PROCEEDINGS OF THE 27TH SYMPOSIUM ON RING THEORY

HELD AT YAMAGUCHI UNIVERSITY, YAMAGUCHI

JULY 23 - 25, 1994

EDITED BY

Yamaguchi University

Kiyoichi OSHIRO

1994

OKAYAMA, JAPAN

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PREFACE

The 27th Symposium on Ring Theory was held at Yamaguchi University, Japan, on July 23-25, 1994. Nearly one hundred participants attended the symposium.

This volume consists of the articles presented at the symposium.

The symposium and the proceedings were financially supported by the Scientific Research Grants of the Educational Ministry of Japan through the arrangements by Professor Yasuo Morita, Tohoku University and Professor Yuji Yoshino, Kyoto University.

I wish to express my hearty thanks to Dr. Hiroaki Komatsu of Okayama University for the publication of the proceedings.

Finally I would like to thank Professor Yukio Tsushima, Osaka City University, and staffs of the Department of Mathematics, Yamaguchi University, for their close cooperation.

November 1994

K. Oshiro

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Preface

PROCEEDINGS OF THE 27TH
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SOCLE EQUIVALENCES AND SOCLE DEFORMATIONS OF SELFINJECTIVE ALGEBRAS

Kunio YAMAGATA

In this note we report two results on structures of selfinjective algebras, which are proved in the joint papers [1] [2] with Skowroński. These results have an important application to the selfinjective artin algebras which have Auslander-Reiten components C with non-periodic generalized standard right stable full translation subquivers closed under successors in C.

All algebras are artin algebras. A selfinjective artin algebra A is said to be *socle equivalent to* a selfinjetive artin algebra B if the factor algebra $A/\mathrm{soc}(A)$ is isomorphic to $B/\mathrm{soc}(B)$ as an algebra. Let I be an ideal and e a residual identity of A/I, that is, e is an idempotent of A such that $1-e$ belongs to I and no proper summand of e is in I. Then the ideal I is called a *deforming* ideal if the ordinary quiver of A/I has no oriented cycles and $r_{eAe}(I) = eIe$ and $\ell_{eAe}(I) = eIe$, where $r_{eAe}(I)$ is the right annihilator of I in eAe, and $\ell_{eAe}(I)$ is the left annihilator of I in eAe.

Theorem 1 *If a selfinjective algebra A has a deforming ideal I then A is a socle equivalent to a split extension algebra of eAe/eIe by I, where e is a residual identity for A/I.*

For an algebra B we denote the repetitive algebra by $\hat{B}$, and by mod B the category of finitely generated right B-mdules.

Theorem 2 *Let A be a selfinjective algebra, I an ideal of A, $B = A/I$, and e a residual identity of B. Assume that the ordinary quiver of B has no oriented cyles, $IeI = 0$, Ie_B or ${}_BeI$ is an injective cogenerator in* mod B *or in* mod B^{op} *respectively. Then A is socle equivalent to $\hat{B}/G$ for some admissible infinite cyclic group G. If R is an algebraically closed field, then the algebra A is isomorphic to $\hat{B}/G$.*

References

[1] A. Skowroński and K. Yamagata, Socle deformations of selfinjective algebras, Mathematical Research Note **94-006**, 1994, University of Tsukuba, Tsukuba, Japan

[2] A. Skowroński and K. Yamagata, Selfinjective algebras and generalized standard components, in preparation

Institute of Mathematics, University of Tsukuba, Tsukuba, Japan

SOME EQUIVALENCES AND SOME DEFORMATIONS OF SELFINJECTIVE ALGEBRAS

Kunio YAMAGATA

In this note, we report two results on [illegible] of selfinjective algebras which are proved in the joint papers [1], [2] with Skowroński. [illegible]

[illegible]

Theorem 1. [illegible]

[illegible]

Theorem 2. [illegible]

References

[1] A. Skowroński and K. Yamagata, [illegible]

[2] A. Skowroński and K. Yamagata, [illegible]

Institute of Mathematics, University of Tsukuba, Tsukuba, Japan

PROCEEDINGS OF THE 27TH
SYMPOSIUM ON RING THEORY, 1994

FROBENIUS EXTENSIONS AND TILTING COMPLEXES

Jun-ichi Miyachi

Introduction

Various results on extensions of algebras and extensions of tilting functors were given in representation theory. Let $0 \to A \to \Lambda$ be an extension of a ring A. When is $T \otimes_A \Lambda$ a tilting Λ- module? What is the relation between $0 \to A \to \Lambda$ and $\mathrm{End}_A(T) \to \mathrm{End}_\Lambda(T \otimes_A \Lambda)$? Tachikawa and Wakamatsu showed that $T \otimes_A \Lambda$ is a classical tilting module, that is a tilting module of projective dimension one, under the condition that T is a classical tilting A-module and that Λ is a trivial extension algebra of A by $\mathrm{D}(tA)$, where tA is a trace ideal of T [14]. In case of $\Lambda = A \ltimes M$, we had the necessary and sufficient condition that $T \otimes_A \Lambda$ is a classical tilting module [6]. Assem and Marmaridis gave the necessary and sufficient condition that $T \otimes_A \Lambda$ is a classical tilting module, in case of split-by-nilpotent extensions of rings [1]. Miyashita introduced the notion of a tilting module of finite projective dimension, and considered a condition that $T \otimes_A \Lambda$ is a tilting module [8]. Hoshino showed the necessary and sufficient condition that $T \otimes_A \Lambda$ is a tilting module, in case of split extensions of rings [5]. Rickard introduced the notion of a tilting complex, and gave a sufficient condition that $T^{\bullet} \otimes_A^{L} \Lambda$ is a tilting complex, in case of $\Lambda = A \ltimes M$ [12], [13]. Also, Rickard showed that a finite

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The detail version of this paper has been submitted for publication elsewhere.

dimensional algebra derived equivalent to a symmetric algebra is itself symmetric, and that if A and B are derived equivalent algebras, then a trivial extension of A by itself and a trivial extension of B by itself are also derived equivalent (see [13] for details). These two cases are close to Frobenius extensions. In case that T is a finitely generated projective generator, Miyashita showed that if $0 \to A \to \Lambda$ is a Frobenius extension, then $0 \to \mathrm{End}_A(T) \to \mathrm{End}_\Lambda(T \otimes_A \Lambda)$ is a also Frobenius extension [7]. In this note, we study extensions of rings and conditions that make $T^{\bullet} \otimes_A^{L} \Lambda$ tilting complexes.

First, in case of split-extensions of rings, we give the necessary and sufficient condition that $T^{\bullet} \otimes_A^{L} \Lambda$ is a tilting complex. Next, in case of a Frobenius extension of a ring, we give a condition that $T^{\bullet} \otimes_A^{L} \Lambda$ is a tilting complex, and show that $0 \to \mathrm{End}_{D(\mathrm{Mod} A)}(T^{\bullet}) \to \mathrm{End}_{D(\mathrm{Mod}\Lambda)}(T^{\bullet} \otimes_A^{L} \Lambda)$ is a also Frobenius extension .

Throughout this note, we assume that all rings have unity and that all modules are unital. For a ring A , Mod A (resp., A-Mod, mod A, A-mod) is the category of right (resp., left, finitely presented right, finitely presented left) A-modules, and Proj-A (resp., A- Proj, $\mathcal{P}_A$, ${}_A\mathcal{P}$) is the category of right (resp., left, finitely generated right, finitely generated left) projective A-modules.

1. Preliminaries

Let $\mathcal{A}$ be an additive category, $C(\mathcal{A})$ the category of complexes of $\mathcal{A}$, $K(\mathcal{A})$ a homotopy category of $C(\mathcal{A})$, and $K^{+}(\mathcal{A})$, $K^{-}(\mathcal{A})$ and $K^{b}(\mathcal{A})$ full subcategories of $K(\mathcal{A})$ generated by the bounded below complexes, the bounded above complexes, and the bounded complexes, respectively. For an abelian category $\mathcal{A}$, a derived category $D(\mathcal{A})$ (resp., $D^{+}(\mathcal{A})$, $D^{-}(\mathcal{A})$, and $D^{b}(\mathcal{A})$ of $\mathcal{A}$ is a quotient of $K(\mathcal{A})$ (resp., $K^{+}(\mathcal{A})$, $K^{-}(\mathcal{A})$ and $K^{b}(\mathcal{A})$) by a multiplicative set of quasi-isomorphisms. For a ring A, Rickard defined a tilting complex $T^{\bullet}$ for A as follows,

(i) $T^{\bullet} \in K^{b}(\mathcal{P}_A)$,

(ii) $\mathrm{Hom}_{K(\mathrm{Mod}A)}(T^{\bullet}, T^{\bullet}[i]) = 0$ for all $i \neq 0$,

(iii) add$T^{\bullet}$, the additive category of direct summands of finite direct sums of $T^{\bullet}$, generates $K^{b}(\mathcal{P}_A)$ as a triangulated category.

Rickard also showed that (iii) can be replaced by

(iii)' For each non-zero object $X^{\bullet}$ of $K^{-}(\mathrm{Proj}\text{-}A)$, there is a some i such that $\mathrm{Hom}_{K(\mathrm{Mod}A)}(T^{\bullet}, X^{\bullet}[i]) \neq 0$.

Then there is a derived equivalent functor $D^{-}(\mathrm{Mod}B) \to D^{-}(\mathrm{Mod}A)$ which sends B to $T^{\bullet}$, where $B = \mathrm{End}_{K(\mathrm{Mod}A)}(T^{\bullet})$ (see [11] for details).

For a tilting complex $T^{\bullet}$ for A, we call $\mathrm{H}^0T^{\bullet}$ a tilting A-module provided that $\mathrm{H}^iT^{\bullet} = 0$ for all $i \neq 0$ ([4] and [8]). In this case, we have $T^{\bullet} \cong \mathrm{H}^0T^{\bullet}$ in $D^b(\mathrm{Mod}A)$. Furthermore, we call a tilting module T a classical tilting module if projective dimension of T is less than or equal to 1. In case that A is a finite dimensional algebra over a field k, there exist two-sided tilting complexes $\Delta^{\bullet}$ in $D^{b}(\mathrm{Mod}(B^{op}\otimes_k A))$ and $\nabla^{\bullet}$ in $D^{b}(\mathrm{Mod}(A^{op}\otimes_k B))$ such that $\Delta^{\bullet}\otimes^{L}_{A}\nabla^{\bullet} \cong {}_BB_B$ and $\nabla^{\bullet}\otimes^{L}_{B}\Delta^{\bullet} \cong {}_AA_A$ (see [13] for details).

Lemma 1.1. *Let* $\mathcal{A}$, $\mathcal{B}$ *and* $\mathcal{C}$ *be additive categories,* $\mathrm{H}, \mathrm{L}: \mathcal{A} \to \mathcal{C}$ *additive functors,* $\eta: \mathrm{H} \to \mathrm{L}$ *a morphism of functors and* $\mathrm{G}: \mathcal{A} \to \mathcal{B}$ *an additive functor which has the right adjoint* $\mathrm{F}: \mathcal{B} \to \mathcal{A}$. *Given* $X^{\bullet}, Z^{\bullet} \in \mathrm{C}(\mathcal{A})$, $Y^{\bullet} \in \mathrm{C}(\mathcal{C})$ *and* $U^{\bullet} \in \mathrm{C}(\mathcal{B})$, *the following results hold.*

(1) $\mathrm{Hom}^{\bullet}_{\mathcal{A}}(X^{\bullet}, Z^{\bullet}) \to \mathrm{Hom}^{\bullet}_{\mathcal{C}}(\mathrm{H}X^{\bullet}, \mathrm{H}Z^{\bullet})$ *induces an* $\mathrm{End}_{K(\mathcal{A})}(X^{\bullet})$-$\mathrm{End}_{K(\mathcal{C})}(Z^{\bullet})$-*homomorphism* $\mathrm{Hom}_{K(\mathcal{A})}(X^{\bullet}, Z^{\bullet}[i]) \to \mathrm{Hom}_{K(\mathcal{C})}(\mathrm{H}X^{\bullet}, \mathrm{H}Z^{\bullet}[i])$ *for all* i.

(2) $\mathrm{Hom}^{\bullet}_{\mathcal{A}}(X^{\bullet}, X^{\bullet}) \to \mathrm{Hom}^{\bullet}_{\mathcal{C}}(\mathrm{H}X^{\bullet}, \mathrm{H}X^{\bullet})$ *induces a ring homomorphism* $\mathrm{End}_{K(\mathcal{A})}(X^{\bullet}) \to \mathrm{End}_{K(\mathcal{C})}(\mathrm{H}X^{\bullet})$.

(3) $\mathrm{Hom}^{\bullet}_{\mathcal{C}}(Y^{\bullet}, \mathrm{H}Z^{\bullet}) \to \mathrm{Hom}^{\bullet}_{\mathcal{C}}(Y^{\bullet}, \mathrm{L}Z^{\bullet})$ *induces an* $\mathrm{End}_{K(\mathcal{A})}(Y^{\bullet})$-$\mathrm{End}_{K(\mathcal{C})}(Z^{\bullet})$-*homomorphism* $\mathrm{Hom}_{K(\mathcal{C})}(Y^{\bullet}, \mathrm{H}Z^{\bullet}[i]) \to \mathrm{Hom}_{K(\mathcal{C})}(Y^{\bullet}, \mathrm{L}Z^{\bullet}[i])$ *for all* i.

(4) $\mathrm{Hom}^\bullet_{\mathscr{B}}(GX^\bullet, U^\bullet) \cong \mathrm{Hom}^\bullet_{\mathscr{A}}(X^\bullet, FU^\bullet)$ *induces an* $\mathrm{End}_{K(\mathscr{A})}(X^\bullet)$-$\mathrm{End}_{K(\mathscr{B})}(U^\bullet)$-*isomorphism* $\mathrm{Hom}_{K(\mathscr{B})}(GX^\bullet, U^\bullet[i]) \cong \mathrm{Hom}_{K(\mathscr{A})}(X^\bullet, FU^\bullet[i])$ *for all* i.

Furthermore, these correspondences are functorial.

2. Ring Morphisms and Tilting Complexes

In this section, we consider the condition of tensor products which induced by ring morphisms. In particular, split extensions of rings induce the necessary and sufficient condition that tensor product of a complex is a tilting complex.

Lemma 2.1. *Let* $A \to \Lambda$ *be a ring homomorphism and* $T^\bullet$ *a tilting complex for* A. *If* $\mathrm{Hom}_{D(\mathrm{Mod}\Lambda)}(T^\bullet, T^\bullet \otimes^{L}_{A} \Lambda_{\Lambda}[i]) = 0$ *for all* $i \neq 0$, *then* $T^\bullet \otimes^{L}_{A} \Lambda$ *is a tilting complex for* Λ.

Corollary 2.2 (Miyashita [8]). *Let* $\mu: A \to \Lambda$ *be a ring homomorphism and* T *a tilting* A-*module. If* $\mathrm{Tor}^{A}_{i}(T, \Lambda) = \mathrm{Ext}^{i}_{A}(T, T \otimes_A \Lambda_A) = 0$ *for all* $i > 0$, *then* $T \otimes_A \Lambda$ *is a tilting* Λ-*module* .

In case of a finite dimensional algebra A over a field k, there exist a duality $D: D^b(\mathrm{mod}A) \to D^b(A\ \mathrm{mod})$, where $D = \mathrm{Hom}_k(-, k)$. Then we can define a cotilting complex $T^\bullet$ as follows,

(i) $T^\bullet \in K^b(\mathscr{I}_A)$, where $\mathscr{I}_A$ is the category of finitely generated injective right A-modules,

(ii) $\mathrm{Hom}_{D(\mathrm{mod}A)}(T^\bullet, T^\bullet[i]) = 0$ for all $i \neq 0$,

(iii) $DA \in \mathscr{T}(\mathrm{add}T^\bullet)$, where $\mathscr{T}(\mathrm{add}T^\bullet)$ is the triangulated subcategory of $K^b(\mathscr{I}_A)$ generated by objects in $\mathrm{add}T^\bullet$.

Happel showed that if $X^\bullet$ belongs to $K^b(\mathscr{P}_A)$, then there exists an Auslander-Reiten translation $\tau_A X^\bullet$ which is isomorphic to $\nu X^\bullet[-1]$, where $\nu_A = - \otimes^{L}_{A} DA$, and then there exists an Auslander-Reiten triangle $\tau_A X^\bullet \to Y^\bullet \to X^\bullet \to \tau_A X^\bullet[1]$ in $D^b(\mathrm{mod}\ A)$ (see [4]). Then $\tau_A T^\bullet$ is a cotilting complex for A if $T^\bullet$ is a tilting complex for A. As well as proposition 1.2 in [6], we have the following result.

Proposition 2.3. *Let $A \to \Lambda$ be a k-algebra homomorphism between finite dimensional k-algebras. If $X^\bullet \in K^b(\mathcal{P}_A)$, then $\tau_\Lambda(X^\bullet \otimes^L_A \Lambda)$ is isomorphic to $\mathbf{R}\,\mathrm{Hom}_A({}_\Lambda\Lambda_A, \tau_A X^\bullet)$ in $D^b(\mathrm{mod}A)$.*

Corollary 2.4. *Let $A \to \Lambda$ be a k-algebra homomorphism between finite dimensional k-algebras, and $T^\bullet$ a tilting complex for A. If $\mathrm{Hom}_{D(\mathrm{mod}A)}(T^\bullet, T^\bullet \otimes^L_A \Lambda_A[i]) = 0$ for all $i \neq 0$, then $\mathbf{R}\,\mathrm{Hom}_A({}_\Lambda\Lambda_A, \tau_A T^\bullet)$ is a cotilting complex for Λ.*

Lemma 2.5. *Let $A \to \Lambda$ a ring homomorphism and $T^\bullet$ an object of $D^-(\mathrm{Mod}A)$. If $T^\bullet \otimes^L_A \Lambda$ is a tilting complex for Λ, then we have $\mathrm{Hom}_{D(\mathrm{Mod}A)}(T^\bullet, T^\bullet \otimes^L_A \Lambda_A[i]) = 0$ for all $i \neq 0$.*

Theorem 2.6. *Let $\mu: A \to \Lambda$ and $\varepsilon: \Lambda \to A$ be ring homomorphisms such that $\varepsilon\mu = \mathrm{id}_A$, and $T^\bullet$ an object of $D^-(\mathrm{Mod}A)$. Then $T^\bullet \otimes^L_A \Lambda$ is a tilting complex for Λ if and only if $T^\bullet$ is a tilting complex for A and $\mathrm{Hom}_{D(\mathrm{Mod}A)}(T^\bullet, T^\bullet \otimes^L_A \Lambda_A[i]) = 0$ for all $i \neq 0$.*
In this case, there exist ring homomorphisms $\eta: B \to \Gamma$ and $\pi: \Gamma \to B$ such that $\pi\eta = \mathrm{id}_B$, where $B := \mathrm{End}_{D(\mathrm{Mod}A)}(T^\bullet)$ and $\Gamma := \mathrm{End}_{D(\mathrm{Mod}\Lambda)}(T^\bullet \otimes^L_A \Lambda)$.

