring and Q is a finitely generated K-module. Hence $Q/P(Q)$ is a module-finite semiprime algebra over $K/P(K)$, which is a von Neumann regular ring. By [1, Theorem 1], $Q/P(Q)$ is von Neumann regular and hence $J(Q) = P(Q)$. If C is a p.p. ring, then K is a von Neumann regular ring [2, Lemma 3.1]. Hence by [15, Proposition 2.2], $J(Q)$ is nilpotent.

Corollary 3. Let R be a ring which is a finitely generated module over its center C. Let $Q = Q_C(R)$.

(1) If C is a generalized p.p. ring without infinite set of orthogonal idempotents, then $Q/J(Q)$ is an Artinian ring.

(2) If C is a p.p. ring without infinite set of orthogonal idempotents, then Q is a semiprimary ring.

Proof. (1) follows from Theorem 1 and Theorem 2. (2) follows from (1) and Theorem 2.

Finally we consider the center of a finitely generated hereditary P.I. algebra.

Proposition 2. Let R be a right hereditary ring which is a finitely generated P.I. algebra over its center C. Suppose that C has no infinite sets of orthogonal idempotents. Then C is a finite direct sum of Dedekind domains.

Proof. By hypothesis there are orthogonal primitive idempotents $e_1,\ldots,e_n$ in C such that $e_1 + \ldots + e_n = 1$. Then we have $C = Ce_1 \oplus \ldots \oplus Ce_n$, which is a Dedekind ring [5, §2, Corollary to Theorem 2]. Each Ce_i is a hereditary ring which has no nontrivial idempotents; hence Ce_i is a Dedekind domain. This completes the proof.

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ON A GENERALIZATION OF MORITA DUALITY AND ITS APPLICATION

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1. Introduction

It seems that there are two ways to generalize Morita duality. The one way is done by defining a Morita duality between categories (especially Grothendieck categories), for example [1], [3] and [5]. The another way is, by observing that a Morita dulity is actually a duality between full subcategories of module categories, to consider a duality between full subcategories of module categories in order that the original Morita duaity can be obtained as a special case of it. In this report, first we consider the latter case and then we consider the former case and also the relation between them.

A full subcategory of an abelian category is called strongly exact if it is closed under finite products, subobjects and quotient objects. Also a full subcategory of an abelian category is called Giraud if the inclusion functor has a kernel preserving left adjoint.

Now let R and S be rings with identity and let R-Mod and Mod-S denote the categories unitary left R- and right S-modules, respectively. Let $\mathcal{A}$ and $\mathcal{B}$ be strongly exact subcategories of R-Mod and Mod-S such that $\mathcal{A} \ni {}_RR$ and $\mathcal{B} \ni S_S$. It is obvious that if there exists a duality between $\mathcal{A}$ and $\mathcal{B}$ then we can say that there exists a Morita duality between R and S. As a generalization of this we say that there exists a localized Morita duality between $\mathcal{A}$ and $\mathcal{B}$ if there exists a duality between Giraud subcategories of $\mathcal{A}$ and $\mathcal{B}$. A localized Morita duality is given by a bimodule. In section 1 we give a necessary and sufficient condition in order that a bimodule induces a localized Morita duality.

In fact we can say that a localized Morita duality is a special type of Morita duality between Grothendieck categories. Let $\mathcal{A}$ and $\mathcal{B}$ be Grothendieck categories.

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Suppose there exists a duality $\mathcal{C} \rightleftarrows \mathcal{D}$, where $\mathcal{C}$ and $\mathcal{D}$ are strongly exact subcategories of $\mathcal{A}$ and $\mathcal{B}$ and moreover each of them contains a generator for $\mathcal{A}$ and $\mathcal{B}$. Then we call such a duality a strong Morita duality between $\mathcal{A}$ and $\mathcal{B}$. In section 2 we will see that a strong Morita duality induces a localized Morita duality.

If R is a QF-3 ring with minimal faithful modules Re and fR then it is well known that the bimodule ${}_{fRf}fRe_{eRe}$ defines a Morita duality. In section3 we generalize this fact.

In section 4 we give an application of section 3 and some examples.

2. Localized Morita duality

First we note that if an abelian category $\mathcal{A}$ is given then there exists a bijective correspondence between Giraud subcategories of $\mathcal{A}$ and strongly hereditary torsion theories in $\mathcal{A}$ ([14]).

