PROCEEDINGS OF THE SECOND JAPAN-CHINA INTERNATIONAL SYMPOSIUM ON RING THEORY AND THE 28TH SYMPOSIUM ON RING THEORY

HELD AT MITSU INTERNATIONAL EXCHANGE HALL

OF OKAYAMA UNIVERSITY OF SCIENCE

OKAYAMA, JAPAN

OCTOBER 24 – 27, 1995

ORGANIZING COMITTEE OF THE SECOND JAPAN-CHINA

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PREFACE

The Second Japan-China International Symposium on Ring Theory was held jointly with the 28-th Symposium on Ring Theory at Okayama University of Science on October 24–28, 1995. The aim of the symposium is to exchange the ideas and results obtained by Japanese and Chinese ring theorists and to encourage each other.

The Organizing Committee of the symposium consists of H. Li (Shaanxi Normal Univ.), S. Liu (Beijing Normal Univ.), H. Marubayashi (Naruto Univ. of Education), T. Nagahara (Okayama Univ.), T. Nakamoto (Kurashiki Univ. of Science and Arts), M. Sato (Yamanashi Univ.), Y. Tsushima (Osaka City Univ.), B. Xie (Jilin Univ.), Y. Xu (Fudan Univ.), K. Yamagata (Tsukuba Univ.) and K. Yokogawa (Okayama Univ. of Science).

We had about one hundred and fifty participants consisting of one hundred Japanese, 19 Chinese, 12 Korean and ring theorists from western countries.

The proceedings contains the articles submitted by the speakers, without referees, who have presented their recent interesting results at the symposium.

The symposium and proceedings were financially supported by the following four: the Scientific Research Grants of Educational Ministry of Japan through the arrangements by professor Masao Koike, Kyushu Univ. and by professor Yuji Yoshino, Kyoto Univ., Okayama University of Science, IBM Company, and private contributors whose names are professors M. Harada, H. Tachikawa, H. Tominaga, T. Nagahara, K. Yamagata, K. Yokogawa and T. Nakamoto.

We would like to express our hearty thanks to them for their financial supports and especially to Okayama University of Science for offering us the symposium place. We would like to express our thanks to professor T. Akiba, Kyoto Univ. for his arrangement for funds and to Dr. H. Komatsu for the publication of the proceedings. Our thanks go to our colleagues for helping us to run the symposium smoothly, in particular, to those who have invited some Chinese to their universities.

It is worthy to mention that the symposium will be extended to three countries, that is, to Japan, China and Korea, and that it will be held at Taegu, Korea in 1999.

Hidetoshi MARUBAYASHI

Naruto, January, 1996

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LIST OF PARTICIPANTS

Akiba, Tomoharu
Kyoto Univ.

Asano, Shigemoto
Takushoku Univ.

Asashiba, Hideto
Osaka City Univ.
e51833@sakura.kudpc.kyoto-u.ac.jp

Azumaya, Goro
Indiana Univ.

Baba, Yoshitomo
Osaka Kyoiku Univ.
ybaba@cc.osaka-kyoiku.ac.jp

Bak, Anthony
Univ. Bielefeld
bak@mathematik.uni-bielefeld.de

Chen, Jianai
Capital Normal Univ.

Chen, Zhizhong
Northern Jiaotong Univ.

Cheng, Fuchang
Guangxi Normal Univ.

Cho, Yong Uk
Busan Women's Univ.

Endoh, Yasuhiro
Univ. of Tsukuba
endoh@math.tsukuba.ac.jp

Enochs, Edgar
Univ. of Kentukey
enochs@ms.uky.edu

Fu, Changlin
Harbin Univ. of Sci. & Tech.

Fukushima, Hiroshi
Gunma Univ.

Fukuzawa, Nobuyuki
Shimane Univ.

Furukawa, Tôru
Sanyo Gakuen Univ.

Haghany, Ahmad
Isfahan Univ. of Tech.
aghagh@cc.iut.ac.ir

Han, Jun Cheol
Koshin Univ.

Hanada, Katsunori
Yamaguchi Univ.

Hanaki, Akihide
Yamanashi Univ.
hanaki@esi.yamanashi.ac.jp

Hida, Akihiko
Saitama Univ.

Hieda, Yoshimasa
Osaka City Univ.
h1773@ocugw.cc.osaka-cu.ac.jp

Hirano, Yasuyuki
Okayama Univ.
yhirano@math.okayama-u.ac.jp

Hong, Chan Yong
Kyung Hee Univ.

Hongan, Motoshi
Tsuyama Nat. Coll. of Tech.
hongan@dnews04.dnet.tsuyama-ct.ac.jp

Horimoto, Hiroshi
Kumamoto Univ.

Hoshino, Mitsuo
Univ. of Tsukuba

Hu, Xianhui
Central Inst. of Nationalities

Huh, Chan
Pusan Nat. Univ.

Iida, Youichi
Sci. Univ. of Tokyo

Ikehata, Shûichi
Okayama Univ.
ikehata@math.ems.okayama-u.ac.jp

Ikeyama, Tamotsu
Hosei Univ.

Inoue, Takayuki
Osaka City Univ.

Ishikawa, Kenta
Chiba Univ.

Ishikawa, Takeshi
Tokyo Metrop. Univ.

Iwanaga, Yasuo
Shinshu Univ.
iwanaga@gipnc.shinshu-u.ac.jp

Iwase, Kenichi
Tennohji Sr. High Scholl of Osaka Kyoiku Univ.

Izawa, Tatsuo
Shizuoka Univ.

Jenda, Overtoun M.G.
Auburn Univ.
jendaov@mail.auburn.edu

Jespers, Eric
Mem. Univ. Newfoundland
EJESPERS@ALBERT.MATH.MUN.CA

Jiang, Zimei
Beijing Normal Univ.

Kado, Jiro
Osaka City Univ.
h1747@ocugw.cc.osaka-cu.ac.jp

Kaita, Tosiko
Univ. of Tsukuba

Kamimura, Hideo
Yamaguchi Univ.

Kanemitsu, Mitsuo
Aichi Univ. of Educ.
mkanemit@auecc.aichi-edu.ac.jp

Kang, Byung Gyun
POSTECH

Kasch, Friedrich
Univ. Munich

Katayama, Hisao
Yamaguchi Univ.
katayama@ccy.yamaguchi-u.ac.jp

Kato, Toyonori
Univ. of Tsukuba

Kawada, Yutaka

Kawamoto, Naoki
Maritime Safety Acad.
kawamoto@ms.ac.jp

Kawata, Shigeto
Osaka City Univ.

Kerner, Otto
Heinrich-Heine-Universität
kerner@mx.cs.uni-duesseldorf.de

Kim, Chol On
Pusan Nat. Univ.

Kim, Eun Sup
Kyungpook Nat. Univ.

Kim, Jin Yong
Kyung Hee Univ.
jykim@nms.kyunghee.ac.kr

Kitamura, Kazuo
Osaka Kyoiku Univ.
kitamura@cc.osaka-kyoiku.ac.jp

Kitamura, Yoshimi
Tokyo Gakugei Univ.
kitamura@u-gakugei.ac.jp

Kobayashi, Shigeru
Naruto Univ. of Educ.
skoba@naruto-u.ac.jp

Koike, Kazutoshi
Ube Coll.
koike@cs.ube-c.ac.jp

Komatsu, Hiroaki
Okayama Univ.
pine@math.okayama-u.ac.jp

Koshitani, Shigeo
Chiba Univ.
koshitan@math.s.chiba-u.ac.jp

Koya, Susumu
Univ. of Tsukuba

Kurata, Yoshiki
Kanagawa Univ.
kurata@info.kanagawa-u.ac.jp

Kutami, Mamoru
Ymaguchi Univ.
kutami@ccy.yamaguchi-u.ac.jp

Lee, Dong Su
Univ. of Ulsan

Li, Huishi
Shaanxi Normal Univ.

Lin, Ya-nan
Xiamen Univ.

Marubayashi, Hidetoshi
Naruto Univ. of Educ.

Masaike, Kanzo
Tokyou Gakugei Univ.
masaike@u-gakugei.ac.jp

Masuoka, Akira
Shimane Univ.

Matsuda, Ryûki
Ibaragi Univ.

Miyachi, Jun-ichi
Tokyo Gakugei Univ.
miyachi@u-gakugei.ac.jp

Miyamoto, Haruo
Anan Nat. Coll. of Tech.

Miyashita, Yôichi
Univ. of Tsukuba

Mogami, Isao
Tsuyama Nat. Coll. of Tech.
mogami@dnews04.dnet.tsuyama-ct.ac.jp

Morimoto, Shoji
Hagi Koen Gakuin

Motose, Kaoru
Hirosaki Univ.
skm@cc.hirosaki-u.ac.jp

Mueller, Bruno J.
McMaster Univ.

Nagahara, Takasi
Okayama Univ.
nagahara@math.okayama-u.ac.jp

Nakajima, Atsushi
Okayama Univ.

Nakamoto, Taichi
Kurashiki Univ. of Sci. & Arts
nakamoto@soft.kusa.ac.jp

Ninomiya, Yasushi
Shinshu Univ.
ysninom@gipac.shinshu-u.ac.jp

Nishida, Kenji
Nagasaki Univ.
kenisida@net.nagasaki-u.ac.jp

Nishida, Koji
Chiba Univ.

Nishinaka, Tsunekazu
Okayama Shoka Univ.
nishinak@math.okayama-u.ac.jp

Niu, Fengwen
Jilin Univ.

Nozawa, Takeshi
Chiba Univ.

Ôhori, Masayuki
Shinshu Univ.

Ohtake, Koichiro
Gunma Univ.
ootake@edu.gunma-u.ac.jp

Okuno, Hiroshi
Juntendo Univ.

Oshiro, Kiyoichi
Yamaguchi Univ.

Otsuka, Kayo
Teikyo Univ.

Park, Jae Keol
Busan Nat. Univ.
jkpark@hyowon.cc.pusan.ac.kr

Park, Young Soo
Kyungpook Nat. Univ.

Peng, Liangang
Sichuan Univ.

Rim, Seog Hoon
Kyungpook Nat. Univ.

Rizvi, S. Tariq
Ohio State Univ.
rizvi@math.ohio-state.edu

Sanada, Katsunori
Sci. Univ. of Tokyo
sanada@rs.kagu.sut.ac.jp

Sasaki, Hiroki
Yamaguchi Univ.
sasaki-h@ccy.yamaguchi-u.ac.jp

Sato, Masahisa
Yamanashi Univ.
sato@kkb.yamanashi.ac.jp

Sekiguchi, Katsusuke
Kokushikan Univ.

Shigenaga, Kazuo
Ube Nat. Coll. of Tech.

Shinjyo, Takashi
Univ. of Ryukyu

Shou, Yu
Chiba Univ.

Sumioka, Takeshi
Osaka City Univ.

Sumiyama, Takao
Aichi Inst. of Tech.
sumiyama@aitech.ac.jp

Tachikawa, Hiroyuki
Ashikaga Inst. of Tech.

Tahara, Ken-ichi
Aichi Univ. of Educ.
Tahara@aichi-edu.ac.jp

Takehana, Yasuhiko
Hakodate Nat. Coll. of Tech.

Takeuchi, Mitsuhiro
Univ. of Tsukuba

Tambara, Daisuke
Hirosaki Univ.

Tanaka, Masaki
Kumamoto Inst. of Tech.

Tang, Gaohua
Guangxi Teacher's Coll.

Tsuda, Kentaro
Okayama Univ.
kentaro@math.okayama-u.ac.jp

Tsunashima, Hironobu
Yamaguchi Univ.

Tsushima, Yukio
Osaka City Univ.

Ueda, Akira
Shimane Univ.
ueda-a@botan.shimane-u.ac.jp

Uematsu, Morio
Jobu Univ.
uematsu@jobu.ac.jp

Uno, Katsuhiro
Osaka Univ.
uno@math.wani.osaka-u.ac.jp

Usami, Yoko
Ochanomizu Univ.

Van Oystaeyen, Fred
Univ. Antwerpen
voyst@hwins.uia.ac.be

Wada, Tomoyuki
Tokyo Univ. of Agri. & Tech.
wada@cc.tuat.ac.jp

Wakamatsu, Takayoshi
Saitama Univ.

Wakita, Tsuneo
Osaka City Univ.

Wang, Zhi-xi
Capital Normal Univ.

Watanabe, Atumi
Kumamoto Univ.

Watanabe, Yutaka
Osaka Women's Univ.

Wu, Pinsan
Beijing Normal Univ.

Yamada, Toshihiko
Sci. Univ. of Tokyo

Yamagata, Kunio
Univ. of Tsukauba
yamagata@sakura.cc.tsukuba.ac.jp

Yamashiro, Yasukazu
Univ. of Ryukyu
yamasiro@u-ryukyu.ac.jp

Yamauchi, Kenichi
Chiba Univ.

Yanai, Tadashi
Niihama Nat. Coll. of Tech.

Yang, Zi-Xu
Sbandong Finance Inst.

Yao, Musheng
Fudan Univ.

Yi, Zhong
Guangxi Teacher's Univ.

Yokogawa, Kenji
Okayama Univ. of Sci.
MHB02225@niftyserve.or.jp

Yoshida, Ken-ichi
Okayama Univ. of Sci.

Yoshida, Tomoyuki
Hokkaido Univ.
yoshidat@math.hokudai.ac.jp

Yoshimura, Hiroshi
Yamaguchi Univ.
yoshimura@ccy.yamaguchi-u.ac.jp

Yoshioka, Akiko
Kobe Design Univ.

Yukimoto, Yoshito
Fukui Univ. of Tech.

Zhang, Changming
Hunan Normal Univ.

Zhang, Yang
Fudan Univ.
zhangy@naruto-u.ac.jp

Zhang, Yingbo
Beijing Normal Univ.
zhangyb@sun.ihep.ac.cn

A Characterization of Coherent Rings in terms of Finite Matrix Functors

Goro Azumaya

Let R be a ring. A right R-module M is called finitely presented if there exists a finitely generated free right module F and a finitely generated submodule K of F such that $M \cong F/K$. The followings are known properties of finitely presented modules: Let M be a right module and N a submodule of M. Then (a) if M is finitely presented and N is finitely generated, then M/N is finitely presented, (b) if M is finitely generated and M/N is finitely presented, then N is finitely generated, (c) if both M/N and N are finitely presented, then M is finitely presented too. Of course, every finitely presented module is finitely generated. But the converse holds, i.e., every finitely generated right R-module is finitely presented if and only if R is right noetherian. This is because right noetherian rings can be characterized as those rings over which every submodule of any finitely generated right module is finitely generated too. Now, R is called right coherent if every finitely generated right ideal of R is finitely presented. Clearly every right noetherian ring is right coherent, but the converse is not true.

Let U be a finite, say $m \times n$-matrix over R. We denote by $U(R)$ the set of those $x \in R$ such that

$$U \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 0 \text{ with } x_1 = x \text{ and for some } x_2, \ldots, x_n \in R.$$

Then $U(R)$ is a right ideal of R. We are now to prove the following.

Theorem *The following conditions are equivalent:*

(1) *R is a right coherent,*

(2) *$U(R)$ is finitely generated for every finite row matrix U over R,*

(3) *$U(R)$ is finitely generated for every finite matrix U over R.*

Proof. Let $a_1, a_2, \ldots, a_n$ be any finite number of elements of R. Let $I_1 = a_1R + a_2R + \ldots + a_nR$ and $I_2 = a_2R + \ldots + a_nR$. Then $I_1 = a_1R + I_2$ and so if we associate each $r \in R$ with the coset $a_1r + I_2 \in I_1/I_2$ we have an epimorphism $R \to I_1/I_2$ as right R-module. The kernel of this epimorphism is clearly $U(R)$, where U is the finite row matrix $[a_1, a_2, \ldots, a_n]$. Thus we have $R/U(R) \cong I_1/I_2$. Therefore, we know that $U(R)$ is finitely generated if and only if I_1/I_2 is finitely presented. Assume that R is right coherent. Then the finitely generated right ideal I_1 is finitely presented. Since I_2 is finitely generated, I_1/I_2 is also finitely presented by (a), which implies that $U(R)$ is finitely generated. Thus (1) $\Rightarrow$ (2) is proved. Conversely assume (2). Then I_1/I_2 is finitely presented. By the same reason, if we put $I_3 = a_3R + \ldots + a_nR, \ldots, I_{n-1} = a_{n-1}R + a_nR, I_n = a_nR$ then $I_2/I_3, \ldots, I_{n-1}/I_n, I_n$ are all finitely presented . Thus we know by (c) that I_1 is finitely presented . This shows the implication (2) $\Rightarrow$ (1).

Now (2) is a particular case of (3), and (3) $\Rightarrow$ (2) is clear. So we need only prove (1) $\Rightarrow$ (3). For this purpose, we make use of the following proposition without proof:

Proposition *Let R be right coherent, and let M be a finitely presented right R-module. Then every finitely generated submodule of M is finitely presented .*

Assume now (1). Let $U = [u_1, u_2, \ldots, u_n]$ be a finite $m \times n$-matrix over R, where $u_1, u_2, \ldots, u_n$ mean the 1st, 2nd, ... n-th columns respectively. Let R^m denote the set of all column vectors of length m over R. Then R^m is a finitely generated free right R-module and so is finitely presented . The n vectors $u_1, u_2, \ldots, u_n$ are in R^m, so if we put $M = u_1R + u_2R + \ldots + u_nR$ M is finitely presented by the above proposition. Let $N = u_2R + \ldots + u_nR$. Then M/N is also finitely presented by (a). If we associate each $r \in R$ with the coset $u_1r + N \in M/N$ then we have an epimorphism $R \to M/N$. The kernel of this epimorphism is $U(R)$, so that we have an isomorphism $R/U(R) \cong M/N$. This completes the proof of our theorem.

Now, we assume that R is both right coherent and left perfect. Then for every finite matrix U over R, $U(R)$ is finitely generated right ideal by preceding theorem. On the other hand, the left perfect ring R satisfies the descending chain condition on finitely generated right ideals by Bass-Björk theorem. Therefore we know that R satisfies the descending chain condition on right ideals of the form $U(R)$ for all finite matrices U over R. This descending chain condition implies, according to Zimmermann theorem, that the left R-module R is Σ-pure-injective, i.e., every direct sum of R, i.e., every free left R-module is pure-injective. Since every projective left R-module is a direct summand of a free left R-module, we have

Theorem *If R is both right coherent and left perfect then every projective left R-module is pure-injective.*

However we point out that the following a kind of converse holds:

Theorem *If every projective left R-module is pure-injective then R is left perfect.*

For, that every projective whence free left R-module is pure-injective implies clearly every free left R-module is Σ-pure-injective. Let L be any flat left R-module. Then it is an epimorphic image of a free left R-module F. The flatness of L then implies that the kernel K of the epimorphism is pure in F. But since F is Σ-pure-injective, its pure submodule K is a direct summand of F by Zimmermann theorem again. Thus $L \cong F/K$ is projective, which shows that R is left perfect due to Bass theorem.

References

[1] H. Bass, *Finitistic dimension and a homological generalization of semi-primary rings*, Trans. Amer. Math. Soc. **95** (1960), 466–488.

[2] S. U. Chase, *Direct product of modules*, Trans. Amer. Math. Soc. **97** (1960), 457–473.

[3] B. Stenström, *Rings and quotients*, Springer-Verlag, Berlin Heidelberg New York, 1975.

[4] W. Zimmermann, *Rein injektive direkte Summen von Moduln*, Comm. Algebra **5** (1977), 1083–1117.

Institute of Mathematics
Indiana University
Bloomington, IN 47405
U.S.A.

Specialization
of
Structure Theorems for QF-3 Rings

Yoshitomo BABA

The following two theorems are ones which we call "Structure Theorems for QF-3 rings".

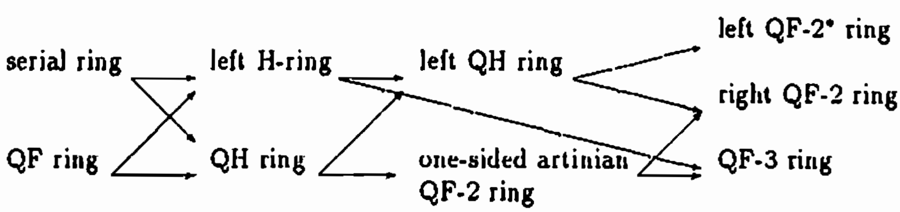
THEOREM A ([8, (5.3)Theorem], [7, (2.1)]). *A ring R is QF-3 maximal quotient if and only if it is the endomorphism ring of a module $_AK$ which is a linearly compact, generator and cogenerator. Then $A \approx fRf$ as rings, where fR is a right minimal faithful ideal of R. Moreover, if $_AK$ and $_{A'}K'$ are such modules and*

$$End(_AK) \approx End(_{A'}K'),$$

then there is a categorical equivalence T *: the left A-modules category $\rightarrow$ the left A'-modules category such that* $\mathrm{T}(_AK) = {}_{A'}K'$.

THEOREM B ([8, (5.4)Theorem], [7,(2.2)]). *The ring R is a QF-3 ring if and only if there exists a QF-3 maximal quotient ring Q with minimal faithful right and left ideals fQ and Qe, respectively, such that*
$< 1, fQ, Qe > \subseteq R \subseteq Q$, *where* $< 1, fQ, Qe >$ *means the smallest subring of Q containing* $\{1, fQ, Qe\}$. *In this case,* $Q = Q(R)$.

There exists the following relations.



The purpose is to specialize Theorems A and B to one-sided artinian QF-2 rings, QH rings, one-sided H-rings and serial rings.

We denote a complete set of orthogonal primitive idempotents of a ring R by $p(R)$ and a set of non-negative integer by N_0.

1. DEFINITION OF H- RINGS

Let R be a left artinian ring. We call that R is a *left H-ring* if one of the following equivalent conditions (1) and (2) is satisfied.

(1) For any indecomposable injective left R-module I, there exists $e \in p(R)$ and $k \in \mathbb{N}_0$ such that ${}_RI \approx {}_RRe/S_k(Re)$, where $S_k(Re)$ means the k-th socle of ${}_RRe$.

(2) For any indecomposable projective right R-module P, there exists $f \in p(R)$ and $l \in \mathbb{N}_0$ such that $P_R \approx fJ^l_R$, where J means the Jacobson radical of R.

LEMMA 1. *Suppose that R is a left H-ring. Then*

(i) the minimal injective cogenerator left R-module is represented as $\oplus_{i=1}^{m}\oplus_{j=0}^{n(i)} Re_i/S_j(Re_i)$, where $m \in \mathbb{N}, e_i \in p(R)$ and $n(i) \in \mathbb{N}_0$ for any $i = 1, \cdots, m$, and

(ii) the minimal projective generator right R-module is represented as $\oplus_{i=1}^{m'}\oplus_{j=0}^{n'(i)} f_iJ^j$, where $m' \in \mathbb{N}, f_i \in p(R)$ with f_iR_R is injective and $n'(i) \in \mathbb{N}_0$ for any $i = 1, \cdots m'$.

Let R be a left artinian ring. Then a left R-module M is called *a module of H-block* (resp. *co-H-block*) if $M = \oplus_{i=1}^{m} M_i$ satisfying

(i) ${}_RM_1$ is indecomposable projective (resp. injective),

(ii) ${}_RM_{i+1} \approx {}_RM_i$ or $\approx {}_RM_i/S(M_i)$ (resp. $\approx {}_RJM_i$) for any $i = 1, \cdots, m-1$, and

(iii) any ${}_RM_i$ ($\not\approx {}_RM_m$) is injective (resp. projective).

By Lemma 1, if R is a left H-ring, then

(i) the minimal injective cogenerator left R-module is represented as a direct sum of left modules of H-block, and

(ii) right regular module R_R is represented as a direct sum of right modules of co-H-block.

Let R be a left (resp. right) H-ring. Then there exists a complete set $\{f_{ij}\}_{i=1}^{m}{}_{j=1}^{n(i)}$ of orthogonal primitive idempotents of R such that

(i) $S(f_{ij}R_R) \approx S(f_{i'j'}R_R)$ (resp. $S({}_RRf_{ij}) \approx S({}_RRf_{i'j'})$) iff $i = i'$, and

(ii) $f_{ij+1}R_R \approx f_{ij}R_R$ or $\approx f_{ij}J_R$ (resp. ${}_RRf_{i,j+1} \approx {}_RRf_{ij}$ or $\approx {}_RJf_{ij}$) for any i, j.

(Then for each i, $\oplus_{j=1}^{n(i)} f_{ij}R_R$ (resp. $\oplus_{j=1}^{n(i)} {}_RRf_{ij}$) becomes a right (resp. left) R-module of co-H-block.)

We call $\{f_{ij}\}_{i=1}^{m}{}_{j=1}^{n(i)}$ a *well-indexed set concerning a left (resp. right) H-ring.*

2. DEFINITION OF QH RINGS

A ring R is called a *left QH ring* if R is left artinian and satisfies one of the following conditions which are equivalent under the assumption that R is left artinian.

(1) For any $f \in p(R)$, there exists $e \in p(R)$ such that $E({}_RRf/Jf) \approx {}_RRe/r_{Re}(fR)$, where $r_{Re}(fR)$ means the right annihilator of fR in Re.

(2) For any $f \in p(R)$, there exists $e \in p(R)$ such that (i) $S(fR_R) \approx eR/eJ$ and (ii) $S({}_{fRf}fRe)$ is simple.

(3) Any indecomposable projective right R-module is quasi-injective.

We call a ring R is a *QH-ring* if R is both left and right QH.

3. Specialization of Theorem A

Now we specialize Theorem A.

THEOREM 1. *A ring R is left artinian, QF-2, maximal quotient if and only if it is the endomorphism ring of an artinian module ${}_AK$ which is a generator, cogenerator and direct sum of colocal local modules with a left artinian endomorphism ring.*

THEOREM 2. *A ring R is QH, maximal quotient if and only if it is the endomorphism ring of an artinian module ${}_AK$ which is a generator, cogenerator and direct sum of quasi-projective quasi-injective modules.*

THEOREM 3. *A ring R is left (resp. right) H- maximal quotient if and only if it is the endomorphism ring of an artinian module ${}_AK$ which is a generator, cogenerator and direct sum of modules of H- (resp. co-H-) block.*

THEOREM 4. *A ring R is serial maximal quotient if and only if it is the endomorphism ring of a module ${}_AK$ over some serial ring A which is a generator, cogenerator, finite direct sum of modules of H-block and finite direct sum of modules of co-H-block.*

4. Specialization of Theorem B

Now we specialize Theorem B.

THEOREM 5. *Let R be a semiperfect ring such that $Q(R)$ is also semiperfect. Then R is a left QF-2, QF-3 ring if and only if there exists a left QF-2, QF-3, maximal quotient ring Q with minimal faithful right and left ideals fQ and Qe, respectively, such that $< p(Q), fQ, Qe > \subseteq R \subseteq Q$. In this case, $Q = Q(R)$.*

THEOREM 6. *The ring R is a QH ring if and only if there exists a QH, maximal quotient ring Q with minimal faithful right and left ideals fQ and Qe, respectively, such that $< \{gQg\}_{g \in p(Q)}, fQ, Qe > \subseteq R \subseteq Q$.*

THEOREM 7. *The ring R is a left H-ring if and only if there exists a left H- maximal quotient ring Q such that $< \{f_{ij}Qf_{ij'}\}_{i=1, j=1, j'=j}^{m \;\; j' \;\; n(i)}, J(Q) > \subseteq R \subseteq Q$, where $\{f_{ij}\}_{i=1, j=1}^{m \;\; n(i)} = p(Q)$ is a well-indexed set concerning a left H-ring.*

THEOREM 8. *The ring R is a right serial, QF-3 ring if and only if there exists a serial maximal quotient ring Q such that $< \{f_{ij}Qf_{ij'}\}_{i=1, j=1, j'=j}^{m\ \ j'\ \ n(i)}, J(Q) > \subseteq R \subseteq Q$, where $\{f_{ij}\}_{i=1, j=1}^{m\ \ n(i)} = p(Q)$ is a well-indexed set concerning a left H-ring.*

References

1. Y. Baba, Injectivity of quasi-projective modules, projectivity of quasi-injective modules, and projective cover of injective modules, *J. Algebra 155, 2 (1993), 415-434*

2. Y. Baba and K. Iwase, On quasi-Harada rings, *Preprint*

3. M. Harada, Non-small modules and non-cosmall modules, *Ring Theory, Proceedings of 1978 Antwerp Conference, F. Van Oystaeyen, ed. New York, Dekker (1979), 669-689*

4. B. J. Mueller, On algebras of dominant dimension one, *Nagoya Math. J. 31 (1968), 173-183*

5. K. Oshiro, On Harada rings I, *Math. J. Okayama Univ. 31 (1989), 161-178*

6. K. Oshiro, On Harada rings II, *Math. J. Okayama Univ. 31 (1989), 179-188*

7. C. M. Ringel and H. Tachikawa, QF-3 rings, *J. Reine Angew. Math. 171 (1975), 49-72*

8. H. Tachikawa, Quasi-Frobenius rings and generalizations. QF-3 and QF-1 rings, "Lecture Note in Math.," Vol. 351, Berlin-New York, Springer-Verlag, 1973

Department of Mathematics
Osaka Kyoiku University
Osaka, 582 Japan

ON GENERALIZED REGULARITY AND THE SIMPLICITY OF PRIME FACTOR RINGS*

Gary F. Birkenmeier
Department of Mathematics
University of Southwestern Louisiana
Lafayette, Louisiana 70504
U. S. A.
e-mail: gfb1127@usl.edu

Jin Yong Kim
Department of Mathematics
Kyung Hee University
Suwon 449–701
South Korea
e-mail: jykim@nms.kyunghee.ac.kr

Jae Keol Park
Department of Mathematics
Busan National University
Busan 609–735
South Korea
e-mail: jkpark@hyowon.cc.pusan.ac.kr

All rings are associative with a unity. All prime ideals are assumed to be proper. A ring is said to satisfy $\mathfrak{pm}$ if *every prime ideal is maximal.*

For a ring R, we use $\mathbf{P}(R)$ to denote the prime radical of R. R is called a *2-primal* ring if $\mathbf{P}(R)$ is the set of all nilpotent elements of R. In [10], Hirano used the term *N-ring* for what we call a 2-primal ring in his investigation of strongly π-regular rings. Also the 2-primal condition was considered independently by Sun [15] with the term *weakly symmetric.* The name *2-primal rings* originally and independently came from the context of left near rings by Birkenmeier, Heatherly and Lee in [3].

For the case of commutative rings, the first clearly established equivalence between condition $\mathfrak{pm}$ and a generalized von Neumann regularity seems to have been made by Storrer [14] in the following result: For a commutative ring R, the following are equivalent: (1) R is π-regular; (2) $R/\mathbf{P}(R)$ is von Neumann regular; (3) R satisfies $\mathfrak{pm}$. This result

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was generalized to PI-rings by Fisher and Snider [8], while to duo rings by Chandran [7]. Next, Chandran's result was generalized by Hirano [10] to right duo rings.

On the other hand, for the case of reduced rings (i.e., rings without nonzero nilpotent elements), the following generalization of Storrer's result was shown by Beidar and Wisbauer [1], Belluce [2], Birkenmeier, Kim and Park [4], and Camillo and Xiao [6]: For a reduced ring R, the following statements are equivalent: (1) R is biregular; (2) R is weakly regular; (3) R is right weakly π-regular; (4) every prime factor ring of R is a simple domain; (5) R satisfies condition $\mathfrak{pm}$.

We introduce some results obtained in [5] which can unify and extend many of the results of previously mentioned papers.

Recall that a ring R is called *right (left) weakly regular* if $I^2 = I$ for each right (left) ideal I of R. R is called *weakly regular* if it is both right and left weakly regular [12]. Note that right (left) weakly regular rings are also called *right (left) fully idempotent.* A ring R is called *right (left) weakly π-regular* if for every $a \in R$ there exists a positive integer $n = n(a)$, depending on a, such that $a^n \in a^nRa^nR$ $(a^n \in Ra^nRa^n)$. R is called *weakly π-regular* if it is both right and left weakly π-regular [9].

Definition 1 [13]. A ring R is called *almost symmetric* if it satisfies the following conditions:

(SI) the right annihilator of each element is an ideal of R;

(SII) for any $a, b, c \in R$, if $a(bc)^n = 0$ for a positive integer n, then $ab^mc^m = 0$ for some positive integer m.

A ring R is called *pseudo symmetric* if it satisfies the following conditions:

(PSI) the factor ring R/I is 2-primal whenever $I = 0$ or I is the right annihilator of aR for some $a \in R$;

(PSII) for any $a, b, c \in R$, if $aR(bc)^n = 0$ for some positive integer n, then $a(RbR)^mc^m = 0$ for some positive integer m.

Lambek [11] calls a ring R *symmetric* provided $abc = 0$ implies $acb = 0$ for any $a, b, c \in R$. Note that commutative rings and reduced rings are symmetric. Symmetric rings are almost symmetric and almost symmetric rings are pseudo symmetric.

Following [13], for a prime ideal P of a ring R, let

$$O_P = \{a \in R \mid ab = 0 \text{ for some } b \in R \setminus P\}.$$

We define

$$\overline{O}_P = \{a \in R \mid a^n \in O_P \text{ for some positive integer } n\}.$$

The equivalence of condition $\mathfrak{pm}$ with generalized von Neumann regularity conditions has been developed in [5]. Indeed, this equivalence condition holds over a large class of 2-primal rings.

Theorem 2 [5]. Let R be a 2-primal ring with all idempotents central. Then the following statements are equivalent:

(1) R is right weakly π-regular;

(2) $R/\mathrm{P}(R)$ is right weakly π-regular;

(3) R satisfies condition $\mathfrak{pm}$;

(4) $R/\mathrm{P}(R)$ is biregular;

(5) $R/\mathrm{P}(R)$ is weakly regular;

(6) $R/\mathrm{P}(R)$ is right weakly regular;

(7) every prime factor ring of R is a simple domain;

(8) for each $a \in R$ there exists a positive integer m such that $R = Ra^m R + r(a^m)$, where $r(a^m)$ is the right annihilator of a^m in R;

(9) for each prime ideal P of R, $P = \overline{O}_P$.

Corollary 3. If R satisfies any of the following conditions, then R is 2-primal with all idempotents central (hence Theorem 2 holds):

(1) R satisfies condition (SI);

(2) R is 2-primal and satisfies condition (SII);

(3) R satisfies condition (PSII);

(4) every nilpotent element is central;

(5) R is a bounded weakly right duo ring.

Corollary 4. If R is a ring which satisfies conditions (SI) and $\mathfrak{pm}$, then for each $a \in R$ there exists a positive integer k (depending on a) such that $(RaR)^k = (RaR)^{k+1}$.

An immediate corollary of Theorem 2 and the fact that in a reduced ring $O_P = \overline{O}_P$ for all prime ideal P is the following result.

Corollary 5 [1, 2, 4 and 6]. Assume that R is a reduced ring. Then the following conditions are equivalent:

(1) R is weakly π-regular;

(2) R is right weakly π-regular;

(3) R satisfies condition $\mathfrak{pm}$;

(4) R is biregular;

(5) R is weakly regular;

(6) R is right weakly regular;

(7) every prime factor ring of R is a simple domain;

(8) $R = RaR + r(a)$ for each $a \in R$;

(9) for each prime ideal P of R, $P = O_P$.

Note that condition (9) of Corollary 5 is the same as condition (3) of Theorem 6 in [6]. Hence Theorem 2 generalizes parts (1), (2), and (3) of Theorem 6 in [6]. Furthermore, if R is reduced, a routine argument shows that condition (8) of Corollary 5 implies that R

is right p.p. and that $RaR = R$ for all a such that $r(a) = 0$ (i.e., part (4) of Theorem 6 in [6]).

REFERENCES

1. K. Beidar and R. Wisbauer, *Properly semiprime self-pp-modules*, Comm. Algebra **23** (1995), 841–861.
2. L. P. Belluce, *Spectral spaces and non-commutative rings*, Comm. Algebra **19** (1991), 1855–1865.
3. G. F. Birkenmeier, H. E. Heatherly and Enoch K. Lee, *Completely prime ideals ans associated radicals*, Proc. Biennial Ohio State-Denison Conference 1992, edited by S. K. Jain and S. T. Rizvi, World Scientific, New Jersey (1993), 102–129.
4. G. F. Birkenmeier, J. Y. Kim and J. K. Park, *A connection between weak regularity and the simplicity of prime factor rings*, Proc. Amer. Math. Soc. **122** (1994), 53–58.
5. G. F. Birkenmeier, J. Y. Kim and J. K. Park, *Regularity conditions and the simplicity of prime factor rings*, preprint.
6. V. Camillo and Y. Xiao, *Weakly regular rings*, Comm. Algebra **22** (1994), 4095–4112.
7. V. R. Chandran, *On two analogues of Cohen's theorem*, Indian J. Pure Appl. Math. **8** (1977), 54–59.
8. J. W. Fisher and R. L. Snider, *On the von Neumann regularity of rings with regular prime factor rings*, Pacific J. Math. **54** (1974), 135–144.
9. V. Gupta, *Weakly π-regular rings and group rings*, Math. J. Okayama Univ. **19** (1977), 123–127.
10. Y. Hirano, *Some studies on strongly π-regular rings*, Math. J. Okayama Univ. **20** (1978), 141–149.
11. J. Lambek, *On the representation of modules by sheaves of factor modules*, Canad. Math. Bull. **14** (1971), 359–372.
12. V. S. Ramamurthi, *Weakly regular rings*, Canad. Math. Bull. **16** (1973), 317–321.
13. G. Shin, *Prime ideals and sheaf representation of a pseudo symmetric ring*, Trans. Amer. Math. Soc. **184** (1973), 43–60.
14. H. H. Storrer, *Epimorphismen von kommutativen Ringen*, Comment Math. Helv. **43** (1968), 378–401.
15. S. H. Sun, *On biregular rings and their duality*, J. Pure and Applied Algebra **89** (1993), 329–337.

MORITA CONTEXTS AND RING EXTENSIONS

JIANAI CHEN AND QUAN HU

We study in this paper two extension problems which may be well integrated into a common scheme by referring to Morita contexts. In §1 we revisit the classical problem posed by Nakayama in 1933 on the relationship between the Galois extension of rings and the Frobenius algebras [N], to which Kasch announced in 1960 his celebrated solution for a broad class of rings [K1,2]. We give a direct proof to a theorem which Zhang proved in [Zh], using a characterization of Frobenius extension by Onodera [O]. In §2 we consider the ideal module condition introduced by Wilke ([Wil], [Wis]) for the $(A\#H, A^H)$-module A. ${}_{A\#H}A$ is exactly an A^H-ideal module if ${}_{A\#H}A$ is an intrinsic projective self-generator (Theorem 2.4). Hence if $A\#H$ is Morita equivalent to A^H, i.e. if ${}_{A\#H}A$ is a progenerator, then ${}_{A\#H}A$ is an A^H-ideal module. We show that the conversion is true if (a) $A|A^H$ is a Galois extension (Theorem 2.7) or (b) $A|A^H$ is an extension with trace-one elements and A is right Noetherian (Theorem 2.9).

1. Galois and Frobenius extensions

A left H-module algebra for finite dimensional H is also a right H^*-comodule algebra with structure map $\rho: A \to A \otimes H^*$. The covariant $A^{\mathrm{co}\,H^*}$ is defined to be $\{a \in A \mid \rho(a) = a \otimes 1\}$ and it is easy to see that $A^{\mathrm{co}\,H^*} = A^H = B$.

(1.1) **Definition.** Let A be a left H-module algebra with H finite dimensional. Then A/B is said to be right H^*-Galois if the map

$$\gamma: \ A \otimes_B A \longrightarrow A \otimes H^*, \quad a \otimes b \longrightarrow (a \otimes 1)\rho(b)$$

is surjective.

(1.2) **Remark.** If S is bijective, then γ is bijective is equivalent to that $\gamma': A \otimes_B A \longrightarrow A \otimes H^*$, $a \otimes b \longrightarrow \rho(a)(1 \otimes b)$ is bijective [KT, Prop 1.2].

(1.3) **Theorem.** Let A and H be as above. Then the following are equivalent:
1) A is a projective left $A\#H$-module.
2) $f: \ A\#H \longrightarrow A$, $a\#h \to \varepsilon(h)a$ is a split $A\#H$-epimorphism.
3) The trace function $\hat{t}: \ A \longrightarrow B$, $A \longrightarrow t \cdot a$, is surjective.
4) The map (,) in the Morita context (see e.g. [CFM, Thm 2.10]) is surjective.
Moreover, if A/B is H^*-Galois, then 1)–4) are equivalent to
5) A is a faithfully flat right (left) B-module.

(1.4) **Definition** [K2]. Let A/B be a ring extension.
(I) A/B is said to be right Frobenius, if
(r1) A_B is f.g. projective,

(r2) ${}_BA_A \cong \mathrm{Hom}_{-B}(A,B)$, which is equivalent to

(r3) there exists a (B,B)-bimodule homomorphism $\psi: A \longrightarrow B$ such that $f: {}_BA_A \longrightarrow \mathrm{Hom}_{-B}(A,B)$, $a \longrightarrow \psi(a-)$ is a (B,A)-bimodule isomorphism.

(II) A/B is said to be left Frobenius, if

(l1) ${}_BA$ is f.g. projective,

(l2) ${}_AA_B \cong \mathrm{Hom}_{B-}(A,B)$, which is equivalent to

(l3) there exists a (B,B)-bimodule homomorphism $\psi': A \longrightarrow B$ such that $f': {}_AA_B \longrightarrow \mathrm{Hom}_{B-}(A,B)$, $a \longrightarrow \psi'(-a)$ is an (A,B)-bimodule isomorphism.

It is well known that A/B is right Frobenius iff it is left Frobenius. Any isomorphism satisfying (r2) or (l2) is called a Frobenius isomorphism and any homomorphism satisfying (r3) or (l3) a Frobenius homomorphism. We also say that A/B is ψ-Frobenius if it is a Frobenius extension with ψ being a Frobenius homomorphism.

(1.5) **Theorem.** Let H be a finite dimensional Hopf algebra and A a left H-module algebra. Then the following are equivalent:

1) A/B is right H^*-Galois.

2) A/B is $\hat{t}$-Frobenius and A is left $A\#H$-faithful.

3) A/B is $\hat{t}$-Frobenius and A is right $A\#H$-faithful.

(1.6) **Corollary** (See [CFM], Question 2.15). Let H be a finite-dimensional Hopf algebra and A a left H-module algebra. If A/B is $\hat{t}$-Frobenius, then the following are equivalent:

1) A is left $A\#H$-faithful.

2) A is right $A\#H$-faithful.

2. Morita equivalence and ideal module condition between $A\#H$ and B

In this section the $(A, \mathrm{End}({}_AM))$-bimodule structure on a left A-module plays a central role in our study. Because of [CFM, Lemma 0.3] the situation applies to the bimodule ${}_{A\#H}A_B$ with $B = \mathrm{End}_{A\#H-}(A)$.

Let $L({}_{A\#H}A)$ denote left $A\#H$-submodule lattice (i.e. H-stable left ideal lattice) of A, $L(B)$ denote the left ideal lattice of B. Define map $\tau: L({}_{A\#H}A) \longrightarrow L(B)$, $U \mapsto \mathrm{Hom}_{A\#H}(A,U)$, and $\theta: L(B) \longrightarrow L({}_{A\#H}A)$, $I \mapsto A \cdot I$. By [ARS, Prop 1.1], $\mathrm{Hom}_{A\#H}(A,U) \cong U^H = U \cap B$ and $\mathrm{End}\,({}_{A\#H}A) \cong B$.

(2.1) **Definition.** ${}_{A\#H}A$ is called an ideal module over B if τ is bijective.

We first characterize the ideal modules by conditions that are weaker than the generator -property and the projectivity in Mod-$A\#H$ respectively. Let R be a ring and M a left R-module. Define $\sigma[M] = \{N \in R - \mathrm{Mod} \mid N$ is a submodule of W, W is an arbitrary left R-module which is generated by $M\}$. If M generates every module of $\sigma[M]$ then we call M a $\sigma[M]$-generator; if M generates every submodule of M then we call M a self-generator. It is obvious that M is a generator $\Longrightarrow$ M is a $\sigma[M]$-generator $\Longrightarrow$ M is a self-generator, but the inverse implications are not true. If for every left R-exact sequence $0 \to K \to M \to N \to 0$, the sequence $0 \to \mathrm{Hom}(P,K) \to \mathrm{Hom}(P,M) \to \mathrm{Hom}(P,N) \to 0$ is exact, then we call P M-projective; if M can be every module of $\sigma[P]$ then P is called $\sigma[P]$-projective; if M can be P itself then P is called self-projective; if M can be $P^k (k \in \mathbb{N})$ then we call P intrinsic projective. It is easy to see that P is projective $\Longrightarrow$ P is $\sigma[P]$-projective $\Longrightarrow$ P is self-projective $\Longrightarrow$ P is intrinsic projective, but the inverse implications are not true.

(2.2) **Proposition.**
τ is injective $\Longleftrightarrow$ ${}_{A\#H}A$ is a self-generator.

(2.3) **Proposition.**
τ is surjective $\Longleftrightarrow$ ${}_{A\#H}A$ is intrinsic projective.

(2.4) **Theorem.** ${}_{A\#H}A$ is an ideal module over B $\Longleftrightarrow$ ${}_{A\#H}A$ is a self-generator and intrinsic projective.

(2.5) **Corollary.** If $A\#H$ and B are Morita equivalent, then ${}_{A\#H}A$ is an ideal module over B.

(2.6) **Theorem** [Wis, 5.9]. Let A be an R-algebra and M be an (A,B)-bimodule with $B = \mathrm{End}_{A-}(M)$. Then the following are equivalent:
(a) ${}_AM$ is a B-ideal module.
(b) ${}_AM$ is a self-generator and M_B is faithfully flat.
(c) ${}_AM$ is a projective self-generator.

(2.7) **Theorem.** If A/B is a right H^*-Galois extension, then:

$${}_{A\#H}A \text{ is an ideal module over } B \Longleftrightarrow A\#H \text{ and } B \text{are Morita equivalent.}$$

Proof. Clear from (1.3) and (2.6).
In case that A/B is an H^*-extension with trace-one elements, our task is to lift the self-generator ${}_{A\#H}A$ to a generator in $A\#H$-Mod.

(2.8) **Lemma.** If ${}_{A\#H}A$ is an ideal module over B and the trace function $\hat{t}$ is surjective, then A is a $\sigma[A]$-generator and ${}_{A\#H}A$ is faithful.

(2.9) **Theorem.** If the trace function is surjective and A is right Noetherian, then:

$$A\#H \text{and } B \text{ are Morita equivalent } \Longleftrightarrow {}_{A\#H}A \text{ is an ideal module over } B.$$

By Lemma 2.8 and [CFM. Cor 3.10] we have the following result:

(2.10) **Corollary.** Assume A is a division algebra, and ${}_{A\#H}A$ is an ideal module over B, then $A\#H$ and B are Morita equivalent and $A \cong B\#_\sigma H$.

REFERENCES

[ARS] M.Auslander, I. Reiten and S. O. Smalϕ, Galois actions and rings and finite Galois coverings, Math. Scand. 65(1989), 5–32.

[CFM] M. Cohen, D. Fischman and S. Montgomery, Hopf Galois extensions, smash products and Morita equivalences, J. Algebra, 133(1990), 351–372.

[K1] F. Kasch, Grundlagen einer Theorie der Frobeniuserweiterungen. Arch. Math. Bd. 127(1954), 453–474.

[K2] F. Kasch, Projective Frobenius-Erweiterungen. Sitzungsber. Heidelberger Akad. Wiss. 1960/61 (4), 89–109.

[Mo] S. Montgomery, Hopf algebras and their actions on rings, CBMS Lectures in Math. Vol.82, AMS, Providence, RI, 1993.

[N] T. Nakayama, On two topics in the structure theory of rings (Galois theory of rings and Frobenius algebras). Proc. of the Int. Congress of Math.

[O] T. Onodera, Some studies on projective Frobenius extensions, J. Fac. Sci. Hokkaido Univ. 18(1964), 89–107.