Corollary 2.7 (Hoshino [5]). *Let $\mu: A \to \Lambda$ and $\varepsilon: \Lambda \to A$ be ring homomorphisms such that $\varepsilon\mu = \mathrm{id}_A$, and T an A-module. Assume that $\mathrm{Tor}_i^A(T, \Lambda) = 0$ for all $i > 0$. Then $T \otimes_A \Lambda$ is a tilting Λ-module if and only if T is a tilting A-module and $\mathrm{Ext}^i_A(T, T \otimes_A \Lambda_A) = 0$ for all $i > 0$.*

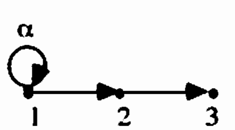
3. Extensions of Rings and Tilting Complexes

In this section, we consider the condition that an extension (not necessary split) of a ring induces a extension of a ring. Furthermore, we show that a Frobenius extension of a ring induces a Frobenius extension of a ring. Next theorem is a generalization of corollary 5.4 in [13].

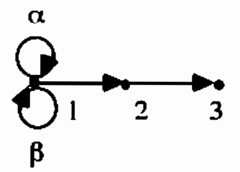
Theorem 3.1. *Let* Λ *be an extension of a ring* A *such that* $0 \to A \to \Lambda \to M \to 0$ *is an exact sequence as* A*-*A*-bimodules. Let* $T^{\bullet}$ *be a tilting complex for* A *such that* $\mathrm{Hom}_{D(\mathrm{Mod}A)}(T^{\bullet}, T^{\bullet}\otimes^{L}_{A} M[i]) = 0$ *for all* $i \neq 0$, and $B := \mathrm{End}_{D(\mathrm{Mod}A)}(T^{\bullet})$, $\Gamma :=$ $\mathrm{End}_{D(\mathrm{Mod}\Lambda)}(T^{\bullet}\otimes^{L}_{A} \Lambda)$ *and* $N := \mathrm{Hom}_{D(\mathrm{Mod}A)}(T^{\bullet}, T^{\bullet}\otimes^{L}_{A} M)$.
Then $T^{\bullet}\otimes^{L}_{A} \Lambda$ *is a tilting complex for* Λ*, and* Γ *is an extension of a ring* B *such that* $0 \to B \to \Gamma \to N \to 0$ *is an exact sequence as* B*-*B*-bimodules.*

Remark. In case that A and B are derived equivalent R-algebras which are projective as R-modules, M is just a A-bimodule which corresponds under the induced equivalence $D^{b}(\mathrm{Mod}\, A^{op}\otimes_{R} A) \to D^{b}(\mathrm{Mod}\, B^{op}\otimes_{R} B)$ to a B-bimodule N (see [13]).

Example. Let A be a finite dimensional algebra over a field k which has the following quiver with relations,



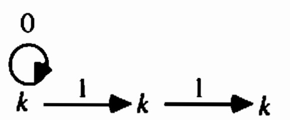
with $\alpha^3 = 0$, and Λ be a finite dimensional algebra over a field k which has the following quiver with relations,



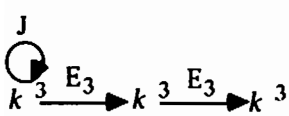
with $\alpha\beta = \beta\alpha = 0$ and $\alpha^2 = \beta^3$. Then Λ is a non split extension of A , and we have the following exact sequence as A-A-bimodules,

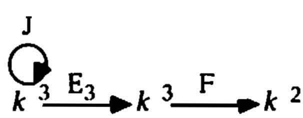
$$0 \to A \to \Lambda \to S(1)^{(2)} \otimes_k X \to 0,$$

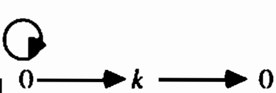
where S(1) is simple left A-module corresponding to vertex 1 and X is a right A-module,



Let T_1, T_2 and T_3 be the following right A-modules, respectively,



, 

and 

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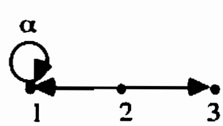
where $J = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $F = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then $T = T_1 \oplus T_2 \oplus T_3$ satisfies the condition of Theorem 3.1, and B has the following quiver with a relation,



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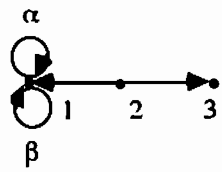
with $\alpha^3 = 0$, Γ has the following quiver with relations,



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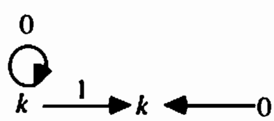
with $\alpha\beta = \beta\alpha = 0$ and $\alpha^2 = \beta^3$, and we get the following exact sequence as B-B-bimodules,

$$0 \to A \to \Lambda \to Y \otimes_k S'(1) \to 0,$$

where $S'(1)$ is a simple right B-module corresponding to vertex 1 and Y is a left B-module,



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Let A be a subring of Λ. Λ is called a Frobenius extension of A provided that Λ_A is a finitely generated projective right A-module, and that ${}_A\Lambda_\Lambda \cong \mathrm{Hom}_A({}_\Lambda\Lambda_A, {}_AA_A)$ as A-Λ- bimodules [2].

Lemma 3.2. *Let A be a ring and $T^\bullet$ a tilting complex for A. Given $X^\bullet \in K^b(\mathcal{P}_A)$, if* $\mathrm{Hom}_{D(\mathrm{Mod}A)}(T^\bullet, X^\bullet[i]) = \mathrm{Hom}_{D(\mathrm{Mod}A)}(X^\bullet, T^\bullet[i]) = 0$ *for all $i \neq 0$, then $X^\bullet$ is isomorphic to a direct summand of a finite direct sum of copies of $T^\bullet$.*

The next theorem is the tilting complex version of the result of Miyashita [7].

Theorem 3.3. *Let Λ be a Frobenius extension of a ring A such that $0 \to A \to \Lambda \to$ $M \to 0$ is an exact sequence as A-A-bimodules. Let $T^\bullet$ be a tilting complex for A such that* $\mathrm{Hom}_{D(\mathrm{Mod}A)}(T^\bullet, T^\bullet \otimes_A^{L} M[i]) = 0$ *for all $i \neq 0$,* and $B := \mathrm{End}_{D(\mathrm{Mod}A)}(T^\bullet)$, $\Gamma :=$

$\mathrm{End}_{D(\mathrm{Mod}\Lambda)}(T^{\bullet}\otimes^{L}_{A}\Lambda)$ *and* $N := \mathrm{Hom}_{D(\mathrm{Mod}A)}(T^{\bullet}, T^{\bullet}\otimes^{L}_{A} M)$.

Then $T^{\bullet}\otimes^{L}_{A}\Lambda$ *is a tilting complex for* Λ, *and* Γ *is a Frobenius extension of a ring* B *such that* $0 \to B \to \Gamma \to N \to 0$ *is an exact sequence as* B*-*B*-bimodules.*

Remark. For a tilting complex $T^{\bullet}$ for A, if $T^{\bullet}\otimes^{L}_{A}\Lambda_{\Lambda}$ is a tilting complex for Λ and $\mathrm{End}_{D(\mathrm{Mod}\Lambda)}(T^{\bullet}\otimes^{L}_{A}\Lambda)$ is an extension of $\mathrm{End}_{D(\mathrm{Mod}A)}(T^{\bullet})$, then we have $\mathrm{Hom}_{D(\mathrm{Mod}A)}(T^{\bullet}, T^{\bullet}\otimes^{L}_{A} M[i]) = 0$ for all $i \neq 0$.

Corollary 3.4. *In the situation of Theorem 3.3,* $\mathrm{End}_{D(\mathrm{Mod}\Lambda)}(T^{\bullet}\otimes^{L}_{A}\Lambda_{\Lambda})$ *is a Frobenius extension of* $\mathrm{End}_{D(\mathrm{Mod}A)}(T^{\bullet}\otimes^{L}_{A}\Lambda_{A})$.

The next proposition is useful in exhibiting examples of Frobenius extensions of algebras which satisfy Theorem 3.3.

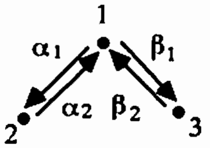
Proposition 3.5. *Let* Λ *be a finite dimensional Frobenius algebra over a field k such that* $0 \to k \to \Lambda \to M \to 0$ *is an exact sequence in mod k. For a finite dimensional k- algebra A and a tilting complex* $T^{\bullet}$, *let* $B = \mathrm{End}_{D(\mathrm{Mod}A)}(T^{\bullet})$, $\Gamma = \mathrm{End}_{D(\mathrm{Mod}\Lambda)}(T^{\bullet}\otimes^{L}_{A}(A\otimes_{k}\Lambda))$ *and* $N = \mathrm{Hom}_{D(\mathrm{Mod}A)}(T^{\bullet}, T^{\bullet}\otimes^{L}_{A}(A\otimes_{k}M))$. *Then* $A\otimes_{k}\Lambda$ *is a Frobenius extension of A which satisfy theorem 3.3 with an exact sequence* $0 \to A \to A\otimes_{k}\Lambda \to A\otimes_{k}M \to 0$, *and* $B\otimes_{k}\Lambda$ *is a Frobenius extension of B such that an exact sequence* $0 \to B \to B\otimes_{k}\Lambda \to B\otimes_{k}M \to 0$ *which is isomorphic to* $0 \to B \to \Gamma \to N \to 0$ *as a B-B-bimodule.*

Remark. Let A be a subring of Λ. Λ is called a quasi-Frobenius extension of A provided that Λ_{A} is a finitely generated projective right A-module, and that ${}_{\Lambda}\Lambda_{\Lambda}$ is a direct summand of a finite direct sum of copies of $\mathrm{Hom}_{A}({}_{A}\Lambda_{\Lambda}, {}_{A}A_{A})$ as A-Λ- bimodules and $\mathrm{Hom}_{A}({}_{A}\Lambda_{\Lambda}, {}_{A}A_{A})$ is a direct summand of a finite direct sum of copies of ${}_{A}\Lambda_{\Lambda}$ as A-Λ- bimodules [9]. Then "a Frobenius extension" in Theorem 3.3 can be replaced by "a quasi-Frobenius extension".

Examples. (1) $k[X]/(X^n)$ and kG satisfy the condition of Proposition 3.5, where G is a finite group and k is a field.

(2) Let A be a finite dimensional k-algebra which has the quiver: $1 \to 3 \leftarrow 2$, $\sigma: A \to A$ a k-algebra automorphism induced by interchanging vertex 1 with vertex 2. For a group $G := \{1, \sigma\}$, we define a strongly G- graded k-algebra $\Lambda := \oplus_{g\in G} A_g$ such that A_g has a natural left action of A and a right action of A which is through g (*i.e.* a crossed product of A with G which has a trivial factor set). Let T = P(1)⊕P(2)⊕I(3), where P(i) (resp., I(i)) is a projective (rep., injective) indecomposable right A-module corresponding to vertex i. Then Λ is a Frobenius extension of A, T satisfies Theorem 3.3.

(3) According to [15], we have the following example (See also Okuyama's lecture in Proceedings of the 4th Symposium on Representaion Theory of Algebras (South Izu, 1993)). Given positive integer n, let A be a finite dimensional algebra over a field k which has the following quiver with relations,

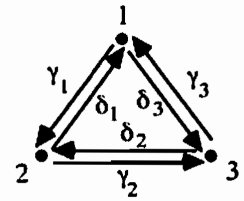


,

with $\alpha_2\alpha_1 = \beta_2\beta_1 = 0$ and $(\alpha_1\alpha_2\beta_1\beta_2)^n = (\beta_1\beta_2\alpha_1\alpha_2)^n$. Let $\sigma: A \to A$ be a k-algebra automorphism induced by interchanging vertex 2 with vertex 3. Let $\Lambda := \oplus_{g\in G} A_g$, where $G := \{1, \sigma\}$, and let $T^{\bullet}$ be the following complex:

$$P(2)^{(2)} \oplus P(3)^{(2)} \xrightarrow{M} P(1),$$

where M = (0 α_2 β_2 0). Then Λ is a Frobenius extension of A, $T^{\bullet}$ satisfies Theorem 3.3. Then $B := \mathrm{End}_{K(\mathrm{Mod}\Lambda)}(T^{\bullet})^{op}$ is a finite dimensional algebra over a field k which has the following quiver with relations,



with $\gamma_1\gamma_2 = \gamma_2\gamma_3 = \gamma_3\gamma_1 = \delta_1\delta_3 = \delta_3\delta_2 = \delta_2\delta_1 = 0$, $\gamma_1\delta_1 = \delta_3\gamma_3$, $\delta_1\gamma_1 = (\gamma_2\delta_2)^n$ and $\gamma_3\delta_3 = (\delta_2\gamma_2)^n$. Let $\sigma: B \to B$ be a k-algebra automorphism induced by interchanging vertex 2 with vertex 3. Then Γ is ring-isomorphic to $\oplus_{g\in G} B_g$, where $G := \{1, \sigma\}$.

References

[1] I. Assem and N. Marmaridis, Tilting modules over split-by-nilpotent extensions, Proc. Third Meeting of Representation Theory, Sherbrooke Preprint Series 94 (1992) 1-15.

[2] F. Kasch, Grundlagen einer theorie der Frobeniuserweiterungen, Math. Annalen 27 (1954) 453-474.

[3] F. Kasch, Projektive Frobenius-Erweiterungen, Sitzungsber. Heidelberger Acad. Wiss (1960/1961) 89-109.

[4] D.Happel, Triangulated categories in the representation theory of finite dimensional algebras, London Mathematical Society Lecture Notes 119 (University Press, Cambridge, 1987).

[5] M. Hoshino, handwritten note, 1993.

[6] J. Miyachi, Tilting modules over trivial extensions of artin algebras, Comm. in Algebra 13 (1985) 1319-1326.

[7] Y. Miyashita, On Galois extensions and crossed products, J. Fac. Sci. Hokkaido Univ., Ser. I 21 (1970), 97-121.

[8] Y. Miyashita, Tilting modules of finite projective dimension, Math. Z. 193 (1986) 113-146.

[9] B. Müller, Quasi-Frobenius-Erweiterungen, Math. Z. 85 (1964) 346-368.

[10] T. Onodera, Some studies on projective Frobenius extensions, J. Fac. Sci. Hokkaido Univ. 18 (1964) 89-107.

[11] J. Rickard, Morita theory for derived categories, J. London Math. Soc. (2) 39 (1989) 436-456.

[12] J. Rickard, Derived categories and stable equivalence, J. Pure and Applied Algebra 61 (1989) 303-317.

[13] J. Rickard, Derived equivalences as derived functors, J. London Math. Soc. (2) 43 (1991), 436-456.

[14] H. Tachikawa and T. Wakamatsu, Extensions of tilting functors and QF-3 algebras, J. Algebra 103 (1986) 662--676.

[15] M. Linckelmann, A derived equivalence for blocks with dihedral defect groups, J. Algebra 164 (1994) 244-255.

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PROCEEDINGS OF THE 27TH
SYMPOSIUM ON RING THEORY, 1994

ON THE RELATIVE HOMOLOGICAL ALGEBRA OF FROBENIUS EXTENSIONS

Takeshi Nozawa

Introduction.

The complete relative homology and cohomology groups which we shall treat in this note are the relativised versions of the complete homology and cohomology groups in [3] and [7] in the sense of [2]. One of the merits of the complete homology and cohomology groups is that there exists an isomorphism between them, but for the complete relative homology and cohomology groups, there doesn't necessarily exist such an isomorphism. The first purpose of this note, which is Theorem 2.1, is to introduce the necessary and sufficient conditions on which a homomorphism between them is isomorphic. By the cup product, the complete cohomology groups becomes a graded ring. In [7], on some assumption it is proved that existing an invertible element in the graded ring is equivalent to that the complete cohomology groups are periodic. The second purpose of this note, which is Theorem 3.1, is to show from the viewpoint of the complete relative cohomology groups that the equivalence holds on the more general assumption.

1. The complete relative homology and cohomology groups of Frobenius extensions.

Let Λ be a ring with unit and Γ a subring. Then it is said that the ring extension Λ/Γ is a Frobenius extension if Λ is left finitely generated Γ-projective and there exists a left Λ and right Γ-isomorphism of $\mathrm{Hom}({}_\Gamma\Lambda, {}_\Gamma\Gamma)$ to Λ. In this note, we shall adopt the equivalent definition, that is, when we say that the ring extension Λ/Γ is a Frobenius extension, we mean that there exist elements $R_1, \cdots, R_n, L_1, \cdots, L_n$ in Λ and a two-sided Γ-homomorphism $H \in \mathrm{Hom}({}_\Gamma\Lambda_\Gamma, {}_\Gamma\Gamma_\Gamma)$ such that $x = \sum_{i=1}^n H(xR_i)L_i = \sum_{i=1}^n R_i H(L_i x)$ for all $x \in \Lambda$. The pair (R_i, L_i) and two-sided Γ-homomorphism H are called the dual projective pair and Frobenius homomorphism of Λ/Γ, respectively.

Let Λ be an algebra over a commutative ring K and denote the enveloping algebra $\Lambda \otimes_K \Lambda^o$ by P, where Λ^o is the opposite ring of Λ. For a subalgebra Γ of Λ let S be the image of the natural homomorphism of $\Gamma \otimes_K \Gamma^o$ to P, namely the image of the homomorphism $x \otimes_K y \in \Gamma \otimes_K \Gamma^o \mapsto x \otimes_K y \in P$. S is a subring of

The final and detailed version of section 3 of this note will be submitted for publication elsewhere.

