Proceedings of the 31th Symposium on Ring Theory and Representation Theory and Japan-Korea Ring Theory and Representation Theory Seminar

November 2–7, 1998
Osaka, Japan

February, 1999
Matsumoto, Japan

第３１回環論および表現論シンポジウム
および
日韓環論および表現論セミナー
報告集

１９９８年１１月２日−７日
大阪市立大学学術情報総合センター

１９９９年２月
信州大学理学部

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序

第３１回環論および表現論シンポジウムは１９９８年１１月４日−６日の日程で大阪市立大学学術情報総合センターで開催されました。本シンポジウムは日本学術振興会日韓科学協力事業セミナー　日韓環論および表現論セミナー（代表：佐藤真久（山梨大工））との共催となり、その第３、４、５日を兼ねました。

この報告集は上記シンポジウム・セミナーの講演に基づき作成されたものです。

このシンポジウムの旅費・出版費等の資金面の援助を文部省科学研究費基盤（A）（1）（研究代表者：石田正典（東北大理））から受けました。ここにあらためて謝意を表します。また会場の準備ならびに運営に関しては、浅芝秀人・加戸次郎（大阪市大理）の両氏に大変お世話になりました。関係諸氏に厚くお礼申し上げます。最後に、講演の「英語化」の要請に快く応じて下さいました講演者の皆様に感謝致します。

1999 年 2 月

西田憲司 (Kenji Nishida)、信州大学理学部
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日本学術振興会日韓科学協力事業セミナー　日韓環論および表現論セミナーは１９９８年１１月２日−７日の日程で大阪市立大学学術情報総合センターで開催されました。韓国と日本の環論研究者の交流の活発化を考慮し、第３１回環論および表現論シンポジウムと共催して行うことを認めて頂きました。

このセミナーの目的は、日本と韓国の代数学（特に環論を中心とした領域）の研究内容の理解を行い研究の更なる向上を図ることにあります。１９９５年に日本で開催された第２回日中環論国際シンポジュームが次回（１９９９年）より日本・中国・韓国の３国の共催となり韓国で開催されるにあたり、この国際会議の準備的段階として両国の研究者の研究状況を把握して、それらの成果を国際会議の場において披露できる土台を築いておくことが大きな狙いでありました。これらはの目的・狙いは参加者の活発な講演や議論で達成出来たと自負しております。参加されました日本および韓国の環論研究者の方々に改めて感謝の意を表したいと思います。また、共催を快く引き受けて下さった環論および表現論シンポジウムの開催責任者である西田憲司氏（信州大・理）、会場のお世話をして頂いた大阪市立大学の皆様, このセミナーの準備に多大な助力を頂きました朴氏（釜山大学）に特に感謝致したく存じます。

1999 年 2 月

佐藤真久、山梨大学工学部

Preface

Japan-Korea Ring Theory and Representation Theory Seminar supported by Japan Society for the Promotion of Science was held at Osaka City University from December 2, 1998 to December 7, 1998. This seminar was the joint seminar with the 31st Symposium on Ring Theory and Representation Theory to activate mathematical exchange in Japan and Korea.

Our purpose is to promote our study in the field of algebra (particularly ring theory and related fields) through mutual understanding of mathematical study and results in Japan and Korea.

At the second Japan-China International Symposium on Ring Theory, we have decided that Korea joins this symposium from next time, which will be held at Korea in 1999. We would like to make this seminar a preliminary step such as we know and understand the contents of our study each other, which will motivate joint works presented in the international symposium.

We believe this seminar finished to succeed to accomplish these aim and purpose. We would like to express our thanks to all the Japanese and Korean participants. Especially we strongly appreciate the acceptance of proposal of joint seminar to Prof. Nishida who was responsible for the 31st Symposium on Ring Theory and Representation Theory, hospitality to all the member of Department of Mathematics at Osaka City University and great help for us to prepare this seminar to Prof. J. K. Park (Pusan National University).

February, 1999

Masahisa Sato
Department of Mathematics, Yamanashi University

List of the Lecturers

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14:00–14:50 Kiyoichi Oshiro (Yamaguchi University)

On QF serial rings

15:10–16:00 Jae Keol Park (Pusan National University)

Semi-central idempotents and Triangular Representations

16:10–17:00 Chan Yong Hong (Kyung Hee University)

On week π-regularity of rings whose prime ideals are maximal and quasi-Baer rings

Nov. 3

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A characterization of AGE-rings

11:00–11:50 Chan Huh (Pusan National University)

On rings in which every maximal one sided ideal contains a maximal ideal

13:30–14:20 Hideto Asashiba (Osaka City University)

From stable equivalences to derived equivalences for some classes of self-injective algebras

14:30–15:20 Hong Kee Kim (Gyeongsang University)

Some results on the the skew polynomial rings over a reduced ring

15:40–16:30 Tai Keun Kwak (Daejin University)

Minimal prime ideals in 2-primal rings

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Quasi-duo rings and 2-primal rings

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Representation theory for fundamental groups of Hakenian manifolds

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Jones polynomial and some separable Frobenius extension

— The introduction of the work of L. Kadison —

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Cancellation of lattices and approximation properties of division algebras

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DTr-variant modules over wild algebras

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On relative projectivity for finite group algebras

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Principal blocks with extra-special defect groups of order 27

13:40–14:30 Akira Masuoka (University of Tsukuba)

Hopf algebra, Lie bialgebra extension and cohomology

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Polynomial rings which are quasi-duo

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Modules with finite Gorenstein dimension

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Noncommutative Poisson algebras and the Gerstenhaber's deformation theory

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9:00–9:50 Naoki Hamaguchi (Okayama University)

Differentially simple rings

10:00–15:50 Yosuke Ohnuki (University of Tsukuba) Kaoru Takeda (University of Tsukuba)

Hochschild extension ring given by 2-cocycle

11:10–12:00 Mitsuhiro Takeuchi (University of Tsukuba)

Kuhn theory – algebraic topology and representation theory

12:10–13:00 Nam Kyun Kim (Pusan National University)

A note on exchange rings

SEMICENTRAL IDEMPOTENTS AND TRIANGULAR REPRESENTATIONS

Gary F. Birkenmeier[1], Henry E. Heatherly[1], Jin Yong Kim[2]

and

Jae Keol Park[3]

Throughout R denotes an associative ring with unity. We say R has a *triangular matrix representation* if the R is ring isomorphic to

$$\begin{pmatrix} R_1 & R_{12} & \cdots & R_{1n} \\ 0 & R_2 & \cdots & R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_n \end{pmatrix},$$

where each R_i is a ring with unity and R_{ij} is a left R_i-right R_j-bimodule for $i < j$.

Triangular matrix representations provide an important tool in the investigation of the structure of a wide range of algebras.

In this expository note extracted from [BHKP], we introduce the concept of a *set of left triangulating idempotents*. These idempotents determine a triangular matrix representation for an algebra. The existence of a set of left triangulating idempotents does not depend on any specific conditions on the algebra (e.g., $\{1\}$ is a set of left triangulating idempotents); however if the algebra satisfies a mild finiteness condition, then such a set can be refined to a "complete" set of left triangulating idempotents in which each "diagonal" subalgebra has no nontrivial triangular matrix representation.

An idempotent $e \in R$ is *left* (resp. *right*) *semicentral* in R if $Re = eRe$ (resp. $eR = eRe$), [Bi2, p.569]. We use $\mathcal{S}_\ell(R)$ and $\mathcal{S}_r(R)$ for the sets of all left and right semicentral idempotents, respectively. Again taking e to be an idempotent of R, observe that $\mathcal{S}_\ell(eRe) = \{0, e\}$ if and only if $\mathcal{S}_r(eRe) = \{0, e\}$; when this occurs we say e is *semicentral reduced*. If 1 is semicentral reduced, then we say R is *semicentral reduced*.

An ordered set $\{e_1, \ldots, e_n\}$ of nonzero distinct idempotents in R is called a set of *left triangulating idempotents* of R if all the following hold:

(i) $1 = e_1 + \cdots + e_n$;

(ii) $e_1 \in \mathcal{S}_\ell(R)$; and

(iii) $e_{k+1} \in \mathcal{S}_\ell(f_k R f_k)$, where $f_k = 1 - (e_1 + \cdots + e_k)$, for $1 \le k \le n - 1$.

Similarly we define a *set of right triangulating idempotents* of R using (i) and $e_1 \in \mathcal{S}_r(R)$, $e_{k+1} \in \mathcal{S}_r(f_k R f_k)$. From part (iii) of the above definition, a set of left (right) triangulating idempotents is a set of pairwise orthogonal idempotents.

The detailed and enlarged version of this note will be published elsewhere with the title "Triangular Matrix Representations". All proofs of results of this note can be found in the forthcoming paper "Triangular Matrix Representations".

A set $\{e_1, \dots, e_n\}$ of left (right) triangulating idempotents of R is said to be *complete* if each e_i is also semicentral reduced. The behavior of a complete set of left triangulating idempotents is "strictly between" that of a complete set of primitive idempotents and a complete set of centrally primitive idempotents.

We use $\mathbf{I}(R)$ and $\mathbf{B}(R)$ for the sets of idempotents and central idempotents of R, respectively. Observe that $\mathcal{S}_r(R) \cap \mathcal{S}_\ell(R) = \mathbf{B}(R)$. The following result is a technical lemma describing the behavior of left semicentral idempotents.

Proposition 1. Let $e \in \mathbf{I}(R)$. Then the following conditions are equivalent:

(i) $e \in \mathcal{S}_\ell(R)$;

(ii) $1 - e \in \mathcal{S}_r(R)$;

(iii) $xe = exe$, for each $x \in R$;

(iv) $(1-e)Re = 0$;

(v) $(1-e)x = (1-e)x(1-e)$, for each $x \in R$;

(vi) eR is an ideal of R;

(vii) $R(1-e)$ is an ideal of R;

(viii) $eR(1-e)$ is an ideal of R and $eR = eR(1-e) \oplus Re$ as a direct sum of left ideals;

(ix) the function defined by $\phi(x) = \begin{pmatrix} exe & ex(1-e) \\ 0 & (1-e)x(1-e) \end{pmatrix}$ is a ring isomorphism from R to $\begin{pmatrix} eRe & eR(1-e) \\ 0 & (1-e)R(1-e) \end{pmatrix}$.

Proposition 2. R has a set of left triangulating idempotents if and only if R has a triangular matrix representation.

It is worth noting that if the set of left triangulating idempotents is complete, then the "diagonal" subalgebras are semicentral reduced. Also note that R is semicentral reduced if and only if R has no nontrivial triangular matrix representation.

Proposition 3. Let $e \in \mathcal{S}_\ell(R) \cup \mathcal{S}_r(R)$ and $f \in \mathcal{S}_\ell(eRe) \cup \mathcal{S}_r(eRe)$. The function $h : R \to fRf$, defined by $h(r) = frf$, for each $r \in R$, is a K-algebra homomorphism.

If R has a matrix representation

$$\begin{pmatrix} R_1 & R_{12} & \cdots & R_{1n} \\ 0 & R_2 & \cdots & R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_n \end{pmatrix},$$

then obviously each R_i is isomorphic to a subring of R. Furthermore, by Proposition 3, each R_i is a ring epimorphic image of R.

For the relationship between the condition of having a (complete) set of left triangulating idempotents and that of having a (complete) set of right triangulating idempotents, we have the following:

Proposition 4. The ordered set $\{b_1, \dots, b_n\}$ is a (complete) set of left triangulating idempotents of R if and only if $\{b_n, \dots, b_1\}$ is a (complete) ordered set of right triangulating idempotents.

The next two results show that a set of left triangulating idempotents can be used to provide an internal characterization of an upper triangular matrix ring. We use $T_n(A)$ for the $n \times n$ upper triangular matrix ring over the ring A.

Proposition 5. $R \cong T_n(A)$ for some ring A if and only if there exists a set of left triangulating idempotents $\{e_1, \dots, e_n\}$ of R such that:

(i) there exist ring isomorphisms $\phi_j : e_jRe_j \to e_1Re_1$ for all j, $1 \le j \le n$; and

(ii) there exist group isomorphisms $\theta_{ij} : e_iRe_j \to e_1Re_1$ such that

α) $e_1re_1 \cdot (\theta_{ij}(e_ise_j)) = \theta_{ij}((\phi_i^{-1}(e_1re_1) \cdot e_ise_j)$, and

β) $(\theta_{ij}(e_ise_j)) \cdot e_1re_1 = \theta_{ij}(e_ise_j \cdot (\phi_j^{-1}(e_1re_1)))$ for all i, j, $1 \le i, j \le n$ and $r, s \in R$.

For a chracterization of rings with a complete set of left triangulating idempotents, we have the following result.

Theorem 6. The following conditions are equivalent:

(i) R has a complete set of left triangulating idempotents;

(ii) $\{eR \mid e \in \mathcal{S}_l(R)\}$ is a finite set;

(iii) $\{Rf \mid f \in \mathcal{S}_r(R)\}$ is a finite set;

(iv) R has a complete set of right triangulating idempotents.

It is worth noting that if $\{e_1, \dots, e_n\}$ is a complete set of left triangulating idempotents of an algebra R and each of the rings e_iRe_i satisfies $\mathrm{I}(e_iRe_i) = \mathrm{B}(e_iRe_i)$, then $\{e_1, \dots, e_n\}$ is a complete set of primitive idempotents. This occurs, for example, if each e_iRe_i is commutative or duo.

We next consider the interplay of upper triangular matrix representations and quasi-Baer rings. In [Cl] Clark called a ring *quasi-Baer* if the right annihilator of every right ideal is generated by an idempotent as a right ideal. Various properties of quasi-Baer rings are extensively studied in [Cl], PZ] and [Bi4].

Proposition 7. A ring R is prime if and only if R is quasi-Baer and semicentral reduced.

Proposition 8. If R satisfies any of the following conditions, then

$$R \cong \begin{pmatrix} R_1 & R_{12} & \cdots & R_{1n} \\ 0 & R_2 & \cdots & R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_n \end{pmatrix},$$

where each R_i is semicentral reduced and satisfies the same condition as R, R_{ij} is a left R_i-right R_j-bimodule, and the K-algebras (rings) $R_1, \dots, R_n$ are uniquely determined by R up to isomorphism (induced by an inner automorphism of R) and permutation:

(i) R has a complete set of primitive idempotents;

(ii) R has no infinite set of orthogonal idempotents;

(iii) R_R has Krull dimension;

(iv) R has DCC on (idempotent generated, principal, or finitely generated) ideals;

(v) R has DCC on (idempotent generated, principal, or finitely generated) right ideals;

(vi) R has ACC on (idempotent generated, principal, or finitely generated) ideals;

(vii) R has ACC on (idempotent generated, principal, or finitely generated) right ideals;

(viii) R has either ACC or DCC on right annihilators;

(ix) R is a semilocal ring;

(x) R is a semiperfect ring;

(xi) R is a semiprimary ring.

Some of our motivating ideas for defining triangulating idempotents originated with Theorem 5 of [Bil]. This result decomposed a ring with a complete set of primitive idempotents in terms of iterated triangular matrix representations involving reduced rings and MDSN rings. Recall that R is MDSN if $0 \neq e \in \mathrm{I}(R)$ implies eR contains a nonzero nilpotent element. By Proposition 8(i), we obtain that if R has a complete set of primitive idempotents, then R has a complete set of left triangulating idempotents $\{e_1, \ldots, e_n\}$ such that each e_iRe_i is either an indecomposable reduced ring or an MDSN ring.

Proposition 9. Let R be a ring. If R has a complete set of left triangulating idempotents and satisfies any of the following conditions, then

$$R \cong \begin{pmatrix} R_1 & R_{12} & \cdots & R_{1n} \\ 0 & R_2 & \cdots & R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_n \end{pmatrix},$$

where each R_i is semicentral reduced and satisfies the same condition as R, R_{ij} is a left R_i-right R_j-bimodule, and the rings $R_1, \ldots, R_n$ are uniquely determined by R up to isomorphism (induced by an inner automorphism of R) and permutation:

(i) R_R has Gabriel dimension;

(ii) R is a (quasi-) Baer ring;

(iii) R is a right semihereditary ring;

(iv) R is a right hereditary ring;

(v) R is an I-ring (i.e., every non-nil right ideal contains a nonzero idempotent element);

(vi) R is a π-regular ring;

(vii) R is a right semiartinian ring;

(viii) R is a PI-ring;

(ix) R is a right PP-ring;

(x) R is a semiregular ring.

Observe in Proposition 9(ii), if R is quasi-Baer, then each R_i is prime by using Proposition 7.

Note that each of the following classes of semiprime rings is closed relative to subrings of the form eRe, where $e = e^2$: (i) von Neumann regular, (ii) biregular, (iii) (right) fully idempotent, (iv) right V-ring. Also if R is semiprime, then R is semicentral reduced if and only if R is indecomposable. Thus we have: if R has a complete set of left triangulating idempotents and is from one of the above classes, then $R = \bigoplus R_i$, where each R_i is indecomposable and from the same class as R.

Recall that R has a "block decomposition" if and only if R has a complete set of centrally primitive idempotents [L, Sections 21, 22]. Our next result states that if R has a complete set of left triangulating idempotents, then R has a "block decomposition".

For $0 \neq e \in \mathrm{B}(R)$, e is said to be *centrally primitive* if 0 and e are the only central idempotents in eR. Also, R is said to have a it complete set of centrally primitive idempotents if there exists a finite set of centrally primitive pairwise orthogonal idempotents whose sum is the unity f R [L, Sections 21 and 22].

Proposition 10. (i) If R has a complete set of primitive idempotents, then R has a complete set of left triangulating idempotents.

(ii) If R has a complete set of left triangulating idempotents, then R has a complete set of centrally primitive idempotents.

Acknowledgments. The first author is grateful for the gracious hospitality he received at Busan National University and at Kyung Hee University. The participation in the 31st Japan Ring Theory Symposium and the 1st Japan-Korea Ring Theory and Representation Theory Seminar jointly held at Osaka City University of the third and fourth authors was supported in part by KOSEF.

REFERENCES

[BHKP] G. F. Birkenmeier, H. E. Heatherly, J. Y. Kim, and J. K. Park, *Triangular matrix representations*, Preprint.

[Bi1] G. F. Birkenmeier, *Indecomposable decompositions and the minimal direct summand containing the nilpotents*, Proc. Amer. Math. Soc. **73** (1979), 11–14.

[Bi2] G. F. Birkenmeier, *Idempotents and completely semiprime ideals*, Comm. Algebra **11** (1983), 567–580.

[Cl] W. E. Clark, *Twisted matrix units semigroup algebras*, Duke Math. J. **34** (1967), 417–423.

[L] T. Y. Lam, *A First Course in Noncommutative Rings*, Springer-Verlag, Heidelberg-New York, 1991.

[PZ] A. Pollingher and A. Zaks, *On Baer and quasi-Baer rings*, Duke Math. J. **37** (1970), 127–138.

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π-REGULAR BAER RINGS WITH COUNTABLY MANY IDEMPOTENTS

GARY F. BIRKENMEIER, JIN YONG KIM AND JAE KEOL PARK

R will denote an associative ring with unity, $J(R)$ its Jacobson radical and $\mathrm{I}(R)$ its set of idempotent elements. From [6], R is called a *π-regular* ring if for every $a \in R$ there exist a positive integer n (depending on a) and an element $x \in R$ such that $a^n = a^n x a^n$. We use $|X|$ and $\mathfrak{c}$ to denote the cardinality of a set X and the cardinality of the continuum, respectively.

