Proceedings of the 30th Symposium on Ring Theory and Representation Theory

October 20–22, 1997
Nagano, Japan

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Editor

February, 1998
Nagano, Japan

第３０回環論及び表現論シンポジウム報告集

１９９７年１０月２０日～２２日
長野市メルパルク

岩永恭雄
編集

１９９８年2月
信州大学教育学部

Contents

Preface

These Proceedings report on talks delivered at the 30th Symposium on Ring Theory and Representation Theory held in Nagano City, Japan during October 20–22, 1997. The Symposium consisted of fifteen talks by twelve speakers including series of lectures by Birge Huisgen-Zimmermann on "*Homological dimensions in representation theory of finite dimensional algebras*", and by Shigeo Koshitani on "*Some topics on modular group algebras of finite groups*" and "*On p-blocks of finite groups*". I'd like to give my gratitude to both of them.

The Symposium on Ring Theory and Representation Theory has been organized by Yasuo Iwanaga (Shinshu University), Kiyoichi Oshiro (Yamaguchi University), Yukio Tsushima (Osaka City University), Hidetoshi Marubayashi (Naruto College of Education) and Kunio Yamagata (Tokyo University of Technology and Agriculture) since 1996. Originally, the Symposium on Ring Theory has started in 1971 by Shizuo Endo, Hiroyuki Tachikawa, Hisao Tominaga and Manabu Harada. We succeed this new-named Symposium.

The Symposium was financially supported by Masahisa Sato (Yamanashi University) through the Grant-in-Aid for Scientific Research from the Ministry of Education, No. 09640022, and the Proceedings by Yukio Tsushima (Osaka City University) through the Grant-in-Aid for Scientific Research from the Ministry of Education, No. 09640056. We wish to express our thanks for their financial support. We wish also to extend our thanks to all speakers of the Symposium.

Nagano, February 1998 — Yasuo Iwanaga

List of the Lectures

Vlastimil Dlab (Carleton Univ.)
Stratified algebras

Yukio Doi (Fukui Univ.)
A survey on separable Hopf algebras

Hisaaki Fujita (Univ. of Tsukuba)
Minimal injective resolution of an order over a local Dedekind domain

Nobuyuki Fukuda (Okayama Univ.)
Gelfand-Kirillov dimension for quantized Weyl algebras

Shiro Goto (Meiji Univ.) and Kenji Nishida (Nagasaki Univ.)
Module-finite algebras over universally catenary rings are catenary

Birge Huisgen-Zimmermann (Univ. of California, Santa Barbara)
Homological dimensions in the representation theory of finite dimensional algebras I, II, III

Osamu Iyama (Kyoto Univ.)
A characterization of finite Auslander-Reiten quivers over orders

Shigeo Koshitani (Chiba Univ.)
Some topics on modular group algebras of finite groups.
On p-blocks of finite groups

Manabu Matsuoka (Nagoya Univ.) and Takao Sumiyama (Aichi Univ. of Technology)
On ring extensions and Everett functions

Jun Morita (Univ. of Tsukuba)
On Kac-Moody groups and certain Gauss decompositions

Mitsuhiro Takeuchi (Univ. of Tsukuba)
Finite Hopf algebras in braided tensor categories

Takayoshi Wakamatsu (Saitama Univ.)
On symmetric algebras

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AN INTRODUCTION TO SEPARABLE HOPF ALGEBRAS

YUKIO DOI

ABSTRACT. The purpose of this note is to give a quick introduction about finite dimensional separable Hopf algebras for ring theorists. In particular I gave a new approach of the trace formulae of Larson and Radford [LR1, 2]: $\mathrm{Tr}(S^2) = \varepsilon(t)\phi(1) = n\mathrm{Tr}(S^2|_{H_y})$, where H is any n-dimensional Hopf algebra with antipode S, t an right integral in H, ϕ an right integral in H^* with $\phi(t) = 1$ and y a specific element in H.

1. H-Frobenius algebras

We first recall a well known fact that any finite dimensional Hopf algebra is Frobenius as an algebra. Suppose that H is a finite dimensional Hopf algebra over a field k with comultiplication Δ and counit ε. We use the following Sweedler notation:

$$\Delta(h) = \sum h_1 \otimes h_2,$$

$$(id \otimes \Delta)\Delta(h) = (\Delta \otimes id)\Delta(h) = \sum h_1 \otimes h_2 \otimes h_3, \ \dots \ (h \in H)$$

For any k-algebra A with augmentation $\varepsilon : A \to k$, the spaces of left and right *integrals* in A are defined, respectively, as follows:

$$\mathrm{Hom}_{A-}(k,\ A) = \{s \in A \mid as = \varepsilon(a)s,\ \forall\, a \in A\},$$
$$\mathrm{Hom}_{-A}(k,\ A) = \{t \in A \mid ta = t\varepsilon(a),\ \forall\, a \in A\}.$$

Here the base field k is a trivial right A-module via $\varepsilon : A \to k$. One sees that if A is Frobenius, i.e., $A \cong A^* := \mathrm{Hom}(A,\ k)$ as right (or equivalently left) A-modules, then the space of left (right) integrals is one dimensional. A is called *unimodular* if $\mathrm{Hom}_{A-}(k,\ A) = \mathrm{Hom}_{-A}(k,\ A)$. Observe that the spaces of left (right) integrals in the dual Hopf algebra H^* coincides the space of all left (right) H-comodule map from H to k, where k is a trivial right H-comodule via unit map $u : k \to H = k \otimes H$, i.e.,

$$\begin{aligned}\mathrm{Hom}_{H^*-}(k,\ H^*) &= \mathrm{Hom}^{H-}(H,\ k)\\ &= \{\psi \in H^* \mid \textstyle\sum h_1\psi(h_2) = \psi(h)1,\ \forall\, h \in H\},\\ \mathrm{Hom}_{-H^*}(k,\ H^*) &= \mathrm{Hom}^{-H}(H,\ k)\\ &= \{\phi \in H^* \mid \textstyle\sum \phi(h_1)h_2 = \phi(h)1,\ \forall\, h \in H\}.\end{aligned}$$

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As H is finite dimensional, there is a canonical k-isomorphism

$$H^* \otimes H \cong \mathrm{End}(H), \quad f \otimes h \;\mapsto\; [x \mapsto f(x)h].$$

We identify these two spaces. We define a right H-Hopf module structure on H^* which is different from Sweedler's:

$$\leftarrow : H^* \otimes H \longrightarrow H^*, \ (f \leftarrow h)(x) = f(hx), \ f \in H, \ h, \ x \in H.$$

$$\rho : H^* \longrightarrow H^* \otimes H (= \mathrm{End}(H)), \ \rho(f)(h) = \sum f(h_1)S(h_2).$$

Thus, if we write $\rho(f) = \sum f_0 \otimes f_1 \in H^* \otimes H$, then $\sum f_0(h) f_1 = \sum f(h_1)S(h_2)$ for all $h \in H$. It is not hard to check that H^* is a right Hopf module, with action $\leftarrow$ and coaction ρ. By the fundamental theorem of Hopf modules (cf. [M, 1.9.4]),

$$(H^*)^{coH} \otimes H \cong H^*, \quad f \otimes h \;\mapsto\; f \leftarrow h$$

and counting dimension we get $\dim(H^*)^{coH} = 1$. Now

$$\begin{aligned} f \in (H^*)^{coH} &\overset{\text{def}}{\iff} \rho(f) = f \otimes 1 \\ &\iff f(h)1 = \sum f(h_1)S(h_2) \text{ for all } h \in H \\ &\iff f(h)1 = \sum f(h_1)h_2 \text{ for all } h \in H \\ &\iff p(1)f = fp \text{ (in } H^*) \text{ for all } p \in H^* \end{aligned}$$

Thus $(H^*)^{coH} = \mathrm{Hom}^{-H}(H, k)$.

Choose (and fix) $0 \neq \phi \in \mathrm{Hom}^{-H}(H, k)$. We then get a Hopf module isomorphism

$$\Theta : H \cong H^*, \quad h \;\mapsto\; \phi \leftarrow h$$

and in particular H is a Frobenius algebra with H-colinear Frobenius map. Choose t in H satisfying $\Theta(t) = \varepsilon$, i.e., $\phi \leftarrow t = \varepsilon$. One can easily check that $\Theta(th) = \Theta(t\varepsilon(h))$ for all $h \in H$. Thus $th = t\varepsilon(h)$ and so t is a right integral in H. Note that if t is a right integral in H with $\phi(t) = 1$, then $\phi(th) = \phi(t\varepsilon(h)) = \phi(t)\varepsilon(h) = \varepsilon(h)$ and so $\phi \leftarrow t = \varepsilon$. Thus

Theorem 1.1. *Let H be a finite dimensional Hopf algebra. Then*

1) *There is a nonzero right integral ϕ in H^* and a right integral t in H with $\phi(t) = 1$.*

2) *The map $\Theta : H \longrightarrow H^*$, $h \mapsto \phi \leftarrow h$, is a right H-module and a right H-comodule isomorphism.*

3) *The antipode S is bijective.*

Proof.. It remains to show the bijectivity of S. For a right H-comodule V, set

$$R(V) = \{\sum \xi(\nu_1)\nu_0 \mid \nu \in V, \ \xi \in V^*\}.$$

We compute $R(H^*)$ for the right H-comodule H^* by two ways. First,

$$\begin{aligned}R(H^*) &= \{\sum \xi(f_1)f_0 | \ f \in H^*, \ \xi \in H^{**}\} \\ &= \{\sum f_1(h)f_0 | \ f \in H^*, \ h \in H\} \\ &= \{\sum f(h_1)S(h_2) | \ f \in H^*, \ h \in H\} \subset S(H).\end{aligned}$$

On the other hand, since Θ is a right H-comodule isomorphism, we have

$$\begin{aligned}R(H^*) &= \{\sum \xi(\Theta(h_1))h_2 | \ h \in H, \ \xi \in H^{**}\} \\ &= \{\sum \zeta(h_1)h_2 | \ h \in H, \ \zeta \in H^*\} \supset \{\sum \varepsilon(h_1)h_2 | \ h \in H\} = H.\end{aligned}$$

Hence S is surjective, thus bijective. □

In case $H = kG$, finite group algebra, one can take $\phi(g) := \delta_{g,1}$, $t := \sum_{g \in G} g$. Note that the above theorem can be generalized to Hopf algebra objects in Yetter-Drinfeld categories (see [D2]). We denote by $\overline{S}$ the composition inverse of S. Since S (hence also $\overline{S}$) is an antialgebra map and an anticoalgebra map, we have

$$\sum \overline{S}(h_2)h_1 = \varepsilon(h)1 = \sum h_2\overline{S}(h_1) \text{ for all } h \in H.$$

Proposition 1.2. *Let H be a finite dimensional Hopf algebra. Then*

1) $h = \sum \overline{S}(t_2)\phi(t_1 h)$ *for all* $h \in H$, *i.e.,* $\{\overline{S}(t_2), t_1\}$ *are dual bases for* ϕ.

2) $id_H = \sum \phi \leftarrow t_1 \otimes \overline{S}(t_2)$ *in* $\mathrm{End}(H) = H^* \otimes H$.

3) $\sum h\overline{S}(t_2) \otimes t_1 = \sum \overline{S}(t_2) \otimes t_1 h$ *(*$\forall\, h \in H$*) in* $H \otimes H$.

4) *The inverse of* $\Theta : H \to H^*$ *is given by* $\Theta^{-1}(p) = \sum p(\overline{S}(t_2))t_1$ *for all* $p \in H^*$.

Proof. 1) We compute for $h \in H$,

$$\begin{aligned}h &= \sum \varepsilon(h_1)h_2 = \sum \phi(th_1)h_2 \ (\text{by } \phi \leftarrow t = \varepsilon) \\ &= \sum \overline{S}(t_3)\phi(t_1h_1)t_2h_2 \ (\text{since } \sum \overline{S}(t_3)t_2 = \varepsilon(t_2)1) \\ &= \sum \overline{S}(t_2)\phi(t_1h) \ (\text{since } \phi(t_1h_1)t_2h_2 = \phi(th)1 \text{ by } \phi \in \mathrm{Hom}^{-H}(H, k)).\end{aligned}$$

2) and 4) follow from 1).

3) These are the same mapping $[x \mapsto hx]$ under the map

$$id \otimes \Theta : H \otimes H \cong H \otimes H^* = \mathrm{End}(H). \quad □$$

Theorem 1.3. *Let H be a finite dimensional Hopf algebra. Then the following are equivalent:*

a) $\varepsilon(t) \neq 0$, b) *H is separable.*, c) *H is semisimple.*

Proof. a)⇒b): Since $\sum \overline{S}(t_2)t_1 = \varepsilon(t)1$ and $\varepsilon(t) \neq 0$ it follows from 1.2.3 that H is separable.

b)⇒c): standard.

c)⇒a): Consider the following exact sequence of left H-modules

$$0 \to \operatorname{Ker} \varepsilon \longrightarrow H \xrightarrow{\varepsilon} k \to 0.$$

Since H is semisimple, there is a $q \in \operatorname{Hom}_{-H}(k,\ H)$ with $\varepsilon q = id_k$. But $q(1)$ is a right integral in H with $\varepsilon(q(1)) = 1$ and the space of right integrals is one dimensional. Hence $\varepsilon(t) \neq 0$. □

Remark. Replacing H by H^* in the above, we obtain

$$\phi(1) \neq 0 \iff H^* \text{ is separable } \iff H^* \text{ is semisimple.}$$

Definition. Since ht is also a right integral for any $h \in H$ and the space of right integrals is one dimensional, we have

$$ht = \alpha(h)t \text{ for some } \alpha \in \operatorname{Alg}(H,\ k).$$

We say α is *the (right) modular function* on H (does not depend on right integrals $t \neq 0$). Dually, since $f\phi[h \mapsto \sum f(h_1)\phi(h_2)]$ is an element of $\operatorname{Hom}^{-H}(H,\ k)$, we have

$$f\phi = f(g)\phi \text{ for some } g \in \mathrm{G}(H).$$

We say g is *the (right) modular element* in H. Clearly we have

Remarks.

1) H (resp. H^*) is *unimodular* $\iff \alpha = \varepsilon$ (resp. $g = 1$).

2) H (resp. H^*) is semisimple $\Longrightarrow H$ (resp. H^*) is unimodular [Apply ε to $ht = \alpha(h)t$].

Hence, $\alpha \neq \varepsilon$ (resp. $g \neq 1$) $\Longrightarrow \varepsilon(t) = 0$ (resp. $\phi(1) = 0$).

Definition. Let A be a right H-comodule algebra. A is said to be *H-Frobenius* if A is finite dimensional over k and there exists a right A-module and a right H-comodule isomorphism $\Theta : A \cong A^*$. Here A^* is a right A-module and a right H-comodule by

$$\leftharpoonup: A^* \otimes A \longrightarrow A^*,\ (f \leftharpoonup a)(x) = f(ax),\ f \in A^*,\ a,\ x \in A,$$

$$\rho : A^* \longrightarrow A^* \otimes H(= \operatorname{Hom}(A,\ H)),\ \rho(f)(a) = \sum f_0(a)f_1 = \sum f(a_0)S(a_1).$$

We have $(A^*)^{coH} = \mathrm{Hom}^{-H}(A,\ k)$. If A is H-Frobenius, then $\Theta(1)$ is clearly a right H-colinear map from A to k, which is called an H-Frobenius map. Conversely one can see that any (right H-comodule) Frobenius algebra with H-colinear Frobenius map is H-Frobenius. Thus H itself is H-Frobenius. For other recent work concerning H-Frobenius extension, see [D1].

The next theorem is essentially the same to [O, Thm 1.1].

Theorem 1.4. *Let H be a finite dimensional Hopf algebra and A a right H-comodule algebra with structure map $\rho_A : A \to A \otimes H$. If H^* is semisimple and A is H-Frobenius, then A^{coH} is a Frobenius algebra.*

Proof. Since $A \cong A^*$ as right H-comodules, we get a right A^{coH}-module isomorphism $A^{coH} \cong \mathrm{Hom}^{-H}(A,\ k)$ by taking $(-)^{coH}$. So it suffices to show that the restriction map

$$\kappa : \mathrm{Hom}^{-H}(A,\ k) \longrightarrow (A^{coH})^*, \quad f \mapsto f|_{A^{coH}}$$

is a right A^{coH}-module isomorphism. To see this, choose $\phi \in \mathrm{Hom}^{-H}(H,\ k)$ with $\phi(1) = 1$ (such a ϕ exists by the above remark since H^* is semisimple). Define

$$\mathrm{tr} : A \longrightarrow A^{coH} \quad , \mathrm{tr}(a) = \sum a_0\phi(S(a_1)),$$

where we write $\rho_A(a) = \sum a_0 \otimes a_1$. Indeed, $\mathrm{tr}(a)$ is in A^{coH} since

$$\begin{aligned}
\rho_A(\mathrm{tr}(a)) &= \sum a_0 \otimes a_1\phi(S(a_2)) \\
&= \sum a_0 a_1\phi(S(a_3))S(a_2) \quad (\text{by } \phi \in \mathrm{Hom}^{-H}(H,\ k)) \\
&= \sum a_0 \otimes \phi(S(a_1)) \\
&= \mathrm{tr}(a) \otimes 1.
\end{aligned}$$

For $F \in (A^{coH})^*$, define $\lambda(F) : A \to k$ by $\lambda(F)(a) = F(\mathrm{tr}(a))$. Then $\lambda(F)$ is right H-linear, since

$$\begin{aligned}
\sum \lambda(F)(a_0)a_1 &= \sum F(\mathrm{tr}(a_0))a_1 \\
&= \sum F(a_0)\phi(S(a_1))a_2 \\
&= \sum F(a_0)S(a_2)\phi(S(a_1))a_3 \\
&\qquad\qquad (\text{since } \phi \text{ is left } H\text{-colinear by } H^* \text{ is unimodular}) \\
&= \sum F(a_0)\phi(S(a_1))1 \\
&= \lambda(F)(a)1.
\end{aligned}$$

Thus λ is a map from $(A^{coH})^*$ into $\mathrm{Hom}^{-H}(A, k)$. We also have

$$\begin{aligned}\lambda(\kappa(f))(a) &= \kappa(f)(\mathrm{tr}(a)) \\ &= \sum f(a_0)\phi(S(a_1)) \\ &= \phi(S(\sum f(a_0)a_1)) \\ &= \phi(S(f(a)1)) \\ &= f(a)\phi(1) = f(a),\end{aligned}$$

and for $b \in A^{coH}$,

$$\kappa(\lambda(F))(b) = \lambda(F)(b) = F(\mathrm{tr}(b)) = F(b) \text{ by } \mathrm{tr}(b) = b.$$

Therefore κ is bijective, which is clearly a right A^{coH}-module map. □

2. Trace formulae

Let $\mathrm{Tr} : \mathrm{End}(H) \longrightarrow k$ denote the (usual) trace map. Then we have $\mathrm{Tr}(f \otimes h) = f(h)$ via identification: $H^* \otimes H \cong \mathrm{End}(H)$, $f \otimes h \mapsto [x \mapsto f(x)h]$.

Proposition 2.1. *Let H be a finite dimensional Hopf algebra and ϕ, t as in* 1.1.

1) $n1 = \mathrm{Tr}(id_H) = \sum \phi(t_1\overline{S}(t_2))$.
2) **(trace formula 1)** $\mathrm{Tr}(S^2) = \varepsilon(t)\phi(1)$.

Proof. 1) follows from 1.2.2.
2) Since $S^2 = \sum \phi \leftarrow t_1 \otimes S(t_2)$ we have

$$\mathrm{Tr}(S^2) = \sum(\phi \leftarrow t_1)(S(t_2)) = \sum \phi(t_1S(t_2)) = \phi(\varepsilon(t)1) = \varepsilon(t)\phi(1). \quad \square$$

Corollary 2.2. *H and H^* are semisimple (= separable)* $\Longleftrightarrow \mathrm{Tr}(S^2) \neq 0$.
In particular, if $S^2 = id$, then H and H^* are separable $\Longleftrightarrow n1 \neq 0$ *(n = dim H).*

Definition. For any $p \in H^*$ we define $p^\sim$, $p^\wedge \in \mathrm{End}(H)$ by

$$p^\sim(h) := \sum p(h_1)h_2, \quad p^\wedge(h) := \sum h_1p(h_2) \quad \text{for all } h \in H.$$

Lemma 2.3.

1) $(p^{\sim})^* = L(p)$, $(p^{\wedge})^* = R(p)$, *where* $L(p)$ *(resp.* $R(p)$*):* $H^* \to H^*$ *denotes the left (right) multiplication by* $p \in H^*$*, i.e.,* $L(p)(q) = pq$ *and* $R(p)(q) = qp$ *in* H^*.

2) $p^{\sim} = \sum \phi \leftarrow t_1 \otimes p(\overline{S}(t_3))\overline{S}(t_2)$, $p^{\wedge} = \sum \phi \leftarrow t_1 \otimes p(\overline{S}(t_2))\overline{S}(t_3)$. *Hence*

$$\mathrm{Tr}(p^{\sim}) = p(\sum \phi(t_1\overline{S}(t_2))\overline{S}(t_3)), \quad \mathrm{Tr}(p^{\wedge}) = p(\sum \phi(t_1\overline{S}(t_3))\overline{S}(t_2)).$$

3) $(\phi \leftarrow h)^{\sim} = \phi \leftarrow h_1 \otimes \overline{S}(h_2)$. *Hence* $\mathrm{Tr}((\phi \leftarrow h)^{\sim}) = \sum \phi(h_1\overline{S}(h_2))$.

Proof. 1) $((p^{\sim})^*(q))(h) := q((p^{\sim})(h)) = q(\sum p(h_1)h_2) = \sum p(h_1)q(h_2) = (pq)(h)$. Hence $(p^{\sim})^*(q) = pq = L(p)(q)$. Similarly $(p^{\wedge})^* = R(p)$.

$$2) \qquad \begin{aligned} p^{\sim}(h) &= p^{\sim}(\sum \phi(t_1 h)\overline{S}(t_2)) \text{ (by 1.2.1)} \\ &= \sum \phi(t_1 h)p(\overline{S}(t_3))\overline{S}(t_2) \quad \text{(since } \overline{S} \text{ is an anticoalgebra map).} \end{aligned}$$

Hence $p^{\sim} = \sum \phi \leftarrow t_1 \otimes p(\overline{S}(t_3))\overline{S}(t_2)$ in $H^* \otimes H$.

3) $(\phi \leftarrow h)^{\sim}(x) = \sum \phi(hx_1)x_2 = \sum \overline{S}(h_3)\phi(h_1x_1)h_2x_2 = \sum \overline{S}(h_2)\phi(h_1x)$.

Definition. Define an element y in H by

$$p(y) = \mathrm{Tr}(p^{\wedge}) \; (= \mathrm{Tr}(p^{\wedge *}) = \mathrm{Tr}(R(p)), \; p \in H^*.$$

As H^* is Frobenius, $\mathrm{Tr}(R(p)) = \mathrm{Tr}(L(p))$, and so the following holds

$$p(y) = \mathrm{Tr}(p^{\sim}) = \mathrm{Tr}(p^{\sim *}) = \mathrm{Tr}(L(p)), \; p \in H^*.$$

The next proposition 2.4 follows from the above lemma.

Proposition 2.4.

1) $y = \sum \phi(t_1\overline{S}(t_3))\overline{S}(t_2) = \sum \phi(t_1\overline{S}(t_2))\overline{S}(t_3)$.

2) $(\phi \leftarrow h)(y) = \mathrm{Tr}((\phi \leftarrow h)^{\sim}) = \phi(\sum h_1\overline{S}(h_2))$.

Proposition 2.5.

1) $\varepsilon(y) = n1$, *where* $n = \dim H$.

2) $S(y) = y$.

3) y *is a cocommutative element, that is,* $\Delta(y) = \sum y_2 \otimes y_1$.

4) *If* c *is a cocommutative element, then* $cy = \varepsilon(c)y = yc$. *In particular* $y^2 = ny$.

5) $\phi(y) = \phi(1)$.

Proof. 1) $\varepsilon(y) = \sum \phi(t_1\overline{S}(t_2))\varepsilon\overline{S}(t_3) = \sum \phi(t_1\overline{S}(t_2)) = n1$.

2) We first show $(p \circ S)^\wedge = \overline{S} \circ p^\sim \circ S$. Indeed,

$$\overline{S}(p^\sim(S(h))) = \overline{S}(\sum p(S(h_2))S(h_1)) = \sum p(S(h_2))h_1 = (p \circ S)^\wedge(h).$$

Now we have

$$p(S(y)) = \mathrm{Tr}(R(p \circ S)) = \mathrm{Tr}((p \circ S)^\wedge) = \mathrm{Tr}(\overline{S} \circ p^\sim \circ S) = \mathrm{Tr}(p^\sim) = p(y).$$

This shows $S(y) = y$.

3) For $p,\ q \in H^*$,

$$\begin{aligned}(pq)^\wedge(h) &= \sum h_1(pq)(h_2)\\ &= \sum h_1 p(h_2)q(h_3)\\ &= p^\wedge(\sum h_1 q(h_2))\\ &= (p^\wedge \circ q^\wedge)(h).\end{aligned}$$

Thus $(pq)(y) = \mathrm{Tr}((pq)^\wedge) = \mathrm{Tr}(p^\wedge \circ q^\wedge) = \mathrm{Tr}(q^\wedge \circ p^\wedge) = \mathrm{Tr}((qp)^\wedge) = (qp)(y)$. Hence y is a cocommutative element.

Another proof: We can check directly the cocommutativity of y:

$$\begin{aligned}\Delta(y) &= \sum \phi(t_1\overline{S}(t_4))\overline{S}(t_3) \otimes \overline{S}(t_2)\\ &= \sum \overline{S}(t_3) \otimes \overline{S}(t_2)\phi(t_1\overline{S}(t_4))\\ &= \sum \overline{S}(t_3) \otimes \overline{S}(t_2)\phi(t_1\overline{S}(t_6)t_2\overline{S}(t_5))\\ &= \sum \overline{S}(t_2) \otimes \phi(t_1\overline{S}(t_4))\overline{S}(t_3)\\ &= \mathrm{twist} \circ \Delta(y).\end{aligned}$$

$$\begin{aligned}4)\quad (\phi \leftharpoonup h)(cy) &= (\phi \leftharpoonup hc)(y)\\ &= \sum \phi(h_1c_1\overline{S}(h_2c_2)) \text{ by 2.4.2}\\ &= \sum \phi(h_1c_1\overline{S}(c_2)\overline{S}(h_2))\\ &= \sum \phi(h_1c_2\overline{S}(c_1)\overline{S}(h_2)) \quad \text{(since } c \text{ is cocommutative)}\\ &= \sum \phi(\varepsilon(c)h_1\overline{S}(h_2))\\ &= (\phi \leftharpoonup h)(\varepsilon(c)y).\end{aligned}$$

It follows from $H^* = \phi \leftharpoonup H$ that $cy = \varepsilon(c)y$. Next notices that $S(c)$ is also a cocommutative element. Hence $S(c)y = \varepsilon(c)y$, and

$$yc = \overline{S}(S(c)S(y)) = \overline{S}(S(c)y) \text{ (by the above 2)} = \overline{S}(\varepsilon(c)y) = \varepsilon(c)y.$$

5) We have $\phi(y) = \mathrm{Tr}(\phi^\sim) = \phi(1)$, since $\phi^\sim(h) = \sum \phi(h_1)h_2 = \phi(h)1$. □

We can now prove the second Trace formula.

Theorem 2.6. **(trace formula 2)** *Let* $n = \dim H$. *Then* $\varepsilon(t)\phi(1) = n\, Tr(S^2|_{Hy})$.

Proof. Notice that $S^2(Hy) = Hy$ by 2.5.2. Now we have from 1.2.1

$$hy = \sum \overline{S}(t_2)\phi(t_1 hy) \text{ for any } h \in H.$$

It follows that $nhy = \sum \overline{S}(t_2)y\phi(t_1 hy)$ (by $y^2 = ny$) and

$$nS^2(hy) = \sum S(t_2)y\phi(t_1 hy) \text{ (by } S^2(y) = y).$$

Hence $nS^2|_{Hy} = \sum \phi \leftarrow t_1 \otimes S(t_2)y$ in $(Hy)^* \otimes Hy$. Since $\phi(y) = \phi(1)$ by 2.5.5, we have

$$n\mathrm{Tr}(S^2|_{Hy}) = \sum(\phi \leftarrow t_1)(S(t_2)y) = \sum \phi(t_1 S(t_2)y) = \varepsilon(t)\phi(y) = \varepsilon(t)\phi(1).$$

□

Corollary 2.7. [LR1, Theorem 2] *H and* H^* *are semisimple* $\Longrightarrow n1 \neq 0$.

3. Applications

In this section we give some applications of trace formulae and the element y. Recall the *Nakayama automorphism* $N : H \to H$ with respect to ϕ, which is defined by the rule

$$\phi(xy) = \phi(yN(x)) \text{ for } x,\ y \in H.$$

We need the following results.

Theorem 3.1. *Let H be a finite dimensional Hopf algebra. Then*

1) $\sum \alpha(h_1)\overline{S}^2(h_2) = N(h) = g(1)^{-1}(\sum S^2(h_1)\alpha(h_2))g$ *for all* $h \in H$. *Here* α *denotes the modular function on H and g the modular element in H.*

2) (Radford, 1976) $S^4(h) = g(\sum \alpha(h_1)h_2\alpha^{-1}(h_3))g$, *for all* $h \in H$.

3) (Larson, 1971) *H and* H^* *are unimodular* $\Rightarrow S^4 = id$.

Proof. We follow a very nice proof given by Schneider [S].

1) We have

$$\begin{aligned} N(h) &= \sum \overline{S}(t_2)\phi(t_1 N(h)) \quad \text{(by 1.2.1)} \\ &= \sum \overline{S}(t_2)\phi(ht_1) \\ &= \overline{S}^2(\sum \phi(ht_1)S(t_2)) \\ &= \overline{S}^2(\sum \phi(h_1t_1)h_2t_2S(t_3)) \\ &= \overline{S}^2(\phi(h_1t)h_2) \\ &= \overline{S}^2(\sum \alpha(h_1)\phi(t)h_2) \\ &= \sum \alpha(h_1)\overline{S}^2(h_2). \end{aligned}$$

We next show that $\{S(t_1)g,\ t_2\}$ are dual bases for ϕ, i.e., $h = \sum S(t_1)g\phi(t_2h)$:

$$\begin{aligned}\sum S(t_1)g\phi(t_2h) &= \sum S(t_1)t_2h_1\phi(t_3h_2) \quad \text{(by the definition of } g\text{)}\\ &= \sum h_1\phi(th_2)\\ &= \sum h_1\varepsilon(h_2) = h.\end{aligned}$$

Now

$$\begin{aligned}N(h) &= \sum S(t_1)g\phi(t_2N(h))\\ &= \sum S(t_1)g\phi(ht_2)\\ &= g^{-1}(\sum gS(t_1)\phi(ht_2))g\\ &= g^{-1}S^2(\sum g\overline{S}(t_1)\phi(ht_2))g\\ &\qquad (g \in G(H) \text{ implies } S(g) = g^{-1} \text{ and so } S^2(g) = g)\\ &= g^{-1}S^2(\sum h_1t_2\phi(h_2t_3)\overline{S}(t_1))g \text{ (by the definition of } g\text{)}\\ &= g^{-1}S^2(\sum h_1\phi(h_2t))g\\ &= g^{-1}S^2(\sum h_1\alpha(h_2)\phi(t))g\\ &= g^{-1}(\sum S^2(h_1)\alpha(h_2))g \text{ (by } \phi(t) = 1).\end{aligned}$$

2) By 1), $\sum \alpha(h_1)\overline{S}^2(h_2) = g^{-1}(\sum S^2(h_1)\alpha(h_2))g$. Hence

$$g(\sum \alpha(h_1)h_2)g^{-1} = \sum S^4(h_1)\alpha(h_2)$$

and so

$$S^4(h) = \sum S^4(h_1)\alpha(h_2)\alpha^{-1}(h_3) = g(\sum \alpha(h_1)h_2\alpha^{-1}(h_3))g^{-1}.$$

3) Since H and H^* are unimodular, we have $\alpha = \varepsilon$ and $g = 1$. So by 2), $S^4 = id$. □

Corollary 3.2. [AN, Proposition 2.14] *Suppose that H is unimodular (i.e., $\alpha = \varepsilon$). Then*

1) $N = \overline{S}^2$.

2) *H is symmetric with respect to ϕ, i.e. $\phi(xy) = \phi(yx)$, $\forall\ x,\ y \in H$ ($\Leftrightarrow$ $N = id$) $\Longleftrightarrow$ $S^2 = id$.*

3) $\phi(xy) = \phi(yS^2(x)) \iff N = S^2 \iff S^4 = id$.

Theorem 3.3. [LR2, Theorem3] *Let k be a field of characteristic zero. Then*

$$H \text{ and } H^* \text{ are semisimple} \iff S^2 = id.$$

Proof. $\Leftarrow$ follows from corollary 2.2.

$\Rightarrow$): The proof follows the lecture note of Schneider [S]. As H and H^* are unimodular, $S^4 = id$ by 3.1.3. Hence the eigenvalues of S^2 and $S^2|_{Hy}$ are all 1 or -1 ($\in \mathbb{Q} \subset k$). Say, respectively, $\mu_1, \mu_2, \cdots, \mu_n$; $\eta_1, \eta_2, \cdots, \eta_m$ ($m = \dim Hy$). Then, by trace formula 2 (and 1),

$$\mu_1 + \mu_2 + \cdots + \mu_n = n(\eta_1 + \eta_2 + \cdots + \eta_m) \neq 0$$

implying

$$n = |\mu_1| + |\mu_2| + \cdots + |\mu_n| \geq n|\eta_1 + \eta_2 + \cdots + \eta_m| > 0.$$

So, $|\eta_1 + \eta_2 + \cdots + \eta_m| = 1$, or $\mu_1 + \mu_2 + \cdots + \mu_n = \pm n$. But, as there is at least one μ_i which equals to 1 (since $S^2(1) = 1$), then all μ_i must equal to 1, thus $S^2 = id$. □

Remark. In [LR2], Larson and Radford proved that finite dimensional cosemisimple Hopf algebras in characteristic 0 are semisimple. Hence the following holds under char $k = 0$,

$$H \text{ is semisimple} \iff H^* \text{ is semisimple} \iff S^2 = id.$$

Theorem 3.4. [LR2, Theorem 4.3]

$$y \text{ is a nonzero left integral} \Longrightarrow H^* \text{ is semisimple and } S^2 = id.$$

Proof. $\Rightarrow$) We have that for any $h \in H$,

$$\begin{aligned}\phi(h_1\overline{S}(h_2)) &= (\phi \leftarrow h)(y) \quad \text{(by 2.4.2)}\\ &= \phi(hy)\\ &= \varepsilon(h)\phi(y) \quad \text{(since } y \text{ is a left integral)}\\ &= \varepsilon(h)\phi(1) \quad \text{(by 2.5.5).}\end{aligned}$$

Since $y \neq 0$ and $\phi \leftarrow H = H^*$, there is an element z in H such that $(\phi \leftarrow z)(y) \neq 0$. Thus $\varepsilon(z)\phi(1) \neq 0$ and hence H^* is semisimple. In particular

H^* is unimodular and $\phi : H \to k$ is left H-colinear, that is, $\phi(h)1 = \sum h_1\phi(h_2)$ or, equivalently $\phi(h)1 = \sum \overline{S}(h_1)\phi(h_2)$, $h \in H$. Now, we compute

$$\begin{aligned}
\phi(1)h &= \sum \varepsilon(h_2)\phi(1)h_1 \\
&= \sum \phi(h_2\overline{S}(h_3))h_1 \quad \text{(by the above)} \\
&= \sum \overline{S}(h_2\overline{S}(h_5))\phi(h_3\overline{S}(h_4))h_1 \quad \text{(since } \phi \text{ is also left } H\text{-colinear)} \\
&= \sum \overline{S}^2(h_5)\overline{S}(h_2)h_1\phi(h_3\overline{S}(h_4)) \\
&= \sum \overline{S}^2(h_3)\phi(h_1\overline{S}(h_2)) \\
&= \sum \overline{S}^2(h_2)\varepsilon(h_1)\phi(1) = \phi(1)\overline{S}^2(h).
\end{aligned}$$

So $\overline{S}^2 = id$, thus $S^2 = id$.

$\Leftarrow$) We have $y \neq 0$ since $\phi(y) = \phi(1) \neq 0$. Next, $S^2 = id$ implies $S = \overline{S}$. So,

$$\begin{aligned}
y &= \sum \phi(t_1\overline{S}(t_2))\overline{S}(t_3) = \sum \phi(t_1S(t_2))S(t_3) \\
&= \sum \phi(\varepsilon(t_1)1)S(t_2) = \phi(1)S(t).
\end{aligned}$$

This shows that y is a left integral, since $S(t)$ is a left integral. □

Corollary 3.5. [LR2, Theorem 4.4] *y is a nonzero left integral and $n1 \neq 0$ if and only if $S^2 = id$ and H and H^* are semisimple.*

Proof. It follows from the above theorem and Corollary 2.2. □

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GELFAND-KIRILLOV DIMENSION FOR QUANTIZED WEYL ALGEBRAS

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ABSTRACT. We obtain an analogue of Bernstein's inequality for quantized Weyl algebras.

1. Introduction

Let k be a field. For an n-tuple $\bar{q} = (q_1, \cdots, q_n) \in (k^\times)^n$ and $n \times n$ matrix $\Lambda = (\lambda_{ij})$ over k such that $\lambda_{ii} = 1$ and $\lambda_{ij} = \lambda_{ji}^{-1}$ for all i, j, the n-th *quantized Weyl algebra* $A_n^{\bar{q},\Lambda}$ is the k-algebra generated by the elements $x_1, \cdots, x_n, y_1, \cdots, y_n$ with the following relations:

$$\begin{aligned} x_i x_j &= q_i \lambda_{ij} x_j x_i, \\ y_i y_j &= \lambda_{ij} y_j y_i, \\ x_i y_j &= \lambda_{ji} y_j x_i, \\ y_i x_j &= q_i^{-1} \lambda_{ji} x_j y_i. \\ x_j y_j - q_j y_j x_j &= 1 + \sum_{l=1}^{j-1} (q_l - 1) y_l x_l, \\ (x_1 y_1 - q_1 y_1 x_1 &= 1), \end{aligned} \tag{1.1}$$

where $1 \leq i < j \leq n$ (see [AD. 3.4]).

This algebra $A_n^{\bar{q},\Lambda}$. appeared in the work of Maltsiniotis on noncommutative differential calculus [Ma], is regarded as a q-analogue of the Weyl algebra A_n.

Bernstein's inequality says that, if M is a nonzero module over the Weyl algebra A_n, then the Gelfand-Kirillov dimension GKdim$(M) \geq n$. The purpose of this note is to obtain an analogue of this result for quantized Weyl algebra $A_n^{\bar{q},\Lambda}$. To this end, a simple localization of $A_n^{\bar{q},\Lambda}$ studied in [J] plays an important role.

1991 *Mathematics Subject Classification*. Primary 16P90; Secondary 17B37.

Key words and phrases. Quantized Weyl algebras, Gelfand-Kirillov dimension, Bernstein's inequality.

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TeX

Throughout this note, let $\bar{q}$, Λ be as above, and suppose that no q_i is a root of unity.

For ring theoretical notions including localizations, filtrations and Gelfand-Kirillov dimension, we refer to [McR].

2. Preliminaries

For $1 \leq i \leq n$, let $z_i = 1 + \sum_{l=1}^{i}(q_l - 1)y_l x_l$. By [J, 2.8] these elements satisfy the following relations:

$$z_j y_i = \begin{cases} y_i z_j & \text{if } j < i, \\ q_i y_i z_j & \text{if } j \geq i, \end{cases} \qquad z_j x_i = \begin{cases} x_i z_j & \text{if } j < i, \\ q_i^{-1} x_i z_j & \text{if } j \geq i, \end{cases} \tag{2.1}$$

$$z_i z_j = z_j z_i.$$

Thus, for each i, the set $\mathcal{Z}_i = \{z_i^s\}_{s \geq 0}$ is an Ore set in $A_n^{\bar{q},\Lambda}$, and the set $\mathcal{Z} = \mathcal{Z}_1 \cdots \mathcal{Z}_n$ is too. We denote by $B_n^{\bar{q},\Lambda}$ the localization of $A_n^{\bar{q},\Lambda}$ at $\mathcal{Z}$.

Proposition 2.2 [J, Theorem 3.2]. *Suppose that no q_i is a root of unity. Then $B_n^{\bar{q},\Lambda}$ is simple. In particular $B_n^{\bar{q},\Lambda}$ has no nonzero finite-dimensional module.*

Let us consider standard filtrations for $A = A_n^{\bar{q},\Lambda}$ and $B = B_n^{\bar{q},\Lambda}$.

Put $V = k + kx_1 + \cdots + kx_n + ky_1 + \cdots + ky_n$. This is a (finite-dimensional) generating subspace of A, that is, $A = \sum_{l \geq 0} V^l$, where $V^0 = k$. Then A has the filtration $\mathcal{B}(A)$ defined by

$$\mathcal{B}_s(A) = \sum_{l=0}^{s} V^l.$$

For the localization B of A, the subspace $W = kx_1 + kx_2 z_1^{-1} + \cdots + kx_n z_{n-1}^{-1} + ky_1 + \cdots + ky_n + kz_1 + \cdots + kz_n + kz_1^{-1} \cdots + kz_n^{-1}$ is a generating subspace. Denote by $\Gamma(B)$ the filtration of B associated with the generating subspace W. Thus

$$\Gamma_s(B) = \sum_{l=0}^{s} W^l.$$

A k-algebra R is called *semi-commutative* if R is generated as a k-algebra by elements $r_1, \cdots, r_m$ such that $r_i r_j = \mu_{ij} r_j r_i$ for $1 \leq i, j \leq n$, where $\mu_{ij} \in k^{\times}$ ([Mc, 3.7]).

Lemma 2.3. *The graded algebra* $\mathrm{gr}_\Gamma B$ *of* $B_n^{\bar{q},\Lambda}$ *associated with the filtration* $\Gamma(B)$ *is semi-commutative.*

Proof. This is clear from the observation that $x_i {z_{i-1}}^{-1} y_i - q_i y_i x_i {z_{i-1}}^{-1} = 1$ for each i. □

By the lemma we can apply [Mc, Theorem 3.8] to B, so the following proposition is obtained. Also see [McP, Section 5] .

Proposition 2.4. *Let notations be as above.*

(1) *For any finitely generated* B*-module* M*, the Gelfand-Kirillov dimension* $\mathrm{GKdim}_B(M)$ *is a nonnegative integer.*

(2) *For any nonzero finitely generated* B*-module* M*, there exists a nonnegative integer* $e_B(M) \geq 1$ *such that*

$$e_B(M) = e_B(L) + e_B(N)$$

for any exact sequence $0 \to L \to M \to N \to 0$ *of finitely generated* B*-modules with* $\mathrm{GKdim}_B(L) = \mathrm{GKdim}_B(M) = \mathrm{GKdim}_B(N)$.

(3) *For a* B*-module* M *of finite length, the endomorphism ring* $\mathrm{End}_B(M)$ *of* M *is algebraic over* k.

In general, $e_B(M)$ is called the multiplicity of M.

3. Main Results

Theorem 3.1. *Suppose that no* q_i *is a root of unity. Let* M *be a nonzero* $B_n^{\bar{q},\Lambda}$*-module. Then*

$$n \leq \mathrm{GKdim}_{B_n^{\bar{q},\Lambda}}(M) \leq 2n.$$

Proof. We modify the proof of [McR, Proposition 5.5] to prove the theorem.

Write $B_n = B_n^{\bar{q},\Lambda}$. Let M be a nonzero B-module. Since $\mathrm{GKdim}(B) = 2n$ by [GL, Proposition 3.4], it follows that $\mathrm{GKdim}_B(M) \leq 2n$.

We will show the inequality $n \leq \mathrm{GKdim}_B(M)$ by induction on n. We can assume that M is finitely generated. If $n = 1$, it is clear from Proposition 2.4 (1) and Proposition 2.2. Assume that the inequality holds for $n-1$. Let $\bar{q}' = (q_1, \cdots, q_{n-1})$, Λ' be the subarray $(\lambda_{ij})_{1\leq i,j\leq n-1}$ of Λ. Then $B_{n-1} = B_{n-1}^{\bar{q}',\Lambda'}$ can be regarded as a subalgebra of B_n. If $\mathrm{GKdim}_{B_n}(M) < n$, then $\mathrm{GKdim}_{B_n}(M) < n-1$. We claim that M has finite length as a B_{n-1}-module. It suffices to show that any finitely generated B_{n-1}-submodule of M has finite length $\leq e_{B_n}(M)$. Let N be a finitely generated nonzero B_{n-1}-submodule of M. By the inductive hypothesis, one sees that $n-1 \leq \mathrm{GKdim}_{B_{n-1}}(N) \leq \mathrm{GKdim}_{B_{n-1}}(M) \leq \mathrm{GKdim}_{B_n}(M) = n-1$, so that $\mathrm{GKdim}_{B_{n-1}}(N) = \mathrm{GKdim}_{B_n}(M) = n-1$.