P. Regard Λ as a left P-module with the usual way. Then by extending the (P, S)-projective resolution of Λ in [2] to the negative direction, we have the following P-exact sequence

$$X : \cdots \to X_r \xrightarrow{d_r} X_{r-1} \to \cdots \xrightarrow{d_1} X_0 \xrightarrow{d_0} X_{-1} \xrightarrow{d_{-1}} \cdots \to X_{-r} \xrightarrow{d_{-r}} X_{-(r+1)} \to \cdots$$
$$\varepsilon \searrow \quad \nearrow \eta$$
$$\Lambda$$

with a P-epimorphism $\varepsilon : X_0 \to \Lambda$ and a P-monomorphism $\eta : \Lambda \to X_{-1}$ such that $\eta \cdot \varepsilon = d_0$, where X has a contracting S-homotopy and the P-module X_r is (P, S)-projective for all $r \in \mathbf{Z}$. We shall call P-exact sequences of this type the complete (P, S)-resolutions of Λ. For any left P-module M and $r \in \mathbf{Z}$ we denote the r-th homology group of the complex $X \otimes_P M$, namely $\mathrm{Ker}\,(d_r \otimes_P 1_M)/\mathrm{Im}\,(d_{r-1} \otimes_P 1_M)$, by $\mathrm{H}_r(\Lambda, \Gamma, M)$ and the r-th cohomology group of the complex $\mathrm{Hom}({}_PX, {}_PM)$, namely $\mathrm{Ker}\,\mathrm{Hom}(d_{r+1}, 1_M)/\mathrm{Im}\,\mathrm{Hom}(d_r, 1_M)$, by $\mathrm{H}^r(\Lambda, \Gamma, M)$, and we shall call them the r-th complete relative homology and cohomology groups with coefficients in M, respectively. Since $X_r \otimes_P M$ and $\mathrm{Hom}({}_PX_r, {}_PM)$ are K-modules, $\mathrm{H}_r(\Lambda, \Gamma, M)$ and $\mathrm{H}^r(\Lambda, \Gamma, M)$ are K-modules. In this note we shall treat the case where Λ/Γ, as a ring extension, is a Frobenius extension since in the case, $\mathrm{H}_r(\Lambda, \Gamma, M)$ and $\mathrm{H}^r(\Lambda, \Gamma, M)$ are independent of the choice of complete (P, S)-resolutions of Λ, that is, they are unique up to isomorphism. When we choose the standard resolution [4, (2)] as a complete (P, S)-resolutin of Λ, the following proposition holds :

Proposition 1.1 ([4, Proposition 1.2] and [5, Proposition 1.1]). *Let the pair (R_i, L_i) be the dual projective pair of the Frobenius extension Λ/Γ. For any left P-module M put* $C_{-1}^{\Lambda/\Gamma}(M) = \{(1 \otimes_\Gamma 1) \otimes_P m \in (\Lambda \otimes_\Gamma \Lambda) \otimes_P M \mid \sum_i (R_i \otimes_\Gamma L_i \otimes_\Gamma 1) \otimes_P m = \sum_i (1 \otimes_\Gamma R_i \otimes_\Gamma L_i) \otimes_P m$ *in* $(\Lambda \otimes_\Gamma \Lambda \otimes_\Gamma \Lambda) \otimes_P M\}$, $B_{-1}^{\Lambda/\Gamma}(M) = \{\sum_i (R_i \otimes_\Gamma L_i) \otimes_P m \in (\Lambda \otimes_\Gamma \Lambda) \otimes_P M \mid m \in M\}$, $M^\Lambda = \{m \in M \mid xm = mx$ *for all* $x \in \Lambda\}$ *and* $N_{\Lambda/\Gamma}(M) = \{\sum_i R_i m L_i \mid m \in M^\Gamma\}$ *where* $M^\Gamma = \{m \in M \mid xm = mx$ *for all* $x \in \Gamma\}$. *When the standard resolution* [4, (2)] *is chosen as a complete (P, S)-resolution of Λ, we have* $\mathrm{H}_{-1}(\Lambda, \Gamma, M) = C_{-1}^{\Lambda/\Gamma}(M)/B_{-1}^{\Lambda/\Gamma}(M)$ *and* $\mathrm{H}^0(\Lambda, \Gamma, M) \simeq M^\Lambda/N_{\Lambda/\Gamma}(M)$.

2. The homomorphism Ψ.

As in section 1, let Λ be an algebra over a commutative ring K and Γ a subalgebra such that Λ/Γ, as a ring extension, is a Frobenius extension with the dual projective pair (R_i, L_i) and Frobenius homomorphism H. In this section, let Γ/K be also a Frobenius extension with the dual projective pair (r_j, l_j) and Frobenius homomorphism h. Then since Λ/Γ and Γ/K are Frobenius extensions, Λ/K is a Frobenius extension, that is, Λ is a Frobenius K-algebra with the dual projective pair $(R_i r_j, L_i l_j)$ and Frobenius homomorphism $h \cdot H$. So we have the Nakayama automorphism Δ, which is an automorphism of Λ over the center of Λ and given by $\Delta(x) = \sum_{i,j} R_i r_j h \cdot H(x l_j L_i)$ for $x \in \Lambda$.

Let M be a left P-module. M can be regarded as a two-sided (Λ, K)-module. Then modifying the structure of the right Λ-module as $m \cdot x = m\Delta(x)$ where $m \in M$ and $x \in \Lambda$, we obtain a left P-

module M^{Δ} from M. We shall denote $m \in M^{\Delta}$ by m^{Δ}. Let X be a complete (P,S)-resolution of Λ. Then so is the sequence $\mathrm{Hom}({}_{\Lambda}X, {}_{\Lambda}\Lambda)$, where the r-th module is $\mathrm{Hom}({}_{\Lambda}X_{-r-1}, {}_{\Lambda}\Lambda)$. Then for any left P-module M, the homomorphism $\psi : \mathrm{Hom}({}_{\Lambda}X_{-r-1}, {}_{\Lambda}\Lambda) \otimes_P M^{\Delta} \to \mathrm{Hom}({}_{P}X_{-r-1}, {}_{P}M)$, given by $\psi(f \otimes_P m^{\Delta}) = [x \mapsto \sum_{i,j} f(xR_i r_j) m l_j L_i]$ for $f \in \mathrm{Hom}({}_{\Lambda}X_{-r-1}, {}_{\Lambda}\Lambda)$, $m^{\Delta} \in M^{\Delta}$ and $x \in X_{-r-1}$, induces the homomorphism

$$\Psi^r_{\Lambda/\Gamma} : \mathrm{H}_r(\Lambda, \Gamma, M^{\Delta}) \to \mathrm{H}^{-r-1}(\Lambda, \Gamma, M)$$

for $r \in \mathbf{Z}$. For the case $\Gamma = K$, $\Psi^r_{\Lambda/\Gamma}$ is an isomorphism for any left P-module M and any $r \in \mathbf{Z}$, but in general cases, homomorphisms of $\mathrm{H}_r(\Lambda, \Gamma, M^{\Delta})$ to $\mathrm{H}^{-r-1}(\Lambda, \Gamma, M)$, including $\Psi^r_{\Lambda/\Gamma}$, aren't necessarily isomorphisms for any left P-module M and any $r \in \mathbf{Z}$. For the homomorphism $\Psi^r_{\Lambda/\Gamma}$ the following theorm holds :

Theorem 2.1 ([5, Theorem 7.1]). *The following conditions are equivalent :*

(1) $\Psi^r_{\Lambda/\Gamma} : \mathrm{H}_r(\Lambda, \Gamma, M^{\Delta}) \to \mathrm{H}^{-r-1}(\Lambda, \Gamma, M)$ *is an isomorphism for any left P-module M and any $r \in \mathbf{Z}$.*

(2) $\Psi^r_{\Lambda/\Gamma} : \mathrm{H}_r(\Lambda, \Gamma, M^{\Delta}) \to \mathrm{H}^{-r-1}(\Lambda, \Gamma, M)$ *is an epimorphism for any left P-module M and any $r \in \mathbf{Z}$.*

(3) *There are elements $\lambda, \xi \in \Lambda$ such that $(1 \otimes_{\Gamma} 1) \otimes_P \lambda^{\Delta} \in C^{\Lambda/\Gamma}_{-1}(\Lambda^{\Delta})$, $\xi \in \Lambda^{\Gamma}$ and $1 = \sum_j r_j \lambda l_j + \sum_i R_i \xi L_i$, where $C^{\Lambda/\Gamma}_{-1}$ and Λ^{Γ} are the same as in Proposition 1.1.*

Outline of the proof. By the Proposition 1.1, $\Psi^{-1}_{\Lambda/\Gamma}$ is regarded as a homomorphism of $C^{\Lambda/\Gamma}_{-1}(M^{\Delta})/B^{\Lambda/\Gamma}_0(M^{\Delta})$ to $M^{\Lambda}/N_{\Lambda/\Gamma}(M)$ such that

$$\Psi^{-1}_{\Lambda/\Gamma}(\overline{(1 \otimes_{\Gamma} 1) \otimes_P m^{\Delta}}) = \overline{\sum_j r_j m l_j}$$

for $(1 \otimes_{\Gamma} 1) \otimes_P m^{\Delta} \in C^{\Lambda/\Gamma}_{-1}(M^{\Delta})$, where $^-$ stands for the residue classes. It is clear that (1) implies (2). Assume that (2) holds. Put $M = \Lambda$. Then since $\Psi^{-1}_{\Lambda/\Gamma}$ is an epimorphism and $1 \in \Lambda^{\Lambda}$, there is an element $\lambda \in \Lambda$ such that $(1 \otimes_{\Gamma} 1) \otimes_P \lambda^{\Delta} \in C^{\Lambda/\Gamma}_{-1}(\Lambda^{\Delta})$ and $\Psi^{-1}_{\Lambda/\Gamma}(\overline{(1 \otimes_{\Gamma} 1) \otimes_P \lambda^{\Delta}}) = \overline{1}$. Thus (3) holds. Assume that (3) holds. For the element $\lambda \in \Lambda$ in (3), when we define the homomorphism $\phi : M^{\Lambda} \to C^{\Lambda/\Gamma}_{-1}(M^{\Delta})$ such that $\phi(m) = (1 \otimes_{\Gamma} 1) \otimes_P (\lambda m)^{\Delta}$, it can be shown that ϕ is well-defined, ϕ induces the homomorphism $\Phi : M^{\Lambda}/N_{\Lambda/\Gamma}(M) \to C^{\Lambda/\Gamma}_{-1}(M^{\Delta})/B^{\Lambda/\Gamma}_0(M^{\Delta})$ and Φ is the inverse isomorphism of $\Psi^{-1}_{\Lambda/\Gamma} : C^{\Lambda/\Gamma}_{-1}(M^{\Delta})/B^{\Lambda/\Gamma}_0(M^{\Delta}) \to M^{\Lambda}/N_{\Lambda/\Gamma}(M)$. So $\Psi^{-1}_{\Lambda/\Gamma}$ is an isomorphism. For any left P-module M and any $r \in \mathbf{Z}$, by the dimension-shifting as in [4, Lemma 3.3], there is a left P-module N such that $\mathrm{H}_r(\Lambda, \Gamma, M^{\Delta}) \simeq \mathrm{H}_{-1}(\Lambda, \Gamma, N^{\Delta})$ and $\mathrm{H}^{-r-1}(\Lambda, \Gamma, M) \simeq \mathrm{H}^0(\Lambda, \Gamma, N)$. Thus (1) holds.

Let G be a finite group and L a subgroup. Then in [2, §4], the relative homology group $\mathrm{H}_r(G, L, M)$ and cohomology group $\mathrm{H}^r(G, L, M)$ are defined for any $r \in \mathbf{Z}$ and any left G-module M. When we put

$\Lambda = \mathbf{Z}G$, $\Gamma = \mathbf{Z}L$ and $K = \mathbf{Z}$, we have $\mathrm{H}_r(G, L, M) \simeq \mathrm{H}_r(\Lambda, \Gamma, M_\epsilon)$ and $\mathrm{H}^r(G, L, M) \simeq \mathrm{H}^r(\Lambda, \Gamma, M_\epsilon)$ where M_ϵ is the module M regarded as a left P-module by using the augmentation map $\epsilon : \mathbf{Z}G \to \mathbf{Z}$ such that $(x \otimes_K y) \cdot m = xm\epsilon(y)$ for $x \otimes_K y \in P$ and $m \in M$. Then the theorem above means the following corollary holds :

Corollary 2.2 ([5, Theorem 8.2]). *The following conditions are equivalent :*

(1) $\Psi^r_{G/L} : \mathrm{H}_r(G, L, M) \to \mathrm{H}^{-r-1}(G, L, M)$ *is an isomorphism for any left G-module M and any $r \in \mathbf{Z}$.*

(2) $\Psi^r_{G/L} : \mathrm{H}_r(G, L, M) \to \mathrm{H}^{-r-1}(G, L, M)$ *is an epimorphism for any left G-module M and any $r \in \mathbf{Z}$.*

(3) *There are $t, z \in \mathbf{Z}$ such that $1 = |L|t + (G : L)z$, where $|L|$ is the order of L and $(G : L)$ is the index of L in G.*

3. The periodicity of the complete relative cohomology groups.

Let C be the center of Λ. In this section we regard the module $\mathrm{H}^r(\Lambda, \Gamma, M)$ as a left and right C-module for any $r \in \mathbf{Z}$ by making use of the action of C on M, but since the action of C on $\mathrm{H}^r(\Lambda, \Gamma, M)$ satisfies $c\alpha = \alpha c$ for $c \in C$ and $\alpha \in \mathrm{H}^r(\Lambda, \Gamma, M)$, it is unnecessary to distinguish the left action of C on $\mathrm{H}^r(\Lambda, \Gamma, M)$ from the right action of C on it. In [4] we define a C-homomorphism $\cup : \mathrm{H}^r(\Lambda, \Gamma, M) \otimes_C \mathrm{H}^s(\Lambda, \Gamma, N) \to \mathrm{H}^{r+s}(\Lambda, \Gamma, M \otimes_\Lambda N)$ for every $r, s \in \mathbf{Z}$. The C-homomorphism $\cup$ is called the cup product and $\cup(\alpha \otimes_C \beta)$ is denoted by $\alpha \cup \beta$ for $\alpha \in \mathrm{H}^r(\Lambda, \Gamma, M)$ and $\beta \in \mathrm{H}^s(\Lambda, \Gamma, N)$. This cup product has the well-known properties : For the case $r = s = 0$, by using Proposition 1.1, the cup product coincides with a map $M^\Lambda / N_{\Lambda/\Gamma}(M) \otimes_C N^\Lambda / N_{\Lambda/\Gamma}(N) \to (M \otimes_\Lambda N)^\Lambda / N_{\Lambda/\Gamma}(M \otimes_\Lambda N)$ $(\overline{m} \otimes_C \overline{n} \mapsto \overline{m \otimes_\Lambda n})$, and for any r, s and $t \in \mathbf{Z}$, the cup product satisfies anti-commutativity, namely $\alpha \cup \beta = (-1)^{rs} \beta \cup \alpha$ for $\alpha \in \mathrm{H}^r(\Lambda, \Gamma, \Lambda)$ and $\beta \in \mathrm{H}^s(\Lambda, \Gamma, M)$, and associativity, namely $(\alpha \cup \beta) \cup \gamma = \alpha \cup (\beta \cup \gamma)$ for $\alpha \in \mathrm{H}^r(\Lambda, \Gamma, M)$, $\beta \in \mathrm{H}^s(\Lambda, \Gamma, N)$ and $\gamma \in \mathrm{H}^t(\Lambda, \Gamma, L)$, where M, N and L are any left P-modules. By this cup product the direct sum $\oplus_{r \in \mathbf{Z}} \mathrm{H}^r(\Lambda, \Gamma, \Lambda)$ is a graded ring whose unit is the image of $\overline{1} \in \Lambda^\Lambda / N_{\Lambda/\Gamma}(\Lambda)$ on the isomorphism $\Lambda^\Lambda / N_{\Lambda/\Gamma}(\Lambda) \simeq \mathrm{H}^0(\Lambda, \Gamma, \Lambda)$ of Proposition 1.1. Then we would like to consider whether the following conditions are equivalent :

(I) There is an element $\alpha \in \mathrm{H}^n(\Lambda, \Gamma, \Lambda)$ which is invertible in the graded ring $\oplus_{r \in \mathbf{Z}} \mathrm{H}^r(\Lambda, \Gamma, \Lambda)$.

(II) For any left P-module M and any $r \in \mathbf{Z}$, $\mathrm{H}^r(\Lambda, \Gamma, M) \simeq \mathrm{H}^{r+n}(\Lambda, \Gamma, M)$ as C-modules.

The problem of the equivalence of the conditions (I) and (II) comes from the theory of the complete cohomology of finite groups. In the theory, it is shown that (I) and (II) are equivalent, for example, as in [1, Chapter VI, Theorem 9.1]. For the complete relative cohomology, it is shown that (I) implies (II). In fact, Assume that (I) holds, that is, there is an element $\beta \in \mathrm{H}^{-n}(\Lambda, \Gamma, \Lambda)$ such that $\alpha \cup \beta = \beta \cup \alpha = 1$. Then the isomorphism $\mathrm{H}^r(\Lambda, \Gamma, M) \xrightarrow{\sim} \mathrm{H}^{r+n}(\Lambda, \Gamma, M)$ is given by $\gamma \mapsto \gamma \cup \alpha$ for $\gamma \in \mathrm{H}^r(\Lambda, \Gamma, M)$ and the inverse isomorphism is given by $\delta \mapsto \delta \cup \beta$ for $\delta \in \mathrm{H}^{r+n}(\Lambda, \Gamma, M)$. But we don't know whether the implication (II) $\Rightarrow$ (I) is true, except the following generalization of [7, Theorem 3.7] :

Theorem 3.1 ([6, Theorem 3.1]). *For the case $\Gamma = K$, the condition* (II) *implies* (I), *that is,* (I) *and* (II) *are equivalent if there is a P-exact sequence* $0 \to \Lambda \to L \to N \to 0$ *such that the following conditions holds.*

(1) $\mathrm{H}^r(\Lambda, K, L) = 0$ *for all* $r \in \mathbf{Z}$.

(2) N^Δ *is the direct summand of* $\mathrm{Hom}({}_K\Lambda, {}_KN^\Delta)$ *as P-modules, where N^Δ is the left P-module defined from N by the same way as in section 2 and the action of P on* $\mathrm{Hom}({}_K\Lambda, {}_KN^\Delta)$ *is given by* $(x \otimes_K y)\cdot f(\ ) = f(y(\ )x)$ *for* $x \otimes_K y \in P$ *and* $f \in \mathrm{Hom}({}_K\Lambda, {}_KN^\Delta)$.