In 1950 [6], Kaplansky proved that every orthogonally finite (i.e., no infinite set of pairwise orthogonal idempotents) π-regular ring is semilocal. In 1967 [10], Small proved that every orthogonally finite p.p.-ring is Baer. As a corollary he obtained that every left perfect left p.p.-ring is semiprimary. In 1974 [9], Rangaswamy established that a countable regular Baer ring of cardinality less than $\mathfrak{c}$ is semisimple Artinian. The foregoing results motivated us to ask the following natural question: Is a countable π-regular Baer ring a semiprimary ring? In this paper we give a positive answer to this question. More generally we will show that if R is a π-regular Baer ring such that $|\mathrm{I}(R)| < \mathfrak{c}$, then $R = A \bigoplus B$ where A is a finite direct sum of division ring and B is a semiprimary ring with $|B| < \mathfrak{c}$. Rangaswamy's theorem then becomes an immediate corollary of this result.

Recall from [1] that an idempotent in a ring R is called *left* (resp. *right*) *semicentral* if $xe = exe$ (resp. $ex = exe$) for all $x \in R$. It can be easily checked that an idempotent e is left (resp. right) semicentral if and only if eR (resp. Re) is an ideal of R. For a ring R, $\mathcal{S}_\ell(R)$ (resp. $\mathcal{S}_r(R)$) denotes the set of all left (resp. right) semicentral idempotents of R. We say that an idempotent $e \in R$ is called *semicentral reduced* if $\mathcal{S}_\ell(eRe) = \{0, e\}$. Note that e is semicentral reduced if and only if $\mathcal{S}_r(eRe) = \{0, e\}$. If 1 is semicentral reduced, then we say that R is semicentral reduced. Recall from [7] and [4] that R is *(quasi-)Baer* if the right annihilator of every (right ideal) nonempty subset of R is generated (as a right ideal) by an idempotent. The study of Baer rings has its roots in functional analysis [7]. Note that a ring R is prime if and only if R is quasi-Baer and semicentral reduced [2, Lemma 4.2].

Definition 1. [2] An ordered set $\{b_1, \ldots, b_n\}$ of nonzero distinct idempotents in R is called a set of *left* (resp. *right*) *triangulating idempotents* of R if all of the following hold:

(i) $1 = b_1 + \cdots + b_n$;

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

(ii) $b_1 \in \mathcal{S}_\ell(R)$ (resp. $b_1 \in \mathcal{S}_r(R)$);

(iii) $b_{k+1} \in \mathcal{S}_\ell(c_k R c_k)$ (resp. $b_{k+1} \in \mathcal{S}_r(c_k R c_k)$), where $c_k = 1 - (b_1 + \cdots + b_k)$ for $1 \leq k \leq n-1$.

Such a set of triangulating idempotents is called a *complete set of left* (resp. *right*) *triangulating idempotents* if each idempotent of the set is semicentral reduced. Note that an orthogonally finite ring has a complete set of primitive idempotents. Moreover, by Proposition 2.14 in [2], a ring with a complete set of primitive idempotents has a complete set of triangulating idempotents.

Lemma 2. Let A be a ring with unity, n a positive integer such that $n \geq 2$, $R = \mathrm{Mat}_n(A)$, and T is the ring of n-by-n upper triangular matrices over A. Then:

(i) R is finite if and only if $\mathcal{S}_\ell(T)$ is finite if and only if $\mathrm{I}(R)$ is finite.

(ii) If R is infinite, then $|A| = |R| = |\mathcal{S}_\ell(T)| = |\mathrm{I}(R)|$.

Theorem 3. Let R be a ring with a complete set of left triangulating idempotents $\{c_1, \ldots, c_n\}$ such that each $c_i R c_i$ is a simple Artinian ring. Then

(i) $R = A \bigoplus B$ (ring direct sum);

(ii) $A = \bigoplus_{i=1}^m A_i$, where $m \leq n$ and A_i is a simple Artinian ring;

(iii) $B \cong \begin{pmatrix} B_1 & B_{12} & \ldots & B_{1k} \\ 0 & B_2 & \ldots & B_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & B_k \end{pmatrix}$, where each B_i is a simple Artinian ring, each B_{ij} is a left B_i-right B_j-bimodule, and $k \leq n$;

(iv) $|B| < \infty$ if and only if $|\mathcal{S}_\ell(B)| < \infty$;

(v) if B is infinite, then $|B| = |\mathcal{S}_\ell(B)| \leq \mathcal{S}_\ell(R)$;

(vi) if $|A_i| > |\mathrm{I}(R)|$, then A_i is a division ring.

The following proposition provides a large class of rings satisfying the hypothesis of Theorem 3 including semiprimary herediatry rings and semiprimary right nonsingular right CS rings [3].

Proposition 4. If R is a semiprimary quasi-Baer ring, then R has a complete set of left triangulating idempotents $\{c_1, \ldots, c_n\}$ such that each $c_i R c_i$ is a simple Artinian ring.

Observe that the following ring is not quasi-Baer, but it satisfies the hypothesis of Theorem 3:

$$R = \begin{pmatrix} F & \mathrm{Mat}_2(F) & \mathrm{Mat}_2(F) \\ 0 & F & \mathrm{Mat}_2(F) \\ 0 & 0 & F \end{pmatrix},$$

where F is a field.

Recall from [5] that a ring R is an *I-ring* if every nonnil right ideal contains a nonzero idempotent. Note that every π-regular ring is an I-ring.

Lemma 5. Let R be an orthogonally finite let p.p.-ring. Then following conditions are equivalent:

(i) R is an I-ring;

(ii) R is a semiprimary ring;

(iii) R is a π-regular ring.

Immediately we have the following corollary which is due to Small [10].

Corollary 6. If R is a left perfect left p.p.-ring, then R is a semiprimary ring.

Lemma 7. If R is a Baer ring $|\mathrm{I}(R) < \mathrm{c}$, then R is orthogonally finite.

Theorem 8. Let R be a π-regular Baer ring with $|\mathrm{I}(R)| < \mathrm{c}$. Then

$$R \cong A \bigoplus \begin{pmatrix} B_1 & B_{12} & \dots & B_{1k} \\ 0 & B_2 & \dots & B_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B_k \end{pmatrix},$$

where

(i) A is a finite direct sum of division rings;

(ii) each B_i is a simple Artinian ring and each B_{ij} is a left B_i-right B_j bimodule;

(iii) either B is finite or $|B| = |\mathrm{I}(R)| < \mathrm{c}$.

Corollary 9. Let R be a π-regular Baer ring with $|R| < \mathrm{c}$. Then R is semiprimary and

$$R \cong \begin{pmatrix} R_1 & R_{12} & \dots & R_{1n} \\ 0 & R_2 & \dots & R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & R_n \end{pmatrix},$$

where each R_i is simple Artinian and each R_{ij} is a left R_i-right R_j bimodule. Moreover gl.dim$R \leq n - 1$.

Since a regular ring is semiprime, Rangaswamy's theorem [9] is an immediate consequence of Corollary 9.

Corollary 10. Let R be a π-regular Baer ring with only countably many idempotents. If R is an algebra over a field K where $|K| > \aleph_0$, then R is a finite direct sum of division rings.

The following examples are provided to illustrate and delimit our results.

Recall that a ring R is called *left* (resp. *right*) *weakly regular* (or *fully idempotent*) if $a \in RaRa$ (resp. $a \in aRaR$) for every $a \in R$. Left and right weakly regular rings are called *weakly regular*.

By Theorem 8, regular Baer rings with only countably many idempotents are semisimple Artinian. So one may raise the following question: Is weakly regular Baer ring with only countably many idempotents a semisimple Artinian ring?

Example 11. There exists a weakly regular Baer ring with only countably many idempotents, but it is not semisimple Artinian. In fact let R be the first Weyl algebra over a field of characteristic zero. Then R is weakly regular Baer with only countably many idempotents, but R is not semisimple Artinian.

Example 12. Theorem 8 cannot be extended to the class of quasi-Baer rings. There exists a regular quasi-Baer ring with only countably many idempotents, but it is not semiprimary. For a finite field F, let

$$R = \{ \begin{pmatrix} A & & & 0 \\ & a & & \\ & & a & \\ 0 & & & \ddots \end{pmatrix} \mid A \in \mathrm{Mat}_n(F), a \in F, n = 1, 2, \dots \}.$$

Then R is prime regular, and so R is quasi-Baer. Also note that R is countable. However R is not orthogonally finite, hence R cannot be semiprimary. Hence, R cannot be a Baer ring because of Lemma 5.

Example 13. The condition "orthogonally finite" in Lemma 5 is not superfluous. There exists a Baer ring which is an I-ring, but not π-regular. Let

$$R = \{(a_n)_{n=1}^{\infty} \in \prod \mathbb{Q} \mid a_n \in \mathbb{Z} \text{ eventually}\},$$

where $\prod \mathbb{Q}$ is the countably infinite direct product of the rationals $\mathbb{Q}$ and $\mathbb{Z}$ is the ring of integers. Then $\prod \mathbb{Q}$ is the maximal ring of quotients of R. Since $Q = \prod \mathbb{Q}$ is regular self-injective, it is Baer. Also note that the set of all idempotents of Q is that of R. Now for a nonempty subset X of R, it follows that $r_R(X) = r_Q(X) \cap R = eQ \cap R$ for some idempotent $e \in R$, where $r(-)$ denotes the right annihilator. Therefore $r_R(X) = eR$, and hence R is a Baer ring. Next, to show that R is an I-ring, let K be a nonzero ideal of R. Then there is a nonzero element,say $x \in K$ with nonzero k-th coordinate, say x_k for some k. Let $y \in \prod \mathbb{Q}$ with the k-th coordinate x_k and 0 for other coordinates. Then $y \in R$ and $xy \in I$ is an nonzero idempotent in R. So R is an I-ring.

Finally, let $\alpha = (2, 2, \dots) \in R$. If R is π-regular, then there are a positive integer n and an element $\beta \in R$ such that $\alpha^n = \alpha^n \beta \alpha^n$. So there is an integer y such that $2^n = 2^n y 2^n$, a contradiction. Thus the ring R cannot be π-regular.

Acknowledgments. The author would like to express his gratitude to Professor Masahisa Sato for his giving the opportunity to present this material on the Japan-

Korea Ring Theory and Representation Theory Seminar jointed with the 31st Japan Ring Theory and Representation Theory Symposium.

REFERENCES

1. G. F. Birkenmeier, *Idempotents and completely semiprime ideals*, Comm. Algebra **11** (1983), 567–580.
2. G. F. Birkenmeier, H. E. Heatherly, J. Y. Kim and J. K. Park, *Semicentral idempotents and triangular representations*, preprint..
3. A. W. Chatters and C. R. Hajarnavis, *Rings in which every complement right ideal is a direct summand*, Quart. J. Math. (2)**28** (1977), 61–81.
4. W. E. Clark, *twisted matrix units semigroup algebras*, Dule Math. J. **34** (1967), 417–423.
5. N. Jacobson, *Structure of Rings*, Amer. Math. Soc. Colloq. Publ. **37**, Providence. 1964..
6. I. Kaplansky, *Topological representation of algebras, II*, Trans. Amer. Math. Soc. **68** (1950), 62–75.
7. I. Kaplansky, *Rings of Operators*, W. A. Benjamin, New York, 1968.
8. J. Y. Kim and J. K. Park, *When is a regular ring a semisimple Artinian ring?*, Math. Japonica **45** (1997), 311–313.
9. K. M. Rangaswamy, *Regular and Baer rings*, Proc. Amer. Math. Soc. **42** (1974), 354–358.
10. L. W. Small, *Semihereditary rings*, Bull. Amer. Math. Soc. **73** (1967), 656–658.

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SOME PROPERTIES ON AGE AND LSD-GENERATED RINGS

YONG UK CHO

1. INTRODUCTION

This paper is a summary which is talked at the Korea-Japan Joint Seminar and 31-Japan Ring Theory and Representation Theoty Symposium in Osaka City University.

In this paper, we will investigate the rings in which all the additive endomorphisms or only the left multiplication endomorphisms are generated by ring endomorphisms. This study was motivated by: the work on the.Sullivan Problem (i.e., characterize those rings in which every additive endomorphism is a ring endomorphism, these rings are called *AE rings*) [12], [1], [5], [6], [7], [9], and the current investigation of LSD-generated algebras [3] and SD-generated algebras [2].

Throughout this paper, R denotes an associative ring not necessarily with unity, $End(R,+)$ the ring of additive endomorphisms of R, and $End(R,+,\cdot)$ the monoid of ring endomorphisms of R. For $X \subseteq R$, we use $gp(X)$ for the subgroup of $(R,+)$ generated by X. For each $x \in R$, ${}_x\tau$ denotes the left multiplication mapping (i.e., $a \longmapsto xa$, for all $a \in R$). Observe ${}_x\tau \in End(R,+)$. $\mathcal{LGE}(R)$ is the set $\{x \in R \mid {}_x\tau \in gp(End(R,+,\cdot))\}$. Note that $\mathcal{LGE}(R)$ is a subring of R. Sometimes, we will use the notations: $End_{\mathbf{Z}}(R)$ instead of $End(R,+)$, $End(R)$ instead of $End(R,+,\cdot)$ and $GE(R)$ instead of $gp < End(R,+,\cdot) >$. Clearly, GE(R) is a subring of $End_{\mathbf{Z}}(R)$.

$\mathcal{L}(R)$ is the set $\{x \in R \mid xab = xaxb\}$. $(\mathcal{L}(R),\cdot)$ is a subsemigroup of $(R,\cdot)$, and $x \in \mathcal{L}(R)$ if and only if ${}_x\tau \in End(R,+,\cdot)$. Also $\mathcal{L}(R) \subseteq \mathcal{LGE}(R)$ and $\mathcal{L}(R)$ contains all one-sided unities of R, the left annihilators of R^2 and all central idempotents.

We use $\mathcal{RGE}(R)$ and $\mathcal{R}(R)$ for the right sided analogs of $\mathcal{LGE}(R)$ and $\mathcal{L}(R)$, respectively. We call that a ring R is an *AGE ring* (*LGE ring*) if

$$End_{\mathbf{Z}}(R) = GE(R) \ (R = \mathcal{LGE}(R)).$$

Similarly we can define the *RGE ring* (i.e.,$R = \mathcal{RGE}(R)$).

Clearly, we see that every AE ring is AGE, LGE and RGE, but not conversely from the following examples. Note if the left regular representation of R into $End(R,+)$ is surjective, then R is an AGE ring.

R is called *LSD* (*LSD-generated*) if $R = \mathcal{L}(R)$ ($R = gp(\mathcal{L}(R))$), and also R is called *RSD* (*RSD-generated*) if $R = \mathcal{R}(R)$ ($R = gp(\mathcal{R}(R))$) [4] and [3]. R is called *SD* (*SD-generated*) if $R = \mathcal{L}(R) \cap \mathcal{R}(R)$ ($R = gp(\mathcal{L}(R) \cap \mathcal{R}(R))$ [2]. The classes of LSD, LSD-generated, SD and SD-generated rings are closed with respect to homomorphisms and direct sums. Observe that the class of LGE rings contains both the class of AGE rings and the class of LSD-generated rings. The class of AGE rings is contained in the class of RGE rings. In the sequel, examples are provided to show that the classes of LGE, AGE and LSD-generated rings are distinct. Although the class of AE rings is a proper subclass of the class of SD rings, the class of AGE rings is not contained in the class of SD-g n particular, since the rings $\mathbb{Z}$ and $\mathbb{Z}_n$ are additively generated by 1, and $End_{\mathbb{Z}}(\mathbb{Z}) \cong \mathbb{Z}, End_{\mathbb{Z}_n}(\mathbb{Z}) \cong \mathbb{Z}_n$, we see that $\mathbb{Z}$ and $\mathbb{Z}_n$ are both AGE, LSD-generated and SD-generated rings. However, $\mathbb{Z}$ and $\mathbb{Z}_n$ are all not AE rings except the cases $\mathbb{Z}_1$ and $\mathbb{Z}_2$, because any nontrivial on $\mathbb{Z}$ or $\mathbb{Z}_n$ is additive endomorphism but which is not ring endomorphism. on the other hand, $x \in \mathcal{L}(R)$ implies $x^3 = x^n$ for $n > 3$, then $\mathcal{L}(S) = \{0\}$ for any nonzero proper subring S of $\mathbb{Z}$. Hence any nonzero proper subring of $\mathbb{Z}$ is an AGE ring which is not LSD-generated and SD-generated.

Proposition 2.2. *For every AGE ring R, and for any positive integer n, we get that* $\bigoplus_{i=1}^{n} R_i$ *is an AGE ring, where* $R_i \cong R$, *for all i=1,2,....,n.*

Proof. We prove the case for $n = 2$, that is, $R \bigoplus R$. Similarly, we can prove for the case $n > 2$. We must show that

$$End_{\mathbb{Z}}(R \bigoplus R) = GE(R \bigoplus R).$$

Since $End_{\mathbb{Z}}(R \bigoplus R) \cong Mat_2(End_{\mathbb{Z}}(R))$, we obtain that

$$End_{\mathbb{Z}}(R \bigoplus R) \cong \begin{bmatrix} End_{\mathbb{Z}}(R) & End_{\mathbb{Z}}(R) \\ End_{\mathbb{Z}}(R) & End_{\mathbb{Z}}(R) \end{bmatrix} = \begin{bmatrix} GE(R) & GE(R) \\ GE(R) & GE(R) \end{bmatrix}.$$

Let $f \in End_{\mathbb{Z}}(R \bigoplus R)$ such that

$$f = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}, \ f_{ij} \in GE(R).$$

Then

$$f_{11} = \sum_i \lambda_i h_i, \ f_{12} = \sum_j \lambda_j h_j, \ f_{21} = \sum_k \lambda_k h_k, \ f_{22} = \sum_t \lambda_t h_t,$$

where, $\lambda's \in \mathbb{Z}$ and $h's \in End(R)$. Thus f is expressed of the form

$$f = \sum_i \lambda_i \begin{bmatrix} h_i & 0 \\ 0 & 0 \end{bmatrix} + \sum_j \lambda_j \begin{bmatrix} 0 & h_j \\ 0 & 0 \end{bmatrix} + \sum_k \lambda_k \begin{bmatrix} 0 & 0 \\ h_k & 0 \end{bmatrix} + \sum_t \lambda_t \begin{bmatrix} 0 & 0 \\ 0 & h_t \end{bmatrix}.$$

Since all $\begin{bmatrix} h_i & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & h_j \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ h_k & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & h_t \end{bmatrix}$ are ring endomorphisms of $R \bigoplus R$. Hence $R \bigoplus R$ is an AGE ring. □

¿From Example 2.1 and Proposition 2.2, there exist numerously many examples of AGE rings and LSD-generated rings.

lemma 2.3. *For any onto ring endomorphism h, $\mathcal{L}(R)$ and $\mathcal{R}(R)$ are all invarant under h.*

Proposition 2.4. *Let R be a ring with unity. If R is an AGE ring with $S \subseteq End(R)$ such that $End_{\mathbb{Z}}(R) = gp < S >$, and each element of S is onto, then R is an LSD-generated, moreover SD-generated.*

Proof. Let $x \in R$. Consider a left translation mapping $\phi_x : R \longrightarrow R$ by $\phi_x(a) = xa$ for all $a \in R$, which is a group endomorphism. Since R is an AGE ring,

$$\phi_x = \sum_i^n \lambda_i h_i,$$

where $\lambda_i \in \mathbb{Z}$ and $h_i \in End(R)$ such that h_i is onto, $i = 1, 2, \cdots, n$. Since $1 \in R$, $\phi_x(1) = \sum_i^n \lambda_i h_i(1)$, that is, $x = \sum_i^n \lambda_i h_i(1)$ and since $1 \in \mathcal{L}(R) \cap \mathcal{R}(R)$ by lemma 2.3, $h_i(1) \in \mathcal{L}(R) \cap \mathcal{R}(R)$. Hence R is $LSD-$generated and $RSD-$generated, so $SD-$generated. □

Example 2.5.