THEOREM 2.1 [11]. *Let R and S be rings with identity. Let $\mathcal{A}$ and $\mathcal{B}$ be strongly exact subcategories of R-Mod and Mod-S such that $\mathcal{A} \ni {}_RR$ and $\mathcal{B} \ni S_S$. Moreover suppose $\mathcal{L}$ and $\mathcal{L}'$ are Giraud subcategories of $\mathcal{A}$ and $\mathcal{B}$, respectively. Then there exists a duality $\mathcal{L} \rightleftarrows \mathcal{L}'$ if and only if there exists a bimodule ${}_RU_S$ which satisfies the following conditions.*
(i) For all $X \in \mathcal{A}$ and $Y \in \mathcal{B}$, $Hom_R(X,U) \in \mathcal{B}$ and $Hom_S(Y,U) \in \mathcal{A}$ hold.
(ii) For all $X \in \mathcal{A}$ and $Y \in \mathcal{B}$, the canonical homomorphisms

$$\eta_X : X \to Hom_S(Hom_R(X,U),U) \quad and$$

$$\eta'_Y : Y \to Hom_R(Hom_S(Y,U),U)$$

give localizations with respect to the torsion theories corresponding to $\mathcal{L}$ and $\mathcal{L}'$, respectively.

Again let $\mathcal{A}$ be an abelian category. An object $A \in \mathcal{A}$ is said to be QF-3″ if for a monomorphism $f : X' \to X$ in $\mathcal{A}$ $Hom_{\mathcal{A}}(X',A) = 0$ whenever $Hom(f,A) = 0$.

LEMMA 2.2. *Let $\mathcal{A}$ be a strongly exact subcategory of Mod-R and $W \in \mathcal{A}$. Let $t(X) = \cap\{Ker f \mid f \in Hom_R(X,W)\}$ for $X \in \mathcal{A}$. Then the following assertions are equivalent.*
(1) W is a QF-3″ object in $\mathcal{A}$.
(2) For any $X \in \mathcal{A}$, $t(X) = \cap\{Ker f \mid f \in Hom_R(X, E(W_R))\}$.
(3) t is a left exact radical of $\mathcal{A}$.
(4) If $W \trianglelefteq Y$ is an essential extension of W_R and $Y \in \mathcal{A}$ then $t(Y) = 0$.

In the preceding lemma, if U is QF-3″ then the radical t corresponds to a hereditary torsion theory in $\mathcal{A}$. This torsion theory is said to be cogenerated by U.

Definition. A bimodule ${}_RU_S$ defines a localized Morita duality if there exist strongly exact subcategories $\mathcal{A} \subset R\text{-}Mod$ and $\mathcal{B} \subset Mod\text{-}S$ which satisfy the following conditions.
(i) ${}_RR, {}_RU \in \mathcal{A}$ and $S_S, U_S \in \mathcal{B}$.
(ii) For all $X \in \mathcal{A}$ and $Y \in \mathcal{B}$, $Hom_R(X,U) \in \mathcal{B}$ and $Hom_S(Y,U) \in \mathcal{A}$ hold.
(iii) For all $X \in \mathcal{A}$ and $Y \in \mathcal{B}$ the canonical homomorphisms

$$\eta_X : X \to Hom_S(Hom_R(X,U),U) \quad \text{and}$$

$$\eta'_Y : Y \to Hom_R(Hom_S(Y,U),U)$$

are localizations with respect to the torsion theories cogenerated by ${}_RU$ in $\mathcal{A}$ and U_S in $\mathcal{B}$, respectively.

${}_RM$ is said to be a QF-3″ module if it cogenerates every finitely ${}_R(M \oplus R)$-generated submodule of $E({}_RM)$. Also a ring R is said to be left QF-3″ if ${}_RR$ is a QF-3″ module ([5]).

THEOREM 2.3 [11]. *A bimodule ${}_RU_S$ defines a localized Morita duality if and only if the following conditions are satisfied.*
(i) ${}_RU$ and U_S are QF-3″ modules.
(ii) ${}_RU$ and U_S are divisible with respect to the hereditary torsion theories cogenerated by $E({}_RU)$ and $E(U_S)$, respectively.
(iii) The canonical homomorphisms $R \to End_S(U)$ and $S \to End_R(U)$ are localizations with respect to the torsion theories described in (ii).