Then it follows from [McP, Proposition 5.7] that $e_{B_{n-1}}(N) \leq e_{B_n}(M)$. Using Proposition 2.4 (2), one sees that N has finite length $\leq e_{B_{n-1}}(N) \leq e_{B_n}(M)$.

Now, by Proposition 2.4 (3), $\mathrm{End}_{B_{n-1}}(M)$ is algebraic over k. From the relations (2.1), left action by $z_n z_{n-1}{}^{-1}$ on M is a left B_{n-1}-module endomorphism of M. Moreover, since M is faithful as a B-module, the k-algebra generated by $z_n z_{n-1}{}^{-1}$ can be regarded as a subalgebra of $\mathrm{End}_{B_{n-1}}(M)$. However it is easy to check that $z_n z_{n-1}{}^{-1}$ is algebraically independent over k, which is a contradiction. □

Corollary 3.2. *Suppose that no q_i is a root of unity. Let M be a finitely generated nonzero $A_n^{\bar{q},\Lambda}$-module. If M is not $\mathcal{Z}$-torsion, then*

$$n \leq \mathrm{GKdim}_{A_n^{\bar{q},\Lambda}}(M) \leq 2n.$$

Proof. Put $A = A_n^{\bar{q},\Lambda}$, $B = B_n^{\bar{q},\Lambda}$. First, we claim that

$$\mathrm{GKdim}_A(M) = \mathrm{GKdim}_B(B \otimes_A M)$$

for any $\mathcal{Z}$-torsionfree finitely generated nonzero A-module M. We modify the proof of [GL, Lemma 3.3] to prove the claim. Let V be the generating subspace of A described before. It is ovbious that $W = V + kz_1^{-1}$ is a generating subspace of the localization $\mathcal{Z}_1^{-1}A$ of A at $\mathcal{Z}_1$. There exists a nonnegative integer t such that $W^m \subset z_1^{-m}V^{mt}$ for each m. Let M_0 be a finite-dimensional generating subspace of the A-module M. Then $W^m M_0 \subset z_1^{-m}V^{mt}M_0$, so that $\dim_k W^m M_0 \leq \dim_k V^{mt}M_0$. Since M is $\mathcal{Z}_1$-torsionfree, we can regard M as an A-submodule of $\mathcal{Z}_1^{-1}M = \mathcal{Z}_1^{-1}A \otimes_A M$ via the map $M \to \mathcal{Z}_1^{-1}M$, $m \mapsto 1 \otimes m$. In particular M_0 is a generating subspace of the $\mathcal{Z}_1^{-1}A$-module $\mathcal{Z}_1^{-1}M$. Thus $\mathrm{GKdim}_{\mathcal{Z}_1^{-1}A}(\mathcal{Z}_1^{-1}M) \leq \mathrm{GKdim}_A(M)$. Clearly $\mathrm{GKdim}_{\mathcal{Z}_1^{-1}A}(\mathcal{Z}_1^{-1}M) \geq \mathrm{GKdim}_A(M)$. Hence $\mathrm{GKdim}_{\mathcal{Z}_1^{-1}A}(\mathcal{Z}_1^{-1}M) = \mathrm{GKdim}_A(M)$. By continuing similar argument, we can prove the claim.

Let M be a nonzero A-module with $T(M) \neq M$. Since $M/T(M)$ is $\mathcal{Z}$-torsionfree nonzero module, it holds that

$$\mathrm{GKdim}_A(M/T(M)) = \mathrm{GKdim}_{\mathcal{Z}^{-1}A}(\mathcal{Z}^{-1}(M/T(M))) \geq n$$

by Theorem 3.1. It follows from [McR, 8.3.2] that $\mathrm{GKdim}_A(M/T(M)) \leq \mathrm{GKdim}_A(M)$, which implies $n \leq \mathrm{GKdim}_A(M)$.

The upper bound is clear since $\mathrm{GKdim}(A) = 2n$ (see [GL, Proposition 3.4]). □

Remark 3.3. The corollary fails without the condition on a $A_n^{\bar{q},\Lambda}$-module M. In fact, for $1 \leq i \leq n$, there exists a $\mathcal{Z}$-torsion finitely generated $A_n^{\bar{q},\Lambda}$-module M with $\mathrm{GKdim}_{A_n^{\bar{q},\Lambda}}(M) = i$. Put $A = A_n^{\bar{q},\Lambda}$. Let $L = Ay_{i+1} + \cdots + Ay_n + Ax_1 + \cdots + Ax_n$. The A-module $M = A/L$ has the filtration $\mathcal{B}'(M)$ induced by the filtration $\mathcal{B}(A)$ of A. Thus $\mathcal{B}'_s(M) = (\mathcal{B}_s(A) + L)/L$ is isomorphic as a vector space to

$$\bigoplus_{\alpha_1+\cdots+\alpha_i \leq s} k y_1^{\alpha_1} \cdots y_i^{\alpha_i}.$$

Hence $\dim_k \mathcal{B}'_s(M) = \binom{i+s}{i}$. This implies that $\mathrm{GKdim}_A(M) = i$.

Another Bernstein's inequality for quantized Weyl algebras has been considered by Demidov in [D].

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CATENARITY IN MODULE-FINITE ALGEBRAS

SHIRO GOTO*AND KENJI NISHIDA

ABSTRACT. The main theorem says that any module-finite (but not necessarily commutative) algebra Λ over a commutative Noetherian universally catenary ring R is catenary. Hence the ring Λ is catenary if R is Cohen-Macaulay. When R is local and Λ is a Cohen-Macaulay R-module, we have that Λ is a catenary ring, $\dim \Lambda = \dim \Lambda/Q + \mathrm{ht}_\Lambda Q$ for any $Q \in \operatorname{Spec} \Lambda$, and the equality $n = \mathrm{ht}_\Lambda Q - \mathrm{ht}_\Lambda P$ holds true for any pair $P \subseteq Q$ of prime ideals in Λ and for any saturated chain $P = P_0 \subset P_1 \subset \dots \subset P_n = Q$ of prime ideals between P and Q.

1. Introduction

Let R be a commutative Noetherian ring and let Λ be an R-algebra which is finitely generated as an R-module. Here we don't assume Λ to be a commutative ring. The purpose of this article is to prove the following

Theorem (1.1). *Any module-finite R-algebra Λ is catenary if R is universally catenary.*

Before going ahead, let us recall the definition of catenarity and universal catenarity ([Ma, p.84]). We say that our ring Λ is catenary if for any pair $P \subseteq Q$ of prime ideals in Λ and for any saturated chain $P = P_0 \subset P_1 \subset \dots \subset P_n = Q$ of prime ideals between P and Q, the length n of the chain is independent of its particular choice and hence equal to $\mathrm{ht}_{\Lambda/P} Q/P$, where $\mathrm{ht}_{\Lambda/P} Q/P$ denotes the height of the prime ideal Q/P in Λ/P. The base ring R is said to be universally catenary if any finitely generated commutative R-algebra is catenary. Hence R is universally catenary if and only if $R/\mathfrak{p}$ is universally catenary for every prime ideal $\mathfrak{p}$ in R. Naturally, homomorphic images and localizations of universally catenary rings are universally catenary. And our theorem (1.1) asserts that not necessarily commutative but module-finite R-algebras are still catenary if R is universally catenary. The converse is also true in some sense if R is a local integral domain (Corollary (3.4)).

Since Cohen-Macaulay rings are necessarily universally catenary, from Theorem (1.1) it follows that

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*Supported by the Grant-in-Aid for Scientific Researches (C)

1991 *Mathematics Subject Classification.* Primary 13E05, 16A18; Secondary 13H10, 16A33.

This paper is not in a final form. The final version will appear in Proceedings of Amer. Math. Soc.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

Corollary (1.2). *Suppose R is a Cohen-Macaulay ring. Then the ring Λ is catenary.*

And in the case where R is local and Λ is Cohen-Macaulay as an R-module, one can improve Theorem (1.1) to get the following, whose consequences will be discussed in the forthcoming paper [GN].

Corollary (1.3). *Assume that $(R, \mathfrak{m})$ is a local ring and Λ is a Cohen-Macaulay R-module. Then Λ is catenary and one has the equality*

$$\dim \Lambda = \dim \Lambda/Q + \mathrm{ht}_\Lambda Q$$

for any prime ideal Q in Λ (here $\dim \Lambda/Q$ and $\dim \Lambda$ respectively denote the Krull dimension of the rings Λ/Q and Λ). The equality $n = \mathrm{ht}_\Lambda Q - \mathrm{ht}_\Lambda P$ holds true for any pair $P \subseteq Q$ of prime ideals in Λ and for any saturated chain $P = P_0 \subset P_1 \subset \ldots \subset P_n = Q$ of prime ideals between P and Q.

The study of catenarity in commutative Noetherian rings dates back to counterexamples of Nagata [N1] in 1956. In it he constructed also examples of catenary but not universally catenary Noetherian local domains (cf. [N2, p. 203, Example 2]). And after Nagata's examples people had focused in vain a great deal of their effort on the problem if Noetherian normal local domains are catenary, until Ogoma [O] constructed in 1980 his celebrated counterexamples. Nevertheless their effort was not entirely useless and as an inheritance it has left deep researches on chain conjectures and quasi-unmixed local rings ([R1, R2, R3, R4, Mc], etc.). See Nishimura [Ni] for the recent developments.

The study of catenarity in non-commutative Noetherian rings is much steadier. We note here the following two strong results. Firstly, Schelter established the catenarity in Noetherian P. I. algebras finitely generated over central subfields. Subsequently in 1978 he succeeded in deleting the Noetherian assumption from his theorem ([S]). And in 1996 Goodearl and Lenagan [GL] showed that certain Auslander-Gorenstein and Cohen-Macaulay quantum algebras finitely generated over central subfields are catenary.

The researches [S] and [GL] study finitely generated algebras over fields, while we are going to explore module-finite algebras over commutative Noetherian rings. And as we shall eventually show in the present paper, comparing with [S] and [GL], our assumption that Λ be module-finite algebras over commutative Noetherian rings R makes the problem simpler and the proof easier. Actually in 1983 Brown, Hajarnavis, and MacEacharn [BHM, 5.2 Theorem] already tried to give a proof of our Corollary (1.3). However it should be mentioned that their argument has contained a serious gap and is not acceptable.

The proof of Theorem (1.1) shall be given in Section 3. In Section 2 we will summarize some basic results on Λ, that we need to prove Theorem (1.1).

In what follows, let R be a commutative Noetherian ring. Let Λ be an R-algebra with $f : R \to \Lambda$ the structure map. The R-algebra Λ is throughout assumed to be finitely generated as an R-module. We denote by $\operatorname{Spec}\Lambda$, $\operatorname{Min}\Lambda$, and $\operatorname{Max}\Lambda$ the set of prime ideals, the set of minimal prime ideals, and the set of maximal ideals in Λ, respectively. Otherwise specified, all Λ-modules stand for left Λ-modules.

2. Some basic results and notation

The purpose of this section is to summarize some basic results on prime ideals in Λ, that we shall freely use in Section 3. We begin with the following, in which for each ideal I in Λ let $I \cap R$ denote the inverse image $f^{-1}(I)$ of I.

Lemma (2.1) ([MR, Chapter 10]). *Suppose that the structure map $f : R \to \Lambda$ is injective. Then the following assertions hold true.*

(1) (**Lying-over**) *For any prime ideal $\mathfrak{p}$ in R there is a prime ideal P in Λ with $\mathfrak{p} = P \cap R$.*
(2) (**Going-up**) *Let $\mathfrak{p} \subseteq \mathfrak{q}$ be prime ideals in R and let $P \in \operatorname{Spec}\Lambda$ with $\mathfrak{p} = P \cap R$. Then there is a prime ideal Q in Λ such that $P \subseteq Q$ and $\mathfrak{q} = Q \cap R$.*
(3) (**Incomparability**) *Let $P \subseteq Q$ be prime ideals in Λ. Then $P = Q$ if and only if $P \cap R = Q \cap R$.*
(4) *Let $P \in \operatorname{Spec}\Lambda$. Then $P \in \operatorname{Max}\Lambda$ if and only if $P \cap R \in \operatorname{Max} R$.*
(5) *For each $P \in \operatorname{Min}\Lambda$ the prime ideal $\mathfrak{p} = P \cap R$ in R consists of zerodivisors for Λ.*
(6) $\dim\Lambda = \dim R = \dim_R \Lambda$, *where $\dim_R \Lambda$ denotes the Krull dimension of Λ as an R-module.*
(7) *For each $P \in \operatorname{Spec}\Lambda$ we have $\operatorname{ht}_\Lambda P = \operatorname{ht}_{\Lambda_\mathfrak{p}} P\Lambda_\mathfrak{p} \le \dim R_\mathfrak{p}$, where $\mathfrak{p} = P \cap R$. In particular $\operatorname{ht}_\Lambda P$ is necessarily finite.*

In the case where $(R, \mathfrak{m})$ is a local ring, for each finitely generated R-module M we put

$$\operatorname{depth}_R M = \inf\{n \in \mathbb{Z} \mid \operatorname{Ext}^n_R(R/\mathfrak{m}, M) \neq (0)\}$$

and call it the depth of M. This invariant $\operatorname{depth}_R M$ equals the length of maximal M-regular sequences contained in $\mathfrak{m}$. See [BH] and [Ma] for detailed investigations about it.

Corollary (2.2). *Assume that $(R, \mathfrak{m})$ is a local ring and $\operatorname{Max}\Lambda \cap \operatorname{Min}\Lambda \neq \emptyset$. Then the following assertions hold true.*

(1) $\operatorname{depth}_R \Lambda = 0$.
(2) $\dim R = 0$ *if Λ is a free R-module.*

Proof. Let $P \in \operatorname{Max}\Lambda \cap \operatorname{Min}\Lambda$. Then because $P \in \operatorname{Max}\Lambda$, we have $\mathfrak{m} = P \cap R$ (see (2.1) (4)), whence by (2.1) (5) the ideal $\mathfrak{m}$ consists of zerodivisors for Λ

since $P \in \operatorname{Min}\Lambda$. Thus $\operatorname{depth}_R \Lambda = 0$. To see assertion (2) let $\mathfrak{p} \in \operatorname{Spec} R$. Then since $P \supseteq \mathfrak{p}\Lambda$, we have $P/\mathfrak{p}\Lambda \in \operatorname{Max}\Lambda/\mathfrak{p}\Lambda \cap \operatorname{Min}\Lambda/\mathfrak{p}\Lambda$. Hence by (1) $\operatorname{depth}_{R/\mathfrak{p}} \Lambda/\mathfrak{p}\Lambda = 0$, which yields $\operatorname{depth}_{R/\mathfrak{p}} R/\mathfrak{p} = 0$, because $\Lambda/\mathfrak{p}\Lambda$ is a free $R/\mathfrak{p}$-module. Thus $\dim R/\mathfrak{p} = 0$ since $R/\mathfrak{p}$ is an integral domain, so that we have $\mathfrak{m} = \mathfrak{p}$. Hence $\dim R = 0$.

Lemma (2.3). *Let $k = \inf\{n \geq 0 \mid t_1, t_2, ..., t_n \in Q \cap R$ such that Q is a minimal prime ideal of $(t_1, t_2, ..., t_n)\Lambda\}$, for each $Q \in \operatorname{Spec}\Lambda$. Then, the equality $\operatorname{ht}_\Lambda Q = k$ holds true.*

Proof. We may assume that the structure map $f : R \to \Lambda$ is injective. Let $h = \operatorname{ht}_\Lambda Q$. To check $h \geq k$, we may assume that $h > 0$ and our inequality holds true for prime ideals with height at most $h-1$. Let $\mathcal{F}$ be the set of minimal prime ideals P of Λ contained in Q. Then since $\mathcal{F}$ is a finite set ([MR, 2.2.17]) and since any $P \in \mathcal{F}$ is strictly contained in Q, by (2.1) (3) and [AM, 1.11] we see the ideal $Q \cap R$ is not contained in $\bigcup_{P \in \mathcal{F}} P \cap R$. Let $t = t_1 \in Q \cap R$ with $t \notin \bigcup_{P \in \mathcal{F}} P \cap R$. Then since $\operatorname{ht}_{\Lambda/t\Lambda} Q/t\Lambda \leq h - 1$ for the choice of t, by the hypothesis on h we may choose elements $t_2, t_3, ..., t_h$ of $Q \cap R$ so that Q is a minimal prime ideal of $(t_1, t_2, ..., t_h)\Lambda$. Thus $h \geq k$ and so we get $\operatorname{ht}_\Lambda Q = k$, because $h \leq k$ by [MR, 4.1.13].

In the case where $(R, \mathfrak{m})$ is a local ring, we denote by R^* the $\mathfrak{m}$-adic completion of R. Recall that R^* is a faithfully flat extension of R, which is a Noetherian local ring with maximal ideal $\mathfrak{m}^* = \mathfrak{m}R^*$ and $\dim R^* = \dim R$. We put $M^* = R^* \otimes_R M$ for each R-module M.

Corollary (2.4). (1) *Suppose $(R, \mathfrak{m})$ is a local ring and let $Q \in \operatorname{Max}\Lambda$. Then $Q^* \in \operatorname{Max}\Lambda^*$ and $\operatorname{ht}_{\Lambda^*} Q^* = \operatorname{ht}_\Lambda Q$.*

(2) *Let $Q \in \operatorname{Spec}\Lambda$ and let $t \in Q \cap R$ be a nonzerodivisor for Λ. Then $\operatorname{ht}_\Lambda Q = \operatorname{ht}_{\Lambda/t\Lambda} Q/t\Lambda + 1$. Consequently, if t is an element in the Jacobson radical $\mathrm{J}(R)$ of R and if t is a nonzerodivisor for Λ, the equality $\dim\Lambda = \dim\Lambda/t\Lambda + 1$ holds true.*

Proof. (1) Since $Q \supseteq \mathfrak{m}\Lambda$ and $R^* \otimes_R R/\mathfrak{m} \cong R/\mathfrak{m}$, we have the isomorphism $\Lambda^*/Q^* = R^* \otimes_R \Lambda/Q \cong \Lambda/Q$ and so certainly $Q^* \in \operatorname{Max}\Lambda^*$. We put $h = \operatorname{ht}_\Lambda Q$. The inequality $\operatorname{ht}_{\Lambda^*} Q^* \leq h$ follows from (2.3). In fact, choose $t_1, t_2, ..., t_h \in Q \cap R$ so that Q is minimal over $I\Lambda$, where $I = (t_1, t_2, ..., t_h)R$. It suffices to see Q^* is minimal over $I\Lambda^*$ too. Let $P \in \operatorname{Spec}\Lambda^*$ with $Q^* \supseteq P \supseteq I\Lambda^*$. Then $Q = Q^* \cap \Lambda \supseteq P \cap \Lambda \supseteq I\Lambda$ clearly. Since $P \cap \Lambda \in \operatorname{Spec}\Lambda$ and since Q is minimal over $I\Lambda$, we have $Q = P \cap \Lambda$ whence $Q^* = Q\Lambda^* \subseteq P$. Thus Q^* is minimal over $I\Lambda^*$ and by (2.3) $\operatorname{ht}_{\Lambda^*} Q^* \leq h$. Let us check that $\operatorname{ht}_{\Lambda^*} Q^* \geq h$ by induction on h. We may assume that $h > 0$ and our inequality holds true for $h - 1$. Let $P_0 \subset P_1 \subset ... \subset P_h = Q$ be a saturated chain of prime ideals in Λ. Then because $\operatorname{ht}_{\Lambda/P_1} Q/P_1 = h - 1$, by the hypothesis on h we see that $\operatorname{ht}_{\Lambda^*/P_1^*} Q^*/P_1^* \geq h - 1$. Choose $\mathfrak{p} \in \operatorname{Spec}\Lambda^*$ such that $Q^* \supseteq \mathfrak{p} \supseteq P_1^*$ and $\mathrm{ht}_{\Lambda^*/\mathfrak{p}} Q^*/\mathfrak{p} = \mathrm{ht}_{\Lambda^*/P_1^*} Q^*/P_1^* \geq h-1$. It is enough to show $\mathfrak{p} \notin \mathrm{Min}\,\Lambda^*$. Let us choose $t \in P_1 \cap R$ so that $t \notin P_0 \cap R$ (by (2.1) (3) this choice is possible). Then since t is a nonzerodivisor for Λ/P_0, the element t must be a nonzerodivisor for Λ^*/P_0^* too (recall R^* is R-flat). Therefore if $\mathfrak{p}$ were a minimal prime ideal in Λ^* then so is the prime ideal $\mathfrak{p}/P_0^*$ in the ring Λ^*/P_0^* and hence by (2.1) (5) the ideal $\mathfrak{p} \cap R^*$ consists of zerodivisors for Λ^*/P_0^*. This is impossible because $t \in \mathfrak{p} \cap R^*$ and t is a nonzerodivisor for Λ^*/P_0^*. Hence $\mathfrak{p} \notin \mathrm{Min}\,\Lambda^*$.

(2) By (2.3) we get $\mathrm{ht}_\Lambda Q \leq \mathrm{ht}_{\Lambda/t\Lambda} Q/t\Lambda + 1$. The opposite inequality $\mathrm{ht}_\Lambda Q \geq \mathrm{ht}_{\Lambda/t\Lambda} Q/t\Lambda + 1$ follows from the fact that for any $P \in \mathrm{Min}\,\Lambda$ the ideal $t\Lambda$ is not contained in P (cf. (2.1) (5)). The second assertion is now clear because $t \in Q \cap R$ for all $Q \in \mathrm{Max}\,\Lambda$ (cf. (2.1) (4)).

Suppose that R is a local ring. Then we say that R is quasi-unmixed if $\dim R^*/\mathfrak{p} = \dim R$ for all $\mathfrak{p} \in \mathrm{Min}\,R^*$. The next epoch-making characterization is due to Ratliff [R2, Theorem 3.6].

Proposition (2.5) ([R2]). *Suppose that R is a local domain. Then the following two conditions are equivalent.*

(1) *R is universally catenary.*

(2) *R is quasi-unmixed.*

When this is the case, the ring $R/\mathfrak{p}$ is also quasi-unmixed and one has the equality $\dim R/\mathfrak{p} + \dim R_\mathfrak{p} = \dim R$ *for all* $\mathfrak{p} \in \mathrm{Spec}\,R$.

We close this section with a few remarks on canonical modules. Let R be a local ring and assume that R is a homomorphic image of a Gorenstein local ring S. Then we put

$$\mathrm{K}_R = \mathrm{Ext}_S^n(R, S) \quad (n = \dim S - \dim R)$$

and call it the canonical module of R. Properties of canonical modules are closely explored in [HK] and [BH]. Here let us pick up two of them, which we will need to prove Theorem (1.1). We indicate a sketch of proof for assertion (2), since we have no direct reference for it.

Lemma (2.6). (1) $\mathrm{K}_{(R_\mathfrak{p})} \cong (\mathrm{K}_R)_\mathfrak{p}$ *for any* $\mathfrak{p} \in \mathrm{Supp}_R\,\mathrm{K}_R$.

(2) *Suppose that* $\dim R \geq 2$ *and let a_1, a_2 be a subsystem of parameters for R. Then the sequence a_1, a_2 is K_R-regular.*

Proof. (1) See [HK, Korollar 5.25] or [A, Corollary 4.3].

(2) Write $R = S/I$ with an ideal I in S. Then I contains an S-regular sequence $f_1, f_2, ..., f_n$ of length n. Passing to the ring $S/(f_1, f_2, ..., f_n)S$, we may assume $n = 0$ whence $\mathrm{K}_R = \mathrm{Hom}_S(R, S)$. Choose an S-regular sequence b_1, b_2 so that $a_1 = b_1 \mod I$ and $a_2 = b_2 \mod I$. (This choice is possible. See [K, Theorem 124].) That the sequence a_1, a_2 is K_R-regular now follows, since $\mathrm{K}_R = \mathrm{Hom}_S(R, S)$ and the sequence b_1, b_2 is S-regular.

3. Proofs of Theorem (1.1) and Corollary (1.3)

The purpose of this section is to prove Theorem (1.1) and Corollary (1.3). We begin with the following.

Lemma (3.1). *Suppose that $(R, \mathfrak{m})$ is a local ring and that R is a homomorphic image of a Gorenstein local ring. Let $P \in \operatorname{Spec} \Lambda$ and put $\mathfrak{p} = P \cap R$, $\Gamma = \Lambda/P$. Then if $\dim \Gamma \geq 2$, there exist a short exact sequence*

$$0 \longrightarrow \Gamma \xrightarrow{\alpha} X \xrightarrow{\beta} Y \longrightarrow 0$$

of finitely generated Γ-modules and an element $t \in \mathfrak{m} \setminus \mathfrak{p}$ satisfying the following conditions.

(1) $\operatorname{depth}_R X \geq 2$.
(2) *t is X-regular and $tY = (0)$.*

Proof. Let $K = \mathrm{K}_{R/\mathfrak{p}}$ be the canonical module of $R/\mathfrak{p}$. We put

$$X = \operatorname{Hom}_{R/\mathfrak{p}}(\operatorname{Hom}_{R/\mathfrak{p}}(\Gamma, K), K)$$

and let $\alpha : \Gamma \to X$ denote the canonical homomorphism of G-modules. Then X is finitely generated and $\operatorname{depth}_R X \geq 2$ by (2.6) (2) because $\dim R/\mathfrak{p} = \dim \Gamma \geq 2$ by (2.1) (6). Since $K_\mathfrak{p} \cong \mathrm{K}_{(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p})} = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ by (2.6) (1), we see the homomorphism $R_\mathfrak{p} \otimes_R \alpha$ is an isomorphism. Hence α is a monomorphism because the $R/\mathfrak{p}$-module Γ is torsionfree. As $Y_\mathfrak{p} = (0)$ where $Y = \operatorname{Coker} \alpha$, we have $tY = (0)$ for some $t \in \mathfrak{m} \setminus \mathfrak{p}$. The element t is X-regular since it is $R/\mathfrak{p}$-regular (cf. (2.6) (2)).

The next result is the key in our proof.

Proposition (3.2). *Let $(R, \mathfrak{m})$ be a universally catenary local ring and assume that Λ is a prime ring. Then $\dim \Lambda = 1$ if Λ contains a maximal ideal Q of $\operatorname{ht}_\Lambda Q = 1$.*

Proof. We may assume that $f : R \to \Lambda$ is injective. Hence R is a local integral domain and $\dim R = \dim \Lambda$ (cf. (2.1) (6)). Firstly we consider the case where R is a homomorphic image of a Gorenstein local ring. Assume that $\dim \Lambda \geq 2$. Then by (3.1) we get an embedding $\Lambda \to X$ of finitely generated Λ-modules and an element $0 \neq t \in \mathfrak{m}$ satisfying the conditions (1) $\operatorname{depth}_R X \geq 2$ and (2) t is X-regular with $tX \subseteq \Lambda$. We put $Z = X/tX$ and $D = \operatorname{End}_\Lambda Z$. Let $I = (0) :_\Lambda Z$. Then since $IX \subseteq tX \subseteq \Lambda$ by condition (2) and since $Q \in \operatorname{Max} \Lambda$ by our assumption, we have $I^2 \subseteq I^2X \subseteq t\Lambda \subseteq Q$ whence $t\Lambda \subseteq I \subseteq Q$. Because $t \notin P \cap R$ for any $P \in \operatorname{Min} \Lambda$ by (2.1) (5) (note that t is Λ-regular) and because $\operatorname{ht}_\Lambda Q = 1$ by our assumption, we have $Q/I \in \operatorname{Min} \Lambda/I \cap \operatorname{Max} \Lambda/I$. Hence $\operatorname{depth}_R \Lambda/I = 0$ by (2.2) (1). On the other hand, from the embedding

$$\Lambda/I \longrightarrow \operatorname{End}_D Z$$

of R-algebras induced from the canonical homomorphism $\lambda : \Lambda \to \mathrm{End}_D Z$, we have $\mathrm{depth}_R \Lambda/I > 0$ because $\mathrm{depth}_R Z = \mathrm{depth}_R X - 1 > 0$ by condition (1). This is a contradiction.

Now we consider the general case. Let R^* denote the $\mathfrak{m}$-adic completion of R. Then since $Q^* = R^* \otimes_R Q$ is a maximal ideal of $\Lambda^* = R^* \otimes_R \Lambda$ with $\mathrm{ht}_{\Lambda^*} Q^* = \mathrm{ht}_\Lambda Q = 1$ (cf. (2.4) (1)), we get $\mathrm{ht}_{\Lambda^*/P} Q^*/P = 1$ for some $P \in \mathrm{Min}\,\Lambda^*$. Hence $\dim \Lambda^*/P = 1$ because R^* is a homomorphic image of a Gorenstein (in fact, regular) local ring. Let $\mathfrak{p} = P \cap R^*$. Then we have

Claim. (1) $P \cap \Lambda = (0)$.
(2) $\mathfrak{p} \in \mathrm{Min}\,R^*$.

Proof of Claim. (1) Note that $P \cap \Lambda \in \mathrm{Spec}\,\Lambda$ and $Q^* \cap \Lambda = Q$. Then if $P \cap \Lambda \neq (0)$, we have $Q = P \cap \Lambda$ since $Q \supseteq P \cap \Lambda \neq (0)$ and $\mathrm{ht}_\Lambda Q = 1$. Hence $Q^* \subseteq P$ which is impossible.
(2) By (1) $\mathfrak{p} \cap R = (0)$ since the map $f : R \to \Lambda$ is a monomorphism. Let $\mathrm{Q}(R)$ denote the quotient field of R. Then since $R^*_\mathfrak{p} \otimes_{R^*} \Lambda^* \cong R^*_\mathfrak{p} \otimes_{\mathrm{Q}(R)} (\mathrm{Q}(R) \otimes_R \Lambda)$, we get $\Lambda^*_\mathfrak{p}$ is a free $R^*_\mathfrak{p}$-module with $P\Lambda^*_\mathfrak{p} \in \mathrm{Max}\,\Lambda^*_\mathfrak{p} \cap \mathrm{Min}\,\Lambda^*_\mathfrak{p}$ ((2.1) (4), (7)). Hence $\dim R^*_\mathfrak{p} = 0$ by (2.2) (2).

By Claim (2) and (2.5) we have $\dim R = \dim R^*/\mathfrak{p} = \dim \Lambda^*/P = 1$. Hence $\dim \Lambda = 1$ since $\dim \Lambda = \dim R$.

We now come to the following

Theorem (3.3). *Suppose that R is a universally catenary local ring and Λ is a prime ring. Then the equality*

$$\dim \Lambda = \dim \Lambda/Q + \mathrm{ht}_\Lambda Q$$

holds true for all $Q \in \mathrm{Spec}\,\Lambda$.

Proof. We may assume $f : R \to \Lambda$ is injective. We first consider the case where $Q \in \mathrm{Max}\,\Lambda$. Let $h = \mathrm{ht}_\Lambda Q$. By (3.2) we may assume that $h \geq 2$ and our assertion holds true for $h-1$. Let $(0) = P_0 \subset P_1 \subset \ldots \subset P_h = Q$ be a saturated chain of prime ideals in Λ. Let $\mathfrak{p} = P_{h-1} \cap R$. Then $\dim \Lambda/P_{h-1} = 1$ by (3.2). On the other hand, since $P_{h-1}\Lambda_\mathfrak{p}$ is a maximal ideal in the prime ring $\Lambda_\mathfrak{p}$ and $\mathrm{ht}_{\Lambda_\mathfrak{p}} P_{h-1}\Lambda_\mathfrak{p} = \mathrm{ht}_\Lambda P_{h-1} = h-1$ ((2.1) (7)), by the hypothesis of induction on h the ring $\Lambda_\mathfrak{p}$ has Krull dimension $h-1$. Consequently, by (2.1) (6) and (2.5) we get $\dim \Lambda = \dim R = \dim R/\mathfrak{p} + \dim R_\mathfrak{p} = \dim \Lambda/P_{h-1} + \dim \Lambda_\mathfrak{p} = 1 + (h-1) = h$. Now let $Q \in \mathrm{Spec}\,\Lambda$ and put $\mathfrak{q} = Q \cap R$. Then since $Q\Lambda_\mathfrak{q} \in \mathrm{Max}\,\Lambda_\mathfrak{q}$, we see $\dim \Lambda_\mathfrak{q} = \mathrm{ht}_{\Lambda_\mathfrak{q}} Q\Lambda_\mathfrak{q} = \mathrm{ht}_\Lambda Q$. Hence $\dim R_\mathfrak{q} = \mathrm{ht}_\Lambda Q$. Because $\dim \Lambda/Q = \dim R/\mathfrak{q}$ and $\dim \Lambda = \dim R = \dim R/\mathfrak{q} + \dim R_\mathfrak{q}$ by (2.5), we get the equality $\dim \Lambda = \dim \Lambda/Q + \mathrm{ht}_\Lambda Q$.

Now we are in a position to prove Theorem (1.1).

Proof of Theorem (1.1). Passing to the ring Λ/P, we may assume $P = (0)$. Let $\mathfrak{p} = Q \cap R$. Then after the localization at $\mathfrak{p}$, we may furthermore assume that R is local and $Q \in \operatorname{Max}\Lambda$. Let $(0) = P_0 \subset P_1 \subset \dots \subset P_n = Q$ be a saturated chain of prime ideals in Λ. We will show $n = \dim\Lambda$ by induction on n. By (3.2) we may assume that $n \geq 2$ and our assertion is true for $n-1$. Then since $\dim \Lambda/P_1 = n-1$ and $\operatorname{ht}_\Lambda P_1 = 1$, we get $n = \dim\Lambda/P_1 + 1 = \dim\Lambda/P_1 + \operatorname{ht}_\Lambda P_1 = \dim\Lambda$ by (3.3).

Proof of Corollary (1.3). We may assume $f : R \to \Lambda$ is injective. Let $d = \dim R$. Let R^* denote the $\mathfrak{m}$-adic completion of R and we get the embedding $R^* \to \Lambda^*$ of R^*-modules too. Hence $\operatorname{Ass}_{R^*} R^* \subseteq \operatorname{Ass}_{R^*}\Lambda^*$ and we have $\dim R^*/\mathfrak{p} = d$ for all $\mathfrak{p} \in \operatorname{Ass}_{R^*} R^*$ (cf. [Ma, p.107]), because Λ^* is a Cohen-Macaulay R^*-module of $\dim_{R^*}\Lambda^* = d$. Therefore for every $\mathfrak{p} \in \operatorname{Spec} R$ the local ring $R/\mathfrak{p}$ is quasi-unmixed ([N2, (34.5)]), whence by (2.5) $R/\mathfrak{p}$ is universally catenary. Thus R is universally catenary, so that by (1.1) Λ is catenary.

Let $Q \in \operatorname{Min}\Lambda$ and put $\mathfrak{q} = Q \cap R$. We will show $\dim\Lambda/Q = d$. The prime ideal $\mathfrak{q}$ consists of zerodivisors for Λ (see (2.1) (5)). Hence $\mathfrak{q} \subseteq \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R \Lambda} \mathfrak{p}$ (cf. [Ma, p.50, Corollary 2]) and we have $\mathfrak{q} \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Ass}_R \Lambda$ (cf. [AM, 1.11]). Note that $\dim R/\mathfrak{p} = d$ (cf. [Ma, p.107]), because Λ is Cohen-Macaulay and $\dim_R \Lambda = d$. And we get $\mathfrak{q} = \mathfrak{p}$ since $\mathfrak{p} \in \operatorname{Min} R$. Hence $\dim R/\mathfrak{q} = d$ so that by (2.1) (6) we have $\dim\Lambda/Q = d$ for all $Q \in \operatorname{Min}\Lambda$.

Thanks to the universal catenarity in R, Theorems (1.1) and (3.3) and this observation readily imply that for any pair $P \subseteq Q$ of prime ideals in Λ with $P \in \operatorname{Min}\Lambda$ and $Q \in \operatorname{Max}\Lambda$, saturated chains of prime ideals between P and Q have the same length equal to $d = \dim\Lambda$. Hence the equality

$$\dim\Lambda = \dim\Lambda/Q + \operatorname{ht}_\Lambda Q$$

holds true for any prime ideal Q in Λ. We also readily have $n = \operatorname{ht}_\Lambda Q - \operatorname{ht}_\Lambda P$ for any pair $P \subseteq Q$ of prime ideals and for any saturated chain $P = P_0 \subset P_1 \subset \dots \subset P_n = Q$ of prime ideals in Λ. Thus Corollary (1.3) is proven.

To conclude this paper, we note the following. The equivalence is a direct consequence of Theorems (1.1), (3.3), and [R2, Theorems 3.1 and 3.6].

Corollary (3.4). *For a commutative Noetherian local integral domain R the following conditions are equivalent.*

(1) *R is universally catenary.*
(2) *R is quasi-unmixed.*
(3) *Every module-finite prime R-algebra Λ is catenary and the equality*

$$\dim\Lambda = \dim\Lambda/Q + \operatorname{ht}_\Lambda Q$$

holds true for any $Q \in \operatorname{Spec}\Lambda$.

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PHANTOMS IN THE REPRESENTATION THEORY OF FINITE DIMENSIONAL ALGEBRAS

BIRGE HUISGEN-ZIMMERMANN

1. INTRODUCTION AND PREREQUISITES

Our principal goal in this overview is to exhibit the importance of phantoms in the process of understanding how a full subcategory $\mathcal{A}$ of Λ-mod is embedded into the latter category in terms of maps leaving $\mathcal{A}$. (Here Λ is a finite dimensional algebra over a field K.) The purpose of phantoms in this connection is twofold: On one hand, they represent the - so far - only tool for systematically tackling the question of whether or not $\mathcal{A}$ is functorially finite in Λ-mod. On the other hand, certain of the $\mathcal{A}$-phantoms of a given object $X \in \Lambda$-mod capture - within a *minimal* frame - the relations of those objects in $\mathcal{A}$ which have nontrivial homomorphisms to X.

For a preview of the central definitions, let $\mathcal{C} \subseteq \mathcal{A}$ and recall that $f \in \operatorname{Hom}_\Lambda(A, X)$ is a $\mathcal{C}$-approximation of X inside $\mathcal{A}$ in case A belongs to $\mathcal{A}$ and all maps in $\operatorname{Hom}_\Lambda(C, X)$ with $C \in \mathcal{C}$ factor through f [20]. In case $\mathcal{C} = \mathcal{A}$, we re-encounter the classical (right) $\mathcal{A}$-approximations of X as introduced by Auslander and Smalø [3]. Note that $\mathcal{C}$-approximations of X inside $\mathcal{A}$ need not exist in general, but are always available when $\mathcal{C}$ is finite. So if $\mathcal{C}$ is countable, for instance, say $\mathcal{C} = \{C_n \mid n \in \mathbb{N}\}$, it is natural to consider minimal $\{C_1, \dots, C_n\}$-approximations of X inside $\mathcal{A}$ and to explore whether they can be glued together to an object in $\mathcal{A}$ which serves as the source of a $\mathcal{C}$-approximation of X inside $\mathcal{A}$. Of course, forming the direct sum of a full collection of $\{C_1, \dots, C_n\}$-approximations of X inside $\mathcal{A}$ will always yield the, not necessarily finitely generated, source of a homomorphism through which all maps $C_n \to X$ will factor. However, such sums will be highly redundant relative to this stipulation, as a rule. The minimality condition which we impose to produce more informative glueings is as follows: An $\mathcal{A}$-phantom of X is a module $H \in \Lambda$-Mod such that, for each finitely generated submodule H' of H, there exists a finite subclass $\mathcal{A}' \subseteq \mathcal{A}$ depending on H' with the property that H' is a subfactor of *each* $\mathcal{A}'$-approximation of X inside A. The class of all $\mathcal{A}$-phantoms of X is closed under subfactors and direct limits (of directed systems) and contains non-finitely generated objects if and only if X fails to have a classical $\mathcal{A}$-approximation. In this negative situation, there always exist non-finitely generated $\mathcal{A}$-phantoms H of X which are *effective* relative to certain subclasses

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX

$\mathcal{C}$ of $\mathcal{A}$ in the sense that H is a direct limit of a directed system in $\mathcal{A}$, and all maps in $\mathrm{Hom}_\Lambda(C, X)$ with $C \in \mathcal{C}$ factor through a suitable homomorphism $H \to X$. More detail on these concepts and results can be found in Section 4.

Our favorite category will be $\mathcal{P}^\infty(\Lambda\text{-mod})$ – for a definition, see under 'Prerequisites' below. To place the concepts of Section 4 into context, we will discuss and exemplify in Sections 2,3 how massive an impact contravariant finiteness of $\mathcal{P}^\infty(\Lambda\text{-mod})$ has on the representation theory of Λ. The most striking results in this context are Theorems 2 and 3 of Section 3 which identify the minimal $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximations of the simples as the basic building blocks of arbitrary objects in $\mathcal{P}^\infty(\Lambda\text{-Mod})$ and as the key objects to consider when determining finitistic dimensions. This program of accessing $\mathcal{P}^\infty(\Lambda\text{-Mod})$ can be carried out in an ideally explicit format when Λ is left serial [7]. This first model situation will be presented at the end of Section 3. The second class of algebras for which the theory developed yields particularly complete and smooth results is the class of string algebras. In that case, answers to essentially all homological questions one might pose can be provided in the form of a finite number of 'characteristic' $\mathcal{P}^\infty(\Lambda\text{-mod})$-phantoms, one for each of the simple left modules. This is the content of ongoing joint work of the author with Smalø [27], and is sketched in Section 5. We include a brief history of insights into the representation theory of string algebras, since these algebras have established their role as excellent display cases of the methods developed during the past decades.

Since we consider examples as the key to an intuitive grasp of the type of information stored in phantoms, we start by setting up a sequence of 'test examples' to which we keep returning as the discussion proceeds.

Content overview.

Prerequisites.

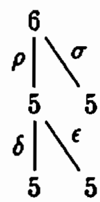
Throughout, $\Lambda = K\Gamma/I$ will be a path algebra modulo relations with Jacobson radical J, and the vertex set of Γ will be identified with a full set of primitive idempotents of Λ. By Λ-Mod we will denote the category of all left Λ-modules and by Λ-mod the full subcategory of finitely generated modules. Moreover, $\mathcal{P}^\infty(\Lambda\text{-Mod})$ and $\mathcal{P}^\infty(\Lambda\text{-mod})$ will be the subcategories of Λ-Mod and Λ-mod, respectively, having as objects the modules of finite projective dimension. The suprema of the projective dimensions attained on $\mathcal{P}^\infty(\Lambda\text{-Mod})$ and $\mathcal{P}^\infty(\Lambda\text{-mod})$ will be labeled $\mathrm{l\,Fin\,dim}\,\Lambda$ and $\mathrm{l\,fin\,dim}\,\Lambda$, respectively.

Given a path p in $K\Gamma$, we will denote by start(p) and end(p) the starting and end points of p, respectively, and our convention for multiplying paths $p, q \in K\Gamma$ is as follows: "qp" stands for "q after p".

Our most important auxiliary devices will be labeled and layered graphs of finite dimensional modules. Since our graphing conventions differ to some extent from those of other authors (in particular, they are akin to but not the same as the module diagrams studied by Alperin [1] and Fuller [14]), we include an informal description of our graphs for the convenience of the reader.

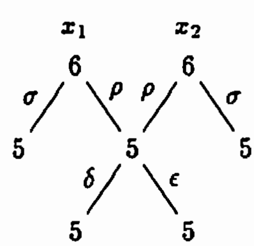
The graphs we use are based on sequences of *top elements* of a module M which are K-linearly independent modulo JM. Here $x \in M$ is a top element of M if $X \notin JM$ and $x = ex$ for one of the primitive idempotents e corresponding to the vertices of Γ.

Let $\Lambda = K\Gamma/I$ be the algebra presented in Example C.1. That the indecomposable projective module Λe_6 have the graph

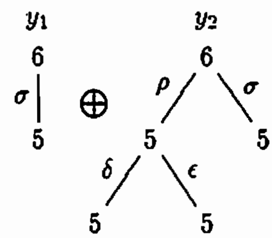


with respect to the top element e_6 means that $J^3e_6 = 0$ and $Je_6/J^2e_6 \cong J^2e_6 \cong S_5 \oplus S_5$, that ρe_6 and σe_6 are K-linearly independent modulo J^2e_6, and that $\delta\rho e_6$ and $\epsilon\rho e_6$ are K-linearly independent (modulo J^3e_6), while $\delta\sigma e_6 = \epsilon\delta e_6 = 0$ in Λe_6.