[Wil] B. Wilke, Zur Modulstruktur von Ringen über Schiefgruppenringen, Inaugural-Dissertation, Heinrich-Heine-Universität, Düsseldorf, 1994.

[Wis] R. Wisbauer, Modules and algebras, the bimodule structure of rings, 1995.

[Zh] Y. Zhang, Hopf Frobenius extensions of algebras, Comm. Alg., 20(7) (1992), 1907–1915.

Mathematics Department
Capital Normal University
Beijing, 100037 China

ON EXTENDING MODULES

CHEN ZHIZHONG

Throughout this paper R is an associative ring with identity. Every module M is unitary right R-module.

The module M is called extending, or a CS-module (for complement submodules are direct summand), provided that it satisfies following condition (C1):

(C1) Every submodule of M is essntial in a direct summand of M. Equivalently, iff every closed submodule is a direct summand.

M is quasi-continuous if and only if it has (C1) and following (C3)

(C3) If M_1 and M_2 are direct summands of M with $M_1 \cap M_2 = 0$, then $M_1 \oplus M_2$ is a direct summand of M.

Let there be given a direct decomposition $M = M_1 \oplus M_2$, and let $\pi_i : M \longrightarrow M_i$ be the projections. If N is a submodule of M with $N \cap M_2 = 0$. Then

$$N = \{ x+y \mid x \in \pi_1(N),\ y=h(x) \in \pi_2(N) \}$$

where $h: \pi_1(N) \longrightarrow \pi_1(N)/(M_1 \cap N) \cong \pi_2(N)/(M_2 \cap N) \cong \pi_2(N)$ is a homomorphism, and $\mathrm{Ker}\ h = M_1 \cap N$.

Let M and N be R-modules and $M' \leq M$, and $i: M' \longrightarrow M$ be the inclusion.

Then N is called almost M-injective if the following (1) or (2) hold for any submodule M' of M and any homomorphism $h: M \longrightarrow N$

(1). There exists $h': M \longrightarrow N$, such that $h'i = h$

or

(2). There exists a non-zero direct summand M_0 of M and $h': N \longrightarrow M_0$, such that $h'h = \pi i$, where $\pi : M \longrightarrow M_0$ is the projection of M onto M_0.

If we have only (1), N is called M-injective.

Proposition 1. [6]. The following statements are equivalent for a module M.

(1). M is quasi-continuous;

(2). Whenever $E(M) = E_1 \oplus E_2$ is a direct sum of submodules E_1, E_2. then $M = (E_1 \cap M) \oplus (E_2 \cap M)$, where $E(M)$ is a injective hull of M.

(3). Whenever N_1 and N_2 are submodules of M such that $N_1 \cap N_2 = 0$, there exists submodules M_1, M_2 of M such that $M = M_1 \oplus M_2$ and $N_i \leqslant M_i$, $(i=1,2)$.

Lemma 2. Let M_1 and M_2 be modules, and let N be a submodule of $M = M_1 \oplus M_2$ with $N \cap M_2 = 0$. Then

(1). N' is an essential extension of N in M iff $\pi_1(N')$ is an essential extension of $\pi_1(N)$ in M.

(2). N is closed in M iff $\pi_1(N)$ is closed in M.

Theorem 3. Let M_1 and M_2 be indecomposible R-modules with local endomorphism rings, and let $M = M_1 \oplus M_2$. For any submodule N of M, if $N \cap M_2 = 0$, $N \cap M_1 \neq 0$. Then the following are equivalent:

(1). M_2 is M_1-injective;

(2). M_2 is almost M_1-injective;

(3). For a submodule N of M, if $N \cap M_2 = 0$, then N is contained in a direct summand of M, such that $M = M' \oplus M_2$ and $N \leqslant M'$.

Proposition 4. Let M_1 and M_2 be R-modules and let $M = M_1 \oplus M_2$. Then M_2 is M_1-injective iff for every submodule N of M such that $N \cap M_2 = 0$, there exists a submodule M' of M such that $M = M' \oplus M_2$ and $N \leqslant M'$

Proposition 5. Let M_1 and M_2 be R-modules with local endomorphism rings and let $M = M_1 \oplus M_2$. Then M_2 is almost M_1-injective iff for every submodule N of M such that $N \cap M_2 = 0$, then N is contained in a proper direct summand of M.

Proposition 6. [1]. Let M_1 and M_2 be uniform modules with local endomorphism rings and $M = M_1 \oplus M_2$. Then M is extending iff $M_1$and M_2 are mutually almost relative injective.

Theorem 7. Let M be an extending module. Then M is quasi-continuous iff whenever $M = M_1 \oplus M_2$ is a diect sum of submodules, then M_1 and M_2 are relative injective.

Proposition 8. Let M_2 be an R-module and M_1 an indecomposable module and let $M = M_1 \oplus M_2$, If M_2 is almost M_1-injective but a not M_1-injective. Then there exists a monomorphism f from a submodule of $M_1$to M_2 and M_2 is embedded in M_1.

Theorem 9. Let M_1 and M_2 be uniform modules with local endomorphism rings and $M = M_1 \oplus M_2$. For any submodule N of M, if $N \cap M_i = 0$, $N \cap M_j \neq 0$, $i \neq j$. Then M is extending iff M is quasi-continuous.

Reference

[1]. Y.Baba and M. Harada: On almost M-projectives and almost M-injectives, Tsukuba Math. J. 14 (1990). 53-69.

[2]. M.Harada: Note on almos relative projectives and almost relative injectives, Osaka J. Math. 29 (1992), 435-446.

[3]. M.Harada and A.Tozaki: Almost M-projectives and Nakayama rings, J. Algebra 122 (1989), 447-474.

[4]. M.A.Kamal: Relative injectivity and CS-modules. Internat. J. Math.& Math. Sci. 17 (1994) 661-666.

[5]. S.H.Mohamed and B.J.Muller: Continuous and Discrete Modules, London Math. Soc. Lec. N, Cambridge Univ. Press., 147 (1990).

[6]. R.Wisbauer: Foundations of Module and Ring Theory,Gordon and Breach Science Publishers, 1991.

Department of Mathematics
Northern Jiaotong University
Beijing, 100044, P.R.China

$\prod$-Coherent Rings and FGT-Flat Dimension

Cheng Fuchang

Let R be a ring, $\Pi=\prod R_R$ be any product of R_R. We say that R is a right Π-coherent ring, if every f.g. submodule of Π is f.p.. The class of such ring is between the class of Noetherian rings and the class of coherent rings. The homological properties and structures of Π-coherent rings have been well studied by many authors [1-4]. The main purpose of this paper is to study the FGT-flat dimension over Π-coherent ring.

All rings are associative with identity, all modules are unital. $(\ )^*=Hom_R(-, R)$.

We first give some characterizations for Π-coherent rings.

Theorem 1 The following statements are equivalent for any ring R :

(1) R is a right Π-coherent ring;

(2) R_R is a right Π-coherent module;

(3) Every f.g. torsionless right R-module is Π-coherent;

(4) Every f.g. torsionless right R-module is f.p. .

Proof (1) <=> (2) and (2) => (3) see [4, theorem 1.1].

(3) => (1) We can finish the proof by [2, theorem 1].

(1) <=> (4) By [2, theorem1] and [5, lemma3], we can imply the results.

By theorem 1, the following facts are easy to prove:

Propositoin 2 Let R be a right Π-coherent ring, $n(\geqslant 2)$ integer, then the following statements are equivalent:

(1) $w.gl.\dim R\leqslant n$;

(2) The dual of any f.g. torsionless right R-module has flat dimension $\leqslant n-2$.

Corollary 3 Let R be a left, right Noetherian ring, then

l. gl. dim$R \leq 2$ if and only if the dual of any f. g. torsionless right R-module is projective module.

Difinition A right R-module M is called right FGT-flat, if $tor_1^R(M, A)=0$ for any f. g. trosionless left R-module A. The number inf{ n | $tor_{n+1}^R(M, A)=0$. for any f. g. torsionless left R-module A } is called FGT-flat dimension of M. denoted it by r. FGT-fd_RM. we call the number sup{ r. FGT-fd_RM. for any right R-module M } right FGT-flat dimension of R. and denoted it by r. FGT-w. dimR.

Theorem 4 Let R be a left $\prod$-coherent, then l. FGT-w. dimR=0 if and only if R is left semihereditary.

Proof If l. FGT-dimR=0, then w gl. dim$R \leq 1$, but R is left coherent ring, so R is left semihereditary .

Conversely. if A is any f. g. torsionless right R-module, there exists an exact sequence 0—>A—>F. where F is f. g. free module. since w. gl. dim$R \leq 1$. then $tor_1^R(A, M)=0$, so l. FGT-w. dimR=0.

Theorem 5 Let R be any ring, then the following statements are equivalent:

(1) l. FGT-w. dimR=1;

(2) There exists a left R-module A such that l. FGT-fd_RA=1, and any closed submodule of f. g. free right R-module is a flat module.

Proof (1) => (2) If 0—>K—>F—>F/K—> 0 is an exact sequence, where F is f. g. free right R-module, F/K f. g. torsionless. Since l. FGT-w. dimR=1, then $fd_R(F/K) \leq 1$, hence K is flat module.

(2) => (1) It holds clearly.

Theorem 6 Let R be a left, right $\prod$-coherent ring. n (≥ 2) integer Then the following statements are equivalent:

(1) r. FGT-w. dim$R \leq n$;

(2) r. $fd_R(M^*) \leq n-2$ for any f. g. torsionless left R-module M;

(3) r. $pd_R(M^*) \leq n-2$ for any f. g. torsionless left R-module M;

(4) r. $fd_R(M^*) \leq n-2$ for any left R-module M.

Proof (2) <=> (3) By theorem 1 (4), so r. $fd_R(M^*)$=r. $pd_R(M^*)$ for any f. g. trosionless left R-module M.

(1) => (2) It holds clearly by theorem 1 and [6, Lemma 3].
(2) => (1), (1) => (4) and (4) => (2) are trivial.

REFERENCES

[1] M. F. Jones, Flatness and f-projectivity of Torsion-Free Modules and Injective Modules, Lecture Notes in Mathematics. Vol. 951 (1982).

[2] V. Camillo, Coherence for Polynoimal Rings, J. Algebra, 132 (1990), 72-76.

[3] Mingyi Wang, Some Studies on Π-Coherent Rings, Proc. Amer. Math. Soc., 119, No. 1 (1993), 71-76.

[4] Cheng Fuchang, Tang Jinyu, Huang Zhaoyong and Wang Mingyi, Π-Coherent Ring and FGT-Injective Dimension Sea Bull. Math. Vol 19 No. 3 (1995), 105-122.

[5] M. F. Jones and M. L. Teply, Coherent Rings of Finit Weak Global Dimension, Comm. Alg. 10 (1982).

Department of Mathematics
Guangxi Normal University
Guilin, 541004 China

Modules over a Local Cohen-Macaulay Ring admitting a Dualizing Module

Edgar E. Enochs
Overtoun M.G. Jenda
Jinzhong Xu

Let R be a (commutative) local Cohen-Macaulay ring admitting a dualizing module Ω. Foxby (see [4]) then defines two classes of R-modules, namely $\mathcal{G}_0(R)$ and $\mathcal{J}_0(R)$. The class $\mathcal{G}_0(R)$ consists of all M such that $\mathrm{Tor}_i(\Omega, M) = 0$ and $\mathrm{Ext}^i(\Omega, \mathrm{Hom}(\Omega, M)) = 0$ for $i \geq 1$ and such that $\Omega \otimes \mathrm{Hom}\,(\Omega, M) \to M$ is an isomorphism. The class $\mathcal{J}_0(R)$ consists of all N such that $\mathrm{Ext}^i(\Omega, N) = 0$ and $\mathrm{Tor}_i(\Omega, \Omega \otimes N) = 0$ for $i \geq 1$ and such that $N \to \mathrm{Hom}\,(\Omega, \Omega \otimes N)$ is an isomorphism.

Foxby proves

Theorem ([5], Theorem 1). A finitely generated R-module M is in $\mathcal{G}_0(R)$ if and only if G-dim $M < \infty$.

Here the G-dimension of a module is as defined in Auslander [2] and in Auslander, Bridger [3].

Motivated by the notion of G-dimension, Enochs and Jenda [7] defined Gorenstein injective and projective modules. These notions provide a generalization of the notion of a G-dimension (namely, the Gorenstein projective dimension) and provide a notion dual to that of the G-dimension (the Gorenstein injective dimension).

We have

Theorem ([8], section 2). A module M is in $\mathcal{G}_0(R)$ if and only if it has finite Gorenstein projective dimension and is in $\mathcal{J}_0(R)$ if and only if it has finite Gorenstein injective dimension.

Auslander announced the result

Theorem ([1]). If R is a local Gorenstein ring and M is a finitely generated R-module, then M has a minimal Cohen-Macaulay approximation $\phi : C \to M$.

This result means that C is finitely generated and is maximal Cohen-Macaulay, that $\mathrm{Hom}(D,C) \to \mathrm{Hom}(D,M) \to 0$ is exact whenever D is also a finitely generated maximal Cohen-Macaulay module and that any $f: C \to C$ such that $f \circ \phi = \phi$ is an automorphism of C (so such approximations are unique up to isomorphism).

We can prove

Theorem ([9]). If R is a local Cohen-Macaulay ring and has a dualizing module, then any finitely generated module of finite G-dimension has a minimal Cohen-Macaulay approximation.

We note that over a local Gorenstein ring, every finitely generated module has finite Gorenstein dimension (see [2]).

To get a non-commutative version of Auslander's theorem we use Iwanaga's definition of a (possibly non-commutative) Gorenstein ring (see [10]). We will call these Iwanaga Gorenstein rings.

Theorem ([9]). If R is Iwanaga Gorenstein and admits Matlis duality, then every finitely generated left R-module M has a finitely generated Gorenstein projective cover $C \to M$ (see [6] for the general definition of a cover).

If R is commutative and local, the condition "admitting Matlis duality" becomes "R is complete" and then the $C \to M$ of the theorem is just a minimal Cohen-Macaulay approximation. The notion "admitting Matlis duality" is defined in [9].

For example, if S is a complete regular local ring, and G is a finite group, the ring $R = SG$ satisfies the hypotheses of the theorem. In this case the C of the conclusion will always be a free S-module, i.e. C will be an SG-lattice.

References

1. M. Auslander, *Minimal Cohen-Macaulay Approximations*, in preparation.

2. M. Auslander, *Anneaux de Gorenstein et torsion en algèbre commutative, séminaire d'algèbre commutative*, Ecole Normale Supérieure de Jeunes Filles, Paris 1966/67.

3. M. Auslander and M. Bridger, *Stable Module Theory*, Memoirs A.M.S. **94**, A.M.S., Providence, R.I., 1969.

4. H.-B. Foxby, *Gorenstein dimensions over Cohen-Macaulay rings*, Proceedings of the international conference on commutative algebra, W. Bruns (editor), Universität Osnabrück, 1994.

5. H.-B. Foxby, *Quasi-perfect modules over Cohen-Macaulay rings*, Math. Nachr. **66** (1975), 103-110.

6. E. Enochs, *Injective and flat covers, envelopes and resolvents*, Israel J. Math. **39** (3), 1981, 189-209.

7. E. Enochs and O.M.G. Jenda, *Gorenstein Injective and Projective modules* (to appear in Math. Zeit.).

8. E. Enochs and O.M.G. Jenda, *Foxby duality and Gorenstein injective and projective modules* (to appear in Trans. Amer. Math. Soc.).

9. E. Enochs and O.M.G. Jenda, *Lifting group representations to maximal Cohen-Macaulay representations* (submitted).

10. Y. Iwanaga, *On rings with finite self-injective dimension II*, Tsukuba J. Math. **4**, 1980, 107-113.

Edgar E. Enochs
Department of Mathematics
University of Kentucky
Lexington, KY 40506-0027
USA
e-mail enochs@ms.uky.edu

Overtoun M.G. Jenda
Department of Discrete and Statistical Sciences
120 Mathematics Annex
Auburn University
Auburn, AL 36849-5307
USA
e-mail jendaov@mail.auburn.edu

Jinzhong Xu
Department of Mathematics
University of Kentucky
Lexington, KY 40506-0027
e-mail abc@ms.uky.edu

GORENSTEIN INJECTIVE ENVELOPES AND ESSENTIAL EXTENSIONS

EDGAR E. ENOCHS and OVERTOUN M. G. JENDA

Throughout this note, R will denote an n-Gorenstein ring, that is, R is left and right noetherian and has self-injective dimension at most n on either side.

An R-module M is said to be *Gorenstein injective* if there exists an exact sequence

$$\cdots \to E^{-2} \to E^{-1} \to E^{0} \to E^{1} \to \cdots$$

of injective R-modules with $M = \text{Ker}(E^0 \to E^1)$ such that the sequence remains exact when $\text{Hom}(E, -)$ is applied to it for any injective R-module E. We note that every injective R-module is Gorenstein injective. Furthermore, it was shown in Enochs-Jenda [2, Theorem 4.2] that every mth cosyzygy is Gorenstein injective whenever $m \geq n$. In particular, over quasi-Frobenius rings, every module is Gorenstein injective.

A *Gorenstein injective envelope* of an R-module M is a linear map $\psi : M \to G$ with G a Gorenstein injective R-module such that

1) for any Gorenstein injective R-module G', the diagram

$$\begin{array}{ccc} M & \xrightarrow{\psi} & G \\ \downarrow & \swarrow & \\ G' & & \end{array}$$

can be completed to a commutative diagram, and

2) the diagram

$$\begin{array}{ccc} M & \xrightarrow{\psi} & G \\ \downarrow{\scriptstyle\psi} & \swarrow & \\ G & & \end{array}$$

can only be completed by an automorphism of G.

If $\psi : M \to G$ satisfies (1) and may be not (2), ψ is called a *Gorenstein injective preenvelope*.

We note that ψ is a monomorphism since injective modules are Gorenstein injective, and that Gorenstein injective envelopes are unique up to isomorphism.

It was shown in Enochs-Jenda [3] and Enochs-Jenda-Xu [4] that Gorenstein injective preenvelopes and envelopes exist over n-Gorenstein rings. The Gorenstein injective envelope of a module M will be denoted by $G(M)$.

The following is often useful.

Lemma. *Let M be an R-module. If $M \subseteq G$ is a Gorenstein injective preenvelope, then $G \cong G(M) \oplus G'$ for some Gorenstein injective R-module G' where the isomorphism leaves M fixed.*

Proof: We have the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & G(M) & \longrightarrow & \frac{G(M)}{M} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & G & \longrightarrow & \frac{G}{M} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & G(M) & \longrightarrow & \frac{G(M)}{M} & \longrightarrow & 0
\end{array}$$

where $G(M) \to G \to G(M)$ is an automorphism. So $G \cong G(M) \oplus G'$ for some Gorenstein injective R-module G'. $\square$

Definition. A submodule N of an R-module M is said to be a *Gorenstein essential submodule* if for each submodule L of M with $pdL < \infty$, $N \cap L = 0$ implies $L = 0$. We will say that $N \subseteq M$ is a *Gorenstein extension* if $pd\frac{M}{N} < \infty$. A Gorenstein extension $N \subseteq M$ is then said to be a *Gorenstein essential extension* if N is a Gorenstein essential submodule of M, and we will say that it is a *Gorenstein injective extension* if M is a Gorenstein injective module.

We have the following interesting properties of Gorenstein injective envelopes.

Proposition. *The following properties hold for an R-module M.*

1) *$M \subseteq G$ is a Gorenstein injective extension if and only if $G \cong G(M) \oplus E$ for some injective R-module E where the isomorphism leaves M fixed.*

2) *$M \subseteq G(M)$ is a Gorenstein essential extension.*

3) *If N is a submodule of M, then $N \subseteq G(M)$ is a Gorenstein essential extension if and only if $G(N) \cong G(M)$ where the isomorphism leaves N fixed.*

4) *M is Gorenstein injective if and only if $M \cong G(M)$.*

Proof: 1) This is Proposition 2.2 of [3] and we provide a proof here for completeness. If $M \subseteq G$ is a Gorenstein injective extension, then $\mathrm{Ext}^1(G/M, H) = 0$ for all Gorenstein injective R-modules H by Enochs-Jenda [2, Corollary 4.4]. Thus $M \subseteq G$ is a Gorenstein injective preenvelope and so $G \cong G(M) \oplus G'$ for some Gorenstein injective G' by the Lemma above. But $pdG/M < \infty$ and so $pdG' < \infty$. But then G' is an injective R-module.

For the converse, we simply note that $M \subseteq G(M)$ is a Gorenstein extension by Enochs-Jenda [3, Proposition 1.8] and E has finite projective dimension by Iwanaga [5].

2) This is part of Theorem 3.3 of [1] and we again provide a proof for completeness. $M \subseteq G(M)$ is a Gorenstein extension as noted above. If N is a submodule of $G(M)$ such that $N \cap M = 0$ and $pdN < \infty$, then $E(N)$ is a submodule of $G(M)$ by Proposition 2.4 of [1]. If $G' \supseteq M$ is maximal in $G(M)$ with respect to $G' \cap E(N) = 0$, then $G(M) \cong E(N) \oplus G'$ and so $M \subseteq G'$ is a Gorenstein injective extension. But then $G' \cong G(M) \oplus E$ for some injective E by part (1). Thus $E(N) = 0$ and so M is a Gorenstein essential submodule of $G(M)$.

3) If $N \subseteq G(M)$ is a Gorenstein extension, then $G(M) \cong G(N) \oplus E$ for some injective E by part (1) where the isomorphism leaves N fixed. But $E \cap N = 0$ and $pdE < \infty$. So $E = 0$ since N is a Gorenstein essential submodule of $G(M)$. Thus the result follows.

If $G(N) \cong G(M)$, then $N \subseteq G(M)$ is a Gorenstein essential extension by part (2).

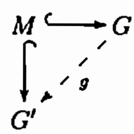
4) If M is Gorenstein injective, then $\mathrm{Hom}(G(M), M) \to \mathrm{Hom}(M, M) \to 0$ is exact and so $G(M) \cong M \oplus G'$ for some Gorenstein injective G'. But then $pdG' < \infty$. So $G' = 0$ since $M \subseteq G(M)$ is Gorenstein essential by (2). The converse is trivial. □

We are now in a position to state the following

Theorem. *The following are equivalent for a submodule M of an R-module G.*

1) $M \subseteq G$ is the Gorenstein injective envelope of M.

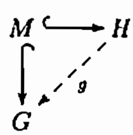
2) $M \subseteq G$ is a Gorenstein injective extension and for every Gorenstein injective extension $M \subseteq G'$, there is a monomorphism $g : G \to G'$ that makes the diagram



commutative.

3) G is a minimal Gorenstein injective extension of M.

4) $M \subseteq G$ is a Gorenstein essential extension and if M is a submodule of an R-module H such that $M \subseteq G(H)$ is a Gorenstein essential extension, then there is a monomorphism $g : H \to G$ that makes the diagram



commutative.

Proof: $1 \Rightarrow 2$. $M \subseteq G$ is a Gorenstein injective extension by the Proposition above. If $M \subseteq G'$ is a Gorenstein injective extension, then it is a Gorenstein injective preenvelope. So there is an isomorphism $g' : G \oplus H \to G'$ that leaves M fixed by the Lemma above. So we set $g = g'_{|G}$.

$2 \Rightarrow 3$ is trivial.

$3 \Rightarrow 1$. $G(M)$ is a summand of G by the Lemma and so $G \cong G(M)$ by minimality.

$1 \Rightarrow 4$. $M \subseteq G$ is a Gorenstein essential extension by the Proposition above. Now if $M \subseteq G(H)$ is a Gorenstein essential extension, then there is an isomorphism $g' : G(H) \to G$ which leaves M fixed by part 3 of the Proposition. We now set $g = g'_{|H}$.

$4 \Rightarrow 1$. Let $H = G(M)$. Then since $M \subsetneq G(M)$ is a Gorenstein essential extension, we have again the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & G(M) & \longrightarrow & \frac{G(M)}{M} & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle g} & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & G & \longrightarrow & \frac{G}{M} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & G(M) & \longrightarrow & \frac{G(M)}{M} & \longrightarrow & 0
\end{array}$$

and so $G \cong G(M) \oplus G'$ for some R-module G' such that $pdG' < \infty$. But $M \cap G' = 0$ and $M \subseteq G$ is a Gorenstein essential extension. So $G' = 0$ and thus $G \cong G(M)$. □

Remark. $G(M)$ can also be characterized as the unique maximal Gorenstein essential extension of M (see Theorem 3.3 of [1]).

References

[1] E. Enochs and O. Jenda: Gorenstein injective modules over Gorenstein rings, submitted.

[2] E. Enochs and O. Jenda: Gorenstein injective and projective modules, Math. Z., to appear.

[3] E. Enochs and O. Jenda: Gorenstein balance of Hom and Tensor, Tsukuba J. Math. **19** (1995), 1-13.

[4] E. Enochs, O. Jenda and J. Xu: Covers and envelopes over Gorenstein rings, Tsukuba J. Math., to appear.

[5] Y. Iwanaga: On rings with finite self-injective dimensions II, Tsukuba J. Math. 4 (1980), 107-113.

Edgar E. Enochs
Department of Mathematics
University of Kentucky
Lexington
KY 40506-0027
USA

Overtoun M. G. Jenda
Department of Discrete and Statistical Sciences
120 Mathematics Annex
Auburn University
AL 36849-5307
USA

A Commutativity Theorem for Rings

Fu Changlin & Wang Qiurong

In this paper R represents a ring, Z(R) is the center of R, $D = \{x \mid \exists y \in R, \ni yx = 0 \text{ or } xy = 0\}$, $f(t_1, \ldots, t_n)$ represents a polynomial of n non-commutative indeterminates with integer coefficients, its coefficients having 1 as their highest common factor. X, Y represents the set of indeterminates with $X \cup Y = \{t_1, \ldots, t_n\}$ and $X \cap Y = \Phi$.

Suppose $f(t_1, \ldots, t_n)$ is an assigned polynomial, X, Y are above mentioned sets of indeterminates. We use $A_{xb}(A_{yb})$ to denote the sum of coefficients of all the terms in $f(t_1, \ldots, t_n)$ which begin with an element of X(Y), and A_{xe} (A_{ye}) to denote the sum coefficients of all the terms in f which are ended with an element in X(Y).

For the assigned sets X,Y, we use B to denote the set of some terms of f in which any two neighboring factors of the terms do not belong to the set X at the same time.

With the similar definition as A_{xb}, A_{xe}, A_{yb}, A_{ye}, we use B_{xb}, B_{xe}, B_{yb}, B_{ye} to denote the sum of the coefficients of terms in B. B_{xx} is the sum of the coefficients of those terms in B with the beginning and the ending indeterminate elements both in the same X. The definitions of the symbols B_{xy}, B_{yx}, B_{yy} are similar.

We use B_1 to denote the subset of the terms in set B which contains only one indeterminate element in X with its degree equal to 1; B_2, the subset of the terms in set B_1 which begin with an element in X; B_3, the subset of the terms in set B_1 which end with an element in X; $\widetilde{B}_i$, the sum of the coefficients of all the terms in B_i; C_1 the set of those terms in f in which no two arbitrary neighboring factors belong to the same set (X or Y); $C_{xb}, \ldots, C_{ye}$, $C_{xx}, \ldots, C_{yy}$; the sum of the coefficients of the terms in set C_1 which are defined in the similar way as $A_{xb}, \ldots, A_{xx}, \ldots$, etc.

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Let $h(t_1,\dots,t_n)$ be a term of $f(t_1,\dots,t_n)$. r_s $(s=1,2,\dots,n)$ is degree of indeterminates t_s in $h(t_1,\dots,t_n)$. If $X=\{t_{k_1},\dots,t_{k_n}\}$, and $Y=\{t_{q_1},\dots,t_{q_n}\}$ then we call $i=r_{k_1}+\dots+r_{k_p}$ the degree of X in $h(t_1,\dots,t_n)$, and $j=r_{q_1}+\dots+r_{q_{n-p}}$ the degree of Y in $h(t_1,\dots,t_n)$. And writing $h(t_1,\dots,t_n)=h_{ij}(t_1,\dots,t_n)$,we call i+j the degree of $h(t_1,\dots,t_n)$.

Suppose polynomial $f(t_1,\dots,t_n)$ has highest degree K_2 and least degree K_1. Writing every $f(t_1,\dots,t_n)$ in the form:

$$f(t_1,\dots,t_n)=\sum_{m=k_1}^{k_2}\sum_{i+j=m} f_{ij}(t_1,\dots,t_n),$$

where f_{ij} denotes the sum of all terms of $f(t_1,\dots,t_n)$ with i degree in X and j in Y.

Definition

$$\text{Let } f(t_1,\dots,t_n)=\sum_{m=k_1}^{k_2}\sum_{i+j=m} f_{ij}(t_1,\dots,t_n),$$

if there exists an integer K $(K_1\le K\le K_2)$ and sets X,Y, such that in $f_{k-1\,1}$ $(f_{1\,k-1})$ only one term has coefficients ±1, and when $m\neq K$, $f_{m-1\,1}=0$ $(f_{1\,m-1}=0)$, then we call $f(t_1,\dots,t_n)$ has property F_k for X(Y).

Let R be an arbitray ring. X, Y are above mentioned sets of indeterminates, and polynomial $f(t_1,\dots,t_n)$has property F_k for X(Y).

If $f(t_1,\dots,t_n)\in Z(R)$ and $f(t_1,\dots,t_n)$ has property F_k for X(Y), obveriously, we can assume that no terms with all indeterminates in Y(X) are contained in f.

We obtain the following results:

Theorem

If for all $X_1,\ldots,X_n \in R$ satisfying $[f(x_1,\ldots,x_n), \Sigma a_s X_{i_s}]=0$ where every $t_{i_s} \in X$ ($t_{i_s} \in Y$) $\Sigma a_s = 1$, if $K=1$ then R is commutative; if $K>1$ but $R \neq D$ and $Z(R) \neq 0$, then R is also commutative.

If $Z(R) \cap D \neq 0$ and cofficients of $f(t_1, \ldots, t_n)$ satisfying one of the condition A'1-A'5 (A"1-A"5), then $f_{k-1,1}$ ($f_{1\,k-1}$) in therom can be in arbitrary form, where A'1-A'5 and A"1- A"5 denote as follows:

A'1, $|A_{xb}| = 1$; A"1, $|A_{yb}| = 1$;

A'2, $|A_{xc}| = 1$; A"2, $|A_{yc}| = 1$;

A'3, $(B_{xb}, B_{xc}) = 1$; A"3, $(B_{yy}, B_{xb} - B_{xo}) = 1$;

A'4, $(C_{xb} - C_{xc}, C_{xx}) = 1$; A"4, $(C_{xc} - C_{xb}, C_{xx}) = 1$;

A'5, $|\widetilde{B}_3| = 1$; A"1, $|\widetilde{B}_2| = 1$.

where $(a_1,\ldots,a_n)$ denotes the greatest common divisor of $a_1,\ldots,a_n$.

It is a generalization of the results in references [4]-[10].

References

[1]. T. P. Kezlan, A note on commutativity of semiprime PI-ring, Math. Japan, 27 (1982),

267-268.

[2]. T. P.Kezlan, a commutativity theorem involving certain polynomial constraints, Math. Japan, 36, No.4 (1991), 785-789.

[3]. T. P. Kezlan, On identities which are equivalent with commutativity, Math. Japan, 34, No.2(1989), 197-204.

[4]. Abujabal H.A.S, Khan.M.S, A commutativity result for rings. Ball Inst. Math. Sinca. 18(1990), 333-337. (MR 92f 16038).

[5]. Nishaakak,Tsunekazu, A commutativity theorem for rings. Rad. Mat 6(1990) No.2 357-359. (MR 92a 16042).

[6]. Quadri Murtaza A, On commutativity of assciative rings, Ball Austral. Math. Soc 38(1988) N0.2 267-271. (MR 89k 16042).

[7]. Ashraf Mohd; Quadri Martaza A, Ali Asma, On commutativity of one sided s-unital rings, Rad. Mat 6(1990) No.1 111-117. (MR 91j 16043).

[8]. Abujabal H.A.S, Khan. M. S, On commutativity theorems for rings, Internet. J. Math. Sci 13(1990) No.1, 87-92. (MR 91c 16043).

[9]. Quadri.M.A, Khan.M.A, Math.Japon, 33 No.2(1988) 275-279.

[10]. Abujabal H.A.S, Two commutativity results for ring, Math. Japon. 38(1993) No.2 297-299. (MR 94b 16042).

Fu Changlin
Harbin University of Science and Technology

Wang Qiurong
Harbin Broadcasting and Television University

On the Transfer of Torsion Theoretic Properties in Morita Contexts

A. HAGHANY

This note contains a brief survey of relationship between torsion theories of context equivalent rings, as well as some new observations.

Rings which are in a Morita context have been investigated from different points of view. Here we consider the transfer of some torsion theoretic properties for a pair of rings which are in a Morita context. First recall that a Morita context (R, V, W, S) consists of a pair of rings R, S, two bimodules ${}_RV_S$, ${}_SW_R$ and two bimodule homomorphisms

$$(-,-) : V \otimes_S W \longrightarrow R \; ; \quad [-,-] : W \otimes_R V \longrightarrow S$$

satisfying the following relations for all v, $v' \in V$; w, $w' \in W$:

$$(v, w)v' = v[w, v'] \; ; \quad [w, v]w' = w(v, w')$$

If the functor $W \otimes_R : R\text{-Mod} \longrightarrow S\text{-Mod}$ and $V \otimes_S : S\text{-Mod} \longrightarrow R\text{-Mod}$ are equivalences of categories then R and S are Morita equivalent rings. In this case W_R, ${}_SW$ are progenerators and $S \simeq \operatorname{End} W_R$, $R \simeq \operatorname{End}_S W$.

Given a Morita context (R, V, W, S) and hereditary torsion theories τ on R-Mod and σ on S-Mod, one may be interested in the relationship of the localized rings R_τ and S_σ. For example if τ and σ are the hereditary torsion theories determined by the trace ideals T_R $(= (V, W)$, the image of $(-,-))$ and $T_S = [W, V]$ then Müller [4] has shown that the quotient categories Mod (R, τ) and Mod (S, σ) are equivalent and the equivalence is induced by the functors $\operatorname{Hom}_R(W, -)$ and $\operatorname{Hom}_S(V, -)$. Some other full subcategories of Mod-R and Mod-S, and their equivalences by Hom, Tensor or their appropriate subfunctors and quotient functors have also been considered. See for example [1], [2], [3], [5], [6].

One may, on the other hand, start with a hereditary torsion theory τ on R-Mod and construct a hereditary torsion theory σ on S-Mod. When R is Morita equivalent to S, if

$\tau = (\underline{T}, \underline{F})$, then $\alpha\tau = (\alpha(\underline{T}), \alpha(\underline{F}))$ is naturally a hereditary torsion theory on S-Mod, where α is the functor $W \otimes_R -$. A classical result is then the following: The maximal rings of quotients of R and S are again Morita equivalent. This is a consequence of the fact that if τ is the Lambek torsion theory then so is $\alpha\tau$. Recall that the Lambek torsion theory is cogenerated by the injective envelope of the ring viewed as a (left) module over itself.

More generally let

$$0 \longrightarrow R \longrightarrow E_0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow \cdots \tag{1}$$

be the minimal injective resolution of ${}_R R$. Suppose τ_n is the torsion theory cogenerated by the injective module $E_0 \oplus \cdots \oplus E_n$. Then $\alpha\tau_n$ is the torsion theory $\tau_n(S)$ cogenerated by $C_0 \oplus \cdots \oplus C_n$ where

$$0 \longrightarrow S \longrightarrow C_0 \longrightarrow C_1 \longrightarrow C_2 \longrightarrow \cdots$$

is the minimal injective resolution of ${}_S S$. The following outlines a proof: By applying α to (1), we obtain

$$0 \longrightarrow \alpha(R) \longrightarrow \alpha(E_0) \longrightarrow \alpha(E_1) \longrightarrow \alpha(E_2) \longrightarrow \cdots$$

and this is the minimal injective resolution of $\alpha(R) = W \otimes_R R \simeq W$. Since W is an S-progenerator the injective modules $C_0 \oplus \cdots \oplus C_n$ and $\alpha(E_0) \oplus \cdots \oplus \alpha(E_n)$ cogenerate the same torsion theory. Thus in particular (for $n = 0$) the Lambek torsion theory is carried by α to the Lambek torsion theory. In general the localized rings $R_{\tau_n(R)}$ and $S_{\tau_n(S)}$ are Morita equivalent. By making use of $\tau_1(R)$ and $\tau_1(S)$ one can show that when R is a prime Goldie ring then an ideal of R is left reflexive if and only if the corresponding ideal in S is left reflexive. The equivalence α preserves some other torsion theoretic properties such as stability, faithfulness, boundedness and being prime if the theory is generalization of Goldie torsion theory. Details will appear elsewhere.

For a general context (R, V, W, S) the functor α is not an equivalence, thus the transfer of properties is not immediate. In [1] a special procedure is described: Let A_τ be the set of right ideals J of S such that $WIV \subseteq J$ for some τ-dense right ideal I of R. (In [1] we used right module categories, right torsion theories, etc.) This set A_τ affords a fundamental system of neighbourhoods of zero for a unique right linear topology on S. Let $\underline{\tau}$ denote the hereditary torsion theory associated with the weakest Gabriel topology, stronger than the above linear topology. Then $\underline{\tau}$ is a generalization of the torsion theory determined by the trace ideal T_S. Consequently there is a hereditary torsion theory σ on Mod-R such that Mod(R, σ) is equivalent to Mod$(S, \underline{\tau})$ under the restriction of $\mathrm{Hom}_S(V, -)$.

We finish by asking whether there is a unified way of associating torsion theories for context equivalent ring?

REFERENCES

1- A. Haghany, Morita contexts and torsion theories, Math. Japonica 42, No. 1 (1995), 137–142.

2- T. Kato, Morita contexts and equivalences II, in Proceedings of the 20th Symposium on Ring Theory, Okayama University (1987), 31–36.

3- T. Kato and K. Ohtake, Morita contexts and equivalences, J. Algebra 61 (1979), 360–366.

4- B.J. Müller, the quotient category of a Morita context, J. Algebra 28 (1974), 389–407.

5- S.K. Nauman, An alternate criterion of localized modules, J. Algebra 164 (1994), 256–263.

6- W.K. Nicholson and J.F. Watters, Morita context functors, Math. Proc. Cambridge Philos. Soc. 103 (3), (1988), 399–408.

Department of Mathematics
Isfahan University of Technology
Isfahan, Iran

On direct sums of extending modules and internal exchange property

KATSUNORI HANADA, JIRO KADO AND KIYOICHI OSHIRO

In this paper, we report several results on direct sums of extending modules, which will appear in [1] and [2].

An R-module M is said to be an extending module if, for any submodule X of M, there exists a direct summand X^* of M which is an essential extension of X. The concept of this module seems to be very primitive, but there are several kind of extending modules. In fact, consider the following conditions for an R-module M:

(A) for any submodule X of M, there is a decomposition $M = X^* \oplus M'$ such that $X \subseteq_e X^*$, where $X \subseteq_e X^*$ means that X^* is an essential extension of X.

(B) for a given decomposition $M = \sum_I \oplus M_i$ and any submodule X of M, there exists a decomposition $M = X^* \oplus \sum_I \oplus M_i'$ with $X \subseteq_e X^*$ and $M_i' \subseteq M_i$.

(C) for any decomposition $M = \sum_I \oplus M_i$ and any submodule X of M, there exists a decomposition $M = X^* \oplus \sum_I \oplus M_i'$ such that $X \subseteq_e X^*$ and $M_i' \subseteq M$.

M with the conditin (A) is, of course, a usual extending module. We say that M is an extending module for $M = \sum_I \oplus M_i$ if M satisfies the condition (B). And we say that M is a normal extending module if M satisfies the condition (C), and say that M is a finite normal extending module if M satisfies the condition (C) for any finite index set I.

Finitely generated torsion free abelian groups are extending modules as a $\mathbb{Z}$-module, but not normal, in general. We emphasize that almost all known extending modules are normal extending modules.

The final detailed version of this paper will be submitted for publication elsewhere.

For extending modules, the following is an open problem:

Let M be an R-module with a decomposition $M = \sum_I \oplus M_i$ with each M_i an extensding module. Then give a characterization for M to be an extending module.

Our main purpose of this paper is to announce some results on this problem by introducing generalizing relative injectivity.

Theorem 1 ([1]). *For an R-module M with a decomposition $M = M_1 \oplus \cdots \oplus M_n$, the followings are equivalent:*

(1) *M is an extending module for $M = M_1 \oplus \cdots \oplus M_n$.*

(2) (a) *each M_i is an extending module,*

(b) *for any M_i and any direct summamd M_j^* of M_j, M_j^* is a generalized M_i-injective, where generalized relative injectivity is defined as follows:*

DEFINITION. Let P and Q are R-modules. Q is said to be a generalized P-injective if, for any submodule A and any homomorphism f from A to Q, there exists decompositions $P = \overline{P} \oplus \overline{\overline{P}}$ and $\overline{Q} \oplus \overline{\overline{Q}}$, and a homomorphism $\overline{f} : \overline{P} \to \overline{Q}$ and a monomorphism $g : \overline{\overline{Q}} \to \overline{\overline{P}}$ such that, for $a \in A$ with $a = \overline{a} + \overline{\overline{a}}$ ($\overline{a} \in \overline{P}$ and $\overline{\overline{a}} \in \overline{\overline{P}}$), $f(a) = \overline{f}(\overline{a}) + \overline{\overline{f}}(\overline{\overline{a}})$, where $\overline{\overline{f}} = g^{-1}$.

Corollary. *Let M be an R-module with decomposition $M = M_1 \oplus \cdots \oplus M_n$. If each M_i is an extending module and M_j-injective for any $i \neq j$, then M is an extending module for $M = M_1 \oplus \cdots \oplus M_n$.*

REMARK. Let P and Q be extending modules and Q is a generalized P-injective. Then

(1) for any direct summand P^* of P, Q is a generalized P^*-injective.

(2) But we do not know that, for a direct summand Q^* of Q, whether Q^* is a generalized P-injective or not, but this holds if Q is a finite normal extending module or a non-singular module.

For normal extending modules, we improve the theorem above as follows:

Theorem 2 ([1]). *For an R-module with a decomposition $M = M_1 \oplus \cdots \oplus M_n$, the followings are equivalent:*

(1) *M is a finite normal extending module.*

(2) (a) *each M_i is a finite normal extending module,*

(b) *for each M_i and M_j ($i \neq j$), M_i is a generalized M_j-injective.*

For the study of infinite direct sums of extending modules, we need the following conditions: Let $\{M_i\}_I$ be a family of R-modules.

(A_2) For any $\{M_1, M_2, \ldots, M_n, \ldots\} \subseteq \{M_i\}_I$ and $\{m_i \in M_i \mid i = 1, 2, \ldots\}$ such that $(0 : m_1) \subseteq (0 : m_i)$ for each $i \geq 2$, the sequence $\{\cap_{i \geq n}(0 : m_i) \mid n = 2, 3, \ldots\}$ terminates.

(A_2') (A_2) holds for $\cap_{i=2}^{\infty} Ker\varphi_i \subseteq_e m_1R$, where φ_i is the canonical homomorphism $: m_1R \to m_iR$.

Theorem 3 ([2]). *Let M be an R-module with a decomposition $M = \sum_{i=1}^{\infty} \oplus M_i$ such that each M_i is finite normal and M_j-injective ($i \neq j$). Then the following conditions are equivalent:*

(1) *M is an extending module for $M = \sum_{i=1}^{\infty} \oplus M_i$.*

(2) (a) *each M_i is an extending module,*

(b) *$\{M_i\}_{i=1}^{\infty}$ satisfies the condition (A_2').*

Theorem 4 ([2]). *Let M be an R-module with a decomposition $M = \sum_I \oplus M_i$, where each M_i is a uniform module. Then the following condition are equivalent:*

(1) *M is a normal extending module.*

(2) (a) *M_i is a generalized M_j injective for each $i \neq j$,*

(b) *$\{M_i\}_I$ satisfies the condition (A_2),*

(c) *there does not exist an infinite squence of non-isomorphic monomorphisms $\{f_k : M_{i_k} \to M_{i_{k+1}}\}$ with all $i_k \in I$ distinct.*

Theorem 5 ([2]). *The following conditions are equivalent for a given ring R:*

(1) *R is a right co-H-ring.*

(2) (a) *R is a left or right perfect ring with the ascending chain condition for annihilator right ideals of R,*

(b) *R is a generalized R-injective as a right R-module.*

We raise the following open problem:

For a family $\{M_i\}_I$ of normal extending modules, when is $M = \sum_I \oplus M_i$ a normal extending module?

This problem seems to be not easy even for a finite index set I.

References

[1] K. Hanada, J. Kado and K. Oshiro, *On finite direct sums of normal extending modules*, preprint.

[2] K. Hanada, J. Kado and K. Oshiro, *On normal extending property of direct sums of uniform modules*, preprint.

[3] S. Mohamed and B.J.Mueller, *Continuous and discrete modules*, London Math. Soc. **147**, 1990.

[4] N.V. Dung, D.V. Huynh, P.F. Smoth and R. Wisbouer, *Extending modules*, Pitman Research Notes in Mathematics Ser. **313**, 1994.

K. Hanada
Department of Mathematics
Yamaguchi University
Yoshida, Yamaguchi 753, Japan

J. Kado
Department of Mathematics
Osaka City University
Osaka 558, Japan

K. Oshiro
Department of Mathematics
Yamaguchi University
Yoshida, Yamaguchi 753, Japan

A NOTE ON SEMIPRIME RINGS WITH DERIVATION

Motoshi HONGAN

Throughout, R will represent a ring, Z the center of R, I a nonzero ideal of R, and $d: R \to R$ a derivation. As usual, for $x, y \in R$, we write $[x,y] = xy - yx$ and $x \circ y = xy + yx$. Given a subset S of R, we put $V_R(S) = \{ x \in R \mid [x,s] = 0 \text{ for all } s \in S \}$. In [1], Daif and Bell showed that a semiprime ring R must be commutative if it admits a derivation d such that (i) $d[x,y] = [x,y]$ for all $x, y \in R$, or (ii) $d[x,y] + [x,y] = 0$ for all $x, y \in R$.

Our present objective is to generalize a theorem of Daif and Bell [1, Theorem 3] as follows.

Theorem 1. *Let R be a 2-torsion free semiprime ring, and I a nonzero ideal of R. If $d[x,y] + [x,y] \in Z$ or $d[x,y] - [x,y] \in Z$ for all $x, y \in I$, then $I \subseteq Z$.*

In preparation for proving our theorem, we state the following two lemmas.

Lemma 1. *Let R be a semiprime ring, I a nonzero ideal of R, and $a \in R$.*

(1) *Let $b \in I$. If $[b,x] = 0$ for all $x \in I$, then $b \in Z$. Therefore, if I is commutative, then $I \subseteq Z$.*

(2) *If $[a,x] \in Z$ for all $x \in I$, then $a \in V_R(I)$.*

(3) *Let R be a 2-torsion free ring. If $[a,[a,x]] = 0$ for all $x \in I$, then $a \in V_R(I)$.*

(4) *Let R be a 2-torsion free ring. If $[a,[a,x]] \in Z$ for all $x \in I$, then $a \in V_R(I)$.*

(5) *Let R be a 2-torsion free ring. If $[a,[x,y]] \in Z$ for all $x, y \in I$, then $a \in V_R(I)$.*

The final version of this paper will be submitted for publication elsewhere.

Lemma 2. *Let R be a semiprime ring, I a nonzero ideal of R, and $d: R \to R$ a nonzero derivation such that $d[x,y] + [x,y] \in Z$ or $d[x,y] - [x,y] \in Z$ for all $x, y \in I$. If $d(I) \subseteq V_R(I)$, then I is commutative, and so $I \subseteq Z$.*

The next is a generalization of [1, Theorem 2].