3. Morita duality for Grothendieck categories

PROPOSITION 3.1 [12]. *Let $\mathcal{G}$ be a Grothendieck category with a generator U and $R = End_{\mathcal{G}}(U)$. Let $\mathcal{A}$ be a strongly exact subcategory of $\mathcal{G}$ such that $U \in \mathcal{A}$. Then $\mathcal{A}$ is category equivalent to a Giraud subcategory of a strongly exact subcategory of $Mod\text{-}R$ which contains R.*

Now let $\mathcal{G}_1$ and $\mathcal{G}_2$ be Grothendieck categories. suppose there exists a strong Morita duality $\mathcal{D}_1 \rightleftarrows \mathcal{D}_2$ between $\mathcal{G}_1$ and $\mathcal{G}_2$. Let $U_i \in \mathcal{D}_i$ be generators for $\mathcal{G}_i$ and let $R_i = End_{\mathcal{G}_i}(U_i)$. By the preceding proposition there exist strongly exact subcategories $\mathcal{A}_i \subset Mod\text{-}R_i$ such that $R_i \in \mathcal{A}_i$ and Giraud subcategories $\mathcal{L}_i$ of $\mathcal{A}i$

such that $\mathcal{L}_i$ are category equivalent to $\mathcal{D}_i$. Thus we have a diagam

$$\begin{array}{ccccccc}
Mod-R_1 & & & & & & Mod-R_2 \\
\cup & & & & & & \cup \\
\mathcal{A}_1 & & \mathcal{G}_1 & & \mathcal{G}_2 & & \mathcal{A}_2 \\
\uparrow\downarrow & & \cup & & \cup & & \uparrow\downarrow \\
\mathcal{L}_1 & \sim & \mathcal{D}_1 & \underset{D_2}{\overset{D_1}{\rightleftarrows}} & \mathcal{D}_2 & \sim & \mathcal{L}_2.
\end{array}$$

We may identify $Mod\text{-}R_1$ as ${R_1}^{op}\text{-}Mod$. Let $V = Hom_{\mathcal{G}_2}(U_2, D_1(U_1))$. Then V is canonically a right R_2-module. But since D_i is contravariant V is canonically a left ${R_1}^{op}$-module. Thus if we look at the proof of Theorem 2.1 we know that ${}_{R_1^{op}}V_{R_2}$ defines a localized Morita duality between $\mathcal{A}_1$ and $\mathcal{A}_2$.

Now we just knew that a strong Morita duality between Grothendieck categories is categorically equivalent to a localized Morita duality. Does the converse hold? The answer is affirmative. In fact every localized Morita duality is a strong Morita duality between Grothendieck categories (see [12, section 3]).

4. Duality induced by ${}_SU \otimes_R V_T$

A Morita context is a couple of bimodules $<{}_R U_S, {}_S V_R>$ together with bilinear maps $(\, , \,) : U \otimes_S V \to R$ and $[\, , \,] : V \otimes_R U \to S$ such that the associativities $(u, v)u' = u[v, u']$ and $[v, u]v' = v(u, v')$ hold.

Throughout this section we fix a 4-tuple of bimodules $< {}_R\Gamma_S, {}_S U_R, {}_R V_T, {}_T\Delta_T >$ which satisfies the following conditions.
(i) $< {}_R\Gamma_S, {}_S U_R >$ forms a Morita context with trace ideals ${}_RI_R$ and ${}_SJ_S$, and $UI = U$ and $Ju \ni u$ $(\forall u \in U)$ hold.
(i)$'$ $< {}_R V_T, {}_T \Delta_R >$ forms a Morita context with trace ideals ${}_RK_R$ and ${}_TL_T$, and $KV = V$ and $vL \ni v$ $(\forall v \in V)$ hold.
(ii) U_R is a QF-3$'$ module with $E(U_R)$-$dom.dim.U_R \geqslant 2$.
(ii)$'$ ${}_RV$ is a QF-3$'$ module with $E({}_RV)$-$dom.dim.{}_RV \geqslant 2$.
(iii) $Ann_U K = 0 = Ann_V I$, where $Ann_U K = \{u \in U \mid uK = 0\}$ and $Ann_V I = \{v \in V \mid Iv = 0\}$.