(3) N^Δ *is injective as a K-module.*

Outline of the proof. When there exists the P-exact sequence in the assumption of this theorem, the homomorphism

$$\zeta : \mathrm{H}^n(\Lambda, K, \Lambda) \to \mathrm{Hom}({}_C\mathrm{H}^{-n}(\Lambda, K, \Lambda), {}_C\mathrm{H}^0(\Lambda, K, \Lambda)),$$

given by $\zeta(\gamma)(\delta) = \gamma \cup \delta$ for $\gamma \in \mathrm{H}^n(\Lambda, K, \Lambda)$ and $\delta \in \mathrm{H}^{-n}(\Lambda, K, \Lambda)$, is an isomorphism. Therefore by putting $M = \Lambda$ and $r = -n$ in the condition (II), there are elements $\alpha \in \mathrm{H}^n(\Lambda, K, \Lambda)$ and $\beta \in \mathrm{H}^{-n}(\Lambda, K, \Lambda)$ such that $\alpha \cup \beta = 1$. Then by the associativity and anti-commutativity of the cup product, we have $\beta = \beta \cup 1 = \beta \cup (\alpha \cup \beta) = (\beta \cup \alpha) \cup \beta = (-1)^{n^2}(\alpha \cup \beta) \cup \beta = (-1)^{n^2} 1 \cup \beta = (-1)^{n^2}\beta$, and so $\beta \cup \alpha = (-1)^{n^2}\beta \cup \alpha = (-1)^{n^2}(-1)^{n^2}\alpha \cup \beta = 1$. Thus the condition (II) implies (I).

References

[1] K. S. Brown : "Cohomology of Groups", Graduate Texts in Mathematiccs Vol. 87, Springer-Verlag, New York-Heidelberg-Berlin, 1982.

[2] G. Hochschild : Relative homological algebra, Trans. Amer. Math. Soc. 82 (1956), 246–269.

[3] T. Nakayama : On the complete cohomology theory of Frobenius algebras, Osaka Math. J. 9 (1957), 165–187.

[4] T. Nozawa : On the complete relative cohomology of Frobenius extensions, Tsukuba J. Math. 17 (1993), 99–113.

[5] T. Nozawa : On the complete relative homology and cohomology of Frobenius extensions, to appear.

[6] T. Nozawa : On the periodicity of the complete relative cohomology of Frobenius extensions, preprint.

[7] K. Sanada : On the cohomology of Frobenius algebras, J. Pure Appl. Algebra 80 (1992), 65–88.

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PROCEEDINGS OF THE 27TH
SYMPOSIUM ON RING THEORY, 1994

Remarks on periodic modules for finite groups*

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1 Introduction

Let G be a finite group and let k be an algebraically closed field of characteristic $p > 0$. Suppose that M is a finitely generated kG-module. Let $\phi : P \longrightarrow M$ be the projective cover of M and let $\Omega(M)$ be the kernel of ϕ. We define inductively as $\Omega^{n+1}(M) = \Omega(\Omega^n(M))$ for $n > 1$. We say that M is periodic if $\Omega^n(M) \simeq M$ for some $n > 0$. If n is the smallest such integer then n is called the period of M. The purpose of this note is to determine the periods of some periodic modules.

Let $V_G(k) = max(H^*(G,k))$ be the maximal ideal spectram of the cohomology ring $H^*(G,k)$. Let $V_G(M)$ be a closed subvariety of $V_G(k)$ determined by the annihilater of $Ext^*_{kG}(M,M)$. If $\zeta(\neq 0) \in H^n(G,k) \simeq Hom_{kG}(\Omega^n(k),k)(n > 0)$, we set L_ζ (or $L(\zeta)) = Ker(\zeta : \Omega^n(k) \longrightarrow k)$. Then the variety $V_G(L_\zeta) = V_G(\zeta)$(the set of maximal ideals containing ζ). Suppose that $\zeta_1, \ldots, \zeta_r$ is a homogeneous set of parameters (h.s.o.p.) for $H^*(G,k)$ (namely, $\zeta_1, \ldots, \zeta_r$ are homogeneous elements in $H^*(G,k)$ and $H^*(G,k)$ is a finitely generated $k[\zeta_1, \ldots, \zeta_r]$-module where $r = dim(H^*(G,k)) =$ p-$rank(G)$). Since $V_G(\otimes_{i=1}^r L(\zeta_i)) = V_G(\zeta_1, \ldots, \zeta_r) = \{0\}(= \{\oplus_{n>0} H^n(G,k)\})$, $\otimes_{i=1}^r L(\zeta_i)$ is projective and so (the non projective part of) $\otimes_{i=1}^{r-1} L(\zeta_i)$ is periodic and the period divides $deg\ \zeta_r$. Here $\otimes$ means the tensor product over k. But in general we can not find a homogeneous element η such that $deg\ \eta =$ period of $\otimes_{i=1}^{r-1} L(\zeta_i)$ and $\zeta_1, \ldots, \zeta_{r-1}, \eta$ is a h.s.o.p. for $H^*(G,k)$. In section 2, we study the relation between the period of $\otimes_{i=1}^{r-1} L(\zeta_i)$ and $deg\ \zeta_r$. In section 3, we study the

*The final version of this note will be submitted for publication elsewhere.

indecomposable summands of $\bigotimes_{i=1}^{r-1} L(\zeta_i)$. We shall refer to [1] for the properties of varieties of modules.

2 Periods of periodic modules

In the rest of this note, we assume that $\zeta_1, \ldots, \zeta_r$ is a h.s.o.p. for $H^*(G,k)$, $r = \dim H^*(G,k) = p\text{-}rank(G)$ $(r \geq 2)$. Under the isomorphism

$$H^n(G,k) \simeq Ext^1_{kG}(\Omega^{n-1}(k),k) \quad \text{for} \quad n>0$$

$\zeta_i (\in H^{n_i}(G,k))$ corresponds to an extension

$$0 \longrightarrow k \xrightarrow{\varphi_i} \Omega^{-1}(L(\zeta_i)) \longrightarrow \Omega^{n_i-1}(k) \longrightarrow 0.$$

In the long exact sequence

$$\xrightarrow{\varphi_{i*}} H^n(G,\Omega^{-1}(L(\zeta_i))) \longrightarrow H^n(G,\Omega^{n_i-1}(k))$$
$$\xrightarrow{\zeta_i} H^{n+1}(G,k) \longrightarrow$$

if ζ_i is not a zero divisor in $H^*(G,k)$ then φ_{i*} is onto for $n \geq \deg \zeta_i$. Inductively we have the following.

Lemma 2.1 *If $\zeta_1, \ldots, \zeta_s$ $(1 \leq s \leq r)$ is a regular sequence for $H^*(G,k)$ then*

$$\varphi_*^{(n)} = (\bigotimes_{i=1}^{s} \varphi_i)_* : H^n(G,k) \longrightarrow H^n(G, \bigotimes_{i=1}^{s} \Omega^{-1}(L(\zeta_i)))$$

is onto for $n \geq \sum_{i=1}^{s} \deg \zeta_i$.

If $\varphi_*^{(n)}$ is onto and $s = r-1$, we have the following.

Theorem 2.2 *Let L be the non projective part of $\bigotimes_{i=1}^{r-1} L(\zeta_i)$. Suppose that $\Omega^m(L) \simeq L$ and $\varphi_*^{(m)}$ is onto for some $m > 0$. Then there exists $\eta \in H^m(G,k)$ such that $\eta \notin \sqrt{(\zeta_1, \ldots, \zeta_{r-1})}$.*

Example 2.3(cf.[2, Lemma 4.4]) Let G be an elementary abelian p-group of order p^r where p is an odd prime. It is well known that $\Omega^2(M) \simeq M$ for every periodic module M. Let $\zeta_1, \ldots, \zeta_r$ be a h.s.o.p. for $H^*(G,k)$. These elements form a regular sequence and the period of the non projective part of $\bigotimes_{i=1}^{r-1} L(\zeta_i)$ is two since every element of odd degree is nilpotent.

Example 2.4($p = 2$) Let G be an extraspecial 2-group. Hence we have a central extension

$$1 \longrightarrow N \longrightarrow G \longrightarrow \bar{G} \longrightarrow 1$$

such that N is a cyclic group of order 2 and $\bar{G}$ is an elementary abelian 2-group. Let E be a maximal elementary abelian 2-subgroup of G. Then the period of a periodic kG-module devides $2^h = |G : E|$. In [3], Benson and Carlson showed that there exist periodic modules of period 2^i for any $0 \leq i \leq h$ using inflated modules and a result of Andrews. Here we give another example. By a result of Quillen([6]), there exists a regular sequence $\zeta_1, \ldots, \zeta_{r-1}, \eta$ such that $H^*(G,k)/\sqrt{(\zeta_1, \ldots, \zeta_{r-1})} \simeq k[\eta]$ and $deg\ \eta = 2^h$ where $r = 2\text{-}rank(G)$. Then by Lemma 2.1 and Theorem 2.2, the period of (the non projective part of) $\bigotimes_{i=1}^{r-1} L(\zeta_i)$ is exactly 2^h. Next we take a normal subgroup H of G satisfying

(i) $G \neq H, H \supseteq N$

(ii) $G = HC_G(H)$

(iii) if F is a maximal elementary abelian 2-subgroup of H, then $|H : F| = 2^h$ or 2^{h-1}.

Suppose that $0 \leq i \leq h-1$. By induction there exists an indecomposable periodic kH-module M of period 2^i. Since $(M^G)_H \simeq \bigoplus M$ (by (ii)), the period of M^G is 2^i where M^G denotes the induced module.

If p is odd then the cohomology ring of an extraspecial p-group is not necessarily Cohen-Macaulay and our method does not work. In [8], Tezuka and Yagita showed that there exist periodic modules of large period using Andrews' theorem.

Next we state a result on groups of p-rank 2. Examples of the following proposition are found in [5],[7].

Proposition 2.5 *Suppose that $r = 2$ and $deg\ \zeta_1 = 1$ or 2. Let m be a positive integer such that $deg\ \zeta_1 | m$. If $L(\zeta_1^i) \simeq \Omega^m(L(\zeta_1^i))$ for any $i \geq 1$ then there exists $\eta \in H^m(G,k)$ such that ζ_1, η is a h.s.o.p. for $H^*(G,k)$.*

3 Decompostion of varieties and modules

Let $V = V_G(\otimes_{i=1}^{s} L(\zeta_i)) = V_G(\zeta_1, \ldots, \zeta_s) = \cup_{j=1}^{t} V_j$ $(1 \leq s < r)$ where each V_j is connected (i.e.if $V_j = W_1 \cup W_2, W_1, W_2$, closed, $W_1 \cap W_2 = \{0\}$, then W_1 or $W_2 = \{0\}$) closed subvariety, $V_j \neq \{0\}$, $V_i \cap V_j = \{0\}$ for $i \neq j$. Then by [4, Theorem 1'] $\otimes_{i=1}^{s} L(\zeta_i) = \oplus_{j=1}^{t} M_j \oplus (proj)$ where $V_G(M_j) = V_j, M_j$ has no projective summand. If $s = 1$ then M_j is indecomposable by [4, Lemma 4.1]. Here we state some results on indcomposable summands of M_j. Note that our results dose not contain Carlson's result.

Theorem 3.1 *If $V = V_1$ is irreducible then M_1 is indecomposable.*

Theorem 3.2 *If p is odd and $V = V_1$ is connected then M_1 is indecomposable.*

Theorem 3.3 *Suppose that every maximal elementary abelian p-subgroup of G has rank r. Then M_j is indecomposable for $1 \leq j \leq t$.*

References

[1] D. J. Benson, Representations and cohomology II, Cambridge studies in advanced mathematics 31,Cambridge University Press,Cambridge, 1991.

[2] D. J. Benson and J. F. Carlson, *Nilpotent elements in the Green ring*, J.Algebra,104(1986),329-350.

[3] D. J. Benson and J. F. Carlson, *Periodic modules with large period*, Quart.J.Math.Oxford(2),43(1992),283-296.

[4] J. F. Carlson, *The variety of indecomposable module is connected*, Invnt.Math.77(1984),291-299.

[5] T. Okuyama and H. Sasaki, *Periodic modules of large periods for metacyclic p-groups*, J.Algebra,144(1991),8-23.

[6] D. Quillen, *The mod 2 cohomology rings of extra-special 2-groups and the spinor groups* Math.Ann. 194(1971),197-212.

[7] H. Sasaki, *Periodic modules of large periods for extra-special p-groups*, Hokkaido Math.J. 19(1990),461-479.

[8] M. Tezuka and N. Yagita *On some results of the cohomology of extra special p-groups*, Representation theory of finite groups and algebras, RIMS Kokyuroku 877,122-132, Kyoto University, 1994.

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PROCEEDINGS OF THE 27TH
SYMPOSIUM ON RING THEORY, 1994

QUASI-CONTINUOUS MODULES AND THE EXCHANGE PROPERTY

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1. Introduction. A module M is said to satisfy the exchange property or the full exchange property, if whenever M is a direct summand of a module $A = \sum_{i \in I} \oplus A_i$ i.e. $A = M \oplus N$, then $A = M \oplus \sum_{i \in I} \oplus B_i$, for some $B_i \subseteq A_i$, where I is an arbitrary index set. If the index set I is restricted to be finite, then M is said to have the finite exchange property. This useful property introduced in [1], was investigated for various well known classes of modules by several authors (e.g. [2], [4–6], [8], [10], [13–14], [18-26]). Warfield [22], showed that injective modules satisfy the full exchange property and Fuchs [2], proved that the same holds true for quasi-injective modules.

In their fundamental paper [1], Crawley and Jónsson raised an important (and still an open) question:

Does the finite exchange property imply the full exchange property?

Important contributions toward an answer to this question were made by Zimmermann-Huisgen and Zimmermann [25], Harada and Ishii [5], Harada [4], Yamagata [19], [20], and others. In most cases, affirmative answers were obtained under special situations. However, the general question remains open.

Continuous and quasi-continuous modules are interesting generalizations of (quasi-) injective modules (see e.g. [3], [9], [11], [16]). These and their duals, namely the discrete

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The final version of this paper will be submitted for publication elsewhere.

and quasi-discrete modules respectivly ([9], [15]), have been studied in greater detail in recent years. The fact that discrete modules satisfy the full exchange property, and that quasi-discrete modules with the finite exchange property have the full exchange property, can be deduced using the results of Oshiro [15], Harada and Ishii [5], Yamagata [19], [20], and Zimmermann-Huisgen and Zimmermann [25].

The question, whether a continuous module satisfies the exchange property was settled in the affirmative by Mohamed and Mueller [8], using a criterion from [25]. It is clear that the class of quasi-continuous modules does not, in general, satisfy even the finite exchange property (consider for example, the ring of integers $\mathbb{Z}$ as a module over itself). The following question therefore, was raised in the list of open problems in the monograph [9], page 105:

Does a quasi-continuous module with the finite exchange property satisfy the full exchange property?

This question is the focus of our present investigations.

As remarked in [10], in the case when a quasi-continuous module M has an indecomposable decomposition $M = \sum_{i \in I} \oplus M_i$, then the assumption of the finite exchange property for M, implies that each indecomposable M_i has a local endomorphism ring ([21], [9]). This in turn, implies that the *ls Tn* condition holds ([12], [20]) and hence the full exchange property for M holds ([25], [5]). Note that if the ring is right noetherian, such a decomposition for a quasi-continuous module, always exists.

Recently, it was shown that a non-singular quasi-continuous module with the finite exchange property has the full exchange property [10, Theorem 6]. In the special case that the underlying ring satisfies the ascending chain condition on essential right ideals, this question for arbitrary quasi-continuous modules, has also been answered in the affirmative [24, Theorem 2.11]. The general case is complex and has been open till now.

We give a complete answer to this open question by proving that an arbitrary quasi-continuous module with the finite exchange property has the full exchange property. In reference to the long standing open problem posed in [1], our result provides another instance, in which the existence of the finite exchange property implies that of the full exchange property, for a large class of modules. We also provide a new proof that continuous modules have the exchange property using alternate techniques.

2. Preliminaries. Throughout, our ring has an identity element and all modules are unital right modules. Let R be a ring. For any two R-modules X and Y, we denote $X \subseteq_e Y$ and $X \subseteq^{\oplus} Y$, to mean that X is an essential submodule of Y and X is a direct summand of Y, respectively. $|I|$ denotes the cardinality of I, for a set I.

Consider the following conditions for an R-module M.

(C_1) Every submodule X of M is essential in a direct summand X^* of M.

(C_2) Every submodule isomorphic to a direct summand is itself a direct summand of M.

(C_3) If X and Y are direct summands of M with $X \cap Y = 0$, then $X \oplus Y$ is a direct summand.

A module M is called continuous if it satisfies conditions (C_1) and (C_2), quasi-continuous if satisfies (C_1) and (C_3), and extending if it satisfies (C_1) only.

It is well known that the hierarchy is as follows:

Injective $\Rightarrow$ quasi-injective $\Rightarrow$ continuous $\Rightarrow$ quasi-continuous $\Rightarrow$ extending.

We refer to [9], for more details on these concepts, however we include in the following, some facts to be used later.

Let M be an R-module with a decomposition $M = \sum_{i \in I} \oplus M_i$. Then consider the following condition on M:

(A) For any choice of $x_{i_n} \in M_{n_i}$ ($n_i \in I$, i distinct) such that the sequence $(0 : x_{i_1}) \subseteq (0 : x_{i_2}) \subseteq \cdots$ stops after a finite number of steps, where $(0 : x)$ denotes the right annihilator of x. (cf. [9], [12])

Proposition 1 [17], [9]. *Let $M = \sum_I \oplus M_i$ be a quasi-continuous module. Then the following hold.*

(1) *For any $J \subseteq I$, $\sum_J \oplus M_i$ is $\sum_{I-J} \oplus M_i$-injective.*

(2) *$M = \sum_I \oplus M_i$ satisfies the condition A.*

(3) *For any direct summand X of M, there exist $N_i \subseteq^{\oplus} M_i$, such that $M = X \oplus \sum_I \oplus N_i$.*

Recall that a module M is called a square module if $M \simeq X \oplus X$, for some module X, and a module is called square free if it does not contain any non-zero square submodules.

Lemma 2 [9, Theorem 2.37]. *Every quasi-continuous module is a direct sum of a quasi-injective module and a square free module.*

Proposition 3 ([3], [17]). *For an R-module X, the following conditions are equivalent:*

(1) *X is quasi-continuous,*

(2) *any decomposition $E(X) = \sum_I \oplus M_i$ implies $X = \sum_I \oplus (M_i \cap X)$, where $E(X)$ is the injective hull of X,*

(3) *for any R-module Y with $X \subseteq_e Y$, any decomposition $Y = \sum_I \oplus Y_i$ implies $X = \sum_I \oplus (Y_i \cap X)$.*

We will need the following useful characterization of the exchange property.