(1) *If S is an LSD-generated ring, then*

$$R = \begin{bmatrix} S & S \\ 0 & S \end{bmatrix}$$

is also LSD-generated by the set

$$\{\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}, \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}, \begin{bmatrix} 0 & y \\ 0 & y \end{bmatrix} \mid v, x, y \in \mathcal{L}(S)\}.$$

(2) *If S is an RSD-generated ring, then*

$$R = \begin{bmatrix} S & S \\ 0 & S \end{bmatrix}$$

is also RSD-generated by the set

$$\{\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}, \begin{bmatrix} y & y \\ 0 & 0 \end{bmatrix} \mid v, x, y \in \mathcal{R}(S)\}.$$

In particular,

$$R = \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{bmatrix}$$

is an LSD-generated ring with the generators: $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$*, and also an RSD-generated ring with the generators:* $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ *and* $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$*, but which is not an SD-generated ring. Clearly,* $\begin{bmatrix} \mathbb{Z} & 0 \\ 0 & \mathbb{Z} \end{bmatrix}$ *is an SD-generated ring.*

Similarly,

$$R = \begin{bmatrix} \mathbb{Z}_n & \mathbb{Z}_n \\ 0 & \mathbb{Z}_n \end{bmatrix}$$

is both LSD-generated and RSD-generated, but which is not SD-generated.

Example 2.6. *Let S be an LSD semigroup (i.e, $xab = xaxb$, for all $x, a, b \in S$). Then the semigroup ring $K[S]$, where K is $\mathbb{Z}$ or $\mathbb{Z}_n$, is an LSD-generated ring. In particular, let S be a nonempty set and define multiplication on S by $st = t$, for each $s, t \in S$. Then $\mathbb{Z}[S]$ and $\mathbb{Z}_n[S]$ are LSD-generated rings. Furthermore if $|S| = 2$, then $\mathbb{Z}_2[S]$ is an LSD ring which is not an AGE ring.*

Proposition 2.7. *Let $Y \subseteq End(R, +, \cdot)$ and $S \subseteq R$ such that $f(S)$, for each $f \in Y$.*

(1) *If R is an LGE ring and for each $x \in R$, ${}_x\tau = \sum_{i \in I} \pm f_i$, where each $f_i \in Y$, then $gp(S)$ is a left ideal of R.*
(2) *If R is an AGE ring and $Y = End(R, +, \cdot)$, then $h(S) \subseteq gp(S)$, for each $h \in End(R, +)$.*

Proof. (1) Let $x \in R$ and $w \in gp(S)$. Then $w = \sum_{j \in J} k_j s_j$, where each $k_j \in \mathbb{Z}$ and each $s_j \in S$. Also there exist $f_i \in Y$ such that

$$ {}_x\tau = \sum_{i \in I} \pm f_i. $$

Hence

$$ xw = {}_x\tau(w) = \sum_{i \in I} \pm f_i(w) = \sum_{i \in I} \pm f_i(\sum_{j \in J} k_j s_j) = \sum_{i \in I} \sum_{j \in J} \pm k_j f_i(s_j) \in gp(S). $$

Thus $gp(S)$ is a left ideal of R.

(2) The proof of this part is similar to that of part (1). □

Proposition 2.7 can be used to show that $R = F[x]$ is not an LGE ring, where F is a field. Assume that $0 \neq f \in End(R, +, \cdot)$. Then $f(1) = 1$ since R is an integral domain. By Proposition 2.7 (1), if R is an LGE ring, then $gp(\mathcal{U}(R)) = R$, where $\mathcal{U}(R)$ is the unit group of R. This is a contradiction.

Corollary 2.8. *Let $S = \mathcal{I}(R)$, $\mathcal{N}(R)$ or the set of quasiregular elements of R.*

(1) *If R and Y are as in Proposition 2.7 (1), then $gp(S)$ is a left ideal of R.*
(2) *If R and Y are as in Proposition 2.7 (2), then $h(S) \subseteq gp(S)$, for each $h \in End(R, +)$.*

Observe that Example 2.5 is an LSD-generated ring which is not an AGE ring. To see this, let $h : R \longrightarrow R$ be defined by

$$ h(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}) = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}. $$

Then $h \in End(R, +)$, but $h(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}) \notin \mathcal{N}(R) = \begin{bmatrix} 0 & V \\ 0 & 0 \end{bmatrix}$. By Corollary 2.8 (2), R is not an AGE ring.

Corollary 2.9.

(1) *If R is an LGE ring with a right unity, then $R = gp(\mathcal{I}(R))$.*

(2) *If R is a simple AGE ring, then $R = gp(\mathcal{N}(R))$.*

It is immediate that the classes of LGE rings and LSD-generated rings are closed with respect to direct sums.

Let $Y \subseteq End(R,+,\cdot)$ and let R^Y denote

$$\{x \in R | f(x) = x, \text{ for each } f \in Y\}.$$

Observe that R^Y is a subring of R (when Y is a group acting as automorphisms on R, then R^Y is called the *fixed ring* under Y).

Remark. *Let $X \subseteq \mathcal{LGE}(R)$. For each $x \in X$, pick a representation of ${}_x\tau = \sum_{i\in I} k_i f_i$ such that $k_i \in \mathbb{Z}$ and $f_i \in End(R,+,\cdot)$. Let Y_x be the set of f_i in this representation. Let $Y = \cup_{x\in X} Y_x$. If $< X >$ is the subring generated by X, then R^Y is a left $< X >$-module. If $< X > = R$, then R^Y is a left ideal of R and R is an LGE ring.*

Proof. Let $x \in X$ and $s \in R^Y$. Then there exists a representation of ${}_x\tau = \sum_{i\in I} k_i f_i$ such that $k_i \in \mathbb{Z}$ and $f_i \in Y$. Hence

$$xs = {}_x\tau(s) = \sum_{i\in I} k_i f_i(s) = (\sum_{i\in I} k_i)s \in R^Y.$$

Since X generates R, then R^Y is a left ideal of R. ☐

Lemma 2.10. *Let $x \in R$ such that ${}_x\tau = k_1 f_1 + k_2 f_2$, where $k_1, k_2 \in \mathbb{Z}$ and $f_1, f_2 \in End(R,+,\cdot)$. Then $k_1 k_2[k_1 + k_2][r(x)]^2 \subseteq ker f_1 \cap ker f_2$.*

By [3, Proposition 2.2], every prime LSD-generated ring is a domain. Our next result shows this result can be extended to some types of LGE rings.

Proposition 2.11. *Let R be a prime ring with zero characteristic. If for each $x \in R$ there exist $k_1, k_2 \in \mathbb{Z}$ with $k_1 + k_2 \neq 0$ and $f_1, f_2 \in End(R,+,\cdot)$ such that ${}_x\tau = k_1 f_1 + k_2 f_2$, then R is a domain.*

Proof. Let $0 \neq x \in R$. There exist $k_1, k_2 \in \mathbb{Z}$ and $f_1, f_2 \in End(R,+,\cdot)$ such that ${}_x\tau = k_1 f_1 + k_2 f_2$. Let $k = k_1 k_2[k_1 + k_2]$ with $k_1 + k_2 \neq 0$. Consider the following cases:

Case 1. Assume $kR \neq 0$. Then

$$x(ker f_1)(ker f_2) = k_1 f_1(ker f_1) f_1(ker f_2) + k_2 f_2(ker f_1) f_2(ker f_2) = 0.$$

Since R is prime and $xR(ker f_1)(ker f_2) = 0$, then either $ker f_1 = 0$ or $ker f_2 = 0$. By Lemma 2.10, $k[r(x)]^2 = 0$. Hence $k[r(x)] = 0$. So $(kR)R[r(x)] = 0$. Since $kR \neq 0$, then $r(x) = 0$.

Case 2. Assume $kR = 0$. Then either $k_1 = 0$ or $k_2 = 0$ (but not both). Without loss of generality, assume $k_2 = 0$. Then ${}_x\tau = k_1 f_1$. Let $\alpha \in r(x)$. Then $k_1\alpha \in ker f_1$. Hence $xRker f_1 = 0$. Since $x \neq 0$, then $ker f_1 = 0$. Hence $(k_1 R)Rr(x) = 0$. Since $k_1 R \neq 0$, then $r(x) = 0$. Therefore, in all cases, R is a domain. ☐

Proposition 2.12. *Let R be a ring and $X \subseteq R$ such that $R = gp(X)$.*

(1) *If I is the ideal generated by $\{axay - axy \mid a, x, y \in X\}$, then R/I is an LSD-generated ring.*
(2) *If J is the ideal generated by $\{xbyb - xyb \mid b, x, y \in X\}$, then R/J is an RSD-generated ring.*

In the class of LGE rings the LSD-generated rings are somewhat well behaved (e.g., closed with respect to homomorphic images and direct sums). Hence the following problems naturally arise:

Problem 1. *Determine conditions which guarantee that a prime AGE (or LGE) ring is a domain.*

Problem 2. *Determine conditions which ensure that an AGE(or LGE) ring is LSD-generated.*

Acknowledgment

the auther is grateful for the kind hospitality while he stayed at the Osaka City University in Japan. Also he was supported (in part) by the Mooryang Hyang Research Foundation.

References

1. G. F. Birkenmeier and H. E. Heatherly, *Rings whose additive endomorphisms are ring endomorphisms*, Bull. Austral. Math. Soc. 42 (1990), 145-152.
2. G. F. Birkenmeier and H. E. Heatherly, *Self-distributively generated algebras*, Proceedings of the 54th Workshop on General Algebra, to appear.
3. G. F. Birkenmeier and H. E. Heatherly, *Left self-distributively generated algebras*, submitted.
4. G. F. Birkenmeier, H. E. Heatherly and T. Kepka, *Rings with left self distributive multiplication*, Acta Math. Hungar. 60 (1992), 107-114.
5. S. Dhompongsa and J. Sanwong, *Rings in which additive mappings are multiplicative*, Studia Sci. Math. Hungar. 22 (1987), 357-359.
6. M. Dugas, J. Hausen and J.A. Johnson, *Rings whose additive endomorphisms are multiplicative*, Period. Math. Hungar. 23 (1991), 65-73.
7. M. Dugas, J. Hausen and J.A. Johnson, *Rings whose additive endomorphisms are ring endomorphisms*, Bull. Austral. Math. Soc. 45 (1992), 91-103.
8. S. Feigelstock, *Rings whose additive endomorphisms are multiplicative*, Period. Math. Hungar. 19 (1988), 257-260.
9. Y. Hirano, *On rings whose additive endomorphisms are multiplicative*, Period. Math. Hungar. 23 (1991), 87-89.
10. A. V. Kelarev, *On left self distributive rings*, Acta Math. Hungar. 71 (1996), 121-122.
11. K. H. Kim and F. W. Roush, *Additive endomorphisms of rings*, Period. Math. Hungar. 12 (1981), 241-242.

12. R. P. Sullivan, *Research problem No. 23*, Period. Math. Hungar. 8 (1977), 313-314.

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Differentially simple rings

Naoki HAMAGUCHI

Abstract

We introduce results obtained in [5] of simple derivations. The article also contains a new example of a simple derivation.

1 Definitions and known results

Let K be a field of characteristic zero, and R a commutative K-algebra. A *K-derivation* d of R is a K-linear map of R into itself such that $d(ab) = d(a)b + ad(b)$ for all $a,\ b \in R$.

Let D be a set of K-derivations of R. An ideal I of R is said to be a *D-ideal* if $d(I) \subseteq I$ for all $d \in D$. If R has no D-ideal other than (0) and R, then R is said to be *D-simple.*

Let (R, M) be a local ring that is the localization of a finitely generated K-algebra. Seidenberg showed that if R is D-simple then R is regular([6]). However, in general, examples of non-regular local $\mathbb{Q}$-algebras that are D-simple for some D have been constructed in [4], [2] and [1]. One of them is as follows:

Example 1. ([1]) Let $\{y_1,\ y_2, \cdots,\ z_1,\ z_2, \cdots\}$ be a set of indeterminates over $\mathbb{Q}$, and $F = \mathbb{Q}(y_1,\ y_2, \cdots,\ z_1,\ z_2, \cdots)$. Let x be another indeterminate over F. Take $m_i,\ n_i \in \{0,\ 1\}$ such that formal power series $z := \sum_{i=1}^{\infty} m_i z_i x^i$ and $y := \sum_{i=1}^{\infty} n_i y_i x^i$ in $F[[x]]$ are algebraically independent over $F(x)$. Let $A = F[[x]] \cap F(x,\ y,\ z)$, $B = F[[x]] \cap F(x,\ z)$, $C = B[y]$, $d = \frac{\partial}{\partial z} :\ F(x,\ y,\ z) \to F(x,\ y,\ z)$, $E = \{\alpha \in C \mid d(\alpha) \in A\}$ and $R = E_{xA\cap E}$. Then R is D-simple local ring (for some D) which is not Cohen-Macaulay, and hence non-regular.

On the other hand, there are some problems about simple derivations. A K-derivation d of R is said to be *simple* if R is $\{d\}$-simple. It is well-known that the Ore extension $R[t,\ d]$ is a simple ring if and only if d is simple. Hence, simple derivations are useful for constructing simple rings.

Now we consider the polynomial ring $K[x,\ y]$. Any K-derivation d of $K[x,\ y]$ has the form

$$f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}$$

where $f,\ g \in K[x,\ y]$. We want to know necessary and sufficient conditions on f and g for d to be simple. However, it is too difficult and we only know partial results and a few examples of simple derivations. One of them is the following example of D.A. Jordan:

The detailed version of this paper will be submitted for publication elsewhere.

Example 2. ([3]) A K-derivation $d_1 = y^3\frac{\partial}{\partial x} + (1-xy)\frac{\partial}{\partial y}$ is simple in $k[x, y]$.

Remark. If d is simple then the ideal $(d(R))$ generated by $d(R)$ is equal to R. The K-derivation d_1 in Example 2 has a considerable property; $d_1(R)$ contains no units.

The following result by A. Nowicki gives a first step of the study of simple derivations d with $d(x) = 1$, and at the same time, gives another form of simple derivations. The condition $d(x) = 1$ means that the restriction $d|_{K[x]}$ is simple in $K[x]$.

Theorem 1.1. ([5]) *Let d be a K-derivation of $K[x, y]$ such that $d(x) = 1$ and $d(y) = ay + b$, where $a, b \in K[x]$, $a \neq 0$.*

(1) *If d is not simple and $\deg a > \deg b$, then $b = 0$.*

(2) *If d is not simple and $\deg a = \deg b$, then $b = \alpha a$ for some $\alpha \in K$.*

(3) *Let δ be a K-derivation of $K[x, y]$ such that $\delta(x) = 1$ and $\delta(y) = ay + \frac{\partial}{\partial x}(c) + r$, where $b = ac + r$, $c, r \in K[x]$, $\deg r < \deg a$. Then d is simple if and only if δ is simple.*

The following is a direct consequence of Theorem 1.1:

Corollary 1.2. ([5]) *Let d_2 be a K-derivation of $K[x, y]$ such that $d_2(x) = 1$ and $d_2(y) = ay + b$ where $a, b \in K[x]$.*

(1) *If $\deg a > \deg b \geq 0$, then d_2 is simple.*

(2) *If $\deg a = \deg b \geq 0$ and $a \neq \alpha b$ for any $\alpha \in K$, then d_2 is simple.*

Remark. If $a = 0$ or $b = 0$ then d_2 is not simple. Moreover, by Theorem 1.1 (3), even if $0 \leq \deg a < \deg b$ we can reduce to the case $\deg a \geq \deg b \geq 0$, and decide whether d_2 is simple or not.

The aim of this paper is to give a new example of a simple derivation.

2 An example of a simple derivation

We begin by the next theorem of Seidenberg.

Theorem 2.1. ([6]) *Let d be a derivation of a commutative noetherian $\mathbb{Q}$-algebra R. Then d is simple if and only if R has no proper prime d-ideal.*

From Theorem 2.1, we obtain the following:

Corollary 2.2. *Let d be a K-derivation of $K[x, y]$ such that $d(f) = 1$ for some $f \in K[x, y]$. Then d is simple if and only if $K[x, y]$ has no principal prime d-ideal other than (0).*

Proof. ($\Rightarrow$): Clear.

($\Leftarrow$): Assume that d is not simple and let I be a proper prime d-ideal of $K[x, y]$. Then I is maximal or of height one. Since $K[x, y]$ is a unique factorization domain, height one prime ideals are principal. Thus, it suffices to show that I can not be maximal.

Suppose that I is maximal. Then there exists a unique K-derivation $\bar{d}$ of $K[x, y]/I$ such that $\pi d = \bar{d}\pi$ where $\pi : K[x, y] \to K[x, y]/I$ is a canonical map. Since $K[x, y]/I$ is algebraic over K, $\bar{d}$ is a zero derivation. Hence, we have $\bar{1} = \pi(1) = \pi(d(f)) = \bar{d}(\pi(f)) = \bar{0}$. This contradiction completes the proof of Corollary 2.2. □

To extend Theorem 1.1, it is natural to consider a derivation δ of $k[x,\ y]$ such that $\delta(x)=1$ and $\delta(y)=y^2+a_1y+a_0$, where $a_1,\ a_0\in K[x]$. Let σ be a K-automorphism of $K[x,\ y]$ defined by $\sigma(x)=x$ and $\sigma(y)=y+a_1/2$. Then

$$\sigma^{-1}\delta\sigma(x)=1,\ \sigma^{-1}\delta\sigma(y)=y^2+a_0+\frac{1}{2}a_1'-\frac{1}{4}a_1^2,$$

where $a_1'=\frac{\partial}{\partial x}(a_1)$. It is known that δ is simple if and only if $\sigma^{-1}\delta\sigma$ is simple. Thus, in this case, it suffices to consider a K-derivation d such that $d(x)=1$ and $d(y)=y^2+g$ where $g\in K[x]$. If $g\in K$ then d is not simple because (y^2+g) is a d-ideal . So we take $g\notin K$.

Now assume that d defined above is not simple. Then, from Corollary 2.2, there exists $0\neq f\in K[x,\ y]$ such that $d(f)=pf$ for some $p\in K[x,\ y]$, and $m=\deg_y f\geq 1$ since $d|_{K[x]}$ is simple. Write $f=f_my^m+\cdots+f_0$ where each $f_i\in K[x]$ and $f_m\neq 0$. Without loss of generality, we may assume that f_m is monic. Let $'$ denote differentiation in $K[x]$ with respect to x. We have

$$\begin{aligned} d(f) &= f_m'y^m+\cdots+f_0'+(mf_my^{m-1}+\cdots+f_1)(y^2+g)\\ &= p\cdot(f_my^m+\cdots+f_0)\end{aligned}$$

and hence $p=my-h$ for some $h\in K[x]$.

Now the following result about h gives an example of a simple derivation.

Lemma 2.3. *Let h be a polynomial in $K[x]$ defined above. If $\mathbb{Q}\subseteq K\subseteq\mathbb{C}$ then $h\notin K$.*

Proof. From $d(f)=(my-h)f$, we have

$$\begin{aligned} f_m' &= f_{m-1}-hf_m, && (*_m)\\ (i+1)f_{i+1}g+f_i' &= (m-i+1)f_{i-1}-hf_i\quad (i=1,\cdots,m-1\ \text{ if }\ m\geq 2), && (*_i)\\ f_1g+f_0' &= -hf_0. && (*_0)\end{aligned}$$

First, we suppose that $m=1$. If $h\in K^\times=K\setminus\{0\}$ then, from $(*_1)$ and $(*_0)$,

$$f_1'=f_0-hf_1,\ f_1g+f_0'=-hf_0.$$

In this case, we have $\deg f_0=\deg f_1$ and $\deg f_0=\deg f_1+\deg g$. Hence $\deg g=0$, which is a contradiction.