Let $Gen({}_SU)$ be the full subcategory of S-Mod consisting of all ${}_SU$-generated modules. $Gen(V_T)$ is defined similarly. Then $Gen({}_SU) = \{{}_SX \mid JX = X\}$ and $Gen(V_T) = \{Y_T \mid YL = Y\}$ hold. Moreover they are Grothendieck categories.

LEMMA 4.1 [12]. *$U \otimes_R V$ is a QF-3$''$ object in both of $Gen({}_SU)$ and $Gen(V_T)$.*

For $X \in S\text{-}Mod$ and $Y \in Mod\text{-}T$ let $t_1(X) = JX$ and $t_2(Y) = YL$. Then by the conditions (i) and (i)$'$ t_1 and t_2 are exact radicals. Let $D_1 : Gen({}_SU) \rightleftarrows$

$Gen(V_T) : D_2$ be functors defined via $D_1(X) = t_2(Hom_S(X, U \otimes_R V))$ and $D_2(Y) = t_1(Hom_T(Y, U \otimes_R V))$.

LEMMA 4.2 [12]. *$D_1 : Gen(_SU) \rightleftarrows Gen(V_T) : D_2$ are (right) adjoint functors to each other.*

Let $\eta : 1_{Gen(_SU)} \to D_2 D_1$ and $\eta' : 1_{Gen(V_T)} \to D_1 D_2$ be adjunctions. Since $U \otimes_R V$ is QF-3$''$ in $Gen(_SU)$ and $Gen(V_T)$ it cogenerates hereditary torsion theories $(\mathcal{T}, \mathcal{F})$ in $Gen(_SU)$ and $(\mathcal{T}', \mathcal{F}')$ in $Gen(V_T)$. The following is the most important result.

THEOREM 4.3 [12]. *Let $\mathcal{A} = \{X \in Gen(_SU) \mid \eta_X$ is a localization with respect to $(\mathcal{T}, \mathcal{F})\}$ and $\mathcal{B} = \{Y \in Gen(V_T) \mid \eta'_Y$ is a localization with respect to $(\mathcal{T}', \mathcal{F}')\}$. Then $\mathcal{A}$ and $\mathcal{B}$ are strongly exact subcategories of $Gen(_SU)$ and $Gen(V_T)$ such that $U \in \mathcal{A}$ and $V \in \mathcal{B}$.*

Next let $\mathcal{L} = \{X \in Gen(_SU) \mid \eta_X$ is an isomorphism $\}$ and $\mathcal{L}' = \{Y \in Gen(V_T) \mid \eta'_Y$ is an isomorphism $\}$. Then $\mathcal{L}$ and $\mathcal{L}'$ are Giraud subcategories of $\mathcal{A}$ and $\mathcal{B}$, respectively. Moreover the duality between $\mathcal{L}$ and $\mathcal{L}'$ induced by D_1 and D_2 is a strong Morita duality between Giraud subcategories of $Gen(_SU)$ and $Gen(V_T)$.

5. Application and examples

The following is a result of combinations of duality and equivalences.

THEOREM 5.1 [12]. *Let $<_R\Gamma_S, {}_SU_R, {}_RV_T, {}_T\Delta_R>$ be the same as section 1. Then ${}_REnd_S(U)_R$ $(\simeq End_T(V))$ defines a localized Morita duality.*

COROLLARY 5.2 [12]. *Let the situation be the same as Theorem 2.3. Then the following assertions are equivalent.*

(1) *U_R is faithful.*
(2) *$End_S(U)$ is a maximal left quotient ring of R.*
(3) *$End_S(U)$ is a maximal right quotient ring of R.*
(4) *$End_S(U)$ is a maximal (left and right) quotient ring of R.*
(1)$'$ *${}_RV$ is faithful.*
(2)$'$ *$End_T(V)$ is a maximal right quotient ring of R.*
(3)$'$ *$End_T(V)$ is a maximal left quotient ring of R.*
(4)$'$ *$End_T(V)$ is a maximal quotient ring of R.*

THEOREM 5.3. *Let the situation be the same as Theorem 5.1. Then ${}_REnd_S(U)_R$ defines a Morita duality if and only if the following conditions are*

satisfied.
(i) $I = K = R$.
(ii) $_SU$ and V_T contain each of their simple factors.
If these conditions hold then $R \simeq End_S(U)$ holds. Hence R is a PF-ring.