Lemma 4 [25]. *A module M satisfies the exchange property if and only if for any $A = M \oplus N = \sum_{i \in I} \oplus M_i$, with each $M_i \simeq M$, there exists $M_i^! \subseteq M_i$ for each $i \in I$, such that $A = M \oplus \sum_{i \in I} \oplus M_i^!$.*

3. The Results. In [7], we introduced the concept of relative continuous modules. A module M is defined to be continuous relative to a fixed module if the conditions (C_1) and (C_2) hold for a special subfamily of submodules of M. Our study of questions related to this concept, has led us to an alternate approach to prove that continuous modules satisfy the full exchange property. This approach has also been useful for the question mentioned in relation to the quasi-continuous modules. Our main contribution in this direction is the following key result (Lemma 5). This lemma also provides useful information about arbitrary submodules of a direct sum of extending modules.

Lemma 5 [17]. *Let P be an R-module with a decomposition $P = \sum_I \oplus M_i$ such that each M_i is extending. We consider the index set I as a well ordered set: $I = \{0, 1, \ldots, \omega, \omega+1, \ldots\}$, and let X be a submodule of M. Then there are submodules $T(i) \subseteq_e T(i)^* \subseteq^{\oplus} M_i$, decompositions $M_i = T(i)^* \oplus N_i$ and a submodule $\sum_I \oplus X(i) \subseteq_e X$ for which the following properties hold:*

1) $X(0) = T(0) \subseteq_e T(0)^*$

2) $X(k) \subseteq T(k) \oplus \sum_{i<k} \oplus N_i$, *for all* $k \in I$.

3) $\sigma(X(k)) = T(k) \subseteq_e T(k)^*$, *and* $X(k) \simeq \sigma(X(k))$ *(via* $\sigma|X(k)$*) for all* $k \in I$, *where* σ *is the projection* $\sigma : P = \sum_I \oplus T(i)^* \oplus \sum_I \oplus N_i \to \sum_I \oplus T(i)^*$

4) $X \simeq \sigma(X)$ *(via* $\sigma|X$*).*

We first use Lemma 5, to show an interesting result: every continuous module which is a submodule of a direct sum of copies of itself, is in fact a *summand* of this direct sum without any additional assumptions. This is the following theorem.

Theorem 6 [17]. *Let X be a submodule of an R-module P. If X is continuous and P has a decomposition $P = \sum_{i \in I} \oplus M_i$, with each $M_i \simeq X$, then there exist direct summands $N_i \subseteq^{\oplus} M_i$ for each $i \in I$ such that $P = X \oplus \sum_{i \in I} \oplus N_i$. Therefore, X is a direct summand of P.*

Proof. By Lemma 5, we obtain submodules $T(i) \subseteq_e T(i)^* \subseteq^{\oplus} M_i$, decompositions $M_i = T(i)^* \oplus N_i$, and a submodule $\sum_I \oplus X(i) \subseteq_e X$ for which the properties listed in Lemma 5 hold. As X is quasi-continuous and $X \simeq \sigma(X) \subseteq_e \sum_I \oplus T(i)^*$, then by Proposition 3 we get $\sigma(X) = \sum_I \oplus (T(i)^* \cap \sigma(X))$.

Setting $X(i)^{\#} = \sigma^{-1}(T(i)^* \cap \sigma(X))$, it is easy to observe that $X = \sum_I \oplus X(i)^{\#}$, $X(i) \subseteq_e X(i)^{\#}$ for all $i \in I$, and that $T(i) \subseteq_e \sigma(X(i)^{\#}) \subseteq_e T(i)^*$ for all $i \in I$.

Now as $X \simeq M_i$ is continuous, and $X(i)^{\#} \subseteq^{\oplus} X$, we obtain that $\sigma(X(i)^{\#}) \subseteq^{\oplus} T(i)^{*}$. Therefore, $\sigma(X(i)^{\#}) = T(i)^{*}$ for all $i \in I$. Consequently, $X \simeq \sigma(X) = \sum_{I} \oplus T(i)^{*}$ (via $\sigma|X$). Hence it follows that $P = X \oplus \sum_{I} \oplus N_i$. □

As an immediate consequence of Theorem 6 we obtain:

Theorem 7 ([9, Theorem 3.24], [17]). *Continuous modules have the exchange property.*

Remark. We note that our proof of Theorem 7 is independent of the existence of the exchange property for quasi-injective modules and does not use the criteria for verifying the exchange property provided in Part 3 of [9, Proposition 3.22]. (Compare our proof to that of [9, Theorem 3.24].)

Finally, we state our main result:

Theorem 8 [17]. *Any quasi-continuous module with the finite exchange property satisfies the full exchange property.*

The proof of this result heavily depends on the alternate approach provided by Lemma 5.

REFERENCES

[1] P. Crawley and B. Jónsson, Refinements for infinite direct decompositions of algebraic systems, Pacific J. Math. 14 (1964), 797–855.

[2] L. Fuchs, On quasi-injective modules, Annali Scoula Normal Sup. Pisa 23 (1969), 541–546.

[3] V. K. Goel and S. K. Jain, π-injective modules and rings whose cyclic modules are π-injective, Comm. Algebra 6 (1978), 59–73.

[4] M. Harada, On the exchange property in a direct sum of indecomposable modules, Osaka J. Math. 12 (1975), 719–736.

[5] M. Harada and T. Ishii, On perfect rings and the exchange property, Osaka J. Math. 12 (1975), 483–491.

[6] M. Kutami and K. Oshiro, An example of a ring whose projective modules have the exchange property, Osaka J. Math. 17 (1980), 415–420.

[7] S. R. López-Permouth, K. Oshiro and S. T. Rizvi, On relative (quasi-) continuous modules, preprint.

[8] S. H. Mohamed and B. J. Mueller, Continuous modules have the exchange property, Proc. Perth. Conf. Abelian Groups, Contemporary Math. (1988), 285–289.

[9] S. H. Mohamed and B. J. Mueller, *Continuous and Discrete Modules*, London Math. Soc. Lect. Note Series 147, Cambridge University Press, 1990.

[10] S. H. Mohamed and B. J. Mueller, The exchange property for quasi-continuous modules, *Ring Theory*, World Scientific, (1993), Eds. S. K. Jain and S. T. Rizvi, 242–247.

[11] B. J. Mueller and S. T. Rizvi, On injective and quasi-continuous modules, J. Pure Applied Algebra 28 (1983), 197–210.

[12] B. J. Mueller and S. T. Rizvi, Direct sums of indecomposable modules, Osaka J. Math. 21 (1984), 365–374.

[13] W. K. Nicholson, Lifting idempotents and exchange rings, Trans. Amer. Math. Soc. 229 (1977), 269–278.

[14] K. Oshiro, Projective modules over von Neumann regular rings have the finite exchange property, Osaka J. Math. 20 (1983), 695–699.

[15] K. Oshiro, Semiperfect modules and quasi-semiperfect modules, Osaka J. Math. 20 (1983), 337–372.

[16] K. Oshiro, Continuous and quasi-continuous modules, Osaka J. Math. 20 (1983), 681–694.

[17] K. Oshiro and S. T. Rizvi, The exchange property of quasi-continuous modules with the finite exchange property, preprint.

[18] K. Yamagata, On projective modules with the exchange property, Sci. Rep. Tokyo Kyoiku Daigaku, Sec. A, 12 (1974), 39–48.

[19] K. Yamagata, The exchange property and direct sums of indecomposable injective modules, Pacific J. Math. 55 (1974), 301–317.

[20] K. Yamagata, On rings of finite representation type and modules with the finite exchange property, Sci. Rep. Tokoyo Kyoiku Daigaku Sec. A 13 (1975), 1–6.

[21] R. B. Warfield, A Krull-Schmidt Theorem for infinite sums of modules, Proc. Amer. Math. Soc. 22 (1969), 460–465.

[22] R. B. Warfield, Decomposition of injective modules, Pacific J. Math. 31 (1969), 263–276.

[23] R. B. Warfield, Exchange rings and decompositions of modules, Math. Ann. 199 (1972), 31–36.

[24] H. Yu, On modules for which the finite exchange property implies the countable exchange property, Comm. Alg. 22 (10) (1994), 3887–3901.

[25] B. Zimmermann-Huisgen and W. Zimmermann, Classes of modules with the exchange property, J. Alg. 88 (1984), 416–434.

[26] B. Zimmermann-Huisgen, Exchanging torsion modules over Dedekind domains, Arch. Math. 55 (1990), 241–246.

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PROCEEDINGS OF THE 27TH
SYMPOSIUM ON RING THEORY, 1994

ON FPF RINGS AND A RESULT

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Dedicated to the memory of Professor HISAO TOMINAGA

This note is to give a brief survey concerning FPF rings, including one new result related to the FPF condition over von Neumann regular rings: the first section makes a rough sketch of the background of FPF rings proceeding from PF rings; the second is to expose some of known results on FPF rings to date; and the last is devoted to nonsingular FPF rings, where we shall present a result determining the structure of von Neumann regular rings R over which every cyclic (or finitely generated) faithful right R-module contains a gnerator for Mod-R.

1. From PF rings to FPF rings

The study of *quasi-Frobenius* (*QF*) algebras was, as well known, initiated by the pioneering work of T. Nakayama [21]. About three decades after, one larger class of the rings — inheriting the relationship between faithful modules and generators, the injectivity and the cogeneratorness from QF rings — was investigated by G. Azumaya [1], B.L. Osofsky [22], and Y. Utumi [30]. Every generator for the category Mod-R of all right modules over any ring R is faithful, while conversely every faithful right module over QF rings is a generator; however, the converse is not necessarily true in general. In the context, Azumaya investigated those rings actually satisfying the converse, and showed that they are precisely the direct sums of indecomposable injective right ideals, each of which contains a minimal right ideal, while Utumi called such rings (i.e., rings over which every faithful right module is a generator) *right pseudo-Frobenius* (*right PF*) rings, and obtained much the same result, independently. On the other hand, Osofsky considered what happens if the chain conditions are dropped from QF rings, but if some of the other properties are kept, and showed that

The final detailed version of the last part of this note will be submitted for publication elsewhere.

if a ring R is an injective cogenerator for Mod-R, then R is a direct sum of right ideals, each of which is simple modulo its radical, and that if R is a one-sided perfect ring which is an injective cogenerator on both sides, then R is QF. It thus turned out that the right PF rings are characterized as follows (c.f. T. Kato [17], [18]):

Theorem (Azumaya [1], Osofsky [22], Utumi [30]). (1) *The following conditions on a ring R are equivalent:*

(a) *R is right PF;*

(b) *R is a right self-injective semiperfect ring with essential right socle;*

(c) *R is an injective cogenerator for* Mod-R.

(2) *If R is a one-sided perfect and two-sided PF ring, then R is QF.*

Now, as noted in H. Tachikawa [29], the faithful injective modules play the key to the proof of the theorem above, although all such modules are, of course, not necessarily finitely generated, so that it seems natural to ask what differences it makes to replace the faithful modules in the PF condition by the *finitely generated* faithful modules. One might thus consider the following condition on a ring R:

(C) *Every finitely generated faithful right R-module is a generator for* Mod-R.

Concerning this, the following two results, from which the study of FPF rings would seem to stem, were at once given by S. Endo [5] and H. Tachikawa [29].

On the one hand, Endo systematically investigated some basic properties of QF algebras, and then determined the structure of algebras satisfying the condition (C) over commutative noetherian rings. 'Especially, he gave a result on commutative noetherian rings satisfying the condition (C).

Theorem (Endo [5]). *The following conditions on a commutative noetherian ring R are equivalent:*

(a) *R satisfies the condition (C);*

(b) *R is isomorphic to a finite direct product of Dedekind domains and a QF ring.*

On the other hand, Tachikawa raised a problem whether an artinian ring R is QF if R satisfies the condition (C), and actually obtained a more general result.

Theorem (Tachikawa [29]). *The following conditions on a left perfect ring R are equivalent:*

(a) *R satisfies the condition (C);*

(b) *R is right PF.*

Consequently, a one-sided artinian ring is QF if and only if it satisfies the condition (C).

2. FPF rings — Some of known results to date

C. Faith [6] simplified the long and difficult proof of Tachikawa's Theorem (for it is actually a constructive proof) and generalized the theorem over a semiperfect ring. There he called the rings R satisfying the condition (C) *right finitely pseudo-Frobenius* (*right FPF*) rings, and proved the following.

Theorem (Faith [6]). (1) *Let R be a semiperfect right FPF ring. Then,*

(a) *Every principal indecomposable right ideal is uniform.*

(b) *Every nonzero right ideal of the basic ring for R contains a nonzero two-sided ideal.*

(c) *If the radical of R is nil, then R is right self-injective.*
(This implies Tachikawa's Theorem)

(2) *If R is a one-sided perfect and two-sided FPF ring, then R is QF.*
(This generalizes Osofsky's Theorem)

In [7], Faith also made further investigations of semiperfect or right self-injective right FPF rings. Once these results had been obtained, Faith and others proceeded to the systematical study of FPF rings under more general assumptions, or in individual cases. Some of known results to date are as follows:

<u>Commutative Case.</u>

In this case, Faith gave the decisive result below.

Theorem (Faith [9], [10], [11]) *Let R be a commutative ring. Then,*

(1) *R is FPF if and only if R satisfies* (i) *the classical ring of quotients of R is self-injective, and* (ii) *every finitely generated faithful ideal of R is projective.*

One of the other interesting results is:

(2) *If R is FPF, then any direct product of R is also FPF.*

Faith also noted in [9] that the same assertion as (2) remains true for any basic semiperfect (not necessarily commutative) rings. Thus, it follows, as a remarkable result, that for a (right) FPF ring R which is commutaive or basic semiperfect, any infinite direct product of R is still (right) FPF, but can be no longer (right) PF.

<u>Noetherian Case.</u>

Endo's Theorem determining the structure of commutative noetherian FPF rings was afterwards generalized in non-commutative case by T.G. Faticoni [13] and S. Page [28], independently.

Theorem (Faticoni [13], Page [28]). *The following conditions on a ring R are equivalent:*

(a) *R is two-sided noetherian and two-sided FPF;*

(b) *R is isomorphic to a finite direct product of bounded Dedekind prime rings and a QF ring.*

Semiperfect or finite Goldie dimensional Case.

After the work of Faith [6], [7], the study of semiperfect (or finite Goldie dimensional) FPF rings was improved in more detail (Faith [8], Faith and Page [12], Faticoni [14], [15], Page [26]). One of the recent results is the following.

Theorem (Faticoni [14], [15]). (1) *Let R be a semiperfect ring such that every right regular element of R is two-sided regular. Then,*

(a) *R is right FPF if and only if R satisfies* (i) *every right ideal of the basic ring for R contains a two-sided ideal,* (ii) *every finitely generated faithful right ideal of R is a generator, and* (iii) *R possesses a right self-injective semiperfect classical ring of quotients.*

(b) *R is right PF if and only if R is right FPF and right Kasch.*

(2) *Let R be a right FPF ring. Then,*

(a) *If R has finite right Goldie dimension, then every finitely generated right R-submodule of the injective hull of R_R can be embedded in R.*

(b) *The maximal right quotient ring of R is right self-injective semiperfect if and only if R satisfies* (i) *R has finite right Goldie dimension, and* (ii) *every right regular element of R is two-sided regular.*

Other topics.

S. Page [25] presented a left FPF ring which is not right FPF (c.f. F. Dischinger and W. Müller [4] constructed a one-sided PF ring).

D. Herbera and P. Menal [16] answered some of questions concerning FPF rings raised in [12] (e.g. semiprime FPF rings, centres and Galois subrings of FPF rings, and group rings over FPF rings).

For most of the basic results on FPF rings, one may consult Faith and Page [12], which includes a list of problems. One of the most interesting problems is:

Are all (right) FPF rings are (right) thin?

where a ring R is *right thin* if there exists a positive integer n such that R contains no direct sums of $n+1$ nonzero pairwise isomorphic right ideals. The problem is settled to be affirmative for all known classes of right FPF rings: e.g. commutative rings, or semiperfect rings, or right nonsingular rings, or right self-injective rings (See [12]).

3. Nonsingular FPF rings and a related result

In this section, we shall consider the FPF condition particularly over nonsingular rings, and present a related result.

A striking difference between the PF rings and the FPF ones is the fact that the injectivity is dropped in general from FPF rings, so that there actually exist non-artinian nonsingular FPF rings. This class of the rings was investigated by G.F. Birkenmeier [2], W.D. Burgess [3], S. Kobayashi [19], [20], S. Page [23], [24], [27], etc. Amongst others, we especially note the following results due to Page, and due to Kobayashi.

Theorem (Page [23], [24]). (1) *If R is a right nonsingular and right FPF ring, then the maximal right quotient ring of R is also right FPF.*

(2) *A (von Neumann) regular ring R is right FPF if and only if R is right self-injective of bounded index of nilpotence. Consequently, the FPF condition is left-right symmetric for regular rings.*

Theorem (Kobayashi [19]). *Let R be a ring with Q the maximal right quotient ring. Then R is right nonsingular and right FPF if and only if R satisfies* (i) *R is right bounded, i.e., every essential right ideal of R contains a two-sided ideal which is essential as a right ideal,* (ii) *the multiplication map $Q \otimes_R Q \to Q$ is an isomorphism, and Q is flat as a right R-module, and* (iii) *for every finitely generated right ideal A of R, it holds that $R = Tr_R(A) \oplus r_R(A)$, where $Tr_R(A)$ and $r_R(A)$ are the trace ideal and the right annihilator ideal of A in R, respectively.*

In view of the theorems above, we shall consider the FPF condition with our attention restricted to regular rings. But then, it may arouse our interest to exist a regular ring R over which, although R fails to be FPF, every cyclic (or finitely generated) faithful right R-module still " *contains* " a generator for Mod-R. For instance, choose a division ring D_n containing a division subring E_n for $n = 1, 2, \dots$, and let k (≥ 2) be an integer. Let us now consider the regular ring R which consists of all sequences (x_n), where each $x_n \in M_k(D_n)$, the ring of all $k \times k$ matrices over D_n, such that $x_n \in M_k(E_n)$ for all but finitely many n. Then, R is FPF only when $D_n = E_n$ for all n, whence in case some E_n is properly contained in D_n, the ring R can be no longer FPF. However, it is shown that every cyclic (finitely generated) faithful right R-module does contain a submodule which is a generator for Mod-R. Such being the case, we shall be concerned with the following two conditions on a ring R:

(A) *Every cyclic faithful right R-module contains a submodule which is a generator for* Mod-R*;*

(B) *Every finitely generated faithful right R-module contains a submodule which is a generator for* Mod-R.