If $h=0$ then

$$f_1'=f_0,\ f_1g+f_0'=0.$$

In this case, we have $f_1g+f_1''=0$, which is impossible because $f_1\neq 0$ and $g\neq 0$.

Next we suppose that $m\geq 2$, and we show that neither $h\in K^\times$ nor $h=0$.

a) Assume that $h\in K^\times$. Put $\varphi_i=\deg f_i$ $(i=0,\cdots,m)$ and $\varphi_g=\deg g$. We denote by θ_i, θ_g and θ_h the arguments of the coefficients of the nonzero leading terms in f_i, g and h, respectively. From $(*_m)$,

$$\varphi_{m-1}=\varphi_m,\ \theta_{m-1}=\theta_m+\theta_h,$$

and from $(*_{m-1})$,

$$\varphi_{m-2}=\varphi_{m-1}+\varphi_g,\ \theta_{m-2}=\theta_{m-1}+\theta_g-\theta_h.$$

By continuous observation, from $(*_i)$ $(i = 1,\cdots,m)$, we have for odd j and even k $(1 \le j,\ k \le m)$

$$\begin{aligned} \varphi_{m-j} &= \varphi_{m-j+1}, && (2.1)\\ \theta_{m-j} &= \theta_{m-j+1} + \theta_h, && (2.2)\\ \varphi_{m-k} &= \varphi_{m-k+1} + \varphi_g, && (2.3)\\ \theta_{m-k} &= \theta_{m-k+1} + \theta_g - \theta_h. && (2.4) \end{aligned}$$

Particularly, $f_i \neq 0$ for $i = 0,\cdots,m$, and m is even from (2.1), (2.3) and $(*_0)$. But, from $(*_0)$, the equation $\theta_0 = \theta_1 + \theta_g - \theta_h$ is impossible.

b) Assume that $h = 0$, and put φ_i, φ_g, θ_i and θ_g as stated above.

If $f_m \neq 1$ then, from $(*_m)$,

$$\varphi_{m-1} = \varphi_m - 1,\ \theta_{m-1} = \theta_m,$$

and from $(*_{m-1})$,

$$\varphi_{m-2} = \varphi_{m-1} + \varphi_g + 1,\ \theta_{m-2} = \theta_{m-1} + \theta_g.$$

By continuous observation, from $(*_i)$ $(i = 1,\cdots,m)$, we have that for odd j and even k $(1 \le j,\ k \le m)$

$$\begin{aligned} \varphi_{m-j} &= \varphi_{m-j+1} - 1, && (2.5)\\ \theta_{m-j} &= \theta_{m-j+1}, && (2.6)\\ \varphi_{m-k} &= \varphi_{m-k+1} + \varphi_g + 1, && (2.7)\\ \theta_{m-k} &= \theta_{m-k+1} + \theta_g. && (2.8) \end{aligned}$$

Particularly, $f_i \neq 0$ for $i = 0,\cdots,m$, and m is even from (2.5), (2.7) and $(*_0)$. But, from $(*_0)$, the equation $\theta_0 = \theta_1 + \theta_g$ is impossible.

Now we suppose that $f_m = 1$. Then, from $(*_m)$, $f_{m-1} = 0$, and applying it to $(*_{m-1})$, we have $\varphi_{m-2} = \varphi_g$ and $\theta_{m-2} = \theta_g$. If $m = 2$ then $f_0' = 0$. It means $g \in K$, a contradiction. Hence $m \ge 3$, and from $(*_i)$ $(i = 1,\cdots,m-2)$, $f_i \neq 0$ for $i = 0,\cdots,m-3$. In the same way as in the case $f_m \neq 1$, we have a contradiction and the proof of Lemma 2.3 is complete. □

We are now in a position to state our example.

Example 3. Let K be a field such that $\mathbb{Q} \subseteq K \subseteq \mathbb{C}$ and d a K-derivation of $K[x,\ y]$ such that

$$d(x) = 1,\ d(y) = y^2 + \alpha x + \beta,$$

where $\alpha,\ \beta \in K,\ \alpha \neq 0$. Then d is simple.

Proof. Suppose that d is not simple. Let f be a polynomial in $K[x,\ y]$ such that

$$d(f) = (my - h)f \quad \text{for some } h \in K[x], \qquad (2.9)$$

where $m = \deg_y f \ge 1$. Put $f = \sum_{i=0}^{n} b_i x^i$ where $b_i \in K[y]$, $i = 0,\cdots,n$, $b_n \neq 0$. From (2.9), we have

$$\begin{aligned} &(my - h)(b_n x^n + \cdots + b_0)\\ &= n b_n x^{n-1} + \cdots + b_1 + \left(\frac{\partial}{\partial y}(b_n)x^n + \cdots + \frac{\partial}{\partial y}(b_0)\right)(y^2 + \alpha x + \beta), \end{aligned}$$

and hence $b_n | \frac{\partial}{\partial y}(b_n)$ in $K[y]$. This means $\frac{\partial}{\partial y}(b_n) = 0$. Then $(my - h)(b_n x^n + \cdots + b_0)$ is of degree n with respect to x, and hence $h \in K$, which is a contradiction by Lemma 2.3. □

References

[1] P. Brumatti, A.M. de Souza Doering and Y. Lequain, *Differential simplicity, Cohen-Macaulayness and formal prime divisors*, J. Pure Appl. Algebra **84** (1993), 1–11.

[2] K.R. Goodearl and T.H. Lenagan, *Constructing bad noetherian local domains using derivations*, J. Algebra **123** (1989), 478–495.

[3] D.A. Jordan, *Differential simple rings with no invertible derivatives*, Quart. J. Math. Oxford (**2**), **32** (1981), 417–424.

[4] Y. Lequain, *Differential simplicity and complete integral closure*, Pacific J. Math. **36** (1971), 741–751.

[5] A. Nowicki, *Polynomial Derivations and Their Rings of Constants*, Toruń UMK, 1994.

[6] A. Seidenberg, *Differential ideals in rings of finitely generated type*, Amer. J. Math. **89** (1967), 22–42.

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NOTES ON CLEAN RINGS

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Abstract. It is shown that if e is an idempotent in a ring ring R with identity 1 such that eRe and $(1\text{-}e)R(1\text{-}e)$ are both clean rings, then R is a clean ring. It is also shown that the ring of all endomorphisms of a countably infinite dimensional free module over a clean ring is clean ring. Clean rings with semicentral primitive idempotents and some extensions of clean rings are investigated.

1. Introduction

An element in a ring R is called *clean* in R if it is expressed as a sum of an idempotent and a unit, and the ring R is called *clean* if every element in R is clean. Nichloson [4] observed that $n \times n$ matrix ring over an algebraically closed field is clean ring, and also Camillo and Yu [1] extended this to unit-regular ring. In section 2, we extend these results to the ring of n × n matrix ring over a clean ring is clean. Nicholson [5] have shown that every linear transformation on a vector space of countable dimension is clean. Also by using the result mentioned above we will show that every endomorphism of a countably infinite dimensional free module over a clean ring is clean ring.

In section 3, it is shown that for any idempotent e in a ring R, eRe is local ring if and only if eRe is clean ring and e is primitive idempotent. An idempotent $e \in R$ is called *left* (resp. *right*) *semicentral* if for all $a \in R$, $ae =$

eae (resp. $eR = eRe$. It is also shown that if R is a clean ring and e is a left (or right) semicentral idempotent in R, then eRe is a clean ring. In section 4, some extensions of clean rings are investigated as follows; First, in general, neigher group rings over clean rings nor tensor product of clean algebras over a field are clean. Secondly, the ring of formal power series over clean ring is clean and the converse is true. Thirdly, if A and B clean rings and ${}_AM_B$ is any bimodule, then A and B are clean rings if and only if the ring R = $\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ is clean. As corollary, we have that any n $\times$ n upper triangular matrix ring over a clean ring is clean ring.

2. Endomorphism Ring over a Clean Ring

The following question is suggested by Prof. Nicholson:

Question If R is a clean ring, is the M_2(R), the matrix ring of all 2 $\times$ 2 matrices over R, clean ring?

The answer to the question is affirmative by the following:

Theorem 2.1. *$e^2 = e \in R$ be such that eRe and (1-e)R(1-e) are both clean rings. Then R is a clean ring.*

Proof. Write $\bar{e} = 1 - e$ and use the Pierce decomosition $R \cong \begin{bmatrix} eRe & eR\bar{e} \\ \bar{e}Re & \bar{e}R\bar{e} \end{bmatrix}$.
If $A = \begin{bmatrix} a & x \\ y & b \end{bmatrix} \in R$ and by hypothesis, let $a = f + v$ in eRe where $f^2 = f$ and v is a unit in eRe with inverse v_1. Then $b - yv_1x \in \bar{e}R\bar{e}$ so that $b - yv_1x$ $= g - w_1$. Hence

$$A = \begin{bmatrix} f+v & x \\ y & g+w+yv_1x \end{bmatrix} = \begin{bmatrix} f & 0 \\ 0 & g \end{bmatrix} + \begin{bmatrix} v & x \\ y & w+yv_1x \end{bmatrix}.$$

Thus it suffices to show that $\begin{bmatrix} v & x \\ y & w+yv_1x \end{bmatrix}$ is unit in R. Compute:

$$\begin{bmatrix} e & 0 \\ -yv_1 & \bar{e} \end{bmatrix}\begin{bmatrix} v & x \\ y & w+yv_1x \end{bmatrix}\begin{bmatrix} e & v_1x \\ 0 & \bar{e} \end{bmatrix} = \begin{bmatrix} v & x \\ 0 & w \end{bmatrix}\begin{bmatrix} e & -v_1x \\ 0 & \bar{e} \end{bmatrix}$$

$$= \begin{bmatrix} v & 0 \\ 0 & w \end{bmatrix}.$$

Since $\begin{bmatrix} e & 0 \\ -yv_1 & \bar{e} \end{bmatrix}$, $\begin{bmatrix} e & -v_1x \\ 0 & \bar{e} \end{bmatrix}$ and $\begin{bmatrix} v & 0 \\ 0 & w \end{bmatrix}$ are all units in R, the proof is complete.

Then we have the following corollaries;

Corollary 2.2. *If $1 = e_1 + \cdots + e_n$ in a ring R where the e_i are orthogonal idempotents and each e_iRe_i is clean, then R is clean. In particular, (see [1, Theorem 9]) every semiperfect ring is clean.*

Corollary 2.3. *R is a clean ring so also is the matrix ring $M_n(R)$.*

Corollary 2.4. *$M = M_1 \oplus + \cdots \oplus M_n$ are modules and $endM_i$ is clean for each i, then end(M) is clean.*

Corollay 2.4 suggests

Question 1. *Does there exists a property χ of modules such that a module M has χ if and only if end(M) is clean?*

Such a property χ must imply the finite exchange property because clean rings are exchange rings and module M has the finite exchange propery if and only if *end*(M) is an exchange ring [3, Theorem 2].

In connection with this question, note that Camillo and Yu [1, Theorem 5] have shown that every unit regular ring R is clean. Here is the following theorem generalized.

Proposition 2.5. *Given a module M_R, the following are eqivalent for $\alpha \in end(M_R)$:*
(1) α is unit regular.
(2) There is an automorphism $\sigma : M \longrightarrow M$ such that $M = \alpha M \oplus \sigma(ker\ \alpha)$. (3) αM and ker α are both summands of M and ker $\alpha \cong M/\alpha M$.

Proof. (1) $\Rightarrow$ (2). Let $\alpha\sigma\alpha = \alpha$ where $\sigma \in aut(M)$. Then $ker\ \alpha = (1 - \sigma\alpha)M = \sigma(1 - \sigma\alpha)\sigma^{-1}M = \sigma(1 - \sigma\alpha)M$ so $(1 - \sigma\alpha)M = \sigma^{-1}(ker\ \alpha)$. Also $\alpha M = \alpha\sigma M$, so we have

M = $\alpha\sigma$M $\oplus$ (1 - $\sigma\alpha)M$ = αM $\oplus$ $\sigma^{-1}(ker\ \alpha)$.

This proves (2). (2) $\Rightarrow$ (3). Given $M = \alpha$M $\oplus$ σ(ker α) we have $M = \sigma^{-1}$M $= \sigma^{-1}\alpha M \oplus ker\ \alpha$, so both αM and $ker\ \alpha$ are summands. Finally, $M/\alpha M \cong \sigma(ker\ \alpha) \cong ker\ \alpha$, proving (3).

(3) $\Rightarrow$ (1). Let $M = \alpha M \oplus K = ker\ \alpha \oplus N$. By hypothesis $ker\ \alpha \cong M/\alpha M \cong K$, so let $\gamma : K \longrightarrow ker\alpha$ be an isomorphism. We have $\alpha M = \alpha(ker\alpha \oplus N) = \alpha N$, so $M = \alpha N \oplus K$. Using this define $\sigma : M \longrightarrow M$ by $\sigma(\alpha n + k = n + \gamma k$. This is well defined because $\alpha M \cap K = 0$. To see that $\alpha\sigma\alpha = \alpha$, let $m \in M = ker\alpha \oplus N$, say $m = k + n$. Then $\alpha\sigma\alpha(m) = \alpha\sigma\alpha(n) = \alpha(n) = \alpha(m)$, as required. Finally $ker\alpha = 0$ because $N \cap \sigma K = N \cap ker\alpha = 0$, so σ is monic, and $\sigma M = N + \sigma K = N + ker\alpha = M$ shows that σ is epic.

Thus the property χ in Question 1 would have to statisfy the properties in Proposition 2.5.

Proposition 2.6. *Let R be a clean ring in which 0 and 1 are the only idempotents. Then R is a local ring.*

Proof. Claim: $ab = 1$ implies $ba = 1$. [Proof. ba is an idempotent, and $ba = 0$ implies that $1 = 1^2 = abab = 0$, a contradiction.]

Let a be a nonunit of R; we must show that $a \in J = J(R)$. We do this by showing that 1 - ra is a unit for all $r \in R$. Because R is clean, $ra = e + u$ where $e^2 = e$ and u is a unit. since ra is a nonunit (by the Claim) it folows that $e = 1$, so 1 - ra is a unit, as required.

Remark. A theorem of Camillo and Yu [1, Theorem 9] asserts that R is semiperfect if and only if it is clean and I-finite (that is, contains no infinite orthogonal set of idempotents). This implies Proposition 1 because semiperfect rings with 0 and 1 as the only idempotents are local. Hence, using Corollary 2.2, we get another proof of the Camillo-yu theorem if we can positively answer

Question 2. *If R is clean and $e^2 = e \in R$ is primitive, is the ring eRe clean?*

Next we have the following question:

Question 3. *If R is a clean ring, is the ring of all endomorphisms of a* *countably infinite dimensional free module over is clean ring?*

The answer to the question could be affirmative. To show this, we will also need some lemmas as given in [5] by Nichloson and Varadarajan.

Let M_R denote a countably infinite dimensional free right R-module over a clean ring R, and $End(M_R)$ denote the ring of all endomorphisms of M_R. If $\{x_1, x_2, . . . \}$ is a basis for M_R, the endomorphism $\sigma : M_R \longrightarrow M_R$ given by $\sigma(x_i) = x_{i+1}$ for each i is called a shift operator on M_R.

Lemma 2.7. *Every shift operator on M_R is clean in End(V_R).*

Proof. The proof is similar to the one as given in [5, Lemma1].

Lemma 2.8. *If $\alpha \in$ End(V_R) is such that M_R is spanned by $\{x, \alpha(x), \alpha^2(x), . . . \}$ for some $x \in M_R$, then α is clean in End(M_R).*

Proof. We may assume that $M \neq 0$. If $\alpha^n(\mathrm{x}) \notin xR + \alpha(x)R + \cdots + \alpha^{n-1}(x)R$ for all $n \geq 1$, then $\{x, \alpha(x), \alpha^2(x), . . . \}$ is a basis for M_R. Since α is the shift operator with respect to this basis, it is clean by Lemma 2.7.

Next, assume that $\alpha^n(\mathrm{x}) \in xR + \alpha(x)R + \cdots + \alpha^{n-1}(x)R$ for fome $n \geq 1$. If n is minimal with this property, then $\{x, \alpha(x), \alpha^2(x), . . . , \alpha^{n-1}(x) \}$ is a basis for M_R. Thus α is clean by Corollary 2.3.

Lemma 2.9. *Let $\alpha \in$ End(M_R) and let U be an α-invariant submodule of M_R. Assume that a vector $x \in M$ - U exists such that $V = U + W$ where $W = xR + \alpha(x)R + \cdots$. If the restriction $\alpha_{|U}$ is clean in End(M_R), then α is clean in End(M_R). More precisely:*

If $\alpha_{|U} = \pi + \sigma$, $\pi^2 = \pi$, σ invertible, then $\alpha = \pi_1 + \sigma_1$ in End(M_R), $\pi_1^2 = \pi_1$, σ_1 invertible, where $(\pi_1)_{|U} = \pi$ and $(\sigma_1)_{|U} = \sigma$.

Proof. It follows from the Lemma 2.8 and the proof given in [5, Lemma 4].

Theorem 2.10. *If M_R is a countably infinite dimensional free right R-module over a clean ring R, then End(M_R) is clean.*

Proof. It follows from Lemma 2.9 and the similar proof given in [5, Theorem].

3. Semicentral Idempotents in Clean Rings

In this section, we start by suggesting the following question:

Question 4. If R is a clean ring and $e \in R$ is idempotent, then is eRe clean?

The answer to the question is still open.

Recall that an idempotent $e \in R$ is *primitive* if eRe contains no idempotents other than 0 and e, and an idempotent $e \in R$ is called *left* (resp. *right*) *semicentral* if $ae = eae$ (resp. $ea = eae$) for all $a \in R$.

Lemma 3.1. *If R is a clean ring and e is a left (or right) semicentral idempotent in R, then $eRe = Re$ (or eR) is also clean ring.*

Proof. If follows from the observation that when $eRe = Re$ the map ϕ : $R \longrightarrow eRe$ defined by $\phi(r) = re$ for all $r \in R$ is onto ring homomorphism.

Theorem 3.2. *Let R be a ring such that $1 \in R$ can be written as $e_1 + e_2 + \cdots + e_n$ where e_i's are orthogonal left (or right) semicentral idempotents in R. Then R is clean ring if and only if e_iRe_i is a clean ring for each $i = 1, 2, \ldots, n$.*

Proof. ($\Rightarrow$) It follows by Lemma 3.1.
($\Leftarrow$) Suppose that e_i is left semicentral idempotent in R and e_iRe_i is a clean ring for each $i = 1, 2, \ldots, n$. For any $x \in R$, $x = 1x = e_1x + e_2x + \cdots + e_nx \in e_1R + e_2R + \cdots + e_nR$. Since $e_iRe_i = e_iR$ is clean ring, $e_ix = e_ia_i + e_iu_i$ where e_ia_i is idempotent and e_iu_i is unit in e_iR for each i = 1, 2, . . . , n. Let $f = e_1a_1 + \cdots + e_na_n$ and $u = e_1u_1 + \cdots + e_nu_n$. Since e_i's are orthogonal idempotents in R, $f^2 = f$. To show that u is unit in R, let e_iv_i be the multiplicative inverse of e_iu_i for each i, i.e., $(e_iu_i)(e_iv_i) = (e_iv_i)(e_iu_i) = e_i$. Thus $uv = (e_1u_1 + \cdots + e_nu_n)(e_1v_1 + \cdots + e_nv_n) = e_1 + e_2 + \cdots + e_n = 1$, and similarly, $vu = 1$. Hence $x = f + u$ $f^2 = f$, u is unit in R, and so R is a clean ring. The case that e_i's are orthogonal right semicentral idempotents in R is also proved by the similar argument.