Proof. Suppose $_REnd_S(U)_R$ defines a Morita duality. In the diagram after Lemma 2.2 of [12], $_RR \in H(\mathcal{L})$ and moreover $H(\mathcal{L})$ is a strongly exact subcategory of $a^{-1}(H(\mathcal{A}))$ (hence of R-Mod) by assumption. Hence, in particular, $R/I \in {}_I\mathcal{L}$. This implies $I = R$. Similarly $K = R$ holds. Since $H(\mathcal{L})$ is a strongly exact subcategory of $H(\mathcal{A})$, $\mathcal{L}$ is also a strongly exact subcategory of $\mathcal{A}$. Hence by [12, Proposition 2.5] $_SU$ contains every simple factor of itself. Similarly V_T also contains every simple factor of itself.

Conversely suppose the conditions hold. Then we can say that $H(\mathcal{L})$ and $H'(\mathcal{L}')$ are strongly exact subcategories of R-Mod and Mod-R by [12, Proposition 2.5 and Proposition 2.7]. We note that U_R is faithful because $Ur = 0$ implies $0 = (\Gamma, Ur) = (\Gamma, U)r = Rr$. Since $_REnd_S(U) \in H(\mathcal{L})$ and $R \to End_S(U)$ is a monomorphism, $R \in H(\mathcal{L})$ holds. Therefore $R \simeq End_S(U)$ and $_RR_R$ defines a Morita duality.

For the rest of this section we give several examples. If a module U_R is given with $S = End_R(U)$ then $< {}_SU_R, {}_RU^*_S >$ forms canonically a Morita context, where $U^* = Hom_R(U, R)$. This context is called the derived context of U_R. From now on we mean every Morita context is a derived context.

Example 5.1 [12]. Suppose U_R and $_RV$ are both faithful projective injective. Then the 4-tuple $< {}_RU^*_S, {}_SU_R, {}_RV_T, {}_TV^*_R >$ satisfies the conditions $(i) \sim (iii)$ of section 3.

Example 5.2 [12]. Let R be left or right artinian and U_R projective injective. Then there exists a projective injective module $_RV$ such that the 4-tuple $< {}_RU^*_S, {}_SU_R, {}_RV_T, {}_TV^*_R >$ satisfies the conditions $(i) \sim (iii)$ of section 3. Thus in particular $_REnd_S(U)_R$ defines a localized Morita duality.

Example 5.3 [12]. Let K be a field and Λ be an infinite set. Let $R = K^\Lambda$ be the direct product of copies of K with index set Λ. Let $U_R = R^R$. Then U_R is injective but not projective. However $< {}_RU^*_S, {}_SU_R, {}_RU_T, {}_TU^*_R >$ satisfies the conditions of section 3.

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ON LEFT DUAL-BIMODULES

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In this talk we shall single out some of results from [6], [7] and [8] to explain the notion of left dual-bimodules.

A ring R with identity in which

$$A = \ell_R r_R(A) \text{ and } B = r_R \ell_R(B)$$

hold for every left ideal A and every right ideal B of R is called a dual ring. As is well-known, a QF-ring is nothing but a dual ring, when it is Artinian. Hence it is interesting to study dual rings.

From the 1930's dual rings have been investigated by many authors including R. Baer[2], M. Hall[3], I. Kaplansky[5] and L. A. Skornjakov[9]. Recently, C. R. Hajarnavis and N. C.Norton [4] have studied dual rings and pointed out that certain properties well-known for QF-rings are also seen to hold without the Artinian assumption. Motivated by the last paper, we have introduced in [6] the notion of left dual-bimodules and tried to give a module-theoretic characterization of dual rings.

Let R and S be rings with identity and ${}_RQ_S$ an (R,S)-bimodule. We shall call Q a left dual-bimodule if

$$\ell_R r_Q(A) = A \text{ and } r_Q \ell_R(Q') = Q'$$

for every left ideal A of R and every submodule Q' of Q_S. A right dual-bimodule is similarly defined. We shall call Q a dual-bimodule if it is a left dual-bimodule and is a right dual-bimodule as well.