Corollary 1. *Let R be a 2-torsion free semiprime ring, Z the center of R and $d: R \to R$ a derivation. If $d[x,y] + [x,y] \in Z$ or $d[x,y] - [x,y] \in Z$ for all $x, y \in R$, then R is commutative.*

Now, we will try to replace $[x,y]$ with $x \circ y$ in [1, Theorem 3].

Proposition 1. *Let R be a 2-torsion free ring with identity 1. Then there is no derivation $d: R \to R$ such that $d(x \circ y) = x \circ y$ or $d(x \circ y) + (x \circ y) = 0$ for all $x, y \in R$.*

Remark. In Theorem 1 and Corollary 1, we can not exclude the condition "2-torsion free" as below.

Example. We denote by Z the integer system. Let $R = \begin{pmatrix} Z/2Z & Z/2Z \\ Z/2Z & Z/2Z \end{pmatrix}$, $a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and d the inner derivation induced by a, that is, $d(x) = [a,x]$ for all $x \in R$. Then R is a non-commutative prime ring with $char\, R = 2$, and $d[x,y] \pm [x,y] \in Z$ for all $x, y \in R$.

Finally, we state two questions.

Let R be a 2-torsion free semiprime ring, $d: R \to R$ a nonzero derivation, and I a nonzero ideal of R. And let n be a fixed positive integer.

Question 1. Does the condition that $d^n[x,y] + [x,y] \in Z$ or $d^n[x,y] - [x,y] \in Z$ for all $x, y \in I$ imply that $I \subseteq Z$?

Question 2. Does the condition that $d^m[x,y]+d^p[x,y] \in Z$ or $d^m[x,y]-d^p[x,y] \in Z$ for some positive integers $m = m(x,y)$ and $p = p(x,y)$, and for all x, $y \in I$ imply that $I \subseteq Z$?

REFERENCE

[1] Daif, M.N. and Bell, H.E.: Remarks on derivations on semiprime rings, Internat. J. Math. & Math. Sci. **15** (1992), 205–206.

Tsuyama College of Technology
Tsuyama, Okayama 708
JAPAN
e-mail address: `hongan@dnews04.dnet.tsuyama-ct.ac.jp`

BOYLE'S CONJECTURE AND RINGS CHARACTERIZED BY CONTINUOUS MODULES

Dinh Van Huynh and S. Tariq Rizvi

1. Introduction. Rings for which every quasi-injective right module is injective were introduced as right QI-rings by Boyle ([1], [2]) and were studied by many authors (see for example, [3], [10], [12], [13]). In Byrd [4], these rings were called right QII-rings.

A ring R is called right hereditary if every right ideal of R is projective. If every simple (resp., singular) right R-module is injective, then R is said to be a right V- (resp., SI-) ring. SI-rings were introduced and investigated by Goodearl in [11]. In particular, any right SI-ring is right hereditary.

It was shown by Boyle [1] that (two-sided) noetherian hereditary V-rings are QI-rings. An example of Cozzens [6] shows the existence of a non-artinian QI-ring which is also an SI-domain. All known examples of QI-rings are hereditary and two-sided QI. Boyle has conjectured that:

Right QI-rings are right hereditary

(cf. Cozzens-Faith [7, p.116] and Faith [10]). It is also unknown whether or not a right QI-ring is left QI. This question is unanswered even if we assume, in addition, that the right QI-ring is right SI.

Our focus, here is to study rings for which all (or some) continuous modules are injective instead of QI-rings. We show that a ring for which all continuous modules are injective is semisimple artinian. Moreover, if we require the injectivity only for (a subfamily of) all singular continuous right R-modules, then R is right SI, in particular, R is right hereditary. Even though the Boyle's Conjecture still remains open, these results may provide an alternative approach to it.

2. The Results. Throughout, we consider associative rings with identity and all modules are unitary modules. For a module M we denote by $Soc(M)$ and $E(M)$, the socle and the injective hull of M, respectively. For a given module M we consider the following conditions:

(C_1) Every submodule of M is essential in a direct summand.

(C_2) Every submodule of M isomorphic to a direct summand of M is itself a direct summand.

(C_3) If H and K are direct summands of M with $H \cap K = 0$, then $H \oplus K$ is a direct summand.

A module is called *continuous* if it satisfies conditions (C_1) and (C_2), *quasi-continuous* if it satisfies (C_1) and (C_3), and *extending* (or CS) if it satisfies (C_1) only. We refer to [8] and [14] for details.

Every quasi-injective module is continuous and the heirarchy is as follows

injective $\Rightarrow$ quasi-injective $\Rightarrow$ continuous $\Rightarrow$ quasi-continuous $\Rightarrow$ extending.

In general, these classes of modules are distinct. We show that, over a ring R, these classes of modules coincide if and only if R is semisimple artinian (Corollary 2).

The following useful lemma provides for the existence of continuous submodules in an indecomposable quasi-injective module.

Lemma 1. *Let M be a quasi-injective right R-module. If H is an essential submodule of M such that M/H is noetherian, then every monomorphism of H into H is an isomorphism. In addition, if M is indecomposable, then H is a continuous module.*

From Lemma 1, it follows for example, that if R is a ring such that $E(R_R)/R$ is noetherian, then R is the classical right quotient ring of itself.

Corollary 2. *A ring R is semisimple artinian if and only if every continuous right R-module is injective.*

It would be interesting to know about the structure of rings whose extending modules are continuous or whose quasi-continuous modules are quasi-injective.

By [3, Theorem 8] every non-singular quasi-injective module over a semiprime right Goldie ring is injective. Moreover, by [15, Corollary 5], every continuous module over a commutative noetherian ring is quasi-injective. Hence, every non-singular continuous module over a commutative noetherian semiprime ring is injective. From this and Corollary 2, *not all* singular continuous modules over such a ring are necessarily injective. The ring of integers is an example which exhibits this conclusion.

For a commutative QI-ring R, [15, Corollary 5] provides the fact that every continuous R-module is injective. Hence, by Corollary 2, R is a direct sum of finitely many fields. This is also a consequence of [4, Proposition 2], or of the fact that a commutative V-ring is von Neumann regular.

Further, [3, Theorem 8] together with [11, Theorem 3.11] shows that a right SI-domain D is right QI. Hence by Corollary 2,if D is not a division ring, then a non-singular continuous right D-module is not necessarily quasi-injective.

A module M is said to satisfy RSSC (restricted semisimple condition) if for each essential submodule E of M, M/E is semisimple. Every semisimple module satisfies RSSC, but the converse is not true in general.

Next, we restrict our consideration to the case when singular continuous modules are injective, and show in Theorem 3 below that this condition characterizes precisely the right SI-rings of Goodearl [11].

Theorem 3. *For a ring R the following conditions are equivalent:*
(a) R is a right SI-ring;
(b) Every singular continuous right R-module is injective;
(c) Every singular continuous right R-module satisfying RSSC is injective.
In this case R is right hereditary.

While a right QI-ring is right noetherian, a ring of Theorem 3 may have infinite right uniform dimension (see [11, Example 3.2]).

A ring R is said to satisfy the restricted right (left) minimum condition if for each essential right ideal E of R, R/E is an artinian right (left) R-module. By Chatters [5], a two-sided noetherian, hereditary ring satisfies the restricted right (and hence left) minimum condition. Hence, as pointed out by Faith in [10], for a two-sided QI-ring R, the presence of the restricted right minimum condition in R is necessary for the truth of Boyle's Conjecture. In this connection, we note that a right noetherian right V-ring R is right SI (and hence right hereditary) if and only if R satisfies the restricted right minimum condition (cf. [11, Propositions 3.1 and 3.3]). Thus, by Theorem 3 and the known fact that a two-sided noetherian right hereditary ring is left hereditary, it follows that a two-sided QI-ring R is hereditary if and only if every singular continuous right R-module is injective if and only if R is a right SI-ring.

However, the question whether a right hereditary right QI-ring is right SI, remains open.

REFERENCES

1. A.K. Boyle, Hereditary QI-rings, Trans. Amer. Math. Soc. 192 (1974) 115-120.

2. A.K. Boyle, Injectives containing no proper quasi-injective submodules, Comm. Algebra 4 (1976), 775-785.

3. A.K. Boyle and K.R. Goodearl, Rings for which certain modules are injective, Pacific J. Math. 58 (1975), 43-45.

4. K.A. Byrd, Rings whose quasi-injective modules are injective, Proc. Amer. Math. Soc. 33 (1972), 235-240.

5. A.W. Chatters, The restricted minimum condition in noetherian hereditary rings, J. London Math. Soc. 76 (1970), 75-79.

6. J.H. Cozzens, Homological properties of the ring of differential polynomials, Bull. Amer. Math. Soc. 76 (1970), 75-79.

7. J.H. Cozzens and C. Faith, *Simple Noetherian Rings,* Cambridge University Press (1975).

8. N.V. Dung, D.V. Huynh, P.F. Smith and R. Wisbauer, *Extending Modules*, Research Notes in Mathematics Series 313, Pitman, London (1994).

9. C. Faith, When are proper cyclics injective? Pacific J. Math., 45 (1973), 97-111.

10. C. Faith, On hereditary rings and Boyle's Conjecture, Arch. Math. 27 (1876), 113-119.

11. K.R. Goodearl, *Singular torsion and the splitting properties,* Memoirs Amer. Math. Soc. 124 (1972).

12. K. Kosler, On hereditary rings and noetherian V-rings, Pacific J. Math. 103 (1982), 467-473.

13. S.R. Lopez-Permouth and S.T. Rizvi, On certain classes of QI-rings, Methods in Module Theory, Lecture Notes Vol. 140, Marcel Dekker (1993), 227-235.

14. S.H. Mohamed and B.J. Müller, *Continuous and Discrete Modules,* London Math. Soc. LN 147, Cambridge Univ. Press (1990).

15. S.T. Rizvi, Commutative rings for which every continuous module is quasi-injective, Arch. Math. 50 (1988), 435-442.

Institute of Mathematics, P.O.Box 631 Boho, Hanoi, Vietnam
and
The Ohio State University at Lima, Lima, OH 45804, USA

Units of Integral Group Rings of Finite Groups

E. Jespers*

The integral group ring $\mathbb{Z}G$ of a finite group is a $\mathbb{Z}$-order in the semisimple rational group algebra $\mathbb{Q}G$. Some of the most important questions on this topic are:

1. Describe the structure of the unit group $\mathcal{U}(\mathbb{Z}G)$ of $\mathbb{Z}G$. In particular, determine the structure in terms of natural invariants associated with $\mathbb{Z}G$.
2. Find an effective method for the construction of the full unit group. In particular, find a presentation of the unit group.

Historically the study of the units was initiated by Gauss, who investigated the unit group of $\mathbb{Z}[i]$. His work culminated in the celebrated Dirichlet Unit Theorem proved in 1840. The latter result describes the structure of the unit group of the ring of integers of an algebraic number field. It is, however, still an open and challenging problem for finding a finite set of generators for the unit group of such rings R, even in the important special case that R is the ring of cyclotomic integers.

For the integral group ring $\mathbb{Z}A$ of a finite abelian group A, Higman proved an analogue of the Dirichlet Unit Theorem, that is

$$\mathcal{U}(\mathbb{Z}A) = \pm A \times F,$$

where F is a finitely generated free abelian group whose rank given in terms of the structure of A. Hence answering question (1) for finite abelian groups. Bass and Milnor constructed finitely many generators, called the *Bass cyclcic units*, for a subgroup of finite index in $\mathcal{U}(\mathbb{Z}A)$.

We recall the definition of a Bass cyclic unit. Let g be an element of order n in a finite group G. Also let i be a number relatively prime to n

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and such that $1 < i < n$. Then the following element in $\mathbf{Z}\langle g\rangle$ is a called a Bass cyclic unit:

$$(1+g+\cdots+g^{i-1})^{\varphi(n)} - \frac{i^{\varphi(n)}-1}{n}(1+g+\cdots+g^{n-1}),$$

where φ denotes the Euler φ-function.

There is another important construction of a unit in the integral group ring $\mathbf{Z}G$ of a finite group G. Let $g, h \in G$ and assume g has order n. Then

$$1+(1-g)h(1+g+\cdots+g^{n-1})$$

is a called a *bicyclic unit.* Of course also

$$1+(1+g+\cdots+g^{n-1})h(1-g)$$

is a unit in $\mathbf{Z}G$.

For noncommutative finite groups G it is well known that $\mathcal{U}(\mathbf{Z}G)$ is finitely presented. However very little is known concerning question (1). Concerning question (2), in recent years there has been given a lot of attention to subgroups of finite index. Sehgal and Ritter constructed finitely many generators for a subgroup of finite index in $\mathcal{U}(\mathbf{Z}G)$ for several classes of groups G. For details and further information we refer the reader to [7].

In joint work with Leal [1, 2], it is shown that the Bass cyclic units together with (both types of) bicyclic units, generate a subgroup of finite index in $\mathcal{U}(\mathbf{Z}G)$ for all finite groups G, provided that G has no non-abelian homomorphic image which is fixed point free and provided that the rational group algebra $\mathbf{Q}G$ does not contain simple components of an exceptional type. The exceptional types are: noncommutative division rings other than totally definite quaternion algebras, two-by-two matrices over $\mathbf{Q}$ or a quadratic imaginary extension of $\mathbf{Q}$ or a noncommutative division ring (so this has to do with the 2-torsion in the group G). In case some of these components appear the above mentioned result is not true in general. The ultimate reason for this is that the celebrated congruence theorems (due to Bass, Milnor, Serre, Vaserstein, Bak and Rehman) fail for the general linear groups defined in the mentioned exceptional components. However, we developed an algorithm to calculate finitely many generators for a subgroup of finite index in $\mathcal{U}(\mathbf{Z}G)$ for any finite nilpotent group (thus even when such exceptional components occur).

Furthermore, in [2], for nilpotent finite groups G, we have characterized, in terms of homomorphic images of the group G, when precisely such exceptional components occur. Even more general, we describe which degree one and two representations occur. In work in preparation [3], we give an upper bound on the index of the subgroup generated by the Bass cyclic units and bicyclic units in the full unit group (this for a large class of groups, including nilpotent groups of odd order).

In [6] finitely many generators for a subgroup of finite index in the centre of the unit group have been given. These generators are products of conjugates of Bass cyclic units.

In this lecture I concentrate on some recent joint work with Leal [4], and Leal and del Rio [5]. We describe when the unit group of the integral group ring of a finite group has a subgroup of finite index which is either a free abelian product of free nonabelian groups or a free product of free abelian groups (hence investigating when Higman's structure theorem can be generalized in an obvious way to the noncommutative case). It follows that this question is completely determined by the rational group algebra structure.

References

[1] E. Jespers, G. Leal, Generators of large subgroups of the unit group of integral group rings, *Manuscripta Mathematica*, 78(1993), 303–315.

[2] E. Jespers, G. Leal, Degree 1 and 2 representations of nilpotent groups and applications to units of group rings, *Manuscripta Mathematica*86 (1995), 479–498.

[3] E. Jespers, G. Leal, in preparation.

[4] E. Jespers, G. Leal, Unit groups of integral group rings and abelian centralisers, in preparation.

[5] E. Jespers, G. Leal, A. Del Rio, Products of free groups in the unit group of integral group rings, *J. Algebra*, to appear.

[6] E. Jespers, M.M. Parmenter, S.K. Sehgal, Central Units of Integral Group Rings of Nilpotent Groups, *Proc. Amer. Math. Soc*, to appear.

[7] S.K. Sehgal, Units in integral group rings, Longman and John Wiley, Essex and New York, 1993.

Department of Mathematics and Statistics
Memorial University of Newfoundland
St.John's, Newfoundland
Canada A1C5S7
e-mail: ejespers@albert.math.mun.ca

ORTHOGONALITY OF THE IDEMPOTENT ELEMENTS WITH RANK ONE IN PRIMITIVE RINGS

JIANG ZIMEI

Let R be a primitive ring with nonzero socle, $\mathfrak{M}$ a faithful irreducible right R-module, Δ the centralizer of $\mathfrak{M}$, and Ω the complete ring of linear transformations of left vector space $\mathfrak{M}$ over division ring Δ, then R can be considered as a dense subring of Ω. If $\{x_\alpha\}_{\alpha\in\Lambda}$ is a basis for $\mathfrak{M}$, a subset $\{e_\alpha\}_{\alpha\in\Lambda}$ of Ω is called a corresponding basis of $\{x_\alpha\}_{\alpha\in\Lambda}$ iff $x_ie_j = \delta_{ij}x_i$ for any $i \in \Lambda$, where δ_{ij} is the Kronecker delta notation. In [1], Jacobson showed that if $L = \sum_{i=1}^{n} \oplus L_i$ is a direct sum of minimal right (left) ideals $L_1, L_2, \cdots, L_n$ in R, then there exists a set $\{e_1, e_2, \cdots, e_n\}$ of orthogonal idempotent elements e_i of rank one in R such that $L = \sum_{j=1}^{n} \oplus e_jR$ $(L = \sum_{j=1}^{n} \oplus Re_j)$. In [2], Xu Yonghua improved this result and obtained that $e_j \in L_j$ for all j. Moreover he indicated that let R be a primitive ring with nonzero socle, and $L = \sum_{i=1,\cdots} \oplus L_i$ be a direct sum of countably minimal right (left) ideals L_i of R. Then there exists a set $\{e_i\}_{i=1,\cdots}$ of countably orthogonal idempotent elements e_i of R such that $L = \sum_{i=1,\cdots} \oplus e_iR$ $(= \sum_{i=1,\cdots} \oplus Re_i)$, where e_iR (Re_i) are minimal right (left) ideals. Then we consider two questions, the first is for the direct sum $L = \sum_{i=1,\cdots} \oplus L_i$ of countably many minimal right ideals of primitive ring R, how many subsets the ring R has such that every subset $\{e_i\}_{i=1,\cdots}$ of R consists of countably many orthogonal idempotent elements e_i of rank one and satisfies $L = \sum_{i=1,\cdots} \oplus L_i = \sum_{i=1,\cdots} \oplus e_iR$? The second is whether $\{e_i\}_{i=1,\cdots}$ can be extended to a corresponding basis of some basis $\{x_\alpha \mid \alpha \in \Lambda\}$ for $\mathfrak{M}$.

Let $R, \mathfrak{M}, \Delta$ and Ω be the same as above, $\mathbb{N}$ be the set of all positive integers. In this paper we show that (1) if $L = \sum_{i\in\mathbb{N}} \oplus L_i$ is a direct sum of countably many minimal right ideals L_i of R, then there exists a family of subsets $I_\alpha = \{e_{\alpha_i} \mid i \in \mathbb{N}\}$ $(\alpha \in W$, $|W|$ is infinite) of R such that $L = \sum_{i\in\mathbb{N}} \oplus e_{\alpha_i}R$ for each $\alpha \in W$, where every $I_\alpha = \{e_{\alpha_i} \mid i \in \mathbb{N}\}$ is a set of countably many orthogonal idempotent elements e_{α_i} of rank one in R; (2) a necessary and sufficient condition of a set $\{e_i \mid i \in \mathbb{N}\}$ of countably many orthogonal idempotent elements e_i with rank one in R can be extended to a corresponding basis $\{e_i \mid i \in \Lambda\}$ of some basis $\{x_i \mid i \in \Lambda\}$ of $\mathfrak{M}$ over Δ; (3) there exists a primitive ring R and $L = \sum_{i\in\mathbb{N}} \oplus L_i$ in R, where the L_i are minimal right ideals of R, but R has no subset $\{e_i \mid i \in \mathbb{N}\}$ of countably many orthogonal idempotent elements of rank one such that $L = \sum_{i\in\mathbb{N}} \oplus e_iR$ and $\{e_i \mid i \in \mathbb{N}\}$ can also be extended to a corresponding basis $\{e_i \mid i \in \Lambda\}$ of some basis $\{x_i \mid i \in \Lambda\}$ of $\mathfrak{M}$ over Δ.

Let $\mathfrak{M}^*$ be the conjugate vector space of $\mathfrak{M}$ over Δ. If Γ is the ring of linear transformations of right vector space $\mathfrak{M}^*$ over Δ, then $\mathfrak{M}^*$ can be considered as a left Ω-module and $\Gamma \supset \Omega$. If $x \in \mathfrak{M}$ and $f \in \mathfrak{M}^*$, we denote by $< x, f >$ the image of x under f. For any

$\omega \in \Omega$ we write $\rho(\omega) = \dim \mathfrak{M}\omega$ and call $\rho(\omega)$ the rank of ω. Let $X = \{u \in \Omega \mid u^2 = u$ and $\rho(u) = 1\}$, then $u \in X$ iff $u\Omega$ (Ωu) is a minimal right (left) ideal in Ω. It is well known that for any $\varepsilon \in X$, $\varepsilon R = \varepsilon\Omega$ is a minimal right ideal of R iff $\varepsilon \in R$. It is easy to prove that if $e \in X$, there are $x \in \mathfrak{M}$ and $x^* \in \mathfrak{M}^*$ such that $xe = x$, $ex^* = x^*$, $\mathfrak{M} = xe\Omega$ and $\mathfrak{M}^* = \Omega ex^*$. The main results of this paper are as follows.

Proposition. (Theorem 3 in [3]) Let $\Delta, \Omega, \mathfrak{M}^*$ and X be the same as above, and $\overline{S}$ be the socle of Ω. Then there exists a lattice isomorphism $\overline{\varphi}$ of the lattice of vector subspaces of $\mathfrak{M}^*$ onto the lattice of right ideals of $\overline{S}$. If N^* is a vector subspace of $\mathfrak{M}^*$, then $\overline{\varphi}(N^*) = \sum_{j\in J} \oplus e_j\Omega$, where $e_j \in X$ for all $j \in J$, and the set $A = \{0 \neq e_j x_j^* = x_j^* \in N^* \mid j \in J\}$ is a basis of N^* over Δ.

Theorem 1. Let R be a primitive ring with nonzero socle, and $\Delta, \mathfrak{M}, \mathfrak{M}^*, \Omega, X$ be the same as above, $\dim \mathfrak{M}$ is infinite. If $L = \sum_{i\in\mathbb{N}} \oplus L_i$ is a direct sum of countably many minimal right ideals L_i of R, then there exists a family of subsets $I_\alpha = \{e_{\alpha_i} \mid i \in \mathbb{N}\}$ ($\alpha \in W$, $|W|$ is infinite) of R such that $L = \sum_{i\in\mathbb{N}} \oplus e_{\alpha_i} R$ for each $\alpha \in W$, where every $\{e_{\alpha_i} \mid i \in \mathbb{N}\}$ is a set of countably many orthogonal elements e_{α_i} in $R \cap X$.

A direct sum of countably many minimal left ideals of a primitive ring R has the similar structure mentioned above.

If $L = \sum_{i\in\mathbb{N}} \oplus e_i R$ is a direct sum of countably many minimal right ideals $e_i R$ of primitive ring R, and $B = \{e_i\}_{i\in\mathbb{N}}$ is the set of countably many orthogonal elements of $X \cap L$, generally, B can not always be extended to a corresponding basis of some basis of $\mathfrak{M}$.

Theorem 2. Let R be a primitive ring with nonzero socle, and $\Delta, \mathfrak{M}, \mathfrak{M}^*, \Omega, X$ be the same as above, $\dim \mathfrak{M}$ is infinite. If $L = \sum_{i\in\mathbb{N}} \oplus e_i R$, where the $e_i R$ are minimal right ideals of R, and $B = \{e_i \mid i \in \mathbb{N}\}$ is a set of countably many orthogonal elements in $L \cap X$. Then B can be extended to a corresponding basis of some basis of $\mathfrak{M}$ iff for any $u \in X$, there exists at most finitely many elements e_{i_j} $(1 \leq j \leq k)$ in B such that $\Omega u \cap e_{i_j}\Omega \cap X \neq \emptyset$.

Theorem 3. Let R be a primitive ring with nonzero socle, and $\Delta, \mathfrak{M}, \Omega, X$ be the same as above, $\dim \mathfrak{M}$ is infinite. If $L = \sum_{i\in\mathbb{N}} \oplus a_i\Omega$, where the $a_i\Omega$ are minimal right ideals of R, $a_i \in R$ for any $i \in \mathbb{N}$. Then a set $B = \{e_i \mid i \in \mathbb{N}\}$ of countably many orthogonal elements of $X \cap L$ can be extended to a corresponding basis of some basis of $\mathfrak{M}$, and $a_1\Omega + a_2\Omega + \cdots + a_n\Omega = e_1\Omega + e_2\Omega + \cdots + e_n\Omega$ for any $n \in \mathbb{N}$ iff there exists a basis $A = \{x_\alpha\}_{\alpha\in\Lambda}$ of $\mathfrak{M}$ over Δ such that for any element a_n, the set $B_n = \{x_\alpha \in A \mid x_\alpha a_n \neq 0\}$ contains at most n elements, and $\left|B_n \setminus \left(\bigcup_{i=1}^{n-1} B_i\right)\right| = 1$ for any $n \geq 2$.

By Theorem 2 and Theorem 3, we can construct a left vector space $\mathfrak{M}$ over a division ring Δ, the ring Ω of linear transformations of $\mathfrak{M}$ over Δ contains a dense subring R and $L = \sum_{i\in\mathbb{N}} \oplus L_i$ in R, where the L_i are minimal right ideals of R. If there exists a set

$B = \{e_i \mid i \in \mathbb{N}\}$ of countably many orthogonal idempotent elements e_i of rank one in R such that $L = \sum_{i\in\mathbb{N}} \oplus L_i = \sum_{i\in\mathbb{N}} \oplus e_i R$, then B can not be extended to a corresponding basis of some basis of $\mathfrak{M}$.

Example. Let V be a right vector space over division ring Δ and $\dim V = \aleph_0$. Let $\mathfrak{M} = \mathrm{Hom}_\Delta(V_\Delta, \Delta)$, then $\mathfrak{M}$ is a left vector space over Δ and $\dim \mathfrak{M}$ is infinite. If $x \in V$ and $f \in \mathfrak{M}$, we denote by $f(x)$ the image of x under f, then $f(x) \in \Delta$. Let $\mathfrak{M}^*$ be the conjugate space of $\mathfrak{M}$, define a map $\psi: V \longrightarrow \mathfrak{M}^*$ by $x \mapsto x^*$, where $< f, x^* >= f(x)$ for all $f \in \mathfrak{M}$. It is clear that the vector spaces V and $\psi(V)$ are isomorphic. Let Ω be the ring of linear transformations of $\mathfrak{M}$ over Δ, $X = \{u \in \Omega \mid u^2 = u$ and $\rho(u) = 1\}$, then $(\mathfrak{M}, \psi(V))$ is a pair of dual vector spaces over Δ. Say $\psi(V) = \mathfrak{M}'$, $R = \mathcal{L}(\mathfrak{M}, \mathfrak{M}') = \{\omega \in \Omega \mid \omega\mathfrak{M}' \subseteq \mathfrak{M}'\}$ and $S = \mathcal{F}(\mathfrak{M}, \mathfrak{M}') = \{\omega \in R \mid \rho(\omega)$ is finite $\}$. By Theorem 2 of [4] R is a dense subring of Ω and has nonzero socle S. Let $\overline{\varphi}$ be the mapping in the Proposition, then by [3] we have $\overline{\varphi}(\mathfrak{M}') = S = \sum_{i\in\mathbb{N}} \oplus L_i$, where the L_i are minimal right ideals of R, and $\mathbb{N}$ is the set of all positive integers. Let $B = \{e_i\}_{i\in\mathbb{N}}$ be a set of countably many orthogonal idempotent elements e_i of rank one in R. If $\overline{\varphi}(\mathfrak{M}') = \sum_{i\in\mathbb{N}} \oplus L_i = \sum_{i\in\mathbb{N}} \oplus e_i R$, choose $0 \neq x_i^* \in \mathfrak{M}'$ such that $e_i x_i^* = x_i^*$ for all $i \in \mathbb{N}$, then by the Proposition $\{x_i^*\}_{i\in\mathbb{N}}$ is a basis of $\mathfrak{M}'$ over Δ. For any $i \in \mathbb{N}$ choose $x_i \in V$ such that $x_i^* = \psi(x_i)$, then $\{x_i\}_{i\in\mathbb{N}}$ is a basis of V over Δ, hence there exists an element $g \in \mathfrak{M}$ such that $g(x_i) = 1$ for all $i \in \mathbb{N}$, then $< g, x_i^* >= 1$ for all $i \in \mathbb{N}$. Therefore there exists $u \in X$ such that $gu = g$ and $ue_i \neq 0$ for all $i \in \mathbb{N}$, we have $\Omega u \cap e_i\Omega \cap X \neq \emptyset$ for all $i \in \mathbb{N}$. By Theorem 2 we obtain that B can not be extended to a corresponding basis of some basis of $\mathfrak{M}$.

References

[1] N. Jacobson: Structure of Rings, Amer. Math. Soc. Colloq. Publ., Vol. 37, 1956.

[2] Xu Yonghua: An orthogonal theorem for primitive rings, Scientia Sinica Ser.A **27**(10) (1984) , 1009-1017.

[3] Jiang Zimei: Partitions of the set of idempotent elements of rank one in a primitive ring and their applications, J. of Math. (PRC) **15**(2) (1995), 175-181.

[4] Jiang Zimei: The Structure of minimal one-sided ideals of a ring and the primitive rings with nonzero socles, Kexue Tongbao **29**(10)(1984), 1284-1287.

Department of Mathematics
Beijing Normal University
Beijing, 100875 China

NOTE ON SEMINORMAL OVERSEMIGROUPS AND OVERRINGS

MITSUO KANEMITSU and RYÛKI MATSUDA

All rings considered are commutative with identity, and semigroups are commutative, cancellative and torsion-free with 0. The operation is written additively.

Let D be an integral domain with quotient field K and let S be a semigroup.

We denote by $q(S) := \{s_1 - s_2 \mid s_1, s_2 \in S\}$ the quotient group of S. T is called an *oversemigroup* of S if T is a subsemigroup of $q(S)$ containing S. D is said to be *seminormal* if, whenever $a \in K$ satisfies $a^2 \in D$ and $a^3 \in D$, then $a \in D$.

Also, we say that S is a *seminormal semigroup* if $2a, 3a \in S$ for $a \in q(S)$, we have $a \in S$.

An element t is said to be *integral* over S if $nt \in S$ for some positive integer n. Let T be a semigroup containing S. T is called an *integral semigroup* over S if each element of T is an integral element over S.

An *ideal* of S is a nonempty subset I of S such that $I \supset s + I := \{s + i \mid i \in I\}$ for each $s \in S$. An ideal I of S is *prime* if $x + y \in I$ implies $x \in I$ or $y \in I$ for $x, y \in S$. Also, set $M = \{m \in S \mid m \text{ is a non-unit element of } S\}$. Then M is the unique maximal ideal of S.

A semigroup S is called a *valuation semigroup* if either $\alpha \in S$ or $-\alpha \in S$ for each $\alpha \in G = q(S)$.

From now on, D denotes a domain, with integral closure $\tilde{D}$ and quotient field K.

D.F.Anderson, D.E.Dobbs and J.A.Huckaba investigated seminormal overrings in [1].

OPEN PROBLEM([1]). *Let D be a subring of a domain T, such that each ring contained between D and T is seminormal. Is either D is a field or T an overring of D ?*

[1, pp.1427-1428] says that we can show the answer is affirmative if D is integrally closed and has a prime ideal of finite positive height. The question remains open, in general ([1]).

But this open problem is solved affirmatively in [2].

[I]([2]). *Let D be a subring of a domain T, such that each ring contained between D and T is seminormal. Then either D is a field or T is an overring of D.*

[A SKETCH OF PROOF]. Suppose the contrary, that is, $D \subsetneq K = q(D)$ and $T \not\subset K$. We may assume that

(1) T is integral over D.

(2) $T \cap K = D$.

(3) (D, M) is a quasi-local domain, where M is the unique maximal ideal of D.

Take $w \in T - K$. Then $T_1 = D[w]$ is integral over D. Therefore T_1 is a finite D-module. Take any non-zero element m of M. Put $x := mw$. Since $D[x^2, x^3]$ is seminormal, we have $x \in D[x^2, x^3]$. Hence we have

$$x = r_0 + r_2x^2 + \cdots + r_nx^n \quad (r_i \in D, i = 0, 2, \cdots, n).$$

We set

$$t_1 := w - r_2mw^2 - \cdots - r_nm^{n-1}w^n \in T \cap K = D.$$

It follows that

$$w - t_1 = r_2mw^2 + \cdots + r_nm^{n-1}w^n \in MT_1.$$

Thus we see that $T_1 = D[w] = D + MT_1$. According to Nakayama's lemma, $T_1 = D$. Hence $w \in D$, a contradiction.

In the proof, Nakayama's lemma is used.

[II] *Let S be a subsemigroup of a semigroup Γ, such that each semigroup between S and Γ is seminormal. Then either S is a group or Γ is an oversemigroup of S.*

[A SKETCH OF PROOF]. Suppose the contrary, that is, $S \subsetneq G = q(S)$ and $\Gamma \not\subset G$.

We may assume that (1) Γ is integral over S and (2) $S = \Gamma \cap G$.

Select $w \in \Gamma - G$. Take any element m of M, where M is the unique maximal ideal of S. Put $x := m + w$. Then $x \in \Gamma - G$. Since $S[2x, 3x]$ is seminormal, we have that $x \in S[2x, 3x]$. Thus $x = s + 2n_1x + 3n_2x$ for $n_1, n_2 \in \mathbf{Z}_0$, where $\mathbf{Z}_0$ is the set of non-negative integers, and $s \in S$. Put $p := 2n_1 + 3n_2 - 1$. Then $p \geq 1$. Hence $0 = s + px = s + pm + pw$. Since $s + pm \in S$, we have that $pw \in \Gamma \cap G = S$. Hence m is a unit of S. This is a contradiction.

The proof of [IV] does not use Nakayama's lemma.

[III] ([1, Corollary 2.4 and Theorem 2.5]) Let $\dim(D) = 1$. Then:

(1) Each overring of D is seminormal if and only if both D is seminormal and $\bar{D}$ is a Prüfer domain.

(2) If D is seminormal and T is an integral overring of D, then T is seminormal.

[IV] is the semigroup version of [III].

[IV] *Let S be a semigroup with* $\dim(S) = 1$. *Then :*

(1) *Each oversemigroup of S is seminormal if and only if both S is seminormal and $\bar{S}$ is a valuation semigroup, where $\bar{S}$ is the integral closure of S.*

(2) *If S is seminormal and T is an integral oversemigroup of S, then T is seminormal.*

The proof of [III] is not easy. But the proof of [IV] is easy.

[V] ([1, Theorem 2.3]) Let L be a field containing D. Then each ring between D and L is seminormal if and only if

(1) $D = K$ and L is algebraic over K; or

(2) $L = K$, each integral overring of D is seminormal, and $\bar{D}$ is a Prüfer domain.

Zariski's main theorem is used in [V].

We recall that Zariski's main theorem.

Let $D \subset T$ be rings and P a prime ideal of T. Then P is said to be *isolated* over $D \cap P$ if P is maximal and minimal with respect to the primes of T whose intersection with D is $D \cap P$.

ZARISKI'S MAIN THEOREM. Let $D \subset T$ with D integrally closed in T such that there exist $t_1, t_2, \cdots, t_n \in T$ with T integral over $D[t_1, \cdots, t_n]$. If a prime ideal P of T is isolated over $P \cap D$, then there exists an $s \in D - D \cap P$ such that $T_s = D_s$.

[V] holds for a semigroup S. Namely:

[VI] *Let S be a subsemigroup of a group L and let $G = q(S)$. Then each semigroup between S and L is seminormal if and only if* (1) *$S = G$ and L is integral over G; or* (2) *$L = G$, each integral oversemigroup of S is seminormal, and the integral closure $\tilde{S}$ of S is a valuation semigroup.*

In [VI] no special tools are used.

REFERENCES

[1] D.F.Anderson, D.E.Dobbs and J.A.Huckaba, On seminormal overrings, *Communications in Algebra* **10** (1982) 1421-1448.

[2] M.Kanemitsu and R.Matsuda, Note on seminormal overrings, *to appear in Houston J. Math.*

Department of Mathematics
Aichi University of Education
Igaya-cho, Kariya-shi, 448 Japan

and

Department of Mathematics
Ibaraki University
Mito, Ibaraki 310, Japan

REGULARITY IN HOM

Friedrich Kasch

1. Let R be a ring with $1 \in R$ and denote by M and N R-rightmodules. Then if $S := End_R(N)$, $T := End_R(M)$, $Hom_R(M,N)$ is a S-T-bimodule. Denote by U a S-T-submodule of $Hom_R(M,N)$. Examples for U are $\Delta(M,N)$ resp. $\nabla(M,N)$ = homomorphisms from M to N with large kernel resp. small image and RAD(M,N) = radical of $Hom_R(M,N)$. In the study of regularity properties of a ring, very often one has to consider a two-sided ideal A of R and the factorring R/A. The similar procedure as in the ring case, that means, to work with $Hom_R(M,N)/U$, is not usefull, since this is not any more a "Hom". But we would like to work still with the good properties of homomorphisms, that are the kernel, the image and the produkt. Therefore we use the following definition.

Definition.

$f \in Hom_R(M,N)$ is called U-regular $:\Longleftrightarrow$ there exist $g \in Hom_R(N,M)$ and $u \in U$ such that

(1) $$f = fgf + u .$$

A subset of $Hom_R(M,N)$ is called U-regular, if all of its elements are U-regular.

We intend to show, that this is a valuable definition for the study of regularity in Hom.

2. It is well-known, that in a ring R, there exists a largest regular two-sided ideal A and R/A has no nonzero regular two-sided ideal. This theorem is also true in our general situation and can even be extendet to the category R-mod. For $f \in Hom_R(M,N)$ we denote by $\langle f \rangle$ the S-T-submodule of $Hom_R(M,N)$ generated by f. Then we define (similar to the ring case)

$$Reg(U) := \{ f \in Hom_R(M,N) \mid \langle f \rangle \text{ is U-regular} \} .$$

Theorem 1.

Reg(U) is the largest U-regular S-T-submodule of $Hom_R(M,N)$ and

$$Reg(Reg(U)) = Reg(U) .$$

3. If U is one of the examples $\Delta(M,N)$, $\nabla(M,N)$ or RAD(M,N), we would be interested to get informations about Reg(U). We give the following example. For the conditions (C1;M) and (C2;M,N) see the abstract of my talk:"Regular and partially invertible elements".

Theorem 2.

(C1;M) and (C2;M,N) imply

$$Reg(\Delta(M,N)) = Hom_R(M,N) .$$

The dual (discretness) conditions imply

$$Reg(\nabla(M,N)) = Hom_R(M,N) .$$

4. Now, we consider the special case M = R. Then $Hom_R(M,N) \cong N$ by the isomorphism φ: Hom (M,N) $\ni f \longmapsto f(1) \in N$. Then N is a S-R-module and $Hom_R(N,R) = N$ is the dual module. This situation was studied by J.Zelmanowitz [1]. He called b ∈ N regular if there exists $g \in N^*$ such that b = bgb. Now, if $U \subsetneq {}_SN_R$, then by applying φ on (1) we get

$$b = bgb + u \quad , \quad u \in U .$$

Then theorem 1 states, that there exists a largest U-regular S-R-submodule Reg(U) of N and Reg(Reg(U)) = Reg(U). Also for U = 0 (case of J.Zelmanowitz), this result is new and may be of some interest. Also theorem 2 can be considered in this special case. If for example R_R is injective, then (C1;R) and (C2;R,N) are satisfied for all N and $f \in \Delta(R,N)$ means, that the annihilater of f(1) is large in R_R.

5. We extend theorem 1 about Reg(U) to the category R-mod of all unitary R-rightmodules. For this, we assume, that W is an ideal in R-mod. Examples for ideals are Δ, ∇ and RAD. If W is an ideal, we denote $W(M,N) := W \cap Hom_R(M,N)$. For $f \in Hom_R(M,N)$ we write $\langle\langle f \rangle\rangle$ for the ideal in R-mod generated by f. We denote f W-regular if it is W(M,N)-regular (U = W(M,N) in (1)). Now let be

$$REG(W)(M,N) := \{ f \in Hom_R(M,N) \mid \ll f \gg \text{ is W-regular} \} .$$

REG(W) is then defined by

$$REG(W) \cap Hon_R(M,N) = REG(W)(M,N) .$$

Theorem 3.

REG(W) is the largest W-regular ideal in R-mod and

$$REG(REG(W)) = REG(W) .$$

If R is semi-simple, then REG(0) = R-mod. Are there other rings R such that

(2) REG(W) = R-mod ,

if W is one of the ideals Δ, ∇ or RAD ? Does there exist rings R such that

$$REG(W) = W$$

for one of the examples ? Does there exist for an arbitrary ring R a smallest (or minimal) ideal W such that (2) is satisfied ? These and other questions can be raised.

It is also possible, to study W-regularity in more general categories than R-mod. And what are the properties of pi-elements and the total for U-regularity resp. W-regularity ? (For this topic see the abstract of my talk: "Regular and partially invertible elements".).

REFERENCES

|1| Zelmanowitz,J.: Regular modules . Transactions of the AMS,163 (1972), 341 - 355.

|2| Mohamed,S.H. and B.J.Müller : Continuous and discrete modules. London M.S., lecture note series 147, 1990 .

|3| Kasch,F. : Moduln mit LE-Zerlegung und Harada-Moduln. Lecture Notes, Universität München, 1982 .

|4| Schneider,W. : Das Total von Moduln und Ringen. Algebra Berichte 55, 1987, Verl.R.Fischer, München.

|5| Kasch,F.: Partiell invertierbare Homomorphismen und das Total. Algebra Berichte 60, 1988,Verl.R.Fischer,München.

|6| Kasch,F.: The total in the category of modules. General algebra 1988. Elsevier Sc.Publ.B.V., 1990 .

|7| Kasch,F. und W.Schneider : The total of modules and rings. Algebra Berichte 69, 1992, Verlag R.Fischer, München.

|8| Kasch,F.: Verallgemeinerung der Sätze von Johnson und Utumi auf Hom. Per.Math.Hungarica, 1995 .

Mathematische Institut der
Universität München
Theresienstr. 39
80333 München
Germany

REGULAR AND PARTIALLY INVERTIBLE ELEMENTS

Friedrich Kasch

1. We consider the definition of a regular element r of a ring R (with $1 \in R$):

(1) $\quad r = rsr \quad , \quad r,s \in R$.

Since we are also interested in regularity in Hom, we give also the definition of a regular homomorphism $f \in \mathrm{Hom}_R(M,N)$:

(2) $\quad f = fgf$,

where M and N are unitary R-rightmodules and $g \in \mathrm{Hom}_R(N,M)$. A subset of R resp. of $\mathrm{Hom}_R(M,N)$ is called regular, if all of its elements are regular. In the following I try to convince the reader, that the elements "in the middle" of (1) and (2) - that are s and g - are not less interesting then the regular elements.

2. By f and g we denote always homomorphisms in $\mathrm{Hom}_R(M,N)$ resp. $\mathrm{Hom}_R(N,M)$, even if they do not satisfy (2).

<u>Lemma</u>. The following properties are equivalent for g :

(i) $\exists f \; [\, 0 \neq f = fgf \,]$,

(ii) $\exists f \; [\, gf =: e = e^2 \neq 0 \,]$,

(iii) $\exists f \; [\, fg =: d = d^2 \neq 0 \,]$,

(iv) $\exists f \; [\, 0 \neq gfg$ is regular $]$,

(v) $\exists A \subseteq^{\oplus} M \, , \, B \subseteq^{\oplus} N \, , \, A \neq 0 \; [\, B \ni b \mapsto g(b) \in A$ is an iso. $]$

If g has the properties of the lemma, we denote it as <u>partially invertible</u> = pi (as abbreviation). We use this notation with respect of (ii) and (iii). If g is not pi, then with respect of (v) we call it a <u>total nonisomorphism</u>. It means, that g does not induce an isomorphism on any nonzero direct summands of N and M.

Assume now $g = gfg \neq 0$, then this implies (ii) in the lemma, hence : If g is regular, then it is pi. So far we know for homomorphisms in $\mathrm{Hom}_R(N,M)$:

(3) $\left\{\begin{array}{l}\text{set of nonzero}\\ \text{regular homomorphisms}\end{array}\right\} \subsetneqq \left\{\begin{array}{l}\text{set of}\\ \text{pi-homomorphisms}\end{array}\right\}$

Further, we see by (i) - (v), that the pi-homomorphisms are all factors of regular homomorphisms $\neq 0$.(To have an easy example in the case of a ring, where the inclusion in (3) is proper, consider the ring $Z/p_1^2p_2Z$ with different primnumbers p_1, p_2.).

Now, we intend to show, that the right set in (3) is "better" then the left !

Lemma. If $g_1 \in \mathrm{Hom}_R(X,M)$, $g_2 \in \mathrm{Hom}_R(N,X)$ for arbitrary R-rightmodules M,N,X and if $g := g_1g_2$ is pi, then g_1, g_2 are pi.

This means, that the right set in (3) is closed under taking factors. Since regular elements $\neq 0$ are pi, the lemma implies, that the factors of nonzero regular elements are pi-elements ! Together with the remark following (3) we have the interesting statement:

> The pi-homomorphisms in $\mathrm{Hom}_R(N,M)$ are exactly all the factors of all nonzero regular homomorphisms in $\mathrm{Hom}_R(N,M)$.

For a ring:

> The pi-elements in R are exactly all the factors of all nonzero regular elements in R.

Now we consider the complements of the sets in (3) in $\mathrm{Hom}_R(N,M)$ (resp. in the ring case in R). The complement of the left set has no (valuable ?) structure, but the complement of the right set has interesting properties. We have for this complement the notation " total from N to M ",abbreviation

$$\mathrm{TOT}(N,M) := \{h \in \mathrm{Hom}_R(N,M) \mid h \text{ is not pi}\} \quad .$$

For a ring R we write

$$\mathrm{TOT}(R) := \{t \in R \mid t \text{ is not pi}\} \quad .$$

Hence the total is the set of all total nonisomorphisms. The total has the following properties :

(I) TOT is a semi-ideal in the categorie R-mod. That means:

1) $\mathrm{TOT}(N,M) \neq \emptyset$, since $0 \in \mathrm{TOT}(N,M)$;

2) If in a product of homomorphisms at least one factor

is in TOT, then the product is in TOT.

(II) $\Delta(N,M) := \{h \in Hom_R(N,M) \mid Ke(h) \text{ is large in } N\}$,
$\nabla(N,M) := \{h \in Hom_R(N,M) \mid Im(h) \text{ is small in } M\}$,
Rad(N,M) are all containd in TOT(N,M) .

(III) If $h \in TOT(N,M)$, $u \in RAD(N,M)$, then $h+u \in TOT(N,M)$.

The properties (I) and (III) suggest the question: Under which conditions is TOT(N,M) additively closed ? If it is additively closed, then it is a T-S-submodule of $Hom_R(N,M)$, where $S := End_R(N)$, $T := End_R(M)$. In the ring case is TOT(R) then a two-sided ideal.

3. We give now examples for conditions such that TOT(N,M) is additively closed. (see |3| - |8|).

1.) If N is selfinjektive or selfprojektive and supplemented, then for arbitrary M both TOT(N,M) and TOT(M,N) are add. closed.

2.) If N is injective or semi-perfekt, then RAD(N,M) = TOT(N,M).

3.) If N is continuous and $S := End_R(N)$, then RAD(S) = TOT(S).

4.) Example 3.) can be generalised by "splitting" the continuous properties on M and N (for cont.prop. see |2|). We need the following properties:

(C1;N) : Every submodule of N is large in a direkt summand of N.

($C2_o$;N): If a submodule of N is isomorphic to N, then it is a direkt summand of N .

(C2;N,M): If a submodule of M is isomorphic to a direct summand of N, then it is a direct summand of N .

(C3;N): If B_1, B_2 are dirct summands of N and $B_1 \cap B_2 = 0$, then $B_1 + B_2$ is a direct summand of N.