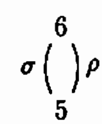
Whenever we present the graph of an indecomposable projective module Λe, we tacitly assume that the corresponding top element of Λe is chosen to be e. In our first example the choice of top element of Λe_6 does not influence the graph, but it will in other situations. For instance, the module $M = (\Lambda e_6 \oplus \Lambda e_6)/U$, where $U = \Lambda(\rho e_6, \rho e_6)$, has graph



relative to the top elements $x_1 = (e_6, 0)$ and $x_2 = (0, e_6)$, while its graph relative to the top elements $y_1 = x_1 + x_2$ and $y_2 = x_2$ is

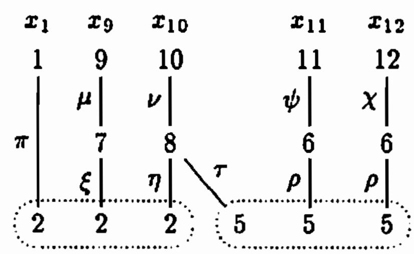


Note that the graph of a module need not determine the module in question, up to isomorphism. For example, each of the modules $M_k = \Lambda e_6/U_k$, where $U_k = \Lambda(\sigma - k\rho)e_6$ with $k \in K^*$, has graph



with respect to the top element $e_6 + U_k$ of M_k, while $M_{k_1} \not\cong M_{k_2}$ iff $k_1 \neq k_2$.

To enlarge the family of objects in Λ-mod which possess labeled and layered graphs relative to suitable sequences of top elements, we also allow for graphs with '*pools*' along the following model. That a module $A \in \Lambda$-mod has graph



is to encode the following information: The module A is generated by top elements x_i of type e_i $(i = 1, 9, 10, 11, 12)$ which are K-linearly independent modulo JA (here automatically satisfied, since $e_i \neq e_j$ for $i \neq j$) such that $J^3A = 0$, and

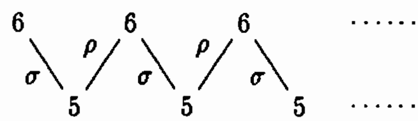
(a) $JA/J^2A \cong S_7 \oplus S_8 \oplus S_6^2$, with μx_9, νx_{10}, ψx_{11}, χx_{12} being K-linearly independent modulo J^2A, and

(b) $J^2A \cong S_2^2 \oplus S_5^2$, and the "pooled elements" πx_1, $\xi\mu x_9$, $\eta\nu x_{10}$ are K-linearly dependent, while any two of these elements are K-linearly independent; analogously, $\tau\nu x_{10}$, $\rho\psi x_{11}$, $\rho\chi x_{12}$ are K-linearly dependent with any subset of two K-linearly independent.

Note that in this particular example, the module A is determined up to isomorphism by its graph, since the coefficients arising in the mentioned linear dependence relations can be adjusted by suitably modifying the top elements by scalar factors.

It is clear that we will not lose information if we omit the labels on edges

with the property that there is a unique arrow $i \to j$ in Γ. Moreover, it should be self-explanatory that certain countably generated left Λ-modules can be communicated by graphs as well. The *infinite* graph



for instance, goes with a module B, uniquely determined up to isomorphism, which is generated by top elements $x_1, x_2, x_3, \ldots$ of type e_6 which are K-linearly independent modulo JB such that $\sigma x_i = \rho x_{i+1}$ for all $i \geq 1$, and such that the elements σx_i, $i \in \mathbb{N}$, are K-linearly independent (modulo $J^2B = 0$).

2. Contravariant finiteness and first examples

In 1980, Auslander and Smalø [3] gave the following definitions and their duals, in connection with their search for conditions ensuring a full, extension-closed subcategory $\mathcal{A}$ of Λ-mod to have almost split sequences. It turned out that this outcome is guaranteed provided that $\mathcal{A}$ is both co- and contravariantly finite in the sense recalled below. Even though their result has been applied since the 1980's, it was not until the 1990's that the concept of contravariant finiteness reached a high level of popularity, due to its links with homology and tilting. The spark in the tinder barrel was a paper by Auslander and Reiten [2]. We will describe their homological results in Section 3.

Credit for the concept of 'approximation', which lies at the heart of contravariant finiteness, should also go to Enochs [12], who introduced and studied it independently under the name 'cover' around the time at which the initial Auslander-Smalø article appeared.

Definitions. Let $\mathcal{A} \subseteq \Lambda$-mod be a full subcategory and $X \in \Lambda$-mod.

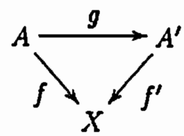
(1) A *right $\mathcal{A}$-approximation of* X is a homomorphism $f : A \to X$ with $A \in \mathcal{A}$ such that each $g \in \mathrm{Hom}(B, X)$ with $B \in \mathcal{A}$ factors through f, or equivalently, such that the following sequence of functors induced by f is exact:

$$\mathrm{Hom}_\Lambda(-, A)|_{\mathcal{A}} \longrightarrow \mathrm{Hom}_\Lambda(,X)|_{\mathcal{A}} \longrightarrow 0.$$

(Since we will hardly mention the dual concept of 'left approximation', we will systematically suppress the qualifier 'right' when discussing approximations.)

(2) The subcategory $\mathcal{A}$ is said to be *contravariantly finite* (in Λ-mod) if each $X \in \Lambda$-mod has an $\mathcal{A}$-approximation, i.e., if each of the functors $\mathrm{Hom}_\Lambda(-, X)|_\mathcal{A}$ is finitely generated in the category of all contravariant functors from $\mathcal{A}$ to **Ab**.

Suppose that X has an $\mathcal{A}$-approximation. By a slight abuse of language, we will then also refer to the source of this map as an approximation. Not surprisingly, the $\mathcal{A}$-approximations of X of minimal length are all isomorphic. Indeed, as was shown by Auslander and Smalø [3], given a minimal $\mathcal{A}$-approximation $f : A \to X$ and any $\mathcal{A}$-approximation $f' : A' \to X$, there exists a split embedding $g : A \to A'$ which makes the following diagram commute:



It is thus justified to refer to the minimal $\mathcal{A}$-approximation of X in case of existence.

If $\mathcal{A}$ is a resolving subcategory of Λ-mod, i.e., if $\mathcal{A}$ contains all projectives in Λ-mod and is closed under extensions and kernels of epimorphisms, then the simples play a prominent role in checking contravariant finiteness. Indeed:

Theorem 1. [2] *Suppose $\mathcal{A}$ is a resolving subcategory of* Λ-mod. *Then $\mathcal{A}$ is contravariantly finite in* Λ-mod *if and only if each of the simple left Λ-modules has an $\mathcal{A}$-approximation.* □

Since, clearly, our favorite category $\mathcal{P}^\infty(\Lambda\text{-mod})$ is resolving, this will provide a convenient test for contravariant finiteness. As we will see in the next section, the minimal $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximations of the simples are the basic structural building blocks for the objects in $\mathcal{P}^\infty(\Lambda\text{-mod})$ in case of existence, whence it is a problem of high priority to understand the structure of these particular approximations.

To provide examples, we begin with some

Well Known Facts.

• A subcategory $\mathcal{A} \subseteq \Lambda$-mod is contravariantly finite in case it is 'very big' or 'very small'. Indeed, if $\mathcal{A} = \Lambda$-mod, then clearly contravariant finiteness is guaranteed, the minimal approximations being the identity maps. So, in particular, $\mathcal{P}^\infty(\Lambda\text{-mod})$ is contravariantly finite provided that Λ has finite global dimension. On the other hand [2], if $\mathcal{A}$ has finite representation type, i.e., if there exist objects $A_1, \dots, A_n \in \mathcal{A}$ such that each object in $\mathcal{A}$ is a direct sum of copies of the A_i, we have contravariant finiteness as well. To construct an $\mathcal{A}$-approximation of a module X, simply add up as many copies of each A_i as the

K-dimension of $\mathrm{Hom}_\Lambda(A_i, X)$ indicates. In particular, if $\mathcal{A}$ is the category of all finitely generated projectives in Λ-mod, the minimal $\mathcal{A}$-approximations are precisely the projective covers. This latter category coincides with $\mathcal{P}^\infty(\Lambda\text{-mod})$ precisely when $\mathrm{l\,fin\,dim}\,\Lambda = 0$.

• [2] If Λ is stably equivalent to a hereditary algebra Λ' (meaning that the stable category Λ-$\underline{\mathrm{mod}}$, obtained as a factor category from Λ-mod by killing the maps that factor through projectives, is equivalent to the corresponding stable category Λ'-$\underline{\mathrm{mod}}$), then $\mathcal{P}^\infty(\Lambda\text{-mod})$ is contravariantly finite. This hypothesis is, in particular, satisfied if $J^2 = 0$, and in that case, the minimal approximation $A(X)$ of any Λ-module X can be readily pinned down as follows [23]: $A(X) = P/(JP)_{\mathrm{fin}}$, where P is the projective cover of X and $(JP)_{\mathrm{fin}}$ is the direct sum of those homogeneous components of JP which have finite projective dimension.

• [7] If Λ is left serial, then $\mathcal{P}^\infty(\Lambda\text{-mod})$ is always contravariantly finite. The minimal approximations of the simples arising in this situation will be described in the next section.

• The first example for which $\mathcal{P}^\infty(\Lambda\text{-mod})$ was shown not to be contravariantly finite is due to Igusa-Smalø-Todorov [28]. It is a monomial relation algebra with $J^3 = 0$ and $\mathrm{l\,fin\,dim}\,\Lambda = 1$ which is closely related to the Kronecker algebra, its K-dimension exceeding that of the latter algebra only by 2. In this example, the right finitistic dimension is 0, which, in view of our first remark, demonstrates that the right-hand category $\mathcal{P}^\infty(\text{mod-}\Lambda)$ is contravariantly finite in mod-Λ. Thus contravariant finiteness of $\mathcal{P}^\infty(-)$ is not left-right symmetric.

• In [20], very general criteria for the failure of contravariant finiteness of $\mathcal{P}^\infty(\Lambda\text{-mod})$ are developed. We refer the reader to the examples worked out there and in [25].

The emerging picture indicates that, while hardly any of the traditionally considered classes of finite dimensional algebras enjoy en bloc the property that $\mathcal{P}^\infty(-)$ is contravariantly finite, the positive case is ubiquitous. In fact, the condition of having contravariantly finite $\mathcal{P}^\infty(-)$ appears to slice diagonally through the prominent classes of algebras of infinite global dimension.

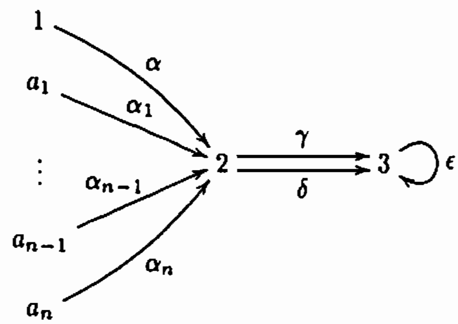
First Installment of Nonstandard Examples.

We will next present a first set of examples which are to communicate the flavor of prototypical phenomena ensuring that a given simple module has a $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximation or that it fails to have such an approximation. These specific algebras will then continue to serve us as illustrations along the way. We will follow with proofs of some of the positive instances, but defer the discussion of the negative instances to Section 4.

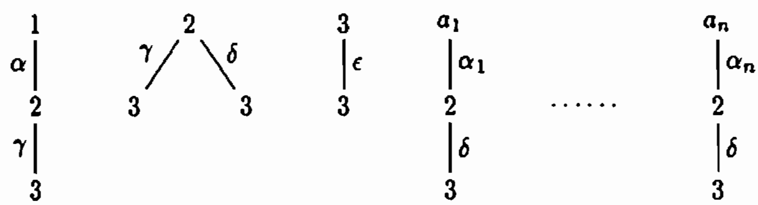
Examples A. Our first example shows that, for each natural number n, there exists a finite dimensional monomial relation algebra Λ and a simple $S \in \Lambda$-mod such that $\mathcal{P}^\infty(\Lambda\text{-mod})$ is contravariantly finite and the minimal $\mathcal{P}^\infty(\Lambda\text{-mod})$-

approximation of S is a direct sum of n distinct nonzero indecomposable components.

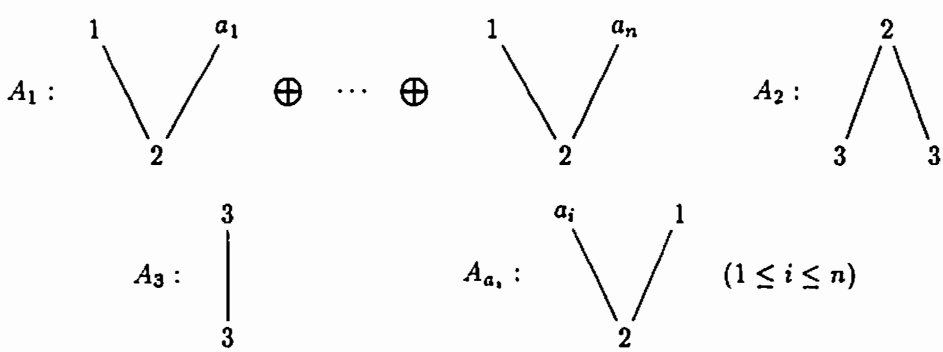
A.1. Fix $n \in \mathbb{N}$, and let $\Lambda = K\Gamma/I$ be the monomial relation algebra with quiver Γ



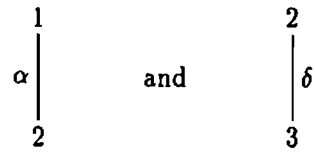
having indecomposable projective left modules with graphs



Then $\mathcal{P}^\infty(\Lambda\text{-mod})$ is contravariantly finite, and the minimal $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximations $A_1, A_2, A_3, A_{a_1}, \ldots, A_{a_n}$ of the simple modules $S_1, S_2, S_3, S_{a_1}, \ldots, S_{a_n}$ have the following graphs (which determine the corresponding modules up to isomorphism).

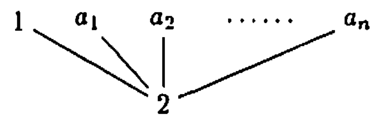


A.2. Now let $\Lambda = K\Gamma'/I'$, where Γ' is obtained from the quiver Γ of A.1 by removing the arrow γ, and the ideal $I' \subseteq K\Gamma'$ is such that the graphs of Λe_3, $\Lambda e_{a_1}, \ldots, \Lambda e_{a_n}$ are as under A.1, whereas Λe_1 and Λe_2 have graphs



respectively.

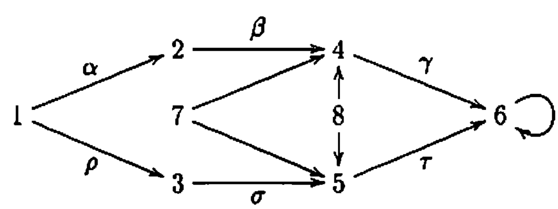
Then $S_1 = \Lambda e_1/Je_1$ has minimal $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximation



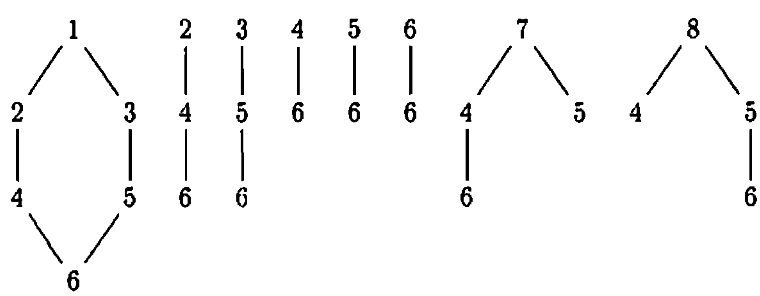
S_2 has minimal $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximation Λe_2, and $S_{a_1}, \ldots, S_{a_n}$ belong to $\mathcal{P}^\infty(\Lambda\text{-mod})$. In particular, $\mathcal{P}^\infty(\Lambda\text{-mod})$ is again contravariantly finite.

Examples B.

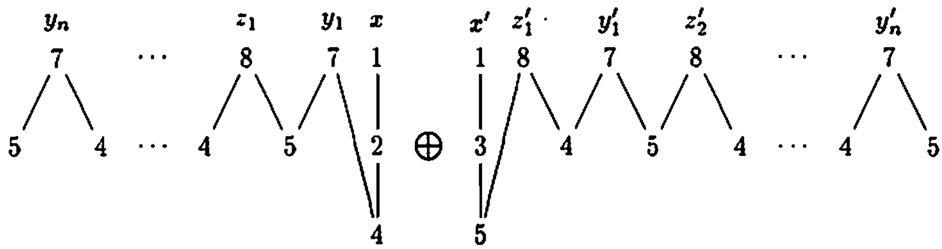
B.1. Let $\Lambda = K\Gamma/I$, where Γ is the quiver



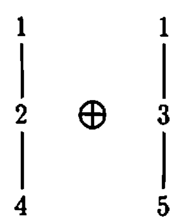
and the ideal $I \subseteq K\Gamma$ contains the relation $\gamma\beta\alpha - \tau\sigma\rho$, together with suitable monomial relations, such that the indecomposable projective left Λ-modules have graphs



Then $S_i \in \mathcal{P}^\infty(\Lambda\text{-mod})$ for $i = 2, 3$, the minimal $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximations of S_4, S_5, S_6 are Λe_4, Λe_5, and Λe_6, respectively, while none of S_1, S_7, S_8 has a $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximation. In fact, in Section 4, we will see that there is no object $A \in \mathcal{P}^\infty(\Lambda\text{-mod})$ such that all of the homomorphisms from the modules in the following subclass of $\mathcal{P}^\infty(\Lambda\text{-mod})$ – call it $\mathcal{C}$ – factor through A.



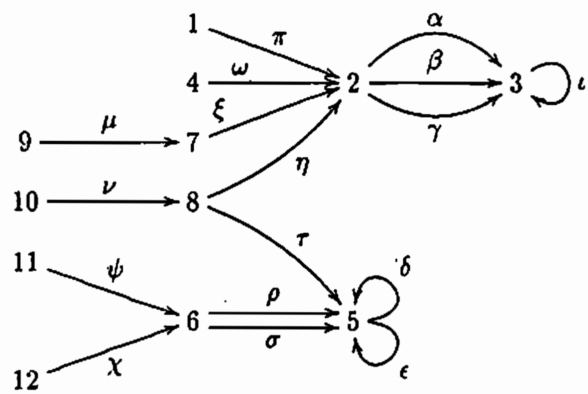
B.2. Now let Λ be the factor algebra of the algebra described in B.1 modulo the ideal generated by e_7 and e_8. Then the graphs of the indecomposable projective modules $\Lambda e_1, \dots, \Lambda e_6$ remain unchanged, but this time $\mathcal{P}^\infty(\Lambda\text{-mod})$ is contravariantly finite. Indeed, the minimal $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximations of $S_2, \dots, S_6$ are as above, while S_1 has the following minimal $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximation:



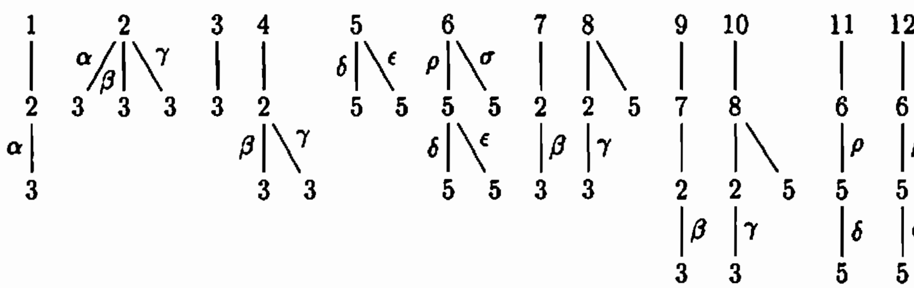
The next example shows that the structure of the minimal $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximations of the simple modules need by no means be as simplistic as in the previous instances, not even in situations where the indecomposable projective modules are of a simplistic makeup.

Examples C.

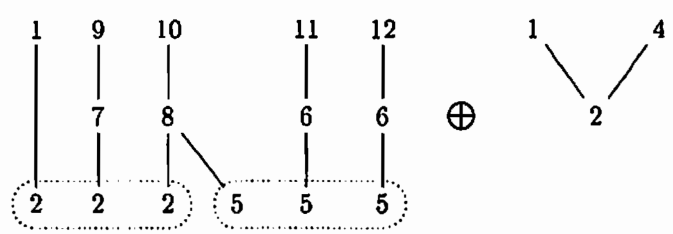
C.1. This time, let $\Lambda = K\Gamma/I$ be the monomial relation algebra with quiver Γ

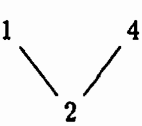


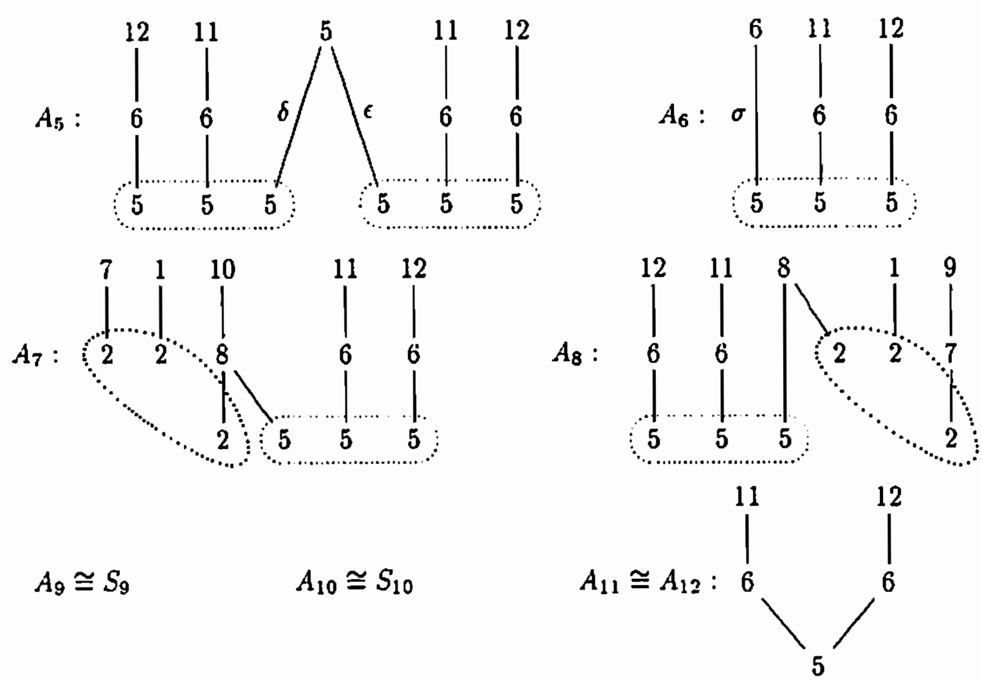
and choose the ideal $I \subseteq K\Gamma$ of relations so that the indecomposable projective left Λ-modules have the following graphs:



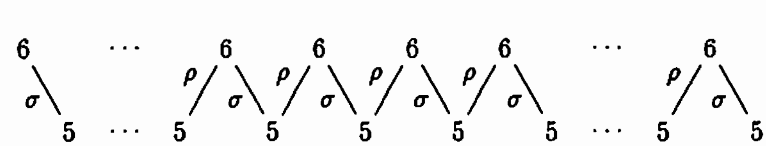
In this example, $\mathcal{P}^\infty(\Lambda\text{-mod})$ is again contravariantly finite, and the minimal approximations $A_1, \dots, A_{12}$ of the simples $S_1, \dots, S_{12}$ are determined by their graphs as follows:

A_1 : 

$A_2 \cong \Lambda e_2$ $\qquad$ $A_3 \cong \Lambda e_3$ $\qquad$ A_4 : 



C.2. Finally, let Λ be the factor algebra of the algebra under C.1, modulo the ideal generated by e_{11} and e_{12}. Note that the graphs of the indecomposable projectives $\Lambda e_1, \dots, \Lambda e_{10}$ remain the same as in C.1, since e_{11} and e_{12} are sources of Γ. This time, $\mathcal{P}^\infty(\Lambda\text{-mod})$ fails to be contravariantly finite, S_1, S_5, S_6, S_8 being precisely those simples which have lost their $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximations in the passage to the smaller algebra. As we will see in Section 4, the homomorphisms onto S_5 from the modules of the following $\mathcal{P}^\infty(\Lambda\text{-mod})$-family can, for instance, not all be factored through a fixed $A \in \mathcal{P}^\infty(\Lambda\text{-mod})$:



(On the other hand, observe that they can be factored through the module A_5 over the algebra in C.1.)

To sketch justifications for some of the minimal approximations we have exhibited, we first spell out an obvious sufficient condition for a simple module $S = \Lambda e/Je$ to have a $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximation. Indeed, this is the case provided that the following is true: There exist indecomposable modules T_1, $\dots$, T_m in $\mathcal{P}^\infty(\Lambda\text{-mod})$ with top elements $x_i \in T_i$ of type e such that for each indecomposable object $X \in \mathcal{P}^\infty(\Lambda\text{-mod})$ having a top element x of type e, there exists a factor module X/Y with $\overline{x} = x + Y \neq 0$ which can be embedded into some T_j in such a fashion that $\overline{x}$ is mapped to x_j. If there exist $T_1, \dots, T_m$ as stipulated, we know moreover that the indecomposable direct summands of the minimal $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximations of S are all recruited from the T_i.

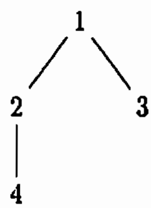
Ad Example A.1. In arguing that A_1 is the minimal $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximation of S_1, we bypass the facts that $\operatorname{l\,fin\,dim}\Lambda = 1$ (use the methods of [21], for instance), that A_1 has finite projective dimension, and that this module has the required minimality, provided that it is an approximation of S_1. To verify the latter, we let $X \in \mathcal{P}^\infty(\Lambda\text{-mod})$ be indecomposable and endowed with a top element x of type e_1. Then $\alpha x \neq 0$, since otherwise the module with graph $\begin{smallmatrix}2\\ |\\ 3\end{smallmatrix}$ would be a direct summand of $\Omega^1(X)$, which is impossible. If $\gamma\alpha x \neq 0$, then $X \cong \Lambda x \cong \Lambda e_1$ (use indecomposability and finite projective dimension), and $X/\operatorname{soc} X$ embeds into each of the modules $\begin{smallmatrix}1 & & a_i\\ & \diagdown\ \diagup & \\ & 2 & \end{smallmatrix}$ for $i = 1, \dots, n$. So suppose that $\gamma\alpha x = 0$. In this case, $J^2X = 0$, the socle of X being homogeneous of type e_2, and $\Omega^1(X) \cong (\Lambda e_2)^r$ for some r. One infers the existence of a factor module X/Y with graph

$$\begin{array}{ccc} \overline{x} & & \\ 1 & & a_j \\ & \diagdown\ \diagup & \\ & 2 & \end{array}$$

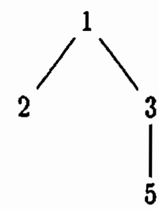
for some j. $\square$

Ad A.2. To see that, also in this example, the exhibited module A_1 is the minimal $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximation of S_1, observe that $\Omega^1(A_1) = \left(\begin{smallmatrix}2\\ |\,\delta\\ 3\end{smallmatrix}\right)^n = (\Lambda e_2)^n$, whence $A_1 \in \mathcal{P}^\infty(\Lambda\text{-mod})$. The rest of the argument is similar to the one given above. $\square$

Ad B.2. Once more, we will show that A_1 is a $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximation of S_1 (minimality being clear then). Any indecomposable module in $\mathcal{P}^\infty(\Lambda\text{-mod})$ containing Λe_1 is clearly isomorphic to Λe_1, and the only proper nonzero factor modules of Λe_1 which embed into indecomposable modules $X \in \mathcal{P}^\infty(\Lambda\text{-mod})$ are the direct summands of A_1, as well as



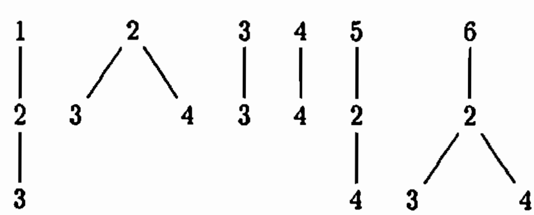
and



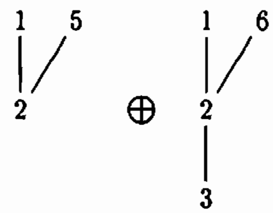
Observe that none of these factors of Λe_1 has a proper extension to an indecomposable in $\mathcal{P}^\infty(\Lambda\text{-mod})$. Since clearly the former factors are in turn factor modules of the latter, our claim follows. □

All direct summands of the minimal approximations of the various simple modules $S = \Lambda e/Je$ exhibited above contain factor modules of Λe which are minimal with respect to the property of being top-embeddable into modules of finite projective dimension. (We say that a monomorphism $A \to B$ is a *top-embedding* if it induces a monomorphism $A/JA \to B/JB$.) This is generally true for the minimal $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximations of simples, whenever Λ is either left serial or a string algebra (see Sections 3 and 5), a fact which greatly facilitates resolving the existence question (always positive in case of a left serial algebra and algorithmically decidable for string algebras) and the construction of such approximations. We conclude this section with an easy example showing that this cannot be expected to hold in general. In Section 5, we will follow up with an example demonstrating that the mentioned asset of string algebras is not shared by arbitrary special biserial algebras either.

Example D. Let $\Lambda = K\Gamma/I$ be the monomial relation algebra with the following indecomposable projective left modules:



Then $\mathcal{P}^\infty(\Lambda\text{-mod})$ is contravariantly finite, the minimal $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximation A_1 of S_1 having the following graph:



Note that the second summand of A_1 contains a copy of Λe_1, whereas $\Lambda e_1/\operatorname{soc}(\Lambda e_1)$ is the (unique in this case) minimal factor module of Λe_1 which can be top-embedded into an object of $\mathcal{P}^\infty(\Lambda\text{-mod})$.

3. Homological importance of contravariant finiteness and a model application

As surfaced in [22], [24] and [34], non-finitely generated modules of finite projective dimension may display structural phenomena which are completely different from those encountered in finitely generated modules of finite projective dimension. (We labeled them 'domino effects' in [22].) In particular, there may be objects in $\mathcal{P}^\infty(\Lambda\text{-Mod})$ whose projective dimension exceeds $\operatorname{l\,fin\,dim}\Lambda$ by any predetermined amount, even when $\operatorname{l\,fin\,dim}\Lambda = 1$ [34]. Moreover, the left cyclic finitistic dimension $\operatorname{l\,cyc\,fin\,dim}\Lambda$, i.e., the supremum of those projective dimensions which are attained on the cyclic modules in $\mathcal{P}^\infty(\Lambda\text{-mod})$, may be strictly smaller than $\operatorname{l\,fin\,dim}\Lambda$. In fact, for each natural number n, there exists a finite dimensional algebra Λ_n such that $\operatorname{l\,fin\,dim}\Lambda_n$ is not attained on any n-generated module (see [24]). However, in case $\mathcal{P}^\infty(\Lambda\text{-mod})$ is contravariantly finite, all of the left finitistic dimensions of Λ coincide, and the objects of the big category $\mathcal{P}^\infty(\Lambda\text{-Mod})$ are as well understood as those of the small $\mathcal{P}^\infty(\Lambda\text{-mod})$.

The following notation will be convenient: Given objects $A_1, \dots, A_n$ in Λ-mod, let $\operatorname{filt}(A_1, \dots, A_n)$ be the full subcategory of Λ-mod the objects of which are those modules which have filtrations with consecutive factors among $A_1, \dots, A_n$. More precisely, X belongs to $\operatorname{filt}(A_1, \dots, A_n)$ if and only if there exists a chain $X = X_0 \supseteq X_1 \supseteq \cdots \supseteq X_m = 0$ such that each of the factors X_i/X_{i+1} is isomorphic to some A_j. Moreover, $\overrightarrow{\operatorname{filt}}(A_1, \dots, A_n)$ will stand for the closure of $\operatorname{filt}(A_1, \dots, A_n)$ under direct limits in Λ-Mod.

Concerning the structure of the *finitely generated* modules of finite projective dimension in case $\mathcal{A} = \mathcal{P}^\infty(\Lambda\text{-mod})$ is contravariantly finite, Auslander and Reiten proved the following result in the more general context of an arbitrary resolving subcategory $\mathcal{A}$.

Theorem 2. [2] *Suppose that $\mathcal{A}$ is a resolving contravariantly finite subcategory of* Λ-mod, *and that $A_1, \dots, A_n$ are the minimal $\mathcal{A}$-approximations of the* *simple left Λ-modules. Then a module X belongs to $\mathcal{A}$ if and only if X is a direct summand of an object in* $\mathrm{filt}(A_1, \ldots, A_n)$. $\square$

Of course, this theorem, applied to $\mathcal{A} = \mathcal{P}^\infty(\Lambda\text{-mod})$, yields the following consequence:

Corollary 3. [2] *If $\mathcal{P}^\infty(\Lambda\text{-mod})$ is contravariantly finite and $A_1, \ldots, A_n$ are the minimal $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximations of the simple left Λ-modules, then*

$$\mathrm{l\,fin\,dim}\,\Lambda = \mathrm{l\,cyc\,fin\,dim}\,\Lambda = \max\{\mathrm{p\,dim}\,A_1, \ldots, \mathrm{p\,dim}\,A_n\}. \quad \square$$

Due to the fact that the requirement that $\mathcal{P}^\infty(\Lambda\text{-mod})$ be contravariantly finite places demands only on the finitely generated modules, the strong impact which this condition has on *non-finitely generated* modules may come as a surprise. In fact, the structure theory for objects in $\mathcal{P}^\infty(\Lambda\text{-mod})$ extends smoothly to $\mathcal{P}^\infty(\Lambda\text{-Mod})$.

Theorem 4. [26] *Again suppose that $\mathcal{P}^\infty(\Lambda\text{-mod})$ is contravariantly finite, and let $A_1, \ldots, A_n$ be as in the corollary. Then*

$$\mathcal{P}^\infty(\Lambda\text{-Mod}) = \overrightarrow{\mathrm{filt}}(A_1, \ldots, A_n),$$

and, in particular,

$$\mathrm{l\,Fin\,dim}\,\Lambda = \mathrm{l\,fin\,dim}\,\Lambda = \max\{\mathrm{p\,dim}\,A_1, \ldots, \mathrm{p\,dim}\,A_n\}. \quad \square$$

Thus, contravariant finiteness of $\mathcal{P}^\infty(\Lambda\text{-mod})$ resolves the notorious quandary of locating objects in Λ-Mod on which $\mathrm{l\,Fin\,dim}\,\Lambda$ is attained; this search may be a very difficult task, even when finiteness of $\mathrm{l\,Fin\,dim}\,\Lambda$ is guaranteed in advance. The helpfulness of the above theory will be displayed to full advantage in our examples.

Examples of Section 2 revisited.

For the moment, we will only determine the finitistic dimensions of those algebras displayed which give rise to contravariantly finite categories $\mathcal{P}^\infty(-)$. By the preceding discussion, the objects of Λ-Mod having finite projective dimension are precisely the direct limits of the objects in $\mathrm{filt}(A_1, \ldots, A_n)$ with the A_i as shown in Section 2.

Ad A.1. Clearly, $\mathrm{p\,dim}\,A_1 = \mathrm{p\,dim}\,A_{a_i} = 1$ for $i = 1, \ldots, n$, whereas $\mathrm{p\,dim}\,A_2 = \mathrm{p\,dim}\,A_3 = 0$. Hence, $\mathrm{l\,Fin\,dim}\,\Lambda = \mathrm{l\,fin\,dim}\,\Lambda = 1$.

Ad A.2. In this example, the minimal $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximation of each simple S_{a_i} centered in the vertex a_i coincides with S_{a_i} and has projective dimension 1, as does the minimal approximation of S_1. The minimal approximations of S_2 and S_3 are again identical with their projective covers. So, once more, $\mathrm{l\,Fin\,dim}\,\Lambda = \mathrm{l\,fin\,dim}\,\Lambda = 1$.

Ad B.2. Here A_1, A_4, A_5, A_6 are projective, while $\operatorname{p\,dim} A_2 = \operatorname{p\,dim} A_3 = 1$, and we obtain the same conclusion as before. ☐

We will briefly digress from our main line of thought for another corollary of the preceding theorem. The existence theorem for internal almost split sequences [3] which we quoted at the outset can be strengthened for the category $\mathcal{A} = \mathcal{P}^\infty(\Lambda\text{-mod})$ as follows.

Corollary 5. *If $\mathcal{P}^\infty(\Lambda\text{-mod})$ is contravariantly finite, then $\mathcal{P}^\infty(\Lambda\text{-mod})$ is also covariantly finite and thus has almost split sequences.*

Proof. By a result of Crawley-Boevey [9, Theorem 4.2], it suffices to show that arbitrary direct products of objects in the category $\mathcal{P}^\infty(\Lambda\text{-mod})$ belong to its closure $\overrightarrow{\mathcal{P}^\infty}(\Lambda\text{-mod})$ under direct limits in Λ-Mod. But in view of the theorem, contravariant finiteness of $\mathcal{P}^\infty(\Lambda\text{-mod})$ entails the equality $\mathcal{P}^\infty(\Lambda\text{-Mod}) = \overrightarrow{\mathcal{P}^\infty}(\Lambda\text{-mod})$ and, in view of the finiteness of $\operatorname{l\,Fin\,dim} \Lambda$, this guarantees closedness of this latter subcategory under direct products. ☐

As a class of examples of algebras Λ with very rich, complex module categories, for which the above program of zeroing in on the structure of the objects in $\mathcal{P}^\infty(\Lambda\text{-Mod})$ works to perfection, we will present the left serial algebras. Recall that a split algebra $\Lambda = K\Gamma/I$ is called left serial in case no more than one arrow leaves any given vertex of Γ; equivalently, this means that the indecomposable projective left Λ-modules are all uniserial. To describe the minimal $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximations of the simple modules in this situation, we require the following definition.

Definition. Suppose that $T_1, \dots, T_m$ is a sequence of nonzero uniserial left Λ-modules, and let p_i be a *mast* of T_i, namely a path in $K\Gamma$ of maximal length with $p_i T_i \neq 0$. A left Λ-module T is called a *saguaro on* $(p_1, \dots, p_m)$ if

(i) $T \cong \left(\bigoplus_{1 \le i \le m} T_i\right)/U$, where $U \subseteq \bigoplus_{1 \le i \le m} JT_i$ is generated by a sequence of elements of the form $q_i t_i - q'_{i+1} t_{i+1}$, $1 \le i \le m-1$, where $t_i \in T_i$ are suitable top elements and q_i, q'_i are right subpaths of the masts p_i such that $q_i t_i \neq 0$, and $q'_{i+1} t_{i+1} \neq 0$; moreover, we require that

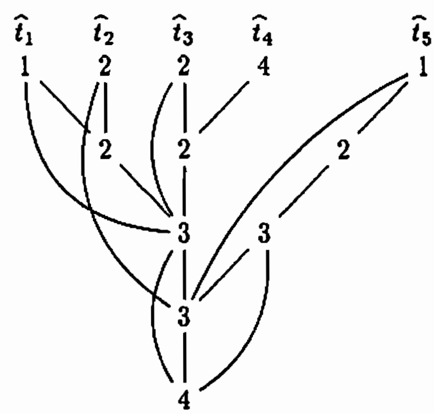
(ii) each T_j embeds canonically into T via

$$T_j \xrightarrow{\text{can}} \left(\bigoplus_{1 \le i \le m} T_i\right) \Big/ U \cong T.$$

The uniserial modules T_i are called the *trunks* of T.

We will identify T with $\left(\bigoplus_{1 \le i \le m} T_i\right)/U$. To avoid ambiguities, we will denote the canonical images of the trunks T_i inside T by $\widehat{T}_i$ and the canonical images of the top elements t_i by $\widehat{t}_i$. Any such sequence $(\widehat{t}_1, \dots, \widehat{t}_m)$ will be called a *canonical sequence of top elements* for T.

Note that saguaros are particularly amenable to graphing, the shape of their graphs explaining their name (they share shape and name with a cactus found in the Sonoran desert, *Cereus giganteus*). In fact, the definition forces them to be glued together in a very straightforward fashion from their uniserial trunks: Layered and labeled graphs relative to a canonical sequence of top elements always exist (not only over left serial algebras), and are built on the pattern illustrated below.



Here $T = \left(\bigoplus_{i=1}^{5} T_i\right)/U$, where the trunks $T_i = \Lambda t_i$ of T have graphs

Theorem 6. [7] *Suppose that $\Lambda = K\Gamma/I$ is a left serial algebra. Then $\mathcal{P}^\infty(\Lambda\text{-mod})$ is contravariantly finite, and the minimal $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximations of the simple left Λ-modules are saguaros with simple socles.*

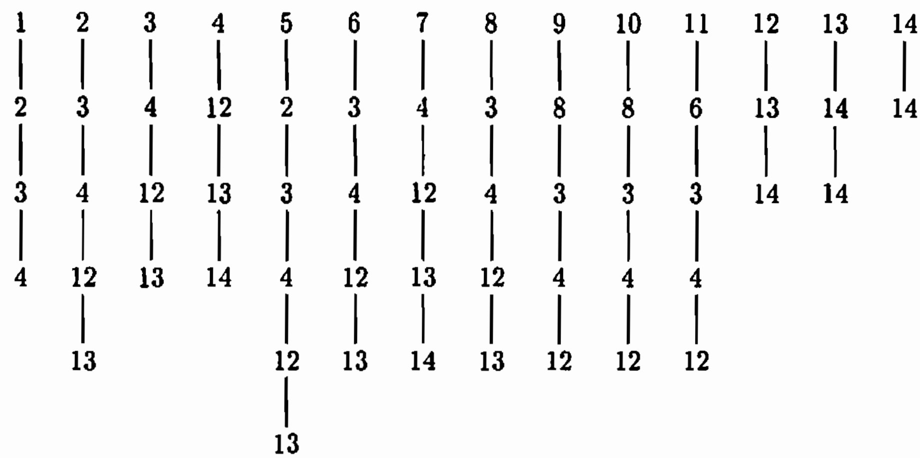
More precisely, the minimal $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximation of a simple left Λ-module $S = \Lambda e/Je$ can be described as follows: If $\Lambda e/C$ is the (unique) minimal nonzero factor module of Λe which has finite projective dimension, there is a unique saguaro $A(S)$ of maximal length in $\mathcal{P}^\infty(\Lambda\text{-mod})$ such that $\Lambda e/C$ is a trunk of $A(S)$ and $\operatorname{soc} A(S)$ is simple. Moreover, the canonical epimorphisms $A(S) \to S$, which map $\Lambda e/C$ onto S and send the other trunks of $A(S)$ to zero, are minimal $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximations. □

To refer back to the concluding remark of Section 2: In the setting of the theorem, $\Lambda e/C$ actually coincides with the nonzero factor module of Λe which is minimal with respect to top-embeddability into a module of finite projective dimension.

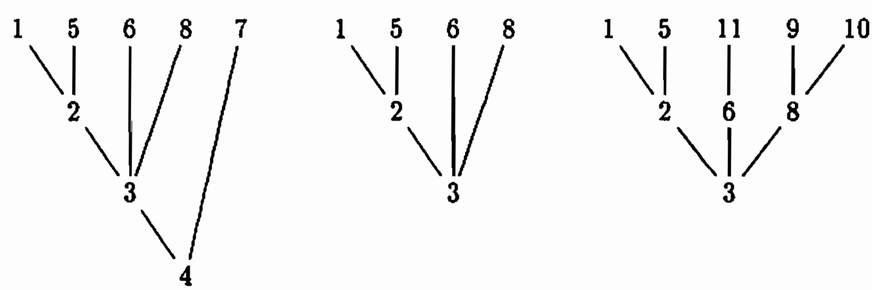
Actually, not only is $\mathcal{P}^\infty(\Lambda\text{-mod})$ always contravariantly finite in the left serial case, but so are the categories $\mathcal{P}^{(d)} = \mathcal{P}^{(d)}(\Lambda\text{-mod})$ consisting of the finitely generated left Λ-modules of projective dimensions at most d. Moreover, the minimal $\mathcal{P}^{(d)}$-approximations of the simples are again saguaros, and the sequences of these saguaros for $1 \le d \le \operatorname{lfin\,dim}\Lambda$ record the homological properties of Λ with high precision, the case $d = \operatorname{lfin\,dim} \Lambda$ leading back to $\mathcal{P}^{\infty}(\Lambda\text{-mod})$.