Theorem 3.3. *Let R be a ring and e is an idempotent in R. Then eRe is* *local ring if and only if eRe is clean ring and e is primitive idempotent, i.e., eRe has only two idempotents 0 and e.*

Proof. ($\Rightarrow$) If eRe is local ring, then eRe is semiperfect ring by [3, Corollary 1, p76], and also is clean ring by [1, Theorem 9]. It is clear that if eRe is local ring, then e is primitive idempotent in R.
($\Leftarrow$) Suppose that eRe is clean ring and e is primitive idempotent in R. Let ere is be an arbitrary nonunit of eRe. By assumption, $ere = e + (ere - e)$ where $ere - e$ is a unit of eRe. Then $u(ere - e) = u(ere) - ue = u(ere) - u$ is unit in eRe for all units u in eRe, and so $uere \in J(eRe) = eJ(R)e$ where $J(R)$ (resp. $J(eRe)$) is the Jacobsonradical of R(resp. eRe) by [3, Proposition 1, p75]. Thus $ere \in J(eRe)$, that is, the set of every nonunit in eRe is equal to $J(eRe)$, and so eRe is a local ring.

Corollary 3.4. *A ring is local if and only if it is clean ring and 1 is a primitive idempotent.*

Proof. It follows from the Theorem 3.3.

Remark. Let F be any field. Then the polynomial ring $F[x]$ is not clean ring with its Jacobson radical 0, and 1 is a primitive idempotent of $F[x]$. But $F[x]$ is not local ring. Hence the condtion of "cleanness" in Corollary 3.4 is not superfluous.

Corolary 3.5. *Let R be a clean ring such that $1 \in R$ can be written as $e_1 + e_2 + \cdots + e_n$ where e_i's are orthogonal primitive idempotents in R. Then R is a direct sum of local rings.*

Proof. By [2, Theorem 22.5], $1 \in$ R can be written as a sum of centrally orthogonal primitive idempotents f_i in R, say $1 = f_1 + f_2 + \cdots + f_k$ for some positive integer k. Then by Theorem 3.3, each $f_i R f_i$ is local ring. Thus we have the result.

4. Some Extension Rings of Clean Rings

Recall that an extension ring S of aring R with the same identity is called a *finite centralizing extension* of Rif there are $x_1, x_2, \ldots, x_n \in S$ with $x_i a$

$= ax_i$ for all $a \in R$ and $S = x_1 R + \cdots + x_n R$.

By Prof. J. K. Park, one may attempt to show that if R is clean then the finite centraling extension S is clean. But this does not hold even group ring of a finite group as the following example:

Example. Let $R = \prod_{i=1}^{\infty} R_i$ where each $R_i = Z_2$ and let $G = \{1, g\}$ be a group with two elements. Then $R[G]$ is not clean ring.

To show this, first we can observe that every idempotent in $R[G]$ is contained in R, and any element $a + bg \in R[G]$ is unit if and only $a \neq b$ and $ab = 0$. Assume that $R[G]$ is a clean ring. Choose an element $\alpha = a + bg \in R[G]$ where $a = (1, 0, 0, \ldots, .)$ and $b = (1, 1, 0, 0, \ldots, .)$ in R. Then α is not unit by the observation. Thus, $\alpha = u + e$ where u is a unit, and $e^2 = e$ $(\neq 0)$ in $R[G]$. Note that $u = (a - e) + bg$ and e in R also by the observation. Since u is unit, by the observation we have $1 + e \neq$ b and $(1 + e)b = 1 + e$ $= 0$, which is a contraction. Therefore $R[G]$ is not a clean ring.

Also the following questions are suggested by Prof. J. K. Park:

Question 5. *Let R be a clean ring and let G be a finite multiplicative group. If the order of G is a unit in R, is the group ring $R[[G]]$ clean?*

Question 6. *Let A and B be algebras over a field F. If A and B are clean, is $A \otimes_F B$, the tensor product of A and B clean?*

The question 5 is still open and the answer to the question 6 is negative by the counter example:

Let $A = \prod_{i=1}^{\infty} R_i$ where each $R_i = Z_2$, $B = Z_2[G]$, $F = Z_2$, and let $G = \{1, g\}$ be a group with two elements.Then A and B are clean, but $A \otimes_F B = \prod_{i=1}^{\infty} R_i[G]$ is not clean by the above example.

The ring $R[[x]]$ is called the ring of fomal power series over the ring R. Then we have the follwing:

Proposition 4.1. *A ring R is clean ring if and only if $R[[x]]$ is clean ring.*

Proof. It follows from the observations; the set of all idempotents of R is equal to the set of all idempotents of $R[[x]]$, and any element $\alpha(x) \in R[[x]]$ is unit if and only if $\alpha(x) = u + xf(x)$ for some unit $u \in R$ and some $f(x) \in R[[x]]$.

Proposition 4.2. *Let A and B be rings and ${}_AM_B$ be a bimodule. Then A and B are clean rings if and only if the ring* $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix} = \{ \begin{bmatrix} a & m \\ 0 & b \end{bmatrix}$ *: $a \in A$, $m \in M$ and $b \in B$ \} is clean ring.*

Proof. ($\Rightarrow$) Suppose that A and B are clean rings. Let $\begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \in \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ be arbitrary. Then $\begin{bmatrix} a & m \\ 0 & b \end{bmatrix} = \begin{bmatrix} u+e & m \\ 0 & v+f \end{bmatrix} = \begin{bmatrix} u & m \\ 0 & v \end{bmatrix} + \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}$ for some unit u and idempotent $e \in A$ (resp. some unit v and idempotent $f \in B$). Note that $\begin{bmatrix} u & m \\ 0 & v \end{bmatrix}$ is unit with its inverse $\begin{bmatrix} u^{-1} & u^{-1}mv^{-1} \\ 0 & v^{-1} \end{bmatrix}$, and clearly $\begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}$ is idempotent in R. Hence R is clean ring.

($\Leftarrow$) Suppose that R is clean ring. Let $a \in A$ be arbitraly. Then $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} u & m \\ 0 & v \end{bmatrix} + \begin{bmatrix} a-u & -m \\ 0 & -v \end{bmatrix}$ where $\begin{bmatrix} u & m \\ 0 & v \end{bmatrix}$ is unit and $\begin{bmatrix} a-u & -m \\ 0 & -v \end{bmatrix}$ is idempotent in R. Observe that $v = 1$, u is unit and a - u is idempotent in A. Hence A is clean ring. By the similar argument, B is clean ring.

Corollary 4.3. *R is clean ring if and only if any $n \times n$ upper triangular matrix ring over R is clean ring.*

Proof. It folows from the Proposition 4.2 and the induction on n.

REFERENCES

1. V. P. Camillo and H. P. Yu, "Exchange rings, Units and Idempotents", Comm. in Algebra, 22(12) (1994) 4737-4749.

2. T. Y. Lam, A First Course in Noncommutative Rings, New York, Springer-Verlag, 1991.

3. J. Lambek, Lectures on Rings and Modules, Blaisdell Pub. Co., 1966.

4. W. K. Nicholson, " Lifting idempotents and exchange rings, Trans. Amer. Math. Soc. 229 (1977), 269-278.

5. W. K. Nicholson and K. varadarajan, " Countable Linear Transformations are Clean", Proc. Amer. Math. Soc., to appear.

Separability and the Jones basic construction

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Abstract. Jones discovered his knot polynomials through his work on type II_1 subfactors of finite Jones index. Recently, Kadison studied algebraic aspects of the Jones basic construction. Using certain separable Frobenius extensions, he produced a tower of algebras and a countable family of idempotents satisfying braid-like relations. In this note, we introduce Kadison's methods to define the Jones polynomials.

§0. Introduction

Jones [5] はブレイド群と von Neumann 代数を関係づけることにより，有向絡み目 L に対する不変量 $V_L(t)$ を作り上げた．最近，Kadison [11] は von Neumann 代数を用いた Jones の構成法の本質を代数的にとらえ直した．この報告集では Kadison [11] による 1 変数の Jones 多項式の構成法を紹介する．

§1. Strongly separable extensions with Markov trace

まず，いくつかの定義を述べよう．以下，$A \supset S$ は環拡大とする．

定義 1.1 (cf.[12]). A_S が有限生成射影的かつ$_SA_A \cong {}_SHom(A_S, S_S)_A$ となることと$_SA$ が有限生成射影的かつ$_AA_S \cong {}_AHom({}_SA, {}_SS)_S$ となることは同値である．これらの同値条件をみたすとき A/S は Frobenius 拡大であるという．

定義 1.2. multiplication map $\mu : A \otimes_S A \to A$, $a \otimes b \mapsto ab$ が $A-A$ 準同型として分解するとき，A/S は分離(的)拡大であるという．このことは $e \in A \otimes_S A$ で $\mu(e) = 1$ かつ任意の $a \in A$ に対して $ae = ea$ をみたすもの (separability element という) が存在することと同値である．

分離拡大については，平田と菅野 [4], そして菅野 [15] において研究されている．

This paper is in a final form and no version of it will be submitted for publication elsewhere.

定義 1.3. $E \in Hom_{S-S}(A, S)$ で $E(1) = 1$ となるものが存在するとき，A/S は split extension であるという．E は $S \subset A$ に対する conditional expectation と呼ばれる．

K を可換環，$S \subset A$ を $K-$多元環の拡大とする．$K-$線形写像 $T : A \to K$ が 任意の $a,\ b \in A$ に対して $T(ab) = T(ba)$ かつ $T(1) = 1$ をみたすとき，T はトレースと呼ばれる．$K^{\times}$ で K の単元のなす群を表す．

定義 1.4. A/S は $K-$多元環の拡大とする．$\tau \in K^{\times}$, conditional expectation $E : A \to S$ および separability element $e = \tau\sum_{i=1}^{n} x_i \otimes y_i$ があり，任意の $a \in A$ に対して，$\sum E(ax_i)y_i = a$ かつ$\sum x_i E(y_i a) = a$ をみたすとき，A/S は強分離拡大であるといわれる．τ^{-1} は E-index とよばれ，$[A : S]_E$ で表される．(x_i, y_i, E, τ) は $A \supset S$ に対する強分離系と呼ばれる．(τ^{-1} と $\sum_{i=1}^{n} x_i \otimes y_i$ はのみに依存することがわかる．) 更に，トレース $T : S \to K$ で $T_A = T \circ E : A \to K$ もトレースになるものが存在するとき，A/S は Markov トレースを持つ強分離拡大であるといわれる．

Remark. (1) A/S が強分離拡大であれば, A/S は Frobenius 拡大である．実際，${}_SA$ は双対基底 $\{y_i, E(-x_i)\}$ を持つので，有限生成射影的である．しかも，${}_AA_S \to {}_AHom({}_SA, {}_SS)_S$, $a \mapsto (x \mapsto E(xa))$ は同型写像である．
(2) 強分離拡大は服部 [3] でも定義されているが，上の定義と少しだけずれがあるようにおもわれる．なお，[1] も参照のこと．

例 1.1. von Neumann algebra II_1 factors $N \subset M$ でN が 有限の Jones index $[M : N]$ を持つ subfactor になっているものは Markov トレース $N \to \mathbf{C}$ を持つ強分離拡大である．$\{4\cos^2 \pi/m \mid m = 3, 4, \cdots\} \cup \{r \in \mathbf{R} \mid r \geq 4\}$ に属する任意の v に対して $\tau^{-1} = [M : N] = v$ となる pair $N \subset M$ が存在することが知られている．

例 1.2. H は群 C の有限指数 n の部分群とし，$g_1, \cdots, g_n$ を剰余類の代表系とする．群環 $A = K[G]$ と $S = K[H]$ を考える．$\pi : A \to S$ で自然な射影を表す (i.e., $\pi(\sum_{g \in G} k_g g) = \sum_{h \in H} k_h h$)．もし，$1/n \in K$ であれば A/S は Markov トレース π_e を持つ強分離拡大であり，$(g_i, g_i^{-1}, \pi_H, 1/n)$ は強分離系である．

P は環 R 上の射影加群で，双対基底 $\{f_i : P \to R\}_{i=1}^{n}$ と $\{x_i\}_{i=1}^{n}$ をもつとする．$[R, R]$ で $\{ab - ba \mid a,\ b \in R\}$ で生成される R の部分加法群を表す．このとき，元$\sum_{i=1}^{n} f_i(x_i) + [R, R] \in R/[R, R]$ を Hattori-Stallings ランクといい，$r({}_RP)$ で表す．

命題 1.1. A/S は Markov トレース $T : S \to K$ を持つ強分離拡大であるとする． このとき E-index $[A : S]_E$ は $T(r({}_SA))$ に等しい．

A を可換環 K-上の多元環で，K-加群として有限生成射影的であるものとする．$\{f_i : P \to K\}_{i=1}^n$ と $\{x_i\}_{i=1}^n$ を A の K 上の双対基底とする．$tr(x) = \sum_i f(xx_i)$ で与えられるトレース写像 $tr = tr_{A/K}$ は $Hom_K(A, K)$ の元であり，$tr(1) = r({}_KA)$ となる．

例 1.3. A を東屋多元環とし，C をその中心とする．ランク $r({}_CA)$ は C の可逆元であると仮定する．このとき A/C は強分離拡大であり，$tr_{A/C}$ をconditional expectation として持つ．恒等写像 $id_C : C \to C$ は Markov トレースである．例えば，複素数体上の全行列環 $M_n(\mathbf{C})$ を考え，E_{ij} を標準的行列単位とする．このとき $M_n(\mathbf{C})/\mathbf{C}$ は強分離拡大であり，$e = \dfrac{1}{n}\sum_{i,j} E_{ij} \otimes E_{ji}$ は separability element である．$E(X) = (1/n)\sum_i X_{ii}$ はconditional expectation であり，$\tau = \dfrac{1}{[M_n(\mathbf{C}) : \mathbf{C}]_E} = \dfrac{1}{n^2}$ である．また，$t = (n^2 - 2 \pm n\sqrt{n^2-4})/2$ は$\tau = \dfrac{t}{(t+1)^2}$ をみたす．

例 1.4. $A \supset S$ は可換環とし，A は S 上分離的かつ有限生成射影的であり，ランク $r({}_SA)$ は S の可逆元であると仮定する．このとき A/S は強分離拡大であり，$tr_{A/S}$ は conditional expectation, 恒等写像 id_S は Markov トレースである．例えば，F/K が次数 n の体の有限分離拡大であり，K の標数と n は互いに素であるとする．このとき，原始的元α が存在し，$F = K(\alpha)$ となる．$p(x) = x^n - \sum_{i=0}^{n-1} c_i x^i$ を α のK 上の最小多項式とし，$E = \dfrac{1}{n} trace : F \to K$ とおく．このとき $\{\alpha^i\}_{i=0}^{n-1}$ と $\{\dfrac{\sum_{j=0}^{i} c_j\alpha^j}{p'(\alpha)\alpha^{i+1}}\}_{i=1}^{n-1}$ は非退化双線形形式 $f(x, y) = E(xy)$ に関する双対基底である．また$\sum_{i=0}^{n-1} \alpha^i \otimes_K (\dfrac{\sum_{j=0}^{i} c_j\alpha^j}{p'(\alpha)\alpha^{i+1}})$ はseparability element である．$(n\alpha^i, \dfrac{\sum_{j=0}^{i} c_j\alpha^j}{p'(\alpha)\alpha^{i+1}}, E, 1/n)$ は強分離系である．

補題 1.1. $(A, S, x_i, y_i, E, \tau)$ を強分離拡大とすると $A \otimes_S A$ は $(a_0 \otimes a_1)(a_2 \otimes a_3) = a_0 \otimes E(a_1a_2)a_3$ と定義することにより K-多元環になる．単位元は $1 = \sum x_i \otimes y_i = \tau^{-1}e$ である．

補題 1.2. 環として $A \otimes_S A \cong End(A_S)$.

証明. a, $x \in A$ に対して $\lambda_a(x) = ax$ とおく．写像 $A \otimes_S A \to End(A_S); a_0 \otimes a_1 \mapsto \lambda_{a_0} \circ E \circ \lambda_{a_1}$ と写像 $End(A_S) \to A \otimes_S A; f \mapsto \sum_{i=1}^n f(x_i) \otimes y_i$ は互いの逆写像.

$A = \{\sum_i ax_i \otimes y_i = a\tau^{-1}e \mid a \in A\} \subset A \otimes_S A$ なる同一視により A を $A \otimes_S A$ の部分環と見なす.

補題 1.3. A/S は Markov トレース T を持つ強分離拡大であり，(x_i, y_i, E, τ) を $A \supset S$ に対する強分離系とする．このとき，$A \otimes_S A/A$ はMarkov トレース $T' = T \circ E$ を持つ強分離拡大であり，$(\tau^{-1} x_i \otimes 1_A, 1_A \otimes y_i, \tau\mu, \tau)$ が強分離系となる．ただし，μ は multiplication map $A \otimes_S A \to A$, $a \otimes b \mapsto ab$ を表す．

証明. $E_1 = \tau\mu : A \otimes_S A \to A$ と $T_1 = T \circ E \circ E_1 : A \otimes_S A \to K$ を考える．このとき，$E_1(1) = \tau \sum x_i y_i = 1$. また，$\sum E_1((a \otimes b)\tau^{-1} x_i \otimes 1)(1 \otimes y_i)) = a \otimes (\sum E((bx_i)y_i) = a \otimes b$. $T_1((a_0 \otimes a_1)(a_2 \otimes a_3)) = \tau T(E(a_3 a_0)E(a_1 a_2)) = T_1((a_2 \otimes a_3)(a_0 \otimes a_1))$ となるので，T_1 は $A \otimes_S A$ 上のトレースである．故に $T' = T \circ E : A \to K$ は Markov トレースである．

$A \supset S$ は強分離拡大であり，$E : A \to S$ が conditional expectation, $[A : S]_E = \tau$, $T : S \to K$ を Markov トレースとする．このとき，$A_1 = A \otimes_S A$ は A の強分離拡大であり，$E_1 = \tau\mu : A_1 \to A$ が conditional expectation, (ここでμ は multiplication map $A \otimes_S A \to A$), $[A_1 : A]_E = \tau$, $T \circ E : A \to K$ が Markov トレース，$e_1 = 1_A \otimes_S 1_A \in A \otimes_S A = A_1$ が separability idempotent となる（ここで，1_A は A の単位元．）同様にして，$A_2 = A_1 \otimes_A A_1 = A \otimes_S A \otimes_S A$ は A_1 の強分離拡大であり，$E_2 = \tau\mu_2 : A_1 \to A$ が conditional expectation (ここでμ_2 は multiplication map $A_1 \otimes_A A_1 \to A_1$), $[A_2 : A_1]_E = \tau$, $T_1 = T \circ E \circ E_1 : A_1 \to K$ が Markov トレース，$e_2 = 1_{A_1} \otimes_A 1_{A_1} \in A_1 \otimes_A A_1 = A_2$ が separability idempotent となる（ここで，1_{A_1} は A_1 の単位元．）帰納的に A_n, E_n, e_n と $T_n = T \circ E \circ E_1 \circ \cdots \circ E_n$ を定義する．$E_n(e_n) = \tau\mu_n(1 \otimes 1) = \tau 1_{A_{n-1}}$ であるので $T_n(e_n) = \tau$ である．従って，次の図式を得る：

$$S \xleftarrow{E} A \xleftarrow{E_1} A \otimes_S A \xleftarrow{E_2} A \otimes_S A \otimes_S A \longleftarrow \cdots \xleftarrow{E_n} A_n = A^{\otimes n+1} \xleftarrow{E_{n+1}} \cdots$$