If all these conditions are satisfied, then

$$\Delta(N,M) = RAD(N,M) = TOT(N,M) .$$

The dual (discretness) assumtions imply

$$\nabla(N,M) = RAD(N,M) = TOT(N,M) .$$

5.) Examples for modules with RT- resp. LE-decompositions. A module A is denoted as RT-module :<=> $RAD(End_R(A))=TOT(End_R(A))$. If $End_R(A)$ is a local ring, then A is called LE-module. Then LE-modules are also RT-modules. A decomposition

$$M = \bigoplus_{j \in J} A_j$$

is called RT- resp. LE-decomposition :⟺ all A_j are RT- resp. LE-modules.

We give now some examples for properties of RT- resp. LE-decomposition, which show the value of our notions (see |3| and |4|):

(i) If M has a RT-decomposition and $T := End_R(M)$, then TOT(T) is an ideal in T.

(ii) If M has a finite RT-decomposition, then M is a RT-module.

(iii) If M has a LE-decompositio, then T/TOT(T) is a product of endomorphism rings of vector spaces over division rings.

(iv) If M has LE-decomposition, then the following conditions (*) and (⁑) are equivalent:

(*) Every LE-decomposition of M complements direct summands,

(⁑) RAD(T) = TOT(T) .

4. For furthes study of the notions "partially invertible" and "total" , we propose the following

Programm:

1) Extend "all" (definitions and properties) from regular elements to pi-elements.

2) Extend "all" from rings to Hom.

3) For the study of regularity in Hom use the "relative" regularity: $f = fgf + u$, $u \in U \subsetneq {}_S Hom_R(M,N)_T$ (see the abstract of my talk about "Regularity in Hom").

4) Extend the notions "pi-element" and "total" to the relative regularity (in 3)).

5) Extend results for LE-decomposition to RT-decompositions.

REFERENCES

See references in the abstract of my talk:
"Regularity in Hom " .

Mathematisches Institut der Universität München
Theresienstr. 39 , 80333 München , Deutschland (Germany)

ON THE STABLE MODULE CATEGORY OF A SELFINJECTIVE ALGEBRA

OTTO KERNER

This talk reports on some results from the joint paper [4] with K. Erdmann.

1. Notations A denotes a finite dimensional, associative, connected k-algebra (with identity 1) over some algebraically closed field k and A-mod denotes the category of finite dimensional left A-modules.

The algebra A is called *self-injective*, if A is injective as a left (and right) A-module. Important examples for self-injective algebras are group algebras kG, where G is a finite group.

If A is self-injective, A-$\underline{\text{mod}}$ denotes the *stable category* of A-mod. Its objects are the A-modules. The morphism spaces are $\underline{\text{Hom}}(X,Y) = \text{Hom}_A(X,Y)/\mathcal{P}(X,Y)$, where $\mathcal{P}(X,Y)$ is the subspace of $\text{Hom}_A(X,Y)$, defined by the maps $f : X \to Y$ factoring through a projective (= injective) module. Stable module categories have played in the past years a crucial role for classification problems of self-injective algebras, see for example [2].

If $\mathcal{C}$ is a class of modules, $\text{add}\,\mathcal{C}$ denotes the full subcategory of A-mod, consisting of direct summands of finite direct sums of objects of $\mathcal{C}$.

2. Concepts from Auslander-Reiten theory If A is a finite dimensional algebra, the Auslander-Reiten quiver $\Gamma(A)$ of A or, more precise of A-mod is defined as follows.

(1) The vertices of $\Gamma(A)$ are the isomorphism classes $[X]$ of the indecomposables X in A-mod.

(2) Let rad(A-mod) denote the Jacobson radical of A-mod. For indecomposables X, Y a morphism $f \in \text{rad}(X,Y) \setminus \text{rad}^2(X,Y)$ is called *irreducible* and $\text{Irr}(X,Y)$ denotes the factor-space $\text{rad}(X,Y)/\text{rad}^2(X,Y)$. The number of arrows from $[X]$ to $[Y]$ is $\dim_k \text{Irr}(X,Y)$.

The quiver (= directed graph) $\Gamma(A)$ is locally finite that is at each vertex there are only finitely many arrows starting and ending. This is proved by the following argument.

If X is indecomposable and not projective (not injective) and $0 \to \tau X \xrightarrow{(f_i)} \oplus Y_i^{r_i} \xrightarrow{(g_i)^t} X \to 0$ $(0 \to X \xrightarrow{(f_i)} \oplus Y_i^{r_i} \xrightarrow{(g_i)^t} \tau^- X \to 0)$ is the Auslander-Reiten sequence ending (starting) in X, where the Y_i are indecomposable and pairwise non-isomorphic, then the r_i components of the maps f_i, g_i define a basis of the corresponding spaces of irreducible maps, see for example [1]. The modules τX and $\tau^- X$ are indecomposable and uniquely determined by X.

τ and τ^- are called the *Auslander-Reiten translations*. They define an additional structure on $\Gamma(A)$, also denoted by τ and τ^-. Hence $\Gamma(A)$ is a translation quiver.

Normally τ and τ^- are not endo-functors of A-mod. If A is self-injective, then τ, τ^- induce equivalences

$$\tau, \tau^- : A\text{-}\underline{\text{mod}} \to A\text{-}\underline{\text{mod}}$$

The Heller functors Ω, Ω^{-1} define equivalences on A-$\underline{\text{mod}}$, too.

The connected components of the quiver $\Gamma(A)$ of a finite dimensional algebra A are called Auslander-Reiten components of $\Gamma(A)$ or A. If $\Gamma(A)$ has a finite component $\mathcal{C}$, then $\mathcal{C} = \Gamma(A)$ holds, and A is of finite representation type, see [1]. Hence we restrict to infinite components.

We will not distinguish between indecomposable modules in A-mod and vertices of $\Gamma(A)$.

3. Stably quasi-serial components A component $\mathcal{C}$ in the Auslander-Reiten quiver $\Gamma(A)$ of A is called *regular*, if it contains neither projective nor injective vertices. Regular components of type $\mathbb{Z}A_\infty$ or $\mathbb{Z}A_\infty/\langle\tau^m\rangle$ (i.e. tubes of rank m) are called *quasi-serial*. The reason for this name is as follows.

a) If A is hereditary and $\mathcal{T}$ is a tube in $\Gamma(A)$, then the category add $\mathcal{T}$ is an abelian serial category.

b) If $\mathcal{C}$ is of type $\mathbb{Z}A_\infty$ or $\mathbb{Z}A_\infty/\langle\tau^m\rangle$, a module X in $\mathcal{C}$ is called *quasi-simple*, if the middle term $X(2)$ of the Auslander-Reiten sequence $0 \to X \to X(2) \to \tau^- X \to 0$ is indecomposable. If X is quasi-simple in $\mathcal{C}$, then there exists a unique infinite sectional path of irreducible monomorphisms

$$X = X(1) \to X(2) \to \cdots \to X(m) \to \cdots$$

If Y is indecomposable in $\mathcal{C}$ there exist unique $r \geq 1$ and Z quasi-simple with $Y \cong Z(r)$. The number r is called the *quasi-length* of Y and Z is the *quasi-socle* of Y.

Normally add $\mathcal{C}$ is not serial, but for X quasi-simple and $1 \leq i, j$ the irreducible maps induce a short exact sequence

$$(*) 0 \to X(i) \to X(i+j) \to \tau^{-i} X(j) \to 0$$

see for example [5], if A is hereditary.

If A is self-injective and $\mathcal{C}$ is an Auslander-Reiten component of A, we get the *stable part* of $\mathcal{C}$ by deleting the projective-injective vertices of $\mathcal{C}$. The stable part of an Auslander-Reiten component is called a *stable component*. An Auslander-Reiten component is called *stably quasi-serial* if its stable part is of type $\mathbb{Z}A_\infty$ or $\mathbb{Z}A_\infty/\langle\tau^m\rangle$. This is justified, since a similar result as (*) holds for stably quasi-serial components, see [4], 2.3, 2.4. It seems that most of the components of the Auslander-Reiten quiver $\Gamma(A)$ for a self-injective algebra A are stably quasi-serial, see for example [3].

4. Some results Here always A denotes a connected self-injective algebra. The presented results deal with $\underline{\mathrm{Hom}}(X, Y)$, where X and Y are in the same stably quasi-serial component of $\Gamma(A)$. An indecomposable module X is called *stable brick* if $\underline{\mathrm{End}}(X) = k$.

One of the basic results is

Proposition 1. *Let $f : X \to X$ be a chain of irreducible maps in a stably quasi-serial components on a sectional path. Then $f \notin \mathcal{P}(X, Y)$.*

For tubes this result implies

Theorem 2. *Let $\mathcal{T}$ be a tube of rank m. Then $\mathcal{T}$ is a stably standard tube, that is the stable category $k(\underline{\mathcal{T}})$ is equivalent to the mesh-category $k\underline{\mathcal{T}}$ of the stable part $\underline{\mathcal{T}}$ of $\mathcal{T}$, if and only if the m quasi-simple modules $X, \ldots, \tau^{m-1}X$ in $\mathcal{T}$ are pairwise stably orthogonal (i.e. $\underline{\mathrm{Hom}}(X, \tau^i X) = 0$ for $0 < i < m$) stable bricks.*

For the stable endomorhism rings $\underline{\mathrm{End}}(M)$ one gets.

Theorem 3. *Let $\mathcal{C}$ be a stably quasi-serial component and X quasi-simple in $\mathcal{C}$.*

(a) $\dim \underline{\mathrm{End}}(X(i)) \leq \dim \underline{\mathrm{End}}(X(i+1))$ *for all i.*

(b) *If $\mathcal{C}$ is a tube, or if $A = kG$ is a group algbra, then* $\{\dim \underline{\mathrm{End}}(X(i)) \mid i \in \mathbb{N}\}$ *is unbounded.*

(c) *If G is a p-group and $A = kG$, then* $\dim \underline{\mathrm{End}}(X(i)) < \dim \underline{\mathrm{End}}(X(i+1))$.

A corresponding result also holds for $\dim \underline{\mathrm{Hom}}(X(i), \tau^m X(i))$ with $m \neq 0$.

It is a central tool for the proofs, that $\underline{\mathrm{Hom}}(X, -)$ behaves well on short exact sequencs, see [4],1.4.

Lemma 4. (a) *If $0 \to U \to V \to W \to 0$ is a short exact sequence in A-mod and M is an A-modules, then the induced sequence*

$$\underline{\mathrm{Hom}}(M, U) \to \underline{\mathrm{Hom}}(M, V) \to \underline{\mathrm{Hom}}(M, W)$$

is exact.

(b) *If $0 \to \tau X \to E \to X \to 0$ is an Auslander-Reiten sequence and M is indecomposable, not isomorphic to X and $\Omega^{-1}X$, then the induced sequence*

$$0 \to \underline{\mathrm{Hom}}(M, \tau X) \to \underline{\mathrm{Hom}}(M, E) \to \underline{\mathrm{Hom}}(M, X) \to 0$$

is exact.

References

[1] M. Auslander, I. Reiten and S. Smalø: Representation theory of artin algebras. Cambridge Studies in Advanced Mathematics,1994.

[2] K. Erdmann: Blocks of tame representation type and related algebras. Springer Lecture Notes in Mathematics **1428**(1990).

[3] K. Erdmann: On Auslander-Reiten components of group algebras. To appear in J. Pure Appl. Algebra.

[4] K. Erdmann and O. Kerner: On the stable module category of a self-injective algebra. Preprint 1995.

[5] C.M. Ringel: Finite-dimensional hereditary algebras of wild representation type. Math. Z. **161**(1978),235-255.

Mathematisches Institut
Heinrich-Heine-Universität
40225 Düsseldorf (Germany)
email kerner@mx.cs.uni-duesseldorf.de

GLOBAL DIMENSION OF TWISTED GROUP RINGS

EUN SUP KIM

Let K be a commutative Noetherian domain with identity, K^* the group of units in K, G a finite group acting on K via a homomorphism $t : G \to Aut(K)$ and let $[\alpha] \in H^2(G, K^*)$. We denote by $K_t^\alpha G$ the crossed product.

Let S be a subgroup of G, and let $[\alpha] \in H^2(G, K^*)$. The restriction of $[\alpha]$ to $H^2(S, K^*)$ is denoted by $Res_S([\alpha])$.

In [1], E. Aljadeff and S. Rosset showed that $gl \dim K_t^\alpha G \leq gl \dim K_t G \leq gl \dim KG$. So it is natural to ask for the following Question.

Question (K. A. Brown)
$gl \dim K_t^\alpha G < \infty$ if and only if for any $S \leq G$ with $Res_S([\alpha]) = 1$, $gl \dim K_t S < \infty$.

The author gave the answer of the Question.

REFERENCES

[1] E. ALJADEFF and S. ROSSET, Global dimension of crossed products, J. Pure Appl. Algebra 40(1986), 103-113.

Department of Mathematics
College of Natural Sciences
Kyungpook National University
Taegu, 702-701, Korea

QUOTIENT RINGS OF FINITE NORMALIZING QF-EXTENSIONS*

YOSHIMI KITAMURA

The purpose of this note is to give a relationship between a maximal quotient ring of a ring A and of its subring B in case A is a finite normalizing quasi-Frobenius extension of B.

Throughout the paper, all rings are associative with identity, all ring extensions contain the common identity and all modules are unital unless otherwise stated.

The notation ${}_RM_S$ stands for M a left R- right S-bimodule. For a right R-module M_R, we denote the injective envelope of M_R by $E(M_R)$ or simply $E(M)$ if there is no ambiguity. We say that M_R has *finite Goldie dimension* if there do not exist infinitely many nonzero submodules whose sum is direct. A ring R is said to have finite right Goldie dimension if R has finite Goldie dimension as a right R-module. A module M is said to be *weakly injective* if every finitely generated submodule of its injective envelope $E(M)$ is contained in a submodule of $E(M)$ isomorphic to M ([3]). Also, M is said to be *tight* if every finitely generated submodule of its injective envelope $E(M)$ can be embedded in M ([2]). A weakly injective module is obviously tight. Conversely, a tight module is weakly injective if it is a finitely generated module of finite Goldie dimension. For other unexplained terminology we refer to [8].

The following extends [1, Theorem 4.1] slightly.

Proposition 1 *Let R be a ring with finite right Goldie dimension. If every right regular element of R is regular and R is weakly injective as a right R-module, then the maximal right quotient ring $Q_{max}(R)$ of R is a right self-injective, classical left quotient ring of R, and conversely.*

Let A/B be a ring extension. A is called a *finite normalizing extension* of B if A_B has a finite subset of generators $\{a_1, \ldots, a_n\}$ each of which

*The detailed version of this paper has been submitted for publication elsewhere.

normalizes B, that is, $a_iB = Ba_i$. Following to Kasch[4], A is a *Frobenius extension* of B if A_B is finitely generated projective and $A \cong A^*$ as (B, A)-bimodules, where $A^* = \mathrm{Hom}(A_B, B_B)$. Also, following to Müller[7], A is called a *right quasi-Frobenius (qF) extension* of B if A_B is finitely generated projective and there exists a (B, A)-bimodule X such that $A \oplus X \cong (A^*)^n$ as (B, A)-bimodules. Similarly, a left qF extension is defined, and a right and left qF extension is called a qF one.

For a qF extension, we have the following inheritance of finite Goldie dimension.

Proposition 2 *Let A/B be a qF extension. Then A has finite right Goldie dimension if and only if B has finite right Goldie dimension.*

The next follows from the proof of Proposition 1.2(2) of [5].

Lemma 3 *Let M be an (S, R)-bimodule with a finite subset $\{m_1, \ldots, m_n\}$ such that $M = \sum_{i=1}^{n} m_iR$ and $Sm_i = m_iR$ for each i. Let X be a right R-module and Y its submodule. If Y_R is essential in X_R, then $\mathrm{Hom}(M_R, Y_R)_S$ is essential in $\mathrm{Hom}(M_R, X_R)_S$.*

Using the above lemma, we have

Lemma 4 *Let A/B be a finite normalizing right qF extension and X a right B-module. If Y is an essential submodule of X, then $Y \otimes_B A$ is essential in $X \otimes_B A$ as a right B, and hence, as a right A-module.*

As a consequence of Lemma 4, we have

Proposition 5 *Let A/B be a finite normalizing right qF extension and E the injective envelope of B_B. Then $E \otimes_B A$ is an essential extension of A_B and is the injective envelope of A_A. Moreover, if B_B is weakly injective (resp. tight), then so is A_A.*

As a consequence of the above proposition, we have

Proposition 6 *Let A/B be a finite normalizing qF extension. If D is a dense right ideal in A, then $D \cap B$ is a dense right ideal in B. Conversely, if K is a dense right ideal in B, then KA is a dense right ideal in A.*

We are now in a position to state the main result of the present paper (cf. [6, Theorem 11]).

Theorem 7 *Let A/B be a finite normalizing qF (resp. Frobenius) extension. Then the maximal right quotient ring $Q_{max}(A)$ of A is a qF (resp. Frobenius) extension of the maximal right quotient ring $Q_{max}(B)$ of B such that*

$$Q_{max}(A) \cong Q_{max}(B) \otimes_B A \cong A \otimes_B Q_{max}(B)$$

canonically. Moreover, when B is a weakly injective module of finite Goldie dimension as a right B-module and every right regular element of B is regular, then A is so as a right A-module and every right regular element of A is regular, and then $Q_{max}(A)$ is a right self-injective, classical left quotient ring of A.

As a direct consequence of the above theorem, we have

Corollary 8 *Let A/B be a finite normalizing qF extension. If $Q_{max}(B)$ is right (resp. left) self-injective, then $Q_{max}(A)$ is right (resp. left) self-injective. Moreover, if A_B is a generator, then the converse is valid.*

By Proposition 1 and Theorem 7, we have

Corollary 9 *Let A/B be a finite normalizing qF extension such that B has finite right Goldie dimension. If the maximal right quotient ring of B is a right self-injective, classical left quotient ring of B, then the maximal right quotient ring of A is a right self-injective, classical left quotient ring of A.*

Recall that a ring R is right FPF if every finitely generated, faithful right R-module is a generator in the category of all right R-modules. By [1], every right FPF ring R of finite right Goldie dimension is weakly injective as a right R-module provided that every right regular element of R is regular. We have the following from the above immediately.

Corollary 10 *Let A/B be a finite normalizing qF extension. If B is a right FPF ring of finite right Goldie dimension and every right regular element of B is regular, then A is weakly injective with finite Goldie dimension as a right A-module and every right regular element of A is regular, and the maximal right quotient ring of A is a right self-injective, classical left quotient ring of A.*

References

[1] T. G. Faticoni, Localization in finite dimensional FPF rings, Pacific J. Math. **134** (1988), 70-99.

[2] J. S. Golan and S. R. López-Permouth, QI-filters and tight modules, Communications in Algebra, **19** (1991), 2217-2229.

[3] S. K. Jain, S. R. López-Permouth and S. Singh, On a class of QI-rings, Glasgow Math. J. **34** (1992), 75-81.

[4] F. Kasch, Projective Frobenius-Erweiterungen, Sitzungsber. Heidelberger Akad. **1960/61**, 89-109.

[5] Y. Kitamura, Quasi-Frobenius extensions with Morita duality, J. Algebra. **73** (1981), 275-286.

[6] K. Louden, Maximal quotient rings of ring extensions, Pacific J. Math. **62** (1976), 489-496.

[7] B. Müller, Quasi-Frobenius-Erweiterungen, Math. Z. **85** (1964),345-368.

[8] B. Stenström, Rings of Quotients, Grundl. der Math. Wiss. **217**, Springer-Verlag, Berlin-Heidelberg-New York, 1975.

DEPARTMENT OF MATHEMATICS AND INFORMATICS
TOKYO GAKUGEI UNIVERSITY
KOGANEI, TOKYO 184, JAPAN

Algorithm Method in Ring Theory

Shigeru KOBAYASHI

In this paper K denotes a fixed field and the term K-algebra is used to denote an associative algebra with unit over K. Given a non-empty set $X = \{x_1, x_2, \cdots, x_n\}$, $\langle X \rangle$ will denote the free monoid with unit, generated by X and $K\langle X \rangle$ will denote the free associative algebra generated by X. We shall consider $\langle X \rangle$ ordered by the degree-lexicographic order $<_X$. We set $x_1 <_X x_2 <_X \cdots <_X x_n$. For any $g = \sum_{i=1}^{m} \alpha_i u_i$, $\alpha_i \in K \backslash \{0\}$, $u_i \in \langle X \rangle$, we denote by $HM(g)$ the highest monomial of g, i.e. $HM(g) = u_i$ if $u_j <_X u_i$, for all $j \neq i$. We say $HM(0) = 0$.

Let F be a set of polynomials in $K\langle X \rangle \backslash \{0\}$. The monomial u is normal (mod F) if it does not contain any of the monomials $HM(f)$, $f \in F$ as a subword. $N(F)$ will denote the set of all normal (mod F) monomials.

Let I be a nontrivial ideal in $K\langle X \rangle$ and let $A = K\langle X \rangle / I$. We put $N(I)$ be the set of all normal (mod I) monomials. There is an equality $K\langle X \rangle = \mathrm{Span} N(I) \oplus I$ as vector spaces. For any $f \in K\langle X \rangle$ one has $f = \bar{f} + g$, where $\bar{f} \in \mathrm{Span} N(I)$ and $g \in I$ are uniquely determined. Clearly there is an isomorphism of vector spaces $A \cong \mathrm{Span} N(I)$, $N(I)$ projects to a K-basis of A.

A (finite) set F of polynomials in $K\langle X \rangle \backslash \{0\}$, generating I as two-sides ideal is called a (finite) Gröbner basis of I if $N(F) = N(I)$.

If A has finite Gröbner basis, then we can show whether an element f of A is zero or not by a finite step. A monomial of the form $x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$, $k_i \geq 0$ $(i = 1, 2, \cdots, n)$ is called *ordered monomial*. T will denote the subset of $\langle X \rangle$ consisting of all ordered monomials.

Let $A = K\langle X \rangle / I$ for some ideal I. In general, it is not known that the relation of $N(I)$ and T. But if I has a finite Gröbner basis F, then $N(I) = N(F) \subseteq T$. So in this case, A become a noetherian. Conversely if A is a noetherian, then in general, $N(I)$ is not contained in T. Nevertheless if

A is a filtered algebra, then we have the following.

Theorem *Lct A be a filtered algebra and generated by X as a filtered algebra, If A is a noetherian, then $N(I) \subseteq T$.*

References

[1] G. M. Bergman, *The diamond lemma for ring theory*, Adv. in Math. **29** (1978), 178-218

[2] T. Gateva-Ivanova, *Noetherian Properties and Growth of Some Associative Algebras*, in "Effective Method in Algebraic Geometry" , Progress in Mathematics 94, Birkhüser, 1991, 143-158.

[3] S. Kobayashi, *On Growth Properties of Algebras*, Proc. of the Sympo. on Ring Theory, (1994), 59-62.

Department of Mathematics
Naruto University of Education
Takashima, Naruto, 772, Japan
e-mail address:skoba@naruto-u.ac.jp

ON QF-3 MODULES

KAZUTOSHI KOIKE

QF-3 rings (rings with minimal faithful modules) have been investigated by many authors. As generalizations of QF-3 rings, QF-3' rings (rings with torsionless injective hull) and QF-3" rings (rings in which every finitely generated submodule of its injective hull is torsionless) are introduced and investigated. Also these two classes of rings are generalized to modules. In this note, we introduce the notion of QF-3 modules as a module version of QF-3 rings and basic results for QF-3 modules, which include several well-known results for QF-3 rings.

The following definition is the starting point of our work.

Definition. A module U_R is called *a QF-3 module* in case U_R has a finitely cogenerated injective submodule that cogenerates U_R.

We recall that a module X is *finitely cogenerated* in case for every index set $(X_i)_I$ of submodules of X, $\bigcap_I X_i = 0$ implies $\bigcap_F X_i = 0$ for some finite subset $F \subset I$. As is well-known, X is finitely cogenerated if and only if $Soc(X)$ is finitely generated and essential in X. Therefore, by using the characterization of minimal faithful modules by Colby-Rutter ([7, Proposition 4.1]), it follows that our definition of QF-3 modules is consistent with that of QF-3 rings.

First we note the following lemma, which is a two-sided dual version of the well-known fact that ${}_SU$ is a generator if and only if U_R is finitely generated projective for a faithfully balanced bimodule ${}_SU_R$. This lemma can be proved by using the similar way of the proof of [7, Theorem 5.1].

Lemma 1. *For a faithfully balanced bimodule ${}_SU_R$, the following conditions are equivalent:*

(1) *U is a cogenerator on both sides.*

(2) *U is finitely cogenerated injective on both sides.*

(3) *${}_SU_R$ defines a Morita duality.*

By using this lemma we can prove the following theorem, which is a generalization of [7, Theorem 5.1].

Theorem 2. *Let ${}_SU_R$ be a faithfully balanced bimodule with finitely cogenerated injective submodules fU_R and ${}_SUe$ that cogenerate U. where $f \in S$ and $e \in R$ are idempotenets. Then ${}_{fSf}fUe_{eRe}$ defines a Morita duality.*

Similar to the case of QF-3 rings, for every faithfully balanced bimodule which is QF-3 on both sides, the right and left localizations coincide and they become faithfully balanced bimodules which are QF-3 on both sides. To state this results, we use the following notation. For an injective module E we denote by k_E the left exact radical corresponding to the hereditary tosion theory cogenerated by E and for a left exact radical τ we denote by Q_τ the localization functor with respect to τ.

Proposition 3. *Let ${}_SU_R$ be a faithfully balanced bimodule which is QF-3 on both sides and let $\tau = k_{E(U_R)}$ and $\sigma = k_{E({}_SU)}$. Then*

(1) *$Q_\tau(U) \cong Q_\sigma(U)(= L$, say) as (S, R)-bimodules.*

(2) *L is faithfully balanced as a $(Q_\sigma(S), Q_\tau(R))$-bimodule.*

(3) *${}_{Q_\sigma(S)}L_{Q_\tau(R)}$ is QF-3 on both sides.*

We recall that a module X is *rationally complete* in case $X = Q_{k_{E(X)}}(X)$. We note that, for a ring R, R_R is rationally complete if and only if R is its own maximal right quotient ring.

Now we can obtain the following characterization of faithfully balanced bimodules which are QF-3 and rationally complete on both sides. This result is a generalization of [7, Theorem 5.3] by Ringel and Tachikawa.

Theorem 4. *There is a bijective correspondence between the following.*

(A) *Isomorphism classes of faithfully balanced bimodules ${}_SU_R$ which are QF-3 and rationally complete on both sides.*

(B) *Morita equivalence classes of pairs of modules (C_A, D_A), where C_A is a linearly compact cogenerator and D_A is a linearly compact generator.*

Here the correspondence (A) $\rightarrow$ (B) *is given by*

$${}_SU_R \text{ (with } {}_SUe) \longmapsto (Ue_{eRe}, Re_{eRe}),$$

where ${}_SUe$ is a finitely cogenerated injective submodule of ${}_SU$ that cogenerates ${}_SU$ and e is an idempotent of R, and the inverse correspondence (B) $\to$ (A) *is given by*

$$(C_A, D_A) \longmapsto {}_{End_A(C)}Hom_A(D,C)_{End_A(D)}.$$

We note that the bijection described in [7, Theorem 5.3] is obtained by restricting the bijection in Theorem 4. In fact, for a pair (C_A, D_A) in (B), if $C = D$, it must be a generator and cogenerator, and (C_A, C_A) corresponds to the endomorphism ring $End_A(C)$.

Theorem 4 asserts that every faithfully balanced bimodule ${}_SU_R$ which is QF-3 and rationally complete on both sides is essentially constructed as the following way.

Example. Let ${}_BV_A$ be a bimodule that defines a Morita duality and let X_A and Y_A be linearly compact modules. Now put $C_A = V \bigoplus X$, $D_A = A \bigoplus Y$, $S = \mathrm{End}_A(C)$, $R = \mathrm{End}_A(D)$ and ${}_SU_R = \mathrm{Hom}_A(D,C)$. Then, by Theorem 4, ${}_SU_R$ is a faithfully balanced bimodule which is QF-3 and rationally complete on both sides. ${}_SU_R$ has the following matrix form:

$${}_{\begin{pmatrix} B & X^* \\ \mathrm{Hom}_A(V,X) & \mathrm{End}_A(X)\end{pmatrix}}\begin{pmatrix} V & Y^* \\ X & \mathrm{Hom}_A(Y,X)\end{pmatrix}_{\begin{pmatrix} A & \mathrm{Hom}_A(Y,A) \\ Y & \mathrm{End}_A(Y)\end{pmatrix}},$$

where $(\ )^*$ denote the V-duals.

The following result asserts that every faithfully balanced bimodule which is QF-3 (but not necessarily rationally complete) on both sides can be realized as certain subbimodule of bimodules which was characterized in Theorem 4.

Theorem 5. *A faithfully balanced bimodule ${}_SU_R$ is QF-3 on both sides if and only if there exists a faithfully balanced bimodule ${}_{\hat{S}}L_{\hat{R}}$ which is QF-3 and rationally complete on both sides, with finitely cogenerated injective submodules ${}_{\hat{S}}Le$ and $fL_{\hat{R}}$ that cogenerate L, where $e \in \hat{R}$ and $f \in \hat{S}$ are idempotents, satisfying the following conditions:*

(1) *$\hat{R}$ is a ring extension of R such that $\hat{R}e \subset R$.*

(2) *$\hat{S}$ is a ring extension of S such that $f\hat{S} \subset S$.*

(3) *${}_SL_R$ is a bimodule extension of ${}_SU_R$ such that $fL \subset U$ and $Le \subset U$.*

As the last result in this note, we point out a connection of a faithfully balanced bimodule ${}_SU_R$ which is QF-3 and rationally complete on both sides and a Morita duality between quotient categories of Mod-R and S-Mod. For a left exact radical τ of Mod-R, we denote by Mod-(R, τ) the quotient category of Mod-R with respect to τ. Similar notation is used for left module categories.

Theorem 6. *Let $_SU_R$ be a faithfully balanced bimodule which is QF-3 and rationally complete on both sides and let $\tau = k_{E(U_R)}$ and $\sigma = k_{E(_SU)}$. Then $\mathrm{Hom}_R(-, U)$ and $\mathrm{Hom}_S(-, U)$ induce a Morita duality between the quotient categories Mod-(R, τ) and (S, σ)-Mod.*

In the theorem above, two quotient categories Mod-(R, τ) and (S, σ)-Mod are equivalent to a full module category. Therefore we can use the terminology "Morita duality" without the ambiguity.

Concluding this note, we remark on a dual concept of QF-3 modules. The definition of QF-3 modules is completely categorical. Therefore we can dualize the definition of QF-3 modules. For this dual concept of QF-3 modules, we can obtain the dual results of Theorem 2 to Theorem 6.

References

[1] F. W. Anderson and K. R. Fuller, *Rings and categories of modules*, Graduate Texts in Math. Springer, New York, 1992, Second Edition.

[2] R. R. Colby and K. R. Fuller, *QF-3' rings and Morita duality*, Tsukuba J. Math. 8, 1984, pp. 183–188.

[3] J. L. Gomez Pardo, *Counterinjective modules and duality*, J. Pure Appl. Algebra 61, 1989, pp. 165–179.

[4] J. L. Gomez Pardo and P. A. Guil Asensio, *Morita dualities associated with the R-dual functors*, preprint.

[5] K. Ohtake, *A generalization of Morita duality by localizations*, preprint.

[6] B. Stenstrom, *Rings of quotients*, Springer, New York, 1975.

[7] H. Tachikawa, *Quasi-Frobenius rings and generalizations*, Lecture Notes in Math. 351, Springer, New York, 1975.

Department of Computer Science
Ube College
5-40 Bunkyou-chou, Ube 755, Japan
E-mail address : koike@cs.ube-c.ac.jp

Hammocks and the algorithms of Zavadskii

YaNan Lin

Hammocks have been considered by Brenner [1] in order to give a numerical criterion for a finite translation quiver to be the Auslander-Reiten quiver of some representation-finite algebra. Ringel and Vossieck [4] gave a combinatorial definition of left hammocks, which generalizes the concept of hammocks in the sense of Brenner, and shown the relationship between thin left hammocks and representations of partical ordered set (abbreviated: poset). An important role in representation theory of poset is played by two differentiation algorithms. One of the algorithms, called "differentiation with respect to maximal element", is due to Nazarova and Roiter [3]. The second is due to Zavadskii [5] and reduces a poset S with a suitable pair (a, b) of elements a, b to a new poset $S' = \partial_{(a,b)}S$ with same representation type. In the paper [2], we disscussed the relationship between hammocks and the algorithm of Nazarova-Roiter. The main purpose of the present paper is to construct some new left hammocks from a given one, and to show the relationship between these new left hammocks and the algorthm of Zavadskii.

A preprojective component $H = (H_0, H_1, \tau)$ is said to be a left hammock provided there exists a function $h_H : H_0 \to N$ such that

(1) H has a unique source ω and $h_H(\omega) = 1$;

(2) h_H is an additional function, that is $h_H(x) + h_H(\tau x) = \sum_{y \to x} h_H(y)$;

(3) if q is an injective vertex then $h_H(q) \geq \sum_{q \to y} h_H(y)$.

When H is a left hammock, the function h_H is said to be its hammock function. A left hammock H is said to be thin provided $h_H(p) = 1$ for any projective vertex p of H.

There is a strong relationship between thin left hammocks and the representation theory of posets: Let S be a finite poset and $\ell(S)$ the category of representations of S, then the preprojective component of the Auslander-Reiten quiver of $\ell(S)$ is a thin left hammock. Conversely, given a thin left hammock H, there exists a unique poset $S := S(H)$ such that $H \cong \mathcal{P}_S$, where $\mathcal{P}_S$ is the preprojective component of the Auslander-Reiten quiver of $\ell(S)$ (see [4]).

Let H be a left hammock, $k(H)$ the mesh category of H. For a given projective vertex

$p(a)$ of H, let ${}_a\mathcal{M}$ be the class of all objects x with $\mathrm{Hom}_{k(H)}(p(a), x) = 0$. For a given injective vertex $q(b)$ of H, let $\mathcal{M}_b$ be the class of all objects x with $\mathrm{Hom}_{k(H)}(x, q(b)) = 0$. Let $\mathcal{M}$ be a class of some objects of $k(H)$, we denote by $\mathrm{Hom}_{k(H)}(x, y)_{\mathcal{M}}$ the subspace of $\mathrm{Hom}_{k(H)}(x, y)$ consisting of all morphismes which factor through some object of $\mathcal{M}$.

Theorem 1 Let H be a thin left hammock, $p(a) \neq p(\omega)$ a projective vertex and $q(b) \neq q(\omega')$ an injective vertex of H. Assume that $\mathrm{Hom}_{k(H)}(p(b), p(a)) = 0$. Then ${}_aH_b = \{x \in H \mid \mathrm{Hom}_{k(H)}(p(a), q(b))_{\{x\}} \neq 0\}$ is a hammock with hammock function $h_{({}_aH_b)} = \dim_k \mathrm{Hom}_{k(H)}(p(a), -) - \dim_k \mathrm{Hom}_{k(H)}(p(a), -)_{\mathcal{M}_b} = \dim_k \mathrm{Hom}_{k(H)}(-, q(b)) - \dim_k \mathrm{Hom}_{k(H)}(-, q(b))_{{}_a\mathcal{M}}$.

Remark. Let H be a thin left hammock. According to [2], We can obtain the poset $\mathcal{S}({}_aH_b)$ corresponding to the hammock ${}_aH_b$ from the poset $\mathcal{S}(H)$ corresponding to the hammock H by a finite sequence of application of the algorithms of Nazarova-Roiter [3].

As we know [2], for a projective vertex $p(a)$ of thin left hammock H, we can construct a new left hammock ${}_aH = \{x \mid \mathrm{Hom}_{k(H)}(p(a), x) \neq 0\}$. And for an injective vertex $q(b)$ of thin left hammock H, we have a new left hammock $H_b = \{x \mid \mathrm{Hom}_{k(H)}(x, q(b)) \neq 0\}$.

Theorem 2 Let H be a thin left hammock, $p(a)$ a projective vertex different from source and $q(b)$ an injective vertex of H different from sink. Assume that $\mathrm{Hom}_{k(H)}(p(b), p(a)) = 0$. Then ${}_aH_b = {}_aH \cap H_b$.

Proposition 3 Let H be a thin left hammock, $p(a), p(c)$ projective vertices of H different from source, and $q(b), q(d)$ injective vertices of H different from sink. Assume that $\mathrm{Hom}_{k(H)}(p(b), p(a)) = 0$, $\mathrm{Hom}_{k(H)}(p(d), p(a)) = 0$, $\mathrm{Hom}_{k(H)}(p(b), p(c)) = 0$ and $\mathrm{Hom}_{k(H)}(p(c), p(a)) \neq 0$, $\mathrm{Hom}_{k(H)}(p(b), p(d)) \neq 0$. Then we have

(1) ${}_aH_b \subseteq {}_aH_d$ and ${}_aH_b = ({}_aH_d)_b$;

(2) ${}_aH_b \subseteq {}_cH_b$ and ${}_aH_b = {}_a({}_cH_b)$;

(3) ${}_aH_b = {}_a({}_cH_d)_b$.

Let H be a left hammock with translation τ and let μ be a projective-injective vertex of H with $\mu^+ = \{\varepsilon\}$. If $\mu^- = \{\tau\varepsilon\}$, then we call the subquiver $H \setminus \{\mu\}$, together with the restriction of τ on it, an "almost" left hammock with respect to ε. If L is an "almost" left hammock obtained from some left hammock H with respect to ε, we write $H = L \cup \{\mu\}$ with $\mu^+ = \{\varepsilon\}$. And we call the vertex μ the additional vertex.

Theorem 4 Let H be a thin left hammock with finitely many projective vertices, let $\mathcal{S} := \mathcal{S}(H)$ be the poset corresponding to H. Let $p(a)$ be a projective vertex different from source and $q(b)$ an injective vertex of H different from sink. Assume that a and b are incomparable in $\mathcal{S}$. Then $H/{}_aH_b = \{x \in H \mid h_H(x) - h_{({}_aH_b)}(x) \neq 0\}$ is an "almost" left hammock with respect to $p(a,b)$. For convenience, we denote by ${}_aH_b^\diamond$ the left hammock $(H/{}_aH_b) \cup \{\mu\}$ where $\mu^+ = \{p(a,b)\}$. Then the hammock function of ${}_aH_b^\diamond$ is

$$h_{({}_aH_b^\diamond)}(x) = \begin{cases} h_H(x) - h_{({}_aH_b)}(x) & x \in H/{}_aH_b, \\ 1 & x = \mu. \end{cases}$$

We recall the algorithm of Zavadskii. Let us fix some notation. Let $\mathcal{S}$ be a poset, we write $\mathcal{S} = A_1 + \cdots + A_n$ if $A_1 \cup \cdots \cup A_n = \mathcal{S}$ and $A_i \cap A_j = \emptyset$ for $i \neq j$ (note that the points from different A_i can be comparable). Let a pair of points a, b be incomparable, we put $\mathcal{S} = a^\vee + b_\wedge + J(a,b)$ and $J := J(a,b) = J_a + J_0 + J_b$ where $J_a = \{x \in J \mid x < a\}$ and $J_b = \{x \in J \mid x > b\}$.

Let $\mathcal{S}$ be a poset. A pair of points (a,b) is called suitable (for a stratification) if a and b are incomparable, and $\mathcal{S} = a^\vee + b_\wedge + J$, where $J = \{z_1 < \cdots < z_n\}$. Following [6], we construct the (a,b)-stratified poset $\partial_{(a,b)}\mathcal{S}$ as follows: The points of $\partial_{(a,b)}\mathcal{S}$ consist of (1) x, for $x \in a^\vee \cup b_\wedge$; (2) $a + x$, for $x \in J_b \cup J_0$ (3) $b \cap x$, for $x \in J_a \cup J_0$. The order relation in $\partial_{(a,b)}\mathcal{S}$ is defined as follows: (1) we keep all relations in $\mathcal{S}$ between elements in $a^\vee \cup b_\wedge$; (2) we set $b \cap x < a + x$ for $x \in J_0$; (3) we set $a + x < a + y$, if $x < y$ in $J_b \cup J_0$; (4) we set $b \cap x < b \cap y$, if $x < y$ in $J_a \cup J_0$; (5) we set $a + x < y$, if $x < y$ for $x \in J_b \cup J_0$ and $y > a$; (6) we set $x < b \cap y$, if $x < y$ for $x \in J_a \cup J_0$ and $y < b$; (7) we add the relation $a < a + x$ for $x \in J_b \cup J_0$, and $b \cap y < b$ for $y \in J_a \cap J_0$; (8) If x and y are such $x > y$ and $x < y$ under the relation above, then we identify x and y.

Theorem 5 Let H be a thin left hammock with finitely many projective vertices, let $\mathcal{S}(H)$ be the poset corresponding to H. Let $p(a)$ be a projective vertex different from source, and $q(b)$ an injective vertex of H different from sink. Assume that a and b are incomparable in $\mathcal{S}(H)$. Denote by $\mathcal{S}({}_aH_b^\diamond)$ the poset corresponding to the left hammock ${}_aH_b^\diamond$. Then $\mathcal{S}({}_aH_b^\diamond)$ is obtained from $\mathcal{S}(H)$ as follows: there is a finite sequence of pairs of points (c_1,d_1), (c_2,d_2), $\cdots$, (c_l,d_l)=(a,b), and a finite sequence of posets $\mathcal{S}_1 = \mathcal{S}(H), \mathcal{S}_2, \cdots, \mathcal{S}_l$ such that

(1) (c_i,d_i) is a suitable pair of points of $\mathcal{S}_i$, for $i = 1, \cdots, l$;

(2) $\mathcal{S}_i = \partial_{(c_{i-1},d_{i-1})}\mathcal{S}_{i-1}$ for $i = 2, \cdots, l$, that is, $\mathcal{S}_i$ is the (c_{i-1},d_{i-1})-stratified poset of $\mathcal{S}_{i-1}$;

(3) $\mathcal{S}_l = \mathcal{S}({}_aH_b^\diamond)$.

In [6], Zavadskii used the two meticulous algorithms, which is called "stratification" and "replenishment", instead of "the differentiation with respect to a pair of points". The following theorem explained completely the replenishment algorithm.

Theorem 6 Let H be a thin left hammock with finitely many projective vertices.

Assume that H has a projective-injective vertex $\mu = p(a) = q(b)$ with $\mu^+ = \{\varepsilon\}$. Let $S(H)$ be the poset corresponding to H. We define $\xi(H) = L$ by sending H to $L = H \setminus \{\mu\}$ and omitting the translation τ on ε in L. Then $S(L)$ is just the replenishment poset $\gamma_{(a,b)}S$ for the specific pair (a, b).

References

1. S. Brenner, A combinatorial characterization of finite Auslander-Reiten quivers, LNM 1177 (1986) 13-49
2. Y. Lin, Hammocks and the algorithm of Nazarova-Roiter, J. London Math. Soc. (to appear)
3. L. A. Nazarova and A. V. Roiter, Representations of partially ordered sets, J. Soviet Math. 3(1975) 585-606
4. C. M. Ringel and D. Vossieck, Hammocks, Proc. London Math. Soc. (3) 54 (1987) 216-246
5. A. G. Zavadskij, Differentiation with respect to a pair of points, in: Matrix Problems, Kiev (1977), 115-121
6. A. G. Zavadskij, The Auslander-Reiten quiver for posets of finite growth, Topic in Algebra, Banach Center Publ. 26 (1990), 569-587

Department of Mathematics
Xiamen University
Xiamen 361005, P.R. China

PRIME AND PRIMARY IDEALS OF NON-COMMUTATIVE PRÜFER RINGS

Hidetoshi MARUBAYASHI and Akira UEDA

Throughout this note, let Q be a simple Artinian ring with finite dimension over its center and let R be an order in Q, that is, R is a prime Goldie ring. R is called a *right Prüfer order* in Q if any finitely generated right R-ideal is a progenerator of Mod-R, that is, projective and a generator of Mod-R. Note that an order R in Q is right Prüfer if and only if $(R:I)_l I = R$, $I(R:I)_l = O_l(I)$ and $(R:I)_l = I^{-1}$ for any finitely generated right R-ideal I, where $(R:I)_l = \{q \in Q \mid qI \subseteq R\}$ and $I^{-1} = \{q \in Q \mid IqI \subseteq I\}$. A *left Prüfer order* is defined similarly. But it is proved that an order R in Q is right Prüfer if and only if R is left Prüfer and so we call an order R to be a *Prüfer order* if R is right and left Prüfer.

Let R be a ring and let A be an ideal of R. We define $\sqrt{A} = \bigcap\{\text{prime ideal of } R \mid P \supseteq A\}$. A is called a *right* ($\sqrt{A}$-)*primary ideal* if $xRy \subseteq A$ and $y \notin \sqrt{A}$, then $x \in A$. It is easily to show that an ideal A is right primary if and only if $BC \subseteq A$ implies that $B \subseteq A$ or $C \subseteq \sqrt{A}$ for ideals B and C of R. Similarly, a *left primary ideal* is defined and an ideal which is right and left primary is called a *primary ideal*. We note that if R is a Prüfer order in Q, then an ideal A of R is right primary if and only if it is left primary and that $\sqrt{A}$ is a prime ideal if A is a primary ideal.

Assume that P is a prime ideal of a Prüfer order R and A and B are P-primary ideals of R. Then we have $AB = BA$ and it is a P-primary ideal of R. Further, if $A \subset B$, we have $\{x \in R \mid Bx \subseteq A\} = \{x \in R \mid xB \subseteq A\}$ and it is also a P-primary ideal of R. Hence we denote this primary ideal by $A : B$.

A prime ideal P of a ring R is said to be *branched* if there exists a P-primary ideal different from P. If P is the only P-primary ideal of R, then P is said to be an *unbranched* prime ideal. By using these concepts, we have the following which are concerned with prime and primary ideals.

Theorem 1. Let R be a Prüfer order in a simple Artinian ring Q with finite dimension over its center. Suppose that the center of R is a Prüfer domain. Let P be a prime ideal of R. Then

(1) If P is branched and $P \neq P^2$, then

(i) $\{P^k \mid k > 0\}$ is the full set of P-primary ideals of R and

(ii) $P_0 = \bigcap_{n=1}^{\infty} P^n$ is a prime ideal and there are no prime ideals P_1 such that $P_0 \subset P_1 \subset P$.

(2) If P is branched and $P = P^2$, then

(i) for any P-primary ideal $A(\neq P)$,

$$P_0 = \bigcap_{n=1}^{\infty} A^n = \bigcap\{A_\lambda \mid A_\lambda : P\text{-primary ideal }\},$$

(ii) P_0 is a prime ideal of R and

(iii) there are no prime ideals P_1 with $P_0 \subset P_1 \subset P$.

(3) The following are equivalent:

(i) P is branched.

(ii) There exists an ideal C of R with $\sqrt{C} = P$ and $C \neq P$.

(iii) There exists $x \in R$ such that P is a minimal prime ideal over RxR.

(iv) $P \neq \bigcup\{P_\lambda \mid P_\lambda \in \mathrm{Spec}(R) \text{ with } P_\lambda \subset P\}$.

(v) There is a prime ideal P_0 of R such that $P_0 \subset P$ and there are no prime ideals P_1 with $P_0 \subset P_1 \subset P$.

(4) P is unbranched if and only if $P = \bigcup\{P_\lambda \mid P_\lambda \in \mathrm{Spec}(R) \text{ with } P_\lambda \subset P\}$.

Next we devide P-primary ideals of a Prüfer order R into four classes, that is, if A is a P-primary ideal of R, then obviously one of the following four possibilities occurs:

I. $AP \subset A \subset A : P$,

II. $AP \subset A = A : P$,

III. $AP = A \subset A : P$,

IV. $AP = A = A : P$.

The *class* of A is meant one of these four possibilities. We shall use the notation $A \in$ III, for example, to denote that A is in class III. Note that P itself is either in class III or I, depending on whether $P^2 = P$ or $P^2 \subset P$. Furthermore, there exist no P-primary ideals A_1 such that $AP \subset A_1 \subset A$. Similarly, if $A \neq P$, then there exist no P-primary ideals A_2 such that $A \subset A_2 \subset A : P$.