In fact, we obtain the following.

Theorem. *Let R be a regular ring. Then,*

(1) *R satisfies the condition (A) if and only if $R \cong \prod_{i=1}^{k} M_{n(i)}(S_i)$, where $n(1) = 1$, and $n(i) \geq 2$ for $i = 2, 3, \ldots, k$, and where each S_i is an abelian regular ring such that for $i = 2, 3, \ldots, k$, every finitely generated faithful right S_i-submodule of the maximal right quotient ring $Q(S_i)$ of S_i contains a unit in $Q(S_i)$.*

(2) *R satisfies the condition (B) if and only if $R \cong \prod_{i=1}^{k} M_{n(i)}(S_i)$, where each S_i is an abelian regular ring such that every finitely generated faithful right S_i-submodule of $Q(S_i)$ contains a unit in $Q(S_i)$.*

The theorem combined with an observation immediately implies the following.

Corollary. *Let R be a regular ring. Then,*

(1) (c.f. Kobayashi [20]) *Every cyclic faithful right R-module is a generator for* Mod-R *if and only if R is isomorphic to a finite direct product of an abelian regular ring and full matrix rings over self-injective abelian regular rings.*

(2) (Page [23]) *R is right FPF if and only if R is isomorphic to a finite direct product of full matrix rings over self-injective abelian regular rings.*

References

[1] G. Azumaya: *Completely faithful modules and self-injective rings*, Nagoya Math. J. **27**(1966), 697-708.

[2] G.F. Birkenmeier: *A generalization of FPF rings*, Comm. Algebra **17**(1989), 855-884.

[3] W.D. Burgess: *On nonsingular right FPF rings*, Comm. Algebra **12**(1984), 1729-1750.

[4] F. Dischinger and W. Müller: *Left PF is not right PF*, Comm. Algebra **14**(1986), 1223-1227.

[5] S. Endo: *Completely faithful modules and quasi-Frobenius algebras*, J. Math. Soc. Japan **19**(1967), 437-456.

[6] C. Faith: *Injective cogenerator rings and a theorem of Tachikawa*, Proc. Amer. Math. Soc. **60**(1976), 25-30.

[7] ______: *Injective cogenerator rings and a theorem of Tachikawa II*, Proc. Amer. Math. Soc. **62**(1977), 15-18.

[8] ______: *Semiperfect Prüfer rings and FPF rings*, Isr. J. Math. **26**(1977), 166-177.

[9] ______: *The genus of a module and generic families of rings*, Lecture Notes in Pure and Appl. Math. **51**, Marcel Dekker, New York, 1979, 613-629.

[10] C. Faith: *Injective quotient rings of commutative rings*, Lecture Notes in Math. **700**, Springer-Verlag, New York, 1979, 151-203.

[11] ______: Injective Modules and Injective Quotient Rings, Lecture Notes in Pure and Appl. Math. **72**, Marcel Dekker, New York, 1982.

[12] C. Faith and S. Page: FPF Ring Theory, London Math. Soc. Lecture Note Ser. **88**, Cambridge Univ. Press, Cambridge, 1984.

[13] T.G. Faticoni: *FPF rings I: The Noetherian case*, Comm. Algebra **13**(1985), 2119-2136.

[14] ______: *Semi-perfect FPF rings and applications*, J. Algebra **107**(1987), 297-315.

[15] ______: *Localization in finite dimensional FPF rings*, Pacific J. Math. **134**(1988), 79-99.

[16] D. Herbera and P. Menal: *On rings whose finitely generated faithful modules are generators*, J. Algebra **122**(1989), 425-438.

[17] T. Kato: *Self-injective rings*, Tôhoku Math. J. **19**(1967), 485-495.

[18] ______: *Some generalizations of QF-rings*, Proc. Japan Acad. **44**(1968), 114-119.

[19] S. Kobayashi: *On non-singular FPF-rings I, II*, Osaka J. Math. **22**(1985), 787-795, 797-803.

[20] ______: *On regular rings whose cyclic faithful modules are generator*, Math. J. Okayama Univ. **30**(1988), 45-52.

[21] T. Nakayama: *On Frobeniusean algebras I, II*, Ann. Math. **40**(1939), 611-633, **42**(1941), 1-21.

[22] B.L. Osofsky: *A generalization of quasi-Frobenius rings*, J. Algebra **4**(1966), 373-387; *Erratum*: J. Algebra **9**(1968), 120.

[23] S. Page: *Regular FPF rings*, Pacific J. Math. **79**(1978), 169-176; *Correction*: Pacific J. Math. **97**(1981), 488-490.

[24] ______: *Semi-prime and non-singular FPF rings*, Comm. Algebra **10**(1982), 2253-2259.

[25] ______: *FPF rings and some conjectures of C. Faith*, Canad. Math. Bull. **26**(1983), 257-259.

[26] ______: *Semiperfect FPF rings*, Proc. Amer. Math. Soc. **89**(1983), 395-401.

[27] ______: *Semihereditary and fully idempotent FPF rings*, Comm. Algebra **11**(1983), 227-242.

[28] ______: *FQF-3 rings*, Math. J. Okayama Univ. **30**(1988), 79-91.

[29] H. Tachikawa: *A generalization of quasi-Frobenius rings*, Proc. Amer. Math. Soc. **20**(1969), 471-476.

[30] Y. Utumi: *Self-injective rings*, J. Algebra **6**(1967), 56-64.

[31] H. Yoshimura: *Rings over which all cyclic faithful modules are generators*, submitted.

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PROCEEDINGS OF THE 27TH
SYMPOSIUM ON RING THEORY, 1994

CLASSIFICATION OF PRIMARY ORDERS OF FINITE REPRESENTATION TYPE

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0. This is a joint work with H. Hijikata.

Let R be a complete discrete valuation ring with the residue field k and the quotient field K. Let Λ be an R-order in a finite dimensional semisimple K-algebra. We call Λ to be primary if it is a local ring. Let $\mathcal{M}^0$ be the category of all R-lattices. Put lat $\Lambda :=$ $\mathcal{M}^0 \cap \text{mod}\Lambda$, where mod Λ is the category of all finitely generated left Λ-modules. We call $M \in$ lat Λ a (left) Λ-lattice. We need two definitions:

(0.1) Λ, FRT(finite representation type) $\overset{def}{\Longleftrightarrow}$ The number of isomorphism classes of ind lat Λ is finite.

(0.2) Λ, WB(well-behaved) $\overset{def}{\Longleftrightarrow}$

0) $\underline{\Lambda} := \cap\{\Gamma : \Gamma$ is a maximal order in A containing $\Lambda\}$ is hereditary;

1) $\mu_\Lambda(\underline{\Lambda}) \leq 3$, where $\mu_\Lambda(X)$ denotes the minimal number of generators of a Λ-module X.

2) $\mu_\Lambda(\text{rad}_\Lambda(\underline{\Lambda}/\Lambda)) \leq 1$.

Let Λ be a primary R-order.

The main theorem of [DK] asserts:

$$\Lambda \text{ is of FRT} \Longleftrightarrow \Lambda \text{ is WB}.$$

Unfortunately, the claim $(\Leftarrow)$ is not correct. However the proof of [DK] can easily be repaired to yield a correct answer:

(0.3) Λ is of FRT $\Longleftrightarrow$ Λ is WB, and $\underline{\Lambda}(\text{rad}\Lambda)$ is a principal $\underline{\Lambda}$-ideal.

Mostly (i.e., unless $A = M_2(D)$ with a sfield D having the maximal order O such that $O/\text{rad}O$ is not separable over k), the added principality condition follows automatically from the other conditions 0) 1) 2). This fact seems to cause some mixed up in the proof of $(\Leftarrow)$ in [DK].

The final version of this paper will be submitted for publication elsewhere.

We shall classify (up to a reasonable sense) all primary orders of FRT, as well as all of their indecomposable lattices. By the elementary part of the method of [DK], together with a classification of Bass orders [HN], we can describe all well-behaved Λ. If $\underline{\Lambda}(\mathrm{rad}\Lambda)$ is principal, we can determine, in a rather unified manner, a connected component of the Auslander-Reiten quiver of Λ, which turns out to be finite. Hence we get also a new proof of ($\Leftarrow$) (0.3). In this report we state the results very briefly.

1. AHP-pairs. A pair (Λ_0,Λ), where Λ is an order and Λ_0 an over order of Λ, is called admissible if $(\mathrm{rad}\Lambda)\Lambda_0 = \Lambda_0(\mathrm{rad}\Lambda)$. Suppose (Λ_0,Λ) is admissible. Put

$$\mathcal{R} := \mathrm{rad}\Lambda, \quad \mathcal{N} := \mathcal{R}\Lambda_0 = \Lambda_0\mathcal{R},$$
$$\Lambda_i := \Lambda + \mathcal{N}^i, \quad \mathcal{R}_i := \mathcal{R} + \mathcal{N}^i \quad (i \geq 1),$$
$$s = s(\Lambda_0, \Lambda) := \inf\{i : \mathcal{N}^i \subset \Lambda\}.$$

Then Λ_i is an R-order such that $\Lambda_0 \supset \Lambda_i \supset \Lambda_{i+1} \supset \Lambda$ and $\mathcal{R}_i = \mathrm{rad}\Lambda_i$. We call s the rank of (Λ_0,Λ). Further, if Λ is primary (i.e. $\Lambda/\mathrm{rad}\Lambda$ is a sfield), then $\Lambda/\mathcal{R} \cong \Lambda_i/\mathcal{R}_i (i \geq 1)$.

Put $\Gamma_{i-1} := O_r(\mathcal{R}_i) = \{x \in A : \mathcal{R}_i x \subset \mathcal{R}_i\}$. We get the following diagram:

$$\begin{array}{ccccccccccc}
\Gamma_0 & \leftarrow & \Gamma_1 & \leftarrow & \dots & \leftarrow & \Gamma_{s-1} & & & & \\
\uparrow & & \uparrow & & & & \uparrow & & & & \\
\Lambda_0 & \leftarrow & \Lambda_1 & \leftarrow & \dots & \leftarrow & \Lambda_{s-1} & \leftarrow & \Lambda_s & = \Lambda & \\
 & & \uparrow & & & & \uparrow & & \uparrow & & \\
 & & \mathcal{R}_1 & \leftarrow & \dots & \leftarrow & \mathcal{R}_{s-1} & \leftarrow & \mathcal{R}_s & = \mathcal{R}, &
\end{array}$$

where $\Lambda_i \to \Gamma_i (0 \leq i < s)$ is an identity or a proper inclusion, all other $\to$'s are proper inclusions.

An admissible pair (Λ_0,Λ) is called an AHP-pair if Λ_0 is hereditary and Λ is primary.

1.1. Remark. i) If Λ is primary WB then $(\underline{\Lambda},\Lambda)$ is an AHP-pair with $\mu_\Lambda(\underline{\Lambda}) \leq 3$.

ii) If $\Lambda = \Lambda_s$ is non-Bass Gorenstein then $\Gamma_{s-1} = \Lambda_{s-1}$ is not Gorenstein, so we assume that Λ is Gorenstein. (The Gorenstein cases will be treated elsewhere.)

2. Let (Λ_0,Λ) be an AHP-pair with $\mu_\Lambda(\Lambda_0) \leq 3$. We shall classify such pairs.
It is easily seen that:
(2.0) $\mu_\Lambda(\Lambda_0) \leq 1 \Leftrightarrow \Lambda_0 = \Lambda_1 = \Lambda$ is a maximal order in a sfield.
(2.1) $\mu_\Lambda(\Lambda_0) = 2 \Leftrightarrow \Lambda$ is a non-hereditary primary Bass order.
Thus there remains the essential case where $\mu_\Lambda(\Lambda_0) = 3$.

2.1. Let Λ_0 be a hereditary order and $\mathcal{N}$ its two-sided ideal. We say that $(\Lambda_0,\mathcal{N})$ extends to an AHP-pair of rank s if and only if there exists an order Λ such that (Λ_0,Λ) is an AHP-pair with $\mathcal{N} = \mathcal{R}_1$ and $s = s(\Lambda_0,\Lambda)$.

2.1.0. A pair $(\Lambda_0,\mathcal{N})$ extends to an AHP-pair (Λ_0,Λ) of rank one and $\mu_\Lambda(\Lambda_0) = 3$ if and only if $\Lambda_0/\mathcal{N}$ contains a sfield $\mathcal{D}$ as an R-subalgebra with $\dim_{\mathcal{D}}\Lambda_0/\mathcal{N} = 3$. Indeed, putting $\Lambda_1 := \varphi^{-1}(\mathcal{D})$, where $\varphi : \Lambda_0 \to \Lambda_0/\mathcal{N}$ is the canonical projection, we see that (Λ_0,Λ_1) is an AHP-pair of rank one with $\mu_{\Lambda_1}(\Lambda_0) = 3$.

2.1.1. If A is not simple, then $(\Lambda_0,\mathcal{N})$ extends to an AHP-pair of rank one if and only if it is one of the following:

(I) $\Lambda_0 = \Omega \oplus M_2(O)$ $\mathcal{N} = \mathrm{rad}\Lambda_0$ $M_2(O/\wp)$ contains $\mathcal{D}$ with index 2;

(II) $\Lambda_0 = \Omega \oplus \begin{pmatrix} O & O \\ \wp & O \end{pmatrix}$ $\mathcal{N} = \mathrm{rad}\Lambda_0$ $O/\wp \cong \mathcal{D}$;

(III) $\Lambda_0 = \Omega \oplus (O \oplus O')$ $\mathcal{N} = \mathrm{rad}\Lambda_0$ $O/\wp \cong O'/\wp' \cong \mathcal{D}$;

(IVa) $\Lambda_0 = \Omega \oplus O$ $\mathcal{N} = \mathrm{rad}\Lambda_0$ $O/\wp$ contains $\mathcal{D}$ with index 2;

(IVb) $\Lambda_0 = \Omega \oplus O$ $\mathcal{N} = \mathrm{rad}\Omega \oplus \wp^2$ $O/\wp \cong \mathcal{D}$ $0 \to \wp/\wp^2 \to O/\wp^2 \to O/\wp \to 0$ splits,

where O(resp. O') is the maximal order in a sfield D(resp. D'), $\wp = \mathrm{rad}O$(resp. $\wp' = \mathrm{rad}O'$), Ω is the maxial order in a sfield, and $\mathcal{D} := \Omega/\mathrm{rad}\Omega$.

2.1.2. If A is simple, then $(\Lambda_0,\mathcal{N})$ extends to an AHP-pair of rank one if and only if it is $A^\times$-conjugate to one of the following:

(Va) $\Lambda_0 = O$ $\mathcal{N} = \mathrm{rad}\Lambda_0$ $O/\wp$ contains $\mathcal{D}$ with index 3;

(Vb) $\Lambda_0 = O$ $\mathcal{N} = \wp^3$ $O/\wp \cong \mathcal{D}$ $0 \to \wp/\wp^3 \to O/\wp^3 \to O/\wp \to 0$ splits;

(VI) $\Lambda_0 = M_3(O)$ $\mathcal{N} = \mathrm{rad}\Lambda_0$ $M_3(O/\wp)$ contains $\mathcal{D}$ with index 3;

(VII) $\Lambda_0 = \begin{pmatrix} O & O & O \\ \wp & O & O \\ \wp & \wp & O \end{pmatrix}$ $\mathcal{N} = \mathrm{rad}\Lambda_0$ $O/\wp \cong \mathcal{D}$;

(VIII) $\Lambda_0 = \begin{pmatrix} O & O \\ \wp & O \end{pmatrix}$ $\mathcal{N} = M_2(\wp)$ $O/\wp \cong \mathcal{D}$,

where O is the maximal order in a sfield D, $\wp = \mathrm{rad}O$, and $\mathcal{D}$ is a sfield.

2.1.3. Remark. The case (VIII) actually occurs. In fact, some Λ_1 can satisfy (0.2) 0), i.e., WB but not of FRT (cf. (0.3)).

3. Let (Λ_0,Λ) be an AHP-pair of rank s with $\mu_\Lambda(\Lambda_0) = 3$. We briefly describe the case of $s \geq 2$.

Suppose that there exists a non-trivial central idempotent $e \in \Lambda_0$. Choosing e properly, we get $\mu_{\Lambda e}(\Lambda_0 e/\Lambda_1 e) = 1$ and $\mu_{\Lambda(1-e)}(\Lambda_0(1-e)/\Lambda_1(1-e)) = 0$. It follows from (2.0) and (2.1) that Λe is non-hereditary primary Bass and $\Lambda(1-e) = \Omega$ is a maximal order in a sfield.

3.1. Assume that there exists a nontrivial central idempotent $e \in \Gamma_1$. Then Λ is a pullback of $\Omega \to \Omega/\mathrm{rad}\Omega \cong \mathcal{D} \cong \Delta_s/\mathrm{rad}\Delta_s \leftarrow \Delta_s$, where Δ_s is a primary Bass order of rank s. Its indecomposable lattices are given by:

$$\text{ind lat } \Lambda_s = \text{ind lat } \Lambda_{s-1} \cup \{ \Lambda_s, \Lambda_s^*, \Delta_s, \tau\Delta_{s-1} \},$$

where $\tau\Delta_{s-1}$ is the Auslander transform of Δ_{s-1} in lat Λ_s, and characterized as the unique (up to isomorphisms) subdirect product such that

$$\Omega^{(2)} \times \Delta_{s-1} \supset \tau\Delta_{s-1} \supset (\mathrm{rad}\Omega)^{(2)} \times (\mathrm{rad}\Delta_s).$$

3.2. Otherwise (i.e., A is simple or A is not simple but Γ_1 contains no non-trivial central idempotent), we can show that $s \leq 2$ and can fully describe Λ_1 and Λ_2.

4. All orders obtained in 3 except type (VIII) are of FRT. This is shown by constructing a finite connected component of the Auslander-Reiten quiver of Λ. Thus we have obtained the classification of non-Bass primary orders of FRT as well as the classification of its indecomposable lattices. Together with the classification of Bass orders[HN], we complete that of primary ones of FRT.

References

[DK] Yu. A. Drozd and V. V. Kirichenko, *Primary orders with a finite number of indecomposable representations*, Math. USSR Izvestija 7(1973), 711-732.

[HN] H. Hijikata and K.Nishida, *Classification of Bass orders*, J. reine angew. Math. 431(1992), 191-220.