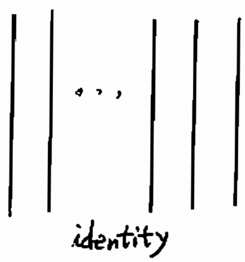
補題 1.4. 各 n に対して，べき等元 $e_1, e_2, \cdots, e_n$ は次の braid like relations をみたす：
$e_i e_{i\pm 1} e_i = \tau e_i$,
$e_i e_j = e_j e_i \ (i - j \geq 2)$.
また $T_n = T \circ E \circ E_1 \circ \cdots \circ E_n : A_n \to K$ はトレースであり，K 及び $e_1, e_2, \cdots, e_i \ (i < n)$ で生成される部分環の元 ω に対して，$T_n(\omega x_{i+1}) = \tau T_n(\omega)$ となる．特に，各 $e_1, e_2, \cdots, e_n$ が多くとも一度しか現れない monomial $e_{i_1} e_{i_2} \cdots e_{i_m} \ (1 \leq i_j \leq n)$ に対して $T_n(e_{i_1} e_{i_2} \cdots e_{i_m}) = \tau^m$ となる．

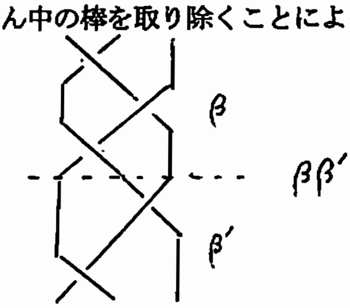
証明. $e_i e_{i-1} e_i = (1_{A_{i-1}} \otimes 1_{A_{i-1}}) e_{i-1} (1_{A_{i-1}} \otimes 1_{A_{i-1}}) = 1 \otimes E_{i-1}(e_{i-1}) 1 = 1_{A_{i-1}} \otimes \tau 1_{A_{i-1}} = \tau e_{i-1}$. 次に $e_i e_{i+1} e_i = \tau e_i$ を示そう．$A_i = A_1$ とおくことにより，$A_2 = A_1 \otimes_A A_1$ の場合を考えれば十分．$A_1 \otimes_A A_1 = A \otimes_S A \otimes_S A$ なる同一視により，積は$(a_0 \otimes a_1 \otimes a_2)(b_0 \otimes b_1 \otimes b_2) = \tau a_0 \otimes a_1 E(a_2 b_0) b_1 \otimes b_2)$ となる．さて，$e_2 = \sum_{i,j} x_i \otimes y_i x_j \otimes y_j$ かつ $e_1 = \sum_k (1 \otimes 1)(\tau^{-1} x_k \otimes 1) \otimes (1 \otimes y_k) = \tau^{-1}(1 \otimes 1 \otimes 1) \in A \otimes_S A \otimes_S A$ となり，よって $e_1 e_2 e_1 = 1 \otimes 1 \otimes 1 = \tau e_1$ となる．次に $i - j \geq 2$ と仮定する．このとき $e_j \in A_{i-1}$ であるので，$e_j e_i = e_j (1_{A_{i-1}} \otimes 1_{A_{i-1}}) = (1_{A_{i-1}} \otimes 1_{A_{i-1}}) e_j = e_i e_j$. 後半の主張は T_n の定義から明らか．

§2. Jones 多項式

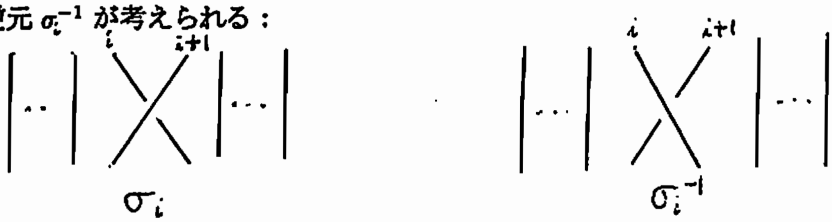
まず絡み目やブレイド群について述べる（詳しくは [13], [14] を参照のこと）. 3 次元球面 S^3 の中に埋め込まれた，向きのついた有限個の滑らかな円周の和を絡み目 (Link) と呼ぶことにする. そして，連結な絡み目を結び目 (Knot) と呼ぶ. また 2 つの絡み目 L_1, L_2 が同値 ($L_1 \approx L_2$) であるとは，S^3 の向きを保つ同相写像 h で $h(L_1) = L_2$ となるものが存在することとする. 絡み目全体の集合を $\mathcal{L}$ で表す. 絡み目 L の同値類を $[L]$ とかき，$[\mathcal{L}] = \mathcal{L}/\approx = \{[\mathcal{L}] \mid \mathcal{L} \in \mathcal{L}\}$ とおく. 絡み目は平面上に 2 重点のみを許して，横断的に交差する曲線から構成される絵として表示できる.

ブレイド(braid) とは，上下に置かれた n 個の留めがねを持つ水平な棒の間に両端がその留めがねにひっかかっている n 本の糸が上から下に単調に下がっているものとする. 端点が同一である2 つのブレイド α と α' が同値であるとは（ここでは，同値であることを $\alpha = \alpha'$ と書く），端点を固定し，上下の棒の外に出ずに α から α' へ連続的に変形できることである. この同値関係により分類した n 本のブレイドの同値類の集合 B_nは群になる. これが ブレイド群 (Artin braid group) である. 単位元は n 本の平行に降りていく紐からなるブレイド（の同値類）である. 2 つのブレイド（の同値類）β, β' の積は下の図のように β の下端点と β' の上端点をつなぎ，その真ん中の棒を取り除くことによって与えられる.





基本的なブレイドとして，入ってくる点の i 番目と $i+1$ 番目のみを入れ換えた σ_i およびその逆元 σ_i^{-1} が考えられる：

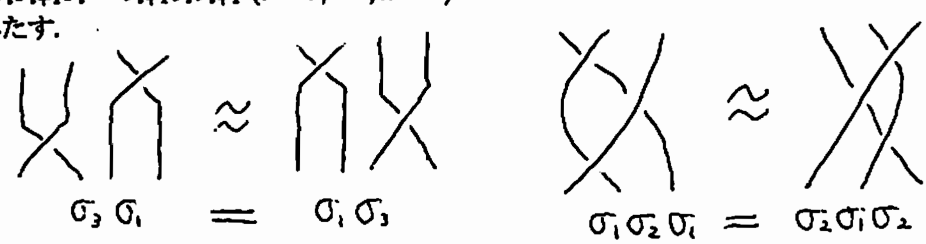


n 次のブレイド群 B_n は $\sigma_1, \cdots, \sigma_{n-1}$ で生成され，これらは関係

(1) $\sigma_i\sigma_j = \sigma_j\sigma_i$ $(i - j \geq 2)$

(2) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ $(i = 1, \cdots, n-2)$

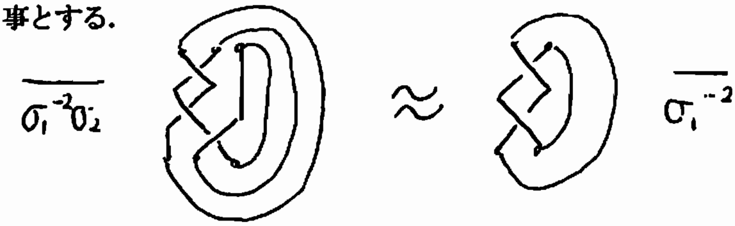
をみたす.



ブレイド β の閉包 $\bar{\beta}$ はブレイドの始点と終点を平行な紐で結んだものである.

Alexander の定理. 任意の絡み目は，ブレイドの閉包の形で表された絡み目に同値である.

1 型の Markov move とは，B_n の元 β を，ある $g \in B_n$ に対して $g\beta g^{-1}(\in B_n)$ と置き換える事とする．2 型の Markov move とは，$\beta \in B_n$ を $\beta\sigma_n \in B_{n+1}$ で置き換えること，あるいはその逆の操作，つまり，β が他にσ_n を含まなければ，$\beta\sigma_n \in B_{n+1}$ を $\beta \in B_n$ に置き換える事とする.



Markov の定理. $\beta_n \in B_n$, $\beta'_m \in B_m$ を，それぞれ，ブレイド群 B_n と B_m 内のブレイドとする．このとき，絡み目 $\bar{\beta}_n$ と $\bar{\beta}'_m$ が全同位になるための必要十分条件は，β'_m が β_n に1 型，2 型の Markov moves を有限回施すことによって得られる事である.

$\sim$ で Markov moves から誘導される同値関係を表すことにすると，Alexander の定理と Markov の定理は$[\mathcal{L}]$ と $(\coprod_{n\geq 1} B_n)/\sim$ の間に 1 対 1 の対応が存在することを示す.

定理 2.1. $A \supset S$ は Markov トレース T を持つ K 上の強分離拡大であり，(x_i, y_i, E, τ) を強分離系とする．e_1, $e_2, \cdots, e_n$ は補題 1.4 のべき等元とする．もし $\tau = t/(t+1)^2$ をみたす $t \in K^\times$ が存在すれば，$\omega_i = (t+1)e_i - 1$ $(i = 1, \cdots, n)$ とおくとき，群準同型 $\Psi_t : B_{n+1} \to A_n^\times$ で $\Psi_t(\sigma_i) = \omega_i$ $(i = 1, \cdots, n)$ をみたすものが存在する.

証明. 群 B_{n+1} は 関係$\sigma_i\sigma_j = \sigma_j\sigma_i$ $(i - j \geq 2)$ と$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ $(i = 1, \cdots, n-1)$ をみたす $\sigma_1, \cdots, \sigma_n$ で生成されるので,
(1) $\omega_i\omega_j = \omega_j\omega_i$ $(i - j \geq 2)$
(2) $\omega_i\omega_{i+1}\omega_i = \omega_{i+1}\omega_i\omega_{i+1}$ $(i = 1, \cdots, n-1)$
を示せば十分である．(1) は関係 $e_ie_j = e_je_i$ $(i - j \geq 2)$ から従う．(2) を示そう.
$\omega_i\omega_{i+1}\omega_i = (t+1)e_i + (t+1)e_{i+1} - (t+1)^2(e_ie_{i+1} + e_{i+1}e_i) - 1 = \omega_{i+1}\omega_i\omega_{i+1}$.

$A \supset S$ は Markov トレース T を持つ K 上の強分離拡大であり，(x_i, y_i, E, τ) を強分離系とする．e_1，$e_2, \cdots, e_n$ は補題 1.4 のべき等元とする．$\tau = t/(t+1)^2$ をみたす $t \in K^\times$ が存在し，更に t は K の中に平方根を持つと仮定する．$\sigma = \sigma_{i_1}^{m_1} \cdots \sigma_{i_r}^{m_r} \in B_{n+1}$ をとり，$L = \bar{\sigma}$ とおく．$\omega_i = (t+1)e_i - 1$, $T_{n+1} = T \circ E \circ E_1 \circ \cdots \circ E_{n+1}$ とおき，$[L]$ の Jones polynomial value を

$$V_L(t) = (-\frac{(t+1)}{\sqrt{t}})^n (\sqrt{t})^{n_1+\cdots+n_r} T_{n+1}(\omega_{i_1}^{n_1} \cdots \omega_{i_r}^{n_r}) = (-\frac{(t+1)}{\sqrt{t}})^n t^{e(\sigma)/2} T_{n+1}(\Psi_t(\sigma))$$

で定義する．ここで Psi_t は前定理で述べた群準同型$\Psi_t : B_{n+1} \to A_n^\times$, $e(\sigma) = n_1 + \cdots + n_r$ は σ を σ_i の和として表したときのべき和であり，ブレイド群 B_{n+1} の定義から，$e(\sigma)$ は σ のみに依存する．

定理 2.2. $V_L(t)$ は絡み目 $L = \bar{\sigma}$ の同値類 $[L]$ にしかよらない．従って，$[\mathcal{L}]$ から K への写像 $[L] \mapsto V_L(t)$ が定義される．

証明．既にのべたように，$[\mathcal{L}]$ と $(\coprod_{n \geq 1} B_n)/\sim$ の間に 1 対 1 の対応が存在するので，$V_L(t) \in K$ が 1 型，2 型の Markov move で不変なことを示せばよい．即ち，σ を $g\sigma g^{-1}$ ($g \in B_{n+1}$) または$\sigma\sigma_{n+2}^{\pm 1}$ で置き換えても $V_L(t)$ が不変なことを示せばよい．T_{n+1} はトレースであるので，$T_{n+1}(\Psi_t(g\sigma g^{-1})) = T_{n+1}(\Psi_t(\sigma)\Psi_t(g^{-1})\Psi_t(g)) = T_{n+1}(\Psi_t(\sigma))$ となる．一方，$T_{n+2}(\Psi_t(\sigma\sigma_{n+2}^{\pm 1})) = T_{n+2}(\Psi_t(\sigma)\omega_{n+2}^{\pm 1}) = T_{n+1}(\Psi_t(\sigma)E_{n+2}(\omega_{n+2}^{\pm 1})) = \tau^{\pm 1} T_{n+1}(\Psi_t(\sigma))$.

定理 2.2 により媒介変数 t で定まる $V_L(t) \in K$ は絡み目 $L = \bar{\sigma}$ の不変量と考えることができる．これが Jones 多項式である．

定義 2.1. 絡み目 L に対して $V_L(t) = (-\frac{(t+1)}{\sqrt{t}})^{N-1} t^{e(\sigma)/2} T_N(\Psi_t(\sigma))$ を L の Jones 多項式とよぶ．ここで σ は $L = \bar{\sigma}$ となるあるブレイド群 B_N の元，Ψ_t は定理 2.1 で述べた群準同型，$e(\sigma)$ はσ を σ_i の和として表したときのべき和である．

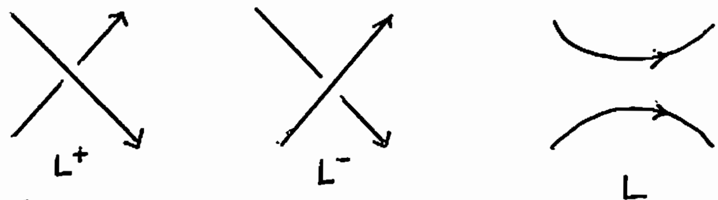
Jones 多項式は次のような性質をもつ(cf. [5])：

命題 2.1. $n(L)$ で絡み目 L の components の数を表す．
(1) $n(L)$ が奇数ならば $V_L(t) \in \mathbf{Z}[t, t^{-1}]$.
(2) $n(L)$ が偶数ならば $V_L(t) \in \sqrt{t}\mathbf{Z}[t, t^{-1}]$.

命題 2.2. 絡み目 L の mirror image を $L^\sim$ で表すとき，$V_{L^\sim}(t) = V_L(t)$.

命題 2.3. $n(L)$ は命題 2.1 と同様とする．このとき $V_L(1)=(-2)^{n(L)-1}$.

絡み目 L^+, L^-, L はある交点の近傍だけが次のように異なるものとする.

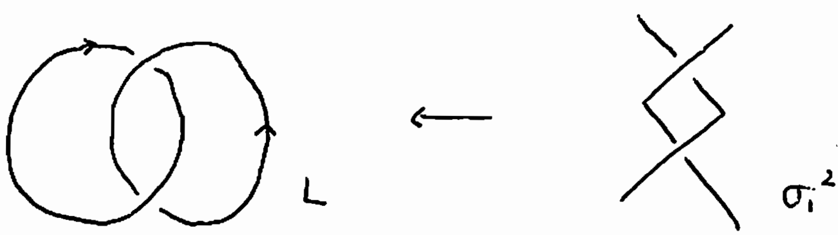


命題 2.4. $\frac{1}{t}V_{L^-}-tV_{L^+}=(\sqrt{t}-\frac{1}{\sqrt{t}})V_L(t)$.

§3. Jones 多項式の計算例

[5] の table 1 で 0_1 から 8_{21} までの結び目に対するブレイド表現と Jones 多項式を見つけることができる．ここでは，いくつかの絡み目の Jones 多項式を計算してみよう.

Hopf link.

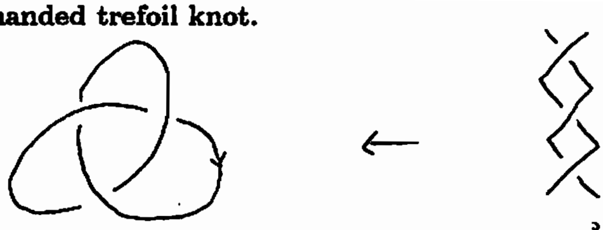


$\tau=t/(t+1)^2$ であるので,

$$V_L(t)=(-\frac{t+1}{\sqrt{t}})^{2-1}(\sqrt{t})^2T_2(\omega_1^2)$$
$$=-(t+1)\sqrt{t}T_2(\{(t+1)e_1-1\}^2)$$
$$=-(t+1)\sqrt{t}T_2((t+1)^2e_1-2(t+1)e_1+1)$$
$$=-(t+1)\sqrt{t}\{(t+1)^2\tau-2(t+1)\tau+1\}$$
$$=-\sqrt{t}(t^2+1).$$

特に $V_L(1)=-2=(-2)^{p-1}$ から $n(L)=p=2$ （命題 2.3）となり，L の components の数が 2 であるという事実とあう.

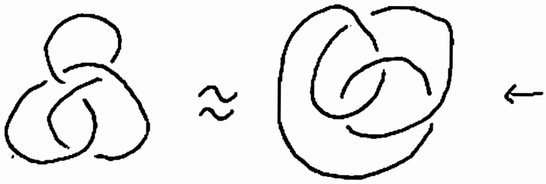
Left-handed trefoil knot.



3_1 σ_1^{3}

$$V_{3_1}(t) = (-\frac{t+1}{\sqrt{t}})^{2-1}(\sqrt{t}^3)T_2(\omega_1^3)$$
$$= -t(t+1)T_2(\{(t+1)e_1 - 1\}^3)$$
$$= -t(t+1)T_2((t+1)^3e_1 - 3(t+1)^2e_1 + 3(t+1)e_1 - 1)$$
$$= -t(T+1)\{(t+1)^3\tau - 3(t+1)^2\tau + 3(t+1)\tau - 1)$$
$$= t + t^3 - t^4.$$

Knot 4_1.



4_1

$\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$

$$V_{4_1}(t) = (-\frac{t+1}{\sqrt{t}})^{3-1}(\sqrt{t})^{1-1+1-1}T_3((\omega_1\omega_2^{-1})^2)$$
$$= \frac{(t+1)^2}{t}T_3((\{(t+1)e_1 - 1\}\{\frac{t+1}{t}e_2 - 1\})^2)$$
$$= \frac{(t+1)^2}{t}T_3(\frac{(t+1)^4}{t^2}e_1e_2e_1e_2 - \frac{(t+1)^3}{t}e_1e_2e_1 - \frac{(t+1)^3}{t^2}e_2e_1e_2 - \frac{(t+1)^2}{t^2}(t^2-t+1)e_1e_2\frac{(t+1)^2}{t}e_2e_1 + (t^2-1)e_1 - \frac{t^2-1}{t^2}e_2 + 1)$$
$$= \frac{(t+1)^2}{t}T_3(\frac{(t+1)^4}{t^2}\tau e_1e_2 - \frac{(t+1)^3}{t}\tau e_1 - \frac{(t+1)^3}{t^2}\tau e_2 - \frac{(t+1)^2}{t^2}(t^2-t+1)e_1e_2\frac{(t+1)^2}{t}e_2e_1 + (t^2-1)e_1 - \frac{t^2-1}{t^2}e_2 + 1)$$
$$= \tau^{-1}\{\frac{(t+1)^4}{t^2}\tau^3 - \frac{(t+1)^3}{t}\tau^2 - \frac{(t+1)^3}{t^2}\tau^2 - \frac{(t+1)^2}{t^2}(t^2-t+1)\tau^2\frac{(t+1)^2}{t}e_2e_1 + (t^2-1)\tau - \frac{t^2-1}{t^2}\tau + 1)\}$$
$$= t^{-2} - t^{-1} + 1 + -t + t^2.$$

REFERENCES

1. F.R. DeMeyer, *The trace map and separable algebras,* Osaka J. Math. 3 (1966), 7-11.

2. F. Goodman, P. de la Harpe and V.F.R. Jones, Coxeter Graphs and Towers of Algebras, M.S.R.I. Publ. 14, Springer, 1989.