For P-primary ideals A, B of R, we use the notation "$(a)\cdot(b) = (c)$" to mean that "If A is in class (a) and B is in class (b), then AB is in class (c)". We use similar notation for the operation ":" in Theorem 3. Then we have the following.

Theorem 2. Let R be a Prüfer order in a simple Artinian ring Q with finite dimension over its center and suppose that the center $Z(R)$ of R is a Prüfer domain. Then

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Theorem 3. Let R be a Prüfer order in a simple Artinian ring Q with finite dimension over its center and suppose that the center $Z(R)$ of R is a Prüfer domain. Assume that $A \subsetneq B$. Then

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REFERENCES

[1] J.H. Alajbegovic and N.I. Dubrovin: Noncommutative Prüfer rings, J. Algebra 135 (1990), 165-176.

[2] N.I. Dubrovin: Noncommutative Prüfer rings, Math. USSR Sbornik 74 (1993), 1-8.

[3] H. Marubayashi and A. Ueda: Prime and primary ideals in a strongly Prüfer order, preprint.

Department of Mathematics
Naruto University of Education
Tokushima, 772 Japan

Department of Mathematics
Shimane University
Shimane, 690 Japan

MAXIMALITY OF PBW EXTENSIONS OF ORDERS

Hidetoshi Marubayashi and Yang Zhang

Department of Mathematics

Naruto University of Education, Takashima, Japan 772

In [1], Bell and Goodearl defined a PBW extension as follows: An over ring S of a ring R is called a (finite) *Poincare-Birkhoff - Witt extension* of R (hereafter called a *PBW extension*, for short) if there exist elements $x_1, x_2, \cdots, x_n \in S$ such that

(1) The ordered monomials $x_1^{v_1} \cdots x_n^{v_n}$ (for nonnegative integers $v_1, \cdots, v_n$) form a basis for S as a free left R-module;

(2) $x_i r - r x_i \in R$ for each $i = 1, \cdots, n$ and any $r \in R$;

(3) $x_i x_j - x_j x_i \in R + Rx_1 + \cdots + Rx_n$ for all $i, j = 1, \cdots, n$.

We refer the reader to [6, 7, 8, 9, 10] for the detailed properties of v-ideals and order theory.

Throughout this paper, we fix some notation as follows: *Let R be an or-*

der in a simple Artinian ring $Q(R)$ and S be the finite PBW extension $R < x_1, x_2, \cdots, x_n >$, $T = S \cdot Q(R) = Q(R) \cdot S = Q(R) < x_1, \cdots, x_n >$.

Lemma 1. *(1) S is a prime ring.*

(2) If gr S is a Noetherian ring, then so is S.

(3) Let I be a Δ-invariant ideal of R. Then I is a Δ-prime ideal of R if and only if IS is a prime ideal of S.

(4) Let M be a regular Ore set of R. Then M is also a regular Ore set of S and $S_M = R_M \cdot S = S \cdot R_M$.

Corollary 2. *If $C = C_R(0)$ is the set of all regular elements of R, then $S_C = Q(R) \cdot S = S \cdot Q(R) = T$.*

Lemma 3. *If $f \in C_S(0)$, then there exists an element $g \in C_S(0)$ such that $g \in fS$ and the leading coefficient of g belongs to $C_R(0)$.*

Theorem 4. *If R is a maximal order in $Q(R)$, then $S = R < x_1, x_2, \cdots, x_n >$ is a maximal order in $Q(S)$.*

Remark: If $S = R < x_1, x_2, \cdots, x_n >$ is a maximal order, then R is a Δ-maximal order. It is still open whether the converse is true or not.

Lemma 5. *(1) Let I be a one-sided Δ-R-ideal, i.e., a Δ-invariant ideal and a one-sided R-ideal. Then $(R : I)_l$, $(R : I)_r$, I_v and ${}_vI$ are Δ-invariant.*

(2) Let I be a Δ-R-ideal. Then IS is an S-ideal, and $O_l(I)$ and $O_r(I)$ are also Δ-invariant.

(3) Let I be a Δ-R-ideal. Then $(IS)_v = I_vS$ and ${}_v(IS) = {}_vIS$.

Lemma 6. *(1) Suppose that R is a Noetherian Δ-maximal order in Q. Then a Δ-v-ideal of R is a maximal Δ-v-ideal if and only if it is a Δ-prime v-ideal. (2) Suppose that R is a Noetherian maximal order in Q. Then a v-ideal of S is a maximal v-ideal if and only if it is a prime v-ideal.*

Corollary 7. *Let A be any maximal Δ-v-ideal of a Noetherian maximal order R. Then AS is a maximal v-ideal of S.*

Lemma 8. *Let R be a Noetherian maximal order in $Q(R)$ and B' be a maximal v-ideal of $T = Q(R) < x_1, x_2, \cdots, x_n >$. Then $B = B' \cap S$ is a maximal v-ideal of S.*

Theorem 9. *Let R be a Noetherian maximal order in $Q(R)$. Then $\{AS, B' \cap S = B \mid A$ is a maximal Δ-v-ideal of R and B' is a maximal v-ideal of $T\}$ is the full set of all maximal v-ideals of S.*

Corollary 10. *$S = \cap_{A \in \mathcal{A}} S_{AS} \bigcap \cap_{B \in \mathcal{B}} S_{B \cap S} \bigcap S(S)$ where $\mathcal{A}$ is the set of maximal v-ideals of R and $\mathcal{B}$ is the set of maximal v-ideals of T, and $S(S) = \bigcup X^{-1}$ where X runs over all v-invertible ideals of S. Moreover, each S_{AS} and each $S_{B \cap S}$ is a local Dedekind prime ring and $S(S)$ is v-simple(i.e., has no proper v-ideals).*

References

[1] A. D. Bell and K. R. Goodearl, *Uniform rank over differential operator rings and Poincare-Birkhoff-Witt extensions*, Pac. J. Math., 131(1)

(1988), 13-37.

[2] M. Chamarie, *Anneaux de Krull non commutatifs,* Thesis, University of Lyon I, 1981.

[3] A. W. Chatters and C. R. Hajarnavis, *Rings with Chain Conditions,* Research Notes in Mathematics 44, Pitman, London, 1980.

[4] C. R. Hajarnavis and T. H. Lenagan, *Localisation in Asano orders,* J. Algebra, 21(1972), 441-449.

[5] K. Kishimoto, H. Marubayashi, and A. Ueda, *An Ore extension over a V-HC order,* Math. J. Okayama Univ., 27(1985), 107-120.

[6] J. C. McConnell and J. C. Robson, *Noncommutative Noetherian Rings,* John Wiley & Sons Ltd., 1987.

[7] H. Marubayashi, *A Krull type generalization of HNP rings with enough invertible ideals,* Comm. Algebra, 11(5)(1983), 469-499.

[8] H. Marubayashi, *A skew polynomial ring over a v-HC order with enough v-invertible ideals,* Comm. Algebra, 12(13)(1984), 1567-1593.

[9] H. Marubayashi, *Remarks on VHC-orders in a simple Artinian ring,* J. Pure Applied Algebra, 31(1984), 109-118.

[10] G. Maury and J. Raynaud, *Ordres Maximaux au sens de K. Asano,* Lecture Notes in Mathematics 808, Springer-Verlag, New York-Berlin, 1980.

Classification of Semisimple Hopf Algebras

Akira Masuoka (増岡　彰)

Beside the quantum groups, which form a class of Hopf algebras usually of infinite dimension, Hopf algebras of finite dimension are also attracting attention from many areas of research including the index theory in operator algebras and the knot theory in topology. Recently some significant structure theorems on finite-dimensional Hopf algebras were proved and hereby it becomes possible to classify semisimple Hopf algebras of small dimension, which before had seemed far out of reach. Here we describe some of the classification results, based on the author's talk at the Japan-China Conference on Ring Theory, 1995, in Okayama, Japan.

We work over a fixed field k which is always assumed to be algebraically closed.

1. Given a finite group G, one can construct immediately the following two kinds of Hopf algebras.

<u>EXAMPLE</u>. 1) the Hopf algebra k^G of all functions $G \to k$, where the product is defined point-wise and the coproduct

$$\Delta(p) = \sum_i p_i \otimes q_i,$$

where $p, p_i, q_i \in k^G$, is determined by

$$p(gh) = \sum_i p_i(g)q_i(h) \quad (g, h \in G).$$

2) the group algebra kG, where the coproduct is defined by

$$\Delta(g) = g \otimes g \quad (g \in G).$$

Note that these are (linearly) dual to each other. We call these kinds of Hopf algebras <u>trivial</u>. One classifies easily all trivial Hopf algebras, noting that, for a finite abelian group G, $k^G \simeq kG$ iff the characteristic $\operatorname{ch} k$ does not divide the order $|G|$ ('the Maschke condition').

<u>PROPOSITION</u>. Let A be a Hopf algebra of finite dimension.

1) A is commutative and semisimple iff A is of the form k^G.

2) A is cocommutative and cosemisimple (that is, the dual A^* is commutative and semisimple) iff A is of the form kG.

We may regard (co)semisimple Hopf algebras of finite dimension as non-commutative version (or quantization, in recent terms) of finite groups.

2. The first classification result was the following:

THEOREM [LR]. A semisimple Hopf algebra of odd dimension ≤ 19 is commutative and cocommutative, and hence it is of the form k^G with G some finite abelian group.

In further results we will make technical assumptions on ch k, which derive from the following result due to Larson and Radford:

THEOREM. Let A be a Hopf algebra of finite dimension. If either i) ch k = 0 or ii) dim A $\leq$ 20 and ch k does not divide dim A, then A is semisimple iff A is cosemisimple.

THEOREM [M1]. Suppose ch k $\neq$ 2, 3. Every semisimple Hopf algebra of dimension 6 is trivial.

In case of characteristic 0, this is generalized in the following result due to the author:

THEOREM. Suppose ch k = 0. Every semisimple Hopf algebra of dimension 2p with p a prime is trivial.

THEOREM [M1]. Suppose ch k $\neq$ 2. There exists only one (up to isomorphisms) non-trivial semisimple Hopf algebra of dimension 8. (We do not construct here this Hopf algebra which had been discovered previously by G. Kac and is popular among operator algebraists.)

3. We cannot expect in a simple way a Sylow type theorem for semisimple Hopf algebras. But we have the useful following:

THEOREM [Z]. Suppose ch k = 0. Let $C(A^*)$ be the subspace of A spanned by all characters of A^* (regarding $A = (A^*)^*$). Then $C(A^*)$ is a semisimple subalgebra of A and for each primitive idempotent e in $C(A^*)$, $\dim eA$ divides $\dim A$.

If $A = k^G$ (resp. $A = kG$), this means that the order of each conjugacy class in G (resp. the degree of each irreducible representation of G) divides $|G|$.

For the application of the theorem above, we suppose henceforth that $\operatorname{ch} k = 0$.

<u>THEOREM</u> [Z]. A Hopf algebra of prime dimension p is the group algebra kC_p of the cyclic group C_p.

This asserts in particular that a Hopf algebra of prime dimension is semisimple, which had been already known [LR].

In order to classify semisimple Hopf algebras of dimension of higher power of prime, we need the following:

<u>THEOREM</u> [M2]. Let A be a semisimple Hopf algebra of dimension p^n, where p is a prime and $n > 0$. There exists a group-like element $g \neq 1$ in A (a non-zero element such that $\Delta(g) = g \otimes g$) which is contained in the center $Z(A)$ of A. Hence there exists a sequence of normal Hopf subalgebras in A,

$$A = A_0 \supset A_1 \supset \ldots . \supset A_r = k$$

such that each quotient Hopf algebra A_{i-1}/A_i is isomorphic with kC_p and is included in $Z(A/A_i)$.

Hence the problem is reduced to the calculation of Hopf algebra extensions.

<u>THEOREM</u> [M2]. A semisimple Hopf algebra of dimension p^2 with p a prime is trivial.

There are examples of non-trivial (and hence non-semisimple) Hopf algebras of dimension p^2.

In order to construct non-trivial semisimple Hopf algebras of dimension p^3, let p be an odd prime and write

$$K = k^{F_p \oplus F_p} = \sum_{i,j} ke_{ij},$$

where F_p is the finite field of p elements and e_{ij} is the characteristic function (or the dual basis) of $(i,j) \in F_p \oplus F_p$.

Define an automorphism $\phi : K \to K$ by $\phi(e_{ij}) = e_{i-j,j}$.

DEFINITION [M3]. Let ζ be a p-th root of 1 in k and g a group-like element in K such that $\phi(g) = g$. Let $A_{\zeta,g}$ be a K-ring (an algebra given an algebra map from K) generated by a symbol x with relations $x^p = g$, $xc = \phi(c)x$ $(c \in K)$. We endow $A_{\zeta,g}$ with a unique Hopf algebra structure such that the canonical map $K \to A_{\zeta,g}$ (which is an injection in fact) is a Hopf algebra map and

$$\Delta(x) = \sum_{ijrs} \zeta^{jr} e_{ij}x \otimes e_{rs}x, \qquad \varepsilon(x) = 1.$$

It is shown that, if $\zeta \neq 1$, $A_{\zeta,g}$ has a remarkable property of self-duality which means $A_{\zeta,g} \simeq (A_{\zeta,g})^*$.

THEOREM [M3]. Apart from seven trivial ones, semisimple Hopf algebras of dimension p^3 with p an odd prime consist of the following p+1 non-trivial ones:

$$A_{\zeta,1},\ A_{\zeta^t,1},\ A_{\zeta,g},\ A_{\zeta^2,g},\dots,\ A_{\zeta^{p-1},g},$$

where ζ, g, t are fixed elements such that $\zeta \neq 1$, $g \neq 1$, $t \in F_p^\times \setminus (F_p^\times)^2$.

REFERENCES

[LR] R.Larson and D.Radford, Semisimple Hopf algebras, J. Algebra 171(1995), 5-35.

[M1] A.Masuoka, Semisimple Hopf algebras of dimension 6, 8, Israel J. Math., in press.

[M2] A.Masuoka, The p^n theorem for semisimple Hopf algebras, Proc. Amer. Math. Soc., in press.

[M3] A.Masuoka, Self-dual Hopf algebras of dimension p^3 obtained by extension, J. Algebra, in press.

[Z] Y.Zhu, Hopf algebras of prime dimension, Internat. Math. Res. Not., No.1(1994), 53-59.

Department of Mathematics
Shimane University (島根大学)
Matsue, Shimane 690 JAPAN

INFINITE GALOIS THEORY OF COMMUTATIVE SEMI-CONNECTED RINGS

TAKASI NAGAHARA

Department of Mathematics
Okayama University
Okayama 700, Japan

Throughout this note, A will mean a commutative ring with an identity 1, and B will mean a ring extension of A with an identity 1 which is the common identity of B and A. If A has finitely many idempotents (resp. no idempotents other than 0 and 1) then A is said to be semi-connected (resp. connected). Moreover, by $\mathrm{Aut}(B)$ (resp. $\mathrm{Aut}(B/A)$), we denote the group of all ring automorphisms of B (resp. all A-ring automorphisms of B). If the fixring of $\mathrm{Aut}(B/A)$ in B coincides with A then the extension B/A will be called to be Galois. For a subgroup H of $\mathrm{Aut}(B/A)$, the extension B/A will be called a Galois extension with a Galois group H if the fixring of H in B coincides with A. The extension B/A will be called to be separable if B is a projective $(B \otimes_A B)$-module. If B/A is separable and projective then B/A will be called to be strongly separable. Moreover, the extension B/A will be called to be locally strongly separable if for any finite subset F of B, there is an intermediate subring T of B/A containing F which is strongly separable over A. Next, we shall introduce a finite topology on $\mathrm{Aut}(B)$ in which the collection of sets of form $\{\tau \in \mathrm{Aut}(B);\ \tau(x_i) = \sigma(x_i)\}$ where $\{x_i\}$ is a finite subset of B and σ is a fixed element of $\mathrm{Aut}(B)$ is a basis of the open sets. Now, let B be semi-connected, and H a subgroup of $\mathrm{Aut}(B)$. As in [4], by H^*, we denote the set of elements σ in $\mathrm{Aut}(B)$ such that for each primitive idempotent e in B, $\sigma|Be = \tau|Be$ for some $\tau \in H$ where $\sigma|Be$ denotes the restriction of σ to Be. Obviously $H^* \supset H$. If $H = H^*$ then H will be called to be fat. As is easily seen, the fixring of H^* in B coincides to that of H in B.

In [4], O.E. Villamayor and D. Zelinsky presented a finite Galois theory of semi-connected rings such that if B is a strongly separable Galois extension of a semi-connected ring A then there is a $1-1$ correspondence between the set of strongly separable A-subalgebras of B containing A and the set of fat subgroups of $\mathrm{Aut}(B/A)$ in the usual sense of Galois theory.

In [1], one of the authors extended the theory of [4], to some non-commutative Galois extensions with finite Galois groups.

Now, if B is a locally strongly separable Galois extension of a connected ring A then, by [2], there exists some Galois correspondence between the set of locally strongly separable A-subalgebras of B containing A and a set of some groupoids.

In this note, we shall present a generalization of the theory of [4] to that of locally strongly separable Galois extension of semi-connected rings, which is a joint work of Nagahara and Narisada [3]. This is the following

Theorem *Let A be a semi-connected ring, and B a semi-connected ring which is a locally strongly separable Galois extension of A. Let $\mathfrak{T}$ be the set of intermediate rings of B/A which are locally strongly separable over A, and $\mathfrak{G}$ the set of subgroups of* $\mathrm{Aut}(B/A)$ *which are fat and closed with respect to the finite topology. Then, there is a* $1-1$ *correspondence between $\mathfrak{T}$ and $\mathfrak{G}$ in the usual sense of Galois theory. Moreover, the group* $\mathrm{Aut}(B/A)$ *is Hausdorff and compact.*

References

[1] K. Kishimoto and T. Nagahara: On G-extensions of a semi-connected rings, Math. J. Okayama Univ. **32** (1990), 25–42.

[2] A. R. Magid: The separable Galois theory of commutative rings, Marcel Dekker, New York, 1974.

[3] T. Nagahara and N. Narisada: Infinite Galois theory of commutative semi-connected rings, *to appear.*

[4] O. E. Villamayor and D. Zelinsky: Galois theory for rings with finitely many idempotents, Nagoya Math. J. **27** (1966) 721–731.

REMARKS ON GROUPS AND CENTRAL ALGEBRAS

Young Soo Park and Eun Sup Kim

Let K be a commutative ring with 1 and let K^* be the group of units of K. Suppose N is a central subgroup of a group G and α is a homomorphism of N into K^*. If $I(\alpha)$ is the ideal of the group algebra KG generated by $\{n - \alpha(n)1 \mid n \in N\}$, then we let $KG\alpha$ denote $KG/I(\alpha)$. In fact, $KG\alpha$ is the algebra obtained from KG by identifying n with $\alpha(n)$ for every $n \in N$, and $I(\alpha) = \sum\{(n - \alpha(n)1)KG \mid n \in N\}$.

In [5], Iskander showed that $KG\alpha$ is a twisted group algebra of G/N over K and every twisted group algebra may be obtained in this way. Also, he characterized a K-basis of the center of $KG\alpha$ of an integral domain K and determined when the algebras are central.

In [4], DeMeyer and Janusz obtained characterizations of group rings which are Azumaya algebras. However, the twisted group ring case does not seem to have an analogous characterization in general. Hence we have a question : when is a twisted group algebra an Azumaya algebra ?

The purpose of this note is to recall our results of a characterization of the center of twisted group algebras over a connected ring and of characterization of such central algebras. Also, we give a partial solution of the above question, which generalizes the result of DeMeyer and Janusz [4].

A commutative ring K with 1 is said to be connected if the topological space $Spec\,K$ is connected. This is equivalent the requirement that $e^2 = e, e \in K$ implies $e = 0$ or $e = 1$. Local rings and integral domains are connected.

The following three theorems are known from our results in [10].

Theorem 1. Suppose K is a connected ring, H is a subgroup of G, N is a central subgroup of G such that $N \subseteq H$, $[G:N] < \infty$ and $[G:N] \in K^*$, $\alpha : N \to K^*$ is an injective homomorphism. Let $D \subseteq G$ and $\{Nd \,|\, d \in D\}$ be a transversal for the H/N-conjugacy classes of G/N. Then the set $\{\sum cl_H(d) + I(\alpha) \,|\, d \in D,\ [d,H] \cap N = \{1\}\}$ is a basis for the centralizer of $KH\alpha$ in $KG\alpha$ as a K-module, where $cl_H(d) = \{h^{-1}dh \,|\, h \in H\}$ and $[d,H] = \{d^{-1}h^{-1}dh \,|\, h \in H\}$.

Theorem 2. Suppose K is a connected ring, N is a central subgroup of G such that $[G:N] < \infty$ and $[G:N] \in K^*$ and $\alpha : N \to K^*$ is an injective homomorphism. Then $KG\alpha$ is a central separable algebra if and $N = Z(G)$ and G is a completely central group, where $Z(G)$ denotes the center of G.

Theorem 3. Let K be a connected ring and let G be a group such that $[G : Z(G)] < \infty$ and $[G : Z(G)] \in K^*$. Then the following statements are equivalent :

(1) There is a homomorphism α of $Z(G)$ into K^* such that $Z(KG\alpha) \cong K$.

(2) There is a central subgroup M of G such that $Z(G/M) = Z(G)/M$ is embeddable into K^* and G/M is completely central.

Remark. It is known in [10] that the hypothesis of a connected ring in Theorem 1 is essential.

In the following, let F be a field and $F^\alpha G$ denote the twisted group algebra of G over F with respect to a 2-cocycle α in G.

Theorem 4. Let F and L be fields with $F \subseteq L$ and let $\alpha \in Z^2(G, F^*)$. Then $F^\alpha G$ is an Azumaya algebra if and only if $L^\alpha G$ is an Azumaya algebra.

Theorem 5. Let $G = Z(G)H$ for some subgroup H of G and let α be a symmetric cocycle in $Z(G)$. Then $F^\alpha G$ is an Azumaya algebra if and only if $F^\alpha H$ is an Azumaya algebra.

Theorem 6. If $[G:Z(G)] < \infty, [G:Z(G)]^{-1} \in F$ and α is a symmetric cocyle in $Z(G)$, then $F^{\alpha}G$ is an Azumaya algebra.

A group G is said to be hypercentral if every nontrivial factor group of G has nontrivial center. One example of a hypercentral group is of course a nilpotent group.

Theorem 7. If G is a torsion-free hypercentral group and $F^{\alpha}G$ is an Azumaya algebra, then the center $Z(G)$ has finite index in G.

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REFERENCES

[1] M. Auslander and O. Goldman, The Brauer group of a commutative ring, Trans. Amer. Math. Soc. 97(1960), 367-409.

[2] G. Azumaya, Separable rings, J. Algebra 63(1980), 1-14.

[3] F. R. DeMeyer and E. Ingraham, Separable algebras over commutative rings, Lecture Notes in Math. 181, Springer-Verlag, Berlin and New York, 1971.

[4] _______ and G. J. Janusz, Group rings which are Azumaya algebras, Trans. Amer. Math. Soc. 279(1983), 389-395.

[5] A. A. Iskander, Groups and central algebras, Illinois J. of Math. 31(1987), 1-16.

[6] G. J. Janusz, Separable algebras over commutative rings, Trans. Amer. Math. Soc. 122(1966), 461-479.

[7] G. Karpilovsky, Projective representations of finite groups, Marcel Dekker Inc., 1985.

[8] ________, Compounding block theory for projective representations of P-nilpotent groups, Comm. in Alg. (3)15(1987), 491-554.

[9] D. H. McLain, Remarks on the upper central series of a group, Proc. Glasgow Math. Assoc. 3(1956), 38-44.

[10] Y. S. Park and E. S. Kim, The center of twisted group algebras over connected rings, Math. Japonica 36(1991), 977-981.

[11] D. S. Passman, The algebraic structure of group rings, Wiley-Interscience, New York, 1977.

[12] D. J. S. Robinson, A course in the theory of groups, GTM80, Springer-Verlag, New York, 1980.

Department of Mathematics
College of Natural Sciences
Kyungpook National University
Taegu, 702-701, Korea

TILTING DECOMPOSITIONS OF SEMISIMPLE COMPLEX LIE ALGEBRAS

LIANGANG PENG AND JIE XIAO

Let A be an associative hereditary algebra over some finite field k of Dynkin type Δ. Let g be the semisimple complex Lie algebra of type Δ with triangular decomposition $\mathbf{g}=\mathbf{n}_-\oplus\mathbf{h}\oplus\mathbf{n}_+$. C. M. Ringel in [1] showed that A, as a directed algebra, has Hall polynomials. So he could make the free abelian group $\oplus_{[M]}\mathbf{Z}u_{[M]}$, indexed by the set of isomorphism classes $[M]$ of all finitely generated A-modules M, into an associative ring by defining:

$$u_{[M]}u_{[N]}=\sum_{[L]}\phi^L_{MN}(1)u_{[L]},$$

where all ϕ^L_{MN} are the Hall polynomials; this ring is called the degenerate Hall algebra of A and denoted by $H_1(A)$. Let $K(\text{mod } A)$ be the Grothendieck group of mod A, the category of all finitely generated (right) A - modules, modulo split exact sequences. Then K (mod A) can be identifed with the subgroup $\oplus_{[X]}\mathbf{Z}u_{[X]}$ of $H_1(A)$, indexed by the set of isomorphhism classes $[X]$ of all indecomposable modules in mod A. He showed that K (mod A) is a Lie subalgebra of $H_1(A)$, and furthermore he has proved in [2] that K (mod A) $\otimes_{\mathbf{Z}}\mathbf{C}$ is isomorphic to $\mathbf{n}_+$ as Lie algebras. So we can think the triangular decomposition of g as the hereditary decomposition.

Let B be a tilted algebra of type Δ. Similarly, we have the degenerate Hall algebra $H_1(B)$ of B and K (mod B) is a Lie subalgebra of $H_1(B)$. In this note, we show that g has another decomposition $\mathbf{g}=\mathbf{a}_-\oplus\mathbf{b}_-\oplus\mathbf{h}\oplus\mathbf{b}_+\oplus\mathbf{a}_+$ with $\mathbf{a}_+\simeq K$ (mod B) $\otimes_{\mathbf{Z}}\mathbf{C}$ as Lie algebras, where $\mathbf{h}$ is the Cartan subalgebra. We can think it as the tilting decomposition of g.

References

[1] Ringel, C. M., Hall algabras, In: Topics in Algebra, Banach Centre. Publ. 26. Part 1(1990)433—447.

[2] Ringel, C. M., Hall polynmials for the representation — finite hereditary algebras, Adv. Math. 84(1990) 137—178.

Liangang Peng
Department of Mathematics,
Sichuan University,
610064 Chengdu, P. R. China.

Jie Xiao
Department of Mathematics,
Beijing Normal University,
100875 Beijing, P. R. China.

GENERALIZED BAER MODULES

Seog Hoon Rim and Mark L. Teply

In 1936, R. Baer [1] raised the question of determining all abelian groups B such that $Ext_Z(B,T) = 0$ for all torsion groups T; i.e., all groups B such that, for every torsion abelian group T, andy exact sequence

$$0 \to T \to X \to B \to 0$$

splits. Baer then showed that if such a B is countable, then B is free. Much later, P. Griffith [5] removed the hyothesis that B is countable from Baer's theorem. More generally, I. Kaplansky [6] proposed characterizing the modules B over a commutative integral domain R such that $Ext_R(B,T) = 0$ for all torsion R-modules T. Modules satisfying this condition are called Baer modules.

In [2], Fuchs and Viljoen described the modules B over a valuation domain R such that $Ext_R(B,X) = 0$ for all bounded torsion and all divisible modules X. This weak form of Baer's splitting problem was considered in [7], [8], [9] for arbitrary torsion theories over an associative ring. As in the valuation ring case, modules playing the role of B in the *"Ext condition"* above are called B^*-modules. (A precise definition is given later.) Under the hypothesis that τ is of finite type (i.e., the filter associated with τ has a cofinal subset of finitely generated left ideals), results in [7] (and [8] gave characterizations of torsion theories τ whose τ-torsionfree mudules are (flat) B^*-modules. The main purpose of this note is to prove a result (Theorem 2) that allows us to remove the restrictive overall hypothesis that τ is of finite type from all the main results of [7] and [8].

For the basic properties of τ and other torsion theoretic terms used in this note, see Golan [3].

Recall that a left R-module E is called *τ-injective* if $Ext_R(T,E) = 0$ for each τ-torsion module T. As in [9], a module D is called τ-divisible if D is a homomorphic image of a direct sum of τ-injective modules. A module M is called a *generalized Baer* *module* if $Ext_R(M, D) = 0$ for each τ-divisible module D. A module M is said to have *τ-bounded order* if M is a submodule of a module N with a set of generators annihilated by a left ideal I of R such that $\tau(R/I) = R/I$. A module M is called B^*-module if $Ext_R(M, X) = 0$ for each τ-divisible X and each X with τ-bounded order.

We use $hd_R M$ to denote the homological dimension of a left R-module M.

Before stating our main result, we need the following minor generalization of [9, Lemma 2.6].

Lemma 1. *If a Q_τ-module B is a generalized Baer module, then $Q_\tau \otimes_R B \cong B$ and B is a projective Q_τ-module.*

The following is the main Theorem :

Theorem 2. *If every τ-torsionfree Q_τ-module is a generalized Baer module, then τ is a perfect torsion theory and Q_τ is a semisimple artinian ring.*

In [7] the question, "When is every τ-torsionfree module a B^*-module ?" is considered. Similarly, in [8] the question, "When is every τ-torsionfree module a flat B^*-module?" is studied. These questions are answered under the hypothesis that τ is of finite type. The answers to these questions show that τ must be closely related to the Goldie torsion theory τ_g; the τ_g-torsionfree modules are precisely the nonsingular modules. The finiteness property of τ is used to prove the following key lemma of [7]:

[7, Lemma 4]. Let τ be of finite type. If every τ-torsionfree module is a B^*-module, then Q_τ is a semisimple artinian ring and τ induces the Goldie torsion theory on $R/\tau(R)$-*mod.*

When Q_τ is semisimple and τ is perfect, then τ automatically induces the Goldie torsion theory on $R/\tau(R)$-*mod.* Hence Theorem 2 shows that [7, Lemma 4] is true without the hypothesis that τ is of finite type. Since [7, Lemma 4] is the only source of the use of the hypothesis that τ is of finite type throughout [7] and [8], all of the main results of [7] and [8] are true without the assumption that τ is of finite type. (In results on the Goldie theory, such as [7, Proposition 11 and Theorem 7] or [8, Theorem

10], this means that the overall hypothesis that R has finite left uniform dimension is not needed.)

Example 3. Let **Z** denote the integers, **Q** the rational numbers, and **R** the real numbers. Consider the matrix ring

$$R = \begin{pmatrix} \mathbf{Z} & \mathbf{R}[x] \\ 0 & \mathbf{Q} \end{pmatrix}.$$

The old versions of the results in [7] and [8] do not apply to Goldie torsion theory for R, as R does not have finite left uniform dimension. But since R has many properties similar to the matrix rings in [7, Theorem 18] and [8, Theorem 14], one might have wondered if every τ_g-torsionfree R-module is a B^*-module. Our Theorem 2 shows immediately that this is not the case.

In addition to generalizing results from [7] and [8], we illustrate the use of Theorem 2 with the following application.

Corollary 4. *If $\tau(R) = 0$, the following statements are equivalent:*

(1) *Every τ-torsionfree Q_τ-module is a generalized Baer module,*

(2) *Every Q_τ-module is a generalized Baer module,*

(3) $hd_R Q_\tau \leq 1$ *and Q_τ is a semisimple artinian ring.*

REFERENCES

[1] R. Baer, The subgroup of the elements of finite order of an abelian group, Ann. Math. 37(1936), 766-781.

[2] L. Fuchs and G. Viljoen, A Weaker Form of Baer's Splitting Problem for Valuation Rings, Quaestiones Math. 14(1991), 227-236.

[3] J. S. Golan, Torsion Theories, Pitman Monographs 29, Longman Scientific and Technical/ John Wiley, New York, 1986.

[4] K. R. Goodearl, Singular Torsion and the Splitting Properties, Mem. Amer. Math. Soc. 124(1972).

[5] P. Griffith, A solution to the splitting mixed problem of Baer, Trans. Amer. Math. Soc. 139(1969), 261-269.

[6] I. Kaplansky, The Splitting of Modules over Integral Domains, Archiv. der Math. 13(1962), 341-343.

[7] M. L. Teply and B. D. Redman, Jr., Torsionfree B^*-modules, Ring Theory: Proceedings of the 21st Ohio State/Denison Conference, World Scientific Publishers, Singapore, 1993, pp.314-328.

[8] M. L. Teply and B. D. Redman, Jr., Flat Torsionfree B^*-modules (to appear).

[9] M. L. Teply and B. Torrecillas, A Weaker Form of Baer's Splitting Problem for Torsion Theories, Czech. J. Math. 43(118) (1993), 663-674.

Department of Mathematics Education
Kyungpook National University
Taegu, 702-701, KOREA

Department of Mathematics
University of Wisconsin
Milwaukee, WI 53201, U.S.A.

GLOBAL DIMENSION OF ENDOMORPHISM RING

MASAHISA SATO

1. PUROPOSE

Let R be a semi primary ring and J its Jacobson radical with nilpotency n. We define

$$A = \mathrm{End}_R(R/J \oplus R/J^2 \oplus \cdots \oplus R/J^{n-1} \oplus R/J^n)$$

Then it holds

$$\mathrm{gl.dim}(A) \leq n.$$

We explain this result by giving the projective resolution of simple A-modules.

By V. Dlab and C.M. Ringel [2], it is proved that A is an quasi-hereditary ring, so it has been known that A has finite global dimension. So our result gives the best upper bound. In the case that R is an artin algebra, Auslander [1] proved this result.

2. NOTATIONS

We put a decomposition of R into indecomposable projective right modules as $R = \sum_{s=1}^{n}\sum_{t=1}^{n_s} \oplus d_{st}R$. Here we assume that d_{st}'s are primitive idenpotents and the lowey length of $d_{st}R$ is s for any t. Also we put $d_i = \sum_{i=s}^{n}\sum_{t=1}^{n_s} d_{st}$ $(1 \leq i \leq n)$, $R_j = R/J^j$ and $J_j = J/J^j$, then $R_j = \sum_{s=1}^{j}\sum_{t=1}^{n_s} \oplus d_{st}R \oplus \sum_{s=j}^{n}\sum_{t=1}^{n_s} \oplus d_{st}R_j$. For any set I and K of R, we denote $(I, K) = \{x \in R \mid xI \subset K\}$, also for a bimodule ${}_RM_R$ and $a \in R$, we denote $O(Ma; I) = \{x \in Ma \mid xI = 0\}$. Further we define for $1 \leq i \leq n$, $\mathrm{Soc}^1(R) = {}_R\mathrm{Soc}(R)$, $\mathrm{Soc}(R/\mathrm{Soc}^i(R)) = \mathrm{Soc}^{i+1}(R)/\mathrm{Soc}^i(R)$, then we have $\mathrm{Soc}^j(R_i) = (J^j : J^i)/J^i$ for any $j \leq i$. For a matrix ring $A = (A_{ij})$, $A_{ij}(x)$ $(x \in A_{ij})$ is a matrix whose (i,j)-component is x and other components are 0. For $X \subseteq A_{ij}$, we denote $A_{ij}(X) = \{A_{ij}(x) \mid x \in X\}$

3. BASIC FORM OF A

The above A is not basic, in fact the matrix form of A is given by the following;

$$A \cong \begin{bmatrix} R_1 & R_1 & \cdots & \cdots & R_1 & R_1 \\ A_{21} & R_2 & \cdots & \cdots & R_2 & R_2 \\ A_{31} & A_{32} & R_3 & \cdots & R_3 & R_3 \\ \cdots & \cdots & \ddots & \ddots & \cdots & \cdots \\ A_{n-1,1} & A_{n-1,2} & \cdots & A_{n-1,n-2} & R_{n-1} & R_{n-1} \\ A_{n,1} & A_{n,2} & \cdots & A_{n,n-2} & A_{n,n-1} & R_n \end{bmatrix}.$$

Here it is not necessary to specify A_{ij}.

[Basic formular] We may assume R is basic since isomorphic indecomposable projective R-modules Re_{st} and $Re_{st'}$ make isomorphic indecomopsable projective A-modules $AA_{ij}(e_{st})$ and $AA_{ij}(e_{st'})$. The basic ring of A is given by $\mathrm{End}(\sum_{i=1}^{n} \oplus d_i R_i)$. Its matrix form is given by

$$\begin{bmatrix} d_1 R_1 d_1 & d_1 R_1 d_2 & \cdots & \cdots & d_1 R_1 d_{n-1} & d_1 R_1 d_n \\ d_2(J/J^2)d_1 & d_2 R_2 d_2 & \cdots & \cdots & d_2 R_2 d_{n-1} & d_2 R_2 d_n \\ d_3 \mathrm{Soc}^1(R_3) d_1 & d_3(J/J^3)d_2 & d_3 R_3 d_3 & \cdots & d_3 R_3 d_{n-1} & d_3 R_3 d_n \\ \cdots & \cdots & \ddots & \ddots & \cdots & \cdots \\ d_{n-1} \mathrm{Soc}^1(R_{n-1}) d_1 & d_{n-1} \mathrm{Soc}^2(R_{n-1}) d_2 & \cdots & d_{n-1}(J/J^{n-1})d_{n-2} & d_{n-1} R_{n-1} d_{n-1} & d_{n-1} R_{n-1} d_n \\ d_n \mathrm{Soc}^1(R_n) d_1 & d_n \mathrm{Soc}^2(R_n) d_2 & \cdots & d_n \mathrm{Soc}^{n-2}(R_n) d_{n-2} & d_n(J/J^n)d_{n-1} & d_n R_n d_n \end{bmatrix}.$$

4. Elementary properties of A

In the following we denote A the basic ring given in section 2. For $1 \le j \le n$, we put $e(j,s,t) = A_{jj}(d_{st} + d_j J^j d_j)$ for $1 \le s \le n,\ 1 \le t \le n_s$ and $e_j = \sum_{s=j}^{n} A_{jj}(d_s + d_j J^j d_j)$, $e_{j1} = \sum_{s=j}^{n-1} A_{jj}(d_s + d_j J^j d_j)$, $e_{j2} = A_{jj}(d_n + d_j J^j d_j)$ and $S(j,s,t) = Ae(j,s,t)/J(A)e(j,s,t)$ for $j \le s \le n,\ 1 \le t \le n_s$.

The structure of A is given by the following proposition.

Proposition 4.1. *It holds the following properties for $T = \mathrm{Rad}(A)$.*

(1) *For $1 \le i \le s \le n$ $1 \le t \le n_s$,*

$$e(i,s,t)(A/T)e(i,s,t) \cong d_{ist}(R/J)d_{ist}$$

as a division ring. Particularly $e_i(A/T)e_i \cong d_i(R/J)d_i$ as ring.

(2) *$e_n(T/T^2)e_{n-1} \cong d_n(J/(J^{n-2} : 0))d_{n-1}$ through the above ring isomorphism.*

(3) *$e_j T e_n = e_n A e_{n-1} A e_n$ for any $j = 1, \ldots, n$.*

(4) *For $1 \le j < k \le n$, $e_k(T/T^2)e_j \cong J_{kj}$. Here we denote δ_{ij} is a Kronecker delta and J_{kj} is*

$$d_k \left((J^j : J^k)/(\delta_{nk} - 1)(J^j : J^{k+1}) + \delta_{nk}(J^{j-1} : J^k) + \sum_{i=j+1}^{k-1} (J^i : J^k) d_i (J^j : J^i) \right) d_j$$

(5) $e_{n-1}(T/T^2)e_n \cong d_{n-1}(R/J)d_n = d_n(R/J)d_n$.

We make the endomophism ring $\overline{A} = \mathrm{End}(\sum_{i=1}^{n-1} \oplus d_i R_i / J^{n-1})$ from R/J^{n-1}. We compare two projective resolutions of simple A-modules and simple $\overline{A}$-modules.

Proposition 4.2. *Let ${}_{\overline{A}}P_1 \xrightarrow{f_1} {}_{\overline{A}}P_0 = \overline{A}e(i,s,t) \xrightarrow{f_0} S \longrightarrow 0$ be a minimal presentation of simple $\overline{A}$- module S and $A \otimes_{\overline{A}} S \xrightarrow{\epsilon} A \otimes_{\overline{A}} S/\mathrm{Rad}(A \otimes_{\overline{A}} S) = S(i,s,t) \longrightarrow 0$ a canonical map. We put $N = Ae_n \mathrm{Rad}(A \otimes_{\overline{A}} P_0)$ and $M = \mathrm{Im}(A \otimes_{\overline{A}} f_1)$. Then*

(1) $\ker(\epsilon \cdot A \otimes_{\overline{A}} f_0) = \mathrm{Rad}(A \otimes_{\overline{A}} P_0) = M + N$.

(2) *$N/\mathrm{Rad}(N)$ is a homomorphic image of Ae_n.*

(3) $\mathrm{Rad}(N) \subset M$.

(4) *Let $h : P \to \mathrm{Rad}(A \otimes_{\overline{A}} P_0)$ be a projective cover. Then*

 (a) $\ker(h) = \ker(A \otimes_{\overline{A}} f_1) \oplus \sum_{t=1}^{n_n} \oplus Ae(n-1, n, t)^{h_t}$ *for some h_t.*

(b) $A \otimes_{\overline{A}} P_1 \subset P$.

(c) *There is a direct summand* P' *of* $A \otimes_{\overline{A}} P_1$ *such that*

$P = P' \oplus \sum_{t=1}^{n_n} \oplus Ae(n,n,t)^{h_t}$.

(5) *There is* $M' \subset \ker(\epsilon \cdot A \otimes_{\overline{A}} f_0)$ *such that* $\ker(\epsilon \cdot A \otimes_{\overline{A}} f_0) = M' \oplus \sum_{t=1}^{n_n} \oplus S(n,n,t)^{h_t}$ *and* $M' \subset M$.

Proposition 4.3. *Let* ${}_{\overline{A}}P_2 \xrightarrow{f_1} {}_{\overline{A}}P_1 \xrightarrow{f_0} {}_{\overline{A}}P_0$ *be a minimal exact sequence of projective* $\overline{A}$*-modules. We put* $N = Ae_n \ker(A \otimes_{\overline{A}} f_0)$ *and* $M = \mathrm{Im}(A \otimes_{\overline{A}} f_1)$. *Then*

(1) $\ker(A \otimes_{\overline{A}} f_0) = M + N$.

(2) $N/\mathrm{Rad}(N)$ *is a homomorphic image of* Ae_n.

(3) $\mathrm{Rad}(N) \subset M$.

(4) *Let* $h : P \to \ker(A \otimes_{\overline{A}} f_0)$ *be a projective cover. Then*

(a) $\ker(h) = \ker(A \otimes_{\overline{A}} f_1) \oplus \sum_{t=1}^{n_n} \oplus Ae(n-1,n,t)^{h_t}$ *for some* h_t.

(b) $A \otimes_{\overline{A}} P_1 \subset P$.

(c) *There is a direct summand* P' *of* $A \otimes_{\overline{A}} P_1$ *such that*

$P = P' \oplus \sum_{t=1}^{n_n} \oplus Ae(n,n,t)^{h_t}$.

(5) *There is* $M' \subset \ker(A \otimes_{\overline{A}} f_0)$ *such that* $\ker(A \otimes_{\overline{A}} f_0) = M' \oplus \sum_{t=1}^{n_n} \oplus S(n,n,t)^{h_t}$ *and* $M' \subset M$.

Corollary 4.4. *Let* $\cdots \to {}_AQ_j \xrightarrow{g_j} \cdots \xrightarrow{g_2} {}_AQ_1 \xrightarrow{g_1} {}_AQ_0 \xrightarrow{g_0} S(i,s,t) \to 0$ *be a minimal projective resolutions of a simple* A*-module* $S(i,s,t)$. *Then it hols that*

(1) $\ker(g_i) = \ker(A \otimes_{\overline{A}} f_i) \oplus \sum_{t=1}^{n_n} \oplus S(n,n,t)^{p_{it}}$ *for some* p_{it}.

(2) $\ker(A \otimes_{\overline{A}} f_i) = \mathrm{Im}(A \otimes_{\overline{A}} f_i) \oplus \sum_{t=1}^{n_n} \oplus S(n,n,t)^{p_{i-1,t}}$.

(3) $A \otimes_{\overline{A}} P_i \subset Q_i$ *for each* i.

(4) *There is a direct summand* Q_i' *of* $A \otimes_{\overline{A}} P_i$ *such that* $Q_i = Q_i' \oplus \sum_{t=1}^{n_n} \oplus Ae(n,n,t)^{p_{i-1,t}}$.

(5) *If radical series of* R *and socle series of* R *coincide, then* $Q_i = A \otimes_{\overline{A}} P_i$.

5. Projective dimension of simple modules

From the calculation in the former section, we have the following results.

Proposition 5.1. *If* R *is not semi-simple, then* $\mathrm{p.d.}(S(n,n,t)) = 1$ *for any* $1 \le t \le n_s$. *i.e., there is an exact sequence* $0 \to Ae(n-1,n,t) \to Ae(n,n,t) \to S(n,n,t) \to 0$.

Theorem 5.2. *It holds that for each* $1 \le i \le n$, $i \le s \le n$, $1 \le t \le n_s$,
$\mathrm{p.d.}\, S(i,s,t) \le n - i + 1$ *and* $\mathrm{gl.dim}(A) \le \mathrm{gl.dim}(\overline{A}) + 1$.
Particularlu $\mathrm{gl.dim}(A) \le n$.

Corollary 5.3. *The following properties are equivalent.*

(1) $\mathrm{gl.dim}(A) = n$.

(2) $\mathrm{p.d.}(S(1,s,t)) = n$ *for some* s, t.

(3) $\operatorname{Ext}_A^{n-1}(S(1,s,t), S(n,n,t')) \neq 0$ *for some* s, t, t'.
(4) $\operatorname{Ext}_A^{n}(S(1,s,t), S(n-1,s',t')) \neq 0$ *for some* s, t, s', t'.

References

1. M. Auslander, *Represenattion theory of artin algebras* i, Comm. in Algebra 1 (1974), 177–258.
2. V. Dlab and C.M. Ringel, *Every semiprimary ring is the endomorphism ring of a projective module over a quasi-hereditary rings*, Proc. Amer. Math. Soc. 107 (1989), 1–5.
3. S.O. Smalφ, *Global dimension of special endomorphism rings over artin algebras*, Illinois J. Math. 22(3) (1978), 414–427.
4. ———, *Structure of special endomorphism rings over artin algebras*, Illinois J. Math. 22(3) (1978), 428–442.

Department of Mathematics
Faculty of Education
Yamanashi University
Kofu, Yamanashi 400 JAPAN
E-mail address: sato@kkb.yamanashi.ac.jp

Enumeration of Finite Rings

Takao SUMIYAMA (Aichi Institute of Technology)

Yasuyuki HIRANO (Okayama University)

By a finite ring we mean an associative ring consisting of only finitely many elements. For a finite ring S , let $|S|$ denote the number of elements of S , which is called the order of S . For a ring R , let $J(R)$ denote the Jacobson radical of R .

As is easily seen, a finite ring is the direct sum of finite rings of prime-power orders. So, we consider only finite rings of prime-power order.

A ring R is called directly indecomposable if there exists no nontrivial ideal decomposition $R = I_1 \oplus I_2$.

In [1], we have proved that, if R is a directly indecomposable finite ring with identity whose Jacobson radical consists of p^m elements, where p is a prime and m is a positive integer, then $|R|$ satisfies the inequality

$p^{m+1} \leq |R| \leq p^{m^2+m+1}$.

Concerning this result, we are interested in the problem what kind of finite rings actually exist in this range.