Example E. [7, Example 7.3]

Let Λ be a left serial algebra whose indecomposable projective modules are represented by the following graphs.



The evolution of the $\mathcal{P}^{(d)}$-approximations of the simple left Λ-module S_1 is graphically represented below. We exhibit the minimal $\mathcal{P}^{(1)}$-, $\mathcal{P}^{(2)}$-, $\mathcal{P}^{(3)}$-approximations of S_1 from left to right; the last candidate coincides with the minimal $\mathcal{P}^{\infty}(\Lambda\text{-mod})$-approximation, since the left finitistic dimension of Λ is 3 in this example.



4. Phantoms. Definitions, existence, and basic properties

The objects discussed in this section were introduced by Happel and the author in [20]. Their purpose is twofold: In the first place, they serve as indicators as to whether or not a given subcategory $\mathcal{A} \subseteq \Lambda\text{-mod}$ is contravariantly finite. Their second role is that of retaining the kind of information which is stored in minimal $\mathcal{A}$-approximations in case of existence, within potentially infinite dimensional frames; this role is played most satisfactorily by the 'effective' phantoms, as defined below.

Since the concept of a phantom is possibly not easily translated into an intuitive picture, we break up the relevant definitions into several parts, and add more extensive discussion than we did in [20]. A detailed analysis of the examples of Section 2 should also aid the recognition that we are dealing with objects arising very naturally in comparisons of the relations of an arbitrary left Λ-module X with those of the objects in a given subcategory $\mathcal{A}$.

Definition, Part I. (Relative approximations) Let $\mathcal{A}$ be a full subcategory of Λ-mod, and $\mathcal{C} \subseteq \mathcal{A}$ a subclass of the object class. Moreover, let $X \in \Lambda$-mod.

A *$\mathcal{C}$-approximation of X inside $\mathcal{A}$* is a homomorphisms $f : A \to X$ with $A \in \mathcal{A}$ such that each map in $\text{Hom}(C, X)$ with $C \in \mathcal{C}$ factors through f. Again, we will loosely refer to the object A as a $\mathcal{C}$-approximation of X inside $\mathcal{A}$.

Clearly, whenever X has a minimal $\mathcal{A}$-approximation, A_0 say, then the $\mathcal{A}$-approximations of X are precisely the $\{A_0\}$-approximations of X inside $\mathcal{A}$. In particular, we obtain A_0 as a $\mathcal{C}$-approximation of X inside $\mathcal{A}$, where $\mathcal{C}$ is a <u>finite</u> subclass of $\mathcal{A}$. Moreover, A_0 is a direct summand of any $\{A_0\}$-approximation of X inside $\mathcal{A}$, and so, a fortiori, is a subfactor of any such approximation. On the other hand, given a finite subclass $\mathcal{C} \subseteq \mathcal{A}$, the module X will have $\mathcal{C}$-approximations inside $\mathcal{A}$ provided that we require $\mathcal{A}$ to be closed under finite direct sums: Just sum up a sufficient number of copies of the objects in $\mathcal{C}$. In other words, approximations relative to <u>finite</u> classes are always available, and in case of existence, minimal $\mathcal{A}$-approximations are always of that ilk. The idea is to find means of efficiently surveying them.

Definition, Part II. (Phantoms) Retain the notation of Part I, and suppose, in addition, that $\mathcal{A}$ is closed under finite direct sums.

A finitely generated module $H \in \Lambda$-mod is an *$\mathcal{A}$-phantom of X* in case

(*) there is a finite subclass $\mathcal{A}' \subseteq \mathcal{A}$ such that H arises as a subfactor of *every* $\mathcal{A}'$-approximation of X inside $\mathcal{A}$.

More generally, an arbitrary module $H \in \Lambda$-Mod will be called an $\mathcal{A}$-phantom of X if each of its finitely generated submodules satisfies (*). Of course, the choices of the finite subclasses $\mathcal{A}' \subseteq \mathcal{A}$ will vary with the finitely generated submodules H' of H.

Clearly, the class of all $\mathcal{A}$-phantoms of X is closed under subfactors, so the two parts of the definition do not conflict with each other. Moreover, this guarantees that the class of phantoms of X is closed under direct limits of direct systems as well. In fact, a module $H \in \Lambda$-Mod is an $\mathcal{A}$-phantom of X if and only if H is the direct limit of a direct system of finitely generated $\mathcal{A}$-phantoms of X. This may make the class of $\mathcal{A}$-phantoms enormous: Indeed, the class of all $\mathcal{A}$-phantoms of a simple module S may encompass the entire class of indecomposables H in Λ-Mod with $S \subseteq H/JH$. This looseness in the definition of phantoms has the advantage of facilitating their construction. Often, the particular structure of phantoms is irrelevant - their sheer size is enough to permit conclusions concerning the existence or non-existence of traditional $\mathcal{A}$-approximations of X. In fact, if X has an $\mathcal{A}$-approximation then the class of $\mathcal{A}$-phantoms of X coincides with the set of subfactors of the minimal such approximation. Consequently, the existence of phantoms of X of unbounded lengths is enough to guarantee non-existence of traditional $\mathcal{A}$-approximations. Of course, in terms of encapsulating information about the relations of objects in $\mathcal{A}$ relative to a set of relations of X, the usefulness of phantoms so generously defined is moderate. Hence, we single out a subclass of phantoms which carry a full complement of information and which are more strongly tied to the category $\mathcal{A}$.

Definition, Part III. (Effective phantoms) Keep the notation of Part II, and denote by $\overrightarrow{\mathcal{A}}$ the closure of $\mathcal{A}$ under direct limits (of direct systems) in Λ-Mod. Moreover, fix a subclass $\mathcal{C} \subseteq \mathcal{A}$.

An $\mathcal{A}$-phantom $H \in \overrightarrow{\mathcal{A}}$ is called *effective relative to* $\mathcal{C}$ if there exists a homomorphism $h : H \to X$ with the property that each map in $\mathrm{Hom}(C, X)$ with $C \in \mathcal{C}$ factors through h. (In other words, h must be a $\mathcal{C}$-approximation of X inside $\overrightarrow{\mathcal{A}}$.)

In case X has a traditional $\mathcal{A}$-approximation, the minimal such approximation is clearly the only effective $\mathcal{A}$-phantom of X relative to $\mathcal{A}$. Otherwise, existence of interesting phantoms, effective or not, is not immediately clear, but is guaranteed by the following result.

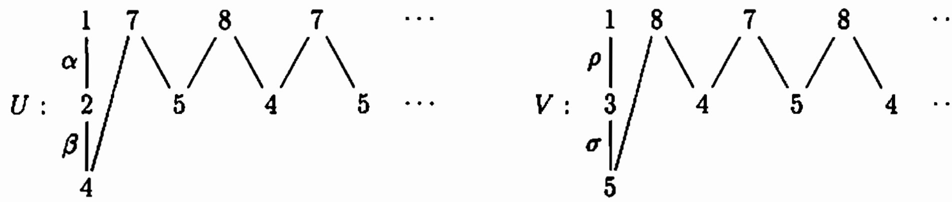
Theorem 7. [20] *Again, let $\mathcal{A} \subseteq \Lambda$-mod be a full subcategory which is closed under finite direct sums. For $X \in \Lambda$-mod, the following conditions are equivalent:*

(1) *X fails to have an $\mathcal{A}$-approximation.*

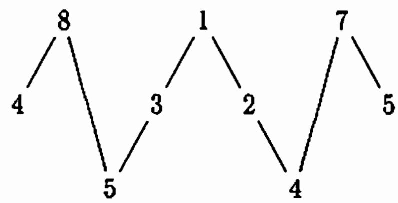
(2) *X has $\mathcal{A}$-phantoms of infinite K-dimension.*

(3) *There exist countable subclasses $\mathcal{C} \subseteq \mathcal{A}$ such that X has infinite dimensional $\mathcal{A}$-phantoms which are effective relative to $\mathcal{C}$.* □

Ad B.1. Here are two infinite dimensional $\mathcal{P}^\infty(\Lambda\text{-mod})$-phantoms of S_1, for example:

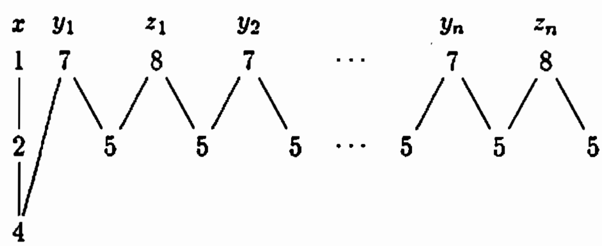


Note that $U \oplus V$ is in turn a $\mathcal{P}^\infty(\Lambda\text{-mod})$-phantom of S_1, since there is no object in $\mathcal{P}^\infty(\Lambda\text{-mod})$ having a graph with subgraph



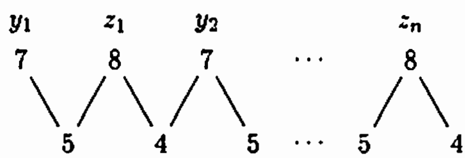
In fact, the phantom $U \oplus V$ is effective relative to the class $\mathcal{C} \subseteq \mathcal{P}^\infty(\Lambda\text{-mod})$ exhibited in B.1.

We will justify only that U is a $\mathcal{P}^\infty(\Lambda\text{-mod})$-phantom of S_1. Consider the class of objects



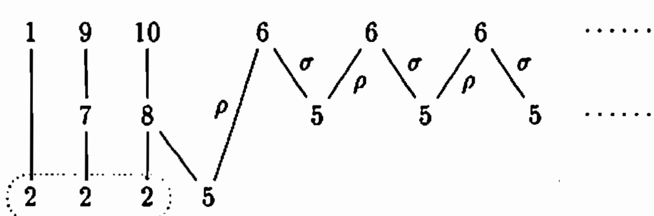
in $\mathcal{P}^\infty(\Lambda\text{-mod})$, and observe that $U = \varinjlim U_n$. Hence, it suffices to show that, for each $n \in \mathbb{N}$, the module U_n is a submodule of each $\{U_n\}$-approximation of S_1 inside $\mathcal{P}^\infty(\Lambda\text{-mod})$. To see this, fix n, and let A be any $\{U_n\}$-approximation of S_1 inside $\mathcal{P}^\infty(\Lambda\text{-mod})$; say $f : A \to S_1$ has the factorization property of the definition, and $g \in \operatorname{Hom}(U_n, A)$ factors the canonical epimorphism $U_n \to S_1$. Let $a = g(\alpha)$. Then a is a top element of A of type e_1. Due to the fact that finiteness of the projective dimension of A entails either $\beta\alpha a \neq 0$ or $\sigma\rho a \neq 0$, our factorization requirement forces $\beta\alpha a$ to be nonzero. Consequently, $g(\beta\alpha x) = \beta\alpha a$. If the arrows $7 \to 4$ and $7 \to 5$ are named χ and ψ, respectively, we deduce $\beta\alpha a = f(\chi y_1) \neq 0$, and hence $f(y_1) \neq 0$. The element $b_1 = f(y_1)$ is a top element of A (necessarily of type e_7), since 7 is a source of Γ. To prevent the syzygy $\Omega^1(A)$ from having a summand S_5 (the latter being incompatible with finite projective dimension), we require $0 \neq \psi(b_1) = \mu f(z_1)$, where μ is the arrow $8 \to 5$. Setting $c_1 = f(z_1)$ and repeating the above argument in spirit, we see that $0 \neq \nu c_1 = f(\chi y_2)$ if ν is the arrow $8 \to 4$. Set $b_2 = f(y_2)$. It is readily checked that the top elements b_1, b_2 are K-linearly independent modulo JA, and an obvious induction on $m \leq n$ gives us sequences of top elements, $b_1, \ldots, b_n$ of type e_7, and $c_1, \ldots, c_n$ of type e_8 in A, both of which are K-linearly independent modulo JA. It is now straightforward to deduce that the submodule of A generated by a and the b_i, c_i has the same graph as U_n. But this graph clearly determines the corresponding module up to isomorphism, which completes the argument.

Each module from the subclass $\mathcal{D} = \{D_n \mid n \in \mathbb{N}\}$ of $\mathcal{P}^\infty(\Lambda\text{-mod})$, with D_n determined by the graph



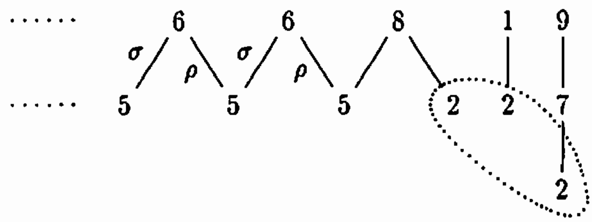
is a $\mathcal{P}^\infty(\Lambda\text{-mod})$-phantom of S_7, and consequently $\varinjlim D_n$ provides us with an infinite dimensional $\mathcal{P}^\infty(\Lambda\text{-mod})$-phantom of S_7 which, moreover, is effective relative to the class $\mathcal{D}$. The simple module S_8 shows analogous behavior.

Ad C.2. An infinite dimensional $\mathcal{P}^\infty(\Lambda\text{-mod})$-phantom of S_1 is determined (uniquely, up to isomorphism) by the graph



This phantom is effective relative to the class of modules in $\mathcal{P}^\infty(\Lambda\text{-mod})$ obtained by chopping suitable 'infinite tails' off the given graph.

An example of an infinite dimensional $\mathcal{P}^\infty(\Lambda\text{-mod})$-phantom of S_8, finally, is



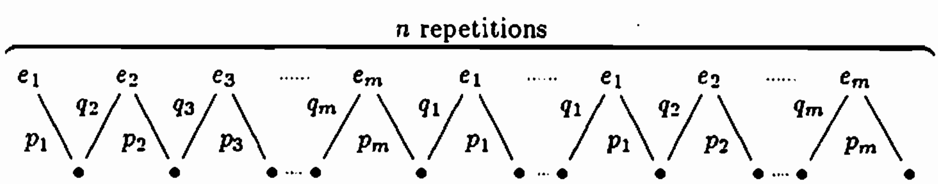
The following criterion from [20] guaranteeing non-existence of $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximations is based on phantoms.

Criterion for failure of contravariant finiteness of $\mathcal{A}$.

Let $\Lambda = K\Gamma/I$ be a split finite dimensional algebra, and, once more, suppose that $\mathcal{A} \subseteq \Lambda$-mod is closed under finite direct sums. The simple module $S = \Lambda e_1/Je_1$ centered in the vertex e_1 fails to have an $\mathcal{A}$-approximation in case the following holds:

The vertex e_1 can be supplemented to a sequence $e_1, \dots, e_m$ of distinct vertices of Γ, together with sequences $p_1, \dots, p_m$, $q_1, \dots, q_m$ in J, where $p_i = p_i e_i$ and $q_i = q_i e_i$ are such that conditions (1) and (2) below are satisfied:

(1) For each $n \in \mathbb{N}$, there exists a module $M_n \in \mathcal{A}$ having a graph that contains a subgraph of the form



(2) Given any object A in $\mathcal{A}$, the top elements of A of type e_1 are not annihilated by p_1, and

$$p_i a = q_{i+1} b \neq 0 \quad \Longrightarrow \quad p_{i+1} b \neq 0$$

folr $a, b \in A$ and $1 \leq i \leq m$; here $p_{m+1} = p_1$ and $q_{m+1} = q_1$. □

In fact, the hypotheses of the criterion yield an infinite dimensional factor module of $\varinjlim M_n$ which is an $\mathcal{A}$-phantom of S and has a subgraph obtained from the graphs under (1) by 'infinite extension'. It is this criterion which hovers in the background of most of the displayed phantoms over our test algebras.

5. Phantoms over string algebras

In this section, we present a preview of joint work with S. O. Smalø which is still in progress.

A special class of string algebras $\Lambda_{m,n}$ – those on the quiver

subject to the relations $\alpha\beta = \beta\alpha = 0$ and relations of the form $\alpha^m = \beta^n = 0$ for suitable integers $m, n \geq 2$ – was first singled out by Gelfand and Ponomarev in the late 1960's, as being intimately related to the representation theory of the Lorentz group [19]; in fact, classifying the finitely generated indecomposable modules of the former amounts to a classification of the Harish-Chandra modules of the latter. Taking this route, Gelfand and Ponomarev gave a hands-on structural description of the finitely generated indecomposable objects in $\Lambda_{m,n}$-mod. In particular, their results show that over an algebraically closed base field the algebras $\Lambda_{m,n}$ are tame. Subsequently, Gabriel presented a categorical reinterpretation of the Gelfand-Ponomarev approach (see [15]), which in turn caused Ringel to recognize that these methods are applicable in a far wider context: In a first round of generalizations, he used them to describe the finite dimensional indecomposable representations of the dihedral 2-groups in characteristic 2 [30]; this work appeared in tandem with a paper of Bondarenko containing roughly the same information [6]. Next, Donovan and Freislich picked up on these methods, proving them applicable to the algebras all of whose indecomposable projective modules P are 'special', i.e., have the property that $\operatorname{rad} P = U + V$ with U, V uniserial and $U \cap V = \operatorname{soc} P$ simple [11]. In due course, this observation triggered the following definitions which cover all the special cases considered at that point.

Definitions. (see [36] and [8]) (1) Λ is called *biserial* if each indecomposable projective left or right Λ-module P has the following property: $\operatorname{rad} P = U + V$, where U, V are uniserial (possibly trivial) with $U \cap V$ either zero or simple.

(2) Λ is *special biserial* provided Λ is of the form $K\Gamma/I$ such that

• Given any vertex e of Γ, there are at most two arrows entering e and at most two arrows leaving e, and

• Given arrows α, β, γ of Γ, either $\beta\alpha$ or $\gamma\alpha$ is zero in Λ, and either $\alpha\beta$ or $\alpha\gamma$ is zero in Λ.

Moreover, Λ is a *string algebra* if, in addition, Λ is a monomial relation algebra, meaning that I can be generated by certain paths in Γ.

Clearly, special biserial algebras are biserial. All finite dimensional biserial algebras over algebraically closed fields are known to be tame: the special biserial case was completed by Wald and Waschbüsch in [36], while the general biserial situation was settled much later by Crawley-Boevey [10] on the basis of an alternate description of biserial algebras due to him and Vila-Freyer [35] and a remarkable result of Geiss [16] (saying that algebras with tame degenerations are always tame).

All the while, special biserial algebras have continued to provide challenges which, in spite of the availability of a highly explicit classification of the indecomposables, were far from resolvable at a glance. We mention only a few such lines, together with a selection of references, not aiming at completeness:

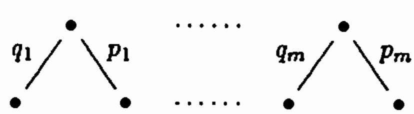
Let Λ be biserial.

- When does Λ have finite representation type? (See [33].)
- What does the Auslander-Reiten quiver of Λ look like? (See, e.g., [8], [13], and [17].)
- Describe the maps between the indecomposables in Λ-mod. (See [29] and [18].)
- Characterize the auto-equivalences of the category Λ-mod. (See [4] and [5].)

New sources of symmetric biserial algebras can be found in [32].

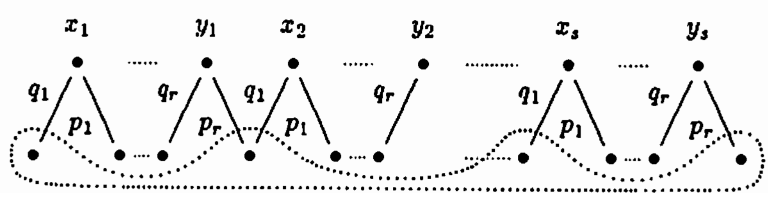
Here we will provide an overview of a complete, constructive solution to the problem as to which string algebras Λ have the property that $\mathcal{P}^{\infty}(\Lambda\text{-mod})$ is contravariantly finite. Our proof of the answer – not given here – makes full use of the description of the indecomposable objects in Λ-mod, as reviewed graphically below.

Theorem 8. (Its evolution can be traced in [19], [30], [11], [36]) *Let $\Lambda = K\Gamma/I$ be a special biserial algebra over an algebraically closed field K. Then each indecomposable object in Λ-mod is either a band module or a string module. Here the string modules are those with graphs of the form*



where the p_i, q_i are paths in $K\Gamma \setminus I$, with q_1 and p_m possibly trivial, such that $\text{firstarrow}(q_i) \neq \text{firstarrow}(p_i)$ *– this condition being void if $i = 1$ and q_1 is trivial or if $i = m$ and p_m is trivial – and* $\text{lastarrow}(p_i) \neq \text{lastarrow}(q_{i+1})$ *for $1 \leq i \leq m-1$.*

The band modules are characterized by their graphs, paired with irreducible vector space automorphisms as follows. The pertinent graphs are of the form



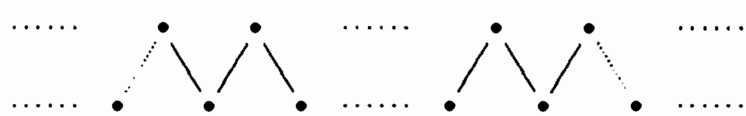
where p_i, q_i are paths of positive length in $K\Gamma\backslash I$ with $\mathrm{firstarrow}(q_i) \neq \mathrm{firstarrow}(p_i)$ *for all i and* $\mathrm{lastarrow}(p_i) \neq \mathrm{lastarrow}(q_{i+1})$ *for $i < r$, and also* $\mathrm{lastarrow}(p_r) \neq \mathrm{lastarrow}(q_1)$. *The nature of the dotted pool is specified by*

$$p_r y = \sum_{i=1}^{s} k_i q_i x_i,$$

where $\begin{bmatrix} 0 & \cdots & 0 & k_1 \\ 1 & \cdots & 0 & k_2 \\ & \ddots & & \vdots \\ 0 & \cdots & 1 & k_s \end{bmatrix}$ *is the Frobenius companion matrix of an irreducible automorphism of K^s.*

Moreover, all modules having one of the above descriptions are indecomposable. ☐

This classification can be completed with a suitable uniqueness statement which we will not require here. For our main theorem, the string modules will be of particular relevance. We will call *generalized string module* any module $X \in \Lambda$-Mod which arises as a direct limit of a countable directed system of string modules, each embedded into its successor, i.e., any module X having a graph of one of the following forms



such that each finite segment is the graph of a string module. (Here the dotted edges may but need not appear, if the graph is one- or two-sided finite.) It should be self-explanatory what we mean by a left and right periodic generalized string module (where the "left" and "right" periods may differ, and termination is regarded as a period). These modules were also considered by Ringel in [31] as "modules associated with $\mathbb{N}$-words of $\mathbb{Z}$-words" over the alphabet $\Gamma_0 \cup \Gamma_0^{-1}$, where Γ_0 is the vertex set of the quiver Γ of Λ.

We are now in a position to state the main new result of this section.

Theorem 9. [27] *Let $\Lambda = K\Gamma/I$ be a finite dimensional string algebra and $S \in$ Λ-mod simple. Then there exists a generalized string module $H = H(S)$ which is uniquely determined by Γ and I, together with a canonical homomorphism $f : H \to S$, having the following properties:*

(I) *H is left and right periodic and can be constructed from Γ and I in fewer than $3|\Gamma_0|$ steps.*

(II) *H belongs to $\mathcal{P}^\infty(\Lambda$-Mod) and is a $\mathcal{P}^\infty(\Lambda$-mod)-phantom of S.*

(III) *The following statements are equivalent:*

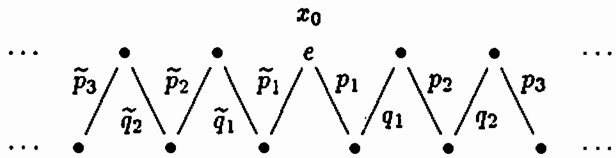
(i) *S has a $\mathcal{P}^\infty(\Lambda$-mod$)$-approximation;*

(ii) $\dim_K H < \infty$;

(iii) *$f : H \to S$ is the minimal $\mathcal{P}^\infty(\Lambda$-mod$)$-approximation of S.*

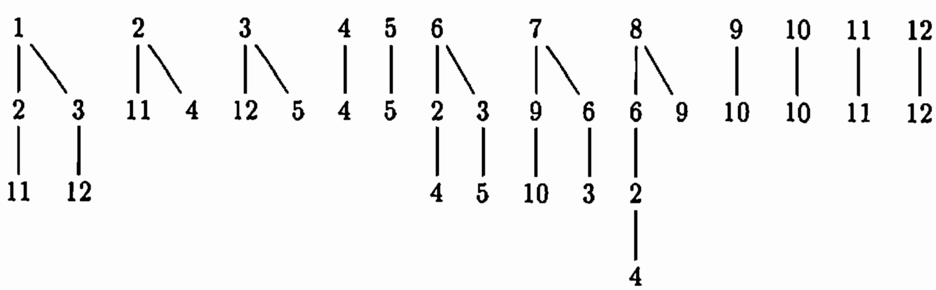
(IV) *The map $f : H \to S$ makes H an effective $\mathcal{P}^\infty(\Lambda$-mod$)$-phantom with respect to the class $\mathcal{S}^\infty(\Lambda$-mod$)$ of all string modules of finite projective dimension.* □

Given a simple module $S = \Lambda e/Je$ over a string algebra Λ, the module $H = H(S)$ of the theorem is called the *characteristic* $\mathcal{P}^\infty(\Lambda$-mod$)$-phantom of S. Following a first example, we will give an inductive description of the finite segments of a graph

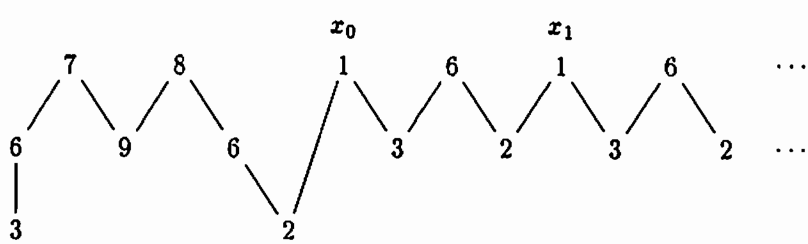


of H which are centered at the top element $x_0 \in H$. Sending this top element to $e + Je$ in S and sending the y_i and z_i to zero will then yield a map $f : H \to S$ as stipulated in the theorem.

Example F. Let $\Lambda = K\Gamma/I$ be the string algebra with the following indecomposable projective left Λ-modules

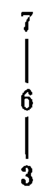


Then the characteristic $\mathcal{P}^\infty(\Lambda$-mod$)$-phantom H_1 of S_1 has graph



and $f : H_1 \to S_1$ is a homomorphism (unique up to a nonzero scalar) which sends x_0 to a nonzero element of S_1 and the x_i for $i \geq 1$ to zero. In particular, S_1 does not have a $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximation.

The characteristic $\mathcal{P}^\infty(\Lambda\text{-mod})$-phantom of S_7, on the other hand, has graph



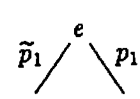
and thus coincides with the minimal $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximation of S_7. These statements are consequences of the algorithm which we describe next. □

Recall that we refer to a module X as a top-embeddable submodule of Y if there exists a monomorphism $f : X \to Y$ which induces a monomorphism $X/JX \to Y/JY$. Dually, we call X a *socle-faithful* factor module of Y if there exists an epimorphism $f : Y \to X$ which induces an epimorphism $\operatorname{soc} Y \to \operatorname{soc} X$.

Description of the characteristic phantom of a simple module $S \in \Lambda$-mod, where Λ is a finite dimensional string algebra.

Let $S = \Lambda e/Je$. The following are the steps of an algorithmic procedure for constructing $H = H(S)$, but here we will not discuss the algorithmic nature, nor prove that the quantities stipulated in the process exist.

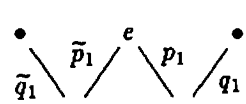
Step 1. Let p_1 and $\widetilde{p}_1$ be paths starting in e which have *minimal* lengths ≥ 0 such that



is the graph of a string module which can be top-embedded into an object in $\mathcal{S}^\infty(\Lambda\text{-mod})$. In particular, we have $p_1 \neq \widetilde{p}_1$ unless both of these paths are trivial, and startarrow$(p_1) \neq$ startarrow$(\widetilde{p}_1)$ if both are nontrivial.

If both p_1 and $\widetilde{p}_1$ are nontrivial, we set $H = S$. Otherwise, we proceed to

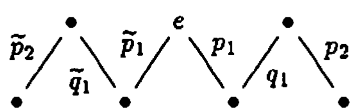
Step 2. Let q_1 and $\widetilde{q}_1$ be paths ending in end(p_1) and end$(\widetilde{p}_1)$, respectively, which have *maximal* lengths ≥ 0 with the property that



is the graph of a string module which arises as a socle-faithful factor module of an object in $\mathcal{S}^\infty(\Lambda\text{-mod})$. In case p_1 is trivial, set $q_1 = e$, and deal symmetrically with $\tilde{q}_1$.

If both q_1 and $\tilde{q}_1$ are trivial, i.e., if the graph of Step 2 coincides with that of Step 1, we let H be the string module having this graph. Otherwise, we proceed to

Step 3. Let p_2 and $\tilde{p}_2$ be paths starting in start(q_1) and start$(\tilde{q}_1)$, respectively, which have *minimal* lengths ≥ 0 with the property that



is the graph of a string module which can be top-embedded into some object in $\mathcal{S}^\infty(\Lambda\text{-mod})$, ... etc.

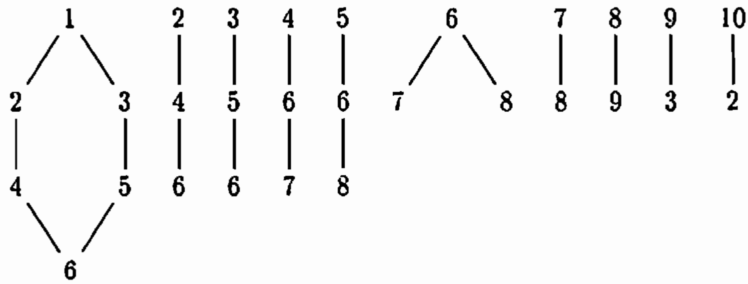
After fewer than $3|\Gamma_0|$ steps, this procedure either terminates or has become periodic on both sides. It is easy to recognize when one has hit a left or right period: Indeed, if startarrow(p_i) = startarrow(p_j) for some $i < j$, then $p_{i+r} = p_{j+r}$ and $q_{i+r} = q_{j+r}$ for all $r \geq 0$, the symmetric criterion holding for the other side. □

As one gleans from this inductive description of the phantoms $H = H(S)$ of the simple modules S, string algebras merit their name also from a homological viewpoint. Indeed the sizes and structures of the modules $H(S)$ depend only on the string modules of finite projective dimension, and the K-dimensions of the phantoms $H(S)$ in turn determine whether or not $\mathcal{P}^\infty(\Lambda\text{-mod})$ is contravariantly finite for a string algebra Λ. One may wonder whether this emphasis of string modules is just dictated by convenience and whether band modules can be attributed a similar homological role. This is not the case: In fact, there exist string algebras of positive little finitistic dimension which have no nontrivial (finitely generated) band modules of finite projective dimension. Another asset of string algebras that arises as a byproduct of our main theorem we record somewhat more formally.

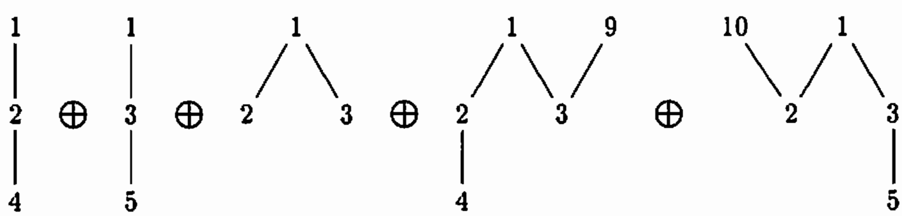
Corollary 10. *Suppose that S is a simple module over a finite dimensional string algebra Λ. If S has a $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximation, then the minimal $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximation of S is a string module and, in particular, is indecomposable.* □

There is no analog for arbitrary special biserial algebras, as the next example demonstrates.

Example G. Let $\Lambda = K\Gamma/I$ be the special biserial algebra with indecomposable projectives



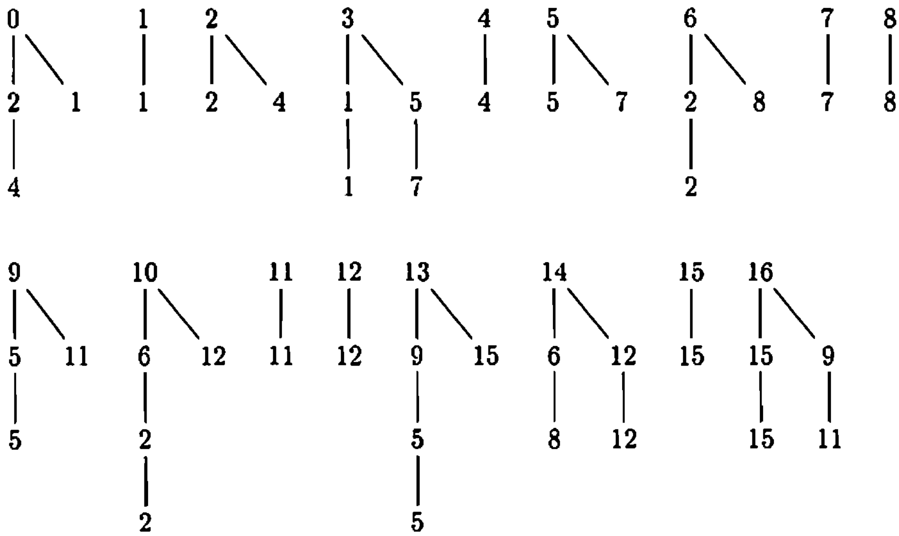
Then the minimal $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximation of S_1 is as follows:



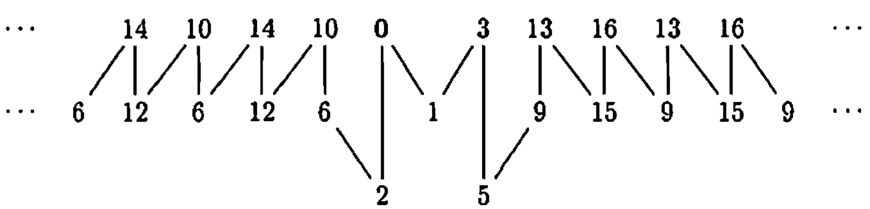
In particular, we observe that the factor modules of Λe_1 contained in the two rightmost summands are not minimal with respect to top-embeddability into objects of $\mathcal{P}^\infty(\Lambda\text{-mod})$. Thus the alternation "choose a minimal factor module embeddable into a module in $\mathcal{P}^\infty(\Lambda\text{-mod})$, then choose a maximal essential extension arising as a factor module of a module in $\mathcal{P}^\infty(\Lambda\text{-mod})$" which leads to the minimal approximations of the simples over string algebras in case of existence, cannot be expected to achieve this goal in the more general situation. □

To give another illustration of our algorithm, we conclude with an example of a string algebra Λ and a simple left Λ-module S, the characteristic phantom of which is twosided infinite with left/right periods reached at different steps of the algorithm.

Example H. Let $\Lambda = K\Gamma/I$ be the string algebra with the following indecomposable projective left modules:



Then the characteristic $\mathcal{P}^{\infty}(\Lambda\text{-mod})$-phantom of $S = \Lambda e_0/Je_0$ has a graph as follows:



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A CHARACTERIZATION OF FINITE AUSLANDER-REITEN QUIVERS OF ORDERS

OSAMU IYAMA

0. Krull-Schmidt categories which have ASS

Let $\mathcal{C}$ be a skeletally small additive category and $\mathcal{C}(X, Y)$ denote the morphism set from X to Y. We adopt the convention that morphisms will be written on the *right side* of the object on which they operate. We denote by Mod $\mathcal{C}$ the category of additive functors $\mathcal{C}^{op} \longrightarrow \mathcal{A}b$, which forms an abelian category. We call $L \in$ Mod $\mathcal{C}$ *finitely generated* if there exists an epimorphism $\mathcal{C}(\ ,X) \longrightarrow L$, and *finitely presented* if there exists an exact sequence $\mathcal{C}(\ ,Y) \longrightarrow \mathcal{C}(\ ,X) \longrightarrow L \longrightarrow 0$. We denote by mod $\mathcal{C}$ the category of finitely presented additive functors. Remark that $L \in$ mod $\mathcal{C}$ is projective if and only if there exists $X \in \mathcal{C}$ such that $L = \mathcal{C}(\ ,X)$. Hence $\mathcal{C}$ is equivalent to the category of projective objects in mod $\mathcal{C}$. For any $L \in$ Mod $\mathcal{C}$, we will denote by $\mathcal{R}L \in$ Mod $\mathcal{C}$ the radical of L, i.e., intersection of all maximal submodules of L. Next, we will define the contravariant functor α : mod $\mathcal{C} \longrightarrow$ mod $\mathcal{C}^{op}$. For $X, Y \in \mathcal{C}$, put $\alpha(\mathcal{C}(\ ,X) \xrightarrow{\cdot f} \mathcal{C}(\ ,Y)) = (\mathcal{C}(Y,\ ) \xrightarrow{f\cdot} \mathcal{C}(X,\ ))$. For general $L \in$ mod $\mathcal{C}$, take a projective resolution $P_1 \xrightarrow{a} P_0 \longrightarrow L \longrightarrow 0$ and put $\alpha L = \operatorname{Ker}(\alpha a)$. Dually, we define β : mod $\mathcal{C}^{op} \longrightarrow$ mod $\mathcal{C}$.

Throughout we assume $\mathcal{C}$ is a *Krull-Schmidt category* (i.e. If $X \in \mathcal{C}$ is indecomposable, then $\mathcal{C}(X, X)$ is a local ring, and each object of $\mathcal{C}$ is isomorphic to some finite direct sum of indecomposable objects). Let $\mathfrak{I}(\mathcal{C})$ be the set of isomorphism classes of indecomposable objects in $\mathcal{C}$. Then we can identify the set of isomorphism classes of objects in $\mathcal{C}$ with the free monoid $\mathbb{N}\mathfrak{I}(\mathcal{C})$ with base set $\mathfrak{I}(\mathcal{C})$. Remark that if $L \in$ Mod $\mathcal{C}$ is finitely generated, then there exists the projective cover (i.e. *essential* epimorphism from a projective object) of L since $\mathcal{C}$ is a Krull-Schmidt category.

In this paper, we study Krull-Schmidt categories which satisfy the following definition. Remark that we do not assume $\mathcal{C}$ is noetherian.

0.1 Definition. We say that $\mathcal{C}$ has *ASS* (resp. *exact ASS*) if the following (1), (2) hold.

The detailed version of this paper will be submitted for publication elsewhere.

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(1) For any simple $S \in \operatorname{Mod} \mathcal{C}$, there exists a minimal projective resolution in mod $\mathcal{C}$: $P_2 \xrightarrow{\nu} P_1 \xrightarrow{\mu} P_0 \longrightarrow S \longrightarrow 0$ (resp. $0 \longrightarrow P_2 \xrightarrow{\nu} P_1 \xrightarrow{\mu} P_0 \longrightarrow S \longrightarrow 0$). Moreover, if $P_2 \neq 0$, then $\alpha P_0 \xrightarrow{\alpha\mu} \alpha P_1 \xrightarrow{\alpha\nu} \alpha P_2 \longrightarrow \operatorname{Cok}(\alpha\nu) \longrightarrow 0$ (resp. $0 \longrightarrow \alpha P_0 \xrightarrow{\alpha\mu} \alpha P_1 \xrightarrow{\alpha\nu} \alpha P_2 \longrightarrow \operatorname{Cok}(\alpha\nu) \longrightarrow 0$) is exact and $\operatorname{Cok}(\alpha\nu)$ is simple in Mod $\mathcal{C}^{op}$.

(2) Mod $\mathcal{C}^{op}$-version of (1).

For $L \in \operatorname{Mod}\mathcal{C}$ and $n \leq m$, put $\mathcal{R}^n L := \mathcal{R}(\mathcal{R}^{n-1}L)$, $\mathcal{R}^\infty L := \bigcap_{n\geq 0} \mathcal{R}^n L$ and $(\mathcal{R}^n/\mathcal{R}^m)L := \mathcal{R}^n L/\mathcal{R}^m L$. We will denote (the first 3-terms of) the minimal projective resolution of $\mathcal{R}^0/\mathcal{R}\mathcal{C}(X, \ )$ and $\mathcal{R}^0/\mathcal{R}\mathcal{C}(\ , X)$ by the following.

$$\mathcal{C}(\ ,\tau_{\mathcal{C}}X) \xrightarrow{\cdot\nu} \mathcal{C}(\ ,\theta_{\mathcal{C}}X) \xrightarrow{\cdot\mu} \mathcal{C}(\ ,X) \longrightarrow \mathcal{R}^0/\mathcal{R}\mathcal{C}(\ ,X) \longrightarrow 0$$

$$\mathcal{C}(\tau_{\mathcal{C}}^- X,\ ) \xrightarrow{\mu\cdot} \mathcal{C}(\theta_{\mathcal{C}}^- X,\ ) \xrightarrow{\nu\cdot} \mathcal{C}(X,\ ) \longrightarrow \mathcal{R}^0/\mathcal{R}\mathcal{C}(X,\ ) \longrightarrow 0$$

$\tau_{\mathcal{C}}$ (resp. $\theta_{\mathcal{C}}$, $\tau_{\mathcal{C}}^-$, $\theta_{\mathcal{C}}^-$) is simply denoted as τ (resp. θ, τ^-, θ^-). Put $\mathcal{P}(\mathcal{C}) := \{X \in \mathfrak{I}(\mathcal{C}) | \tau X = 0\}$ and $\mathcal{I}(\mathcal{C}) := \{X \in \mathfrak{I}(\mathcal{C}) | \tau^- X = 0\}$. It is easily shown that τ gives a bijection from $\mathfrak{I}(\mathcal{C}) - \mathcal{P}(\mathcal{C})$ to $\mathfrak{I}(\mathcal{C}) - \mathcal{I}(\mathcal{C})$. $X \in \mathfrak{I}(\mathcal{C})$ is called *weakly projective* (resp. *weakly injective*) if $0 \longrightarrow \mathcal{C}(X,\ ) \longrightarrow \mathcal{C}(\theta X,\ )$ (resp. $0 \longrightarrow \mathcal{C}(\ ,X) \longrightarrow \mathcal{C}(\ ,\theta^- X)$) is not exact. Put w-proj $\mathcal{C} := \{X \in \mathfrak{I}(\mathcal{C})|\ X$ is weakly projective$\}$ and w-inj $\mathcal{C} := \{X \in \mathfrak{I}(\mathcal{C})|\ X$ is weakly injective$\}$.

0.2 Examples

(1) Let Λ be an Auslander-Gorenstein artin algebra with gl.dim.$\Lambda \leq 2$. Then the category of finitely generated projective Λ-modules has exact ASS.

(2) Let R be a complete discrete valuation ring and Λ an Auslander order over R (4.1). Then the category of finitely generated projective Λ-modules has exact ASS.

(3) Let R be a 2-dimensional integrally closed complete local noetherian domain, Λ a *tame* R-*order* (4.5.2), and $\mathcal{C}$ the category of finitely generated Λ-modules which are reflexive as R-module. Then $\mathcal{C}$ has exact ASS [**RV**].

0.3 Remarks about quotient categories Let $\mathcal{S}$ be a subset of $\mathfrak{I}(\mathcal{C})$ and $I_{\mathcal{S}}$ the ideal of $\mathcal{C}$ generated by morphisms which factor through some object in $\mathcal{S}$. We define the quotient category $\mathcal{C}/\mathcal{S}$ as follows. The objects of $\mathcal{C}/\mathcal{S}$ are the same as $\mathcal{C}$. $\mathcal{C}/\mathcal{S}(X,Y) := \mathcal{C}(X,Y)/I_{\mathcal{S}}(X,Y)$ for each pair of objects X, Y. There exists the natural full functor $\overline{} : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{S}$.