Trivially a dual ring is a dual-bimodule. A bimodule which defines a Morita duality is a dual-bimodule. Furthermore, a dual-bimodule is a quasi-Frobenius bimodule in the sense of G. Azumaya[1].

1. We shall begin with a more concrete example of a dual-bimodule.

Example 1. Let p be a prime number and

$$R = \{a/b \in \underline{Q} \mid (a,b) = 1, p \nmid b\},$$

where $\underline{Q}$ is the field of rational numbers. Then R is a local ring with the unique maximal ideal Rp and nonzero proper ideals of R are exhausted by Rp^n, $n > 0$.

Let $Q = \underline{Q}/R$. Then Q is an (R,R)-bimodule and the only nonzero proper submodules of Q_R are those of the form $p^{-n}R/R$ for some $n > 0$. Since $r_Q(Rp^n) = p^{-n}R/R$ and $\ell_R(p^{-n}R/R) = Rp^n$. Q is a left dual-bimodule. Likewise it is a right dual-bimodule.

For some kind of rings, for example, semisimple rings and simple Artinian rings, left dual-bimodules have more specific forms. Indeed, if R is semisimple, for any R-module $_RQ$ with $S = \mathrm{End}(_RQ)$, $_RQ_S$ is a left dual-bimodule if and only if $_RQ$ is a cogenerator. Furthermore, every nonzero left R-module with its endomorphism ring S is a left dual-bimodule if and only if R is simple Artinian.

The notion of left dual-bimodules is closed under Morita equivalence, i.e. if $_RQ_S$ is a left dual-bimodule and if T is a ring equivalent to S via an equivalence H: mod-S $\to$ mod-T, then $_RH(Q)_T$ is also a left dual-bimodule. Hence, for any left dual-bimodule $_RQ_S$ and any $n > 0$, $_RQ^n_{(S)_n}$ is also a left dual-bimodule.

2. For an (R,S)-bimodule $_RQ_S$, consider the full subcategory $\underline{M}$ of R-mod of finitely generated Q-torsionless R-modules and the full subcategory $\underline{N}$ of mod-S whose objects are all the S-modules N such that there exists an exact sequence of the form $0 \to N \to Q^n \to Q^I$ for some $n > 0$ and a set I. Let (H', H'') be a pair of functors

$$H' = \mathrm{Hom}_R(-,Q)\colon \underline{M} \to \underline{N} \text{ and } H'' = \mathrm{Hom}_S(-,Q)\colon \underline{N} \to \underline{M}$$

and let $\lambda : R \to \mathrm{End}(Q_S)$ be the natural homomorphism.

Theorem 2. For an (R, S)-bimodule ${}_RQ_S$, consider the following conditions:

(1) Q_S is quasi-injective and λ is surjective.

(2) The pair (H', H'') defines a duality between $\underline{M}$ and $\underline{N}$.

Then (1) implies (2). If Q is a left dual-bimodule with ${}_RQ$ finitely generated, then (2) implies (1).

As is seen from the following example, (2) $\Rightarrow$ (1) is not true in general without the assumption that Q is a left dual-bimodule.

Example 3. Let R be the ring of 2×2 upper triangular matrices over a field and let $Q = {}_RR_R$. Then Q is neither a left dual-bimodule nor quasi-injective. However, the pair (H', H'') defines a duality between $\underline{M}$ and $\underline{N}$, since $\underline{M}$ is the full subcategory of R-mod of finitely generated R-modules and $\underline{N}$ is the full subcategory of mod-R of finitely generated R-modules.

Theorem 4. Let ${}_RQ_S$ be a left dual-bimodule. Then the following conditions are equivalent:

(1) Q_S is quasi-injective and λ is surjective.

(2) Every cyclic left R-module is Q-reflexive.

(3) Every finitely generated Q-torsionless left R-module is Q-reflexive.

Moreover, if each one of these conditions holds, then R is semiperfect.

If Q is a dual-bimodule and Q_S is Noetherian, then Q satisfies the equivalent condition of Theorem 4. However, without the assumption that Q is a left dual-bimodule, (3) $\Rightarrow$ (1) in Theorem 4 is not true in general, as is seen form Example 3. In that example R is left and right Artinian and is hereditary. Hence, every Q-torsionless left R-module is projective and thus every finitely generated Q-torsionless left R-module is Q-reflexive.