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PROCEEDINGS OF THE 27TH
SYMPOSIUM ON RING THEORY, 1994

THE LAST TERM OF A MINIMAL INJECTIVE RESOLUTION FOR A GORENSTEIN RING

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Introduction.

In view of the work of Bass [1] for commutative Gorenstein rings, which we will state later, we may consider two different conditions for the definition of non-commutative Gorenstein rings.

The first : Noetherian rings with finite self-injective dimension, say n.

The second : Noetherian rings R satisfying the property that each term E_i of a minimal injective resolution for ${}_RR$ is of flat dimension at most i for every $i < k$. Here k is fixed, and is allowed to be ∞.

The first condition is a direct generalization for non-commutative rings. But in that case it has not been settled in general whether or not the left self-injective dimension coincides with the right one. On the other hand, the second condition has been shown by Auslander to be left-right symmetric ([4]). The first condition does not imply the second condition. But it is unknown whether or not the second condition implies that the ring is of finite self-injective dimension even though the ring considered is an artin algebra.

If both the left and the right self-injective dimensions are finite, then they coincide (Zaks[15]). In this paper, such a ring is called a *Gorenstein* ring. On the other hand, a noetherian ring satisfying the second condition is usually called a

The detailed version of this paper has been submitted for publication elsewhere

k-Gorenstein ring. But, to avoid confusion, we won't use the combined term "a k-Gorenstein and Gorenstein ring" . We will instead call such a ring a *k-Gorenstein ring of finite self-injective dimension.*

Watanabe(K.-i.)says in [14] that in the commutative case the Gorenstein property is computable. On the contrary, in the non-commutative case, we cannot usually compute the Gorenstein property for a given ring. This is why the study of Gorenstein rings has tended to be limited to low-dimensional cases. Since about twenty years ago, differential operator algebras over algebraic varieties has been studied and it has been found that many of those rings are ∞-Gorenstein rings of finite self-injective dimension. (See Björk[2] and Levasseur[12]). Of course such examples are worth studying in themselves and, at the same time, the authors believe that they give *raison d'être* to the abstract study of higher dimensional Gorenstein rings. For a more detailed survey of differential operator algebras, refer to Iwanaga [9].

We are greatly interested in the last term of a minimal injective resolution for a Gorenstein ring and we will consider conditions which imply that the socle of the last term is nonzero or essential in the last term. Our purpose in this paper is to discuss the importance of the last term as well as to state our results.

Finally we should remark that some of our statements here are not necessarily in the most general form to make our subject clear. Refer to our paper [11] for them.

Notation and Setting

Throughout this paper, R stands for a left and right noetherian ring. For a (left or right) R-module M, we denote the projective dimension of M by $pd(M)$, the injective dimension by $id(M)$ and the flat dimension by $fd(M)$. When we specify its side, for example, the projective dimension of the left R-module M, we denote it by $pd({}_RM)$. If $id({}_RR) = id(R_R)$ then we denote it by $id(R)$. Also we denote the socle of M by $soc(M)$ and the injective hull of M by $E(M)$. Furthermore we denote a minimal injective resolution for ${}_RR$ by

$$0 \to {}_RR \to E_0 \to E_1 \to \cdots \to E_i \to E_{i+1} \to \cdots .$$

When $id({}_RR) = n$, its resolution is of the form:

$$0 \to {}_RR \to E_0 \to E_1 \to \cdots \to E_{n-1} \to E_n \to 0,$$

and then we will call E_n the last term for a minimal injective resolution for ${}_RR$, or the last term for a Gorenstein ring ${}_RR$ in short.

§1. Generalities and Commutative Gorenstein Rings

Let us recall the work of Bass, which is our motive to consider non-commutative Gorenstein rings.

Theorem [1]. *Let R be a commutative noetherian ring such that $id(R_P) < \infty$ for any $P \in Spec(R)$. Then we have the following*

(1) $E(R/P)$ is a direct summand of E_j if and only if the height of P is exactly j for $P \in Spec(R)$.

(2) Any direct summand of E_j has flat dimension exactly j.

(3) R is a Gorenstein ring in our sense, that is, $id(R) = n < \infty$ if and only if Krull-dim(R), the Krull dimension of R, is exactly $n < \infty$.

Making use of our definitions in the Introduction, we see the following immediately.

Corollary. *Let R be a commutative Gorenstein ring with $id(R) = n < \infty$. Then we have the following.*

(1) $fd(E_j) = j$ for any j, in particular, R is ∞-Gorenstein.

(2) The direct sum $W = E_0 \oplus E_1 \oplus \cdots \oplus E_n$ is a finitely embedding cogenerator, that is, any finitely generated R-module can be embedded into a finite direct sum of

copies of W. or equivalently. every injective indecomposable R-module is isomorphic to a direct summand of W.

(3) E_i and E_j have no isomorphic indecomposable summand for $i \neq j$.

(4) $soc(E_n)$ is essential in (E_n) . in particular $soc(E_n) \neq 0$.

In the rest of this section. we would like to consider whether the corresponding statements in the corollary above hold or not for the non-commutative case.

The statement corresponding to (1) in the corollary does not hold in the non-commutative case. as mentioned in the Introduction. For the statement corresponding to (2), Iwanaga showed the following.

Theorem[8]. *Let R be a left noetherian ring with $id(R_R) = n < \infty$. Then $W = E_0 \oplus E_1 \oplus \cdots \oplus E_n$ is an injective cogenerator.*

Also Colby and Fuller generalized Iwanaga's theorem above and gave a quite useful formulation for it. as follows.

Theorem[3]. *Let R be a left noetherian ring with $id(R_R) = n < \infty$ and M any finitely generated left R-module. If $Ext^k_R(M, R) = 0$ for all $k \geq 0$, then we have $M = 0$.*

But, in the non-commutative case, it is not known yet whether or not $W = E_0 \oplus E_1 \oplus \cdots \oplus E_n$ is a finitely embedding injective cogenerator in general. Here we should mention the fundamental theorem due to Auslander for k-Gorenstein rings.

Theorem [4]. *The following are equivalent for a (left and right) noetherian ring R and an integer $n \geq 1$:*

(1) $fd(E_i) \leq i$ for any i $(0 \leq i < n)$.

(2) For any finitely generated right R-module X_R and any j $(1 \leq j \leq n)$. we have $Ext^i_R(M, R) = 0$ if ${}_RM$ is a submodule of $Ext^j_R(X, R)$ and if $i < j$.

(3) The dual of (1).

(4) The dual of (2).

The conditons (2) and (4) in the above are usually called "Auslander's condition", and a noetherian ring satisfying Auslander's condition is usually called "a k-Gorenstein ring". Here k may be ∞, and an ∞-Gorenstein ring is sometimes called an Auslander ring.

Now let us return to the problem whether or not $W = E_0 \oplus E_1 \oplus \cdots \oplus E_n$ is a finitely embedding injective cogenerator for an n-Gorenstein ring. Hoshino has shown the following.

Theorem[6]. *Let R be an ∞-Gorenstein ring. Then we have*

(1) $id(R_R) < \infty$ implies $id({}_RR) < \infty$, and so $id({}_RR) = id(R_R)$.

(2) $W = E_0 \oplus E_1 \oplus \cdots \oplus E_n$ is a finitely embedding cogenerator.

This concludes our discussion of the statement (2) in the Corollary.

Next, the statement (3) in the Corollary above does not hold in general even under the assumption that the ring considered is ∞-Gorenstein. Nevertheless we have shown the following.

Theorem[10]. *Let R be a Gorenstein ring with $id({}_RR) = n > 0$. Then E_0 and E_n have no isomorphic indecomposable summand. In particular, every finitely generated submodule of the last term E_n cannot be torsionless.*

It is to be noted that the first half of our statement in the Theorem just above does hold without the assumption that R is a noetherian ring. Refer to our paper [11] for it.

The above is almost all of the facts which hold in the non-commutative case concerning the corresponding statements in the Corollary except (4). In the rest of this paper, we will concentrate our consideration on the last term of a minimal injective resolution for a Gorenstein ring.

§2. The Last Term for a Gorenstein Ring

In this section, R stands for a Gorenstein ring with $id(R) = n$. As stated in the Introduction, we will consider the following questions all of which Iwanaga took up in [9] :

Questions

(a) Does it hold that any nonzero direct summand of E_n is of flat dimension exactly n ?

(b) Does it hold that $soc(E_n)$ is essential in E_n ?

(c) Is it true that $soc(E_n)$ is nonzero ?

(As is trivially seen, (b) implies (c)).

The first result follows from Iwanaga's theorem in §1 and some of the elementary properties of the functors Ext^n_R and Tor^R_n, and it shows that the answer to (a) is affirmative for any Gorenstein ring, and that the last term E_n reflects the features of Gorenstein rings.

Theorem A. *Let R be a Gorenstein ring with $id(R) = n$. Then we have the following.*

(1) For any nonzero submodule X of E_n, we have $Ext^n_R(X, R) \neq 0$ and so $pd(X) \geq 0$.

(2) For any finitely generated submodule U of E_n, there exists a simple right R-module S_R and finitely generated submodules V_1, V_2 of $E(S)$ satisfying

$$Tor^R_n(E(S), U) \neq 0, \ Tor^R_n(E(S), E(U)) \neq 0$$

and

$$Tor^R_n(V_1, U) \neq 0, \ Tor^R_n(V_2, E(U)) \neq 0.$$

Thus $fd(E(S)) = fd(E(U)) = n$.

(3) For any nonzero direct summand E of E_n, we have $pd(E) = fd(E) = n$.

We immediately have the following from Theorem A.

Corollary. *Let R be an n-Gorenstein ring with $id(R) = n$. Then R is ∞-Gorenstein, and hence $E_0 \oplus E_1 \oplus \cdots \oplus E_{n-1}$ and E_n have no isomorphic indecomposable summand.*

Next, we will investigate the socle of the last term E_n of a minimal injective resolution for a Gorenstein ring R with $id(R) = n$. We have not settled in general the questions (b) and (c) stated in the beginning of this section. So we will impose some suitable conditions on a Gorenstein ring.

We will consider the folloing three classes of Gorenstein rings:

(a): ∞-Gorenstein rings with finite self-injective dimension.

(b): Fully bounded noetherian rings with finite self-injective dimension.

(c): Noetherian rings with finite global dimension.

Let us recall the definition of a fully bounded noetherian ring.

A prime noetherian ring is said to be bounded in case every one-sided essential ideal contains some nonzero two-sided ideal, and a noetherian ring is said to be fully bounded in case each of its prime factor rings is bounded.

It is well known that a module-finite algebra over a commutative noetherian ring is always fully bounded.(Cf.[5]) Therefore any commutative Gorenstein ring belongs to the class (b), and also to the class (a). (See Corollary in §1.)

It is not too hard to show that a noetherian ring with finite global dimension, say n, is of finite self-injective dimension exactly n and so it is a Gorenstein ring with $id(R) = n$.

Our main result is the following.

Theorem B. *Let R be an n-Gorenstein ring with $id(R) = n$. Then $soc(E_n)$ is essential in E_n. Here E_n is the last term of a minimal injective resolution of ${}_RR$.*

Theorem B above is a direct consequence of the proposition below.

Proposition. *Let R be an n-Gorenstein ring with $id(R) = n$. Then $Ext_R^n(M, R)$ is an artinian right R-module for any finitely generated left R-module M.*

The proposition just above can be proved by using the thorem of Colby and Fuller and Auslander's condition stated in §1.

Now, as a result of Theorems A and B, we have the following.

Corollary. *Let R be an n-Gorenstein ring with $id(R) = n$, and E an injective indecomposable left R-module. Then $fd(E) = n$ if and only if there exists a simple submodule of E_n such that $E = E(S)$.*

We will now turn our attention to fully bounded noetherian rings with finite self-injective dimension. As mentioned before, the class of such rings contains all commutative Gorenstein rings. But such a ring is not always ∞-Gorenstein in the non-commutative case. Next, noetherian rings with finite global dimension n (say) are of finite self-injective dimension exactly n, and are not always ∞-Gorenstein in the non-commutative case. Nevertheless we have the following.

Theorem C. *Let R be a noetherian ring satisfying one of the following conditions:*

(a) R is a fully bounded ring with $id(R) = n$, or,

(b) R has finite global dimension n.

Then the last term E_n for R has the nonzero socle.

Besides the above, we know another sufficient condition so that the last term E_n has the nonzero socle, which was shown by Hoshino.

Proposition [7]. *Let R be a Gorenstein ring with $id(R) = n \leq 2$. Then the last term E_n has nonzero socle.*

It has not been settled in general whether the last term for a Gorenstein ring has nonzero socle or not.

§3. Applications and Remarks

Concerning Iwanaga's theorem in §1, we will take up the following question:

Question. For a Gorenstein ring R of finite self-injective dimension n, when is E_0 or E_n a cogenerator?

In this section R stands for a Gorenstein ring of finite self-injective dimension n. At first, we will consider the case where the first term E_0 is a cogenerator. The following is immediate by the last theorem in §1.

Proposition. *Let R be a ring as above. Assume that E_0 is a cogenerator and $soc(E_n) \neq 0$. Then R is a quasi-Frobenius ring.*

Let R be a Gorenstein ring with $id(R) \leq 2$. Then $soc(E_n) \neq 0$ by the last proposition in §2. If E_0 is a cogenerator, then it follows from the proposition above that R is a quasi-Frobenius ring. We see that R is its own maximal quotient ring if and only if E_0 is a cogenerator in case $id(R) \leq 1$. Taking account of these facts as well as Theorem C in §2, we have the following.

Corollary. *Let R be a Gorenstein ring of finite self-injective dimension n. Then the following statements hold.*

(1) If R is its own maximal quotient ring and $n \leq 1$, then R is a quasi-Frobenius ring.

(2) If E_0 is a cogenerator and $n \leq 2$, then R is a quasi-Frobenius ring.

(3) If R is fully bounded and E_0 is a cogenerator, then R is a quasi-Frobenius ring.

(4) If R has finite global dimension and E_0 is a cogenerator, then R is a semi-simple artinian ring.

For an n-Gorenstein ring, we can slightly weaken the condition in the Corollary just above, as follows.

Corollary. *Let R be an n-Gorenstein ring with $id(R) = n$. Assume that $E_0 \oplus E_1 \oplus \cdots \oplus E_{n-1}$ is a cogenerator. Then R is a quasi-Frobenius ring.*

Next, we will consider the case where the last term E_n is a cogenerator. Then we see $soc(R) = 0$ by the last theorem in §1. Moreover, if $id(R) = 1$, then the converse is true by Iwanaga's theorem in §1. As is the case, we have shown the following.

Theorem[13]. *Let R be a 1-Gorenstein ring with $id(R) = 1$, and assume $soc(R) = 0$. Then R has a classical two-sided quotient ring Q which is a quasi-Frobenius ring, and we have $Krull\text{-}dim(R) = 1$, that is to say, R has Krull dimension exactly one.*

For the case of general dimension, however, we do not yet have a corresponding result even though such a ring is ∞-Gorenstein.

Finally we will mention a problem concerning global dimensions for right and left noetherian rings.(Cf.[5];p.287)

Problem. Is the global dimension of a noetherian ring R equal to the supremum of the projective dimensions of the simple R-modules?

This was established for noetherian rings of finite global dimension by Bhatdekar and Goodearl, and for fully bounded noetherian rings as well as noetherian rings of Krull dimension ≤ 1 by Rainwater. We can give another proof for noetherian rings of finite global dimension.

In fact, assume R is a noetherian ring of finite global dimension n. It follows from Theorem C in §2 that there exists a simple submodule S in the last term E_n. By Theorem A in §2, we see $pd(S) \geq n$, and hence $pd(S) = n = gl.dim(R)$.

References

1. Bass,H., *On the ubiquity of Gorenstein rings*, Math.Z. **82** (1963), 8 28.
2. Björk,J.E., *Rings of differential operators* North Holland, 1979.
3. Colby,R.R. and Fuller,K.R., *A note on the Nakayama conjecture*, Tsukuba J. Math. **14** (1990), 343-352.

4. Fossum,R.M., Griffith,P.A. and Reiten,I., *Trivial Extensions of Abelian Categories*, Lect. Notes in Math. **456**, (1975), Springer.
5. Goodearl,K.R. and Warfield,R.B., "*An introduction to noncommutative noetherian rings*, Cambridge Univ. Press, 1989 London Math. Soc. Student Texts 16.
6. Hoshino,M., *On self-injective dimensions of artinian rings*, Tsukuba J. Math. **18** (1994), 1–8.
7. ———, *Noetherian rings of self-injective dimension two*, Comm. in Alg. **21** (1993), 1071–1094.
8. Iwanaga,Y., *On rings with finite self-injective dimension*, Comm. in Alg. **7** (1978), 33–46.
9. ———, *Minimal injective resolution and syzygy modules over a non-commutative Gorenstein ring*, Proc. 4th. Symposium on Representations on Algebras (1993), 8–27.
10. Iwanaga,Y. and Sato,H., *Minimal injective resolutions of Gorenstein rings*, Comm. in Alg. **18** (1989), 3835–3856.
11. ———, *On Auslander's n-Gorenstein rings,(preprint)*.
12. Levasseur,T., *Grade des modules sur certains anneax filtres*. Comm. in Alg. **9** (1981), 1519–1532.
13. Sato,H., *On maximal quotient rings of QF-3 1-Gorenstein rings with zero socle*, Osaka J. Math. **16** (1979), 383–394.
14. Watanabe,K.-i., *Ten characterizations of Gorenstein rings*. Proc. 4th. Symposium on Representations on Algebras (1993), 139–157.
15. Zaks,A., *Injective dimension of semiprimary rings*, J. of Alg. **13** (1969), 73–86.

PROCEEDINGS OF THE 27TH
SYMPOSIUM ON RING THEORY, 1994

On growth properties of algebras *

Shigeru Kobayashi

1 Gröbner basis

Let K be a field, $X = \{x_1, \cdots, x_t\}$ be a set of indeterminates, and X^* denote the free monoid over X. An element x of X^* is called a word on X and $|x|$ denotes the length of x. Let $<_X$ be a total order on X. The order $<_X$ is extended to a total order on X^* denoted by the same symbol $<_X$ as follows;

Let x, y be in X^*. First, if $|x| < |y|$, then $x <_X y$. Next suppose $|x| = |y|$ and $x = x_1x'$, $y = y_1y'$ with $x_1, y_1 \in X$ and $x', y' \in X^*$. Then if $x_1 <_X y_1$ or both $x_1 = y_1$ and $x' <_X y'$, then $x <_X y$.