Proposition. *Let $\mathcal{S}$ be a subset of $\mathfrak{I}(\mathcal{C})$ and $\overline{\mathcal{C}} := \mathcal{C}/\mathcal{S}$. If $\mathcal{C}$ has ASS, then $\overline{\mathcal{C}}$ also has ASS. Moreover, for any $X \in \mathfrak{I}(\mathcal{C}) - \mathcal{S}$, the minimal projective resolution of $\mathcal{R}^0/\mathcal{R}\overline{\mathcal{C}}(\ ,X) \in$ Mod $\overline{\mathcal{C}}$ is given by the following sequence.*

$$\overline{\mathcal{C}}(\ ,\overline{\tau_{\mathcal{C}}X}) \xrightarrow{\cdot\overline{\nu}} \overline{\mathcal{C}}(\ ,\overline{\theta_{\mathcal{C}}X}) \xrightarrow{\cdot\overline{\mu}} \overline{\mathcal{C}}(\ ,\overline{X}) \longrightarrow \mathcal{R}^0/\mathcal{R}\overline{\mathcal{C}}(\ ,\overline{X}) \longrightarrow 0 \quad (\text{if } \overline{\theta_{\mathcal{C}}X} \neq 0)$$

$$0 \longrightarrow \overline{\mathcal{C}}(\ ,\overline{X}) \longrightarrow \mathcal{R}^0/\mathcal{R}\overline{\mathcal{C}}(\ ,\overline{X}) \longrightarrow 0 \quad (\text{if } \overline{\theta_{\mathcal{C}}X} = 0)$$

1. A characterization of finite Auslander-Reiten quivers of orders

Main application of results in this paper is to characterize finite Auslander-Reiten quivers of orders over complete discrete valuation rings. This can be seen as one-dimensional version of Igusa-Todorov theorem [**IT3**] which characterizes finite Auslander-Reiten quivers of artin argebras. But most difficult point in our proof is to show that some (infinite dimensional) algebra over a field becomes an order over some complete discrete valuation ring. This is a peculiar problem to one-dimensional case.

1.1 Definition. (A) Let Q be a set. $(Q, \mathcal{P}(Q), \mathcal{I}(Q), \tau, d, d')$ is called a *pre-translation quiver* if the following conditions (1), (2), (3) are satisfied.

(1) $\mathcal{P}(Q)$ and $\mathcal{I}(Q)$ are subsets of Q and $\tau : Q - \mathcal{P}(Q) \longrightarrow Q - \mathcal{I}(Q)$ is a bijection.

(2) $d : Q \times Q \longrightarrow \mathbb{N} \cup \{0\}$ and $d' : Q \times Q \longrightarrow \mathbb{N} \cup \{0\}$ are maps.

(3) $d'(\tau Y, X) = d(X, Y)$ for any $X \in Q$, $Y \in Q - \mathcal{P}(Q)$.

(B) A pre-translation quiver is called a *translation quiver* if there exists a map $n : Q \longrightarrow \mathbb{N}$ which satisfies the following equations (4), (5).

(4) $n(\tau X) = n(X)$ for any $X \in Q - \mathcal{P}(Q)$.

(5) $n(X)d(X, Y) = d'(X, Y)n(Y)$ for any $X, Y \in Q$.

(C) A translation quiver is called a *finite translation quiver* if Q is a finite set.

1.2 (A) Let Q be a translation quiver.

(1) Let $\mathbb{Z}Q$ be a free abelian group with base set Q. On Q, we introduce the inner product $\langle\ ,\ \rangle$ taking Q as an orthonormal base. We shall also introduce an ordering in $\mathbb{Z}Q$ by $X \leq Y \Leftrightarrow \langle X, L\rangle \leq \langle Y, L\rangle$ for any $L \in Q$. For $L \in \mathbb{Z}Q$, let (L_+, L_-) be elements of $\mathbb{Z}Q$ such that $L = L_+ - L_-$, $L_+ \geq 0$, $L_- \geq 0$ and $\langle L_+, L_-\rangle = 0$.

(2) For $X \in Q$, define θX, $\theta^- X \in \mathbb{Z}Q$ by

$$\theta X := \sum_{Y \in Q} d(Y, X)Y, \quad \theta^- X := \sum_{Y \in Q} d'(X, Y)Y.$$

Let $\tau^- : Q - \mathcal{I}(Q) \longrightarrow Q - \mathcal{P}(Q)$ be the inverse of τ. For $X \in \mathcal{P}(Q)$ (resp. $X \in \mathcal{I}(Q)$), put $\tau X := 0$ (resp. $\tau^- X := 0$). Thus we obtain τX, $\tau^- X \in \mathbb{Z}Q$ for any $X \in Q$. θ, θ^-, τ and τ^- uniquely extend to elements of $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}Q)$.

(3) For a subset S of Q, let $i_S : \mathbb{Z}(Q-S) \longrightarrow \mathbb{Z}Q$ (resp. $p_S : \mathbb{Z}Q \longrightarrow \mathbb{Z}(Q-S)$) be the natural injetion (resp. natural projection). For $f \in \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}Q)$, define $f_{Q/S} \in \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}(Q - S))$ by $f_{Q/S} := p_S \circ f \circ i_S$.

(B) Let $\mathcal{C}$ have ASS. Define $\mathbb{Z}\mathfrak{I}(\mathcal{C})$, inner product and ordering in $\mathbb{Z}\mathfrak{I}(\mathcal{C})$ and θ, $\theta^-, \tau, \tau^- \in \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}\mathfrak{I}(\mathcal{C}))$ by similar way in (A).

1.3 Theorem. *For a finite translation quiver Q, the following conditions are equivalent.*

(1) *There is a complete discrete valuation ring R and R-order Λ such that $Q = \mathfrak{A}(\mathrm{lat}\Lambda)$ where $\mathfrak{A}(\mathrm{lat}\Lambda)$ is the Auslander-Reiten quiver of Λ (3.2).*

(2)(C1) *For any $I \in \mathcal{I}(Q)$, put $X_1 := \theta^- I$, $X_2 := \theta\theta^- I - I$ and $X_i := \theta X_{i-1} - \tau X_{i-2}$ for $i \geq 3$. Then $X_i \geq 0$ for any i, and there exists $n \geq 1$ such that $X_n \in \mathcal{P}(Q)$, $X_{n+1} = 0$. Moreover, $X_1, \dots, X_{n-1}$ have no $\mathcal{P}(Q)$-summand.*

(C2) *Put $s(I) := \bigcup_{i=1}^{n} \{$non-zero components of $X_i\}$, then $Q = \bigcup_{I \in \mathcal{I}(Q)} s(I)$.*

(C3) *For any $X \in \mathcal{P}(Q)$, put $Y_0 := 0$, $Y_1 := X$ and $Y_i := (\theta_{Q/\mathcal{I}(Q)} Y_{i-1} - \tau_{Q/\mathcal{I}(Q)} Y_{i-2})_+$ for $i \geq 2$. Then there exists $n \geq 1$ such that $Y_n = 0$.*

1.4 Remark. (A) More strong than (2)$\Longrightarrow$(1), we show the following assertion $(*)$.

$(*)$ Let k be any fileld and $\mathcal{Q}$ a k-modulated finite translation quiver (3.1). If $|\mathcal{Q}|$ satisfies (C1), (C2), (C3), then there exists a $k[[x]]$-order Λ such that $\mathcal{Q} = \mathfrak{A}_m(\mathrm{lat}\Lambda)$.

(B) The correspondence $\mathcal{I}(C) \ni I \mapsto X_n \in \mathcal{P}(C)$ gives the Nakayama functor of Λ. $s(I)$ gives the set of indecomposable Λ-lattices L such that the projective cover $P \longrightarrow L$ satisfies $X_n \leq P$.

2. Minimal projective resolution of some special modules

First important step is to compute the minimal projective resolution of some special $L \in \mathrm{mod}\, C$. This is a key lemma to prove 6.1.

2.1 Theorem. *Assume C has ASS and fix $A \in \mathfrak{I}(C)$. Let $C(\theta^- A, \ ) \xrightarrow{\nu^{\cdot}} C(A, \ ) \longrightarrow \mathcal{R}^0/\mathcal{R}C(A, \ ) \longrightarrow 0$ be the minimal projective resolution and define $L \in \mathrm{mod}\, C$ by $C(\ , A) \xrightarrow{\cdot\nu} C(\ , \theta^- A) \longrightarrow L \longrightarrow 0$. Assume $L \neq 0$.*

(A) (1) *There exists $X_i \xrightarrow{a_i} Y_i$ such that $C(\ , X_i) \xrightarrow{\cdot a_i} C(\ , Y_i) \longrightarrow \mathcal{R}^i L \longrightarrow 0$ gives (the first 2-terms of) the minimal projective resolution of $\mathcal{R}^i L$.*

(2) *$(X_0, Y_0) = (A, \theta^- A)$, $(X_1, Y_1) = (\tau Y_0 - (\theta Y_1 - \tau Y_0)_-, \ \theta\theta^- A - A)$ and $(X_i, Y_i) = (\tau Y_{i-1} - (\theta Y_i - \tau Y_{i-1})_-, \ (\theta Y_{i-1} - \tau Y_{i-2})_+)$ for any $i \geq 2$.*

(B) *Assume C has exact ASS and there exists the minimal $n \geq 1$ such that $\mathcal{R}^n L = 0$.*

(1) *$(X_i, Y_i) = (\tau Y_{i-1}, \theta Y_{i-1} - X_{i-1})$ for any $i = 1, \dots, n-1$.*

(2) *$A \notin$ w-inj C if and only if $Y_{n-1} \in \mathrm{add}\mathcal{P}(C)$.*

2.1.1 Corollary. *Assume C has ASS. Fix $A \in C$ and $i \geq 0$.*

(1) *There exists $\tau_i A \xrightarrow{a_i} \theta_i A \xrightarrow{b_i} A$ such that $C(\ , \tau_i A) \xrightarrow{\cdot a_i} C(\ , \theta_i A) \xrightarrow{\cdot b_i} \mathcal{R}^i C(\ , A) \longrightarrow 0$ gives (the first 2-terms of) the minimal projective resolution of $\mathcal{R}^i C(\ , A)$.*

(2) *$(\tau_0 A, \ \theta_0 A) = (0, A)$, $(\tau_1 A, \theta_1 A) = (\tau A, \ \theta A)$ and $(\tau_i A, \ \theta_i A) = (\tau\theta_{i-1} A - (\theta\theta_i A - \tau\theta_{i-1} A)_-, \ (\theta\theta_{i-1} A - \tau\theta_{i-2} A)_+)$ for any $i \geq 2$.*

2.2 This is proved by showing the existence of 'ladder'. Ladder is a special case of deformation of complexes [I]. In [IT1], it is shown for $C = \text{mod } A$ where A is an artin algebra of finite representation type (resp. an algebra over an algebraically closed field). But their proof heavily depends on the fact that for the Auslander-Reiten quiver of such category, each arrow has a valuation $(1,n)$, $(n,1)$ where $n = 1, 2$ or 3 (resp. (n,n)). Here, we will give a general (and easy) proof.

Lemma. (Existence theorem of ladders) *Assume that C has ASS and $a_0 \in \mathcal{R}C(X_0, Y_0)$ satisfies $a_0 + \mathcal{R}^2C(X_0, Y_0) \subseteq \text{Aut}(X_0)a_0\text{Aut}(Y_0)$. Define $L \in$ mod C by $C(\ ,X_0) \xrightarrow{\cdot a_0} C(\ ,Y_0) \longrightarrow L \longrightarrow 0$. Then the following (1), (2), (3) hold.*

(1) *For any $i \geq 0$, there exists $\begin{pmatrix} f_i \\ 0 \end{pmatrix} : P_{i,1} \oplus Q_i \longrightarrow P_{i,0}$ such that the following sequences give the minimal projective resolutions.*

$$P_{i,1} \xrightarrow{f_i} P_{i,0} \longrightarrow \mathcal{R}^i L \longrightarrow 0$$

$$P_{i+1,1} \oplus Q_{i+1} \xrightarrow{\begin{pmatrix} f_{i+1} & * \\ 0 & * \end{pmatrix}} P_{i+1,0} \oplus P_{i,1} \xrightarrow{\binom{*}{f_i}} P_{i,0} \longrightarrow P_{i,0}/\mathcal{R}P_{i,0} \longrightarrow 0$$

(2) *If a_0 is a monomorphism and C has exact ASS, then $Q_i = 0$ for any $i \geq 1$.*

(3) *Define $M \in$ mod C^{op} by $C(Y_0,\ ) \xrightarrow{a_0\cdot} C(X_0,\ ) \longrightarrow M \longrightarrow 0$. If M is semi-simple, then $\mathcal{R}^{i+1}\alpha Q_i = 0$ for any $i \geq 0$.*

2.2.1 Sketch of proof. Fix $X, X', Y \in C$ and take the minimal projective resolution $C(\ ,\tau Y) \xrightarrow{\cdot\nu} C(\ ,\theta Y) \xrightarrow{\cdot\mu} C(\ ,Y) \longrightarrow \mathcal{R}^0/\mathcal{R}C(\ ,Y) \longrightarrow 0$. Let ${}_1\mathcal{R}C(X,Y)$ be a subset of $\mathcal{R}C(X,Y)$ formed by α such that there exists a split monomorphism $i : X \longrightarrow \theta Y$ such that $\alpha = i\mu$. Also, let $\mathcal{R}_1'C(\tau Y, X')$ be a subset of $\mathcal{R}C(\tau Y, X')$ formed by α such that there exists a split epimorphism $p : \theta Y \longrightarrow X'$ such that $\alpha = \nu p$. It is easily shown that ${}_1\mathcal{R}C(X,Y)$ (resp. $\mathcal{R}_1'C(\tau Y, X')$) is union of cosets modulo $\mathcal{R}^2C(X,Y)$ (resp. $\mathcal{R}^2C(\tau Y, X')$).

For any decomposition $\theta Y = X \oplus X'$, we will define the map

$$j_{X,Y} : \text{Aut}(X)\backslash{}_1\mathcal{R}C(X,Y)/\text{Aut}(Y) \longrightarrow \text{Aut}(\tau Y)\backslash\mathcal{R}_1'C(\tau Y, X')/\text{Aut}(X').$$

For $\alpha \in {}_1\mathcal{R}C(X,Y)$, by definition, there exists an isomorphism $\begin{pmatrix} f \\ g \end{pmatrix}$: $X \oplus X' \longrightarrow \theta Y$ which makes the following diagram commutative. Then put $j_{X,Y}([\alpha]) := [\beta']$.

$$\begin{array}{ccccc}
C(\ ,\tau Y) & \xrightarrow{\cdot(\beta,\beta')} & C(\ ,X \oplus X') & \xrightarrow{\cdot\binom{\alpha}{\alpha'}} & C(\ ,Y) \\
\downarrow 1 & & \downarrow \cdot\binom{f}{g} & & \downarrow 1 \\
C(\ ,\tau Y) & \xrightarrow{\cdot\nu} & C(\ ,\theta Y) & \xrightarrow{\cdot\mu} & C(\ ,Y)
\end{array}$$

Then the following (1)–(4) are easily verified.

(1) $j_{X,Y}$ is well defined.

(2) $j_{X,Y}$ is surjective. If Y has no $\mathcal{P}(\mathcal{C})$-summand, then $j_{X,Y}$ is bijective.

(3) For $n \geq 2$, $j_{X,Y}$ induces

$$j^n_{X,Y} : \mathrm{Aut}(X)\backslash({}_1\mathcal{R}/\mathcal{R}^n\mathcal{C}(X,Y))/\mathrm{Aut}(Y) \longrightarrow \\ \mathrm{Aut}(\tau Y)\backslash(\mathcal{R}'_1/\mathcal{R}^n\mathcal{C}(\tau Y, X'))/\mathrm{Aut}(X').$$

(4) For $n \geq 2$, if $j_{X,Y}([\alpha]) = [\beta']$ and $\beta' - \beta'_1 \in \mathcal{R}^n\mathcal{C}(\tau Y, X')$, then there exists α_1 such that $j_{X,Y}([\alpha_1]) = [\beta'_1]$ and $\alpha - \alpha_1 \in \mathcal{R}^n\mathcal{C}(X,Y)$.

2.2.2 We will prove 2.2 (1) and (2). By induction, we only have to prove the following (∗).

(∗) There exists a decomposition $a_0 = \begin{pmatrix} b_0 \\ 0 \end{pmatrix} : X_0 = Z_0 \oplus U_0 \longrightarrow Y_0$ such that $b_0 \in {}_1\mathcal{RC}(Z_0, Y_0)$. If $a_1 \in \mathcal{R}'_1\mathcal{C}(X_1, Y_1)$ satisfies $[a_1] = j_{Z_0,Y_0}([b_0])$, then $a_1 + \mathcal{R}^2\mathcal{C}(X_1, Y_1) \subseteq \mathrm{Aut}(X_1)a_1\mathrm{Aut}(Y_1)$. Moreover, if a_0 is a monomorphism, then a_1 is also a monomorphism.

In general, it is easily shown that there exists a decomposition $a_0 = \begin{pmatrix} b_0 \\ c_0 \end{pmatrix}$: $X_0 = Z_0 \oplus U_0 \longrightarrow Y_0$ such that $b_0 \in {}_1\mathcal{RC}(Z_0, Y_0)$ and $c_0 \in \mathcal{R}^2\mathcal{C}(U_0, Y_0)$. Since $a_0 - \begin{pmatrix} b_0 \\ 0 \end{pmatrix} \in \mathcal{R}^2\mathcal{C}(X_0, Y_0)$, $\begin{pmatrix} b_0 \\ 0 \end{pmatrix} \in \mathrm{Aut}(X_0)a_0\mathrm{Aut}(Y_0)$ by our assumption. Hence the first part of (∗) is proved. To prove the second part, take an element $a'_1 \in a_1 + \mathcal{R}^2\mathcal{C}(X_1, Y_1)$. By 2.2.1 (4), there exists $b'_0 \in {}_1\mathcal{RC}(Z_0, Y_0)$ such that $j_{Z_0,Y_0}([b'_0]) = [a'_1]$ and $b_0 - b'_0 \in \mathcal{R}^2\mathcal{C}(Z_0, Y_0)$. By our assumption, $\mathrm{Aut}(X_0)\begin{pmatrix} b_0 \\ 0 \end{pmatrix}\mathrm{Aut}(Y_0) = \mathrm{Aut}(X_0)\begin{pmatrix} b'_0 \\ 0 \end{pmatrix}\mathrm{Aut}(Y_0)$, hence $\mathrm{Aut}(Z_0)b_0\mathrm{Aut}(Y_0) = \mathrm{Aut}(Z_0)b'_0\mathrm{Aut}(Y_0)$. Since j_{Z_0,Y_0} is well defined, $\mathrm{Aut}(X_1)a_1\mathrm{Aut}(Y_1) = [a_1] = j_{Z_0,Y_0}([b_0]) = j_{Z_0,Y_0}([b'_0]) = [a'_1] = \mathrm{Aut}(X_1)a'_1\mathrm{Aut}(Y_1)$. We will prove the third part. Since a_0 is a monomorphism, $b_0 = a_0$ and $U_0 = 0$. Since $\mathcal{C}$ has exact ASS, $0 \longrightarrow \mathcal{C}(\ , X_1) \xrightarrow{\cdot(a_1\ *)} \mathcal{C}(\ , Y_1 \oplus Z_0) \xrightarrow{\cdot\binom{*}{b_0}} \mathcal{C}(\ , Y_0) \longrightarrow \mathcal{R}^0/\mathcal{RC}(\ , Y_0) \longrightarrow 0$ gives the minimal projective resolution by the definition of j_{Z_0,Y_0}. Since b_0 is a monomorphism, it is easily shown that a_1 is also a monomorphism.

2.2.3 Fix $X, Y, Y' \in \mathcal{C}$ and take the minimal projective resolution $\mathcal{C}(\tau^- X,\ ) \xrightarrow{\mu\cdot} \mathcal{C}(\theta^- X,\ ) \xrightarrow{\nu\cdot} \mathcal{C}(X,\ ) \longrightarrow \mathcal{R}^0/\mathcal{RC}(X,\ ) \longrightarrow 0$. Let $\mathcal{R}_1\mathcal{C}(X,Y)$ be a subset of $\mathcal{RC}(X,Y)$ formed by α such that there exists a split epimorphism $p : \theta^- X \longrightarrow Y$ such that $\alpha = \nu p$. Also, let ${}_1\mathcal{R}'\mathcal{C}(Y', \tau^- X)$ be a subset of $\mathcal{RC}(Y', \tau^- X)$ formed by α such that there exists a split monomorphism i :

$Y' \longrightarrow \theta^- X$ such that $\alpha = i\mu$. For any decomposition $\theta^- X = Y \oplus Y'$, we can define the map

$$j^-_{X,Y} : \mathrm{Aut}(X)\backslash \mathcal{R}_1 \mathcal{C}(X,Y)/\mathrm{Aut}(Y) \longrightarrow \mathrm{Aut}(Y')\backslash {}_1\mathcal{R}'\mathcal{C}(Y', \tau^- X)/\mathrm{Aut}(\tau^- X)$$

by dual argument of 2.2.1.

2.2(3) is easily shown by the following lemma.

Lemma. *Assume that $\mathcal{C}$ has ASS and $f \in \mathcal{R}_1(X,Y)$, $f' \in {}_1\mathcal{R}'(X',Y')$, $g \in {}_1\mathcal{R}(A,B)$ and $g' \in \mathcal{R}'_1(A',B')$ satisfy the following conditions* (1), (2).

(1) $j^-_{X,Y}([f]) = [f']$ *and* $j_{A,B}([g]) = [g']$.

(2) *There exist $t \in \mathcal{C}(Y,B')$ and a split monomorphism $s \in \mathcal{C}(X,A')$ such that $ft = sg'$.*

Then there exist $t' \in \mathcal{C}(Y',B)$ and a split monomorphism $s' \in \mathcal{C}(X',A)$ such that $f't' = s'g$.

2.4 The following corollary is a key lemma to prove the structure theorem 3.5.

Corollary. (Radical layers theorem) *Assume $\mathcal{C}$ has exact ASS. Then for each $X \in \mathcal{C}$ and $n \geq -1$, the following is exact.*

$$0 \to \mathcal{R}^n/\mathcal{R}^{n+1}\mathcal{C}(\ ,\tau X) \xrightarrow{\cdot \nu} \mathcal{R}^{n+1}/\mathcal{R}^{n+2}\mathcal{C}(\ ,\theta X) \xrightarrow{\cdot \mu} \mathcal{R}^{n+2}/\mathcal{R}^{n+3}\mathcal{C}(\ ,X) \to 0$$

2.5 We call $\mathcal{C}$ *left artinian* (resp. *right artinian*) if $\mathcal{C}(\ ,X)$ (resp. $\mathcal{C}(X,\ )$) is finite length for any $X \in \mathcal{C}$. $\mathcal{C}$ is called *artinian* if $\mathcal{C}$ is 2-sided artinian.

Corollary. (Criterion for $\mathcal{C}$ to be artinian) *Assume $\mathcal{C}$ has ASS.*

(1) *$\mathcal{C}$ is left artinian if and only if for any $A \in \mathcal{C}$, there exists $n \geq 0$ such that $\theta_n A = 0$.*

(2) *Assume $\mathfrak{A}(\mathcal{C})$ (3.2) is connected, $\mathfrak{I}(\mathcal{C})$ is a finite set and $\mathcal{I}(\mathcal{C}) \neq \emptyset$. Then $\mathcal{C}$ is left artinian if and only if for any $A \in \mathcal{I}(\mathcal{C})$, there exists $n \geq 0$ such that $\theta_n A = 0$.*

2.6 Corollary. (Criterion for $\mathcal{C}$ to have exact ASS) *Assume $\mathcal{C}$ has ASS and $\mathcal{R}^\infty\mathcal{C}(\ ,\ ) = 0$. Then the following conditions are equivalent.*

(1) *$\mathcal{C}$ has exact ASS.*

(2) $\mathcal{I}(\mathcal{C}) \supseteq$ *w-inj$\mathcal{C}$.*

(3) $\theta\theta_{i+1}A \geq \tau\theta_i A$ *for any $A \in \mathcal{C}$ and $i \geq 0$.*

3. A construction of categories which has ASS

In this section, we introduce Igusa-Todorov construction of mesh categories of modulated translation quiver [**IT2**].

3.1 Definition. (A) $\mathcal{Q} = (Q, \mathcal{P}(Q), \mathcal{I}(Q), \tau, D_X, {}_XM_Y, a, b)$ is called a *modulated translation quiver* if the following conditions are satisfied.

(1) $\mathcal{P}(Q)$ and $\mathcal{I}(Q)$ are subsets of Q and $\tau : Q - \mathcal{P}(Q) \longrightarrow Q - \mathcal{I}(Q)$ is a bijection.

(2) D_X is a skew field for any $X \in Q$.

(3) ${}_XM_Y$ is a (D_X, D_Y)-bimodule for any $X, Y \in Q$.

(4) $a_X : D_X \longrightarrow D_{\tau X}$ is a ring isomorphism for any $X \in Q - \mathcal{P}(Q)$.

(5) $b_{X,Y} : {}_YM_X \otimes_{D_X} {}_{\tau X}M_Y \longrightarrow D_Y$ is a non-singular (D_Y, D_Y)-homomorphism for any $X \in Q - \mathcal{P}(Q)$ and $Y \in Q$.

(6) $\sum_{Y \in Q} \dim_{D_X} {}_XM_Y$ and $\sum_{Y \in Q} \dim_{D_X} {}_YM_X$ are finite for any $X \in Q$.

(B) Let $\mathcal{Q}$ be a modulated translation quiver and k be a field. $\mathcal{Q}$ is called a *k-modulated translation quiver* if D_X is a finite dimensional division k-algebra, k acts ${}_XM_Y$ centrally and any a_X and $b_{X,Y}$ are k-linear.

(C) Let $\mathcal{Q} = (Q, \mathcal{P}(Q), \mathcal{I}(Q), \tau, D_X, {}_XM_Y, a, b)$ be a modulated translation quiver. We define a pre-translation quiver $|\mathcal{Q}| = (Q, \mathcal{P}(Q), \mathcal{I}(Q), \tau, d, d')$ by putting $d(X,Y) := \dim_{D_X} {}_XM_Y$ and $d'(X,Y) := \dim_{D_Y} {}_XM_Y$ for any $X, Y \in Q$. Remark that if $\mathcal{Q}$ is a k-modulated translation quiver, then $|\mathcal{Q}|$ becomes a translation quiver by putting $n(X) := \dim_k D_X$ for any $X \in Q$.

3.2 Definition. Let $\mathcal{C}$ have ASS.

(A) We define a modulated translation quiver $\mathfrak{A}_m(\mathcal{C})$ called the *modulated Auslander -Reiten quiver* of $\mathcal{C}$ as follows.

(1) $Q := \mathfrak{I}(\mathcal{C})$, $\mathcal{P}(Q) := \mathcal{P}(\mathcal{C})$, $\mathcal{I}(Q) := \mathcal{I}(\mathcal{C})$ and $\tau : Q - \mathcal{P}(Q) \longrightarrow Q - \mathcal{I}(Q)$ is defined in §0.

(2) $D_X := \mathcal{R}^0/\mathcal{R}\mathcal{C}(X,X)$ and ${}_XM_Y := \mathcal{R}/\mathcal{R}^2\mathcal{C}(X,Y)$ for any $X, Y \in \mathfrak{I}(\mathcal{C})$.

(3) For any $X \in Q - \mathcal{P}(Q)$, take the minimal projective resolution of $\mathcal{R}^0/\mathcal{R}\mathcal{C}(\ ,X)$. $a_X(f)$ is induced from the following commutative diagram.

$$\begin{array}{ccccccc}
\mathcal{C}(\ ,\tau X) & \xrightarrow{\cdot\nu} & \mathcal{C}(\ ,\theta X) & \xrightarrow{\cdot\mu} & \mathcal{C}(\ ,X) & \longrightarrow & \mathcal{R}^0/\mathcal{R}\mathcal{C}(\ ,X) \\
\downarrow{\scriptstyle \cdot a_X(f)} & & \downarrow & & \downarrow{\scriptstyle \cdot f} & & \downarrow \\
\mathcal{C}(\ ,\tau X) & \xrightarrow{\cdot\nu} & \mathcal{C}(\ ,\theta X) & \xrightarrow{\cdot\mu} & \mathcal{C}(\ ,X) & \longrightarrow & \mathcal{R}^0/\mathcal{R}\mathcal{C}(\ ,X)
\end{array}$$

(4) For any $X \in Q - \mathcal{P}(Q)$ and $Y \in Q$, $b_{X,Y}(f \otimes g) := \tilde{f}\tilde{g}$ is induced from the following commutative diagrams.

$$\begin{array}{ccccc}
 & & \mathcal{C}(Y,\ ) & & \\
 & \swarrow 0 & \downarrow g\cdot & \tilde{g}\cdot \searrow & \\
\mathcal{R}^0/\mathcal{R}\mathcal{C}(\tau X,\ ) & \longleftarrow & \mathcal{C}(\tau X,\ ) & \xleftarrow{\nu\cdot} & \mathcal{C}(\theta X,\ ),
\end{array}$$

$$\begin{array}{ccccc}
 & & \mathcal{C}(\ ,Y) & & \\
 & \swarrow \cdot\tilde{f} & \downarrow \cdot f & 0 \searrow & \\
\mathcal{C}(\ ,\theta X) & \xrightarrow{\cdot\nu} & \mathcal{C}(\ ,X) & \longrightarrow & \mathcal{R}^0/\mathcal{R}\mathcal{C}(\ ,X)
\end{array}$$

(B) We call $\mathfrak{A}(\mathcal{C}) := |\mathfrak{A}_m(\mathcal{C})|$ the *Auslander-Reiten quiver* of $\mathcal{C}$, which is a pre-translation quiver.

3.3 Definition. Let $\mathcal{Q} = (Q, \mathcal{P}(Q), \mathcal{I}(Q), \tau, D_X, {}_XM_Y, a, b)$ be a modulated translation quiver.

(A) We define an additive category $P_\mathcal{Q}$ called the *path category* of $\mathcal{Q}$ as follows.

(1) The objects of $P_\mathcal{Q}$ are $\mathbb{N}Q$.

(2) For $X, Y \in Q$ and $n \geq 1$,

$$P_{\mathcal{Q},0}(X,Y) := \begin{cases} 0 & (X \neq Y) \\ D_X & (X = Y) \end{cases}$$

$$P_{\mathcal{Q},n}(X,Y) := \bigoplus_{Z_0,\dots,Z_n \in Q} {}_XM_{Z_1} \otimes_{D_{Z_1}} \cdots \otimes_{D_{Z_n}} {}_{Z_n}M_Y$$

$$P_\mathcal{Q}(X,Y) := \bigoplus_{n \geq 0} P_{\mathcal{Q},n}(X,Y).$$

(B) We define an additive category $\widehat{M}_\mathcal{Q}$ called the *mesh category* of $\mathcal{Q}$ as follows.

(1) For any $X \in Q - \mathcal{P}(Q)$ and $Y \in Q$, take a D_Y-basis $u_Y^1, \dots, u_Y^d$ of ${}_YM_X$. By 3.1 (5), take its dual basis $v_Y^1, \dots, v_Y^d$ of ${}_{\tau X}M_Y$. Put $\gamma X(Y) := \sum_{i=1}^{d} v_Y^i \otimes u_Y^i \in {}_{\tau X}M_Y \otimes_{D_Y} {}_YM_X$ and $\gamma X := \sum_{Y \in Q} \gamma X(Y) \in P_\mathcal{Q}(\tau X, X)$.

(2) Let $I_\mathcal{Q} = \bigoplus_{n \geq 0} I_{\mathcal{Q},n}$ be the homogeneous ideal of $P_\mathcal{Q}$ generated by γX for any $X \in Q - \mathcal{P}(Q)$.

$$M_{\mathcal{Q},n} := P_{\mathcal{Q},n}/I_{\mathcal{Q},n}$$

$$\widehat{M}_\mathcal{Q} := \prod_{n \geq 0} M_{\mathcal{Q},n}$$

Remark that $P_\mathcal{Q}$ is not necessarily a Krull-Schimdt category, but $\widehat{M}_\mathcal{Q}$ becomes always a Krull-Schimdt category.

3.4 Definition. For any Krull-Schmidt category $\mathcal{C}$, define the additive category $\widehat{\mathrm{Gr}}(\mathcal{C})$ as follows. The objects of $\widehat{\mathrm{Gr}}(\mathcal{C})$ are the same as those of $\mathcal{C}$. $\widehat{\mathrm{Gr}}(\mathcal{C})(X,Y) := \prod_{n \geq 0} \mathcal{R}^n/\mathcal{R}^{n+1}\mathcal{C}(X,Y)$ for each pair of objects X, Y. Composition is given by

$$(f_i)_{i \geq 0}(g_i)_{i \geq 0} = (\sum_{j=0}^{i} f_j g_{i-j})_{i \geq 0}.$$

It is easily seen that $\widehat{\mathrm{Gr}}(\mathcal{C})$ becomes a Krull-Schmidt category.

3.5 Now, we can state the main theorem in this section. This gives one-to-one correspondence between modulated translation quivers and complete graded Krull-Schmidt categories which have ASS.

Theorem. *For any modulated translation quiver Q, its mesh category $\widehat{M_Q}$ has ASS and $\mathfrak{A}_m(\widehat{M_Q}) = Q$. Conversely, if C has ASS, then $\widehat{Gr}(C)$ is equivalent to $\widehat{M}_{\mathfrak{A}_m(C)}$. In particular, $\widehat{Gr}(C)$ has ASS.*

4. A characterization of Auslander orders

Throughout this section, let R be a *complete discrete valuation ring* and Λ an *R-order*, i.e., R-algebra which is finitely generated torsion free R-module. We denote by latΛ the category of *Λ-lattices*, i.e., Λ-modules which are finitely generated torsion free as R-module. Also, we denote by proj Λ (resp. inj Λ) the set of isomorphism classes of indecomposable projective (resp. injective) Λ-lattices.

4.1 Definition. Λ is called a *generalized Auslander order* (resp. *Auslander order*) if the following (1), (2) (resp. (0), (1), (2)) are satisfied.

(0) $K \otimes_R \Lambda$ is a semi-sinple K-algebra where K is the field of quotients of R.

(1) gl.dim.$\Lambda \leq 2$.

(2) There exists an exact sequence $0 \longrightarrow \Lambda \longrightarrow I_0 \longrightarrow I_1 \longrightarrow 0$ such that I_0 is a projective injective lattice and I_1 is an injective lattice.

4.2 We can characterize generalized Auslander orders as follows.

Theorem. *Let Λ be an R-order with* gl.dim.$\Lambda \leq 2$. *Then the following conditions are equivalent.*

(1) *Λ is a generalized Auslander order.*

(2) *The category of finitely generated projective Λ-lattices has ASS.*

(3) *For any $I \neq I' \in$ inj Λ^{op}, $\Omega(I)$ and $\Omega(I')$ do not have common direct summands.*

(4) *For any $P \in$ proj Λ, if* $\mathrm{pd}(P/\mathrm{rad}P) = 1$, *then* $\mathrm{Hom}_\Lambda(P, \Lambda) \in$ inj Λ^{op}.

(i^*) *Λ^{op}-version of (i) ($i = 1, 2, 3, 4$).*

4.3 On the other hand, Auslander and Roggenkamp show that the category of lattices over an order of finite representation type can be characterized as the category of projective modules of Auslander order.

Proposition.([AR]) The following conditions are equivalent.

(1) Λ is an Auslander order.

(2) There exists an R-order Δ and an additive generator M of latΔ such that $\mathrm{End}_\Delta(M) \cong \Lambda$.

4.4 Combining 4.2 and 4.3, we obtain the following main theorem in this section. This is the second important step to prove 1.3.

Corollary. Assume that $\mathcal{C}$ is a Krull-Schmidt category with an additive generator M and $\mathcal{C}(M, M)$ is an R-order. Then the following conditions are equivalent.

(1) $\mathcal{C}$ has ASS.

(2) There exists an R-order Δ such that $\mathcal{C} \simeq \text{lat}\Delta$.

4.5 Theorem 4.2 can be applied to prove the following results in **[I2]**.

4.5.1 Corollary.([I2]) *Let R be a complete discrete valuation ring with field of quotient K, Λ an Auslander order and ε a central idempotent of $K \otimes_R \Lambda$. Then $\varepsilon\Lambda$ is also an Auslander order.*

4.5.2 In this subsection, let R be a 2-dimensional integrally closed complete local noetherian domain.

(A) An R-algebra Λ is called *tame R-order* if the following conditions (1), (2) are satisfied.

(1) Λ is a finitely generated reflexive R-module.

(2) $\Lambda_{\mathfrak{p}}$ is a hereditary $R_{\mathfrak{p}}$-order for any height 1 prime of R.

(B) Let Λ be a tame R-order. We denote by ref Λ the category of Λ-modules which are reflexive as R-module.

Corollary. ([I2]) *Let k be a field, $R = k[[x, y]]$, Λ a tame R-order and Λ' a tame overorder of Λ. Assume that $\mathcal{C} :=$ ref Λ/ref Λ' has an additive generator M. Then there exists a complete discrete valuation ring R' such that $\mathcal{C}(M, M)$ is a generalized Auslander order over R'.*

5. A generalization of overorders

This section is, in a sense, a central part of this paper. Results in this section are used to prove the lemmas in §6. Throughout this section, subcategories are assumed to be full and closed under isomorphisms, direct sums and direct summands.

5.1 We introduce the concept of well behaved subcategories. By 5.2, this can be seen as a generalization of 'category of lattices over some overorder'.

Definition. Let $\mathcal{C}$ have exact ASS. For a subcategory $\mathcal{C}'$ of $\mathcal{C}$, we define $\langle \mathcal{C}' \rangle$ as the minimal subcategory of $\mathcal{C}$ which satisfies the following conditions.

(1) $\langle \mathcal{C}' \rangle \supseteq \mathcal{C}'$.

(2) If $0 \longrightarrow \mathcal{C}(\ , X) \longrightarrow \mathcal{C}(\ , Y) \longrightarrow \mathcal{C}(\ , Z)$ is exact and $Y \in \mathcal{C}'$, then $X \in \mathcal{C}'$.

(3) If $0 \longrightarrow \mathcal{C}(X,\ ) \longrightarrow \mathcal{C}(Y,\ ) \longrightarrow \mathcal{C}(Z,\ )$ is exact and $Y \in \mathcal{C}'$, then $X \in \mathcal{C}'$.

A subcategory $\mathcal{C}'$ of $\mathcal{C}$ is called *well behaved* if $\langle \mathcal{C}' \rangle = \mathcal{C}'$ and $\mathcal{C}/\mathcal{C}'$ is artinian (2.5).

5.2 Proposition. Let Λ be an order over complete discrete valuation ring (resp. artin algebra), $\mathcal{C} = \mathrm{lat}\Lambda$ (resp. mod Λ) and $\mathcal{C}'$ a subcategory of $\mathcal{C}$. Then the following conditions are equivalent.

(1) $\langle \mathcal{C}' \rangle = \mathcal{C}'$.

(2) There exists an overring (resp. quotient algebra) Λ' of Λ such that $\mathcal{C}' = \mathrm{lat}\Lambda'$ (resp. $\mathcal{C}' = \mathrm{mod}\ \Lambda'$).

5.3 The most remarkable properties of well behaved categories are given by the following theorems. They obviously hold for the cases in 5.2.

Theorem. *Assume that $\mathcal{C}$ has exact ASS and $\mathcal{C}'$ is a well behaved subcategory of $\mathcal{C}$. Then $\mathcal{C}'$ also has exact ASS.*

5.3.1 For the cases in 5.2, we have a natural functor $\underset{\cdot}{(\ )} : \mathcal{C} \to \mathcal{C}'$, $\underset{\cdot}{X} = \mathrm{Hom}_\Lambda(\Lambda', X)$. We can prove that there exists a similar functor in our general cases.

Theorem. *Assume that $\mathcal{C}$ has exact ASS and $\mathcal{C}'$ is a well behaved subcategory of $\mathcal{C}$. Then there exists an functor $\underset{\cdot}{(\ )} : \mathcal{C} \longrightarrow \mathcal{C}'$ and a natural transformation $\alpha : \underset{\cdot}{(\ )} \longrightarrow 1_{\mathcal{C}}$ which satisfy the following properties.*

(1) $\underset{\cdot}{(\ )}|_{\mathcal{C}'} = 1_{\mathcal{C}'}$ *and* $\alpha_X = 1_X$ *for any* $X \in \mathcal{C}'$.

(2) $\cdot\alpha_Y : \mathcal{C}(X, \underset{\cdot}{Y}) \longrightarrow \mathcal{C}(X, Y)$ *is a bijection for any* $X \in \mathcal{C}'$ *and* $Y \in \mathcal{C}$.

(3) *For* $X \in \mathfrak{I}(\mathcal{C}')$, $X \in \mathcal{I}(\mathcal{C}')$ *if and only if there exists* $I \in \mathcal{I}(\mathcal{C})$ *such that* $X \leq \underset{\cdot}{I}$.

5.4 In [I], for the cases in 5.2, Rejection Lemma (i.e. a characterization of well behaved subcategories by the language of Auslander-Reiten quivers) is established. Here, we can generalize Rejection Lemma to our general cases.

5.4.1 Theorem. *Assume that $\mathcal{C}$ has exact ASS, $\mathcal{C}'$ is a subcategory of $\mathcal{C}$, and $\overline{\mathcal{C}} := \mathcal{C}/\mathcal{C}'$ is artinian. Then the following conditions are equivalent.*

(1) *$\mathcal{C}'$ is a well behaved subcategory of $\mathcal{C}$.*

(2) w-proj $\overline{\mathcal{C}} \subseteq \mathcal{P}(\mathcal{C})$ *and* w-inj $\overline{\mathcal{C}} \subseteq \mathcal{I}(\mathcal{C})$.

(3) (i) *For any* $X \in \mathfrak{I}(\mathcal{C}) - \mathcal{P}(\mathcal{C})$, *put* $X_1 := \theta_{\overline{\mathcal{C}}}X$, $X_2 := \theta^-_{\overline{\mathcal{C}}}\theta_{\overline{\mathcal{C}}}X - X$ *and* $X_n := \theta^-_{\overline{\mathcal{C}}}X_{n-1} - \tau^-_{\overline{\mathcal{C}}}X_{n-2}$ *for* $n \geq 3$. *Then for any* $n \geq 0$, $X_n \geq 0$.

(ii) *For any* $X \in \mathfrak{I}(\mathcal{C}) - \mathcal{I}(\mathcal{C})$, *put* $X_1 := \theta^-_{\overline{\mathcal{C}}}X$, $X_2 := \theta_{\overline{\mathcal{C}}}\theta^-_{\overline{\mathcal{C}}}X - X$ *and* $X_n := \theta_{\overline{\mathcal{C}}}X_{n-1} - \tau_{\overline{\mathcal{C}}}X_{n-2}$ *for* $n \geq 3$. *Then for any* $n \geq 0$, $X_n \geq 0$.

5.4.2 Theorem. *Assume that $\mathcal{C}$ has exact ASS, $\mathcal{C}'$ is a subcategory of $\mathcal{C}$, and both $\overline{\mathcal{C}} := \mathcal{C}/\mathcal{C}'$ and $\mathcal{C}/\mathcal{I}(\mathcal{C})$ are artinian. Then the following conditions are equivalent.*

(1) $\langle \mathcal{C}' \rangle = \mathcal{C}$.

(2) *For any $P \in \mathcal{P}(\mathcal{C})$ and $I \in \mathcal{I}(\mathcal{C})$, $\overline{\mathcal{C}}(P, I) = 0$.*

(3) *For any $P \in \mathcal{P}(\mathcal{C})$, put $X_0 := 0$, $X_1 := P$ and $X_n := (\theta^-{}_{\overline{\mathcal{C}}} X_{n-1} - \tau^-{}_{\overline{\mathcal{C}}} X_{n-2})_+$ for $n \geq 2$. Then $\langle X_n, I \rangle = 0$ for any $I \in \mathcal{I}(\mathcal{C})$ and $n \geq 1$.*

6. Sketch of proof of theorem 1.3

6.1 By the following lemma, we can translate the combinatorial conditions (C1), (C2), (C3) to some categorical conditions. We call this categorical condition (C). Remark that (C) is easily checked for $\mathcal{C} = \mathrm{lat}\Lambda$ where Λ is an R-order of finite representation type and R is a complete discrete valuation ring. Hence 1.3(1)$\Longrightarrow$1.3(2) is proved.