When p is a prime, $C(p^e)$ denotes the finite cyclic group of order p^e . Let $\langle a \rangle$ denote the cyclic group generated by a . An Abelian group A is called of type $(p^{f_1}, n_1)(p^{f_2}, n_2)\cdots(p^{f_t}, n_t)$, where $n_1, n_2, \cdots, n_t$ and $f_1 < f_2 < \cdots < f_t$ are positive integers, if A is isomorphic to the direct sum of n_1 copies of $C(p^{f_1})$, n_2 copies of $C(p^{f_2}), \cdots$, and n_t copies of $C(p^{f_t})$.

When R is a ring, R^+ denotes the additive group of R , $J(R)$ the Jacobson radical of R , and $(R)_{n\times n}$ the ring of all $n \times n$ matrices having entries in R .

Let $e_1 \leq e_2 \leq \cdots \leq e_n$ be a nondecreasing sequence of positive integers. Let $S_n = \{(a_{ij}) \in \mathbf{Z}_{n\times n} \mid a_{ij} \equiv 0 \mod p^{e_j - e_i} \text{ for } i < j\}$. It is easy to check that S_n is a subring of $(\mathbf{Z})_{n\times n}$.

Let $\pi : S_n \longrightarrow (\mathbf{Z}/p\mathbf{Z})_{n\times n}$ be the natural homomorphism given by $(a_{ij}) \longmapsto (\bar{a}_{ij})$. An element (a_{ij}) of S_n is called to be non-singular if $\det \pi(a_{ij}) \neq 0$ (in $\mathbf{Z}/p\mathbf{Z}$).

Theorem 1. ([3, Satz 2])

Let R be a finite p-ring whose additive group is

$R^+ = \langle a_1 \rangle \oplus \langle a_2 \rangle \oplus \cdots \oplus \langle a_n \rangle$,

where $\langle a_i \rangle \cong C(p^{e_i})$ $(1 \leq i \leq n)$ and $1 \leq e_1 \leq e_2 \leq \cdots \leq e_n$. Let us write

(1) $\quad a_i a_k = \sum_{j=1}^{n} \alpha_{ijk} a_j \quad (1 \leq i, k \leq n)$,

where α_{ijk} are integers such that

(2) $\quad 0 \leq \alpha_{ijk} \leq p^{e_j} - 1 \quad (1 \leq i, j, k \leq n)$.

Then it holds that

(3) $\quad \alpha_{ijk} \equiv 0 \mod p^{e_j - e_k}$ for $1 \leq k < j \leq n$,

(4) $\quad \alpha_{ijk} \equiv 0 \mod p^{e_j - e_i}$ for $1 \leq i < j \leq n$,

and

(5) $\quad \sum_{k=1}^{n} \alpha_{rki} \alpha_{kjs} \equiv \sum_{k=1}^{n} \alpha_{iks} \alpha_{rjk} \mod p^{e_j} \quad (1 \leq i, j, r, s \leq n)$.

Conversely, let

(6) $\quad A = \langle a_1 \rangle \oplus \langle a_2 \rangle \oplus \cdots \oplus \langle a_n \rangle$

$(\langle a_i \rangle \cong C(p^{e_i}), 1 \leq e_1 \leq e_2 \leq \cdots \leq e_n)$

be a finite Abelian p-group. If α_{ijk} $(1 \leq i, j, k \leq n)$ are integers which satisfy (2), (3), (4) and (5), then we can make A into a ring by defining the multiplication on A by (1). By this manner we can construct all rings which have the Abelian group A as their additive group.

By Theorem 1, for a given prime power p^e, we can get all rings of order p^e, since a ring of order p^e has the additive group

$R^+ \cong C(p^{e_1}) \bigoplus C(p^{e_2}) \bigoplus \cdots \bigoplus C(p^{e_n}) \quad (e_1 + e_2 + \cdots + e_n = e)$.

When α_{ijk} $(1 \leq i, j, k \leq n)$ are integers which satisfy (2), (3), (4) and (5), we call $\{\alpha_{ijk}\}_{i,j,k=1}^{n}$ a set of structure constants for the Abelian group (6).

Let $\{\alpha_{ijk}\}_{i,j,k=1}^{n}$ and $\{\beta_{ijk}\}_{i,j,k=1}^{n}$ be two sets of structure constants for the Abelian group (6). We shall say that $\{\alpha_{ijk}\}_{i,j,k=1}^{n}$ and $\{\beta_{ijk}\}_{i,j,k=1}^{n}$ are equivalent if there exists a non-singular element $(t_{ij}) \in S_n$ such that

$\sum_{j=1}^{n} \beta_{ijk} t_{js} \equiv \sum_{j=1}^{n} \sum_{r=1}^{n} t_{ij} t_{kr} \alpha_{jsr} \mod p^{e_s} \quad (1 \leq i, k, s \leq n)$.

Theorem 2. ([3, Satz 5])

Let $\{\alpha_{ijk}\}_{i,j,k=1}^{n}$ and $\{\beta_{ijk}\}_{i,j,k=1}^{n}$ be two sets of structure constants for the Abelian group (6). Let R be the ring whose additive group is (6) and whose multiplication is defined by

$a_i a_k = \sum_{j=1}^{n} \alpha_{ijk} a_j \quad (1 \leq i, k \leq n)$.

Let R' be the ring whose additive group is (6) and whose multiplication $\circ$ is defined by

$a_i \circ a_k = \sum_{j=1}^{n} \beta_{ijk} a_j \quad (1 \le i, k \le n)$.
Then R and R' are isomorphic if and only if $\{\alpha_{ijk}\}_{i,j,k=1}^{n}$ and $\{\beta_{ijk}\}_{i,j,k=1}^{n}$ are equivalent.

Let $\{\alpha_{ijk}\}_{i,j,k=1}^{n}$ be a set of structure constants for the Abelian group (6). We shall say that $\{\alpha_{ijk}\}_{i,j,k=1}^{n}$ is decomposable if there exists a partition $\{1, 2, \cdots, n\}$ $J_1 \cup J_2$ such that (i) $J_1 \cap J_2 = \phi$, (ii) $J_1 \ne \phi, J_2 \ne \phi$, (iii) if $i \in J_1, j \in J_2$ and $e_i = e_j$, then $i < j$, and (iv) if, $i \in J_1$ and $j \in J_2$, or, $i \in J_2$ and $j \in J_1$, or, $j \in J_1$ and $k \in J_2$,or, $j \in J_2$ and $k \in J_1$, then $\alpha_{ijk} = 0$. By the following theorem, we can see whether a ring with given structure constants is indecomposable or not.

Theorem 3. ([2, Theorem 3])

Let $\{\alpha_{ijk}\}_{i,j,k=1}^{n}$ be a set of structure constants for the Abelian group (6). Let R be the ring whose additive group is (6) and whose multiplication is defined by

$a_i a_k = \sum_{j=1}^{n} \alpha_{ijk} a_j \quad (1 \le i, k \le n)$.
Then R is indecomposable if and only if there exists no set of structure constants for the Abelian group (6) which is decomposable and equivalent to $\{\alpha_{ijk}\}_{i,j,k=1}^{n}$.

Theorem 4. ([2, Theorem 4])

Let $\{\alpha_{ijk}\}_{i,j,k=1}^{n}$ be a set of structure constants for the Abelian group (6). Let R be the ring whose additive group is (6) and whose multiplication is defined by

$a_i a_k = \sum_{j=1}^{n} \alpha_{ijk} a_j \quad (1 \le i, k \le n)$.
Then:

(I) R has a left (right) identity if and only if there exist integers $c_1, c_2, \cdots, c_n$ such that $0 \le c_i \le p^{e_i} - 1 \quad (1 \le i \le n)$ and
$\sum_{i=1}^{n} c_i \alpha_{ijk} \equiv \delta_{jk}$ (resp. $\sum_{i=1}^{n} c_i \alpha_{kji} \equiv \delta_{jk}$) mod p^{e_j}
$(1 \le j, k \le n)$.

(II) R has an identity if and only if there exist integers $c_1, c_2, \cdots, c_n$ such that $0 \le c_i \le p^{e_i - 1} \quad (1 \le i \le n)$ and
$\sum_{i=1}^{n} c_i \alpha_{ijk} \equiv \sum_{i=1}^{n} c_i \alpha_{kji} \equiv \delta_{jk}$ mod p^{e_j}
$(1 \le j, k \le n)$.

Theorem 5. ([2, Theorem 5])

Let $\{\alpha_{ijk}\}_{i,j,k=1}^{n}$ be a set of structure constants for the Abelian group (6). Let R be the ring whose additive group is (6) and whose multiplication is defined by

$a_i a_k = \sum_{j=1}^{n} \alpha_{ijk} a_j \quad (1 \leq i, k \leq n)$.

Then, $b = \sum_{i=1}^{n} u_i a_i \quad (0 \leq u_i \leq p^{e_i} - 1, 1 \leq i \leq n)$ belongs to $J(R)$ if and only if, for any n integers $x_1, x_2, \cdots, x_n$ satisfying $0 \leq x_i \leq p^{e_i} - 1 \quad (1 \leq i \leq n)$, there exist n integers $y_1, y_2, \cdots, y_n$ such that $0 \leq y_i \leq p^{e_i} - 1 \quad (1 \leq i \leq n)$ and

$$\sum_{i,j=1}^{n} u_i x_j \alpha_{irj} + y_r - \sum_{i,j,k,t=1}^{n} u_i x_j \alpha_{ikj} y_t \alpha_{krt} \equiv 0 \mod p^{e_r}$$
$(1 \leq r \leq n)$.

By making use of above results, we completed a program which determines, for a given prime-power p^n, all finite rings of order p^n, determines decomposability and existence of identity elements for them, and counts the orders of Jacobson radicals of them.

References

[1] Hirano, Y. and Sumiyama, T., On orders of directly indecomposable finite rings, Bulletin of the Australian Mathematical Society 46 (1992), No. 3, 353 - 359.

[2] Sumiyama, T., On algorisms for finite rings (to appear in Mathematica Japonica, vol. 44, No. 2)

[3] Wiesenbauer, J., Über die endlichen Ringe mit gegebener additiver Gruppe Monatshefte für Mathematik 78 (1974), 164 - 173.

COCYCLE DEFORMATIONS OF BIALGEBRAS AND HOPF ALGEBRAS

Mitsuhiro TAKEUCHI

We work over a field k. Let A be a bialgebra over k with comultiplication Δ and counit ε. We use the usual sigma notation: $\Delta(a) = \sum a_1 \otimes a_2$, $(\Delta \otimes I)\Delta(a) = \sum a_1 \otimes a_2 \otimes a_3$, etc., a in A.

Let $\sigma: A \times A \to k$ be a bilinear form. We may consider σ as an element in the dual algebra $(A \otimes A)^*$ of $A \otimes A$. It is called a 2-cocycle on A if

(1) σ is invertible in $(A \otimes A)^*$,

(2) $\sum \sigma(x_1,y_1)\sigma(x_2y_2,z) = \sum \sigma(y_1,z_1)\sigma(x,y_2z_2)$, x, y, z in A,

(3) $\sigma(1,x) = \varepsilon(x) = \sigma(x,1)$, x in A.

If this is the case, the following multiplication

$$a \cdot b = \sum \sigma(a_1,b_1)a_2b_2\sigma^{-1}(a_3,b_3), \quad a,\ b \text{ in } A$$

makes a new bialgebra A^σ together with the original unit and coalgebra structure. If A has antipode, then so does A^σ. The cocycle deformation A^σ was introduced by Doi [1] and studied in [2] in detail.

Example 1. Let H be a finite dimensional Hopf algebra and $A = H^{*cop} \otimes H$, where $(\ )^{cop}$ means to make the coalgebra structure opposite. Define

$$\sigma(a \otimes x, b \otimes y) = \langle a,1\rangle\langle b,x\rangle\langle\varepsilon,y\rangle, \quad a,\ b \text{ in } H^{*cop},\ x,\ y \text{ in } H.$$

Then σ is a 2-cocycle on A and the cocycle deformation A^σ

coincides with Drinfeld's quantum double D(H) [2].

The first example suggests that the idea of cocycle deformation would possibly play some important role in quantum group theory. Here is a motivation for our study.

Given a bialgebra or a Hopf algebra, it is interesting to know whether essentially new bialgebras or Hopf algebras are obtained as its cocycle deformations. In the following two cases, we have a negative answer in contrast with the previous case.

Example 2. Let H_4 be Sweedler's 4-dimensional Hopf algebra. It is defined by two generators x, y and the following relations:

$$x^2 = 1,\ y^2 = 0,\ xy + yx = 0.$$

The comultiplication is given by

$$\Delta(x) = x \otimes x,\ \Delta(y) = 1 \otimes y + y \otimes x.$$

We can show that any cocycle deformation H_4^{σ} is isomorphic to H_4 itself unless char(k) = 2.

Example 3. Let q be a non-zero element in k with $q^2 \neq 1$. The quantum Hopf algebra $U_q(sl_2)$ is defined by generators K, K^{-1}, E, F and the following relations

$$KEK^{-1} = q^2E,\ KFK^{-1} = q^{-2}F,\ EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$$

with comultiplication given by

$$\Delta(K) = K \otimes K,\ \Delta(E) = 1 \otimes E + E \otimes K,\ \Delta(F) = K^{-1} \otimes F + F \otimes 1.$$

We can show that $U_q(sl_2)$ has only two cocycle deformations up to isomorphisms. One is $U_q(sl_2)$ itself, and the other is the Hopf algebra defined quite similarly as above with the third relation modified as follows: $EF - FE = 0$. The comultiplication is given similarly.

Note that the Hopf algebras appearing in the two examples above are pointed. It does not seem that essentially new Hopf algebras arise from cocycle deformations of pointed Hopf algebras.

We say two bialgebras (or Hopf algebras) A, B are cocycle deformations of each other if $B \simeq A^{\sigma}$ for some 2-cocycle σ on A. This is an equivalence relation. In quantum group theory we often encounter a family of bialgebras or Hopf algebras A_q. It is interesting to know when two of them A_q and $A_{q'}$ are cocycle deformations of each other.

Example 4. Let $O_q(GL(n))$ (resp. $O_q(M(n))$) be the coordinate algebra of the quantum general linear group $GL_q(n)$ (resp. the quantum matrix semigroup $M_q(n)$), for $q \neq 0$ in k. $O_q(M(n))$ is a quadratic matrix bialgebra, and $O_q(GL(n))$ is its localization at a central group-like element called the quantum determinant. These quantizations are studied by many authors, and see [4] for definitions. We can prove the following results.

Theorem. If $q^2 \neq 1$, then $O_q(GL(n))$ (resp. $O_q(M(n))$) is never isomorphic to cocycle deformations of commutative Hopf algebras (resp. commutative bialgebras).

Theorem. Let $q, q' \neq 0$ in k. $O_q(M(n))$ and $O_{q'}(M(n))$

are cocycle deformations of each other iff $q'^2 = q^2$ or $q'^2 = q^{-2}$.

The second result generalizes to the 2-parameter quantization.

Example 5. Let α, β be non-zero elements in k. We have introduced 2-parameter matrix semigroup $M_{\alpha,\beta}(n)$ and general linear group $GL_{\alpha,\beta}(n)$ [3]. Let $O_{\alpha,\beta}(M(n))$ be the coordinate algebra of $M_{\alpha,\beta}(n)$. We have:

Theorem. $O_{\alpha,\beta}(M(n))$ and $O_{\alpha',\beta'}(M(n))$ are cocycle deformations of each other iff $\alpha'\beta' = \alpha\beta$ or $\alpha'\beta' = (\alpha\beta)^{-1}$.

We conjecture that a similar result holds for $O_{\alpha,\beta}(GL(n))$.

References

[1] Y.Doi, Braided bialgebras and quadratic bialgebras, Comm. Algebra 21(1993), 1731-1749.

[2] Y.Doi and M.Takeuchi, Multiplication alteration by 2-cocycles, (The quantum version), Comm. Algebra 22(1994), 5715-5732.

[3] M.Takeuchi, A 2-parameter quantization of GL(n), Proc. Japan Acad. 66 (Ser.A)(1990), 112-114.

[4] M.Takeuchi, Some topics on $GL_q(n)$, J. Algebra 147(1992), 379-410.

Institute of Mathematics
University of Tsukuba
Tsukuba 305 JAPAN

Finte Injective Dimension

Tang Gaohua
(Guangxi Teachers' College, P.R.China)

ABSTRACT

It is well known that projective dimension, injective dimension and weak dimension are very important for inverstigating rings and modules. In this paper, finite injective dimension will be invoked and discussed.

1. Finite injective dimension of modules

Definitionl.1 Let E be a left R-module. E is said to be finite injective if for every $f:I\to E$ from any finitely generated left ideal I of R, there exists a map $g:R\to E$ such that the following diagram commutes.

$$\begin{array}{ccc} 0\to I\to R \\ f\downarrow \quad g \\ E \end{array}$$

Definition 1.2 A finite injective resolution of a module M is an exact sequence $0\to M\to E_0\to E_1\to\cdots\to E_n\to E_{n+1}\cdots$ in which each E_i is finite injective.

Definition 1.3 If M is a left R-module, then $l\cdot fid(M)\leqslant n$ ($l\cdot fid$ abbreviate left finite injective dimension) if there is a finite injective resolution

$0\to M\to M_0\to E_1\to\cdots\to E_n\to 0$. If no such finite resolution exists, define $l\cdot fid(M)=\infty$. Otherwise, if n is the least such integer, define $l\cdot fid(M)=n$.

Lemma 1.1 Let M be a left R-module, then the following statements are equivalent:

(i) M is finite injective module;

(ii) $Ext_R^1(R/I,M)=0$ for all finitely generated left ideals I of R;

(iii) $Ext_R^k(R/I,M)=0$ for all finitely generated left ideals I of R and all $k\geqslant 1$.

Lemma 1.2 Let $0\to M\to E\to N\to 0$ be a short exact sequence in which E is finite injective. Then $Ext_R^{n+1}(R/I,M)\cong Ext_R^n(R/I,N)$ for all finitely generated left ideals I of R.

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Lemma 1.3 If $0 \to M \to E_0 \to E_1 \to \cdots \to E_{n-1} \to N \to 0$ is an exact seqnence with every E_i is finite injective. Then $Ext^{n+1}_R(R/I,M) \cong Ext^1_R(R/I,N)$ for all finitely generated left ideals I or R.

Theorem 1.1 Let M be a left R-module, then the following statements are equivalent.

(i) $l \cdot fid(M) < n$;

(ii) $Ext^{n+1}_R(R/I,M)=0$ for all finitely generated left ideals I of R;

(iii) $Ext^k_R(R/I,M)=0$ for all finitely generated left ideals I of R and all $k > n+1$;

(iv) For any exact sequence $0 \to M \to E_0 \to E_1 \to \cdots \to E_{n-1} \to N \to 0$ in which every E_i is finite injective, then N is also injective.

Proof. (i)=>(ii) Since $l \cdot fid(M) < n$, then there extists a finite injective resolution: $0 \to M \to E_0 \to E_1 \to \cdots \to E_n \to 0$, then $Ext^{n+1}_R(R/I,M) \cong Ext^1_R(R/I,E_n)$ for all finitely generated left ideals I of R by Lemma 1.3. and so $Ext^{n+1}_R(R/I,M)$ $=0$ by Lemma 1.1.

(ii)=>(iii) is obvious.

(iii)=>(iv) Let $0 \to M \to E_0 \to E_1 \to \cdots \to E_{n-1} \to N \to 0$ be an exact sequence in which every E_i is finite injective. Then $Ext^1_R(R/I,N) \cong Ext^{n+1}_R(R/I,M)=0$ for every finitely generated left ideal I of R by Lemma1.3. Using Lemma1.1 we get that N is finite injective.

(iv)=>(i) is very clear.

Theorem1.2 Let $0 \to A' \to A \to A'' \to 0$ be an exact sequence. Then

(i) $l \cdot fid(A'') < \max\{l \cdot fid(A),\ l \cdot fid(A')\}$;

(ii) $l \cdot fid(A') < 1+\max\{l \cdot fid(A),\ l \cdot fid(A'')\}$;

(iii) $l \cdot fid(A) \leq \max\{l \cdot fid(A'),\ l \cdot fid(A'')\}$;

(iv) If A is finite injective, then A′ and A″ both are finite injective or $l \cdot fid(A')=l \cdot fid(A'')+1$.

2. Finite injective dimension of rings

Definition2.1 Let R be a ring with identity, The left finite injective dimension of R is defined by $l \cdot fid(R)=\sup\{l \cdot fid(M) \mid M \text{ a left } R\text{-module}\}$.

If one considers right R-modules, he may define the right finite injective dimension of R: $r \cdot fid(R)$.

Theorem2.1 Let R be a ring, then

$l \cdot fid(R) = \sup\{l \cdot pd_R(R/I) \mid I$ a finitely generated left ideal of $R\}$.

Where $l \cdot pd_R(M)$ denotes the projective dimension of M.

Proof. First, assume that $l \cdot fid(R) = n < \infty$, then $l \cdot fid(M) \leq n$ for every left R-module M, Therefore $l \cdot pd_R(R/I) \leq n$ for every finitely generated left ideal I of R, hence

$\sup\{l \cdot pd_R(R/I) \mid I$ a finitely generated left ideal of $R\} \leq n$. that is

$l \cdot fid(R) \geq \sup\{l \cdot pd_R(R/I) \mid I$ a finitely generated left ideal of $R\}$.

To prove the converse inequality, clearly we may assume that $\sup\{l \cdot pd_R(R/I) \mid I$ a finitely generated left ideal of $R\} = n < \infty$, then $l \cdot pd_R(R/I) \leq n$ for all finitely generated left ideals I of R, and hence $Ext^{n+1}_R(R/I, M) = 0$ for all finitely generated left ideals I of R and every left R-module M by Theorem1.1, and hence $l \cdot fid(R) \leq n$, that is

$l \cdot fid(R) \leq \sup\{l \cdot pd_R(R/I) \mid I$ a finitely generated left ideal of $R\}$

For the above reasons, we have the equality.

Theorem2.2 Let R be a ring, then

(i) $WD(R) \leq l \cdot fid(R) \leq l \cdot D(R)$, and there exists a ring R, such that $WD(R) < l \cdot fid(R) < l \cdot D(R)$;

(ii) If R is a left noetherian, then $WD(R) = l \cdot fid(R) = l \cdot D(R)$;

(iii) If R is a left coherent ring, then $WD(R) = l \cdot fid(R)$. Furthmore, if R is a left and right coherent ring, then $l \cdot fid(R) = r \cdot fid(R) = WD(R)$.

Where $WD(R)$ denotes the weak dimension of R, $l \cdot D(R)$ denotes the left global dimension of R, $fd_R(M)$ denotes the flat dimension of the R-module M.

Proof. (i) Using [5. 0.10 Lemma] and [6. Theorem1.3.8] we have $WD(R) = \sup\{fd_R(R/I) \mid I$ a finitely generated left ideal of $R\} \leq \sup\{l \cdot pd_R(R/I) \mid I$ a finitely generated left ideal of $R\} = l \cdot fid(R)$. by Theorem2.1. The other inequality $l \cdot fid(R) \leq l \cdot D(R)$ is very obvious.

Now we prove that there is a ring R, such that $WD(R) < l \cdot fid(R) < l \cdot D(R)$. In [7], L.W.Small gave a ring T which is right hereditary but it is not left semihereditary with $l \cdot D(T) = 3$. Then $0 \leq WD(T) \leq r \cdot fid(T) \leq 1$. Since T is not left semihereditary. then T is not Von Neumman regular ring, and so that $WD(T) \neq 0$, Thus $WD(T) = r \cdot fid(R) = 1$. Seeing the proof of Theorem1 in [7], there exists a principal left ideal N of T with $l \cdot pd_T(N) = 2$, then $l \cdot pd_T(T/N) = 3$, and hence $l \cdot fid(T) = l \cdot D(T) = 3$, and $WD(T) < l \cdot fid(T)$. In [8],

we know, for any integer n(0<n<∞), there exists a Valuation ring R with l·D(R)=n. Since every Valuation ring must be a left semihereditary ring, then R is left coherent ring and WD(R)≤1. Hence
l·fid(R)=WD(R)≤1 (here we have used the result (iii) of Theorem2.2).
Then we can get a Valuation ring ∧, such that l·fid(∧)=WD(∧)≤1 and l·D(∧)=n>3. Now we set R=T⊕∧ then

WD(R)=max{WD(T), WD(∧)}=1.

l·fiD(R)=max{l·fiD(T), l·fiD(∧)}=l·fiD(T)=3.

l·D(R)=max{l·D(T), l·D(∧)}=l·D(∧)=n>3.

It implis that WD(R)<l·fiD(R)<l·D(R).

(ii). is clear.

(iii). Since WD(R)=Sup{fd_R(R/I) | I a finitely generated left ideal of R}. R/I is a finitely related R-module for every finitely generated left ideal I of R then fd_R(R/I)=l·pD_R(R/I) by [9,Theorem1.3], hence WD(R)=l·fiD(R).

If R is a right coherent ring, we can also show that WD(R)= r·fiD(R). Hence we get the last result.

Referenees.

[1]. Chen Fuchang. Homological Algebra. Guangxi Normal University Press. Guilin.1989.

[2]. Joseph J.Rotman. An Introduction to Homological Algebra. Academic Press. INC.1979.

[3]. Chen JiaLai. Rings and Modules. Beijing Teachers' College Press. Beijing.1989.

[4]. Frank W. Anderson, Kent R. Fuller. Rings And Categories of Modules. Springer-Verlag, New York.INC.1974.

[5]. Wolmer V. Vascancelos. The Rings of Dimension Two. Marcel Dekker .INC.1976.

[6]. Sarah Glaz. Commutative Coherent Rings. Springer-Verlag, Beilin Heidelberg.1989.

[7]. L.W. Small. Hereditary Rings. Proc. Nat. Acad. Sci. U.S.A. 55(1966).

[8]. B.L. Osofsky. Global dimension of Valuation Rings. Trans. Amer. Math. Soc.127(1967).

[9]. Tang Gaohua. Homlogical Properties of Coherent Rings. M.S. thesis .1989.

Schur algebras and their centers

Yukio TSUSHIMA

§1. Background

Let R be a commutative ring. We fix a positive integer n and let $E = E_R$ be the set of column vectors of size n over R. Let Σ_r be the symmetric group on r letters. Remember that Σ_r acts on $E^{\otimes r}$ by place permutations, namely for $\sigma \in \Sigma_r$ we have:

$$(v_1 \otimes \cdots \otimes v_r)\sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(r)} .$$

Now we define the Schur algebra $S_R = S_R(n, r)$ as follows:

Definition 1. $S_R = S_R(n, r) = \mathrm{End}_{R\Sigma_r}(E^{\otimes r})$.

There is a diagonal action of the general linear group GL(n, R) on $E^{\otimes r}$ and we have a map:

$$\psi_R : GL(n, R) \longrightarrow S_R(n, r).$$

We let K be an infinite field throughout.

(1.1) ψ_K induces a categorical equivalence between mod $S_K(n, r)$ and the category of homogeneous polynomial representations of GL(n, K) of degree r.

(1.2)(Weyl, de Concini-Processi) Natural map

$$\phi_L : L\Sigma_r \longrightarrow \mathrm{End}_{S_L}(E^{\otimes r})$$

is surjective for any field L.

Definition 2.

$$\Lambda^+(n, r) = \{\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}^n ; \alpha_1 \geqq \cdots \geqq \alpha_n \geqq 0, \alpha_1 + \cdots + \alpha_n = r\}$$

(1.3)(Weyl)

$$\mathrm{IRR}(S_K(n, r)) = \{F_\lambda ; \lambda \in \Lambda^+(n, r)\}$$

where $F_\lambda = hd(V_\lambda)$ and $V_\lambda \subset E^{\otimes r}$ is the Weyl module with the highest weight λ when considered as the rational $GL(n, K)$-module via ψ_K. In particular we have:

$$\# IRR(S_K(n, r)) = \# \Lambda^+(n, r).$$

(1.4)(Weyl) If ch K=0, then

$$F_\lambda = V_\lambda = E^{\otimes} S^\lambda$$

where S^λ denotes the dual Specht module over Σ_r corresponding to the partition λ of r.

Remark(James). If ch K=p>0, the above result holds provided λ is column p-regular, i.e., $0 \leq \lambda_i - \lambda_{i+1} \leq p-1$ for all $i(1 \leq i \leq n-1)$.

Since $S_R = End_{R\Sigma_r}(E^{\otimes r})$, we have a natural map between the centers:

(1.5) $\rho_R : Z(R\Sigma_r) \longrightarrow Z(S_R)$.

(1.6) If ch K=0, then S_K as well as $K\Sigma_r$ is a split semisimple algebra over K, so that we have the following from (1.2):

(1) ρ_K is surjective.

(2) $\dim_K Z(S_K) = \# IRR(S_K(n, r)) = \# \Lambda^+(n, r)$.

Furthermore we have

(3)(Weyl)Let λ be a partition of r and e_λ be the central primitive idempotent of $K\Sigma_r$ corresponding to λ. Then $\rho_K(e_\lambda) \neq 0$ if and only if $\lambda \in \Lambda^+(n, r)$. In particular, ρ_K is an isomorphism, whenever $n \geq r$.

From now on, we let (K, R, k) be a p-modular system, that is, R is a complete discrete valuation ring with K as the quotient field of characteristic 0 and k is the residue field of R of prime characteristic p.

(1.7)(Dade) The canonical map $S_R \to S_k$ induces a bijection between the central primitive idempotents of S_R and S_k.

This is generally true for any R-order S_R which is free of finite rank over R. So we can speak of the p-block distribution of the simple modules of the semisimple algebra S_K.

If $n \geq r$, then $E^{\otimes r}$ is projective over S_R and we get the following result:

(1.8) Suppose that $n \geq r$. Then we have the isomorphisms:

$$\phi_R : R\Sigma_r \simeq \mathrm{End}_{S_R}(E^{\otimes r});$$

$$\rho_R : Z(R\Sigma_r) \simeq Z(S_R).$$

(1.9)(Donkin) Suppose that $n \geq r$. We identify $Z(R\Sigma_r) = Z(S_R)$ and let

$$R\Sigma_r = \bigoplus R\Sigma_r \varepsilon$$

be the block decomposition of $R\Sigma_r$ with block idempotents ε. Then

$$S_R = \bigoplus S_R \varepsilon$$

is the block decomposition of S_R. Also, $V_\lambda \mapsto S^\lambda$ gives rise to the bijection:

$$\mathrm{IRR}(S_K \varepsilon) \longrightarrow \mathrm{IRR}(K\Sigma_r \varepsilon).$$

As ch K=0, we have by 1.6 that:

(1.10) $Z(S_K)$ is a K-span of $\{\rho_K([C]);\ C \in CL(\Sigma_r)\}$, where $[C]$ denotes the sum of the elements of C in $K\Sigma_r$. If $n \geq r$, these form a K-basis of $Z(S_K)$.

As a direct consequence of 1.6(2), we have:

(1.11) $\dim_k Z(S_k(n, r)) \geq \sharp\Lambda^+(n, r)$.

If $n \geq r$, we have the equality sign in the above. However we do not know whether this is true in general.

§2 Description of $Z(S_K)$

We continue to assume that ch K=0.

Let $\{e_1, \cdots, e_n\}$ be a K-basis of E_K and let

$$I = I(n, r) = \{i = (i_1, \cdots, i_r);\ 1 \leq i_k \leq n\}.$$

For $i \in I$, we set

$$e_i = e_{i_1} \otimes \cdots \otimes e_{i_r}.$$

Then $\{e_i;\ i \in I\}$ forms a K-basis of $E^{\otimes r}$. Also, we let Σ_r act on I by place permutations. Let Ω be a set of representatives of the Σ_r-orbits on IxI. Now, we know from the general theory of permutation modules that S_K

has a basis $\{\xi_{ij}; (i,j) \in \Omega\}$, where ξ_{ij} is defined by

$$\xi_{ij}(e_k) = \sum e_h$$

where h runs over I such that (h, k) lies in the same Σ_r-orbit as (i, j).

We shall here describe $\rho_K([C])$ in terms of ξ_{ij}'s. For $j \in I$, let $G_j = \{\sigma \in \Sigma_r; j\sigma = j\}$. Each $C \in CL(\Sigma_r)$ splits into the disjoint union of G_j-orbits C_α:

$$C = \bigcup_{1 \leq \alpha \leq m} C_\alpha.$$

For $\sigma_\circ \in C_\alpha$, let

$$f(\sigma_\circ, j) = [G_j^{\sigma_\circ} \cap G_j : C_{G_j}(\sigma_\circ)]$$

and

$$f(j, C) = \sum_{1 \leq \alpha \leq m} f(\sigma_\circ, j)\, \xi_{j\sigma_\circ, j}.$$

Then we have:

Theorem. $\rho_K([C]) = \sum_{j \in \Theta} f(j, C)$, where Θ is a set of representatives of Σ_r-orbits on I.

The proof is a bit lengthy but straightforward.

REFERENCES

[1] de Concini and C. Procesi: A characteristic-free approach to invariant theory, Adv. Math. 21(1976), 330-354

[2] E. C. Dade: Block extensions, Illinois J. Math. 17(1973), 198-272

[3] S. Donkin: On Schur algebras and related algebras II, J. Algebra, 111(1987), 354-364

[4] J. A. Green: Polynomial representations of GL_n: Springer L. N. M. 830(1980)

[5] G. D. James: The decomposition of tensors over fields of prime characteristic, Math. Z. 172(1980), 161-178

Department of Mathematics
Osaka City Universiy
558 Osaka, Japan

On Frobenius Algebras

Takayoshi Wakamatsu
Faculty of Education
Saitama University
338 Urawa, Saitama
Japan

1 Introduction

Let K be a field which we usually assumed to be algebraically closed for simplicity. Throuout this note, all algebras and modules are assumed to be finite dimensional over K. The usual duality functor $Hom_K(?, K)$ is denoted by D.

An algebra Λ is said to be Frobenius if $\Lambda_\Lambda \cong D(\Lambda)_\Lambda$. This condition is left-right symmetric. In the previous paper [6], we proved that any Frobenius algebra can be described as a direct sum $A \oplus M \oplus {}_\sigma D(A)_A$ for an algebra A and a bimodule ${}_AM_A$ and an automorphism $\sigma \in Aut_K(A)$, with its multiplication

$$(a, m, f) \cdot (a', m', f') = (aa', am' + ma' + \varphi(m \otimes m'), \sigma(a)f' + fa' + \psi(m \otimes m'))$$

for suitable maps $\varphi : {}_AM \otimes_A M_A \to {}_AM_A$ and $\psi : {}_AM \otimes_A M_A \to {}_\sigma D(A)_A$. So, in order to get a Frobenius algebras, we have to construct such a system

$$(\sigma \in Aut_K(A), {}_AM_A, \varphi, \psi).$$

In this note, we will give a simple construction of such a system for a graded Frobenius algebra.

2 Frobenius System

Let A be an algebra, ${}_AX_A$ a bimodule and σ a K-algebra automorphism of A. We suppose that there is an isomorphism

$$\gamma : {}_AX_A \to {}_\sigma X_\sigma.$$

We consider a map

$$\varphi : {}_A(X^{\otimes d})_A \to {}_\sigma D(A)_A$$

with the property

$$\varphi(x_1 \otimes x_2 \otimes \cdots \otimes x_d) = \varphi(x_2 \otimes \cdots \otimes x_d \otimes \gamma(x_1)),$$

where $\otimes$ means $\otimes_A$. we call $\Phi = (\sigma, \gamma, \varphi)$ a Frobenius system, in this case.

Remark For any bimodule ${}_AM_A$, we have an isomorphism

$$Hom_{A^e}({}_AM_A, {}_\sigma D(A)_A) \cong D(M/\langle\sigma^{-1}(a)m - ma \mid a \in A,\ m \in M\rangle) \subseteq D(M).$$

Therefore, for any map $\varphi : {}_AM_A \to {}_\sigma D(A)_A$, we have the corresponding map θ in $D(M/\langle\sigma^{-1}(a)m - ma\rangle)$. Denote by $\theta \in D(X^{\otimes d})$ the map corresponding to φ of the Frobenius system Φ. Then, this satisfies

$$\theta(x_1 \otimes x_2 \otimes \cdots \otimes x_d) = \theta(x_2 \otimes \cdots \otimes x_d \otimes \gamma(x_1)).$$

For each $0 \le i \le d$, we can define a map

$$\varphi_i : {}_A(X^{\otimes i})_A \to {}_\sigma Hom_{mod-A}({}_A(X^{\otimes(d-i)})_A, {}_\sigma D(A)_A)_A$$

by $\varphi_i(y)(z) = \varphi(y \otimes z)$. Since there is an isomorphism

$${}_\sigma Hom_{mod-A}({}_A(X^{\otimes(d-i)})_A, {}_\sigma D(A)_A)_A \cong {}_\sigma D(X^{\otimes(d-i)})_A,$$

we also have a map

$$\theta_i : {}_A(X^{\otimes i})_A \to {}_\sigma D(X^{\otimes(d-i)})_A,$$

defined by $\theta_i(y)(z) = \varphi(y \otimes z)(1) = \theta(y \otimes z)$. Similarly, the map

$$\varphi_i' : {}_AX_A^{\otimes i} \to Hom_{A-mod}({}_A(X^{\otimes(d-i)})_A, {}_\sigma D(A)_A)$$

is defined by $\varphi_i'(y)(z) = \varphi(z \otimes y)$ with the corresponding map

$$\theta_i' : {}_AX_A^{\otimes i} \to {}_AD(X^{\otimes(d-i)})_{\sigma^{-1}},$$

where we have the relations $\varphi_i(y)(z) = \varphi_{d-i}'(z)(y)$ and $\theta_i(y)(z) = \theta_{d-i}'(z)(y)$. Further, it is easy to see that $Ker(\varphi_i) = Ker(\theta_i)$ and $Ker(\varphi_i') = Ker(\theta_i')$.

Proposition 2.1 *For each $0 \le i \le d$, $Ker(\varphi_i) = Ker(\varphi_i')$ and $\gamma^{\otimes i}$ induces the automorphism of $Ker(\varphi_i)$ ($\gamma^{\otimes 0} = \sigma$).*

We put $R_i = R_i(\Phi) = Ker(\varphi_i)$ for $0 \le i \le d$ and $R_i = X^{\otimes i}$ for $d + 1 \le i$. Then $R = R(\Phi) = \bigoplus_{i \ge 0} R_i$ becomes an ideal in the tensor algebra $T(\Phi) = A \oplus X \oplus X^{\otimes 2} \oplus \cdots$. We consider the factor algebra

$$\Lambda(\Phi) = T(\Phi)/R(\Phi) = A/R_0 \oplus X/R_1 \oplus \cdots \oplus X^{\otimes d}/R_d.$$

Let us put $B = A/R_0$ and $Y = X/R_1$. Then, it is proved that σ, γ and φ induce $\sigma' \in Aut_K(B)$, $\gamma' : {}_BY_B \xrightarrow{\sim} {}_{\sigma'}Y_{\sigma'}$ and $\varphi' : {}_BY_B^{\otimes d} \to {}_{\sigma'}D(B)_B$. Further, $\Phi' = (\sigma', \gamma', \varphi')$ becomes again a Frobenius system.

Proposition 2.2 $\Lambda(\Phi) \cong \Lambda(\Phi')$.

By this result, we may assume that the Frobenius system satisfies the conditions

$$R_0 = 0, \quad R_1 = 0.$$

We call such a system a reduced Frobenius system. It should be noted that the map φ is surjective if Φ satisfies the above conditions and ${}_A(X^{\otimes d}/R_d)_A$ can be identified with ${}_\sigma D(A)_A$.

Theorem 2.3 *$\Lambda(\Phi)$ is a Frobenius algebra.*

3 Frobenius System over K

In the case of $A = K$, a Frobenius system is given very simply. First, σ must be the identity map and X is just a K-vector space. γ is an element of $GL(X)$. The map $\varphi = \theta : X^{\otimes d} \to K$ satisfies

$$\varphi(x_1 \otimes x_2 \otimes \cdots \otimes x_d) = \varphi(x_2 \otimes \cdots \otimes x_d \otimes \gamma(x_1)).$$

So, a Frobenius system is simply a pair $\Phi = (\gamma, \varphi)$ satisfying the above condition. Furthermore, since we consider only reduced system, we may assume that the map φ satisfies the following condition also:

$$\varphi(x \otimes X^{\otimes(d-1)}) = 0 \text{ implies } x = 0.$$

It is easily proved that the map γ is uniquely determined by φ. Therefore, we denote the algebra $\Lambda(\Phi)$ by $\Lambda(\varphi)$, usually.

Theorem 3.1 *Let K be algebraically closed. Then, any (radical-)graded local Frobenius (basic) algera Λ is isomorphic to $\Lambda(\varphi)$ for a Frobenius system (γ, φ).*

Theorem 3.2 *Let (γ, φ) be a Frobenius system, $s \in GL(X)$ and $\alpha \in K$ a non-zero element. Then, $(s^{-1} \cdot \gamma \cdot s, \alpha\varphi \cdot s^{\otimes d})$ is a Frobenius system. Moreover, for two reduced Frobenius system (γ, φ) and (γ', φ'), $\Lambda(\varphi)$ is isomorphic to $\Lambda(\varphi')$ if and only if $(\gamma', \varphi') = (s^{-1} \cdot \gamma \cdot s, \alpha\varphi \cdot s^{\otimes d})$ for some s and α.*

Theorem 3.3 *Let (γ, φ) be a Frobenius system. Suppose $s \in GL(X)$ satisfies $\varphi \cdot s^{\otimes d} = \varphi$. Then, by putting $\varphi_s = \varphi \cdot (1 \otimes s \otimes s^2 \otimes \cdots \otimes s^{d-1})$, $(s^{-d} \cdot \gamma, \varphi_s)$ becomes a Frobenius system.*

4 Examples

Let X be a n-dimensional K-vector space. We fix a K-basis $v_1, v_2, \cdots, v_d$ of X. Consider the elements in $X^{\otimes d}$ of the form $x_1 \otimes \cdots \otimes x_d$. Asuume that each element

x_i is expressed as $v_1 c_{1,i} + v_2 c_{s,i} + \cdots + v_d c_{d,i}$. Then, we may identify the element $x_1 \otimes \cdots \otimes x_d$ with the (d,d)-matrix

$$M(x_1, \cdots, x_d) = \begin{bmatrix} c_{1,1} & \cdots & c_{1,d} \\ & \ddots & \\ c_{d,1} & \cdots & c_{d,d} \end{bmatrix}.$$

We can define a map $det : X^{\otimes d} \to K$ by the correspondence $x_1 \otimes \cdots \otimes x_d \mapsto det M(x_1, \cdots, x_d)$. It is easy to see that $((-1)^{d-1}, det)$ is a Frobenius map with the exterior algebra $\bigwedge X \cong \Lambda(det)$. $s \in GL(X)$ satisfies $det \cdot s^{\otimes d} = det$ if and only if $s \in SL(X)$.

Theorem 4.1 *For elements s, $t \in SL(X)$, The isomorphism $\Lambda(det_s) \cong \Lambda(det_t)$ holds if and only if $t = \omega g^{-1} \cdot s \cdot g$ for some $g \in GL(X)$ and $\omega \in K$ such that $\omega^d = 1$.*

Theorem 4.2 *$\Lambda(det_s)$ is always Koszul with the dual algebra*

$$\Lambda(det_s)^! = Ext^{\bullet}_{\Lambda(det_s)}(K, K),$$

which is noetherian.

References

[1] M. Artin and W. Schelter, Graded algebras of global dimension 3, Advances in Math., 66(1987), 171-216.

[2] A. Beilinson, V. Ginsburg and W. Soerger, Koszul duality patterns in representation theory, preprint, 1992.

[3] Yu. I. Manin, Some remarks on Koszul algebras and quantum groups, Ann. Inst. Fourier, 37(1987), 191-205.

[4] S. B. Priddy, Koszul resolutions, Trans. Amer. Math. Soc., 152(1970), 39-60.

[5] S. P. Smith, Some finite dimensional algebras related to elliptic curves, preprint, 1994.

[6] T. Wakamatsu, Tilting theory and selfinjective algebras, in "Finite Dimensional Algebras and Related Topics", Nato ASI Series, Series C, 424(1993), 361-390.

HOMOLOGICAL DIMENSIONS OF INVARIANTS FOR HOPF ALGEBRA ACTIONS

ZHIXI WANG

ABSTRACT. Let H be a Hopf algebra acting on a ring A, and let A^H denote the invariant subring of A. It is shown that $\mathrm{Id}A^H \leq \mathrm{pdim}_H k + \mathrm{Id}A + \mathrm{fdim}(A_{A^H})$ in the case that A is a projective left $A\#H$-module.

Let H be a Hopf algebra over a field k and A an $H-$module algebra. We let A^H denote the ring of invariants under the $H-$action and $A\#H$ the smash product. Our main interest is to investigate homological dimensions of A^H. When A is a projective left $A\#H-$module, we establish the following estimates for the injective dimension of A^H in terms of that of A and other related data:

$$\mathrm{Id}A^H \leq \mathrm{pdim}_H k + \mathrm{Id}A + \mathrm{fdim}(A_{A^H}).$$

We fix our notation, following the references [M] and [Rot]. In particular, we will keep the following notations:

· $\mathrm{Id}R$: denotes the injective dimension of R as a left $R-$module.

· $\mathrm{pdim}M$ and $\mathrm{fdim}M$ denote the projective dimension and the flat dimension of the module M, respectively.

· $R-$mod(resp. mod$-R$) denotes the category of left $R-$modules (right $R-$modules).

· For $M \in A\#H-$mod, $M^H = \{m \in M \mid h \cdot m = \varepsilon(h)m, \quad \text{all} \quad h \in H\}$.

Lemma 1. *(1) $Id_{A\#H}(A) \leq pdim_H k + IdA$.*

(2) If H is semisimple, then ${}_{A\#H}A$ is a projective module.

Proof. Let ${}_{A\#H}M$ be a left $A\#H-$module. For left $A\#H-$modules M and A, Applying ([LL], Proposition 2.3(a)), there is a spectral sequence

$$\mathrm{Ext}^p_H(k, \mathrm{Ext}^q_A(M, A) \underset{p}{\Longrightarrow} \mathrm{Ext}^n_{A\#H}(M, A),$$

proving (1).

If H is semisimple, the spectral sequence of ([LL], Proposition 2.3(a)) collapses on the $p-$axis, yielding isomorphisms

$$\mathrm{Hom}_H(k, \mathrm{Ext}^n_A(V, W)) \cong \mathrm{Ext}^n_{A\#H}(V, W)$$

for any left $A\#H-$ modules V and W.

Taking $V = A$, we obtain that $\mathrm{Ext}^n_{A\#H}(A, W) = 0$ for any left $A\#H-$ module W and all $n \geq 1$, and so A is a projective left $A\#H-$ module. □

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Lemma 2. *If H is semisimple, then*

$$IdA = Id_{A\#H}(A) = Id(A\#H).$$

Proof. If ϕ: $A \longrightarrow A\#H$ is the ring embedding map, then from ([Rot], Theorem 11.65) there are isomorphisms

$$\mathrm{Ext}^n_{A\#H}(A\#H \otimes_A M, A) \cong \mathrm{Ext}^n_A(M, A)$$

for any left A-module M. It follows that $\mathrm{Id}A \leq \mathrm{Id}({}_{A\#H}A)$. By Lemma 1 (1) $\mathrm{Id}A\#H(A) \leq \mathrm{Id}A$ since H is semisimple. Hence the equality holds. By [S, Corollary 2.7], H must be finite dimensional. Taking $W = A\#H$ in the spectral sequence of ([LL], Proposition 2.3(a)), we obtain a spectral sequence

$$\mathrm{Ext}^p_H(k, \mathrm{Ext}^q_A(V, A\#H) \underset{p}{\Longrightarrow} \mathrm{Ext}^n_{A\#H}(V, A\#H),$$

where V is an arbitrary left $A\#H$-module. Since H is semisimple, this spectral sequence collapses on the p-axis, yielding isomorphisms

$$\mathrm{Hom}_H(k, \mathrm{Ext}^n_A(V, A\#H)) \cong \mathrm{Ext}^n_{A\#H}(V, A\#H).$$

Hence $\mathrm{Id}(A\#H) \leq \mathrm{Id}_A(A\#H)$. Recall that $A\#H$ is a finitely generated free left A-module. It yields that $\mathrm{Id}_A(A\#H) = \mathrm{Id}A$, and so $\mathrm{Id}(A\#H) \leq \mathrm{Id}A$.