Theorem 5. Let ${}_RQ_S$ be a left dual-bimodule with Q_S quasi-injective and λ surjective. Then

$$\underline{M} = \{{}_RM \mid M \text{ is finitely generated } Q\text{-reflexive}\},$$
$$\underline{N} = \{N_S \mid N \text{ is finitely cogenerated } Q\text{-reflexive}\}.$$

Moreover, $\underline{N}$ coincides with

$$\{N_S \mid 0 \to N \to Q^n \text{ is exact for some } n > 0\}.$$

3. The following theorem points out that the linearly compactness of R is closely related to the existence of some kind of left dual-bimodules.

Theorem 6. A ring R is linearly compact if and only if there exists a left dual-bimodule ${}_RQ_S$ such that Q_S is linearly compact quasi-injective and λ is surjective.

A subcategory of the module category is called finitely closed if it is closed under submodules, factor modules and finite direct sums. Let $\underline{\mathrm{FG}}$ be the full subcategory of finitely generated left R-modules and let $\overline{\underline{\mathrm{FG}}}$ be the smallest one of the finitely closed subcategory containing $\underline{\mathrm{FG}}$. Likewise $\overline{\underline{\mathrm{N}}}$ denotes the smallest one of the finitely closed subcategory containing $\underline{\mathrm{N}}$. Then those left dual-bimodules mentioned in Theorem 6 can be characterized by means of a duality.

Theorem 7. Let ${}_RQ_S$ be a left dual-bimodule with ${}_RQ$ finitely generated and λ surjective. Then the following conditions are equivalent:

(1) Q_S is linearly compact and quasi-injective.

(2) The pair (H', H'') defines a duality between $\overline{\text{FG}}$ and $\overline{\text{N}}$.

(3) ${}_RQ$ is an injective cogenerator.

4. Finally we shall refer to the endomorphism ring of left dual-bimodules. First we shall remark the following:

Theorem 8. Let ${}_RQ_S$ be a left dual-bimodule with Q_S quasi-injective and λ surjective. Then (H', H'') defines a duality between the finitely generated left ideals of R and the finitely cogenerated factor modules of Q_S.

Using this theorem, we can give a necessary and sufficient condition for R to be left semihereditary or left coherent.

Theorem 9. Let ${}_RQ_S$ be a left dual-bimodule with Q_S quasi-injective and λ surjective. Then the following conditions are equivalent:

(1) R is left semihereditary.

(2) Every finitely cogenerated factor module of Q_S is Q-injective.

(3) For every finitely generated left ideal A of R, $A^* = \mathrm{Hom}_R(A, Q)$ is Q-injective.

In particular, if R is a dual ring, R is left semihereditary if and only if R is semisimple.

Theorem 10. Let ${}_RQ_S$ be a left dual-bimodule with Q_S quasi-injective and λ surjective. Then the following conditions are equivalent:

(1) R is left coherent.

(2) For every finitely cogenerated factor module Q/Q' of Q_S, there exist integers $n, m > 0$ such that

$$0 \to Q/Q' \to Q^n \to Q^m$$

is exact.

(3) For every finitely generated left ideal A of R, there exist integers $n, m > 0$ such that

$$0 \to A^* \to Q^n \to Q^m$$

is exact.

(4) For every integer $n > 0$ and every S-homomorphism $f : Q \to Q^n$ there exist an integer $m > 0$ and an S-homomorphism $g : Q^n \to Q^m$ such that

$$Q \to^f Q^n \to^g Q^m$$

is exact.

(5) For every integer $n > 0$ and every R-homomorphism f: ${}_RR^n \to {}_RR$ there exist an integer $m > 0$ and an R-homomorphism g: ${}_RR^m \to {}_RR^n$ such that

$$R^m \to^g R^n \to^f R$$

is exact.

For example, in Example 1, let $Q' = p^{-1}R/R$. Then ${}_{\bar{R}}Q'_{\bar{R}}$ is a left dual-bimodule, where $\bar{R} = R/Rp$. In this case, $\bar{R}$ is trivially left semihereditary and is left coherent.

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