Lemma. *Assume that $\mathcal{C}$ has ASS, $\mathfrak{I}(\mathcal{C})$ is a finite set and $\mathcal{R}^\infty\mathcal{C}(\ ,\ ) = 0$. Then the following (1), (2) are equivalent. In this case, $\mathcal{C}$ has exact ASS.*

(1) *$Q := \mathfrak{A}(\mathcal{C})$ satisfies (C1), (C2) and (C3).*

(2) (i) *For any $X \in \mathcal{I}(\mathcal{C})$, $0 \longrightarrow \mathcal{C}(\ , X) \xrightarrow{\cdot\nu} \mathcal{C}(\ , \theta^- X) \longrightarrow M \longrightarrow 0$ is exact where M is finite length.*

(ii) *For any $X \in \mathcal{P}(\mathcal{C})$, $0 \longrightarrow \mathcal{C}(X,\ ) \xrightarrow{\mu\cdot} \mathcal{C}(\theta X,\ ) \longrightarrow M \longrightarrow 0$ is exact where M is finite length.*

(iii) *For any non-zero $X \in \mathcal{C}$, there exists $I \in \mathcal{I}(\mathcal{C})$ such that $\mathcal{C}(X, I) \neq 0$.*

(iv) *$\mathcal{C}/\mathcal{I}(\mathcal{C})$ is artinian.*

6.2 Maximal case Let $\mathcal{C}$ have exact ASS. $\mathcal{C}$ is called *maximal* if $\mathcal{C}$ does not have a well behaved subcategory except 0 and $\mathcal{C}$. This is a generalization of 'categories of maximal overorders'. Similar to the cases of overorders, the following property holds.

6.2.1 Lemma. *Assume that $\mathcal{C}$ has exact ASS and is maximal. If $\mathcal{C}$ satisfies (C), then $\mathfrak{A}(\mathcal{C})$ has the following form.*

$$\circlearrowleft \quad \circlearrowleft \quad \cdots \quad \circlearrowleft$$

6.2.2 Lemma. *Assume that a k-algebra Λ and an idempotent $e \in \Lambda$ satisfy the following conditions.*

(1) $\Lambda = \prod_{i \geq 0} \Lambda_i$, $\mathrm{rad}\Lambda = \prod_{i \geq 1} \Lambda_i$ *and $\Lambda_i \Lambda_j \subseteq \Lambda_{i+j}$ for any i, j.*

(2) *$e\Lambda e$ has ASS and $\mathfrak{A}(e\Lambda e) = \circlearrowleft$.*

(3) *$e\Lambda_0 e$ is a finite extension of k.*

Then $e\Lambda e$ is a $k[[x]]$-order and $e\Lambda e \otimes_{k[[x]]} k((x))$ is a semi-simple $k((x))$-algebra.

6.3 Reduction to well behaved subcategories

Lemma. *Assume that $\mathcal{C}$ has exact ASS and $\mathcal{C}'$ is a well behaved subcategory of $\mathcal{C}$.*

(1) *If $\mathcal{C}$ satisfies* (C), *then $\mathcal{C}'$ also satisfies* (C).

(2) *Assume that $\mathcal{C}$ satisfies* (C). *$\mathcal{C}$ has an additive generator M and $\mathcal{C}'$ has an additive generator M'. Let k be a field. If $\mathcal{C}(M,M)$ is a k-algebra and $\mathcal{C}'(M'e,M')$ is a $k[[x]]$-order, then $\mathcal{C}(M,M)$ is a $k[[x^N]]$-order for some $N>0$. Moreover, if $\mathcal{C}'(M',M')\otimes_{k[[x]]}k((x))$ is a semi-simple $k((x))$-algebra, then $\mathcal{C}(M,M)\otimes_{k[[x^N]]}k((x^N))$ is a semi-simple $k((x^N))$-algebra.*

6.4 Proof of 1.3(2)$\Longrightarrow$1.3(1) Take a finite translation quiver Q which satisfies (C1), (C2), (C3). In general, it is easily shown that for any translation quiver Q, there exists a k-modulated translation quiver $\mathcal{Q}$ such that $|\mathcal{Q}|=Q$ (3.1). Take such a $\mathcal{Q}$, put $\mathcal{C}:=\widehat{M_{\mathcal{Q}}}$ and take an additive generator M of $\mathcal{C}$. By 3.5 and 6.1, $\mathcal{C}$ satisfies (C) and has exact ASS. By 4.4, we only have to show that there exists a complete discrete valuation ring R with field of quotients K such that $\mathcal{C}(M,M)$ is an R-order and $\mathcal{C}(M,M)\otimes_R K$ is a semi-simple K-algebra.

Take a maximal well behaved subcategory $\mathcal{C}'$ of $\mathcal{C}$ and an additive generator M' of $\mathcal{C}'$. By 6.3(1), $\mathcal{C}'$ also satisfies (C). By 6.2, $\mathcal{C}'(M',M')$ is a $k[[x]]$-order and $\mathcal{C}'(M',M')\otimes_{k[[x]]}k((x))$ is a semi-simple $k((x))$-algebra. By 6.3(2), $\mathcal{C}(M,M)$ is a $k[[x^N]]$-order for some $N>0$ and $\mathcal{C}(M,M)\otimes_{k[[x^N]]}k((x^N))$ is a semi-simple $k((x^N))$-algebra. Hence 1.3(2)$\Longrightarrow$1.3(1) is proved.

6.5 Summarizing above argument, we obtain the following corollary.

Corollary. *Let k be a field and Λ a k-algebra which satisfies the following conditions* (1)–(5). *Then* Cen(Λ) *contains a subring R which is isomorphic to $k[[x]]$, and Λ becomes an R-order.*

(1) $\Lambda=\prod_{i\geq 0}\Lambda_i$, $\mathrm{rad}\Lambda=\prod_{i\geq 1}\Lambda_i$ *and* $\Lambda_i\Lambda_j\subseteq\Lambda_{i+j}$ *for any* i,j.

(2) $\dim_k\Lambda_0<\infty$.

(3) *Any simple module S satisfies* $\mathrm{Ext}^0_\Lambda(S,\Lambda)=0$ *and has the projective resolution*

$$0\longrightarrow P_2\longrightarrow P_1\longrightarrow P_0\longrightarrow S\longrightarrow 0$$

where P_0, P_1 and P_2 are finitely generated.

(i) *If* $\mathrm{pd}S\leq 1$, *then* $\mathrm{Ext}^1_\Lambda(S,\Lambda)$ *is a finite length Λ^{op}-module.*

(ii) *If* $\mathrm{pd}S=2$, *then* $\mathrm{Ext}^1_\Lambda(S,\Lambda)=0$ *and* $\mathrm{Ext}^2_\Lambda(S,\Lambda)$ *is a simple Λ^{op}-module.*

(4) *Λ^{op}-version of* (3).

(5) *Let $\mathcal{E}$ be a complete set of orthogonal primitive idempotents of Λ and put* $f:=\sum_{e\in\mathcal{E},\mathrm{pd}(\Lambda/\mathrm{rad}\Lambda e)\leq 1}e$. *Then $\Lambda/\Lambda f\Lambda$ is artinian.*

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SOME TOPICS ON MODULAR GROUP ALGEBRAS OF FINITE GROUPS

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0. Introduction and notation

In the representation theory of finite groups, especially, modular representation theory of finite groups, there are several problems and conjectures (or questions) given by, first of all, of course, R. Brauer, then by J.L. Alperin, E.C. Dade, M. Broué, L. Puig, P. Donovan and so on (see [1], [2], [4], [5], [7], [8], [9], [10], [12], [13], [14] [21], [22]). Actually, the conjectures by Alperin (Alperin weight conjecture) [4], which has been changed more precisely by Dade [13], [14], and that by Broué [8], [9], [10], [11] seem in the center of modular representation theory nowadays (see a survey [22, pp.95–96] by Linckelmann, too). In this note, however, we are not going into these conjectures. Probably, it will be treated in part in another article in these proceedings by the same author. As a matter of fact, nevertheless, we are going to consider several topics on structure of group algebras of finite groups over an algebraically closed field of prime characteristic. It should be remarked that the subjects we will treat here in the article are just only from the author's point of view, although it seems that they are still so important objectively.

Anyway, let's go into the subject. The main problem we treat throughout till the end is the following well-known problem which was announced by R. Brauer (1901–1977) who actually was a founder of modular representation theory of finite groups. Namely,

Problem 16 (R. Brauer, 1963, [7, p.145]). *Obtain classes of groups G by imposing group theoretical conditions which can also be characterized by algebra-theoretic conditions imposed on the group algebra kG, where k is a field.*

$$G \longleftrightarrow kG$$

1991 *Mathematics Subject Classification.* Primary 20C05, 20C20, 16G10, 16G60, 16S34.

Key words and phrases. block, group algebra, isomorphism problem, Morita equivalence, radical.

This paper is in a final form and no version of it will be submitted for publication elsewhere.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Because of a well-known theorem by H. Maschke, 1898 [23], we will consider group algebras over a field which has prime characteristic. So, let's start by giving some notation we need here, and they will be fixed throughout this paper.

Notation. We use the following notation.
k = an algebraically closed field of prime characteristic p
p = a prime number such that $\mathrm{char}(k) = p$
G = a finite group
kG = the group algebra of G over k
$\mathrm{Syl}_p(G)$ = the set of all Sylow p-subgroups of G
k_G = the trivial kG-module of k-dimension one
$B_0(kG)$ = the principal block (ideal) of kG
A = a finite dimensional k-algebra with unit element 1_A
$J(A)$ = the Jacobson radical of A
a module = a finitely generated left module

Perhaps, it would be better to mention on blocks of the group algeba kG. Namely, we say that A is a *block* of kG if A is an indecomposable direct summand of kG as a two-sided ideal. In this case A can be regarded also as an indecomposable left $k[G \times G]$-module where the action by $G \times G$ on A is given by

$$(g_1, g_2)a = g_1 a g_2^{-1}$$

for $g_i \in G$ and $a \in A$. Then every indecomposable kG-module M belongs to a unique block A of kG in the sense that $1_A \cdot M \neq 0$. So, in the case, M can be considered just a left A-module. Now, for any finite group G there always exists at least one simple kG-module, say, the *trivial* module which is afforded by the following trivial k-representation

$$G \longrightarrow GL_1(k) = k^\times, \qquad g \mapsto 1.$$

As in the list of notation above, we denote by k_G the trivial kG-module. Then we say that a block A of kG is the *principal block* of kG if k_G is contained in A as a left kG-module (equivalently, if k_G is an epimorphic image of A as a left kG-module). Anyhow, this particular and important block, say, the principal block of kG, is denoted here by $B_0(kG)$ as in the list of notation.

For other notation and terminology, see the standard next text books by Alperin [3], Nagao-Tsushima [26] and Thévenaz [30].

1. Isomorphism problems for kG

In this section we will discuss on isomorphism problems for *modular* group algebras kG over an algebraically closed field k of prime characteristic p. There are, of course, several papers (articles) on this subject. But, the ingredient there looks much different from ours. Therefore, we believe that it seems meaningful to give the following material we discuss. Anyway, the first result we want to describe here was the following (it seems that it was due to R. Brauer).

(1.1) Theorem. *The following two assertions are equivalent.*

(1) $\dim_k S = 1$ *for any simple kG-module S.*

(2) *G has a normal Sylow p-subgroup P, and moreover the factor group G/P is abelian.*

Remark on (1.1). It sure is well-known that G is abelian if, and only if every irreducible complex representation of G has degree one. So, (1.1) can be regarded as a generalization of this fact.

Definition. We say that A is *primary* if the factor algebra $A/J(A)$ is a simple algebra. We call G *p-nilpotent* if G has a normal p'-subgroup K such that G/K is isomorphic to a Sylow p-subgroup P of G, (and hence G is a semi-direct product of K by P). Of course, nilpotent groups G (which means that G is just a direct product of Sylow ℓ-subgroups for all primes ℓ which divides the order $|G|$ of G) are p-nilpotent, always.

It looks that the next one was also obtained early 40's by M. Osima (1912 - 1995).

(1.2) Theorem (M. Osima, 1942 [29], K. Morita, 1951 [25, Theorem 7]). *The following two assertions are equivalent.*

(1) *For any block A of kG, A is primary.*

(2) *G is a p-nilpotent group.*

Almost ten years later, as mentioned above already, K. Morita (1915 - 1995) gave the following result, which was a generalization of (1.2). K. Morita is, of course, famous because of *Morita duality* and *Morita equivalence*, by the way (see [5]). Before we state his result we need one more definition.

Definition. As a genaralized notion of p-nilpotent groups, we call G a *p-solvable group of p-length one* or just a group of *p-length one* if G has a normal p'-subgroup K such that for a Sylow p-subgroup P of G, KP is a normal subgroup of G.

(1.3) Theorem (K. Morita, 1951 [25, Theorem 6], see (1.2) and [12, §62C]). *The following three conditions are equivalent.*

(1) *For any block A of kG, it follows that* $\dim_k S = \dim_k T$ *if S and T are simple kG-modules in A.*

(1') *For any simple kG-module S in the principal block $B_0(kG)$ of kG, it follows* $\dim_k S = 1$.

(2) *G is p-solvable of p-length one (namely, G has a normal p'-subgroup K such that KP is normal in G, where $P \in \mathrm{Syl}_p(G)$), and furthermore G/KP is abelian.*

In the same paper as above, Morita gave the following result, too.

(1.4) Theorem (K. Morita, 1951 [25, Theorem 8], see [12, (62.29) Theorem and (62.26) Theorem]). *The following three conditions are equivalent.*

(1) *kG satisfies the condition* (1) *in* (1.3), *and moreover kG is a Nakayama algebra (say, a generalized uniserial algebra, that is, every projective indecomposable left kG-module has a unique composition series).*

(1') *There exist elements $a, b \in kG$ such that $J(kG) = kG{\cdot}a = b{\cdot}kG$.*

(2) *G is p-solvable of p-length one, and Sylow p-subgroups of G are cyclic.*

Remark. The equivalence (1) $\leftrightarrow$ (1') actually holds for not only the group algebras kG but also general finite dimensional k-algebras (see [25, Theorem 1] and [12, (62.26) Theorem]). Of course, nowadays, everybody knows that the group algebra kG is of finite representation type if and only if Sylow p-subgroups of G are cyclic. And it is not difficult that Nakayama algebras are of finite representation type, so that Sylow p-subgroups of G are cyclic by a well-known result obtained by D.G. Higman [17] in 1954 and F. Kasch-M. Kneser-H. Kupisch [18] in 1957. So, a part of (1.4) above is an easy consequence from it. But, let's consider that the year. It was in 1951. Namely,

(1.5) Theorem (D.G. Higman, 1954 [17], and F. Kasch-M. Kneser-H. Kupisch, 1957 [18]). *The following are equivalent.*

(1) *kG is of finite representation type, that is, there are only finitely many non-isomorphic indecomposable left kG-modules.*

(2) *Sylow p-subgroups of G are cyclic.*

The two papers above are not so long, in particular, the second one by Kasch-Kneser-Kupisch has just one or two pages. It is, however, very fundamental and important.

Now, let's skip 30 years!

Around mid 80's G. Michler prove the following interesting result which is related to the first theorem (1.1). Namely,

(1.6) Theorem (G.O. Michler, 1986 [24, Theorem 5.5]). *By making use of the classification of finite simple groups* [15], [16], *for any odd prime p, the following are equivalent.*

(1) *For any simple kG-module S, $p \nmid \dim_k S$.*

(2) *G has a normal Sylow p-subgroup.*

(1.7) Remark. For the case $p = 2$, the same thing in (1.6) holds. It was proved by T. Okuyama [27], probably, early 80's. It should be mentioned that T. Okuyama didn't use the classification of finite simple groups.

So far, we have not mentioned clearly isomorphism problems for kG. However, we have already treated it actually. Namely,

(1.8) Theorem. *Let G and H be finite groups. In each case* (1.1) - (1.7), *if we assume that $kG \cong kH$ as k-algebras and that G satisfies the condition* (2), *then H satisfies the condition* (1) *as well.*

(1.9) Corollary to (1.6) and (1.7). *Assume that $kG \cong kH$ as k-algebras and that G is of p-length one. Then H is of p-length one, too.*

Remark on (1.9). Because of (1.6), the above (1.9) for p odd depends on the classification of finite simple groups [15], [16].

Then, let's conclude this section by giving the following problem.

(1.10) Problem (see (1.4)). Give a necessary and sufficient condition on G under which kG is a Nakayama algebra. That is,

$$(1)\ kG \text{ is a Nakayama algebra } \leftrightarrow\ (2)\ G\ ?$$

Remark on (1.10). The author was told very roughly that the above (1.9) can be proved by making use of the classification of finite simple group.

2. Loewy length of projectives for kG

In this section we discuss on the Loewy length of the projective indecomposable kG-module corresponding to the trivial kG-module k_G. In the proceedings of this meeting (which was held in 1993), the author actually discussed on the same topic more or less, and he gave there a conjecture (question). And this is, as a matter of fact, true. That is to say, the next year in 1994 the author could affirmatively solve the conjecture given in 1993, which will be mentioned in the following (2.4).

So, first of all, we have to give notation here.

Notation. Throughout this section we use the following.

$P(k_G)$ = the projective cover of the trivial kG-module k_G
namely, $P(k_G)/J(kG)\cdot P(k_G) \cong k_G$ as left kG-modules

$j = j(P(k_G))$ the Loewy length of $P(k_G)$,
namely, j is the least positive integer i such that $J(kG)^i\cdot P(k_G) = 0$.

In this section we consider the following problem.

Problem. Determine G when j is given, especially for small j.

We start here assume only that k is an algebraically closed field, though we assume that k has prime characteristic p in a minute. Then, the starting point of the subject discussed here in §2 is, again, the following Maschke's theorem.

(2.1) Theorem (H. Maschke, 1898 [23]). *The following are equivalent.*
(1) $j = 1$.
(2) char(k) = 0, *or* char(k) = $p > 0$ *and* $p \nmid |G|$.

Because of this, we can assume here again that k has prime characteristic p. Namely,

Assumption and notation. We use also the following notation.
p = characteristic of k
P = a Sylow p-subgroup of G
Σ_n = the symmetric group on n letters
C_n = the cyclic group of order n

The next result was, firstly, announced (published) by D.A.R. Wallace.

(2.2) Theorem (D.A.R. Wallace, 1962 [31]). *The following are equivalent.*
(1) $j = 2$.
(2) $p = 2$ *and Sylow 2-subgroups of G are cyclic of order* 2.

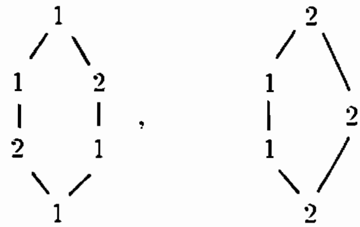
Then, of course, we might be interested in the next step. Unfortunately, however, finite groups which satisfy $j = 3$ have not been determined yet as far as we know. T. Okuyama, however, gave the following nice result which is a partial answer to the problem for $j = 3$ under the assumption $p = 2$. Namely,

(2.3) Theorem (T. Okuyama, 1986 [28, Theorem 2]). *Assume that $p = 2$ and $j = 3$. Then, Sylow 2-subgroups of G are dihedral (can be an elementary abelian group $C_2 \times C_2$ of order 4).*

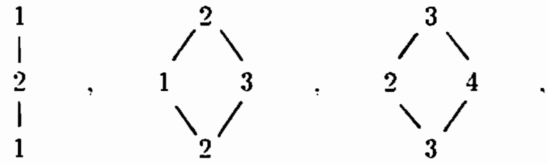
On the theorem (2.3) above, it should presumably be noticed the following a couple of remarks. That is,

(2.4) Remark on (2.3) (see T. Okuyama [28]).

(i) If $p = 2$ and $G = \Sigma_4$, then $j = 4$. Actually, in this case, the structure of $P(k_G)$ and $P(2)$ are the following, where $P(2)$ is the projective cover of the another simple kG-module, which has k-dimension 2. In the following we write 1 for k_G.



(ii) If p is odd and $G = \Sigma_p$ then the projective indecomposable kG-modules have the following structure.



$\cdots$ stops in finite steps

Hence, $j = 3$ in this case.

Now, let's go to the next step, say, the case $j = 4$, though the case $j = 3$ has not been completely settled. Actually, as the author wrote in the introduction of [20], the condition $j = 4$ could be stronger. As we have already conjectured in1993 in [19], we fortunately could give an affirmative answer. Namely,

(2.5) Theorem (S. Koshitani, 1996 [20, Corollary]). *If $j = 4$, then $p = 2$.*

(2.6) Remark on (2.5). We did not use the classification of finite simple groups to prove the above (2.5). It is just a sort of feeling. But, as far as we feel, the reason why the condition $j = 4$ implies $p = 2$ would be the fact that the group algebras kG of finite groups G over a field k are symmetric algebras.

As in §1, we finish this section by giving a few problems (questions) we are interested in. That is,

(2.7) Problems.

(i) Determine finite groups G which satisfy $j = 3$.

(ii) Determine finite groups G which satisfy $j = 4$. Thanks to (2.5), we have to think of this only in the case that $p = 2$.

(2.8) Question. Is it true that the condition j is even would imply that $p = 2$? This is true at least for the case $j = 2, 4$. However, there are, of course, infinitely many even numbers still. What can we do then?

Acknowledgements. The author would like to thank Professor Y. Iwanaga for giving him an opportunity to give two talks in the meeting, the 30th Symposium on Ring Theory and Representation Theory.

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ON p-BLOCKS OF FINITE GROUPS

SHIGEO KOSHITANI

0. Introduction and notation

In this note we try to give two results we got recently, and their atmosphere, on p-blocks of finite groups in modulrar representation theory. These two results are not going to be accompanied with their proofs, because one of them will be published in other journal with complete proofs [12] and the other will probably be submitted somewhere else. We will try, however, to give the readers to understand what's going on there, anyhow.

In representation of finite gorups, or even of finite dimensional algebras, a notion *Morita equivalence* is so important. As M. Broué says in his article [5, Remark p.10], Morita equivalence between blocks of finite groups does not occur so frequently. Especially in oreder to solve Broué's conjecture on derived equivalence between two blocks with the same abelian defect groups. As Broué writes there, Morita equivalence seems too strong practically, (see Broué's paper [3], [4], [5], [6]). However, here in this article, we will give two results which are both on Morita equivalence between two blocks of finite groups. In particular, the second one implies an affirmative answer to what is called *Broué's conjecture* on derived equivalent blocks with abelian defect groups for a type of finite Chevalley groups, say $\mathrm{PSU}(3, q^2)$ in non-defining characteristic case, which contains infinitely many simple groups.

Throughout this note we use the following notation.

Notation. We use the following notation.
p = a prime number
k = an algebraically closed field of prime characteristic p
G = a finite group
P = a Sylow p-subgroup of G
$\mathrm{Syl}_p(G)$ = the set of all Sylow p-subgroups of G
kG = the group algebra of G over k
k_G = the tivial kG-module of k-dimension one

1991 *Mathematics Subject Classification.* Primary 20C05, 20C20, 16G10, 16G60, 16S34.

Key words and phrases. block, group algebra, Morita equivalence, Broué conjecture, derived equivalence.

A part of this paper will be published in detailed version elsewhere, and a part of this paper will prabably submitted for publication somewhere else with complete proofs.

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$B_0(kG)$ = the principal block (ideal) of kG
A = a finite dimensional k-algebra with unit element 1_A
$J(A)$ = the Jacobson radical of A
a module = a finitely generated left module
$\mathrm{Irr}(B)$ = the set of all irreducible ordinary characters of a block B
$\mathrm{IBr}(B)$ = the set of all non-isomorphic simple kG-modules in a block B of kG
$M_n(A)$ = the full matrix algebra over A of degree n for $n \in \mathbb{N}$

For other notation and terminology see the books of Alperin [2], Nagao-Tsushima [21] and Thévenaz [23].

1. Isomorphic blocks

In this section we discuss on isomorphic blocks of finite groups, which is much stronger than even Morita equivalent blocks, and however, a fundamental and a starting point on the subject we will discuss.

When we consider blocks of finite groups, the following should be placed at first. Namely,

(1.1) Theorem (see [7, (62.21) Theorem]). *Let A be a block of kG. Then A is of finite representation type (that is, there are only finitely many non-isomorphic indecomposable left kG-modules in A) if, and only if defect groups of A are cyclic.*

Now, let's begin with giving the following two old results.

(1.2) Proposition. *If $G = P$ is a finite p-group, then kG has only one simple module, say k_G, so that kG is a local algebra and $kG = B_0(kG)$, namely, kG itself is a block.*

For the proof of (1.2), see, for example, [21, Chapter 5, Problem 2.10] or [2, §3 Corollary 3 (p.14)].

(1.3) Theorem (K. Morita [20, Theorem 2]). *Assume that G is a p-nilpotent group with Sylow p-subgroup P. Then*

$$B_0(kG) \cong kP \qquad \text{as } k\text{-algebras.}$$

Now, as a kind of generalization of Morita's theorem (1.3), I.M. Isaacs and S.D. Smith gave the following result, which characterizes finite groups of p-length one in terms of representation theory. (see [17, (1.3)] for the definition). Namely,

(1.4) Theorem (I.M. Isaacs and S.D. Smith, 1976, [14] ;and S. Koshitani 1981 [15], 1990 [16]). *Assume that $P \in \mathrm{Syl}_p(G)$, and let $H = N_G(P)$, $A = B_0(kG)$ and $B = B_0(kH)$. Then the following four conditions are equivalent.*

(1) *G is p-solvable of p-length one.*

(2) $\mathrm{Irr}(A) \longrightarrow \mathrm{Irr}(B)$, $\chi \mapsto \chi_{\downarrow H}$ *(restriction) gives a bijection.*

(3) $\mathrm{IBr}(A) \longrightarrow \mathrm{IBr}(B)$, $S \mapsto S_{\downarrow H}$ *(restriction) gives a bijection.*

(4) $B \longrightarrow A$, $b \mapsto b \cdot 1_A$ *gives an isomorphism of k-algebas (so that $1_B \cdot 1_A = 1_A$).*

It is, of course, natural to consider relations between the principal blocks $B_0(kG)$ and $B_0(kH)$ in the case that H is a normal subgroup of G. There are actually results, due to Alperin and Dade, which are on isomorphic principal blocks of a finite group and its normal subgroup.

(1.5) Theorem (J.L. Alperin 1976, [1] and E.C. Dade 1977 [8]). *Let H be a normal subgroup of G satisfying $p \not| |G : H|$. Thus the Sylow p-subgroup P of G is also that of H. Assume, furthermore, that G has a factorization $G = H \cdot C_G(P)$ (note that the condition $G = H \cdot N_G(P)$ always holds by the Frattini argument). Then, the principal blocks $B_0(kG)$ and $B_0(kH)$ of kG and kH, respectively, are isomorphic as k-algebras. More precisely, it follows the following. Let $A = B_0(kG)$, and $B = B_0(kH)$. Then*

$$B \longrightarrow A, \qquad b \mapsto b \cdot 1_A$$

gives an isomorphism of k-algebras and

$$\mathrm{Irr}(A) \longrightarrow \mathrm{Irr}(B), \qquad \chi \mapsto \chi_{\downarrow H} \quad (\text{restriction})$$

is a bijection.

2. Naturally Morita equivalent blocks

In (1.4) and (1.5) in the last section we gave two results on isomorphic principal blocks of kG and kH for a subgroup H of G, in the cases that $H = N_G(P)$ and that $H \lhd G$. In this section, we actually treat these two cases together at the same time, and even under weakened situation that Morita equivalent blocks instead of isomorphic blocks.

Anyway the motivation here was, as a matter of fact, given by B. Külshammer [19], where he gave a new, but very natural (canonical) notion he calls *naturally Morita equivalent* blocks. So, let's start the second section by giving the definition of it. It should be, however, noticed that the notion we state below is a little bit different from Külshammer's because he gave it for a normal subgroup H of G, while we consider it for any subgroup H of G. That's the difference. Of course, they are the same once we assume that the subgroup H we consider is normal in G.

Definition(B. Külshammer, 1990, [19]). Let H be any subgroup of G (not necessarily normal in G), and let A and B be blocks of kG and kH, respectively. We assume that n is a positive integer. Then we say that A and B are *naturally Morita equivalent of degree n* when the next conditions hold:
There exists a k-subalgebra S of A which satisfies

$$1_A \in S \cong M_n(k), \qquad \text{as } k\text{-algebras}$$

$$B \bigotimes\nolimits_k S \longrightarrow A, \qquad b \otimes s \mapsto bs$$

gives an isomorphism of k-algebras (so that $1_B \cdot 1_A = 1_A$).

Remark. It should be noted that if two blocks A and B are naturally Morita equivalent of degree one, then it means that A and B are isomorphic as k-algebras via restriction just as in (1.4) and (1.5).

In the above situation, say, if two blocks A and B are naturally Morita equivalent of degree n, then it holds that $A \cong M_n(B)$, so that A and B are Morita equivalent, of course.

There had been, actually, one result by B. Külshammer [18], which was a generalization of a theorem on isomorphic principal blocks due to Alperin and Dade (see (1.5)) and, at the same time, which was a motivation of having notion "naturally Morita equivalent" blocks. Namely,

(2.1) Theorem (B. Külshammer, 1984 [18], see also M.E. Harris, 1994 [10]). *Let H be a normal subgroup of G and let $Q \in \mathrm{Syl}_p(H)$ and $B = B_0(kH)$. Suppose that G has a factorization $G = H \cdot C_G(Q)$. Then, for any block A of kG which satisfies that $1_A \cdot 1_B \neq 0$ (which means A covers B) and that Q is a defect group of A as well, there is an isomorphisms*

$$B \bigotimes_k \pi(A) \longrightarrow A \qquad \text{and} \qquad \pi(A) \cong M_n(k)$$

of k-algebras for some positive integer n, where π is a k-algebra-epimorphism $\pi : kG \to k[G/H]$ induced by the canonical epimorphism $G \to G/H$. In particular, $A \cong M_n(B)$, which implies that A and B are naturally Morita equivalent of degree n.

Recently, A. Hida and the author got results [12], which are mainly on generalization of ones by B. Külshammer [19]. Actually, our joint paper [12] have several theorems. So, readers who are interested in our results stated below, please look at it. It has much more things there.

(2.2) Theorem (A. Hida and S. Koshitani, 1997 [12, (2.10) Theorem, (3.6) Theorem]. See also B. Külshammer 1990 [19, Theorem 8 and Proposition 10]). *Let H be any subgroup of G and let A and B, respectively, be blocks of kG and kH. Let D be a defect group of B. Assume that A and B are naturally Morita equivalent of degree n. (Then, it follows that D is a defect group of A, too). So, let $N = N_G(D)$ and $M = N_H(D)$. Then it is well-known that there are two blocks $\tilde{A}$ and $\tilde{B}$ of kN and kM, respectively, such that A and $\tilde{A}$ correspond via the Brauer correspondence with respect to D, and so are B and $\tilde{B}$.*

Then, The blocks $\tilde{A}$ and $\tilde{B}$ are also naturally Morita equivalent of degree n.

Remark. The above was proved by B. Külshammer [19] in the case that H is normal in G. It should be, presumably, noted that the above was related to a theorem due to M.E. Harris and R. Knörr, 1985 [11]. The converse of (2.2) holds as well, roughly speaking, if H is normal in G. One more thing. Everything mentiond so far in this section here (so that in §1, as well) can hold for a suitable complete discrete valuation ring $\mathcal{O}$ not only for k (see [12]).

3. Morita equivalent 3-blocks with abelian defect groups

In this section we consider very particular cases and treat very special cocrete examples. It looks, however, important to investigate such concrete examples because we have had only a few (or a little bit more, maybe, but not many, anyway), where so-called Broué's conjecture on derived equivalent blocks of finite groups holds. Last year 1996, T. Okuyama actually announced very nice and interesting results. He gave several examles where Broué's conjecture holds [22]. We almost forgot, by the way, to tell what the Broué's conjecture is. That is,

Broué's conjecture([3, 6.2 Question], [5, 4.9 Conjecture]). *Let A and B be principal blocks of kG and $k[N_G(P)]$, respectively, where $P \in \mathrm{Syl}_p(G)$. If P is abelian, then A and B would be derived equivalent.*

We are not going into detail about derived equivalent blocks (see Broué's articles [3], [4], [5], [6]). The point here is just that Morita equivalence implies derived equivalence.

In the remainder of this section we assume the following.

Assumption and remarks. We assume that $\mathrm{char}(k) = p = 3$. Let $G = \mathrm{PSU}(3, q^2)$, the projective special unitary group of degree 3 over a finite field $\mathrm{GF}(q^2)$ of q^2 elements, where q is a power of a prime such that $q \equiv 2$ or $5 \pmod 9$. In this case, a Sylow 3-subgroup P of G is elementary abelian of order 9, say $C_3 \times C_3$. Let $H = N_G(P)$, the normalizer of P in G. Then, it is known that $H \cong \mathrm{PSU}(3, 2^2) \cong PQ_8$, the semi-direct product of P by the quaternion group of order 8 (see [13, II §10]). Finally, let $A = B_0(kG)$ and $B = B_0(kH) = kH$, the principal blocks of kG and kH, respectively. It should be noted that G is a simple group if $q > 2$, and that there are infinitely many such q's by Dirichlet's theorem.

It had been known from a result by M. Geck [9, pp.571–573] that the principal block A of G has a decomposition matrix and a Cartan matrix for a prime 3, which are independent from q if q satisfies the above condition. So, it could have been imagined that for any such q the block A would have the same structure, say, uniquely determined up to Morita equivalence. Actually, this is true! It is a recent joint work with N. Kunugi, which is the following.

(3.1) Theorem (N. Kunugi and S. Koshitani, 1997). *For any power q of a prime which satisfies $q \equiv 2$ or $5 \pmod 9$, the principal 3-block $A = B_0(kG)$ of the projective special unitary group $G = PSU(3, q^2)$ is determined uniquely, up to Morita equivalence. Therefore, the two principal blocks $A = B_0(kG)$ and $B = B_0(kH) = kH$ are Morita equivalent, where $H = N_G(P) = PSU(3, 2^2)$, $P \in \mathrm{Syl}_3(G)$ (so that $P \cong C_3 \times C_3$).*

(3.2) Corollary to (3.1). *Broué's conjecture holds for the case $p = 3$ and $G = PSU(3, q^2)$ when $q \equiv 2$ or $5 \pmod 9$.*

Remark. (3.1) (so that (3.2) as well) can hold for a suitable complete discrete valuation ring $\mathcal{O}$ not only for k.

Acknowledgements. The author would like to thank Professor Y. Iwanaga for giving him an opportunity to give two talks in the meeting, the 30th Symposium on Ring Theory and Representation Theory.

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KAC-MOODY GROUPS AND SOME GAUSS DECOMPOSITIONS

JUN MORITA

ABSTRACT. We will give an introductory explanation of a paper "On some Gauss decomposition of a Kac-Moody group" by E. Plotkin and the author. In such a paper, an axiomatic approach to a Gauss decomposition of a Kac-Moody group has been given, and a certain prescribed version has been discussed (cf. [13]).

1. Introduction

In 1968, the so-called Kac-Moody theory was born. V. G. Kac and R. V. Moody have independently constructed the theory of a certain class of infinite dimensional Lie algebras (cf. [5], [9]). The year 1998 is the 30th anniversary of the Kac-Moody theory. We shall celebrate it! (In another sense we must remember the year 1998 as the year of Nagano Olympic Game.) And also the theory of associated Kac-Moody groups has been studied. It contains the theory of semisimple algebraic groups. If we consider a Kac-Moody group over a commutative ring, then it contains the theory of Chevalley groups. It is very interesting to study these objects comparing the infinite dimensional case to the finite dimensional case. Lots of theorems in the finite dimensional case have been generalized to those in the infinite dimensional case. However, there are many open questions about infinite dimensional Kac-Moody groups. One of the most difficult questions is to study the simplicity of a Kac-Moody group over a field in the non-affine case. We need much more explicite information about the structure of Kac-Moody groups. In the paper [13], we have begun to study a certain Gauss decomposition of a Kac-Moody group, which might be helpful to get an explicite group structure of a Kac-Moody group. To do so, recent successive papers [1], [2], [3], [4] by E. W. Ellers and N. Gordeev gave us a new shed of light. They gave a Gauss decomposition (with a prescribed torus element) of a simple group of Lie type. And also they discussed an application to the O. Ore conjecture and the J. Thompson conjecture (cf. [4], [14]). To discuss a Gauss decomposition in our situation, we will give an axiom of a triangular system. This is a general setting. But mainly we are interested in an application to Kac-Moody groups. Then we will deal with a Gauss decomposition (with a prescribed torus element) of an infinite dimensional Kac-Moody group of rank two.

In Section 2, we will give a quick review of Kac-Moody groups. We will present an axiom of a triangular system in Section 3. And, in Section 4, we will study a Gauss decomposition with a prescribed torus element in the rank two case.

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2. Kac-Moody Lie algebras and Kac-Moody groups

We will review the definitions of Kac-Moody Lie algebras, root systems and Kac-Moody groups (cf. [5], [6], [7], [9], [10], [11], [15], [17]).

An $n \times n$ integral matrix $A = (a_{ij})_{1 \le i,j \le n}$ is called a generalized Cartan matrix if $a_{ii} = 2$ $(1 \le i \le n)$, $a_{ij} \le 0$ $(1 \le i \ne j \le n)$, $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$ $(1 \le i,j \le n)$. A triplet $(\mathfrak{h}, \Pi, \Pi^\vee)$ is called a realization of A if $\mathfrak{h}$ is a vector space over $\mathbb{C}$ of dimension $2n - \mathrm{rank}(A)$; $\Pi = \{ \alpha_1, \cdots, \alpha_n \}$ is a set of n linearly independent elements of $\mathfrak{h}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$; $\Pi^\vee = \{ h_1, \cdots, h_n \}$ is a set of n linearly independent elements of $\mathfrak{h}$; and $\alpha_i(h_j) = a_{ji}$ for $1 \le i,j \le n$. Let $\mathfrak{g}$ be the Lie algebra over $\mathbb{C}$ generated by the so-called Cartan subalgebra $\mathfrak{h}$ and the so-called Chevalley generators $e_1, \cdots, e_n$ and $f_1, \cdots, f_n$ with the following defining relations:

$$[h, h'] = 0 \ (h, h' \in \mathfrak{h}), \quad [e_i, f_j] - \delta_{ij} h_i = 0 \ (1 \le i,j \le n),$$
$$[h, e_i] - \alpha_i(h) e_i = [h, f_i] + \alpha_i(h) f_i = 0 \ (h \in \mathfrak{h}, 1 \le i \le n),$$
$$(\mathrm{ad}\ e_i)^{n(i,j)} e_j = (\mathrm{ad}\ f_i)^{n(i,j)} f_j = 0 \ (1 \le i \ne j \le n),$$

where $n(i,j) = -a_{ij} + 1$. This Lie algebra is called a Kac-Moody Lie algebra.

Let w_i be the involutive automorphism of $\mathfrak{h}^*$ defined by $w_i(\mu) = \mu - \mu(h_i)\alpha_i$ for all $\mu \in \mathfrak{h}^*$. The subgroup of $\mathrm{GL}(\mathfrak{h}^*)$ generated by $w_1, \cdots, w_n$ is called the Weyl group and denoted by W. Let ω be the involutive automorphism of the Lie algebra $\mathfrak{g}$ defined by $\omega(e_i) = -f_i$, $\omega(f_i) = -e_i$ and $\omega(h) = -h$ for all $1 \le i \le n$ and $h \in \mathfrak{h}$. Then ω is called the Chevalley involution.

Under the adjoint action, $\mathfrak{h}$ is diagonalizable on $\mathfrak{g}$, that is, $\mathfrak{g} = \oplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}^\alpha$, where $\mathfrak{g}^\alpha = \{ x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h} \}$. Let $\Delta = \{ \alpha \in \mathfrak{h}^* \mid \mathfrak{g}^\alpha \ne 0 \}$. We call Δ the root system of $\mathfrak{g}$, and Π the set of simple roots of $\mathfrak{g}$. We see that $\mathfrak{g}^{\alpha_i} = \mathbb{C}e_i$, $\mathfrak{g}^{-\alpha_i} = \mathbb{C}f_i$ and $\mathfrak{g}^0 = \mathfrak{h}$, hence $\{0\} \cup \{\pm\alpha_i\}_{i=1}^n \subset \Delta$. Let $\Delta_+ = \Delta \cap (\sum_{i=1}^n \mathbb{Z}_{\ge 0}\, \alpha_i \setminus \{0\})$, the set of positive roots, and $\Delta_- = -\Delta_+$, the set of negative roots. Then $\Delta = \Delta_+ \cup \{0\} \cup \Delta_-$. One sees that Δ is W-stable. Therefore, we can define $\Delta^{\mathrm{re}} = \bigcup_{i=1}^n W\alpha_i$, the set of real roots, as a subset of Δ. For each $\alpha \in \Delta$, we obtain $\alpha \in \Delta^{\mathrm{re}} \Leftrightarrow \mathbb{Z}\alpha \cap \Delta = \{0, \pm\alpha\}$. Furthermore, $\dim \mathfrak{g}^\alpha = 1$ if $\alpha \in \Delta^{\mathrm{re}}$. For each $\alpha = w\alpha_i \in \Delta^{\mathrm{re}}$ with $w \in W$ and $\alpha_i \in \Pi$, we put $h_\alpha = wh_i$ by the contragradient action of w on $\mathfrak{h}$. The h_α are well-defined.

For each $\alpha \in \Delta^{\mathrm{re}}$, a pair $(e_\alpha, e_{-\alpha}) \in \mathfrak{g}^\alpha \times \mathfrak{g}^{-\alpha}$ is called a Chevalley pair for α if $[e_\alpha, e_{-\alpha}] = h_\alpha$ and $\omega(e_\alpha) + e_{-\alpha} = 0$. There are presicely two Chevalley pairs for each $\alpha \in \Delta^{\mathrm{re}}$. If one is $(e_\alpha, e_{-\alpha})$, then $(-e_\alpha, -e_{-\alpha})$ is the other. We choose and fix a Chevalley pair for each $\alpha \in \Delta_+^{\mathrm{re}} = \Delta_+ \cap \Delta^{\mathrm{re}}$ with $e_{\alpha_i} = e_i$ and $e_{-\alpha_i} = f_i$ for $1 \le i \le n$. Then we obtain the set $C = \{ e_\alpha \mid \alpha \in \Delta^{\mathrm{re}} \}$, which is called a Chevalley basis for Δ^{re}.

Now we define the numbers $\eta_{\alpha\beta}$ and $N_{\alpha\beta}$ by the following:

$$(\exp \mathrm{ad}\ e_\alpha)(\exp - \mathrm{ad}\ e_{-\alpha})(\exp \mathrm{ad}\ e_\alpha) e_\beta = \eta_{\alpha\beta} e_{\beta'}$$

for all $\alpha, \beta \in \Delta^{\mathrm{re}}$, and

$$[e_\alpha, e_\beta] = N_{\alpha\beta} e_{\alpha+\beta}$$

for all $\alpha, \beta \in \Delta^{re}$ with $\alpha + \beta \in \Delta^{re}$, where $\beta' = \beta - \beta(h_\alpha)\alpha \in \Delta^{re}$. Then, $\eta_{\alpha\beta} = \pm 1$ and $N_{\alpha\beta} = \pm(p+1)$, where p is the largest integer satisfying $\beta - p\alpha \in \Delta^{re}$.