3. A. Hattori, *On strongly separable algebras, Osaka J. Math. 2 (1965), 369-372.*

4. *K. Hirata and K. Sugano, On semisimple extension and separable extensions over non commutative rings,* J. Math. Soc. Japan 18 (1964), 360-373.

5. V.F.R. Jones, *A polynomial invariant for knots via von Neumann algebras,* Bull. A.M.S. 12 (1985), 103-111.

6. V.F.R. Jones, *Braid groups, Hecke algebras and type II_1 factors,* in Geometric Methods in Operator Algebras, Proc. US-Japan Seminar (1986), 242-273.

7. V.F.R. Jones, *Hecke algebra representations of braid groups and link polynomials,* Ann. Math. 126 (1987), 335-388.

8. L. Kadison, *Algebraic aspects of the Jones basic construction,* C.R. Math. Rep. Acad. Sci. Canada 15 (1993), 223-228.

9. L. Kadison, *Separability and the Jones Polynomial,* in Rings, extensions, and cohomology, ed. A.R. Magid, Lect. Notes in Pure and Applied Math. 132 (1994), 123-137.

10. L. Kadison, *On split, separable subalgebras with counitality condition,* Hokkaido Math. J. 24 (1995), 527-549.

11. L. Kadison, *The Jones polynomial and certain separable Frobenius extensions,* J. Algebra 186 (1996), 461-475.

12. F. Kasch, *Dualitätseigenschaften von Frobenius-Erweiterungen,* Math. Zeit. 77 (1961), 219-227.

13. L. H. Kauffmann, Knots and Physics, World Sci. Pub., 1991. （日本語訳：結び目の数学と物理，培風館）

14. 小林毅, 絡み目理論の新しい不変量—作用素環に由来する Jones 多項式とその一般化—, 数学 38(1) (1986), 1-14.

15. K. Sugano, *Note on separability of endomorphism rings,* Hokkaido Math. J. 21 (1971), 196-208.

THE CONNECTIONS BETWEEN THE WEAK π-REGULARITY AND THE MAXIMALITY OF PRIME IDEALS OF RINGS

CHAN YONG HONG*, NAM KYUN KIM, TAI KEUN KWAK AND YANG LEE

The relationship between various generalizations of von Neumann regularity and the condition that every prime ideal is maximal have been investigated by many authors [3, 4, 7, 9, 13, 15, and 16]. The first clearly established equivalence between a generalization of von Neumann regularity and the maximality of prime ideals seems to have been made by Storrer [13] in the following result: If R is a commutative ring with identity then R is π-regular if and only if every prime ideal of R is maximal. Storrer's result was extended to PI-rings [7, Theorem 2.3], right duo rings [9, Corollary 1] and bounded weakly right duo rings [15, Theorem 3], respectively. On the other hand, Hirano [9] proved that if R is a 2-primal ring, then R is π-regular if and only if every prime ideal of R is a maximal one-sided ideal. Recently, Birkenmeier, Kim and Park [3] showed that if R is a 2-primal ring, then $R/P(R)$ is right weakly π-regular if and only if every prime ideal of R is maximal. The π-regularity of rings is extended to the weak π-regularity. In general, π-regular rings are weakly π-regular rings but the converse is not hold.

We investigate the connections between the results of previously mentioned papers and weak π-regularity in 2-primal rings, right quasi-duo rings and PI-rings, respectively.

Throughout this paper the letter R denotes an associative ring with identity and all prime ideals of R are assumed to be proper. $P(R)$, $J(R)$ and $N(R)$ denote the prime radical, the Jacobson radical and the set of all nilpotent elements of R, respectively.

We begin with the following definitions.

Definition 1. (1) A ring R is said to be *(strongly) π-regular* if for every $x \in R$ there exists a natural number n, depending on x, such that ($x^n \in x^{n+1}R$) $x^n \in x^nRx^n$. Strong π-regularity is right-left symmetric [6].

(2) A ring R is said to be *right (left) weakly π-regular* if for every $x \in R$ there exists a natural number n, depending on x, such that $x^n \in x^nRx^nR$ ($x^n \in Rx^nRx^n$). R is *weakly π-regular* if it is both right and left weakly π-regular [8].

Definition 2. A ring R is called *2-primal* if $P(R) = N(R)$ [2].

The term *2-primal* was come upon originally by Birkenmeier, Heatherly and Lee. But Hirano [9] used the term *N-ring* for what we call a 2-primal ring. The 2-primal condition was taken up independently by Sun [14], where in the setting of rings he

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introduced a condition to be called *weakly symmetric*, which is equivalent to the 2-primal condition for rings.

Hirano showed the following proposition.

Proposition 3. *[9, Theorem 1] Let R be a 2-primal ring. Then the following statements are equivalent:*

(a) *R is strongly π-regular.*
(b) *R is π-regular.*
(c) *$R/J(R)$ is π-regular and $J(R)$ is nil.*
(d) *Every prime ideal of R is a maximal one-sided ideal.*

The following example shows that in a 2-primal ring "weak π-regularity" is not the same "π-regularity". Hence in Proposition 3, the condition "(b)" can not be replaced by the condition "R is right weakly π-regular".

Example 4. Let D be a simple domain which is not a division ring. We consider the ring

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in D \right\}.$$

Then R is a 2-primal right weakly π-regular ring which is not π-regular.

As a parallel result to this Proposition 3, we obtain the following result.

Proposition 5. *Let R be a 2-primal ring. Then the following statements are equivalent:*

(a) *$R/J(R)$ is right weakly π-regular and $J(R)$ is nil.*
(b) *Every prime ideal of R is maximal.*

Related to this Proposition 5, we noted that there exists a 2-primal ring whose prime ideals are maximal but neither right nor left weakly π-regular [3, Example 12].

However, we have the following theorem.

Theorem 6. *Let R be a 2-primal ring whose primitive factor rings are Artinian. Then the following statements are equivalent:*

(a) *R is strongly π-regular.*
(b) *R is π-regular.*
(c) *R is weakly π-regular.*
(d) *R is right weakly π-regular.*
(e) *$R/J(R)$ is right weakly π-regular and $J(R)$ is nil.*
(f) *Every prime ideal of R is maximal.*

Recall that a ring R is said to be *right* (*left*) *quasi-duo* if every maximal right (left) ideal of R is two-sided. A ring R is said to be *of bounded index* (*of nilpotency*) if there exists a positive integer n such that $a^n = 0$ for all nilpotent elements a of R.

The following theorem extends [4, Proposition 2.14].

Theorem 7. *Let R be a right quasi-duo ring. Then the following statements are equivalent:*

(a) *R is right weakly π-regular.*
(b) *R is strongly π-regular.*

The fact that a right duo ring (i.e., a ring whose right ideals are two-sided) implies 2-primal lead one to conjecture that a right quasi-duo ring is 2-primal. However in the following examples, the rings R are right quasi-duo but not 2-primal.

Example 8. Let T be the n-by-n upper triangular matrix ring over a field F, where n is an infinite cardinal number. Then by [16, Proposition 2.1] T is right quasi-duo. Now consider $R = T[[x]]$, where $T[[x]]$ denotes the ring of formal power series over T. Then R is right quasi-duo which is not 2-primal.

On the other hand, Yu [16] proved the following.

Proposition 9. *Let R be a right quasi-duo ring. If every prime ideal of R is maximal, then R is strongly π-regular.*

However, the converse of this Proposition 9 is not hold by the following example.

Example 10. [4, Example 3.3] Let G be an abelian group which is the direct sum of a countably infinite number of infinite cyclic groups; and denote by $\{b(0), b(1), b(-1), \ldots, b(i), b(-i), \ldots\}$ of basis of G. Then there exists one and only one homomorphism $u(i)$ of G, for $i = 1, 2, \ldots$ such that $u(i)(b(j)) = 0$ if $j \equiv 0 (\text{mod } 2^i)$ and $u(i)(b(j)) = b(j-1)$ if $j \not\equiv 0 (\text{mod } 2^i)$. Denote U the ring of endomorphisms of G generated by the endomorphisms $u(1), u(2), \ldots$. Now let A be the ring obtained from U by adjoining the identity map of G. Let $R = A \otimes_{\mathbb{Z}} \mathbb{Q}$, where $\mathbb{Z}$ is the ring of integers and $\mathbb{Q}$ is the field of rationals. Then R is a right quasi-duo ring of no bounded index and $J(R)$ is nil but R is not 2-primal. R is also a semiprime strongly π-regular ring, but there exists a prime ideal of R which is not maximal.

Note that the ring R in Example 8 is of no bounded index and $J(R)$ is not nil.

Proposition 11. *Let R be a right quasi-duo ring of bounded index with $J(R)$ nil. Then R is a 2-primal ring.*

Corollary 12. *Let R be a right quasi-duo ring of bounded index. Then the following statements are equivalent:*

(a) *R is strongly π-regular.*
(b) *R is π-regular.*
(c) *$R/J(R)$ is π-regular and $J(R)$ is nil.*
(d) *R is weakly π-regular.*
(e) *R is right weakly π-regular.*
(f) *$R/J(R)$ is right weakly π-regular and $J(R)$ is nil.*
(g) *$R/P(R)$ is right weakly π-regular.*
(h) *Every prime ideal of R is maximal.*

Note that in Proposition 11, the conditions "R is of bounded index" (Example 10) and "$J(R)$ is nil" are not superfluous.

Example 13. Let F be a field and S denote the full ring of 2-by-2 matrices over F and $T = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in S \mid a \in F \right\}$. We consider the ring $R = T + xS[[x]]$, where $S[[x]]$ denotes the ring of formal power series over S. Then R is a right quasi-duo, semiprime PI-ring of bounded index 2 and $J(R) = xS[[x]]$ is not nil. But R is not 2-primal.

In Corollary 12, the condition "R is of bounded index" is not superfluous (Example 10).

Remark. In Example 4, the ring R is 2-primal right weakly π-regular which is of bounded index 2. But it is not π-regular. The ring in [3, Example 12] is a 2-primal ring whose prime ideals are maximal, and it is of bounded index 2. But it is neither right nor left weakly π-regular.

Now we investigate the connection between the weak π-regularity and the maximality of prime ideals in PI-rings.

For a PI-ring, Fisher and Snider [7] proved the following theorem.

Theorem 13. *Let R be a PI-ring. Then the following statements are equivalent:*

(a) *R is strongly π-regular.*
(b) *R is π-regular.*
(c) *$R/P(R)$ is π-regular.*
(d) *Every prime ideal of R is maximal.*

Hirano [9] also obtained the following result.

Theorem 14. *Let R be a PI-ring. If R is right weakly π-regular, then R is strongly π-regular.*

Thus the concept of right weak π-regularity, weak π-regularity, strong π-regularity and the maximality of prime ideals for a PI-ring are equivalent.

However, we have the following.

Theorem 15. *Let R be a PI-ring. Then the following statements are equivalent:*

(a) *R is right (left) weakly π-regular.*
(b) *$R/J(R)$ is right weakly π-regular and $J(R)$ is nil.*

Consequently, for PI-rings we have the same corollary as 2-primal rings and right quasi-duo rings without any extra conditions. The following Corollary 16 also includes [7, Theorem 2.3] and [9, Theorem 4].

Corollary 16. *Let R be a PI-ring. Then the following statements are equivalent:*

(a) *R is strongly π-regular.*
(b) *R is π-regular.*
(c) *$R/P(R)$ is π-regular.*
(d) *$R/J(R)$ is π-regular and $J(R)$ is nil.*
(e) *R is weakly π-regular.*
(f) *R is right (left) weakly π-regular.*

(g) *$R/J(R)$ is right weakly π-regular and $J(R)$ is nil.*
(h) *Every prime factor ring of R is right (left) weakly π-regular.*
(i) *Every prime ideal of R is maximal.*
(j) *Every prime factor ring of R is simple Artinian.*

Since all primitive factor rings of a PI-ring and a right quasi-duo ring are Artinian, we have the following proposition related to Corollary 12 and Corollary 16.

Proposition 17. *Let R be of bounded index whose primitive factor rings are Artinian. Then the following statements are equivalent:*

(a) *R is strongly π-regular.*
(b) *R is π-regular.*
(c) *$R/P(R)$ is π-regular.*
(d) *Every prime ideal of R is maximal.*

The conditions of Proposition 17 (1) "R is of bounded index" and (2) "every primitive factor ring of R is Artinian" are not superflous.

Example 18. (1) In Example 10, the ring R is a right quasi-duo ring, so every primitive factor ring of R is Artinian. R is also a strongly π-regular ring of no bounded index. But there exists a prime ideal of R which is not maximal.
(2) [3, Example 13] Let $W = W_1[F]$ be the first Weyl algebra over a field F of characteristic zero. Now we consider the ring

$$R = \{(a_i)_{i=1}^{\infty} \mid a_i \in Mat_2(W) \text{ is eventually a constant upper triangular matrix } \},$$

where $Mat_2(W)$ denotes the full ring of 2-by-2 matrices over W. Then R is a semiprime ring of bounded index 2 whose prime ideals are maximal. But R is not π-regular and so R does not satisfy the condition (2) "every primitive factor ring of R is Artinian".

Acknowledgements

The authors would like to thank Professor Jae Keol Park and Professor Masahisa Sato for their giving the opportunity to present this material on The Korea-Japan joint ring theory seminar.

References

[1] A. Badawi, On abelian π-regular rings, Comm. Algebra 25(4) (1997) 1009-1021.

[2] G.F. Birkenmeier, H.E. Heatherly and E.K. Lee, Completely prime ideals and associated radicals, in: S.K. Jain and S.T. Rizvi, eds., Proc. Biennial Ohio State-Denison Conference 1992 (World Scientific, New Jersey, 1993) 102-129.

[3] G.F. Birkenmeier, J.Y. Kim and J.K. Park, A connection between weak regularity and the simplicity of prime factor rings, Proc. Amer. Math. Soc. 122 (1994) 53-58.

[4] G.F. Birkenmeier, J.Y. Kim and J.K. Park, Regularity conditions and the simplicity of prime factor rings, J. Pure and Appl. Algebra 115 (1997) 213-230.

[5] A.W. Chatters and C.R. Hajarnavis, Rings with chain conditions (Pitman Advanced Publishing Program, 1980).

[6] F. Dischinger, Sur les anneaux fortment π-reguliers, C.R. Acad. Sci. Paris Ser. A-B 283 (1976) 571-573.

[7] J.W. Fisher and R.L. Snider, On the von Neumann regularity of rings with regular prime factor rings, Pacific J. Math. 54(1) (1974) 135-144.

[8] V. Gupta, Weakly π-regular rings and group rings, Math. J. Okayama Univ. 19 (1977) 123-127.

[9] Y. Hirano, Some studies on strongly π-regular rings, Math. J. Okayama Univ. 20 (1978) 141-149.

[10] Y. Hirano, Some characterizations of π-regular rings of bounded index, Math. J. Okayama Univ. 32 (1990) 97-101.

[11] P. Menal, On π-regular rings whose primitive factor rings are Artinian, J. Pure and Appl. Algebra 20 (1981) 71-78.

[12] E.C. Posner, Prime rings satisfying a polynomial identity, Proc. Amer. Math. Soc. 11 (1960) 180-183.

[13] H.H. Storrer, Epimorphismen von kommutativen Ringen, Comment Math. Helv. 43 (1968) 378-401.

[14] S.H. Sun, Noncommutative rings in which every prime ideal is contained in a unique maximal ideal, J. Pure and Appl. Algebra 76 (1991) 179-192.

[15] Xue Yao, Weakly right duo rings, Pure and Appl. Math. Sci. XXI(1-2) (1985) 19-24.

[16] H.P. Yu, On quasi-duo rings, Glasgow Math. J. 37 (1995) 21-31.

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On modules of finite Gorenstein dimension

by

Mitsuo Hoshino

Abstract. In this note, there is given a left-right symmetric condition (*) for left and right coherent rings, which is weaker than the Auslander condition. It is shown that, if A is a left and right noetherian ring satisfying the condition (*), then for any integer $n \geq 0$ the following statements are equivalent: (1) A has injective dimension at most n on both sides; (2) every finitely generated left A-module has Gorenstein dimension at most n; and (3) every finitely generated right A-module has Gorenstein dimension at most n.

1. 未解決の問題

最初に、扱う問題の説明をする。以下に於て、話を簡単にするために、A は両側ネター環とする。左 A-加群としての A を ${}_AA$ で、右 A-加群としての A を A_A で表す。

加群 X の入射次元を inj dim X で表す。次の未解決の問題がある。

問題 A. inj dim ${}_AA$ = inj dim A_A か?

知る限りでは反例はない。肯定的な結果も殆どないが、それは結果が出ないというだけであって誰も研究していないという意味ではない。個人的には、少なくとも現時点では、肯定的であると信じるだけの積極的な理由がない。

上の問題に関連して次の事実がある ([8] を参照せよ)。

命題 1.1. inj dim ${}_AA < \infty$ かつ inj dim $A_A < \infty$ なら inj dim ${}_AA$ = inj dim A_A である。

従って、問題 A は次の様に言い換えられる。

問題 B. inj dim $A_A < \infty$ なら inj dim ${}_AA < \infty$ か?

加群 X の平坦次元を flat dim X で表す。問題 B に関連して次の事実がある (例えば、[2] を参照せよ)。

命題 1.2. inj dim $A_A < \infty$ なら次の (1), (2) は同値である。

(1) inj dim ${}_AA < \infty$ である。

(2) (a) 任意の単純左 A-加群 S に対して $\mathrm{Hom}_A(S, I^i) \neq 0$ を満たす $i \geq 0$ が存在する。

(b) 極小入射分解 $A_A \to I^{\bullet}$ に於て、全ての $i \geq 0$ に対して flat dim $I^i < \infty$ である。

This note is not in final form. A detailed version may be submitted for publication elsewhere.