Let $K\langle X\rangle = KX^*$ be the free algebra generated by X over K. For any

$$g = \sum_{i=1}^{m} \alpha_i u_i, \quad (\alpha_i \in K \setminus \{0\}, u_i \in X^*)$$

in $K\langle X\rangle$, we denote by $HM(g)$ the highest monomial of g, i.e. $HM(g) = u_i$ if $u_j <_X u_i$ for all $j \neq i$. We say $HM(0) = 0$.

Let F be a set of polynomials in $K\langle X\rangle \setminus \{0\}$. The monomial u is normal (modulo F) if it does not contain any of the monomials $HM(f)$, $f \in F$, as a subword. $N(F)$ will denote the set of normal (mod F) monomials. Let I be a nontrivial ideal in $K\langle X\rangle$, $A = K\langle X\rangle/I$. We shall assume, without loss of generality, that the presentation $A = K\langle X\rangle/I$ is minimal, this means that I does not contain polynomials of degree one. Let $N(I)$ be the set of all normal (mod I) monomials. Then $X \subseteq N(I)$ and there is an equality $K\langle X\rangle = \mathrm{Span}N(I) \oplus I$ as vector spaces. For any $f \in K\langle X\rangle$, one has $f = \tilde{f} + g$ where $\tilde{f} \in \mathrm{Span}N(I)$ and $g \in I$ are uniquely determined. The polynomial $\tilde{f}$ is called the normal form of f. Obviously, there is an isomorphism of vector spaces $A \cong \mathrm{Span}N(I)$, so $N(I)$ projects to a K-basis of A. For a set F of polynomials in $K\langle X\rangle$, $HM(F)$ denotes the ideal of $K\langle X\rangle$ generated by $HM(f), f \in F$. We now give the definition of Gröbner basis.

Definition 1 *A finite set G of polynomials in $K\langle X\rangle \setminus \{0\}$, generating I as a two-sided ideal is a Gröbner basis of I if one of the following equivalent conditions holds*

*This paper is in final form and no version of it will be submitted for publication elsewhere.

: *(1)* $HM(G) = HM(I)$; *(2)* $N(F) = N(I)$.
A Gröbner basis

$$G = \{f_p = w_p - g_p \mid p \in P, w_p \in X^*, g_p \in K\langle X\rangle, HM(g_p) <_X w_p\}$$

is reduced if for any $p \in P$ *the polynomial* f_p *is in normal form modulo* $F \setminus \{f_p\}$

It is well known that if I is an ideal of $K\langle X\rangle$ such that I has a Gröbner basis G, then there exists a uniquely determined reduced Gröbner basis. Further it is well known that any ideal of finitely generated commutative K-algebra has a Gröbner basis. However in non-commutative case, there are many ideals does not have Gröbner basis, so we want to find the condition that I has a Gröbner basis. To this end, we shall use growth properties of algebras.

2 Growth of algebras

Let I be an ideal of $K\langle X\rangle$ and assume that I has a Gröbner basis G. Then $HM(G) = HM(I)$. In this section, we compute the growth of $A = K\langle X\rangle/I$. Let V be a generating subspace of A. By assumption of I, we can take $V = K + K\tilde{x}_1 + \cdots + K\tilde{x}_t$, where $\tilde{x}_i = x_i \bmod I$. Set $V^0 = k$, and V^i is the subspace of A spanned by all the products of i elements of V. Then we define the growth function with respect to V as

$$d_V(n) = \dim\left(\sum_{i=0}^{n} V^i\right)$$

for $n \geq 0$.

Definition 2 *(1) A is said to have polynomial growth if* $d_V(n) \sim O(n^\alpha)$ *for some real number* $\alpha \geq 0$.
(2) A is said to have intermediate growth if $d_V(n) \sim O\left(\alpha^{n^\beta}\right)$ *for some real number* $\alpha \geq 1$ *and* $0 \leq \beta < 1$.
(3) A is said to have exponential growth if $d_V(n) \sim O(\alpha^n)$ *for some* $\alpha \geq 1$.

By the assumption, I has a Gröbner basis G, so $HM(I) = HM(G)$ and equivalently, $N(I) = N(G)$. Further since $A \cong \mathrm{Span}N(I)$ as a vector space, $A \cong K\langle X\rangle/HM(I)$ as a vector space (cf.[B]). Hence we can see that the growth type of A and $K\langle X\rangle/HM(I)$ are same. Now we consider the algebra $\tilde{A} = K\langle X\rangle/HM(I)$. Since $HM(I) = HM(G)$, $\tilde{A}$ is a finitely presented monomial algebra. Let $HM(I) = HM(G) = (w_1, w_2, \cdots, w_l)$ and $m = \max\{|w_i| \mid 1 \leq i \leq l\}$. We can associate a labeled directed graph $G(\tilde{A})$ defined as follows (cf.[U]). The set of vertices of $G(\tilde{A})$ is $M = \{w \in N(I) \mid |w| = m - 1\}$; for $u, v \in M$, there is an edge $u \xrightarrow{a} v$ with label $a \in X$ if $ua \in M$ and $ua = vb$ for some $b \in X$. The following result is basic.

Proposition 1 *([KK],[U]) The algebra $\tilde{A}$ has either a polynomial growth or an exponential growth. More precisely*
(1) $\tilde{A}$ has an exponential growth if $G(\tilde{A})$ has a multiple vertex
(2) $\tilde{A}$ has a polynomial growth if $G(\tilde{A})$ has only simple vertices.

This proposition implies that A has either a polynomial growth or an exponential growth when I has a Gröbner basis.

3 Growth theoretical properties of algebras

In this section, we consider when an ideal I of $K\langle X\rangle$ has a Gröbner basis. First we assume that $A = K\langle X\rangle/I$ is a graded algebra, i.e. I is a homogeneous ideal. Then the following proposition is useful to measure the growth of A.

Proposition 2 *([GS]) Let $d_i = \sharp\{f \in I \mid \deg f = i\}$. Then if $d_i \leq \frac{(t-1)^2}{4}$, A is infinite dimensional.*

From the view point of growth, we have the following corollary.

Corollary 1 *If $t \geq 2$ and $d_i \leq \frac{(t-1)^2}{4}$, then A has an exponential growth.*

This corollary implies the following.

Theorem 1 *Let $A = K\langle X\rangle/I$ be a graded algebra. Asuume that $gl.dim A \leq 3$ and A is a domain, then A has a polynomial growth or has an exponential growth. Further if A has a polynomial growth, then I has a* Gröbner *basis.*

In general case, in order to obtain a similar result, we need more assumption.
Finally, we shall give an unsolved problem.

Problem 1 *Let A be a finitely generated filtered algebra over K. Assume that A is a Noetherian domain, then does A have a polynomial growth ?*

If we have a stable range theorem on A, then this theorem can be proved. But we can not yet have a stable range theorem.

References

[B] G.M. Bergman, The diamond lemma for ring theory, Adv.in Math. **29** (1978),178-218

[GS] E.S. Golod and I.R. Shafarevich, On the class field tower, Izv.Akad.Nauk.SSSR Ser.Mat. **28** (1964), 261-272

[KK] S. Kobayashi and Y. Kobayashi, On algebras with Gelfand-Kirillov dimension one, Proc.Amer.M.S. **119** (1993), 1095-1104

[K1] S. Kobayashi, On finitely generated graded domain with $gl.dim \leq 2$, To appear in Comm.in Alg.

[K2] S. Kobayashi, On Ore extension over polynomial rings, Preprint

[U] V. Ufnarovskij, A growth criterion for graphs and algebras defined by words, Mat.Zametki. **31** (1982), 465-472

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PROCEEDINGS OF THE 27TH
SYMPOSIUM ON RING THEORY, 1994

WEAK HOPF GALOIS EXTENSIONS

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0. Introduction.

The notion of a Hopf algebra action is able to generalize to the notion of a comodule algebra action and it has an application to the Hopf Galois theory([N1]). In this note, we define a new type of module algebra and a weak Hopf Galois extension which is a generalization of a Hopf Galois extension in the sense of [CS], [KT] and [Y]. After that we give some fundamental properties of weak Galois extensions without proof.

Throughout this note, we use the following notations: R is a commutative ring with identity and we always work over R. B is an algebra and a coalgebra with structure maps μ, η and Δ, ε, respectively. We use the Sweedler's sigma notation $\Delta(b) = \sum_{(b)} b_{(1)} \otimes b_{(2)}$ and omit the summation index (b) in case it is clear. For an R-module M, we set $M^* = \mathrm{Hom}(M, R)$ and the other notations and terminologies used here, we refer [CS], [M] and [S].

1. Actions.

Let J be an augmented algebra with augmentation $\varepsilon_J : J \to R$ and a left B-comodule with structure map $\rho : J \to B \otimes J$.

The detailed version of this paper has been submitted for publication elsewhere.

Definition 1.1. Let A be an algebra. We say that (B, J) *measures* A if the following conditions are satisfied:

(1) A is a left B-module and B measures A, that is, for any $b \in B$ and $x, y \in A$, there hold

$$b(xy) = \sum b_{(-2)}(x)b_{(-1)}(y) \quad \text{and} \quad b(1) = \varepsilon(b)1,$$

where $\Delta(b) = \sum b_{(-2)} \otimes b_{(-1)} \in B \otimes B$.

(2) A is a left J-module such that, for any $\Omega \in J$,

$$\Omega(xy) = \sum \Omega_{(-1)}(x)\Omega_{(0)}(y) \quad \text{and} \quad \Omega(1) = \varepsilon_J(\Omega)1,$$

where $\rho(\Omega) = \sum \Omega_{(-1)} \otimes \Omega_{(0)} \in B \otimes J$.

Moreover, if Δ and ρ are algebra maps, then we say that A is a (B, J)-*module algebra*.

Note that the map $J \ni \Omega \to \sum \varepsilon_J(\Omega_{(0)})\Omega_{(-1)} \in B$ is an algebra map. If it is a monomorphism, we see that J is a B-subcomodule of B.

Now, for a (B, J)-module algebra A, we can construct the *smash product algebra* $A \sharp J$ as usual.

Definition 1.2. Let A be a (B, J)-module algebra and C a subalgebra of A with common identity. A ring extension A/C is called a *weak* (B, J)-*Galois extension* if the following two conditions are satisfied.

(1) The invariant subalgebra $A^J = \{a \in A \mid \Omega(a) = \varepsilon_J(\Omega)a$ for any $\Omega \in J\}$ is equal to C.

(2) The map $\varphi : A \sharp J \to \mathrm{Hom}(A_C, A_C)$ defined by $\varphi(a \sharp \Omega)(x) = a\Omega(x)$ is an isomorphism.

Moreover if

(3) A is finitely generated projective right C-module,

then A/C is called a *finite weak* (B, J)-*Galois extension.*

We give an example of a finite weak Galois extension which is not a Hopf Galois extension.

Let R be a field with $2 \neq 0$ and let $R[x]$ be a free R-module with basis $\{1, x\}$. For a non-zero fixed element $r \in R$, we set $x^2 = 2rx - r^2$. Here we take $B = < 1, \sigma, \delta, \sigma\delta >$, the Sweedler's 4-dimensional Hopf algebra and

$J =< 1, \delta >$. Define action of B by $\sigma(x) = -x + 2r$ and $\delta(x) = 1$. Since the 2-dimensional Hopf algebras were completely determined by [K], we have

Example 1.3. *$R[x]/R$ is a finite weak (B, J)-Galois extension which is not a Hopf Galois extension.*

In the following, we assume that A is a (B, J)-module algebra and B has a *twisted antipode*, that is, there is a map $\lambda : B \to B$ such that

$$\sum b_{(-1)}\lambda(b_{(-2)}) = \sum \lambda(b_{(-1)})b_{(-2)} = \varepsilon(b) \quad \text{for any} \quad b \in B.$$

The following lemma is useful in our study.

Lemma 1.4. (1) $A\sharp J = (1\sharp J)(A\sharp 1)$.
(2) *If φ is an isomorphism and $A^J = C$, then*

$$\varphi((1\sharp J^J)(A\sharp 1)) = \mathrm{Hom}(A_C, C_C).$$

Moreover, C_C is a direct summand of A_C if and only if there exist x_1, x_2, ..., $x_k \in A$ and Ω_1, Ω_2, ..., $\Omega_k \in J^J$ such that $\sum_{i=1}^k \Omega_i(x_i) = 1$.

Using this Lemma, we have

Theorem 1.5. *Let $A^J = C$. Then the following statements are equivalent.*

(1) *A/C is a finite (B, J)-Galois extension.*
(2) *There exist finite elements $\Omega_{ij} \in J^J$ and x_i, $y_{ij} \in A$ such that*

$$\sum_{i,j,(\Omega_{ij})} x_i\Omega_{ij(-1)}(y_{ij})\sharp\Omega_{ij(0)} = 1\sharp 1 \in A\sharp J.$$

(3) *A is a finitely generated projective right C-module and the map*

$$\beta : A \otimes_C A \to A \otimes J^* \quad \text{defined by} \quad \beta(x \otimes y) = \sum \Omega_i(x)y \otimes \Omega_i^*$$

is an isomorphism as right A-modules, where $\{\Omega_i, \Omega_i^*\}$ *is a projective coordinate system of* J *and the right A-module structures of* $A \otimes_C A$ *and* $A \otimes J^*$ *are given by*

$$(x \otimes y)a = x \otimes ya \quad \text{and} \quad (x \otimes \Omega^*)a = xa \otimes \Omega^*,$$

respectively.

Moreover, if B *is cocommutative and* J *is a subcomodule algebra of* B *via* $\Delta : J \to B \otimes J$*, then these statements are equivalent to the following.*

(4) *The map* $\gamma : A \otimes_C A \to A \otimes J^*$ *defined by* $\gamma(x \otimes y) = \sum x\Omega_i(y) \otimes \Omega_i^*$ *is an isomorphism.*

Corollary 1.6. *Let* $A^J = C$*. Assume* J *has a free basis* $\{\Omega_1 = 1, \Omega_2, ..., \Omega_n\}$ *and* $J^J = R\Omega_k$ *for some* k $(1 \leq k \leq n)$*. Then* A/C *is a finite* (B, J)*-Galois extension if and only if there exist* $x_1, x_2, ..., x_m, y_1, y_2, ..., y_m \in A$ *such that*

$$\sum_{i=1}^{m} x_i \omega_{k_i}(y_i) = 1 \quad \text{and} \quad \sum_{i=1}^{m} x_1 \omega_{k_j}(y_i) = 0 \quad (j = 2, ..., n),$$

where $\rho(\Omega_k) = \sum_{j=1}^{n} \omega_{k_j} \otimes \Omega_j \in B \otimes J$.

$\{x_i, y_i\}$ was known as a *Galois coordinate system*(cf. [CHR]), and we can get some other results for finite weak Galois extensions([N2]).

2. Coactions.

The notions in section 1 dualize to coactions. Let J be a coalgebra with algebra map $R \to J$ and a left B-module with structure map $\omega : B \otimes J \to J$ such that ω is a coalgebra map.

Definition 2.1. We say that A is a (B, J)-*comodule algebra* if the following conditions are satisfied:

(1) A is a right B-comodule with map $\rho_B : A \to A \otimes B$ such that

$$\rho_B(xy) = \sum x_{(0)} y_{(0)} \otimes x_{(B)} y_{(B)} \quad \text{and} \quad \rho_B(1) = 1 \otimes 1,$$

where $\rho_B(x) = \sum x_{(0)} \otimes x_{(B)} \in A \otimes B$.

(2) A is a right J-comodule with map $\rho_J : A \to A \otimes J$.

(3) The following diagram is commutative:

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\rho_B \otimes \rho_J} & A \otimes B \otimes A \otimes J \\ {\scriptstyle \mu_A}\big\downarrow & & \big\downarrow{\scriptstyle (\mu_A \otimes \omega)(1 \otimes t \otimes 1)} \\ A & \xrightarrow[\rho_J]{} & A \otimes J, \end{array}$$

where $\mu_A : A \otimes A \to A$ is the multiplication of A and $t : h \otimes a \to a \otimes h$ the twist map.

Definition 2.2. Let A be a (B, J)-comodule algebra and C a subalgebra of A with common identity. A ring extension A/C is called a *weak* (B, J)*-Galois object* if the following conditions are satisfied:

(1) $C = A^{coJ} = \{a \in A \mid \rho_J(a) = a \otimes 1\}$.

(2) $\gamma : A \otimes A \to A \otimes J$ defined by $\gamma(x \otimes y) = (x \otimes 1)\rho_J(y)$ is an isomorphism.

These are dual notions of actions. As usual, we have the following

Theorem 2.3. *Let A/C be a weak (B, J)-Galois object. Assume that B has an antipode and J is a subcoalgebra of B which is finitely generated projective R-module. Then A is a (B^*, J^*)- module algebra and the following hold.*

(1) $A^{J^*} = C$.

(2) *A is a finitely generated projective right C-module.*

(3) *$\varphi : A \# J^* \to \mathrm{Hom}(A_C, A_C)$ defined by $\varphi(a \# \Omega^*)(x) = a\Omega^*(x)$ is an isomorphism.*

We have some other results which are similar to the results of Hopf Galois extensions([N2]).

References

[CHR] S.U.Chase, D.K.Harrison and A.Rosenberg : Galois Theory and Galois cohomology of commutative rings, Mem. Amer. Math. Soc. 52(1965), 15-33.

[CS] S.U.Chase and M.E.Sweedler : Hopf Algebras and Galois Theory, Lecture Notes in Math. 97(1969), Springer-Verlag, Berlin.

[K] H.F.Kreimer: Quadratic Hopf algebras and Galois extensions, Contemporary Math. 13(1982), 353-361.

[KT] H.F.Kreimer and M.Takeuchi: Hopf algebras and Galois extensions of an algebra, Indiana Univ. Math. J. 30(1981), 675-692.

[M] S.Montgomery: Hopf Algebras and Their Actions of Rings, CBMS No.82(1993).

[N1] A.Nakajima: Weak Hopf Galois extensions and P-Galois extensions of a ring, to be submitted.

[N2] A.Nakajima : Fundamental properties of weak Galois extensions and smash products, in preparation.

[S] M.E.Sweedler: Hopf Algebras, Benjamin, New York, 1969.

[Y] K.Yokogawa: Non-commutative Hopf Galois extensions, Osaka J. Math. 18(1981), 67-73.

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PROCEEDINGS OF THE 27TH
SYMPOSIUM ON RING THEORY, 1994

SURVEY

1. MODULAR REPRESENTATION OF SYMMETRIC GROUPS

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2. RINGS WHOSE PROPER CYCLICS ARE QUASI-INJECTIVE

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