A $\mathfrak{g}$-module M is called integrable if $M = \oplus_{\mu \in \mathfrak{h}^*} M^\mu$ and each e_α is locally nilpotent on M, where $M^\mu = \{ v \in M \mid hv = \mu(h)v \text{ for all } h \in \mathfrak{h} \}$ and $\alpha \in \Delta^{re}$. A basis of M is called standard if it consists of the bases of the M^μ and its $\mathbb{Z}$-span is stable under $e_\alpha^{(m)}$ for all $\alpha \in \Delta^{re}$ and $m \in \mathbb{Z}_{\geq 0}$, where

$$e_\alpha^{(m)} = \frac{e_\alpha^m}{m!}.$$

We take an integral $\mathfrak{g}$-module M with a standard basis. Let $M_\mathbb{Z}$ be the $\mathbb{Z}$-span of this standard basis in M. For a commutative ring, R, with 1, we put $M(R) = R \otimes M_\mathbb{Z}$. Then, for each $\alpha \in \Delta^{re}$ and $t \in R$, we can consider the R-linear operator $x_\alpha(t)$ on $M(R)$ defined by

$$x_\alpha(t)(r \otimes v) = \sum_{m=0}^{\infty} t^m r \otimes e_\alpha^{(m)} v$$

for all $r \in R$ and $v \in M_\mathbb{Z}$. Since $x_\alpha(-t)$ is the inverse of $x_\alpha(t)$, the operator $x_\alpha(t)$ lies in $\mathrm{GL}(M(R))$. Now we define the associated (standard or elementary) Kac-Moody group $G_M(A, R)$ as the subgroup of $\mathrm{GL}(M(R))$ generated by $x_\alpha(t)$ for all $\alpha \in \Delta^{re}$ and $t \in R$. Then, in $G_M(A, R)$, we obtain the following relations (cf. [12], [16], [17]):

(A) $x_\alpha(s)x_\alpha(t) = x_\alpha(s+t)$,

(B1) $[x_\alpha(s), x_\beta(t)] = 1$
if $Q_{\alpha\beta} = \emptyset$,

(B2) $[x_\alpha(s), x_\beta(t)] = x_{\alpha+\beta}(N_{\alpha\beta}st)$
if $Q_{\alpha\beta} = \{ \alpha + \beta \} \subset \Delta^{re}$,

(B3) $[x_\alpha(s), x_\beta(t)] = x_{\alpha+\beta}(\pm st)x_{\alpha+2\beta}(\pm st^2)$
if $Q_{\alpha\beta} = \{ \alpha + \beta, \alpha + 2\beta \} \subset \Delta^{re}$,

(B4) $[x_\alpha(s), x_\beta(t)] = x_{\alpha+\beta}(\pm 2st)x_{2\alpha+\beta}(\pm 3s^2t)x_{\alpha+2\beta}(\pm 3st^2)$
if $Q_{\alpha\beta} = \{ \alpha + \beta, 2\alpha + \beta, \alpha + 2\beta \} \subset \Delta^{re}$,

(B5) $[x_\alpha(s), x_\beta(t)] = x_{\alpha+\beta}(\pm st)x_{2\alpha+\beta}(\pm s^2t)x_{3\alpha+\beta}(\pm s^3t)x_{3\alpha+2\beta}(\pm 2s^3t^2)$
if $Q_{\alpha\beta} = \{ \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta \} \subset \Delta^{re}$,

(B') $w_\alpha(u)x_\beta(t)w_\alpha(-u) = x_{\beta'}(\eta_{\alpha\beta}u't)$,

(C) $h_\alpha(u)h_\alpha(v) = h_\alpha(uv)$,

where $s, t \in R$, $\alpha \in \Delta^{re}$, $Q_{\alpha\beta} = (\mathbb{Z}_{>0}\alpha + \mathbb{Z}_{>0}\beta) \cap \Delta$, $\beta' = \beta - \beta(h_\alpha)\alpha$, $u' = u^{-\beta(h_\alpha)}$, and $w_\alpha(u) = x_\alpha(u)x_{-\alpha}(-u^{-1})x_\alpha(u)$ and $h_\alpha(u) = w_\alpha(u)w_\alpha(-1)$ with u in the multiplicative group, $R^\times$, of R.

There are several ways to construct Kac-Moody groups - for example, Marcuson-type Kac-Moody groups (cf. [8]) or Tits-type Kac-Moody groups (cf. [17]). We take here standard (or elemantary) Kac-Moody groups, which are just corresponding to the elementary subgroups of Chevalley groups over commutative rings in finite dimensional case [16].

3. Triangular systems

Here we call $(G, U, T, V, \{\phi_1, \cdots, \phi_n\})$ a triangular system (or a Gauss system) if

(1) G is a group, and $U, T, V \leq G$ are subgroups.
(2) $\phi_i : \mathrm{SL}_2(K) \longrightarrow G$ is a group homomorphism of SL_2 over a field K into G with

$$\text{(a)}\quad \phi_i\left\{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mid a \in K\right\} = U_i \leq U;$$

$$\text{(b)}\quad \phi_i\left\{\begin{pmatrix} t & 0 \\ 0 & \frac{1}{t} \end{pmatrix} \mid t \in K^\times\right\} = T_i \leq T;$$

$$\text{(c)}\quad \phi_i\left\{\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \mid a \in K\right\} = V_i \leq V,$$

(3) $G = \langle U, V_1, \cdots, V_n \rangle$, and $TU_i = U_iT$ for $1 \leq i \leq n$,
(4) there exist the subgroups of G called U_i' and V_i' for $1 \leq i \leq n$ such that

$$U = U_i'U_i = U_iU_i',$$
$$V = V_i'V_i = V_iV_i',$$
$$V_iU_i' = U_i'V_i,$$
$$U_iV_i' = V_i'U_i.$$

Then we can establish

$$G = UVTU = \bigcup_{u \in U} u(VTU)u^{-1}$$

(see below). We call this decomposition a Gauss decomposition of G.

Theorem 1. *Every triangular system has a Gauss decomposition.*

Let K be a field, and $G = G_M(A, K)$ a Kac-Moody group over K. Then, we put

$$U = \langle x_\alpha(a) \mid \alpha \in \Delta_+^{re},\ a \in K \rangle,$$
$$T = \langle h_\alpha(t) \mid \alpha \in \Delta^{re},\ t \in K^\times \rangle,$$
$$V = \langle x_\alpha(a) \mid \alpha \in \Delta_-^{re},\ a \in K \rangle,$$

$$\begin{aligned}
\phi_i &: \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mapsto x_{\alpha_i}(a), \\
\phi_i &: \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mapsto h_{\alpha_i}(t), \\
\phi_i &: \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \mapsto x_{-\alpha_i}(a), \\
U_i' &= \langle\, x_{\alpha_i}(s)x_\beta(t)x_{\alpha_i}(-s) \mid s,t \in K,\ \beta \in \Delta_+^{re} \setminus \{\alpha_i\} \,\rangle, \\
V_i' &= \langle\, x_{-\alpha_i}(s)x_\beta(t)x_{-\alpha_i}(-s) \mid s,t \in K,\ \beta \in \Delta_-^{re} \setminus \{-\alpha_i\} \,\rangle.
\end{aligned}$$

Then $(G, U, T, V, \{\phi_1, \cdots, \phi_n\})$ is a triangular system. Hence,

$$\begin{aligned} G &= UVTU \\ &= \bigcup_{u \in U} u(VTU)u^{-1}. \end{aligned}$$

4. Prescribed Gauss decompositions

Let $(G, U, T, V, \{\phi_1, \cdots, \phi_n\})$ be a triangular system. Then, as in Section 3, we obtain

$$G = \bigcup_{u \in U} u(VTU)u^{-1}.$$

We now take an element $h^* \in T$. And we put

$$G(h^*) = Z(G) \cup \bigcup_{g \in G} g(Vh^*U)g^{-1},$$

where $Z(G)$ is the center of G. Then we want to consider whether $G = G(h^*)$ or not. If $G = G(h^*)$ for all $h^* \in T$, then we say that G has a Gauss decomposition with prescribed elements in T. This is equivalent to the fact that a non-central element $g \in G$ can be written as the form

$$g = g_1 v h u g_1^{-1}$$

with $g_1 \in G$, $v \in V$, $u \in U$ if we fix any prescribed element h in T.

Here, we will check this in the case when $G = SL_2(K)$, which is the easiest but important in our discussion. We choose and fix

$$h^* = \begin{pmatrix} t & 0 \\ 0 & \frac{1}{t} \end{pmatrix} \in T.$$

Let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in VTU \subset SL_2(K).$$

Then $a \neq 0$, and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix} \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix}.$$

If $b \neq 0$, then

$$\begin{pmatrix} 1 & 0 \\ \frac{a-t}{b} & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{t-a}{b} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{c'}{t} & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & \frac{1}{t} \end{pmatrix} \begin{pmatrix} 1 & \frac{b}{t} \\ 0 & 1 \end{pmatrix},$$

where

$$c' = \frac{1}{b}\{ta + td - t^2 - ad + bc\}.$$

If $c \neq 0$, then

$$\begin{pmatrix} 1 & \frac{t-a}{c} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & \frac{a-t}{c} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{c}{t} & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & \frac{1}{t} \end{pmatrix} \begin{pmatrix} 1 & \frac{b'}{t} \\ 0 & 1 \end{pmatrix},$$

where

$$b' = \frac{1}{c}\{ta + td - t^2 - ad + bc\}.$$

If $b = c = 0$, $a \neq \pm 1$, then

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & a - \frac{1}{a} \\ 0 & \frac{1}{a} \end{pmatrix},$$

which arrives at the case of $b \neq 0$ above. If $b = c = 0$, $a = \pm 1$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \in Z(G).$$

Therefore, we obtain

$$G = G(h^*)$$

for all $h^* \in T$ when $G = \mathrm{SL}_2(K)$. Hence, $\mathrm{SL}_2(K)$ has a Gauss decomposition with prescribed elements in T. Similarly we can obtain the following result.

Theorem 2. *Let* $A = \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}$ *be a generalized Cartan matrix with* $ab \geq 4$. *Put* $m = \max\{a, b\}$. *Let* K *be a field with* $|K| > m + 3$. *Then every standard Kac-Moody group* G *over* K *of type* A *has a Gauss decomposition with prescribed elements in* T.

It remains to consider the same problem for (infinite dimensional) standard Kac-Moody groups of rank ≥ 3.

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ON EVERETT RING EXTENSIONS AND EVERETT FUNCTIONS

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Abstract. Concerning Schreier's extension problem for rings, C. J. Everett considered four functions, which are called Everett functions. For a ring R and an ideal I of R, if there exists a multiplication-preserving right inverse of the natural ring homomorphism of R to R/I, we can take one of the four Everett functions trivial. This paper is concerned with existence and extensions of such right inverses.

Introduction

When we refer an extension of a ring, it has two different interpretations. In the first case, when R is a ring and R' is an abelian subgroup of R which is closed under multiplication, we say that R is an extension of R'. In the second case, when R is a ring and I is an ideal of R, we say that R is an extension of I. When we refer Everett ring extensions, we mean the second case. By an Everett extension of a ring I by a ring A, we understand any ring R with the properties that I is an ideal of R and there exists an isomorphism $R/I \cong A$.

Let I be an ideal of R, and $\pi : R \longrightarrow A = R/I$ be the natural ring homomorphism. A mapping $f : A \longrightarrow R$ is called a right inverse of π if $\pi \circ f = id_A$. In [2], for two rings I and A, to construct all Everett extensions of I by A, C. J. Everett directed his attention to right inverses and considered four mappings, which are called Everett functions. The detailed discussion on this subject will be found in [4, §52].

A multiplicative system of representatives is a right inverse which preserves multiplication. If there exists such a right inverse f, then, among the four Everett functions, we can take $\langle a, b\rangle \equiv 0$ (see [4, §52]).

For instance, let R be a commutative ring with 1. Assume that R is Hausdorff and complete with respect to the topology defined by a decreasing sequence $I_1 \supset I_2 \supset \cdots \supset I_i \supset I_{i+1} \supset \cdots$ of ideals such that $I_i I_j \subset I_{i+j}$. Let $\pi : R \longrightarrow R/I_1$ be the natural ring homomorphism. If the residue ring R/I_1 is a perfect ring of characteristic p (p a prime), then by [5, Chapter II, §4, Proposition 8], there exists a multiplicative system of representatives $f : R/I_1 \longrightarrow R$.

The detailed version of this paper will be submitted for publication elsewhere.

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In what follows, we shall consider conditions for existence and extensions of multiplicative systems of representatives.

Throughout this paper, all rings are associative rings (not necessarily with 1). Let R be a ring, and I an ideal of R. Let $A = R/I$, and $\pi : R \longrightarrow A$ be the natural ring homomorphism. A mapping $f : A \longrightarrow R$ is called a multiplicative system of representatives from A to R if f satisfies the following:

(1) $\pi \circ f = id_A$,
(2) $f(ab) = f(a)f(b) \quad (a, b \in A)$,
(3) $f(0) = 0$.

To begin with, we shall define double homothetisms of rings. Let N be a ring. Let $E_1(N)$ denote the endomorphism ring of N as a right N-module, and $E_2(N)$ denote the endomorphism ring as a left N-module. Elements of $E_1(N)$ or $E_2(N)$ act on N from the left. The abelian group $E_1(N) \oplus E_2(N)$ together with the multiplication

$$(\phi^1,\ \phi^2)(\psi^1,\ \psi^2) = (\phi^1 \circ \psi^1,\ \psi^2 \circ \phi^2) \quad (\phi^1, \psi^1 \in E_1(N),\ \phi^2, \psi^2 \in E_2(N))$$

is a ring, which will be denoted by $E'(N)$. An element $\phi = (\phi^1, \phi^2)$ of $E'(N)$ is called a double homothetism of N if it satisfies

(4) $x(\phi^1 y) = (\phi^2 x)y$,
(5) $\phi^1(\phi^2 x) = \phi^2(\phi^1 x) \quad (x, y \in N)$.

Let us denote by $\mathrm{DH}(N)$ the set of all double homothetisms of N.

Let K and N be rings. Let $[\ ,\ ] : (\alpha, \beta) \longmapsto [\alpha, \beta]$ be a mapping from $K \times K$ to N and $d : \alpha \longmapsto d_\alpha = (d^1_\alpha, d^2_\alpha)$ be a mapping from K to $\mathrm{DH}(N)$. The couple $([\ ,\ ], d)$ of these mappings will be called an Everett couple for K and N if the following (6)-(15) are satisfied:

(6) $[\alpha, \beta] = [\beta, \alpha]$,
(7) $[\alpha, \beta] + [\alpha + \beta, \gamma] = [\alpha, \beta + \gamma] + [\beta, \gamma]$,
(8) $[0, \alpha] = 0$,
(9) $d^1_{\alpha+\beta}x + [\alpha, \beta]x = d^1_\alpha x + d^1_\beta x$,
(10) $d^2_{\alpha+\beta}x + x[\alpha, \beta] = d^2_\alpha x + d^2_\beta x$,
(11) $d^1_\gamma([\alpha, \beta]) = [\gamma\alpha, \gamma\beta]$,
(12) $d^2_\gamma([\alpha, \beta]) = [\alpha\gamma, \beta\gamma]$,
(13) $d^1_\alpha(d^2_\beta x) = d^2_\beta(d^1_\alpha x)$,
(14) $d^1_0 x = d^2_0 x = 0$,
(15) $d_{\alpha\beta} = d_\alpha d_\beta \quad (\alpha, \beta, \gamma \in K, x \in N)$.

Let us assume that $([\ ,\ ], d)$ is an Everett couple for K and N. The set $K \times N$ together with the operations

(16) $(\alpha, x) + (\beta, y) = (\alpha + \beta,\ [\alpha, \beta] + x + y)$,

(17) $(\alpha, x)(\beta, y) = (\alpha\beta,\ d^1_\alpha y + d^2_\beta x + xy)$

is a ring, which will be denoted by $K \bullet N_{([\ ,\ ],d)}$. The ring N is regarded as an ideal of $K \bullet N_{([\ ,\ ],d)}$ by the embedding $\iota : x \longmapsto (0, x)$. Let $\pi : (\alpha, x) \longmapsto \alpha$ be the projection of $K \bullet N_{([\ ,\ ],d)}$ onto K. Then we get an exact sequence of rings:

$$0 \longrightarrow N \xrightarrow{\ \iota\ } K \bullet N_{([\ ,\ ],d)} \xrightarrow{\ \pi\ } K \longrightarrow 0$$

Theorem 1. *Let R be a ring, I an ideal of R, and $A = R/I$. Let $\iota_0 : I \longrightarrow R$ be the natural embedding, and $\pi_0 : R \longrightarrow A$ the natural ring homomorphism. Then there exists a multiplicative system of representatives from A to R if and only if there exists an Everett couple $([\ ,\ ], d)$ for A and I, and there exists a ring isomorphism σ from $A \bullet I_{([\ ,\ ],d)}$ to R which commutes the following diagram* (18):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \xrightarrow{\ \iota\ } & A \bullet I_{([\ ,\ ],d)} & \xrightarrow{\ \pi\ } & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & I & \xrightarrow{\ \iota_0\ } & R & \xrightarrow{\ \pi_0\ } & A & \longrightarrow & 0 \end{array} \qquad (18)$$

Proof. Let $([\ ,\ ], d)$ be an Everett couple for A and I, and let $\sigma : A \bullet I_{([\ ,\ ],d)} \longrightarrow R$ be a ring isomorphism which commutes the diagram (18). Then we see that the mapping $f : A \longrightarrow R$ given by $\alpha \longmapsto \sigma(\alpha, 0)$ is a multiplicative system of representatives.

Conversely, assume that there exists a multiplicative system of representatives $f : A \longrightarrow R$. We can define $[\ ,\ ] : A \times A \longrightarrow I$ and $d : A \longrightarrow DH(I)$ as follows:

(19) $[\alpha, \beta] = f(\alpha) + f(\beta) - f(\alpha + \beta) \quad (\alpha, \beta \in A)$,

(20) $d_\alpha = (d^1_\alpha, d^2_\alpha)$,

where $d^1_\alpha x = f(\alpha)x$, $d^2_\alpha x = xf(\alpha) \quad (\alpha \in A, x \in I)$.

We see that this couple $([\ ,\ ], d)$ satisfies the conditions (6)-(15), and the mapping $\sigma : A \bullet I_{([\ ,\ ],d)} \longrightarrow R$ defined by $(\alpha, x) \longmapsto f(\alpha) + x$ is a ring isomorphism which commutes the diagram (18). □

In [6], it plays an important role to extend multiplicative systems of representatives. In what follows, we shall consider extensions of multiplicative systems of representatives and their uniqueness.

Let R^* be a ring, and I^* an ideal of R^*. Let R be a subring of R^*, and I an ideal of R. We say that (R^*, I^*) dominates (R, I) if $I = I^* \cap R$ (cf.[3, p.14]).

In this case, $A = R/I$ is naturally regarded as a subring of $A^* = R^*/I^*$. Let $f : A \longrightarrow R$ be a multiplicative system of representatives. If there exists a multiplicative system of representatives $f^* : A^* \longrightarrow R^*$ such that $f^*|_A = f$, then f^* is called an extension of f. If $f^* : A^* \longrightarrow R^*$ is the only multiplicative system of representatives from A^* to R^* which is an extension of f, then we say that f is uniquely extended to f^*.

In what follows, we assume that (R^*, I^*) dominates (R, I). Let $A^* = R^*/I^*$, and $A = R/I$. Let $\iota' : I^* \longrightarrow R^*$ be the natural embedding, and $\pi' : R^* \longrightarrow A^*$ the natural ring homomorphism.

Let f be a multiplicative system of representatives from A to R. Let us assume that there exists a multiplicative system of representatives from A^* to R^* which is an extension of f. Then there exists an Everett couple $([\ ,\ ]^*, d^*)$ for A^* and I^* which is defined by the following:

(21) $[\alpha, \beta]^* = f^*(\alpha) + f^*(\beta) - f^*(\alpha + \beta)$,

(22) $d^*_\alpha = (d^{*1}_\alpha, d^{*2}_\alpha)$,

where $d^{*1}_\alpha x = f^*(\alpha)x$, $d^{*2}_\alpha x = xf^*(\alpha) \quad (\alpha \in A^*, x \in I^*)$.

By Theorem 1 and its proof, we see that there exists a ring isomorphism $\sigma : R^* \longrightarrow A^* \bullet I^*_{([\ ,\ ]^*, d^*)}$ and $\sigma(R) = \{(\alpha, x) \in A^* \bullet I^*_{([\ ,\ ]^*, d^*)} \mid \alpha \in A, x \in I\}$.

Let us assume that there exist an Everett couple $([\ ,\ ], d)$ for A and I, and an Everett couple $([\ ,\ ]^*, d^*)$ for A^* and I^*. We say that $([\ ,\ ]^*, d^*)$ is an extension of $([\ ,\ ], d)$ if it satisfies the following:

(23) If $\alpha, \beta \in A$, then $[\alpha, \beta]^* = [\alpha, \beta]$;

(24) If $\alpha \in A$, then $d^{*1}_\alpha(I) \subseteq I$, $d^{*2}_\alpha(I) \subseteq I$, $d^{*1}_\alpha x = d^1_\alpha x$, and $d^{*2}_\alpha x = d^2_\alpha x \quad (x \in I)$.

In this case, $A \bullet I_{([\ ,\ ], d)}$ is regarded as a subring of $A^* \bullet I^*_{([\ ,\ ]^*, d^*)}$ by the embedding $(\alpha, x) \longmapsto (\alpha, x)$.

Let us assume that there exist two Everett couples $([\ ,\ ]^*, d^*)$ and $([\ ,\ ]^{**}, d^{**})$ for A^* and I^* which are extensions of $c = ([\ ,\ ], d)$. Then $A \bullet I_{([\ ,\ ], d)}$ is regarded as a subring of $A^* \bullet I^*_{([\ ,\ ]^*, d^*)}$ and a subring of $A^* \bullet I^*_{([\ ,\ ]^{**}, d^{**})}$. Let $\iota^* : I^* \longrightarrow A^* \bullet I^*_{([\ ,\ ]^*, d^*)}$ be $x \longmapsto (0, x)$, $\iota^{**} : I^* \longrightarrow A^* \bullet I^*_{([\ ,\ ]^{**}, d^{**})}$ be $x \longmapsto (0, x)$, $\pi^* : A^* \bullet I^*_{([\ ,\ ]^*, d^*)} \longrightarrow A^*$ be $(\alpha, x) \longmapsto \alpha$, and $\pi^{**} : A^* \bullet I^*_{([\ ,\ ]^{**}, d^{**})} \longrightarrow A^*$ be $(\alpha, x) \longmapsto \alpha$. Two Everett couples $([\ ,\ ]^*, d^*)$ and $([\ ,\ ]^{**}, d^{**})$ which are extensions of $c = ([\ ,\ ], d)$ are said to be c-equivalent if there exists a ring isomorphism $\tau : A^* \bullet I^*_{([\ ,\ ]^*, d^*)} \longrightarrow A^* \bullet I^*_{([\ ,\ ]^{**}, d^{**})}$ which satisfies $\tau|_{A \bullet I_{([\ ,\ ], d)}} = id$ and commutes the following diagram (25).

$$
(25) \qquad
\begin{array}{ccccccccc}
0 & \longrightarrow & I^* & \xrightarrow{\iota^*} & A^* \bullet I^*_{([\ ,\ ]^*, d^*)} & \xrightarrow{\pi^*} & A^* & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle id} & & \\
0 & \longrightarrow & I^* & \xrightarrow{\iota^{**}} & A^* \bullet I^*_{([\ ,\ ]^{**}, d^{**})} & \xrightarrow{\pi^{**}} & A^* & \longrightarrow & 0
\end{array}
$$

This means that two rings $A^* \bullet I^*_{([\ ,\ ]^*,d^*)}$ and $A^* \bullet I^*_{([\ ,\ ]^{**},d^{**})}$ are equivalent as ring extensions of $A \bullet I_{([\ ,\ ],d)}$ by a ring isomorphism which leaves the elements of $A \bullet I_{([\ ,\ ],d)}$ fixed and maps every class modulo I^* onto itself (see [4, p. 196]).

Proposition 2. (cf.[5, p.196, Satz 114 and Korollar]) *Let $e = ([\ ,\ ], d)$ be an Everett couple for A and I. Let $([\ ,\ ]^*, d^*)$ and $([\ ,\ ]^{**}, d^{**})$ be two Everett couples for A^* and I^* which are extensions of $e = ([\ ,\ ], d)$. Then $([\ ,\ ]^*, d^*)$ and $([\ ,\ ]^{**}, d^{**})$ are e-equivalent if and only if there exists a mapping $\lambda : A^* \longrightarrow I^*$ which satisfies the following* (26)-(30).

(26) $\lambda(\alpha + \beta) - \lambda(\alpha) - \lambda(\beta) = [\alpha, \beta]^{**} - [\alpha, \beta]^*$,
(27) $\lambda(\alpha\beta) - \lambda(\alpha)\lambda(\beta) = d^{**1}_\alpha(\lambda(\beta)) + d^{**2}_\beta(\lambda(\alpha))$,
(28) $\lambda(\alpha)x = d^{*1}_\alpha x - d^{**1}_\alpha x$,
(29) $x\lambda(\alpha) = d^{*2}_\alpha x - d^{**2}_\alpha x$,
(30) $\lambda|_A \equiv 0 \qquad (\alpha, \beta \in A^*, x \in I^*)$.

Proof. Let $\tau : A^* \bullet I^*_{([\ ,\ ]^*,d^*)} \longrightarrow A^* \bullet I^*_{([\ ,\ ]^{**},d^{**})}$ be a ring isomorphism which satisfies $\tau|_{A\bullet I_{([\ ,\ ],d)}} = id$ and commute the diagram (25). We can define the mapping $\lambda : A^* \longrightarrow I^*$ by $\lambda(\alpha) = x$, where $\tau(\alpha, 0) = (\alpha, x)$. We see that λ satisfies (26)-(30), and $\lambda(\alpha) = 0 \ (\alpha \in A)$.

Conversely, suppose that there exists a mapping $\lambda : A^* \longrightarrow I^*$ which satisfies (26)-(30). We can define $\tau : A^* \bullet I^*_{([\ ,\ ]^*,d^*)} \longrightarrow A^* \bullet I^*_{([\ ,\ ]^{**},d^{**})}$ by $\tau(\alpha, x) = (\alpha, x + \lambda(\alpha))$. This mapping τ is a ring isomorphism which satisfies $\tau|_{A\bullet I_{([\ ,\ ],d)}} = id$ and commutes the diagram (25). □

If there exists a multiplicative system of representatives f from A to R, we can define an Everett couple $e = ([\ ,\ ], d)$ for A and I as follows:

(31) $[\alpha, \beta] = f(\alpha) + f(\beta) - f(\alpha + \beta)$,
(32) $d_\alpha = (d^1_\alpha, d^2_\alpha)$,
where $d^1_\alpha x = f(\alpha)x$, $d^2_\alpha x = xf(\alpha) \ (\alpha, \beta \in A, x \in I)$.

Theorem 3. *Let $f : A \longrightarrow I$ be a multiplicative system of representatives, and $e = ([\ ,\ ], d)$ be the Everett couple for A and I which is defined by (31) and (32). Then:*

(I) *There exists a multiplicative system of representatives $f^* : A^* \longrightarrow R^*$ which is an extension of f if and only if there exists an Everett couple $([\ ,\ ]^*, d^*)$ for A^* and I^* which is an extension of e, and there exists a ring isomorphism $\sigma : A^* \bullet I^*_{([\ ,\ ]^*,d^*)} \longrightarrow R^*$ which satisfies $\sigma(\alpha, 0) = f(\alpha)$ $(\alpha \in A)$ and commutes the following diagram* (33):

$$
(33) \qquad
\begin{array}{ccccccccc}
0 & \longrightarrow & I^* & \xrightarrow{\iota^*} & A^* \bullet I^*_{([\ ,\ ]^*,d^*)} & \xrightarrow{\pi^*} & A^* & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle id} & & \\
0 & \longrightarrow & I^* & \xrightarrow{\iota'} & R^* & \xrightarrow{\pi'} & A^* & \longrightarrow & 0
\end{array}
$$

(II) *Let* $f^* : A^* \longrightarrow R^*$ *be a multiplicative system of representatives which is an extension of* f. *Let* $([\ ,\ ]^*, d^*)$ *be the Everett couple for* A^* *and* I^* *defined as follows:*

(34) $[\alpha,\beta]^* = f^*(\alpha) + f^*(\beta) - f^*(\alpha+\beta)$.
(35) $d_\alpha = (d^1_\alpha, d^2_\alpha)$,
where $d^{*1}_\alpha x = f^*(\alpha)x, \quad d^{*2}_\alpha x = xf^*(\alpha) \quad (\alpha,\beta \in A^*, x \in I^*)$.

Then f *is uniquely extended to* f^* *if and only if, for each Everett couple* $([\ ,\ ]^{**}, d^{**})$ *for* A^* *and* I^* *which is e-equivalent to* $([\ ,\ ]^*, d^*)$, $\lambda \equiv 0$ *is the only mapping of* A^* *to* I^* *which satisfies* (26)-(30).

Proof. (I) Let $f^* : A^* \longrightarrow R^*$ be a multiplicative system of representatives which is an extension of f.

Then we can define an Everett couple $([\ ,\ ]^*, d^*)$ for A^* and I^* by (34) and (35). This Everett couple $([\ ,\ ]^*, d^*)$ is an extension of e. The mapping $\sigma : A^* \bullet I^*_{([\ ,\ ]^*, d^*)} \longrightarrow R^*$ defined by $(\alpha, x) \longmapsto f^*(\alpha) + x$ is a ring isomorphism which commutes the diagram (33) and $\sigma(\alpha, 0) = f^*(\alpha) = f(\alpha) \quad (\alpha \in A)$.

Conversely, suppose that there exists an Everett couple $([\ ,\ ]^*, d^*)$ for A^* and I^* which is an extension of e, and there exists a ring isomorphism σ : $A^* \bullet I^*_{([\ ,\ ]^*, d^*)} \longrightarrow R^*$ which satisfies $\sigma(\alpha, 0) = f(\alpha) \quad (\alpha \in A)$ and commutes the diagram (33). Then we can define $f^* : A^* \longrightarrow R^*$ by $\alpha \longmapsto \sigma(\alpha, 0)$. This mapping f^* is a multiplicative system of representatives from A^* to R^* which is an extension of f.

(II) Let $([\ ,\ ]^{**}, d^{**})$ be an Everett couple for A^* which is e-equivalent to $([\ ,\ ]^*, d^*)$. Suppose that there exists a non-zero mapping $\lambda : A^* \longrightarrow I^*$ which satisfies (26)-(30). Let $\sigma : A^* \bullet I^*_{([\ ,\ ]^*, d^*)} \longrightarrow R^*$ be the isomorphism stated in (I). Two mappings $f^* : \alpha \longmapsto \sigma(\alpha, 0)$ and $g^* : \alpha \longmapsto \sigma(\alpha, -\lambda(\alpha))$ are multiplicative systems of representatives from A^* to R^* which are extensions of f, and we see that $g^* \neq f^*$.

Conversely, let g^* be a multiplicative system of representatives different from f^* which is an extension of f. We can define an Everett couple $([\ ,\ ]^{**}, d^{**})$ for A^* and I^* as follows:

$$[\alpha,\beta]^{**} = g^*(\alpha) + g^*(\beta) - g^*(\alpha+\beta),$$
$$d^{**1}_\alpha x = g^*(\alpha)x,$$
$$d^{**2}_\alpha x = xg^*(\alpha) \quad (\alpha,\beta \in A^*, x \in I^*).$$

Then the mapping $\lambda : A^* \longrightarrow I^*$ defined by $\alpha \longmapsto f^*(\alpha) - g^*(\alpha)$ satisfies (26)-(30), and is not the zero mapping. □

Example. Let $A = \mathrm{GF}(2) = Z/(2)$, $A^* = \mathrm{GF}(4) = \mathrm{GF}(2)[X]/(X^2+X+1)$ and σ be an automorphism of A^* defined by $x \longmapsto x^2$. The Abelian group $V = A^* \oplus A^*$ together with the multiplication

$$(a_1, b_1)(a_2, b_2) = (a_1 a_2,\ a_1 b_2 + \sigma(a_2) b_1)$$

is a ring, which is denoted by R^*. We see that $I^* = \{(0,b) \in R^* \mid b \in A^*\}$ is an ideal of R^*. The natural homomorphism $\pi^* : R^* \longrightarrow A^*$ given by $(a,b) \longmapsto a$ yields $R^*/I^* \cong A^*$. Moreover, $R = \{(a,b) \in R^* \mid a,b \in A\}$ is a subring of R^*, and $I = \{(0,b) \in R \mid b \in A\}$ is an ideal of R. We get $R/I \cong A$ by $\pi^*|_R$. Let γ be a generator of A^* which satisfies the relation $\gamma^2+\gamma+1=0$. The mapping $f^* : A^* \longrightarrow R^*$ defined by $a \longmapsto (a,0)$ is a multiplicative system of representatives. As $f = f^*|_A$ is a multiplicative system of representatives from A to R, so f^* is an extension of f.

Let $e = ([\ ,\ ], d)$ be the Everett couple for A and I defined by (31) and (32). That is,

$$[\alpha,\beta] \equiv 0,$$
$$d_\alpha^1(0,x) = d_\alpha^2(0,x) = (0,\alpha x) \quad (\alpha,\beta \in A, (0,x) \in I).$$

Let $([\ ,\ ]^*, d^*)$ be the Everett couple for A^* and I^* defined by (34) and (35). That is,

$$[\alpha,\beta]^* \equiv 0,$$
$$d_\alpha^{*1}(0,x) = (0,\alpha x),\ d_\alpha^{*2}(0,x) = (0,\sigma(\alpha)x) \quad (\alpha,\beta \in A^*, (0,x) \in I^*).$$

Let $([\ ,\ ]^{**}, d^{**})$ be another Everett couple for A^* and I^* defined as follows:

$$[\alpha,\beta]^{**} \equiv 0,$$
$$d_0^{**1}(0,x) = d_0^{**2}(0,x) = (0,0),$$
$$d_1^{**1}(0,x) = d_1^{**2}(0,x) = (0,x),$$
$$d_\gamma^{**1}(0,x) = (0,\gamma x),$$
$$d_\gamma^{**2}(0,x) = (0,\gamma^2 x),$$
$$d_{\gamma^2}^{**1}(0,x) = (0,\gamma^2 x),$$
$$d_{\gamma^2}^{**2}(0,x) = (0,\gamma x) \quad (\alpha,\beta \in A^*, (0,x) \in I^*).$$

Let $\lambda : A^* \longrightarrow I^*$ be the mapping defined as follows:

$$1 \longmapsto (0,0), \quad \gamma \longmapsto (0,1), \quad \gamma^2 \longmapsto (0,1).$$

This mapping λ satisfies (26)-(30), so $([\ ,\ ]^*, d^*)$ and $([\ ,\ ]^{**}, d^{**})$ are e-equivalent by Proposition 2. Since λ is not the zero mapping, by Theorem 3 (II), we see that f is not uniquely extended to f^*.

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FINITE HOPF ALGEBRAS IN BRAIDED TENSOR CATEGORIES

Mitsuhiro Takeuchi

The notion of Hopf algebras in a braided tensor category has been introduced by S. Majid [M] under the name 'braided groups'. Later, V. Lyubashenko [L] has generalized the notions of Hopf modules and integrals to this context and has proved the fundamental structure theorem of Hopf modules, originally due to Larson and Sweedler [LS]. He has also claimed that the space of left or right integrals in a finite Hopf algebra is an invertible object. However, his proof does not seem complete.

On the other hand, Y. Doi [D] studies Hopf modules for Hopf algebras in the Yetter-Drinfeld category and proves bijectivity of the antipode of a finite-dimensional Hopf algebra.

Motivated by these works, I talked about how to approach Hopf algebras in a braided tensor category with diagrammatic methods and how to prove correctly the uniqueness of integrals for finite Hopf algebras, with main source in [T]. In the following, I reproduce the transparency sheets I used in my talk so that the reader can imagine and enjoy the atmosphere of my talk.

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[D] Y. Doi, *Hopf modules in Yetter-Drinfeld categories*, preprint.

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[T] M. Takeuchi, *Finite Hopf algebras in braided tensor categories*, to appear in J. Pure and Appl. Algebra.

The detailed version of this paper will be published in J. of Pure and Applied Algebra.

Finite Hopf algebras in braided tensor categories

by M. Takeuchi

1. What's a braided tensor category

$\mathcal{M}$ a monoidal tensor category

$V \otimes W$ tensor product

k unit object

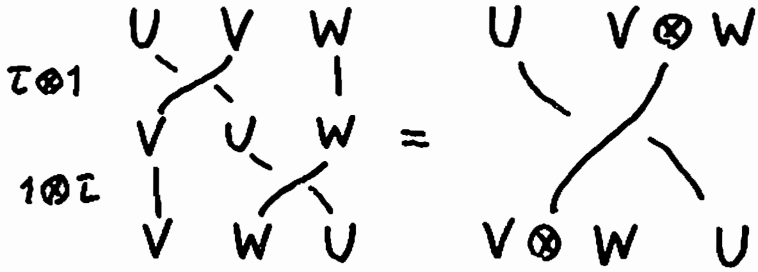
$(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$

$k \otimes V \cong V \cong V \otimes k$

$\tau : V \otimes W \xrightarrow{\cong} W \otimes V$ braiding

τ

τ^{-1}



Braiding Axiom

<u>Examples</u> 1) G an abelian group

$\chi : G \times G \longrightarrow k^{\times}$ a bi-character

$\mathcal{M}^{kG}$ G-graded k-vector spaces

$\tau : V \otimes W \longrightarrow W \otimes V$

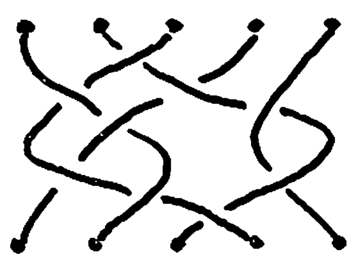
$\tau(v \otimes w) = \chi(x, y)\, w \otimes v$

if v, w homogeneous of degree x, y.

2) objects : $0, 1, 2, \cdots$

morphisms : braids

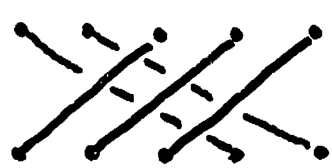
$5 \downarrow 5$



$m \otimes n = m + n$

$\tau_{m,n} : m \otimes n \longrightarrow n \otimes m$

$\tau_{2,3}$:



3) H a quasi-triangular Hopf algebra

$R \in H \otimes H$ (invertible)

$R\Delta(h)R^{-1} = \Delta^{op}(h)$, $h \in H$,

$(\Delta \otimes 1)(R) = R_{13}R_{23}$,

$(1 \otimes \Delta)(R) = R_{13}R_{12}$.

${}_H\mathcal{M}od$ left H modules

$\tau : V \otimes W \longrightarrow W \otimes V$

$$\tau(v \otimes w) = \sum_i e_i' w \otimes e_i v$$

where $R = \sum_i e_i \otimes e_i'$

4) L a Hopf algebra with bijective antipode

${}_L^L\mathcal{YD}$ Yetter Drinfeld modules

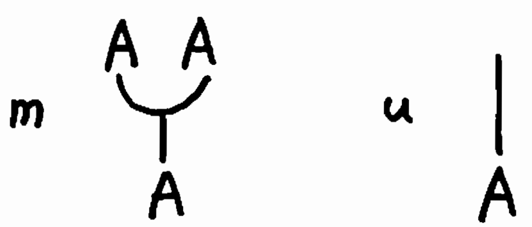
left L modules and left L comodules .....

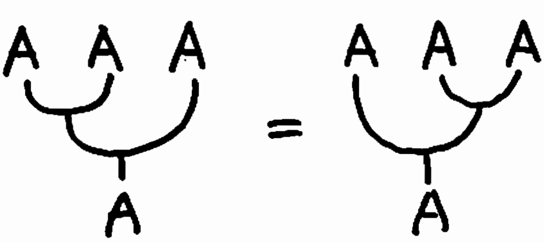
2. Algebras, coalgebras, bialgebras and Hopf algebras

$\mathcal{M}$ a braided tensor category

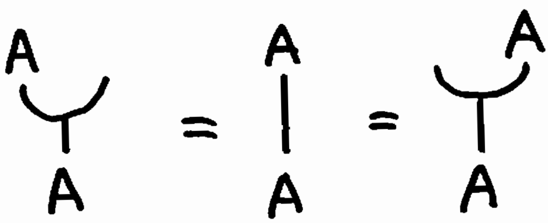
an algebra in $\mathcal{M}$:

$A \in \mathcal{M}$, $m : A \otimes A \longrightarrow A$, $u : k \longrightarrow A$





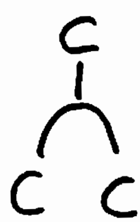
Associativity



Unit Axiom

a coalgebra in $\mathcal{M}$:

$C \in \mathcal{M}, \ \Delta : C \longrightarrow C \otimes C, \ \varepsilon : C \longrightarrow k$

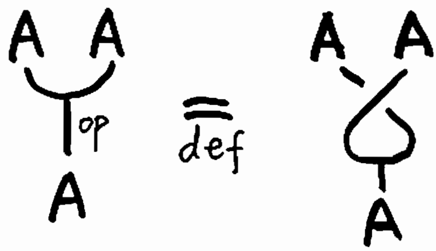
Δ :  , ε : 

a left A module in $\mathcal{M}$:

$V \in \mathcal{M}$, action : $A \otimes V \longrightarrow V$

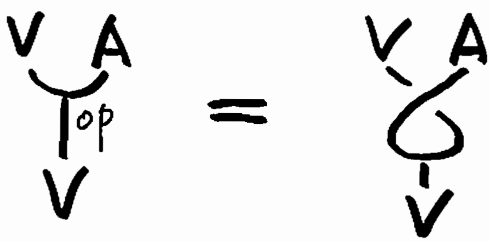
right A module, left or right C comodule defined similarly

A^{op} <u>opposite</u> algebra

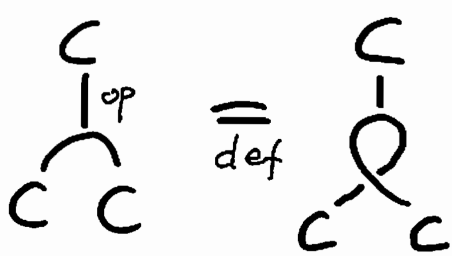


i.e., $m^{op} = m\tau$

V left A module $\Rightarrow$ right A^{op} module



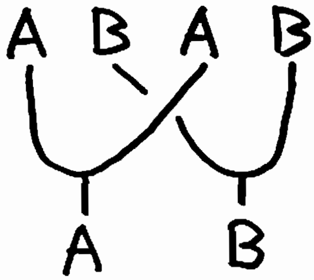
C^{cop} opposite coalgebra



i.e., $\Delta^{op} = \tau^{-1}\Delta$

M left (right) C comodule $\Rightarrow$ right (left) C^{cop} comodule

A, B algebras $\Longrightarrow$ $A \otimes B$ algebra



C, D coalgebras $\Longrightarrow$ $C \otimes D$ coalgebra

$\mathcal{M}(C, A)$ forms a semigroup :

$$f * g : C \xrightarrow{\Delta} C \otimes C \xrightarrow{f \otimes g} A \otimes A \xrightarrow{m} A$$

unit $= u\varepsilon$

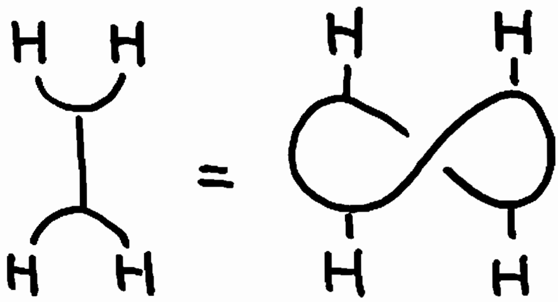
a <u>bialgebra</u> in $\mathcal{M}$ is

an algebra and a coalgebra H

s.t. $\Delta : H \longrightarrow H \otimes H$, $\varepsilon : H \longrightarrow k$

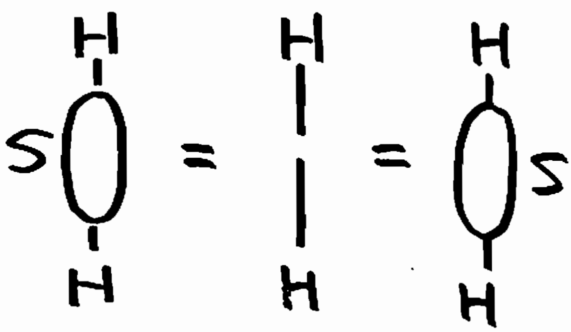
are algebra maps

($\Longleftrightarrow$ m, u are coalgebra maps)



Bialgebra Axiom

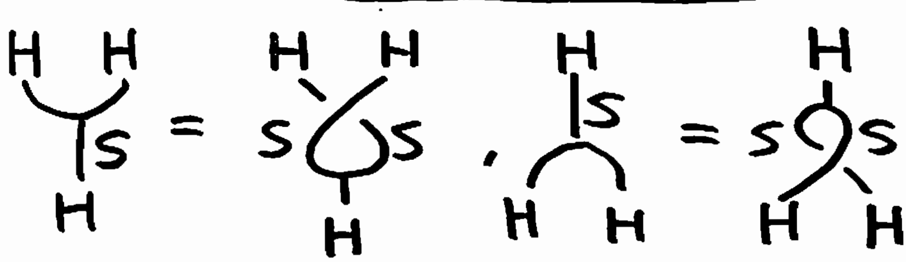
$S : H \to H$ antipode if $S * I = I * S = u\varepsilon$