On the other hand, $\mathrm{Id}A \leq \mathrm{Id}(A\#H)$. Indeed, by Lemma 1(2) ${}_{A\#H}A$ is a projective module and so the exact sequence $A\#H \longrightarrow_{A\#H} A \longrightarrow 0$ splits. This implies that $\mathrm{Id}_{A\#H}(A) \leq \mathrm{Id}(A\#H)$ and hence $\mathrm{Id}A \leq \mathrm{Id}(A\#H)$. □

Theorem 3. *If ${}_{A\#H}A$ is projecitve, then*

$$IdA^H \leq pdim_H k + IdA + fdim(A_{A^H}).$$

Proof. For rings A^H and $A\#H$, consider the situation $({}_{A^H}M, {}_{A\#H}A_{A^H}, {}_{A\#H}A)$. Then we have the adjoint isomorphism

$$\mathrm{Hom}_{A\#H}(A \otimes_{A^H} M, A) \cong \mathrm{Hom}_{A^H}(M, \mathrm{End}({}_{A\#H}A)).$$

Note that $\mathrm{End}({}_{A\#H}A) \cong A^H$ as algebras([CFM]). Hence

$$\mathrm{Hom}_{A\#H}(A \otimes_{A^H} M, A) \cong \mathrm{Hom}_{A^H}(M, A^H)$$

for any left A^H-module.

Define functors

$$F: \quad A\#H - \mathrm{mod} \longrightarrow Ab(\text{ the category of Abelian groups}), \quad \text{via}$$

$$F(M) = \mathrm{Hom}_{A\#H}(M, A)$$

and

$$G : A^H - \text{mod} \longrightarrow A\#H - \text{mod}, \quad \text{via}$$

$$G(N) = A \otimes_{A^H} N.$$

Then FG is equivalent with the functor $\text{Hom}_{A^H}(-, A^H)$, and so the right derived functors $R^n(FG)$ are equivalent with $\text{Ext}^n_{A^H}(-, A^H)$.

Moreover, if P is a projective left A^H−module, then we have isomorphisms ([Rot], Exercise 9.20)

$$\text{Ext}^n_{A\#H}(A \otimes_{A^H} P, A) \cong \text{Hom}_{A^H}(P, \text{Ext}^n_{A\#H}(A, A)).$$

However, ${}_{A\#H}A$ is projective, so

$$\text{Ext}^n_{A\#H}(A \otimes_{A^H} P, A) = 0 \quad \text{all} \quad n \geq 1,$$

and so $G(P)$ is right F−acyclic.

Applying the Grothendieck spectral sequence ([Rot], Theorem 11.40), there is a third quadrant spectral sequence

$$\text{Ext}^p_{A\#H}(Tor_q^{A^H}(A, M), A) \underset{p}{\Longrightarrow} \text{Ext}^n_{A^H}(M, A^H).$$

This directly implies the required estimates for $\text{Id}A^H$. □

References

[CFM] M. Cohen, D. Fischman and S. Montgomery, *Hopf Galios extensions, smash products, and Morita equivalence*, J. Algebra, 133(1990), 351–372.

[LL] M. E. Lorenz and Martin Lorenz, *On crossed products of Hopf algebras*, Proc. Amer. Math. Soc, 123(1995), 33–38.

[M] S. Montgomery, *Hopf Algebras and Their Actions on Rings*, CBMS Lectures in Math. Vol.82 AMS, Providence, RI 1993.

[S] M. E. Sweedler, *Integrals for Hopf algebras*, Ann. Math. 2(1969), 323-335.

[Rot] J. J. Rotman, *An Introduction to Homological Algebra*, New York, 1979.

Department of Mathematics
Capital Normal University
Beijing 100037, China

SOME NEW CHARACTERIZATIONS OF V-RINGS

Wu Pinsan
Department of Mathematics, Beijing
Normal University, Beijing, 100875
Hu Xianhui
Department of Mathematics
The Central University of Nationalities, Beijing, 100081

In this paper, a ring R always means an associative ring which does not necessarily contain the multiplicative identity. R is called a left V-ring (or a cosemisimple ring) if every nonzero left ideal of R is the intersection of maximal left ideals. See [1] [2]. In this paper, we will give some new characterizations of this ring. Here we define a cyclic R-module as following: A left R-module M is called a cyclic module if there exists a R-module $\overline{M}$ which contains M as a submodule, and there exists an element m of $\overline{M}$ such that $M = Rm$. M is called a strictly cyclic module if there exists an element m of M such that $M = Rm$. A R-module M is called a weak simple if M does not contain proper submodule. A ring R is called to have property (V) if for every left ideal $A \neq R$ and $a \in R \setminus A$, then every left ideal L of R that is maximal with respect to the property that contains A but does not contain the element a is surely a maximal left ideal of R. The main result in this paper is:

Theorem. The following statements are equivalent:

(a) R is a left V-ring.

(b) Every nonzero cyclic left R-module always can be embedded in the direct product of weak simple modules.

(c) R has property (V).

Proof.

(a)$\Longrightarrow$(b)

Let M be a nonzero cyclic left R-module. then $M = Rm$. Let f denote the mapping: $r \mapsto rm$ for every $r \in R$, then f is a R-homomorphism of R onto M, and the $\ker f = L$ is a nonzero left ideal of R. By (a) $L = \cap B_\alpha$, where B_α is maximal left ideal of R, and R/B_α is weak simple R-module. Hence $M = Rm$ can be embedded in the direct product of weak simple modules:

$$0 \longrightarrow Rm \simeq R/\cap B_\alpha \longrightarrow \prod R/B_\alpha.$$

(b)$\Longrightarrow$(c).

Let A be any left ideal of R and $A \neq R$. Assuming that L is a left ideal of R which contains A but does not contain an element a of R, and L is the maximal with respect to this property. We will prove that L is surely a maximal left ideal of R. It is sufficient to show that any left ideal of R which contains L and a is R itself.

Let $B = L + (Ra + \mathbb{Z}a)$, then B is a left ideal of R which contains L and a. It is easy to see that B/L as a R-module is a weak simple module. Let $R^1 = R + \mathbb{Z}$ and define multiplication in R by

$$(a, m)(b, n) = (ab + na + mb, mn)$$

for $a, b \in R$ and $m, n \in \mathbb{Z}$. Then R^1 is a ring with unit $(0, 1)$.

For $r \in R$, we define

$$r(a, m) = (r, 0)(a, m) = (ra + mr, 0)$$

then R^1 is a left R-module and it contains L as a submodule. Here we do not distinguish r and $(r, 0)$. Let $M = R/L$, $\overline{M} = R^1/L$, then there exists $\overline{1} = (0, 1) + L$ in $\overline{M}$ such that $M = R\overline{1}$, thus M is a cyclic R-module. We denote the weak simple R-module B/L by S, it is easy to show that the injective hull of S exists, we denote it by $\hat{S}$. Consider the following commutative diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & S = B/L & \xrightarrow{\varphi} & M = R/L = R\overline{1} \\ & & \downarrow & \swarrow f & \\ & & \hat{S} & & \end{array}$$

here i is the inclusion mapping. Next we prove that

$$f \circ \varphi = \mathrm{id}_S.$$

Since $f(M) = f(R\overline{1}) = Rf(\overline{1}) = Rm \subseteq \hat{S}$, and $f(M) \supseteq f(S) = i(S) = S \neq 0$, therefore $Rm \neq 0$ and $Rm \cap S \neq 0$. Hence $Rm \supseteq S$.

By (b), Rm can be embedded in the direct product of weak simple modules, hence we have

$$0 \longrightarrow Rm \xrightarrow{g} \prod_{\alpha \in \Gamma} S_\alpha \xrightarrow{\varphi_\alpha} S_\alpha$$

here φ_α are projections of $\prod S_\alpha$ to S_α. We prove that there exists an index $\alpha \in \Gamma$ such that $\varphi_\alpha \circ g$ is a monomorphism.

Assuming, on the contrary, that for every $\alpha \in \Gamma$, $\mathrm{Ker}(\varphi_\alpha \circ g) \neq 0$, then $K_\alpha = \mathrm{Ker}(\varphi_\alpha \circ g)$ is nonzero submodule of $\hat{S}$ and $K_\alpha \cap S \neq 0 \Longrightarrow K_\alpha \supseteq S \Longrightarrow \bigcap_{\alpha \in \Gamma} K_\alpha \supseteq S$. But $\cap \mathrm{Ker}(\varphi_\alpha \circ g) = g^{-1}(\cap \varphi_\alpha) = g^{-1}(0) = 0 \Longrightarrow S = 0$, this is a contradiction. Therefore there exists an index $\alpha \in \Gamma$ such that $\mathrm{Ker}(\varphi_\alpha \circ g) \neq 0 \Longrightarrow Rm \simeq S_\alpha$. We conclude that Rm is a weak simple module $\Longrightarrow Rm = S$. We obtain $f \circ \varphi = \mathrm{id}_S$, so f is spilt. Therefore $R/L = B/L \dotplus C/L$. If $C/L \neq 0$, by the maximal property of L, we know that $a \in C \Longrightarrow a \in B \cap C \Longrightarrow a + L \in B/L \cap C/L$, this is impossible Hense $C/L = 0$, thus $B = R$ as claimed.

(c)$\Longrightarrow$(a).

Given any left ideal L of R and $L \neq R$, there exists an element $a \in R$, $a \notin L$. Let A be a maximal left ideal which contains L but does not contain a, then A is a maximal left ideal of R. This is to say that every left ideal L of R can be embedded in a maximal left ideal of R. Taken all maximal left ideal of R which contains L, we denote those by $\{A_\alpha\}$ and $\cap A_\alpha \supseteq L$. We prove that $\cap A_\alpha = L$.

Suppose that $\cap A_\alpha \neq L$, then there exists an element $x \in \cap A_\alpha$, $x \notin L$. Let B be a maximal left ideal which contains L but does not contain x, then by (c), B is a maximal left ideal of R which contains L, hence $B \supseteq \cap A_\alpha$ and $x \in B$. This is a contradiction.

References

[1] F. W. Anderson and K. R. Fuller, Rings and categories of modules, Springer-Verlag, Berlin-Heidelberg-New York, 1974.

[2] C. Faith, Algebra I Rings, Modules, and Categories, Springer-Verlag, Berlin-Heidelberg-New York, 1981.

A filtration problem for algebras of finite global dimension

Kunio Yamagata

In this note we consider a problem to compare global dimensions of two algebras, an artin algebra A and a factor algebra A/I of A by a specific ideal I filtered by some indecomposable modules. The class of those ideals contains heredity ideals, and practically any artin algebra of finite global dimension seems to have those ideals. In the first section we introduce a property related to filtrations and minimal projective resolutions of modules. In the second section we show some facts on the global dimension of factor algebras.

Throughout this paper all algebras are artin algebras and modules are finitely generated left modules. For an algebra A, $\mathrm{mod}\,A$ denotes the category of finitely generated left A-modules, $\mathcal{S}(A)$ denotes the set of simple left A-modules, and $\mathrm{pd}_A(X)$ is the projective dimension of a left A-module X.

1. Definition

Let A be an algbra and I an ideal of A, and decompose the ideal I as a left A-module to a direct sum of two submodules, say I_0 and I_1, such that

$$I_0 \subseteq \mathrm{rad}\, P_0 \text{ and } I_1 = P_1$$

where P_0, P_1 are projective modules with $A = P_0 \oplus P_1$. Since I is an ideal of A, this is always possible.

Let Δ be a finite set of nonisomorphic indecomposable A-modules and let $\mathcal{F}(\Delta)$ be the class of A-modules with filtrations in Δ, that is, $0 \in \mathcal{F}(\Delta)$, and a non-zero module M belongs to $\mathcal{F}(\Delta)$ if and only if there is a chain of submodules

$$M = M_0 \supset M_1 \supset \ldots \supset M_s \supset M_{s+1} = 0$$

such that M_i/M_{i+1} is isomorphic to a module in Δ for any i. (We say simply that M_i/M_{i+1} belongs to Δ and denote it by $M_i/M_{i+1} \in \Delta$.) We assume that I_0 and I_1 belong to $\mathcal{F}(\Delta)$.

A left A-module M belonging to Δ is said to be **Δ-simple** if any nonzero morphisms $f : X \to M$ and $g : M \to Y$ are isomorphic for any X and Y from Δ.

Let $P.(M)$ be a minimal projective resolutuion of a module M, say

$$\cdots \longrightarrow P_n \xrightarrow{f_n} P_{n-1} \longrightarrow \cdots \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0.$$

Then $P.(M)$ is said to be **Δ-closed** if $\mathrm{Ker}(f_n|_I)$ and $\mathrm{Im}(f_n|_I)$ belong to $\mathcal{F}(\Delta)$ for any $n > 0$, where $f_n|_I : IP_n \to IP_{n-1}$ is a restriction of f_n. A Δ-closed minimal projective resolution $P.(M)$ is said to be **Δ-splittable** if $f_n|_I$ is splittable for any $n > 0$.

In [4] it is shown a construction of algebras with two simple modules and with arbitrarily large global dimension such that minimal projective resolutions of any modules are Δ-splittable by some ideals. In fact, for those algebras and ideals contained in the radicals, say A and I, the restriction $f|_I : IP \to IQ$ is splittable for any morphism $f : P \to Q$ between projective A-modules, and moreover the factor algebra A/I has smaller global dimension than $\mathrm{gl\,dim}\, A$.

2. Comparison of global dimensions

In this section, as in the above, I is an ideal of an algebra A and Δ is a finite set of nonisomophic indecomposable A-modules. In the case when there are modules X from Δ annihilated by I, we put

$$\mathrm{pd}_{A/I}(\Delta) = \max\{\mathrm{pd}_{A/I}(X) \mid X \in \Delta \textit{ and } IX = 0\}$$

and in the other case, $\mathrm{pd}_{A/I}(\Delta) = -1$.

The following theorem is proved in [5].

Theorem 2.1 *Let Δ be a finite set of indecomposable modules over an algebra A.*
(1) *If I be an ideal of A satisying the following two conditions:*

(a) *$I_0{}^2 = 0$, and any indecomposable summand of $_AI_1$ is Δ-simple,*

(b) *every simple A-module has a Δ-closed minimal projective resolution,*

then

$\mathrm{gl\,dim}\, A - \mathrm{pd}_A(I) - 2 \le \mathrm{gl\,dim}\, A/I$ *and*

$$\mathrm{gl\,dim}\, A/I \le \max\{\mathrm{pd}_A(S) \mid S \in \mathcal{S}(A) \textit{ and } IP(S) \subsetneq \mathrm{rad}\, P(S)\} + \mathrm{pd}_{A/I}(\Delta) + 1,$$

where $P(S)$ is a projective cover of an A-module S.

(2) *If an ideal I is projective as a left A-module and satisfies the condition* (a) *above and* $\mathrm{Hom}(I, \mathrm{rad}\, I) = 0$, *then*

$$\mathrm{gl\,dim}\, A - 2 \le \mathrm{gl\,dim}\, A/I \le \max\{\mathrm{pd}_A(S) \mid S \in \mathcal{S}(A) \textit{ and } IP(S) \subsetneq \mathrm{rad}\, P(S)\}.$$

This implies a well known inequality for a heredity ideal I, namely, $\mathrm{gl\,dim}\, A - 2 \le \mathrm{gl\,dim}\, A/I \le \mathrm{gl\,dim}\, A$ [1].

Proposition 2.2 *Let A be an algebra and D a semisimple algebra. Let $_AM_D$ and $_DN_A$ be bimodules such that there is an A-bimodule monomorphism $\varphi : M \otimes_D N \to A$ and put* $I = \mathrm{Im}(\varphi)$. *Then any A-module has a Δ-splittable minimal projective resolution, where*

Δ is the set of nonisomorphic direct summands of the left A-module I. Moreover it holds that ${I_0}^2 = 0$ and any indecomposable summand of ${}_AI_1$ is Δ-simple.

Proof. Let P, Q be projective A-modules and $f : P \to Q$ an A-homomorphism. For isomorphisms

$$\sigma : P \xrightarrow{\sim} \oplus_{i=1}^m Ae_i,\ \beta : Q \xrightarrow{\sim} \oplus_{j=1}^n Ae'_j,$$

where e'_i, e'_j are primitive idempotents, let

$$\beta f \alpha^{-1} = (f_{ji}) : \oplus_i Ae_i \xrightarrow{\alpha^{-1}} P \xrightarrow{f} Q \xrightarrow{\beta} \oplus_j Ae'_j.$$

Here each homomorphism $f_{ji} : Ae_i \to Ae'_j$ is a right multiplication of an element of A, say $a_{ij} = e_i a_{ij} e'_j \in A$. Now, take A-homomorphisms g_{ji}, h_{ji} such that the following diagram is commutative:

$$\begin{array}{ccccccc}
M \otimes_D (\oplus_i Ne_i) & \xrightarrow{nat} & \oplus_i (M \otimes_D Ne_i) & \xrightarrow{\oplus \varphi e_i} & \oplus_i Ie_i & \longrightarrow & IP \\
{\scriptstyle (h_{ji})}\downarrow & & {\scriptstyle (g_{ji})}\downarrow & & \downarrow{\scriptstyle (f_{ji}|_I)} & & \downarrow{\scriptstyle f_I} \\
M \otimes_D (\oplus_j Ne'_j) & \xrightarrow{nat} & \oplus_j (M \otimes_D Ne'_j) & \xrightarrow{\oplus \varphi e'_j} & \oplus_j Ie'_j & \longrightarrow & IQ
\end{array}$$

Since φ is a right A-homomorphism, it is seen that each g_{ji} is induced by right multiplication of a_{ij}. Let $\psi_{ij} : Ne_i \to Ne'_j$, $xe_i \mapsto xa_{ij}$. Then it is easily seen that $h = 1_M \otimes \psi$, where $h = (h_{ji})$ and $\psi = (\psi_{ij}) : \oplus_i Ne_i \to \oplus_j Ne'_j$. This implies that f_I is splittable because so is h.

For example, if A is an algebra with an heredity ideal AeA for an idempotent e, then eAe is semisimple and the natural morphism $f : Ae \otimes_{eAe} eA \to A$ is monomorphic [1].

Corollary 2.3 *Let A be an algebra and D a semisimple algebra. Let ${}_AM_D$ and ${}_DN_A$ be (A, D)-bimodules such that there is an A-bimodule monomorphism $\varphi : M \otimes_D N \to A$. Let $I = \operatorname{Im}(\varphi)$. Then*

$$\operatorname{gl\,dim} A - 2 \leq \operatorname{gl\,dim} A/I \leq \operatorname{gl\,dim} A + p(M) + 1,$$

where $p(M)$ is a maximal projective dimension of indecomposable summands of ${}_AM$ annihilated by I.

Proof. It follows from the proposition above that the ideal I satisfies the coditions in Theorem 2.1. Moreover, ${}_AM \otimes_D N \in \operatorname{add}({}_AM)$ because D is semisimple, and hence ${}_AI \in \operatorname{add}({}_AM)$ and $\operatorname{pd}_{A/I}(\Delta) \leq p(M)$, where Δ is the set of all nonisomorphic indecomposable summands of ${}_AI$. The required inequality hence follows from Theorem 2.1.

The existence of nonzero ideals satisfiyng the two conditions in the theorem above is not known in general for algebras of finite global dimension.

Problem 2.4 Let A be an algebra of finite global dimension. Then, find a chain of ideals

$$A = I_0 \supset I_1 \supset \cdots \supset I_m \supset I_{m+1} = 0$$

satisfying the following conditions for any i.

a) I_i/I_{i+1} satisfies the conditions in Theorem 2.1 (1) for a set Δ_i in mod A/I_{i+1},

b) gl dim A/I_{i+1} is finite,

c) A/I_i is a quasi-hereditary, and

d) $m \leq$ gl dim A.

A module is said to be homogeneous if it is isomorphic to a direct sum of copies of an indecomposable module which is called a type of the module. A semihomogeneous module of rank r is, by definition, a direct sum of r homogeneous modules of orthogonal types. The following are considered as a special case of the problem above.

Problem 2.5 What algebra A has a module M such that $\mathrm{End}_A(A \oplus M)$ is quasi-hereditary and M is a direct sum of m semihomogeneous modules with $m \leq$ gl dim A.

Problem 2.6 Find a semihomogeneous module M with $\mathrm{End}_A(A \oplus M)$ quasi-hereditary for a non-quasi-hereditary algebra A of gl dim $A = 3$.

Finally we put a simple remark on global dimensions of factor algebras in the following where the assumption holds for algebras, over an algebraically closed field, with finite global dimension [2, 3].

Remark *Let A be a basic and connected algebra whose radical is not simple as an A-module, and assume that* $\mathrm{Ext}^1_A(S, S) = 0$ *for any simple A-module S. Then there is a nonzero ideal I properly contained in* rad A *such that* gl dim A/I *is finite.*

Referenses

[1] V. Dlab and C. M. Ringel, Quasi-hereditary rings. Illinois J. Math. **33**, 280-291 (1989).

[2] K. Igusa, Notes on the no looop conjecture. J. Pure and Applied Algebra **69**, 161-176 (1990).

[3] H. Lenzing, Nilpotente Elemente in Ringen von endlicher globaler Dimension. Math. Z., **108**, 313-329 (1969).

[4] K. Yamagata, Remarks on a family of algebras with the same number of simple modules. Proc. the 24th Symposium on Ring Theory, Osaka, 1991, 85-89 (1991).

[5] K. Yamagata, Global dimensions of factor algebras. in preparation.

Kunio Yamagata
Institute of Mathematics, University of Tsukuba
Tsukuba, Ibaraki 305, Japan

ACTIONS OF POINTED HOPF ALGEBRAS ON PRIME RINGS

Tadashi YANAI

In this note, we report some results on the "outer" actions of pointed Hopf algebras on prime rings and Galois correspondence theory obtained by applying Milinski's idea in [M] to the Galois correspondence theory of Kharchenko [K].

Throughout, R represents a prime ring and $\mathcal{F}$ the set of all nonzero ideals of R. $R_{\mathcal{F}} := \varinjlim_{I\in\mathcal{F}} \mathrm{Hom}({}_RI, {}_RR)$ denotes the left Martindale quotient ring of R and K the center of $R_{\mathcal{F}}$. A subring $U \subseteq R$ is called *rationally complete* if for x in R and a nonzero ideal A of U, $Ax \subseteq U$ implies $x \in R$.

Let H be a pointed Hopf algebra and we assume that $R_{\mathcal{F}}$ is a left H-module algebra via $H \otimes R_{\mathcal{F}} \ni h \otimes x \mapsto h \cdot x \in R_{\mathcal{F}}$. $R_{\mathcal{F}}\#H$ represents the smash product algebra. In this case, $R_{\mathcal{F}}$ is a left $R_{\mathcal{F}}\#H$-module via $a\#h \to x = a(h \cdot x)$ for $a, x \in R_{\mathcal{F}}$ and $h \in H$. We say that the action of H on $R_{\mathcal{F}}$ is *continuous* if for any $h \in H$ and $J_1 \in \mathcal{F}$, there exists $J \in \mathcal{F}$ with $h{\cdot}J \subseteq J_1$. On the other hand, the action of H is called *outer* if $(R_{\mathcal{F}}\#H)^R := \{\xi \in R_{\mathcal{F}}\#H | r\xi = \xi r \text{ for any } r \in R\} = K$.

Henceforth, we assume that H is a finite dimensional pointed Hopf algebra acting on $R_{\mathcal{F}}$ and the action is continuous and outer. We set $R^H := \{r \in R | h \cdot r = \varepsilon(h)r \text{ for all } h \in H\}$. A subalgebra $U \subseteq R$ is said to be an *intermediate* subalgebra if $R^H \subseteq U$. Since H is a finite dimensional Hopf algebra, it has a nonzero left integral t. We write $K\#H$ instead of $K \otimes H$.

An intermediate subalgebra U is characterized as follows.

Lemma 1. ([M,Y])

(1) U is a prime ring.

(2) $\{x \in R_{\mathcal{F}} | xs = sx$ for all $s \in U\} = K$.

(3) (The bimodule property) For any $0 \neq {}_R M_U \subseteq {}_R R_U$, there exists $I \in \mathcal{F}$ such that $I \subseteq M$.

For an intermediate subalgebra $U \subseteq R$, we set $\Phi(U) := \{\xi \in R_{\mathcal{F}}\#H | s\xi = \xi s$ for all $s \in U\}$ $(\subseteq R_{\mathcal{F}}\#H)$ and for a left K-subspace $\Lambda \subseteq R_{\mathcal{F}}\#H$, $\Psi(\Lambda) := \{r \in R | \xi r = r\xi$ for all $\xi \in \Lambda\}$ $(\subseteq R)$. Then, we have the followings.

Lemma 2.

(1) $\Phi(R) = K$ and $\Psi(K) = R$.

(2) $\Phi(R^H) = K\#H$ and $\Psi(K\#H) = R^H$.

Lemma 3. ([Y])

(1) For any left K-subspace $\Lambda \subseteq K\#H$, $\Psi(\Lambda)$ is a rationally complete subalgebra of R.

(2) For any intermediate subalgebra U, $\Psi(U)$ is a subalgebra of $R_{\mathcal{F}}\#H$ containing K.

So, Φ and Ψ give a correspondence between the set of all rationally complete intermediate subalgebras of R and the set of all subalgebras of $R_{\mathcal{F}}\#H$ containing K.

(3) For any intermediate subalgebra U, $U_1 := \{s \in U | h \cdot s \in R$ for all $h \in H\}$ satisfies $\Phi(U_1) = \Phi(U)$.

Now, we consider the following problems which generalize the "X-outer" Galois correspondence theory of Kharchenko:

Question.

1. *When $\Psi(\Phi(U)) = U$ holds for a rationally complete subalgebra $U \subseteq R$?*

2. *When $\Phi(\Psi(\Lambda)) = \Lambda$ holds for a subalgebra K-subspace $\Lambda \subseteq K\#H$ containing K?*

In the remaining, we give the sufficient conditions so that $\Psi(\Phi(U)) = U$ holds and some concrete examples such that those conditions are satisfied. The solution of the second question is a future problem.

Condition A. There exist some $r_i \in R, s_i \in U (i = 1, \ldots, n)$ and $\xi_j \in \Phi(U), q_j \in R_{\mathcal{F}} (j = 1, \ldots m)$, so that $\sum_{i=1}^{n} t \cdot (xr_i)s_i = \sum_{j=1}^{m} \xi_j \rightarrow (xq_j)$ for all $x \in R$.

Let $R_{\mathcal{F}}{}^{\text{op}}$ be an algebra opposite to $R_{\mathcal{F}}$ with multiplication $a \circ b = ba$ for $a, b \in R_{\mathcal{F}}{}^{\text{op}}$ and H^{cop} the Hopf algebra which is co-opposite to H. In this case, $R_{\mathcal{F}}{}^{\text{op}}$ is a left H^{cop}-module algebra and $R_{\mathcal{F}}{}^{\text{op}}$ is a left $R_{\mathcal{F}}{}^{\text{op}}\#H^{\text{cop}}$-module via $a\#h \rightarrow' x = a \circ (h \cdot x)$ for $a, x \in R_{\mathcal{F}}{}^{\text{op}}$ and $h \in H^{\text{cop}}$. For an intermediate subalgebra $U^{\text{op}} \subseteq R^{\text{op}}$, we set $\Phi'(U^{\text{op}}) := \{\xi \in R_{\mathcal{F}}{}^{\text{op}}\#H^{\text{cop}} | \xi s = s\xi \text{ for all } s \in U^{\text{op}}\}$.

Condition B. There exist some $r'_i \in R^{\text{op}}, s'_i \in U^{\text{op}} (i = 1, \ldots, n')$ and $\xi'_j \in \Phi'(U^{\text{op}}), q'_j \in R_{\mathcal{F}}{}^{\text{op}} (j = 1, \ldots, m')$, so that $\sum_{i=1}^{n'} t \cdot (x \circ r'_i) \circ s'_i = \sum_{j=1}^{m'} \xi'_j \rightarrow' (x \circ q'_j)$ for all $x \in R^{\text{op}}$.

The following is a partial generalization of Kharchenko's theorem.

Theorem 4. *If the Conditions A. and B. are satisfied, then $\Psi(\Phi(U)) = U$ is valid.*

Examples.

Let U be a rationally complete subalgebra.

(1) If $\Phi(U) \subseteq K\#G(H)$, where $G(H) := \{g \in H | \Delta(g) = g \otimes g\}$, then Conditions A. and B. are satisfied and $\Psi(\Phi(U)) = U$. In this case $U = R^{G'}$ for some subgroup $G' \subseteq G(H)$.

(2) If $H = \mathbf{k}G$, where G is a group of automorphisms, then $\Psi(\Phi(U)) = U$ for any rationally complete subalgebra U and in this case $U = R^{G'}$ for some subgroup

$G' \subseteq G$ ([K: Theorem 3.10.2], [Y]).

(3) If Char $\mathbf{k} > 0$ and $H = u(L)$, the restricted enveloping algebra of L, where L is a finite dimensional restricted Lie algebra of derivations of $R_{\mathcal{F}}$, then $\Psi(\Phi(U)) = U$ for any rationally complete intermediate subalgebra U ([K: Theorem 4.5.2], [Y]).

In this case, U is a subalgebra consisting of constants of a certain left K-subspace of $K \otimes L$.

(4) We define $H_{n^2} := \mathbf{k} < g, x | g^n = 1, x^n = 0, xg = \zeta_n gx >$, $\Delta(g) = g \otimes g$, $\Delta(x) = 1 \otimes x + x \otimes g$, where ζ_n is a root of n-th cyclotomic polynomial of 1 over $\mathbb{Z}$. (In this case, we assume that $\mathbf{k}$ contains ζ_n.)

If $H = H_4$ ([Y]) or H_9, then $\Psi(\Phi(U)) = U$ holds for any rationally complete subalgebra U. In both cases, U is one of the following four subalgebras:

(i) R,

(ii) $R_g = \{r \in R | g \cdot r = r\}$

(iii) $R_\alpha = \{r \in R | x \cdot r = \alpha(r - g \cdot r)\}$ $(\alpha \in K)$

(iv) R^H.

References

[K] V. K. Kharchenko, *Automorphisms and Derivations of Associative Rings*, Kluwer, Dordrecht, 1991.

[M] A. Milinski, *Actions of pointed Hopf algebras on prime algebras*, Comm. Algebra **23** (1995), 313–333.

[Y] T. Yanai, *Correspondence theory of Kharchenko and the actions of pointed Hopf algebras* (submitted).

Niihama National College of Technology

7-1, Yagumo-cho, Niihama, Ehime, 700 Japan

The Structure of Involutive Rings Which in
the Residue Class Rings

Yang Zixu

let R be an arbitrary associativde ring with a unit element, and let us denote the unit group of R by U(R) which consists of all the units of R. To study the structure of R and to definite R by making use of U(R) is one of the most important methods on research of ring theory. For example, in [3], the author has proved that the ring R is a noncommutative division ring if and only if U(R)=R-{0} and U(R) is not a solvable group. The primary purpose, in this paper, is to study the structure of residue class ring R if U(R) is an involutive group.

Definition 1. Let G be a group, and e an identity of G. If element of G satisfies equation $x^2=e$, that is to say, the inverse elements of every elements of G are itself, then we call G an Involutive Group.

Clearly, the involutive groups are commutalive groups.

Definition 2. Let R be a ring with a unit element. We call R the Involutive Ring if the unit group U(R) of R is an involutive group.

It is clear that the integral ring Z and the ring $Z_3=\{0, 1, 2\}$ are involutive rings.

Theorem 1. Suppose that R is a ring with a unit element, and R_i ($i=1, 2, \cdots, n$) are the ideals of R, and $R=R_1 \oplus R_2 \oplus \cdots \oplus R_n$. Then R is an involutive ring if and only if the every ideals R_i ($i=1, 2, \cdots, n$) are both the involutive rings.

Proof Let $a \in R$, and

$$a=a_1+a_2+\cdots+a_n, a_i \in R_i, \quad i=1, 2, \cdots, n.$$

Then

$$e=e_1+e_2+\cdots+e_n, e_i \in R_i, \quad i=1, 2, \cdots, n.$$

where e and e_i are the unit elements of R and R_i respectively.

From this, we may prove that

$$\phi: a=a_1+a_2+\cdots+a_n \to (a_1, a_2, \cdots, a_n)$$

is an isomorphic mapping from group U(R) to group $U(R_1)\times U(R_2)\times\cdots\times U(R_n)$.

Therefore

$$U(R) = U(R_1)\times U(R_2)\times\cdots\times U(R_n).$$

It follows that, if every $U(R_i)$ are the involutive groups, then for every

$$a=a_1+a_2+\cdots+a_n \in U(R), a_i \in U(R_i), i=1, 2, \cdots, n$$

we have $a_i^2=e_i$, and

$$a^2=a_1^2+a_2^2+\cdots+a_n^2=e_1+e_2+\cdots+e_n=e,$$

thus U(R) is an involutive group, and hence that R is an involutive ring.

The necessity of the theorem is obvious.

Theorem 2. The polynomial ring on an involutive ring is an involutive ring.

Since the unit group of ring R coincides with the group of polynomial ring $R[x_1, x_2, \cdots, x_n]$ on the R, the proof of the theorem is obvious.

The following is mainly to study the Involutive ring which in the residue class rings.

Example 1. Residue class ring Z_{12} modulo 12 is an involutive ring.

In fact, since $U(R)=\{1, 5, 7, 11\}$, and since

$$1^2=1, 5^2=25=1, 7^2=49=1, 11^2=121=1,$$

it follows that $U(Z_{12})$ is an involutive group, and hence Z_{12} is an involutive ring.

Example 2. Residue class ring Z_{10} modulo 10 is not an involutive ring.

In fact, since $U(Z_{10})=\{1, 3, 7, 9\}$, and since $3^2=9\neq 1$, therefore $U(Z_{10})$ is not an involutive group, and hence Z_{10} is not an Involutive ring.

Since Z is an involutive ring, and since $Z/\langle 10\rangle = Z_{10}$, therefore $Z/\langle 10\rangle$ is not an involutive ring. It follows that, in general, the quotient ring of an involutive ring is not an involutive ring.

The following aim theorem will make it very clear that among residue class rings which rings are earth involutive rings.

Theorem 3. Asufficient and necessary condition that the residue ring Z_n modulo $n>1$ is an involutive ring is that n are the following integer:

$$2, 3, 4, 6, 8, 12, 24. \quad (1)$$

Proof We may immediate calculate that Z_n are the involutive rings, if n is the anyy integer in (1).

We prove that. in the following, for every positive integer n which is not among the(1), the rings Z_n are not involutive rings. Such positive integer n has five cases as follows:

1) $n=2^s 3^t$, where $s>4$, $t=0$ or 1.

If $t=0$, then $n=2^s$. Since $3\in U(Z_n)$, $s>4$, hence

$$3^2=9<2^4<2^s,$$

thus $3^2=9\neq 1$ in the group $U(Z_n)$. Therefore $U(Z_n)$ is not an involutive group;

If $t=1$, then $n=2^s.3$. Since $5\in U(Z_n)$, $s>4$, hence

$$5^2=25<2^4\ 3<2^s\ 3,$$

thus $5^2=25\neq 1$. Therefore $U(Z_n)$ is not also an involutive group.

2) $n=2^s 3^t$, where $s>0$, $t>2$.

If $s=0$, then $n=3^t$. Because $2\in U(Z_n)$, $t>2$, and so

$$2^2=4<3^2<3^t,$$

thus $2^2=4\neq 1$, and hence that $U(Z_n)$ is not an involutive group;

If $s=1$ and $t=2$, the>n $n=2\ 3^2$, Since $5\in U(Z_n)$, it follows that $5^2=25=7\neq 1$ in the group $U(Z_n)$, and hence that $U(Z_n)$ is not an inbolutive group;

If $s=1$ and $t>2$, then $n=2\ 3^t$. In this case, since $5\in U(Z_n)$, and

$$5^2=25<2\ 3^3<2\ 3^t,$$

it follows that $5^2=25\neq 1$, and hence that $U(Z_n)$ is not also an involutive group.

3) $n=p^k$, where p is a prime number which greater than 3, $k>1$.

Since $2\in U(Z_n)$ and $2^2=4<5<p^k$, it follows that $2^2=4\neq 1$, and hence that $U(Z_n)$ is not an

involutive group.

4) $n=2^s p^t$, where $s>1$, $t>1$, and p is a prime which greater than 3.

Since $3\in U(Z_n)$ and $3^2=9<2\ 5<2^s p^t$, it follows that $3^2=9\neq 1$, and hence that $U(Z_n)$ is not an involutive group.

5) $n=2^s p_1^{t_1}p_2^{t_2}\cdots p_m^{t_m}$, where $s>0$, $m>2$ and $p_1, p_2, \cdots, p_m$ are m distince odd prime numbers each other, and $t_i>1$, $i=1,2,\cdots,m$.

Because of the order of group $U(Z_n)$ is $\Phi(n)$(Euler's function), it is clear that in order to prove that $U(Z_n)$ is not an involutive group, it is sufficient to show that the number of solutions of the residue class equation

$$x^2 \equiv 1 \pmod n \tag{2}$$

is less that $\phi(n)$.

If $s=0$ ir $s=1$, then since the every residue class equations

$$x^2\equiv 1 \pmod{p_i^{k_i}} \quad i=1,2,\cdots,m$$

has two solutions, and hence that the residue class equation (2) has 2^m solutions. But since $m>2$, this implies that

$$2^m<p_1^{t_1-1}p_2^{t_2-1}\cdots p_m^{t_m-1}(p_1-1)(p_2-1)\cdots(p_m-1)=\phi(n);$$

When $s=2$, we have $n=2^2\ p_1^{t_1}p_2^{t_2}\cdots p_m^{t_m}$. Since

$$x^2 \equiv 1 \pmod 4$$

has two solutions, and hence that the residue class equation (2) has 2^{m+1} solutions. Consequenly, since $m>2$, we have

$$2^{m+1}<2p_1^{t_1-1}p_2^{t_2-1}\cdots p_m^{t_m-1}(p_1-1)(p_2-1)\cdots(p_m-1)=\phi(n);$$

When $s>2$, then since the residue class equation

$$x^2\equiv 1 \pmod{2^s}$$

has four solutions, and hence that equation (2) has 2^{m+2} solutions. Similarly, since $m>2$, we have also

$$2^{m+2}<2^{s-1}p_1^{t_1-1}p_2^{t_2-1}\cdots p_m^{t_m-1}(p_1-1)(p_2-1)\cdots(p_m-1)=\phi(n).$$

Suppose that $C_n=\langle a\rangle$ is a cyclic group of order n which generated by element a, and aut C_n is its automorphism group. Then the group Aut C_n and $U(Z_n)$ are both of order $\Phi(n)$, and clearly

$$\psi: \sigma_m \to m$$

is an isomorphism mapping from groups Aut C_n to $U(Z_n)$, where $\sigma_m\in$ Aut C_n and $\sigma_m(a)=a^m$, but $(m, n)=1$. Therefore we have

Theorem 4. Aut $C_n\cong U(Z_n)$.

In view of Theorem 3 and Theorem 4, we may obtain at once the following

Corollary 1. A sufficient and necessary condition that the automorphism group of the cyclic group of order $n>1$ is an involutive group is that n are the following integers:

2, 3, 4, 6, 12, 24.

The author, in [4] gives the concept of cyclic rings, that is the ring R is called a cyclic ring if the additive group $(R, +)$ of R is a cyclic group.

Lemma. Let R be a cyclic ring of order n. Then R has an unit element if and only if $R=Zn$.

Proof Let $R=\langle a\rangle=\{1, a, 2a, \cdots, (n-1)a\}$, and the order of a in additive group $(R, +)$, in n, and $a^2=ka$.

The sufficiency of Lemma is obvious. In the following, we prove that the necessity of Lemma.

Suppose that R has an unit element. Then, from [4], we have $(k,n)=1$ and it follows that there exist integers s, t such that

$$ks+nt=1 . \tag{3}$$

Let $\phi: rsa \to r$. If $r_1sa=r_2sa$, then $(r_1-r_2)sa=0$, and hence $n \mid (r_1-r_2)s$. But from (3) we have $(n,s)=1$, therefore $n \mid (r_1-r_2)$, that is $r_1=r_2$ in ring Z_n. Hence the ϕ is a mapping from the ring R to ring Z_n. Similarly, it is easily verified that ϕ is a one-to-one mapping and holds add operation. Moreover, we have

$$\begin{aligned}\phi(r_1sa\ r_2sa)&=\phi(r_1r_2s^2a^2)=\phi(r_1r_2s^2ka)\\&=\phi(r_1r_2ks\ sa)=r_1r_2ks\\&=r_1r_2(1-nt)=r_1r_2\\&=\phi(r_1sa)\ \phi(r_2sa),\end{aligned}$$

and hence that ϕ is isomorphic mapping from the ring R onto the ring Z_n. Therefore $R=Z_n$.

From this Lemma, we may easy obtain the following

Theorem 5. The cyclic ring R with an unit element is an involutive ring if and only if $R = Z$ or R is finite and $|R|$ is any integer in (1).

Definition 3. A ring R with an unit element is said to be an U-cyclic ring, if the unit group of R is cyclic group.

By the theory of promitive root in the theory of number, we have at once the following

Theorem 6 The residue class ring Z_n modulo n is a cyclic ring if and only if n are the following integers:

$$2,\ 4,\ p^k,\ 2p^k,$$

where p is an arbitrary odd prime number, and k is any positive integer.

By Theorem 3 and Theorem 6, we have at once the following

Corollary 2 The involutive ring $Z_n(n>1)$ is an U-cyclic ring if and only if n are the following integers: 2, 3, 4, 6.

References

[1] Hua Lookang; Introduction to theory of number, Beijing, Science publishing house, 1979.

[2] John Shout; Automorphisms of cyclic groups, Amer. Math. Monthly 71(1964), 568.

[3] Hua Lookang; On the multiplicative groups of a field, Annals of Sciences, Vol. 3, 1(1950), 1~6.

[4] Yang Zixu; On cyclic rings and their idempotent elements, Math. in practice and Theory, 1985, No. 3, 73~76.

[5] Yang Zixu; The nilpotent radicals of cyclic rings, Quart. J. Math., 3(1988), No. 2, 105~107.

Shandong Finance Institute
Jinan, Shandong
People's Republic of China

MORITA EQUIVALENCE OF FUNCTOR CATEGORIES

YAO MUSHENG YIN CUI

Department of Mathematics,Fudan University, Shanghai,200433,China

INTRODUCTION

Since K.Morita established the well-known theory of equivalences of categories of modules, many people have developed this theory. K.R.Fuller extended this theory to equivalence between full subcategory of the category of modules and the category of modules. M. Sato extended it to equivalence between module subcategories in [2]. In [3], Abrams established the theory of Morita equivalence of rings with local units. P. N.Ánh and L. Márki [4] simplified this theory,covered a wider range of rings, and transfered more of the classical Morita theory. H. Komatsu[5] developed the theory of Morita equivalence for the category of S-unital modules. In 1990, Angel Del Río gave a description of the equivalence of categories of $gr - R$ and $gr - A$, where R and A are graded rings respectively (see [6]).

In this paper, we'll give further generalizations. Morita equivalence of functor categories will be developed, and many known results may be regarded as special cases of the theory.

1.PRELIMINARIES

Let $\mathcal{C}$ and $\mathcal{C}'$ be two preadditive categories and let $\mathcal{C}$ be small. We denote by $[\mathcal{C},\mathcal{C}']$ the class of all covariant functors from $\mathcal{C}$ to $\mathcal{C}'$. For any $S, T \in [\mathcal{C},\mathcal{C}']$, a morphism from S to T is a functorial morphism u from the functor S to the functor T With the law of composition of functorial morphisms , $[\mathcal{C},\mathcal{C}']$ is a category.

We denote by $(\mathcal{C},\mathcal{C}')$ the full subcategory of $[\mathcal{C},\mathcal{C}']$, which consists of all additive functors from $\mathcal{C}$ to $\mathcal{C}'$. Also we denote by $(\mathcal{C}^o,\mathcal{C}')$ the category of additive contravariant functors from $\mathcal{C}$ to $\mathcal{C}'$. Paticularly, if $\mathcal{C}$ is a small preadditive category, we denote $\mathbf{Mod}\mathcal{C}$ as the category $(\mathcal{C},\mathbf{Ab})$ and $\mathbf{Mod}\mathcal{C}^o$ as $(\mathcal{C}^o,\mathbf{Ab})$, where, $\mathbf{Ab}$ is the category of abelian groups. An object of $\mathbf{Mod}\mathcal{C}$ is called a left module over $\mathcal{C}$, and an object of $\mathbf{Mod}\mathcal{C}^o$ is called a right module over $\mathcal{C}$.

An important contravariant functor $h : \mathcal{C} \to \mathbf{Mod}\mathcal{C}$ is defined as follows: for any $X \in ob\mathcal{C}$, $h(X) = h^X$, such that $h^X(Y) = Hom_{\mathcal{C}}(X,Y)$. Similarly, a functor $h^0 : \mathcal{C} \to \mathbf{Mod}\mathcal{C}^0$ is defined by the equality $h^0(X) = h_X$, such that $h_X(Y) = Hom_{\mathcal{C}}(Y,X)$.

It is well-known that the category of modules over a given ring is a special functor category. It is not difficult to see that the category of graded modules

$R - gr$ $(gr - R)$ is also a functor category. In fact, let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a graded ring, where G is a group. Define a category C as follows: the objects of C are $\{g \mid g \in G\}$ and $Hom(g,h)$ is the additive group $R_{hg^{-1}}$. If $M \in (C, \mathbf{Ab})$, we denote $M(g) = M_g$, for any $g \in obC$. For any $a \in R_g = Hom(h, gh)$, the left multiplication of a defines a map

$$M(a) : M_h \to M_{gh}.$$

¿From this point of view, it is easy to see that $(C, \mathbf{Ab}) \cong R - gr$. Similaly, we have $(C^o, \mathbf{Ab}) \cong gr - R$.

2. Bimodules, Hom and Tensor products

Definition 1. Let C and C' be preadditive categories, a bimodule is a bifunctor $P : C \times C'^o \to \mathbf{Ab}$. That is , for any $X \in C$, $Y \in C'^o$, $P(X,Y)$ is an abelian group, and for every $X \in C, P(X,-)$ is a right module over C'^o while $P(-,Y)$ is a left C-module for every $Y \in C'$, and for $f : X \to X'$ in C, $g : Y' \to Y$ in C'^o the following diagram is commutative:

$$\begin{array}{ccc} P(X,Y) & \xrightarrow{P(X,g)} & P(X,Y') \\ {\scriptstyle P(f,Y)}\downarrow & & \downarrow{\scriptstyle P(f,Y')} \\ P(X',Y) & \xrightarrow{P(X',g)} & P(X',Y') \end{array}$$

Definition 2. Given $M \in \mathbf{Mod}C^o$ and $N \in \mathbf{Mod}C$, let $F = \bigoplus_{X \in C} M(X) \otimes_{\mathbf{Z}} N(X)$, where $\mathbf{Z}$ is the ring of integers. Let K be the subgroup of F generated by

$$\{mf \otimes n - m \otimes fn | m \in M(Y), n \in N(X), f \in hom_C(X,Y), \}$$

Define the tensor product of M and N as the quotient group $M \otimes N = F/K$.

Remark. Assume that N is a C-C' - bimodule, that is, $N \in \mathbf{Mod}(C \times C'^o)$, then for every $X \in C$, $N(X) \in \mathbf{Mod}C'^o$. Let $F : C'^o \to \mathbf{Ab}$, such that $F(X') = \bigoplus_{X \in C} M(X) \otimes_{\mathbf{Z}} N(X, X')$ for any $X' \in C'^o$. $F(f') = \bigoplus_{X \in C} M(X) \otimes_{\mathbf{Z}} N(X, f')$ for any $f' : Y' \to X'$ in C'^o. K is a submodule of F and restriction of $F(f')$ to $K(X')$ gives a morphism from $K(X')$ to $K(Y')$, hence $M \otimes N = F/K$ is a right C' - module. Similarly, if $M \in \mathbf{Mod}(C' \times C^o)$, then $M \otimes N$ is a left C' -module.