Antipode Axiom

a Hopf algebra = a bialgebra with antipode

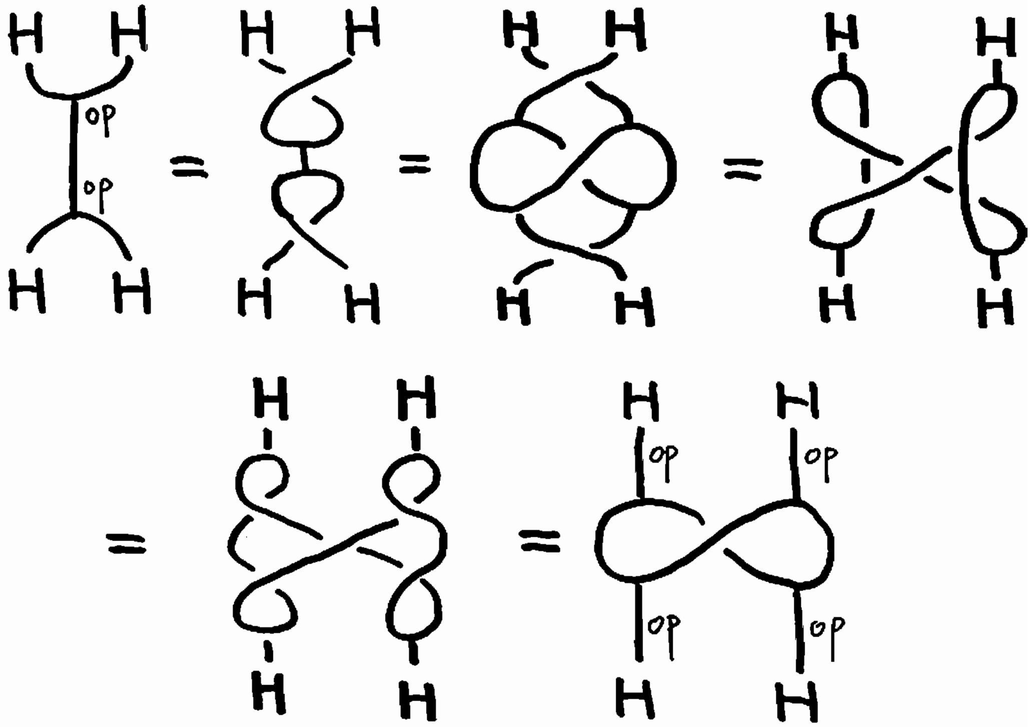
Relations of S and m, Δ



$Sm = m\tau(S \otimes S)$ $\Delta S = (S \otimes S)\tau\Delta$

$S : H \longrightarrow H^{op,cop}$ bialgebra map

Bialgebra axiom for $H^{op,cop}$



3. Finite objects and dual Hopf algebra

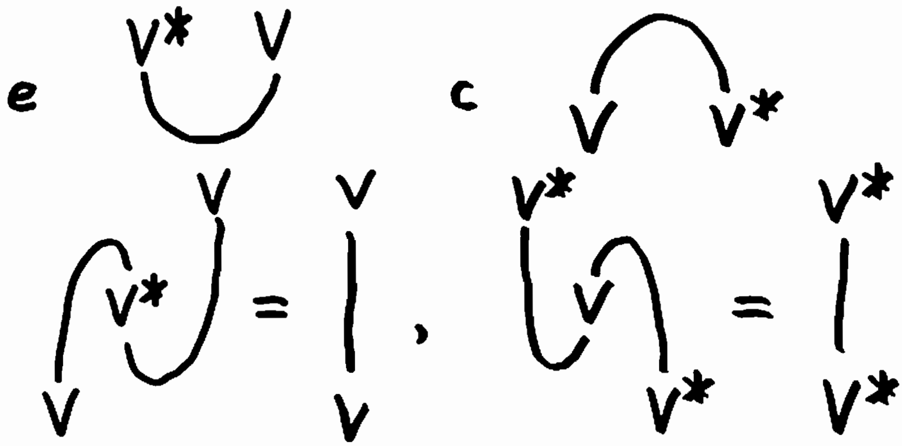
Def $V \in \mathcal{M}$ is *finite* if

$\exists\, V^* \in \mathcal{M}$, $\exists\, e : V^* \otimes V \longrightarrow k$,

$\exists\, c : k \longrightarrow V \otimes V^*$ s.t.

$$1_V : V \xrightarrow{c \otimes 1} V \otimes V^* \otimes V \xrightarrow{1 \otimes e} V$$

$$1_{V^*} : V^* \xrightarrow{1 \otimes c} V^* \otimes V \otimes V^* \xrightarrow{e \otimes 1} V^*$$



<u>Ex.</u> In both ${}^{L}_{L}\mathcal{YD}$ and ${}_{H}\mathcal{M}od$, finite objects are those modules which are finite-dim. over k, the base field

<u>Properties of finite objects</u>

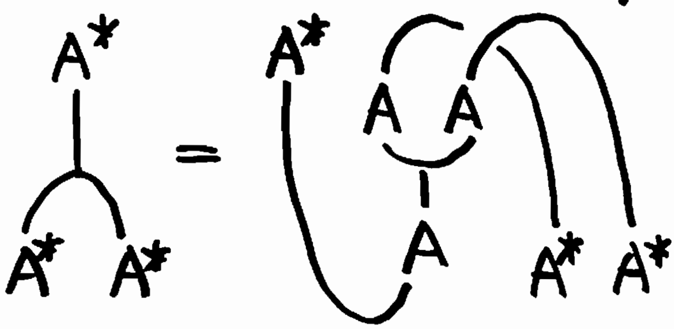
i) k is finite.

ii) V is finite $\Rightarrow$ V^* is finite and $V^{**} \cong V$.

iii) V, W finite $\Longrightarrow$ $V \otimes W$ is finite and $V^* \otimes W^* \cong (V \otimes W)^*$

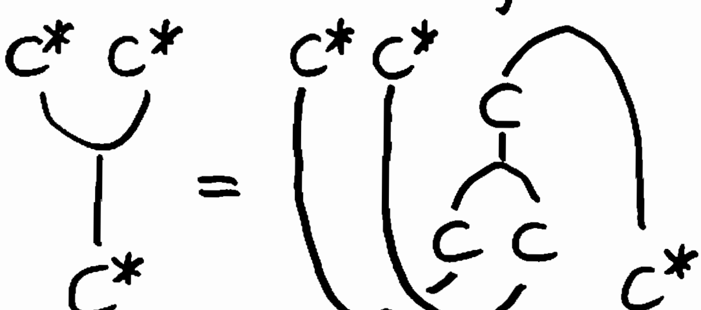
If A is a finite algebra,

A^* has dual coalgebra str. :



If C is a (finite) coalgebra,

C^* has dual algebra str. :



<u>Prop</u> H finite Hopf algebra w. antipode S

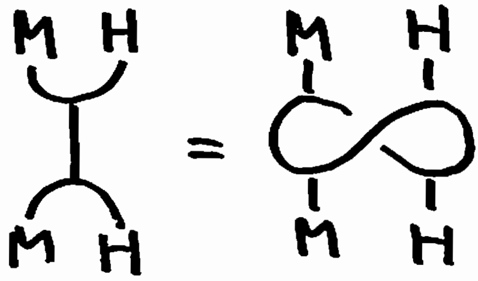
$\Rightarrow$ H^* finite Hopf algebra w. antipode S^*

4. Hopf modules

H a Hopf algebra in $\mathcal{M}$

M a right H module and a right H comodule

M is a right H **Hopf module** if



Hopf Module Axiom

Assume $\mathcal{M}$ has **equalizers**

$$M^{coH} \underset{def}{=} \ker\left(M \overset{\text{coaction}}{\underset{1\otimes u}{\rightrightarrows}} M\otimes H\right)$$

the space of **co-invariants**

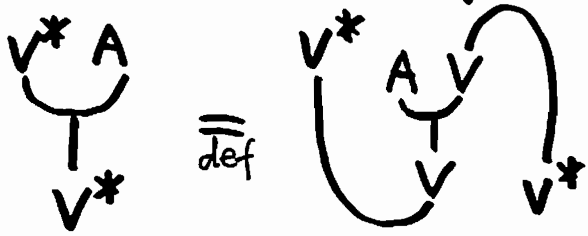
Thm M a right H Hopf module

$$\Rightarrow \quad M^{coH}\otimes H \xrightarrow[\cong]{\text{action}} M$$

5. The antipode and integrals

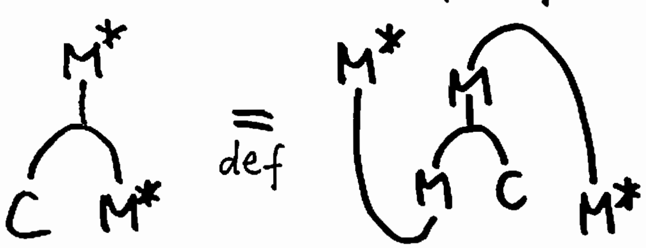
V a (finite) left A module

$\Rightarrow$ V^* is a right A module



M a finite right C comodule

$\Rightarrow$ M^* is a left C comodule



H a finite Hopf algebra

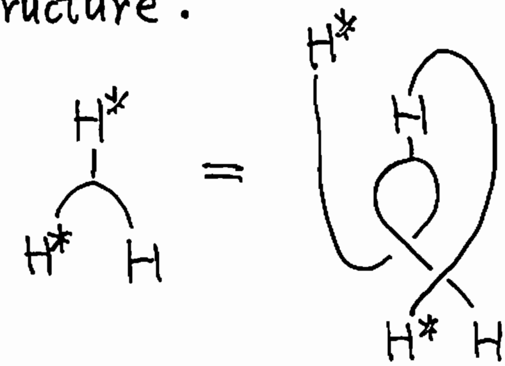
$$H_H \longleftrightarrow {}_{H^{op}}H \longrightarrow H^*_{H^{op}} \longrightarrow H^*_H \quad (S : H \to H^{op})$$

$${}^H H \longleftrightarrow H^{H^{cop}} \longrightarrow {}^{H^{cop}}H^* \longleftrightarrow H^{*H}$$

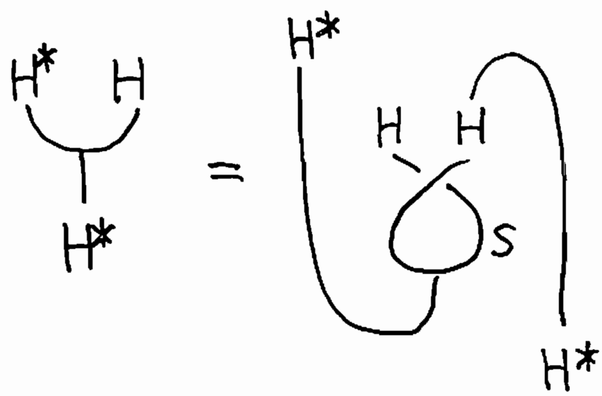
<u>Thm</u> H^* is a right H Hopf module

Hopf module axiom for H^*

Structure :

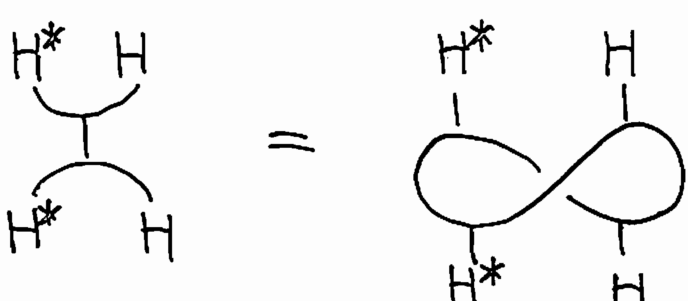


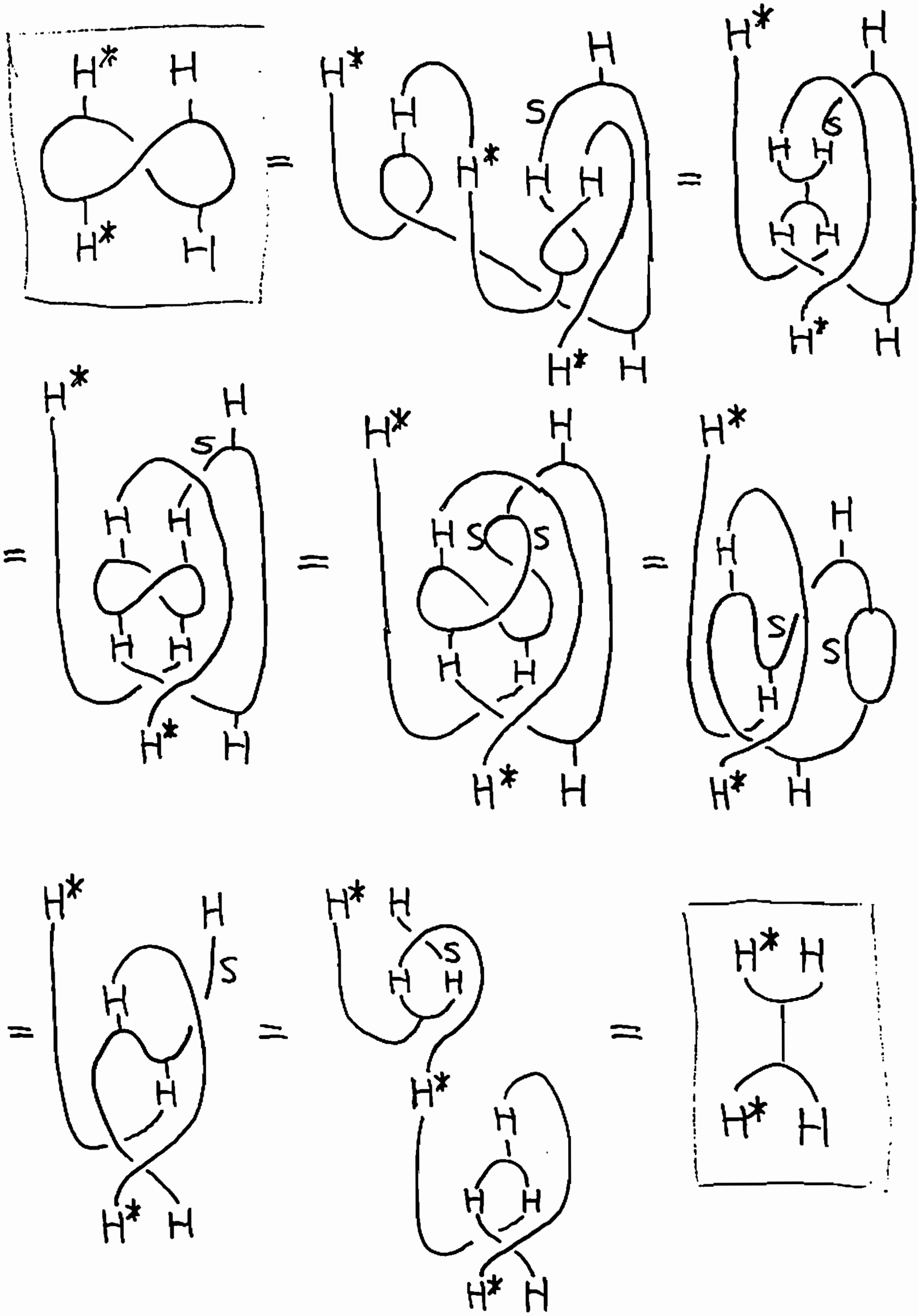
right H comodule



right H module

We claim :




H*
H
S
=

Thm. Antipode S of a finite Hopf algebra H is an isomorphism.

Proof. $H^* \cong I \otimes H$, $I = H^{*\,\mathrm{coH}}$

$$\begin{array}{ccc} H^* & & I\otimes H \\ \exists \nwarrow & & \swarrow 1\otimes S \\ & I \otimes H & \end{array}$$

i.e., $1\otimes S$ has a left inverse.

Applying $H \otimes -$, $1 \otimes S : H^* \otimes H \longrightarrow H^* \otimes H$ has a left inverse, since $H^* \cong H \otimes I$.

$$\begin{array}{ccc} H & \xrightarrow{\varepsilon^* \otimes 1} & H^* \otimes H \\ S \downarrow & & \downarrow 1\otimes S \\ H & \xrightarrow{\varepsilon^* \otimes 1} & H^* \otimes H \end{array}$$

Maps $\bigcirc$ have left inverses

$\therefore$ S has a left inverse.

Applying to H^*, S^* has a left inverse.

$\therefore$ S has a right inverse.

$\therefore$ S is an isomorphism. //

If H is an ordinary Hopf algebra,

$$I_\ell(H) = \{y \in H \mid xy = \varepsilon(x)y \quad \forall x \in H\}$$

$$I_r(H) = \{x \in H \mid xy = x\varepsilon(y) \quad \forall y \in H\}$$

the spaces of left, right integrals

H a finite-dim. Hopf algebra

$\Rightarrow I_\ell(H), I_r(H)$ are 1-dimensional

Let H be a finite Hopf algebra in $\mathcal{M}$.

$I_\ell(H)$, $I_r(H)$ are defined

if $\mathcal{M}$ has equalizers.

Thm

$I_\ell(H)$ and $I_r(H)$ are invertible.

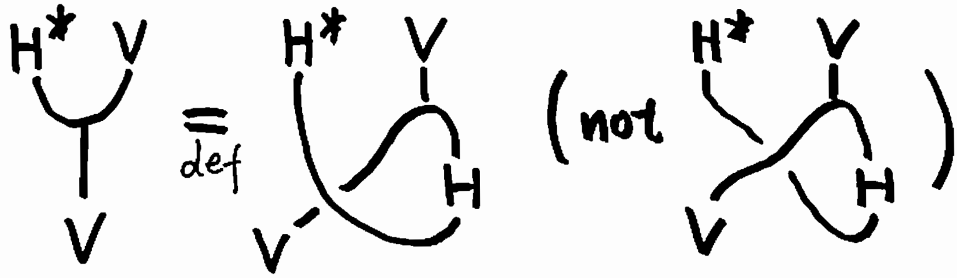
$V \in \mathcal{M}$ is invertible if $\exists\, W \in \mathcal{M}$

s.t. $V \otimes W \cong k$.

(Invertible $\Rightarrow$ finite)

We may identify :

right H comodule = left H^* module



H , H^* right H comodules

$\rightsquigarrow$ left H^* modules

The left H^* module H^* is not equal

but isoc. to the <u>regular</u> module

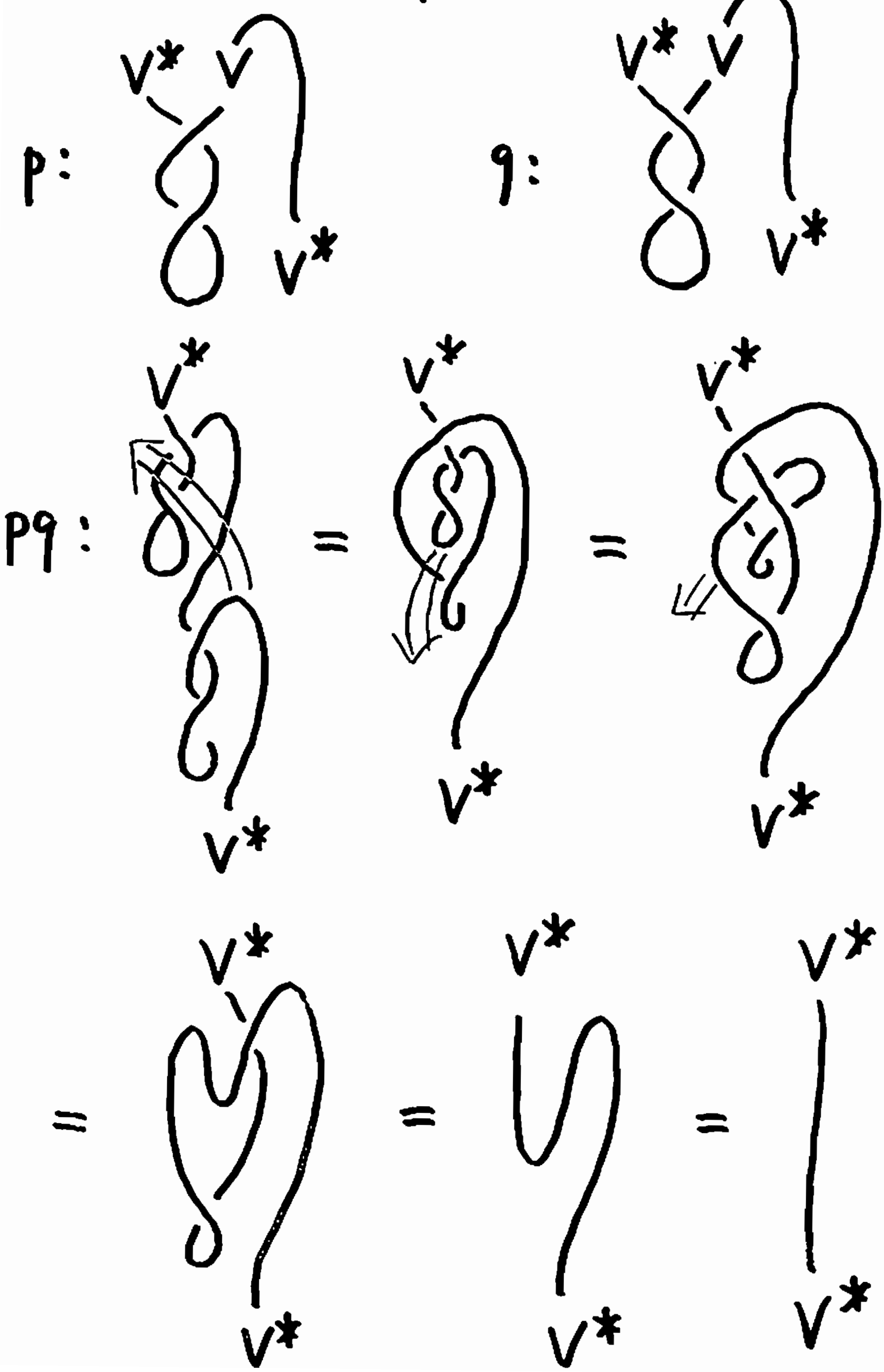
$\exists \ p : H^* \xrightarrow{\cong} H^*$ left H^*-iso.

H^* : regular $\downarrow$ $I_\ell(H^*)$

H^* : Hopf module $\downarrow$ I

co-invariants

What's p ?



Thus we have

$$H^*(\mathrm{reg}) \cong H \otimes I_\ell(H^*) \quad \text{left } H^*\text{-iso.}$$

There is also

same left H^*module

$$H \cong H^*(\mathrm{reg}) \otimes J \quad \text{left } H^*\text{-iso.}$$

$$\therefore \; H^* \cong H^* \otimes J \otimes I_\ell(H^*)$$

as left H^*modules

This yields $k \cong J \otimes I_\ell(H^*)$

$\therefore I_\ell(H^*)$ is invertible.

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ON LOCAL SYMMETRIC ALGEBRAS

TAKAYOSHI WAKAMATSU

1. Nilpotent Selfinjective Algebras

Throughout this note, all algebras and modules are assumed to be finite dimensional vector spaces over a fixed algebraically closed field K. The dual functor $\mathrm{Hom}_K(?,\ K)$ is always denoted by D.

In the previous paper [1], we have proved that any selfinjective algebras can be constructed by using nilpotent selfinjective algebras. So, we start by recalling the definition of nilpotent selfinjective algebras. Let A be an algebra and ${}_AM_A$ and ${}_AS_A$ bimodules. The module ${}_AS_A$ is called an injective cogenerator if the functor $\mathrm{Hom}_A(?,\ {}_AS_A)$ induces a duality between A-mod and mod-A. We consider a pair of linear maps

$$\varphi : {}_AM \otimes_A M_A \longrightarrow {}_AM_A$$

and

$$\psi : {}_AM \otimes_A M_A \longrightarrow {}_AS_A.$$

The map φ is called an associative map if the equality

$$\varphi(\varphi(x \otimes y) \otimes z) = \varphi(x \otimes \varphi(y \otimes z))$$

holds for any elements $x, y, z \in M$. If the map φ is associative, then we get an associative algebra $\Lambda(\varphi) = A \bigoplus M$ with the multiplication

$$(a, m) \cdot (a', m') = (aa', am' + ma' + \varphi(m \otimes m'))$$

for $(a,m), (a', m') \in A \bigoplus M$. It is easy to see that M is always an ideal in the algebra. We call an associative map φ is nilpotent if M in $A \bigoplus M$ is nilpotent. In this case, we say that φ is a nilpotent algebra. The map ψ is called φ-associative if the equality

$$\psi(\varphi(x \otimes y) \otimes z) = \psi(x \otimes \psi(y \otimes z))$$

The final version of this paper will be submitted for publication elsewhere.

hold for any x, y, z in M. We call a system (φ, ψ) a nilpotent selfinjective algebra if the following conditions are satisfied;

(1) φ is a nilpotent associative map and ψ a φ-associative map,

(2) ${}_AS_A$ is an injective cogenerator, and

(3) ψ is non-degenerate in the sense that one of the conditions $\psi(m \otimes M) = 0$ or $\psi(M \otimes m) = 0$ implies $m = 0$ for an element $m \in M$.

From a nilpotent selfinjective algebra (φ, ψ), we can form the usual selfinjective algebra $A(\varphi, \psi) = A \bigoplus M \bigoplus S$ by giving its multiplication as

$$(a, m, s) \cdot (a', m', s') = (aa', am' + ma' + \varphi(m \otimes m'), as' + sa' + \psi(m \otimes m'))$$

for $(a, m, s), (a', m', s') \in A$. It is proved that any selfinjective algebra A without semisimple part is of the form $A(\varphi, \psi)$ for some nilpotent selfinjective algebra (φ, ψ). If A is Frobenius then we may suppose that the injective cogenerator ${}_AS_A$ is of the form ${}_\sigma D(A)_A$ for some K-algebra automorphism $\sigma \in \mathrm{Aut}_K(A)$. Moreover, if A is symmetric then we may suppose that ${}_AS_A = {}_AD(A)_A$ and that the map ψ has the property

$$\psi(m \otimes m')(1_A) = \psi(m' \otimes m)(1_A).$$

We call a nilpotent selfinjective algebra (φ, ψ) a nilpotent Frobenius algebra (resp. a nilpotent symmetric algebra) if ${}_AS_A = {}_\sigma D(A)_A$ (resp. ${}_AS_A = {}_AD(A)_A$ and the map ψ posesses the above property).

Any nilpotent symmetric algebra (φ, ψ) gives the isomorphism

$$\chi\ {}_AM_A \xrightarrow{\approx} {}_AD(M)_A$$

defined by $\chi(m)(m') = \psi(m \otimes m')(1_A)$, which satisfies the relations $\chi(m)(m') = \chi(m')(m)$ and $\chi(\varphi(m \otimes m'))(m'') = \chi(m)(\varphi(m' \otimes m''))$. Conversely, such an isomorphism χ determines the map ψ by $\psi(m \otimes m')(a) = \chi(m)(m'a)$. So, we may use the isomorphism χ to denote nilpotent symmetric algebra and we write sometimes (φ, χ) instead of (φ, ψ).

As studied in the paper [2], in order to construct graded Frobenius algebras, we start from a bimodule ${}_AX_A$, an automorphism $\sigma \in \mathrm{Aut}_K(A)$, a bimodule isomorphism $\gamma : {}_AX_A \xrightarrow{\approx} {}_\sigma X_\sigma$, and a surjective bimodule map

$$\theta : {}_AX^{\otimes n}{}_A \longrightarrow {}_\sigma D(A)_A$$

satisfying the following conditions;

(1) $\theta(x_1 \otimes x_2 \otimes \cdots \otimes x_n)(1_A) = \theta(x_2 \otimes \cdots \otimes x_n \otimes \gamma(x_1))(1_A)$, and

(2) $\theta(x \otimes X^{\otimes(n-1)}) = 0$ implies $x = 0$.

Define the map

$$\theta_i : {}_AX^{\otimes i}{}_A \longrightarrow {}_\sigma\mathrm{Hom}_{\mathrm{mod}\text{-}A}({}_AX^{\otimes(n-i)}{}_A, {}_\sigma D(A)_A)_A$$

by $\theta_i(y)(z) = \theta(y \otimes z)$. Then, the space

$$R(\theta) = \mathrm{Ker}(\theta_2) \oplus \cdots \oplus \mathrm{Ker}(\theta_n) \oplus X^{\otimes(n+1)} \oplus \cdots$$

becomes an ideal in the tensor algebra

$$T_A(X) = A \oplus X \oplus X^{\otimes 2} \oplus \cdots \oplus X^{\otimes n} \oplus X^{\otimes(n+1)} \oplus \cdots,$$

and we have a graded Frobenius algebra $\Lambda(\theta) = T_A(X)/R(\theta)$. If we put

$${}_AM_A = X \oplus X^{\otimes 2}/\mathrm{Ker}(\theta_2) \oplus \cdots \oplus X^{\otimes(n-1)}/\mathrm{Ker}(\theta_{n-1}),$$

then the algebra-structure of $\Lambda(\theta)$ gives two maps

$$\varphi : {}_AM \otimes_A M_A \longrightarrow {}_AM_A, \quad \psi : {}_AM \otimes_A M_A \longrightarrow X^{\otimes n}/\mathrm{Ker}(\theta) \cong {}_\sigma D(A)_A,$$

and we get a nilpotent Frobenius algebra (φ, ψ).

Finally in this section, we would like to mention that the Krull-Schmidt property for nilpotent algebras holds true.

Theorem 1. *Any nilpotent algebra is a direct product of indecomposable nilpotent algebras, and such a decomposition is essentially unique.*

2. Local Symmetric Algebras

In this section, we study only local symmetric algebras. Since we are assuming that the field K is algebraically closed, we may suppose that the underlyin algebra A is K itself, and we have to construct nilpotent symmetric algebra (φ, χ) over a vector space V, where $\varphi : V \otimes V \to V$ is an associative map and $\chi : V \to D(V)$ a bijective map satisfying

(1) $\chi(v_1)(v_2) = \chi(v_2)(v_1)$, and

(2) $\chi(\varphi(v_1 \otimes v_2))(v_3) = \chi(v_1)(\varphi(v_2 \otimes v_3))$.

By the following result, we know that all nilpotent symmetric algebras are obtained by combining only nilpotent symmetric indecomposable algebras.

Theorem 2. *Let $(V,\varphi) = (V_1,\varphi_1) \times (V_2,\varphi_2)$ be a direct product decomposition of a nilpotent algebra. Suppose that the map $\chi : V \to D(V)$ has a matrix expression as*

$$\begin{bmatrix} \chi_{11} & \chi_{12} \\ \chi_{21} & \chi_{22} \end{bmatrix} : V_1 \oplus V_2 \longrightarrow D(V_1) \oplus D(V_2).$$

Then, the following assertions are equivalent;

(1) *(V,φ,χ) is a nilpotent symmetric algebra,*

(2) *Both $(V_1,\varphi_1,\chi_{11})$ and $(V_2,\varphi_2,\chi_{22})$ are nilpotent symmetric algebras, $\chi_{12}(v_2)(v_1) = \chi_{21}(v_1)(v_2)$ holds for any $v_1 \in V_1$ and $v_2 \in V_2$, and further $\chi_{12}(V_2^2) = 0 = \chi_{21}(V_1^2)$.*

In order to construct graded symmetric algebra, we start from a linear surjective map

$$\theta : X^{\otimes n} \longrightarrow K$$

satisfying conditions (1) $\theta(x_1 \otimes x_2 \otimes \cdots \otimes x_n) = \theta(x_2 \otimes \cdots \otimes x_n \otimes x_1)$, and (2) $\theta(x \otimes X^{\otimes(n-1)}) = 0$ implies $x = 0$. By setting

$$V = X \oplus X^{\otimes 2}/\mathrm{Ker}(\theta_2) \oplus \cdots \oplus X^{\otimes(n-1)}/\mathrm{Ker}(\theta_{n-1}),$$

we get two maps

$$\varphi : V \otimes V \longrightarrow V, \quad \psi : V \otimes V \longrightarrow X^{\otimes n}/\mathrm{Ker}(\theta) \cong K,$$

and (φ,ψ) becomes a nilpotent symmetric algebra, and the algebra $\Lambda(\theta) = \Lambda(\varphi,\psi)$ is graded and symmetric. In the paper [2], we have proved that, for such linear maps θ and θ', $\Lambda(\theta) \cong \Lambda(\theta')$ as K-algebras if and only if $\theta' = \theta s^{\otimes n}$ for some $s \in GL(X)$. A similar result for general symmetric algebras is given by the following.

Theorem 3. *Let (φ,ψ) and (φ',ψ') be nilpotent symmetric algebras. Then, the algebras $\Lambda(\varphi,\psi)$ and $\Lambda(\varphi',\psi')$ are K-isomorphic if and only if we may suppose $\varphi' = \varphi$ and $\psi' = (\psi + f\varphi)s^{\otimes 2}$ holds for an element $s \in GL(V)$ such that $\varphi s^{\otimes 2} = s\varphi$ and a linear map $f \in D(V)$ with the property $f([V,V]_\varphi) = 0$, where $[V,V]_\varphi$ is the subspace of V generated by all elements of the form $[v_1,v_2]_\varphi = \varphi(v_1 \otimes v_2 - v_2 \otimes v_1)$.*

For a nilpotent symmetric algebra (V,φ,χ) with $V^{n+1} = 0$ and $V^n \neq 0$, we define subspaces

$$\mathrm{soc}^i(V) = \{v \in V \mid \varphi(v \otimes V^i) = 0\},$$

for all $1 \leq i \leq n$. Further, we can choose subspaces $\{V_{ij}\}_{1\leq i\leq n, 1\leq j\leq i}$ in such a way that the equalities

In fact, using the notation of the above theorem, we have the following characterization of standardly $\mathcal{X}$-stratified algebras.

Theorem 2. ([ADL1]) *Let $(A, \mathbf{e})$ be a K-algebra. Let $\mathcal{X} = (X(1), X(2), \ldots, X(n))$, where for each $1 \le i \le n$, either $X(i) \simeq \Delta(i)$ or $X(i) \simeq \overline{\Delta}(i)$. Then the following statements are equivalent.*

(i) *$(A, \mathbf{e})$ is standardly $\mathcal{X}$-stratified.*

(ii) $\mathrm{Ext}^2_A(X(i), \mathrm{D}Y(j)) = 0$ *for all* $1 \le i, j \le n$, $X(i) \in \mathcal{X}$, $Y(j) \in \mathcal{X}^o$.

(iii) $\{M \mid M$ *is filtered by modules from* $\mathcal{X}\} = \{M \mid \mathrm{Ext}^1_A(M, \mathrm{D}Y(j)) = 0$ *for all* $1 \le j \le n$, $Y(j) \in \mathcal{X}^o\}$.

Here, $\mathrm{D}Y$ denotes the dual A-module $\mathrm{Hom}_K(Y, K)$ of Y. We can also formulate quite generally the following Bernstein-Gelfand-Gelfand reciprocity law.

Theorem 3. *Let $(A, \mathbf{e})$ be a $(\mathcal{Y}, \mathcal{X})$-stratified algebra. Write $d_i = \dim_K \mathrm{End}(S(i)) = \dim_K \mathrm{End}(S^o(i))$, $1 \le i \le n$. Then*

$$d_j\big[P^o(i) : Y(j)\big] = d_i\big[X(j) : S(i)\big] \text{ and}$$

$$d_j\big[P(i) : X(j)\big] = d_j\big[Y(i) : S^o(i)\big] \text{ for all } 1 \le i, j \le n.$$

Moreover, in reference to [D], we can formulate the following theorem.

Theorem 4. *A $(\mathcal{Y}, \mathcal{X})$-stratified algebra $(A, \mathbf{e})$ is a quasi-hereditary algebra if and only if it is of finite global dimension.*

One of the immediate goals of the theory should be to determine under what conditions (?) for the sequence $\mathcal{X}$ and the algebra $(A, \mathbf{e})$, one can assert the following general statement: *Given a right stratifying sequence $\mathcal{X} = (X(1), X(2), \ldots, X(n))$ of an algebra $(A, \mathbf{e})$ satisfying the conditions (?), there is a right stratifying sequence $\mathcal{Z} = ((Z(1), Z(2), \ldots, Z(n))$ and a left stratifying sequence $\mathcal{Y} = (Y(1), Y(2), \ldots, Y(n))$ such that $X(i)$ is filtered by $Z(i)$ for each $1 \le i \le n$, and $(A, \mathbf{e})$ is a $(\mathcal{Y}, \mathcal{Z})$-stratified algebra.*

We have already pointed out that the statement holds for $\mathcal{X}$ with $X(i) \simeq \Delta(i)$ or $\overline{\Delta}(i)$ for each $1 \le i \le n$. It is also easy to verify it in the case when $(A, \mathbf{e})$ is a monomial K-algebra over a finite (connected) quiver which has a stratifying sequence $\mathcal{X}$ such that the filtration of each projective module $P(j)$ by modules from $\mathcal{X}^{(j)}$ is compatible with the K-basis of A consisting of the paths. In such situation, proceeding by induction on the trace filtration of A_A, we deal with a K-basis $\mathcal{B}_n$ of $P(n)$ which satisfies the following properties:

$$\mathcal{B}_n = \{c_n = g_1^{(n)} = h_1^{(n)}, g_2^{(n)}, \ldots, g_{r_n}^{(n)}, h_2^{(n)}, \ldots, h_{s_n}^{(n)}\} \cup$$

$$\cup \{a_{pq}^{(n)} = g_p^{(n)} h_q^{(n)} \mid 2 \le p \le r_n,\ 2 \le q \le s_n\},$$

$$P(n) = H_1 = \sum_{p=1}^{r_n} g_p A \supset \cdots \supset H_r = \sum_{p=r}^{r_n} g_p A \supset \cdots \supset H_{r_n} = g_{r_n}^{(n)} A \supset H_{r_n+1} = 0$$

is a trace filtration of $P(n)$ by $X(n)$, i.e. $H_r/H_{r+1} \simeq X(n)$ for all $1 \leq r \leq r_n$ and

$$H_1/H_2 \supset \cdots \supset (\sum_{q=s}^{s_n} h_q^{(n)} A + H_2)/H_2 \supset \cdots \supset (h_{s_n}^{(n)} A + H_2)/H_2 \supset 0$$

is a composition series of $H_1/H_2 = X(n)$.

Similarly, for every $1 \leq j \leq n-1$, we have (compatible) K-basis of $e_j A e_n A$:

$$\mathcal{B}_j = \{g_p^{(j)} \mid 1 \leq p \leq r_j\} \cup \{a_{pq}^{(j)} = g_p^{(j)} h_q^{(n)} \mid 1 \leq p \leq r_j,\ 2 \leq q \leq s_n\},$$

such that

$$e_j A e_n A \supset \cdots \supset \sum_{p=r}^{r_j} g_p^{(j)} A \supset \cdots \supset g_{r_j}^{(j)} A \supset 0$$

is a filtration of $e_j A e_n A$ by $X(n)$.

We have

$$g_p^{(j)} = e_j g_p^{(j)} e_n \ \text{ for all } 1 \leq j \leq n \text{ and } 1 \leq p \leq r_j,$$

and

$$a_{pq}^{(i)} = e_j a_{pq}^{(i)} e_{i_q} \ \text{ for all } 1 \leq j \leq n,\ 1 \leq p \leq r_j,\ 1 \leq q \leq s_n \text{ and a certain } i_q;$$

here,

$$a_{1q}^{(n)} = h_q^n \ (2 \leq q \leq s_n).$$

Now, if the K-subspace $V(n)$ of the vector space Ae_n generated by the subset

$$\{a_{pq}^{(j)} \mid a_{pq}^{(j)} e_n = a_{pq}^{(j)}\}$$

of the K-basis $\mathcal{B} = \bigcup_{j=1}^{n} \mathcal{B}_j$ of $Ae_n A$ is a left ideal of A (i.e. if $V(n)$ is an A-submodule of $P^o(n) = Ae_n$), then the K-space $Y(n)$ generated by

$$\{g_p^{(j)} \mid 1 \leq j \leq n,\ 1 \leq p \leq r_j\}$$

is a factor module of $P^o(n), Y(n) = P^o(n)/V(n)$ and the partition of the sequence $(h_1^{(n)} = h_1^{(n)} e_n, h_2^{(n)} = h_2^{(n)} e_{i_2}, \ldots, h_{r_n}^{(n)} e_{i_{r_n}})$ into n subsequences (some of which may be void) of elements with equal right annihilators (i.e. with the same e_{i_q}'s) determine filtrations of $Ae_n Ae_j$ by $Y(n)$. Observe that

$$Y(n) \otimes_K X(n) \simeq Ae_n A$$

and thus

$$\dim_K Y(n) \cdot \dim_K X(n) = \dim_K Ae_nA.$$

Let us conclude our report with a few examples.

Example 1. Let $(A, \mathbf{e})$ be the monomial algebra KQ/I, where

$$Q: \quad \overset{1}{\bullet} \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \overset{2}{\bullet} \circlearrowright \gamma \qquad \text{and} \qquad I = \langle \alpha\gamma, \beta\alpha, \gamma\beta, \gamma^3 \rangle.$$

Thus the right regular representation (the left one looks similarly) is as follows:

$$A_A = \begin{array}{c} 1 \\ 2 \\ 1 \end{array} \oplus \begin{array}{c} 2 \\ 1\,2 \\ 2 \end{array}$$

Here, Ae_2A is filtered by $X(2) = P(2)/\langle \gamma^2 \rangle$, but $P(2)$ is not filtered. There is no right (and no left) stratifying sequence of $(A, \mathbf{e})$.

Example 2. Let $(A, \mathbf{e})$ be the monomial algebra KQ/I, where

$$Q: \quad \overset{1}{\bullet} \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \overset{3}{\bullet} \underset{\gamma}{\overset{\delta}{\rightleftarrows}} \overset{2}{\bullet} \qquad \text{and} \qquad I = \langle \alpha\delta, \beta\alpha, (\gamma\delta)^2, \delta\gamma\beta \rangle.$$

Thus the right and left regular representations are as follows:

$$A_A = \begin{array}{c} 1 \\ 3 \\ 1 \end{array} \oplus \begin{array}{c} 2 \\ 3 \\ 1\,2 \\ 3 \end{array} \oplus \begin{array}{c} 3 \\ 1\,2 \\ 3 \\ 2 \\ 3 \end{array} \quad \text{and} \quad {}_AA = \begin{array}{c} 1 \\ 3 \\ 1\,2 \end{array} \oplus \begin{array}{c} 2 \\ 3 \\ 2 \\ 3 \end{array} \oplus \begin{array}{c} 3 \\ 1\,2 \\ 3 \\ 2 \\ 3 \end{array}$$

Here, Ae_3A is filtered by $X(3) = P(3)/\langle \delta\gamma\delta \rangle$, but neither e_1Ae_3A nor e_3A is filtered. There is no left filtration of Ae_3A by a local module.

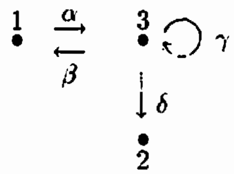
Example 3. In order to illustrate the statement following Theorem 4, let $(A, \mathbf{e})$ be the 72-dimensional monomial algebra KQ/I, where

$$Q: \quad \overset{1}{\bullet} \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \overset{3}{\bullet} \underset{\gamma}{\overset{\delta}{\rightleftarrows}} \overset{2}{\bullet} \qquad \text{and} \qquad I = \langle \beta\alpha\beta, \alpha\beta\alpha\delta\gamma, \beta\alpha(\delta\gamma)^3, \gamma\beta\alpha\delta\gamma, (\delta\gamma)^4 \rangle.$$

(a) Let $\mathcal{X} = (S(1), S(2), P(3)/\langle \delta\gamma \rangle)$; then $\mathcal{Y} = (S^o(1), S^o(2), P^o(3)/\langle \delta\gamma, \beta\alpha \rangle)$ is a left stratifying sequence and $(A, \mathbf{e})$ is a $(\mathcal{Y}, \mathcal{X})$-stratified, i.e. $\mathcal{Z} = \mathcal{X}$.

(b) Let $\mathcal{X} = (S(1), S(2), P(3)/\langle \beta\alpha\delta\gamma, (\delta\gamma)^2 \rangle)$; then $\mathcal{Z} = (S(1), S(2), P(3)/\langle \delta\gamma \rangle)$, $\mathcal{Y}$ as in (a) and $(A, \mathbf{e})$ is a $(\mathcal{Y}, \mathcal{Z})$-stratified algebra.

Example 4. (E. Lukács) Let $(A, \mathbf{e})$ be the algebra KQ/I, where

$Q:$  and $I = \langle \beta\alpha\delta, \gamma\beta, \gamma^2, \alpha\beta\alpha - \alpha\gamma \rangle.$

Here, $(A, \mathbf{e})$ has a right stratifying sequence $\mathcal{X} = (S(1),\ S(2),\ e_3A/\langle \beta\alpha\beta, \gamma\delta, \beta\alpha - \gamma \rangle)$. On the other hand, considering the left regular representation of A, one can see immediately that there is no left stratifying sequence of $(A, \mathbf{e})$:

$$
{}_AA = \begin{array}{c} 1 \\ 3 \\ 1 \\ 3 \end{array} \oplus \begin{array}{c} 2 \\ 3 \\ 1\ 3 \end{array} \oplus \begin{array}{c} 3 \\ 1 \quad \\ \quad 3 \\ 3 \quad \\ 1 \\ 3 \end{array}
$$

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