Proposition 1. *Let M be a right module over C, then for any $X \in C$, there exists a map*

$$\eta_X : M(X) \to M \otimes h^X$$

which is an isomorphism. Moreover, $\eta = \{\eta_X | X \in C\} : M(-) \to M \otimes h^-$ is a functorial isomorphism.

Proposition 2. *If $\{M_\alpha, \alpha \in I\}$ are right modules over C and N is a left module over C, then we have an isomorphism*

$$\eta_N : (\bigoplus_{\alpha\in I} M_\alpha) \otimes N \to \bigoplus_{\alpha\in I}(M_\alpha \otimes N)$$

Moreover, $N \to \eta_N$ is natural.

Proposition 3. *Let $M \in \mathbf{Mod}C^0, N \in \mathbf{Mod}C$, then the functors $M\otimes - : \mathbf{Mod}C \to \mathbf{Ab}$ and $- \otimes N : \mathbf{Mod}C^0 \to \mathbf{Ab}$ are right exact.*

Theorem 1. *Let $M \in \mathbf{Mod}C^0$, then the functor $M \otimes - : \mathbf{Mod}C \to \mathbf{Ab}$ has a right adjoint*

$$Hom(M, -) = Hom_{\mathbf{Z}}(M(-), -) : \mathbf{Ab} \to \mathbf{Mod}C$$
$$B \to Hom_{\mathbf{Z}}(M(-), B) : C \to \mathbf{Ab}$$
$$X \to Hom_{\mathbf{Z}}(M(X), B)$$

To simplify notations,now we write $Hom_C(M, N)$ instead of $Hom_{\mathbf{Mod}C}(M, N)$ and $Hom_{C'}(L, N)$ instead of $Hom_{\mathbf{Mod}C'}(L, N)$.

Proposition 4. *Let C, C' be preadditive categories and let ${}_{C'}L \in \mathbf{Mod}C', M_C \in \mathbf{Mod}C^0$, ${}_{C'}N_C \in \mathbf{Mod}(C' \times C^o)$. Then there is a natural isomorphism*

$$Hom_{C'}(L, Hom_C(M, N)) \cong Hom_C(M, Hom_{C'}(L, N))$$

3. Generators, Projective generators and small generators

Definition 3. A subset $\{U_i\}_{i\in I}$ of objects of C is called a set of generators of C if for any couple (X, Y) of objects from C and for any two distinct morphisms $f, g \in Hom_C(X, Y)$, there is $i_0 \in I$ and $k \in Hom_C(U_{i_0}, X)$ such that $fk \neq gk$. An object U_i is called a generator if U_i is an element of a set of generators.

Definition 4. An object X of the additive category C is called small if the functor $h^X : C \to \mathbf{Ab}$ commutes with direct sums. (See [9], P_{90})

Proposition 5. *Let C, C' be preadditive categories and let ${}_{C'}P_C \in \mathbf{Mod}(C^o \times C')$, ${}_{C'}M_C \in \mathbf{Mod}(C^o \times C')$, ${}_CN \in \mathbf{Mod}C$. Then there is a natural homomorphism*

$$\eta : Hom_{C'}(P, M) \otimes_C N \to Hom_{C'}(P, M \otimes_C N).$$

Moreover, if ${}_{C'}P$ is projective in $\mathbf{Mod}C'$, then η is an isomorphism.

Proposition 6. *Let ${}_{C'}P_C \in \mathbf{Mod}(C^o \times C')$, ${}_{C'}M_C \in \mathbf{Mod}(C^o \times C')$, ${}_CN \in \mathbf{Mod}C$. Then there is a natural homomorphism*

$$\phi : P \otimes_{C'} Hom_C(M, N) \to Hom_C(Hom_{C'}(P, M), N)$$

Moreover, if P_C is projective in $\mathbf{Mod}C$, then η is an isomorphism.

4. MAIN THEOREMS

Theorem 2. *Let C, C' be two preadditive categories and let*

$$F : \mathbf{Mod}C \to \mathbf{Mod}C', \quad G : \mathbf{Mod}C' \to \mathbf{Mod}C$$

be inverse equivalences. Let $P(X) = F(h^X)$, for any $X \in obC$ and $Q(X') = G(h^{X'})$, for any $X' \in obC'$. We denote

$$P = P(-) = F(h^-), \quad Q = Q(-) = G(h^-),$$

Where $P : C^o \times C' \to \mathbf{Ab}$ such that $P(X, X') = F(h^X)(X')$ and $Q : C \times C'^o \to \mathbf{Ab}$ such that $Q(Y, Y') = G(h^{Y'})(Y)$. Then ${}_{C'}P_C$ and ${}_CQ_{C'}$ are natural bimodules such that

(1) $\{{}_{C'}P(X)\}_{X \in obC}, \{{}_CQ(X')\}_{X' \in obC'}$ are all small projective generator sets;
(2) $F \cong Hom_C(Q, -)$ and $G \cong Hom_{C'}(P, -)$;
(3) ${}_{C'}P_C \cong Hom_C(Q, h^-)$ and ${}_CQ_{C'} \cong Hom_{C'}(P, h^-)$;
(4) $F \cong P \otimes_C -$ and $G \cong Q \otimes_{C'} -$

Definition 5. A C'-C-bimodule ${}_{C'}P_C \in \mathbf{Mod}(C' \times C^o)$ is called balanced, if $Hom_C({}_{C'}P_C, {}_{C'}P_C) \cong h^-$, for any $h^- \in \mathbf{Mod}(C' \times C'^o)$ and $Hom_{C'}({}_{C'}P_C, {}_{C'}P_C) \cong h^-$, for any $h^- \in \mathbf{Mod}(C \times C^o)$.

Theorem 3. *Let C, C' be two preadditive categories and let*

$$F : \mathbf{Mod}C \to \mathbf{Mod}C', G : \mathbf{Mod}C' \to \mathbf{Mod}C$$

be additive functors. If there exists a bimodule ${}_{C'}P_C$ such that

(1) $\{{}_{C'}P(X)\}_{X \in obC}$ and $\{P(X')_C\}_{X \in obC}$ are small projective generator sets;
(2) ${}_{C'}P_C$ is balanced;
(3) $F \cong (P \otimes_C -)$ and $G \cong Hom_{C'}(P, -)$.
Then F and G are inverse equivalences.

References

1. F.W.Anderson and K.R.Fuller, *Rings and Categories of Modules*, Graduate Texts in Mathematics, 13, Springer, Berlin, 1974.
2. M. Sato, *On equivalences between module subcategories*, J.Algebra **59** (1979), 412 – 420.
3. G.D.Abrams, *Morita equivalence for rings with local units*, Commun. Algebra (1983), 801 – 837.
4. P.N.Ánh and L.M Árki, *Morita equivalence for rings without identity*, Tsukuba J.Math **11** (1987), 1 – 16.
5. H.Komatsu, *The category of S -unital modules*, Okayama Journal of Mathematics, Japan.
6. Angel Del Río, *Graded rings and equivalences of categories*, Communications in Algebra **19(3)** (1991), 997 – 1012.
7. Horst Schubert, *Categories*, Springer -Verlag Berlin Heidelberg, New York, 1972.
8. N. Popescu, *Abelian Categories with Applilcations to Rings and Modules*, Academic Press, London & New York, 1973.

NOTES ON HOMOLOGICAL DIMENSION OF GROUP GRADED RINGS

Zhong Yi
Department of Mathematics, Guang Xi Teachers University
Guilin, Guangxi, 541004, P. R. China

1. TRACES AND GLOBAL DIMENSION

In this section, we would like to study the relationships between the global dimension and the trace maps for strongly group graded rings and skew group rings. For the definitions and basic properties about group graded rings and skew group rings, see [Pa] and [Mo].

Let G be a finite group and let $R = \oplus_{g \in G} R_g$ be a strongly G-graded ring. For each $g \in G$, since $1 \in R_{g^{-1}} R_g = R_1$, we may fix a decomposition of the identity

$$1 = \sum_{i \in I_g} v_{g^{-1}}^{(i)} u_g^{(i)}, \tag{1}$$

where I_g is a finite set, $u_g^{(i)} \in R_g$ and $v_{g^{-1}}^{(i)} \in R_{g^{-1}}$. Thus for each $g \in G$, we can define a map $(\ )^g$ from R_1 to R_1 by

$$r^g = \sum_{i \in I_g} v_{g^{-1}}^{(i)} r u_g^{(i)} \tag{2}$$

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for all $r \in R_1$. we call the following map

$$tr : z(R_1) \rightarrow z(R_1); r \rightarrow \sum_{g \in G} r^g,$$

for all $r \in z(R_1)$, where $(\ )^g$ is defined as in (2), the *trace* of G on $z(R_1)$; see [CVV, Section 1] for details. This is a generalization of the usual trace maps in group actions. Let R be a strongly G-graded ring. Suppose that M is a right R-module. Using (1) we can define the concept of $R-$*regular* modules; see [Na] for details. By some direct calculations, we have the following

1.1 THEOREM *Let G be a finite group and let R be a strongly G-graded ring with coefficient ring R_1. Suppose that there exists an element $c \in z(R_1)$ such that $tr(c) = 1$. Then every right R-module is R-regular. In particular, we have*

(i) for each right R-module M,

$$pr.dim._R(M) = pr.dim._{R_1}(M); inj.dim._R(M) = inj.dim._{R_1}(M);$$

(ii) $r.gl.dim.(R) = r.gl.dim.(R_1)$.

Using 1.1 Theorem, we can deduce some results of [NV].

1.2 Proposition. *Let G be a finite group and lat R be a strongly G-graded ring with coefficient ring R_1.*

(i) If R is right hereditary, then so is R_1. (This part is valid for arbitrary groups.)

(ii) If R is right semihereditary, then so is R_1.

Suppose that there exists an element $c \in z(R_1)$ such that $tr(c) = 1$. Then we have

(i)' if R_1 is right hereditary, then so is R;

(ii)' if R_1 is right semihereditary, then so is R.

In the specially case of a skew group ring over a commutative coefficient ring. We obtain

1.3 COROLLARY. *Let G be a finite group acting on a commutative ring R and let $S = R * G$ be the skew group ring. Then the following are equivalent:*

(i) $r.gl.dim.(R) < \infty$;

(ii) (a) $r.gl.dim.(R) < \infty$;

(b) R is projective as a principal right $R * G$-module;

(iii) (a) $r.gl.dim.(R) < \infty$;

(b) there exists an element $c \in R$ such that $tr(c) = 1$.

1.4 REMARK. Refer to [Al, Proposition3.4] and [GD] for related results.

2. FINITISTIC DIMENSIONS OF STRONGLY GROUP GRADED RINGS

Finitistic dimensions are useful tools to study rings of infinite global dimension as shown in the literature. We refer to [Ba] for the basic properties and definitions of finitistic dimensions. In this section we would like to remark that the finitistic dimensions of a group graded ring and that of its coefficient ring are always stable.

2.1 DEFINITION. [Ba, Section 5] *Let R ba a ring. The right finitistic dimensions of R are defined as follows;*
rFPD(R)=sup{ pr.dim.(A)| A is a right R-module with pr.dim.(A) $< \infty$ *}.*
rFWD(R)=sup{ w.dim.(A)| A is a right R-module with w.dim.(A) $< \infty$ *},*
where w.dim.(A) denotes the flat dimension of A.
rFID(R)=sup{ inj.dim.(A) | A is a right R-module with inj.dim.(A)$< \infty$ *}.*
rfPD(R)=sup{ pr.dim.(A) | A is a finitely generated right R-module and pr.dim.(A)$< \infty$ *}.*

The following proposition, which is an analogue of [Na, Theorem 2.1], can be proved by a similar argument as the proof of that result.

2.2 PROPOSITION. *Let G be a finite group and let* $S = R(G)$ *be a strongly G-graded ring with coefficient ring R. Let M be a right S-module and let* $N = \oplus_{g\in G} N_g$ *be a graded left S-module. Then*
(i) for each $g \in G$ *and each non-zero n,*

$$Tor_n^S(M,N) \cong Tor_n^R(M,N_g),$$

as Abelian groups;
(ii) $w.dim._R(M) \leq w.dim._S(M)$*, and the equality holds if* $w.dim._S(M)$ *is finite.*

Using 2.2 Proposition and [Na, Theorem 2.1], we can obtain the following theorem, which describes the relationships between the finitistic dimension of a strongly group graded ring and those of its coefficient ring.

2.3 THEOREM. *Let G be a finite group and let* $S = R(G)$ *be a strongly G-graded ring with coefficient ring R. Then*

(i) $rFPD(R) = rFPD(S)$;
(ii) $rFWD(R) = rFWD(S)$;
(iii) $rFID(R) = rFID(S)$;
(iv) $rfPD(R) = rfPD(S)$.

References

[Al] E. Aljadeff, On the surjectivity of some trace maps, preprint, Technion-Israel Insitute of Technology, 1992.

[Ba] H. Bass, Finitistic dimension and a homological generalization of semiprimary rings, Tran. Amer. Math. Soc. 95(1960), 466-488.

[CVV] S. Caenepeel, M. Van Den Bergh and F. Van Oystaeyen, Generalized crossed products applied to maximal orders, Brauer groups and related exact sequences, J. Pure Applied Algebra 33(1984) 123-149.

[GD] J. J. Garcia and A. Del Rio, Group acting on rings, projectivity and global dimension, preprint, Universidad de Murcia, Spain, 1993.

[Mo] S. Montgomery, Fixed Rings of Finite Automorphism Groups of Associate Rings, Lecture Notes in Math. 818, Springer, (Berlin), 1980.

[Na] C. Nastaesescu, Strongly graded rings of finite groups, Comm. Algebra 11(10) (1983), 1033-1071.

[NV] E. Naewelaerts and F. Van Oystaeyen, Finite generalized crossed products over tame and maximal orders, J. Algebra 101(1986), 61-68.

[Pa] D. S. Passman, Infinite Crossed Products, Academic Press, (San Diego), 1989.

[Yi] Z. Yi, Homological dimension of skew group rings and crossed products, J. Algebra, 164(1)(1994), 101-123.

Monoidal Categories and Hecke-like Categories

Tomoyuki YOSHIDA
Dept. Mathematics, Hokkaido Univ.,
Sapporo 060, JAPAN
yoshidat@math.hokudai.ac.jp

1.Introduction. The classical Hecke category related to a group G is the category of permutation kG-modules. This category has many important properties such as additivity, self-duality and monoidality, and it has played essential parts not only in representation theory of finite groups but also in many areas of mathematics, for exmple, automorphic function theory, Galois theory, algebraic topology, and so on.

There are many categories with properties similar to those of classical Hecke categories. Historically speaking, the first Hecke-like catetory is the one of spans which appeared in the theory of 2-categories before 1960's. This category is related to abstract transfer-induction theory of finite groups developed by J.A.Green and A.Dress at the begining of 1970's. In fact, it was not long before abstract transfer-induction theory is noticed to be nothing but representation theory of the category of spans.

On the other hand, the concept of monoidal categories or more generally higher dimensional categories is very recently recognized to be essential in topological quantum field theory(TQFT). It is interesting that the monoidal 2-category of cobordisms appeared in TQFT is very like the one of spans excepting the direction of arrows. Here is the motivation to resume studying of the old abstract transfer-induction theory.

2.The category of spans. Let $\mathcal{E}$ be a finitely complete category. Then the category of spans, $\mathbf{Sp}(\mathcal{E})$, is defined by $\mathrm{Obj}(\mathbf{Sp}(\mathcal{E})) := \mathrm{Obj}(\mathcal{E})$ and

$$\mathrm{Hom}(Y,X) := \{[X \xleftarrow{l} A \xrightarrow{r} Y] \mid A \in \mathrm{Obj}(\mathcal{E})\}/\cong$$

with composition

$$[X \longleftarrow A \longrightarrow Y] \circ [Y \longleftarrow B \longrightarrow Z] := [X \longleftarrow A \times_Y B \longrightarrow Z],$$

where $A \times_Y B$ is the pullback of A and B along Y.

The category of spans is self-dual and has finite bi-products. Furthermore, it has the structure of a monoidal 2-category with tensoring by product. A representation of the category of spans is called a *Mackey functor* ([Dr 73]). When $\mathcal{E}$ is the category of finite G-sets for a finite group G, a Mackey functor is nothing but a G-functor ([Gr 71]). The category of spans can be called a Hecke-like category by the following proposition:

Proposition: *Assume that the above category $\mathcal{E}$ has a finite set of generators. Then* $\mathbf{Sp}(\mathcal{E})$ *can be embedded into a product of classical Hecke categories through Yoneda embedding, and both categories are Morita equivalent each other after tensoring with $\mathbf{Q}$.*

We can further develop the theory of the category of spans and its representation theory. For example, we can define the concept of bilinear maps (pairings) and then the concepts of *ring*, *modules* over a ring([Yo 87]). As a representation category of such modules, we again obtain a Hecke-like category.

3.Hecke categories on a monoidal category. After the representation theory of Hecke-like categories (abstranct transfer-induction theory), we can develop the representation theory of monoidal categories. Let $\mathcal{A} = (\mathcal{A}, \otimes, I, a, l, r)$ be a monoidal category with a tensor product $\otimes$, a unit object I and an associator a, and so on([Ke 82]).

Let $\mathcal{A}$ and $\mathcal{V}$ be monoidal categories and let $L, M, N : \mathcal{A} \longrightarrow \mathcal{V}$ a (*non-tensor*)functors. The a *pairing* $\rho : L \otimes M \longrightarrow N$ is a family of natural maps

$$\rho_{A,B} : L(A) \otimes M(B) \longrightarrow N(A \otimes B).$$

A *ring* R is a functor $R : \mathcal{A} \longrightarrow \mathcal{V}$ equipped with a pairing $\mu : R \otimes R \longrightarrow R$ satisfying a kind of associativity. Similarly, a *module* M over R is defined as a functor equipped with a pairing $\alpha : R \otimes M \longrightarrow M$.

Theorem: *Let $\mathcal{A}, \mathcal{V}$ be monoidal categories with $\mathcal{A}$ closed. Let $R : \mathcal{A} \longrightarrow \mathcal{V}$ be a ring. Then the category of R-modules is represented by a Hecke-like category* **Hec**$(\mathcal{A}, R)$. *Here an object of $Hec(\mathcal{A}, R)$ is an object of $\mathcal{A}$ and a hom-set $Hom(A, B)$ is defined to be $R(B^A)$.*

References

[BD95] J.Baez–J.Dolan, Higher-dimensional algebra and topological quantum field theory, *J. Math. Phys.*, **36** (1995), 6073–6105.

[Dr 73] A.Dress, Contributions to the theory of induced representatins, in *Algebrai K-theory II*, Lecture Notes in Math. **342**(1973), 283–240, Springer.

[Gr 71] J.A.Green, Axioomatic representation theory of finite groups, J. Pure Appl. Algebra, 1 (1971), 41–77.

[JS 93] A.Joyal–R.Street, Braided tensor categories, *Adv. Math.*, **102** (1993), 20–78.

[KV 94] M.Kapranov–V.Voevodsky, 2-categories and Zamolodchiov tetrahedra equations, *Proc. Symp. Pure Math.*, **56** (1994) Part 2, 177–259.

[Ke82] G.M.Kelly, "Basic Concepts of Enriched Category Theory ", London Math. Soc. Lecture Notes, **64** (1982), Cambridge univ.

[Qu 95] F.Quinn, Lectures on axiomatic topological quantum field theory, in IAS/Park City Mathematics Series, vol 1, 1995, 325–459.

[Yo 87] T.Yoshida, Fisher's inequality for block designs with finite group action, *J.Fac.Sci.Univ. Tokyo*, **34** (1987), 513–544.

ON A SPECIAL TYPE OF QUASIFROBENIUS RINGS

Yoshito YUKIMOTO[1]

Fukui University of Technology[2]

Throughout this note QuasiFrobenius is abbreviated to QF.

Though generalizations of QF rings are studied by many authors, the class of QF rings is big from a point of view. For instance, almost nothing is known for me about the composition series of a projective indecomposable module over a QF ring. Hence I will study a special type of QF rings.

Let us consider the following condition on a ring R:

(C) eRe is QF for every idempotent $e \in R$.

This condition is due to K.Oshiro & S.H.Rim[3], and it is a Morita invariant condition. In this note I will give a partial answer to the question what the ring R with condition (C) is. If R satisfies the condition (C), then $R = 1R1$ itself is QF. Therefore R has the Nakayama permutation ν defined by

$$\mathrm{soc}(e_i R) \cong \mathrm{top}(e_{\nu(i)} R) \quad (i \in I),$$

where $\{e_i\}_{i \in I}$ is the basic set of primitive idempotents for R. The partial answer is the following factorization theorem:

Theorem. *If a basic ring R satisfies the condition (C), $\{e_1, e_2, \ldots, e_n\}$ is the basic set of primitive idempotents for R, and $\{1, 2, \ldots, n\} = \coprod_{k=1}^{m} I_k$ is the orbit decomposition by the Nakayama permutation ν of R with $\# I_k = 1 (1 \leq k \leq a)$ and $\# I_k > 1 (a+1 \leq k \leq m)$, then*

$$R = fRf \times f_{a+1}Rf_{a+1} \times f_{a+2}Rf_{a+2} \times \cdots \times f_m R f_m,$$

where $f_k = \sum_{i \in I_k} e_i$ and $f = \sum_{1 \leq k \leq a} f_k$.

I will prove the above theorem in a very special but essential case of $n = 3$ and

$$\nu = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}.$$

Proof for the special case. The Jacobson radical of R is denoted by J here.

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[1] 雪本義人

[2] 福井工業大学

[3] 大城紀代市 & 林碩勳

First we show

$$\operatorname{Ext}(e_1R/e_1J, e_3R/e_3J) = 0.$$

Suppose that $\operatorname{Ext}(e_1R/e_1J, e_3R/e_3J) \neq 0$. Then there exists an extension M of e_3R/e_3J by e_1R/e_1J which is not semisimple. The extension M is uniserial, $\operatorname{soc}(M) \cong e_3R/e_3J$ and $\operatorname{top}(M) \cong e_1R/e_1J$. Since e_2R is isomorphic to an injective hull of e_3R/e_3J, there is a submodule $N \leq e_2R$ which is isomorphic to M. We have $NJ(e_1 + e_2) \cong \operatorname{Hom}((e_1 + e_2)R, NJ) = 0$ because $NJ \cong \operatorname{top}(e_3R)$, and we have $N(e_1 + e_2) = Ne_1$ because $Ne_2 \cong \operatorname{Hom}(e_2R, N) \cong \operatorname{Hom}(e_2R, NJ) = 0$. Hence Ne_1 is a simple right $(e_1 + e_2)R(e_1 + e_2)$-submodule of $e_2R(e_1 + e_2)$.

By a similar way it is shown that $\operatorname{soc}(e_1R)e_1$ is also a simple right $(e_1 + e_2)R(e_1 + e_2)$-submodule of $e_1R(e_1 + e_2)$.

Both Ne_1 and $\operatorname{soc}(e_1R)e_1$ are isomorphic to the right $(e_1 + e_2)R(e_1 + e_2)$-module $\operatorname{top}(e_1R(e_1 + e_2))$, which contradicts to a part of (C) that $(e_1 + e_2)R(e_1 + e_2)$ is QF. Therefore $\operatorname{Ext}(e_1R/e_1J, e_3R/e_3J) = 0$.

We can show $\operatorname{Ext}(e_1R/e_1J, e_2R/e_2J) = 0$ by the same way.

Second we show that e_1J^n/e_1J^{n+1} is e_1R/e_1J-homogeneous for every $n = 0, 1, 2, \dots$ by induction on n. The statement is evident if $n = 0$. Suppose the statement is true for n. Then, if we express $e_1J^n = \sum_{i=1}^{k} L_i$ as a sum of local submodules L_i of e_1J^n, the top of L_i is isomorphic to e_1R/e_1J by the hypothesis of induction. Because of $\operatorname{Ext}(e_1R/e_1J, e_3R/e_3J) = 0$ and $\operatorname{Ext}(e_1R/e_1J, e_2R/e_2J) = 0$, the second top L_iJ/L_iJ^2 is e_1R/e_1J-homogeneous. Hence e_1J^{n+1}/e_1J^{n+2} is also e_1R/e_1J-homogeneous. This completes the induction.

The above statement is equivalent to $e_1R(e_2 + e_3) = 0$. By the duality of QF ring R, we have also $(e_2 + e_3)Re_1 = 0$. Hence $R = e_1Re_1 \times (e_2 + e_3)R(e_2 + e_3)$.

Remark. The first factor of the factorization in the general statement is not necessarily indecomposable as ring, while the other factors are indecomposable as ring.

Examples. 1. A ring is called weakly symmetric if it is QF and its Nakayama permutation is an identity permutation. Every weakly symmetirc ring satisfies the condition (C).

2. If R is a basic serial ring (indecomposable as ring), $\{e_1, e_2, \dots, e_n\}$ is the basic set of primitive idempotents for R, and $|e_1R| = |e_2R| = \cdots = |e_nR| = mn$ or $mn + 1$ for some integer $m > 0$, then R satisfies the condition (C).

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GENERALIZED PLANAR NEAR—RINGS

Zhang Changming
Hunan Normal University PRC

In this paper, we introduce the concept of generalized planar near—rings and extend basic properties of planar near—rings, which provides a way of proving whether there is a unique solution in the near—ring N to the equation $xa=xb+c$.

We denote a right near—ring by $N=(N, +, \cdot)$. The other concepts and symboles are taken from [1] ~ [4]

1. Basic concepts

Definition 1. 1 Let N be a near—ring, I be a nonzero ideal of N, a, $b\in N$,

$a\equiv b \Longleftrightarrow \forall\, n\in I$: $na=nb$

Obviously, $\equiv$ is a relation on N and it is an equivalence relation on N, called a right multiplicative relation of N decided by I.

Definition 1. 2 A near—ring N is said to be a planar near—ring if right multiplicative relation $\equiv$ of N decided by N satisfies condition $|N/\equiv|\geqslant 3$ and if there is a unique solution in N to equation $xa=xb+c$, $\forall$ a, $b\in N$, $a\not\equiv b$

Definiton 1. 3 Let N be a near—ring, I be a nonzero ideal of N. If the right multiplicative relation $\equiv$ of N decided by I satisfies condition $|N/\equiv|\geqslant 3$ and if there is the unique solution in N to equation. $xa=xb+c$, $\forall$ a, b, $c\in N$, $a\not\equiv b$, $a+b\notin I$, N is said to be a generalized planar near—ring on I (abbreviation by GPNR), I is called a relation ideal of N.

It is clear that any planar near—ring N is a GPNR itself and to any near—ring N, if right multiplicative relation $\equiv$ of N decided by N, satisfies condition $|N/\equiv|\geqslant 3$, N is a GPNR itself, too.

Example 1. 1 Let $N=Z_6=\{0, 1, 2, 3, 4, 5\}$, then N is a GPNR on the non zero ideal $I=\{0, 2, 4\}$ of N.

Proof $Z_6/_\equiv=\{\{0, 3\}, \{1, 4\}, \{2, 5\}\}$, $|Z_6/\equiv|\geqslant 3$. Go through test and verify, $\forall$ a, b, $c\in Z_6$, $a\not\equiv b$, $a+b\notin I$, equation $xa=xb+c$ has a unique solution in Z_6, therefore, $N=Z_6$ is a GPNR on the I. But, Z_6 is not planar near—ring, because there exists the equation $x\cdot 2=x\cdot 4+0$ $(2\not\equiv 4)$ which two solutions $x=0. 3$.

2 Basic properties of GPNR

Let N be a GPNR with a relation ideal I. Let $A=\{n\in N \mid n\equiv 0 \text{ or } n\in I\}$, denote $N\backslash A$ by $N^{\#}$. If $N^{\#}=\varnothing$ and N is non planar near-ring, N is called natural GPNR. Otherwise it is non-natural GPNR, which is discussed below.

Proposition 2. 1 Every generalized planar near-ring is zero-symmetric.

Proof Let N be a GPNR with a relation ideal I. $\forall\, n\in N$, let $a\in N^{\#}$, then $a\not\equiv 0$ and $a\notin I$, hence equation $xa=x0+0$ has a unique solution in N. It is clear 0 is a solution to $xa=x0+0$ and

$(n0)\,a=n\,(0a)=n0,\ (n0)\,0=n\,(00)=n0$

$\Rightarrow (n0)\,a=(n0)\,0+0$, thus $n0$ is also a solution to $xa=x0+0$, then 0 and $n0$ are both solutions to equation $xa=x0+0$, hence $n0=0$.

Proposition 2. 2 Let N be a GPNR with a relation ideal I.

(i) If $a\in N$ is a right zero divisor, that is to say, the existing $n\in N$, $n\neq 0$, which makes $na=0$, $a\in A$. Conversely, if $a\in A$ and $a\notin I$, a is a right zero divisor.

(ii) $\forall\, n\in N^{\#}$, $m\in N$, then there is a unique $x\in N$, which makes $xn=m$.

Proof (i) If $a\notin A\Rightarrow a\not\equiv 0$ and $a\notin I$, equation $xa=x0+0$ has a unique solution in N. $na=0$ and $n0=0$, implies n and 0 are hoth solutions to equation $xa=x0+0$, a contradiction.

(ii) If $n\in N^{\#}\Rightarrow n\not\equiv 0$ and $n\notin I$, equation $xn=x0+m$ has a unique solution. Thus, $\exists\, 1\, x\in N$, which makes $xn=m$.

Corollary 2. 1 $\forall\, n,\ m\in N^{\#}$, $\exists\, 1\, x\in N^{\#}$, which makes $xn=m$.

Proof By proposition 2. 2 (ii), there is the unique $x\in N$, which makes $xn=m$. Since $m\in N^{\#}$, $x\notin I$. If $x\equiv 0\Rightarrow\forall\, i\in I$; $ix=i0=0\Rightarrow im=i\,(xn)=(ix)\,n=0n=0\Rightarrow m$ is a right zero divisor, by proposition 2. 2 (i) $\Rightarrow m\in A$, which is a contradiction. Hence $x\in N^{\#}$.

Let N be a GPNR, I is the relation ideal of N. $\forall\, a\in N^{\#}$, let l_a be the unique solution to $xa=a$, $B_a=\{x\in N^{\#} \mid l_a x=x\}$, then the following proposition and theorem are correct:

Proposition 2. 3 To any $a\in N^{\#}$, there is $a\in B_a$, $l_a\in N^{\#}$ and $N^{\#}=\bigcup_{a\in N^{\#}} B_a$.

Proof It is clear $a\in B_a$.

If $l_a\notin N^{\#}\Rightarrow l_a\in A\Rightarrow l_a\equiv 0$ or $l_a\in I$. If $la\equiv 0\Rightarrow\forall\, i\in I$; $il_a=i0$ and $l_a a=a$

$\Rightarrow ia=i\,(l_a a)=(il_a)\,a=0a=0\Rightarrow a\in A$, a confradiction. If $l_a\in I\Rightarrow l_a a\in I\Rightarrow a\in I$, a confradiction, too. Hence $l_a\in N^{\#}$.

Any $a\in N^{\#}\Rightarrow a\in B_a\Rightarrow a\in\bigcup_{a\in N^{\#}} B_a$, any $x\in\bigcup_{a\in N^{\#}} B_a\Rightarrow x\in B_a\Rightarrow x\in N^{\#}$.

Hence, $N^{\#}=\bigcup_{a\in N^{\#}} B_a$.

Proposition 2. 4 Let $a\in N^{\#}$, l_a be a unique solution to equation $xa=a$, then $l_a\in B_a$.

Proof By proposition 2. $3\Rightarrow l_a\in N^{\#}$, again $l_a l_a$ and l_a are both solutions to $xa=x0+a$, hence $l_a l_a=l_a$, thus $l_a\in B_a$.

Proposition 2. 5 $\forall\, b\in B_a$, Let $\bar{b}$ be a unique solution to $xb=l_a$, that is $\bar{b}b=l_a$, then $\bar{b}\in N^{\#}$, further $\Rightarrow\bar{b}\in B_a$.

Proof Because b, $l_a\in N^{\#}$, as a result $\bar{b}\in N^{\#}$, by corollary 2. 1. Again $(l_a\,\bar{b})$ b

$=1_a$ $(\bar{b}b)$ $=1_a1_a=1_a$, hence $1_a\bar{b}$ is a solution to equation $xb=1_a$, too, so $1_a\bar{b}=\bar{b}$. Thus $\bar{b}\in B_a$.

Propostion 2. 6 To any $a\in N^{**}$, the unique solution 1_a to equation $xa=a$ is a right identity for $(N, +, \cdot)$.

Proof For each $n\in N$, $n1_a$ and n are both solutions to $x1_a=x0+n1_a$, hence $n1a=n$. Thus 1_a is a right identity for $(N, +, \cdot)$.

Proposition 2. 7 Let $b\in B_a$, if there is $\bar{b}b=1_a$, there is $b\bar{b}=1_a$, too.

Proof By corollary 2. 1, $\bar{b}\in N^{**}$. Let b' be a unique solution to $x\bar{b}=1_a$, that is $b'\bar{b}=1_a$, then $b'\in N^{**}$, thereupon.

$(b'\bar{b})$ $(b\bar{b})$ $=1_a$ $(b\bar{b})$ $=$ (1_ab) $\bar{b}=b\bar{b}$, again

$(b'\bar{b})$ $(b\bar{b})$ $=b'$ $[(\bar{b}b)\bar{b}]$ $=b'$ $(1_a\bar{b})$ $=b'\bar{b}=1_a$,

therefore $b\bar{b}=1_a$, that is $\bar{b}b=b\bar{b}=1_a$.

Theorem 2. 1 To any $a\in N^{**}$, let 1_a be a unique solution to $xa=a$, then come to conclusions:

(i) For each $a\in N^{**}$, $(B_a, \cdot)$ is a group with identity 1_a.

(ii) A and the Ba's $(a\in N^{**})$ form a partition of N.

(iii) $\forall$ $a\in N^{**}$, there is $B_aN^{**}=B_a$.

(iv) If a, $b\in N^{**}$, $\Phi:\begin{matrix} B_a \longrightarrow B_b \\ x \longrightarrow 1_bx \end{matrix}$ is a (group) isomorphism.

(v) If $S\subseteq N^{**}$ and $SN^{**}\subseteq S$, $S=\bigcup_{a\in S}B_a$

Proof (i) By proposition 2. 4, there is $a\in B_a$, therefore 1_a is a left identity in $(B_a, \cdot)$. By proposition 2. 5, to any $b\in B_a$, there is $\bar{b}\in B_a$, which makes $\bar{b}b=1_a$, that is, $\bar{b}$ is a left inverse element of b.

$\forall$ b_1, $b_2\in B_a$, we prove $b_1b_2\in B_a$.

First prove $b_1b_2\in N^{**}$. If $b_1b_2\notin N^{**}\Rightarrow b_1b_2\equiv 0$ or $b_1b_2\in I$. a) If $b_1b_2\equiv 0\Rightarrow\forall$ $i\in I$; i (b_1b_2) $=i0=0\Rightarrow$ (ib_1) $b_2=$ (ib_1) $0+0$, that is ib_1 is a solution to equation $xb_2=x0+0\Rightarrow ib_1=0\Rightarrow$ $ib_1=i0+0$, that is i is a solution to $xb_1=x0+0\Rightarrow i=0$, $\forall$ $i\in I$, a contradiction. Thus, $b_1b_2\not\equiv 0$. b)If $b_1b_2\in I$, Let $\bar{b}_2$ be a unique solution to equation $xb_2=1_a$, that is $\bar{b}_2b_2=1_a$, by proposition 2. 7, $b_2\bar{b}_2=1_a$, too, as a result $(b_1b_2)\bar{b}_2=b_1(b_2\bar{b}_2)=b_11_a=b_1$. But $(b_1b_2)\bar{b}_2\in I\Rightarrow b_1\in I$, a contradiction, too. Hence $b_1b_2\in N^{**}$.

Secondly, 1a (b_1b_2) $=$ (1_ab_1) $b_2=b_1b_2$. Hence $b_1b_2\in B_a$. Thus it is proved that $(B_a, \cdot)$ is a group with identity 1_a.

(ii) It is enough to show that $\forall$ a, $b\in N^{**}$ either $B_a\cap B_b=\varnothing$ or $B_a=B_b$. In fact, if $n\in B_a\cap B_b\Rightarrow 1_an=n=1_bn$, hence 1_a and 1_b are both solutions to equation $xn=x0+n$, so $1_a=1_b$ and $B_a=B_b$.

(iii) Let $a\in N^{**}$, $\forall$ $b\in B_a$, $n\in N^{**}$, then $bn\in N^{**}$, once again 1_a (bn) $=$ (1_ab) $n=bn$, hence $bn\in B_a$. Conversely, $\forall$ $b\in B_a$, then $b=1_ab\in B_aN^{**}$. Thus $B_aN^{**}=B_a$.

(iv) $\forall$ $x\in B_a$, there is $x\in N^{**}$, $\Phi(x)=1_bx$, because $1_bx\in B_b$, therefore $1_bx\in B_bN^{**}=B_b$, hence Φ $(x)=1_bx\in B_b$ and 1_bx is the unique determiner in B_a, that is, Φ is a mapping of from B_a to B_b.

$\forall$ a', $a''\in B_a$, if Φ (a') $=\Phi$ (a''), $1_ba'=1_ba''$, and by proposition 2. 6, $a'=1_aa'=$ (1_a1_b) a' $=1_a$ $(1_ba')$ $=1_a$ $(1_ba'')$ $=$ (1_a1_b) $a''=1_aa''=a''$, that is $a'=a''$, hence Φ is a injection.

If $b' \in B_b$, there is $1_a b' \in B_a$, which makes $\Phi(1_a b') = 1_b(1_a b') = (1_b 1_a) b' = 1_b b' = b'$, hence, Φ is a surjection.

$\forall a', a'' \in B_a$, $\Phi(a'a'') = 1_b(a'a'') = [(1_b a') 1_b] a'' = ((1_b a')(1_b a'') = \Phi(a')\Phi(a'')$.

Therefore Φ is an isomorphism.

(v) To any $s \in S \Rightarrow s \in B_s \Rightarrow s \in \bigcup_{a \in S} B_a$

Conversely, to any $x \in \bigcup_{a \in S} B_a \Rightarrow x \in B_a$ $(a \in S)$ $\Rightarrow x \in N^{*}$ and $1_a x = x$. Because B_a is a group and 1_a is an identity, there is $\bar{a} \in B_a$ which makes $a\bar{a} = 1_a$, of course, $\bar{a} \in N^{*}$, hence $\bar{a}x \in N^{*}$. Thus.

$x = 1_a x = (a\bar{a})x = a(\bar{a}x) \in SN^{*} \subseteq S \Rightarrow x \in S$. Hence $S = \bigcup_{a \in S} B_a$.

The further study of generalized planar near—rings can solve the problem: whether, to some elements pair (a, b) (a, b∈N) and ∀ c∈N, equation $xa = xb + c$ has a unique solution in the near—ring N.

REFERENCES

[1] Günter Pilz, Near—rings the theory and its application. North—Holland Publishing compang, 1983.

[2] Zhang Changming, J—radical of type 5/2 for near—rings. Acta Sci Nat Univ Norm Hunan, 1988, 11 (1): 14—18

[3] Zhang Changming, J—radical of type * and class of J—type radical for the near—rings. Acta Sci Nat Univ Norm Hunan, 1988, 11 (3), 189—192

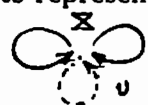
[4] Zhang Changming, Class of J—type radical for the weakly symmetric Near—rings. Acta Sci Nat Univ Norm Hunan, 1991, 14 (2), 96—101.

THE REPRESENTATION CATEGORY OF A WILD BOCS

ZHANG YINGBO

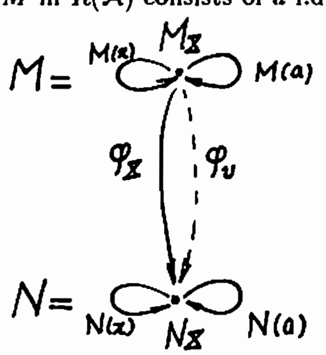
This is a joint work with Dr. Lei Tiangang and Professor R. Bautista.

§1. Bocs $\mathcal{A}$ and its representation category $R(\mathcal{A})$.

Let bocs $\mathcal{A} = x$  a

over an algebraically closed field with differentials $\delta(x) = 0$, $\delta(a) = xv - vx$, $\delta(v) = 0$. Roughly speaking, $\mathcal{A}$ consists of a free k-algebra A generated by x and a, x  a which is of course infinite dimension. And a free bimodule V generated by dotted dot u_X and dotted arrow v over A. Then we construct a coalgebra structure from this free bimodule. The counit and comultiplication are uniquely determined by above differentials.

The finitely dimensional representation category $R(\mathcal{A})$ of $\mathcal{A}$ is as follows: the object M in $R(\mathcal{A})$ consists of a f.d. vector space M_X and two linear transformations $M(x)$ and $M(a): M_X \longrightarrow M_X$. Denote it by $M = (M_X; M(x), M(a))$. If $N = (N_X, N(x), N(a))$ is also an object in $R(\mathcal{A})$, then a morphism $\varphi: M \longrightarrow N$ in $R(\mathcal{A})$ consists of two linear transformations φ_X, $\varphi_v: M_X \longrightarrow N_X$, such that $N(x)\varphi_X - \varphi_X M(x) = 0$ from $\delta(x) = 0$, $N(a)\varphi_X - \varphi_X M(a) = N(x)\varphi_v - \varphi_v M(x)$ from $\delta(a) = xv - vx$. If $\psi: N \longrightarrow L$ is another morphism in $R(\mathcal{A})$, then the composition $\psi\varphi: M \longrightarrow L$ consists of $(\psi\varphi)_X = \psi_X\varphi_X$, and $(\psi\varphi)_v = \psi_X\varphi_V - \psi_v\varphi_X$ from $\delta(v) = 0$.



§2. The matrix expressions of objects and morphisms.

2.1. **Lemma.** Let M be an object in $R(\mathcal{A})$. If M is indecomposable,then $M(x)$ has only one unique eigenvalue.

2.2. **Proposition.** Let M be an indecomposable object. Then M has an expression $M(x) = \lambda + J_{\underline{d};\underline{p}}$, where $d = (d_1, \cdots, d_s)$, $d_1 < \cdots < d_s$, $\underline{p} = (p_1, \cdots, p_s)$, $p_1, \cdots, p_s > 0$, $J_{\underline{d};\underline{p}} = \operatorname{diag}(J_{d_1;p_1}, \cdots, J_{d_s;p_s})$ a partitional diagonal matrix with

$$J_{d,p} = \begin{pmatrix} 0 & I & & & \\ & 0 & I & & \\ & & \ddots & \ddots & \\ & & & 0 & I \\ & & & & 0 \end{pmatrix}$$

the $d \times d$ partitioned matrix, I $p \times p$ identity matrix and O $p \times p$ zero matrix.

$J_{d;p}$ is similar to diag $(J(0,d),\cdots,J(0,d))_{p\times p}$, $J(0,d)$ is a $d\times d$ Jordan form with eigenvalue 0.

$M(a)=(M_{ij})_{s\times s}$ a partitioned matrix with

$$M_{ij}=\begin{pmatrix} M_{ij}^1 & 0 & \cdots & 0\\ M_{ij}^2 & 0 & \cdots & 0\\ \cdots & \cdots & & \\ M_{ij}^{d_i} & 0 & \cdots & 0\end{pmatrix}_{d_i\times d_j}$$

and $M_{ij}^1=\cdots=M_{ij}^{d_i-d_j}=0$ if $d_i>d_j$.

2.3. **Lemma.** Let M,N be two objects in $R(\mathcal{A})$ with $M(x)=\lambda+J_{\underline{d};\underline{p}}$ and $N(x)=\mu+J_{\underline{e};\underline{q}}$. If $\lambda\neq\mu$, then there exists only zero map from M to N.

2.4. **Proposition.** Let $\varphi: M\longrightarrow N$ be a morphism in $R(\mathcal{A})$, with $M(x)=\lambda+J_{\underline{d};\underline{p}}$ and $N(x)=\lambda+J_{\underline{e};\underline{q}}$. Then $\varphi_X=(F_{ij})_{t\times s}$ with

$$F_{ij}=\begin{pmatrix} F_{ij}^{(1)} & F_{ij}^{(2)} & \cdots & F_{ij}^{(d_j)}\\ 0 & F_{ij}^{(1)} & \cdots & F_{ij}^{(d_j-1)}\\ \cdots & \cdots & & \\ 0 & 0 & \cdots & F_{ij}^{(1)}\\ 0 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0\end{pmatrix}_{e_i\times d_j} \qquad \text{when } e_i\geq d_j$$

$$F_{ij}=\begin{pmatrix} 0 & \cdots & 0 & F_{ij}^{(d_j-e_i+1)} & F_{ij}^{(d_j-e_i+2)} & \cdots & F^{(d_j)}\\ 0 & \cdots & 0 & 0 & F_{ij}^{(d_j-e_i+1)} & \cdots & F^{(d_j-1)}\\ \cdots & \cdots & & & & & \\ 0 & \cdots & 0 & 0 & 0 & \cdots & F_{ij}^{(d_j-e_i+1)}\end{pmatrix}_{e_i\times d_j} \qquad \text{when } e_i<d_j.$$

where $F_{ij}^{(k)}$ are $e_i\times p_j$-matrices.

§3. Bocs $\mathcal{A}$ has almost split sequences and strong Homogeneous property

3.1. **Definition.** (1) Let $\mathcal{B}$ be a layered bocs. We say that $\mathcal{B}$ has almost split sequences, if for any non-proper injecture M in $R(\mathcal{B})$, there exists a proper almost split sequence $M\longrightarrow E\longrightarrow N$ in $R(\mathcal{B})$; and for any non proper projective N in $R(\mathcal{B})$, there exists a proper almost split sequence $M\longrightarrow E\longrightarrow N$ in $R(\mathcal{B})$.

(2) Let $\mathcal{B}$ be a layered bocs, we call an indecomposable object M in $R(\mathcal{B})$ homogeneous, if there exists a proper almost split sequence $M\xrightarrow{L}E\xrightarrow{\pi}M$ in $R(\mathcal{B})$.

(3) Strong homogeneous property. Let $\mathcal{B}$ be a layered bocs having almost split sequences. If there exist neither proper-projective nor proper injective in $R(\mathcal{B})$). And for every indecomposable M in $R(\mathcal{B})$, M is homogeneous. Then $\mathcal{B}$ is called to have strong homogeneous property.

3.2. Theorem. Let M be an indecomposable object in $R(\mathcal{A})$, with $M(x) = \lambda + J_{\underline{d};\underline{p}}$, $\underline{d} = (d_1, \cdots, d_s)$, $d_1 < \cdots < d_s$, $\underline{p} = (p_1, \cdots, p_s)$, $\varphi : M \longrightarrow M$ be a morphism with $\varphi_X = (F_{ij})_{s\times s}$. Then $F_{ii}^{(1)}$ has only one unique eigenvalue and when i runs over $1, \cdots, s$, the eigenvalues are all equal. Moreover φ_X has only one unique eigenvalue.

3.3. Main theorem. Bocs $\mathcal{A}$ has almost split sequences and strong homogeneous property in case ch $k = 0$. But $\mathcal{A}$ is of wild type.

REFERENCES

[CB] W. W. Crawley-Boevey, On ttame algebras and bocses, Proc. Math. Soc. (3) 56(1988), 451-483.

[ZLB] Y. B. Zhang, T. G. Lei and R. Bautista, The representation category of a bocs I-IV, J. Beijing Normal Univ. Vol.31, No.3,4, 1995 and Vol.32, No.1,2, 1996.

Department of Mathematics
Beijing Normal University
Beijing, 100875 China