上の命題に於ける条件 (2) の (a) は Generalized Nakayama Conjecture の特別な場合である。他方で、Generalized Nakayama Conjecture は解けそうな状況から遥かに遠い。この観点からは、問題 B を解くのは絶望的と思われる。ところが、実際は必ずしもそうではなく、むしろ上の命題に於ける条件 (2) の (b) の方が難しい様である。例えば、次の事実がある ([7] を参照せよ)。

命題 1.3. A を両側アルティン環とする。$\text{inj dim}\, A_A < \infty$ なら次の (1), (2) は同値である。
(1) $\text{inj dim}\, {}_A A < \infty$ である。
(2) 極小入射分解 $A_A \to I^\bullet$ に於て、全ての $i \geq 0$ に対して $\text{flat dim}\, I^i < \infty$ である。

加群 X の Gorenstein 次元を $\text{G-dim}\, X$ で表す。次の事実がある ([1] を参照せよ)。

命題 1.4. $n \geq 0$ を整数とする。次の (1)–(3) について、(1) $\Rightarrow$ (2) $\Rightarrow$ (3) が成り立つ。
(1) $\text{inj dim}\, {}_A A = \text{inj dim}\, A_A \leq n$ である。
(2) 全ての有限生成左 A-加群 X に対して $\text{G-dim}\, X \leq n$ である。
(3) $\text{inj dim}\, {}_A A \leq n$ である。

上の命題に於ける条件 (1)–(3) について、(3) $\Rightarrow$ (1) が成り立つか？が問題 B に他ならない。本稿では、(2) $\Rightarrow$ (1) が成り立つか？を問題にする。即ち、次の問題を考察する。

問題 C. 任意の整数 $n \geq 0$ に対して次の (1)–(3) は同値か？
(1) $\text{inj dim}\, {}_A A = \text{inj dim}\, A_A \leq n$ である。
(2) 全ての有限生成左 A-加群 X に対して $\text{G-dim}\, X \leq n$ である。
(3) 全ての有限生成右 A-加群 M に対して $\text{G-dim}\, M \leq n$ である。

有限生成左 A-加群の圏が enough injectives (例えば、A がアルティン多元環のとき) なら上の問題は任意の $n \geq 0$ に対して肯定的である ([6] を参照せよ)。また、$n \leq 2$ のときには、上の問題は任意の A に対して肯定的である ([4] を参照せよ)。
更に、次の事実がある (例えば、[1], [6] 等を参照せよ)。

命題 1.5. 任意の整数 $n \geq 0$ に対して次の (1), (2) は同値である。
(1) $\text{inj dim}\, {}_A A = \text{inj dim}\, A_A \leq n$ である。
(2) (a) 全ての有限生成右 A-加群 M に対して $\text{G-dim}\, M \leq n$ である。
  (b) 極小入射分解 $A_A \to I^\bullet$ に於て、全ての $0 \leq i \leq n$ に対して $\text{flat dim}\, I^i < \infty$ である。

従って、問題 C を考察するにあたっては極小入射分解 $A_A \to I^\bullet$ に条件を設定しては意味がなくなる。
本稿では、両側ネター環についての左右対称な条件 (*) を与え、その条件 (*) を満たす両側ネター環に対しては問題 C が任意の $n \geq 0$ に対して肯定的であることを示す。また、条件 (*) は命題 1.5 に於ける条件 (2) の (b) とは無関係であることを注意しておく。

2. 射影的に安定な加群論

主結果を述べるために必要な概念を復習する (詳しくは [1] を参照せよ)。以下に於て、話を簡単にするため、A は両側ネター環とする。有限生成左 A-加群全体の成す圏を $\text{mod}\, A$ で表す。A の反

転環を A^{op} で表し右 A-加群を左 A^{op}-加群とみなす。また、$(\ )^* = \mathrm{Hom}_A(-, A)$ と置く。

定義 2.1. 任意の $X, Y \in \mathrm{mod}\,A$ に対して、有限生成射影左 A-加群を経由する $f \in \mathrm{Hom}_A(X, Y)$ 全体から成る $\mathrm{Hom}_A(X, Y)$ の部分アーベル群を $\mathrm{Phom}_A(X, Y)$ で表し

$$\underline{\mathrm{Hom}}_A(X, Y) = \mathrm{Hom}_A(X, Y)/\mathrm{Phom}_A(X, Y)$$

と置く。このとき、$\mathrm{Ob}(\underline{\mathrm{mod}}\,A) = \mathrm{Ob}(\mathrm{mod}\,A)$ とし、任意の $X, Y \in \underline{\mathrm{mod}}\,A$ に対して X から Y への射集合を $\underline{\mathrm{Hom}}_A(X, Y)$ とする加法圏 $\underline{\mathrm{mod}}\,A$ を得る。

定義 2.2. 任意の $X \in \underline{\mathrm{mod}}\,A$ に対して、P を射影的とする完全列 $0 \to \Omega X \to P \to X \to 0$ に依って $\Omega X \in \underline{\mathrm{mod}}\,A$ を定義する。このとき、加法的共変函手を得る

$$\Omega : \underline{\mathrm{mod}}\,A \to \underline{\mathrm{mod}}\,A,\ X \mapsto \Omega X.$$

定義 2.3. 各 $X \in \mathrm{mod}\,A$ に対して、有限表示 $P_1 \to P_0 \to X \to 0$ (即ち、P_0, P_1 を有限生成射影加群とする完全列) を取り完全列 $P_0^* \to P_1^* \to \mathrm{Tr}X \to 0$ に依って $\mathrm{Tr}X \in \mathrm{mod}\,A^{op}$ を定義する。このとき、加法的反変函手を得る

$$\mathrm{Tr} : \underline{\mathrm{mod}}\,A \to \underline{\mathrm{mod}}\,A^{op},\ X \mapsto \mathrm{Tr}X.$$

更に、反変函手の対

$$\mathrm{Tr} : \underline{\mathrm{mod}}\,A \to \underline{\mathrm{mod}}\,A^{op}, \quad \mathrm{Tr} : \underline{\mathrm{mod}}\,A^{op} \to \underline{\mathrm{mod}}\,A$$

は $\underline{\mathrm{mod}}\,A$ と $\underline{\mathrm{mod}}\,A^{op}$ との間に双対を定める。

定義 2.4. 各 $X \in \mathrm{mod}\,A$ に対して

$$\mathrm{grade}\ X = \min\{i \geq 0 \mid \mathrm{Ext}_A^i(X, A) \neq 0\}$$

と置く。但し、全ての $i \geq 0$ に対して $\mathrm{Ext}_A^i(X, A) = 0$ のときには、$\mathrm{grade}\ X = \infty$ とする。また、

$$\text{reduced grade } X = \min\{i \geq 1 \mid \mathrm{Ext}_A^i(X, A) \neq 0\}$$

と置く。但し、全ての $i \geq 1$ に対して $\mathrm{Ext}_A^i(X, A) = 0$ のときには、reduced grade $X = \infty$ とする。

定義 2.5. $X \in \mathrm{mod}\ A$ に対して, reduced grade $X = \infty$ かつ reduced grade $\mathrm{Tr}X = \infty$ のとき $\mathrm{G\text{-}dim}\,X = 0$ と定義する。次に、全ての $0 \leq i \leq n$ に対して $\mathrm{G\text{-}dim}\,X_i = 0$ である完全列

$$0 \to X_n \to \cdots \to X_0 \to X \to 0$$

が存在するとき、$\mathrm{G\text{-}dim}\,X \leq n$ と定義する。

命題 2.1. 任意の $X \in \mathrm{mod}\,A$ および $n \geq 0$ に対して次の (1), (2) は同値である。
(1) $\mathrm{G\text{-}dim}\,X \leq n$.
(2) $\mathrm{G\text{-}dim}\,\Omega^n(X) = 0$.

系 2.2. 任意の $X \in \mathrm{mod}\, A$ に対して $\mathrm{G\text{-}dim}\, X \le \mathrm{proj\ dim}\, X$ が成り立つ。但し、$\mathrm{proj\ dim}\, X$ は X の射影次元を表す。

3. 左右対称性

以下に於いて、話を簡単にするため、A は両側ネター環とする。有限生成左 A-加群全体の成す圏を $\mathrm{mod}\, A$ で表す。また、A の反転環を A^{op} で表し右 A-加群を左 A^{op}-加群とみなす。
先ず、次の注意をする。

注意 3.1. 極小入射分解 ${}_AA \to I({}_AA)^\bullet$ と極小入射分解 $A_A \to I(A_A)^\bullet$ との間の対称性は Auslander condition ([3] を参照せよ) 以外には存在しない。即ち、$\mathrm{flat\ dim}\, I({}_AA)^0 = 1$ かつ $\mathrm{flat\ dim}\, I(A_A)^0 = \infty$ を満たす A が存在する ([7] を参照せよ)。

極小入射分解に関係しない左右対象な条件がある。

定理 3.1. 任意の $n \ge 2$ に対して次の (1)-(4) は同値である。
(1) 全ての $X \in \mathrm{mod}\, A$ および $2 \le i \le n$ に対して $\mathrm{grade}\ \mathrm{Ext}_A^i(X, A) \ge i-1$ である。
(2) 全ての $X \in \mathrm{mod}\, A$ および $2 \le i \le n$ に対して reduced grade $\mathrm{Tr}(\Omega^i(X)) \ge i+1$ である。
(3) 全ての $M \in \mathrm{mod}\, A^{\mathrm{op}}$ および $2 \le i \le n$ に対して $\mathrm{grade}\ \mathrm{Ext}_A^i(M, A) \ge i-1$ が成り立つ。
(4) 全ての $M \in \mathrm{mod}\, A^{\mathrm{op}}$ および $2 \le i \le n$ に対して reduced grade $\mathrm{Tr}(\Omega^i(M)) \ge i+1$ である。

証明は $n \ge 2$ に関する帰納法に依る。複雑ではあるが、事実に気付きさえすれば、証明するのはそう難しくない。

定義 3.1. 整数 $n \ge 2$ に対して、命題 3.1 の同値条件が満たされるとき、A は条件 (C_n) を満たすと言うことにする。

また、Auslander condition と (C_n) の中間に位置する非対称な条件がある ([5] を参照せよ)。

命題 3.2. 任意の $n \ge 0$ に対して次の (1), (2) は同値である。
(1) 全ての $X \in \mathrm{mod}\, A$ および $0 \le i \le n$ に対して $\mathrm{grade}\ \mathrm{Ext}_A^{i+1}(X, A) \ge i+1$ である。
(2) 極小入射分解 ${}_AA \to I^\bullet$ に於て、全ての $0 \le i \le n$ に対して $\mathrm{flat\ dim}\, I^i \le i+1$ である。

注意 3.2. $n \ge 0$ を整数とする。命題 3.2 の同値条件が満たされるなら A は条件 (C_{n+2}) を満たす。

A がアルティン環の場合には問題 B に対する部分的解答がある ([7] を参照せよ)。

命題 3.3. A を両側アルティン環とする。$n \ge 0$ を整数とし、$n \ge 2$ の場合には、A は条件 (C_n) を満たすと仮定する。このとき、次の (1), (2) は同値である。
(1) $\mathrm{inj\ dim}\, {}_AA \le n$ である。
(2) $\mathrm{inj\ dim}\, A_A \le n$ である。

最後に、問題Cに対する部分的解答を与える。

定理 3.4. $n \geq 0$ を整数とし、$n \geq 3$ の場合には、A は条件 (C_{n-1}) を満たすと仮定する。このとき、次の (1)–(3) は同値である。

(1) $\mathrm{inj\,dim}\,_{\Lambda}A = \mathrm{inj\,dim}\,A_{\Lambda} \leq n$ である。

(2) 全ての $X \in \mathrm{mod}\,A$ に対して $\text{G-dim}\,X \leq n$ である。

(3) 全ての $M \in \mathrm{mod}\,A^{op}$ に対して $\text{G-dim}\,M \leq n$ である。

参考文献

[1] M. Auslander and M. Bridger, *Stable module theory*, Mem. Amer. Math. Soc. vol. **94**, 1969.

[2] H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press, Princeton, 1956.

[3] R. M. Fossum, Ph. A. Griffith and I. Reiten, *Trivial extensions of abelian categories*, Springer LNM vol. **456**, 1976.

[4] M. Hoshino, *Reflexive modules and rings with self-injective dimension two*, Tsukuba J. Math. **13**(1989), 419-422.

[5] M. Hoshino, *Syzygies and Gorenstein rings*, Arch. Math. **55**(1990), 355-360.

[6] M. Hoshino, *Algebras of finite self-injective dimension*, Proc. Amer. Math. Soc. 112(1991), 619-622.

[7] M. Hoshino, *On self-injective dimensions of artinian rings*, Tsukuba J. Math. **18**(1994), 1-8.

[8] A. Zaks, *Injective dimension of semiprimary rings*, J. Algebra **13**(1969), 73-86.

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ON A GENERALIZATION OF QUASI-DUO CONDITION

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Throughout this paper, all rings are associative with identity. Given a ring R, the Jacobson radical of R, the polynomial ring over R and the formal power series ring over R are denoted by $J(R)$, $R[x]$ and $R[[x]]$, respectively. In this paper we study a condition that is a Morita invariant property and is a generalization of following two conditions: (1) the quasi-duo condition, which was initiated by Yu in [12] and is related to the Bass' conjecture in [2]; (2) the pm condition that was studied by Birkenmeier-Kim-Park in [3]. Also we study some conditions under which the Jacobson conjecture [5, p. 241] holds by using the properties of the condition (*). A ring R is called *right (left) quasi-duo* if every maximal right (left) ideal of R is two-sided. Commutative rings are clearly right quasi-duo and right quasi-duo rings satisfy (*). The n by n full matrix ring over a division ring, with n any positive integer $\geqq 2$, is not right quasi-duo; but the ring satisfies (*). However the ring of infinite row finite matrices over a division ring, say R, does not satisfy (*) because there exist maximal right ideals of R that do not contain the nonzero proper ideal $\{f \in R \mid \text{rank}(f) \text{ is finite}\}$ of R. A ring R is said to satisfy *pm* if every prime ideal of R is maximal. Such rings satisfy (*) obviously. However there exist rings which satisfy (*) but do not satisfy pm as in [3, Example 3.3].

We first observe some properties of rings which satisfy (*).

Proposition 1. *For a ring R, the following statements are equivalent:*

(1) *R satisfies (*);*

(2) *Every right primitive ideal of R is maximal.*

Remark. We may obtain the same result for the left version of (*) by replacing "right" by "left" in the preceding proposition.

A ring R is called a *PI-ring* if R satisfies a polynomial identity with coefficients in the ring of integers.

Corollary 2. *Rings whose right primitive factor rings are artinian satisfy (*); especially, PI-rings satisfy (*). Hence if R is a division ring that is finite dimensional over its center, then $R[x]$ satisfies (*).*

This is a part of the joint paper "On rings in which every maximal one-sided ideal contains a maximal ideal, to appear in Comm. Algebra" with Yang Lee

By this corollary the polynomial ring, over the Hamilton quaternions over the real field, satisfies (*).

As an elementary fact, a commutative semiprimitive ring is a subdirect product of fields. In the next corollary, we obtain a similar result for rings which satisfy (*).

Corollary 3. *A semiprimitive ring that satisfies (*) is a subdirect product of simple rings.*

Lemma 4. *If a ring R satisfies (*), then every homomorphic image of R satisfies (*).*

Lemma 5. *[7, Lemma 4] Let R be a ring and $0 \neq e^2 = e \in R$. If I is a maximal ideal of R, then either $eIe = eRe$ or eIe is a maximal ideal of eRe.*

In the preceding lemma, if $ReR = R$, then $eJe \subsetneqq eRe$ for every proper ideal J of R; hence eIe is a maximal ideal of eRe for each maximal ideal I of R.

Lemma 6. *Let R be a ring and $0 \neq e^2 = e \in R$. If I is a maximal right (left) ideal of R, then either $eIe = eRe$ or eIe is a maximal right (left) ideal of eRe.*

Lemma 7. *Let R be a ring and $0 \neq e^2 = e \in R$. If R satisfies (*), then so does eRe.*

Remark. The converse of Lemma 7 does not hold in general. Let R be the infinite row finite matrix ring over a division ring D, and let e be the idempotent in R such that $(1,1)$-entry of e is 1_D and other entries of e are 0_D. Then $eRe(\cong D)$ satisfies (*), but R does not satisfy (*). However if $eIe \subsetneqq eRe$ for each proper ideal I of a ring R and $e^2 = e \in R$ (for example, if $ReR = R$), then the converse is also true as in the following.

Note that if R is a primitive ring, then eRe is also a primitive ring for every nonzero idempotent $e \in R$. Following theorem is one of our main results of this paper.

Theorem 8. *Let R be a ring and $0 \neq e^2 = e \in R$. Suppose that $eIe \subsetneqq eRe$ for each proper ideal I of R. Then the following statements are equivalent:*

(1) *R satisfies (*);*
(2) *eRe satisfies (*).*

We may compare the following result with [12, Proposition 2.1].

Corollary 9. *For a ring R the following statements are equivalent:*

(1) *R satisfies (*);*
(2) *Every n by n upper triangular matrix ring over R satisfies (*);*
(3) *Every n by n lower triangular matrix ring over R satisfies (*),*

where n is any finite or an infinite cardinal number.

Corollary 10. *Suppose that R is a ring and e is a central idempotent of R that is neither 0 nor 1. Then the following statements are equivalent:*

(1) *R satisfies (*);*
(2) *eRe and $(1-e)R(1-e)$ satisfy (*).*

Furthermore we observe some properties of polynomial rings and formal power series rings that satisfy (*).

Proposition 11. *For a ring R, the following statements are equivalent:*

(1) *R satisfies (*);*
(2) *$R[[x]]$ satisfies (*).*
(3) *$R[[x;\theta]]$ satisfies (*) for each endomorphism $\theta : R \to R$.*
(4) *$R[[x;\theta]]$ satisfies (*) for some endomorphism $\theta : R \to R$,*

where $R[[x;\theta]]$ is the skew power series ring over R by θ, subject to $xa = \theta(a)x$ for each $a \in R$.

Similarly we obtain the following result.

Proposition 12. *For a ring R, if $R[x]$ satisfies (*), then R satisfies (*).*

Corollary 13. *For a ring R the following statements are equivalent:*

(1) *$R[x]$ satisfies (*) and R is simple;*
(2) *$R[x]$ satisfies (*) and R is right primitive.*

Based on Proposition 11 and Proposition 12, we may raise the following question.

Question. For a ring R, does $R[x]$ satisfy (*) if R satisfies (*)?

Answer. Negative by the following Example 14.

Example 14. Let $W = W_1[\mathbb{Q}]$ be the first Weyl algebra over the field $\mathbb{Q}$ of rationals, which is subject to $yx = xy + 1$, and let R be the right quotient division ring of W. Then the center of R is $\mathbb{Q}$, and since R is purely transcendental over $\mathbb{Q}$, it follows that $A = R \otimes_{\mathbb{Q}} \mathbb{Q}(t)$ is not a division ring by [4, Theorem 3. 21], where $\mathbb{Q}(t)$ is the quotient field of the polynomial ring $\mathbb{Q}[t]$ in an indeterminate t. Hence $A \neq R(t)$; so $R[t]$ is right primitive by [4, Theorem 3. 21], where $R[t]$ is the polynomial ring over R in t and $R(t)$ is the right quotient division ring of $R[t]$. Clearly R satisfies (*). But the zero ideal of $R[t]$ is right primitive which is not maximal. Therefore $R[t]$ does not satisfy (*) by Proposition 1.

In the following lemma we use the method in the proof of [4, Proposition 3.19].

Lemma 15. *If a ring R is simple, then every ideal of $R[x]$ is generated by a central monic polynomial in $R[x]$.*

Following theorem is also one of our main results in this paper.

Theorem 16. *For a simple ring R, the following statements are equivalent:*

(1) *$R[x]$ satisfies (*);*
(2) *$R[x]$ is not right primitive.*

Remark. We may obtain the same result for the left version of (*) by replacing "right" by "left" in the preceding theorem.

Corollary 17. *Let R be a division ring with its center F. If $R[x]$ satisfies (*), then R is algebraic over F and hence the Jacobson conjecture (i.e., if R is a central algebraic division algebra over F, then is $R[x]$ not right primitive?) holds. Moreover if F is algebraically closed, then $R = F$.*

We do not know whether the condition (*) is left-right symmetric. But if R is a division ring, then $R[x]$ satisfies (*) if and only if $R[x]$ satisfies the left version of (*) as in the following results.

Lemma 18. *Let R be a division ring. Then $R[x]$ is right primitive if and only if $R[x]$ is left primitive.*

Proposition 19. *Let R be a division ring. Then the following statements are equivalent:*

(1) *$R[x]$ satisfies (*);*
(2) *$R[x]$ is not right primitive;*
(3) *$R[x]$ is not left primitive;*
(4) *$R[x]$ satisfies the left version of (*).*

Recall that quasi-duo rings satisfy (*) by [7, Proposition 1]. In the following proposition we have a connection for left-right symmetry between the condition (*) and the quasi-duo condition.

Proposition 20. *If the condition (*) is left-right symmetric for a ring R, then the quasi-duo condition is also left-right symmetric for R.*

Recall that for a chain $R \subseteq S$ of rings, S is called a finite normalizing (centralizing) extension of R if S_R has a finite set of generators each of which normalizes R (centralizes R). Clearly every centralizing extension is a normalizing extension. In Example 14 the ring R satisfies (*), but $R[t]$, which is an infinite centralizing extension of R, does not satisfy (*). So we obtain following results.

Lemma 21. *Let S be a prime ring and let $R \subseteq S$ be a chain of rings. Suppose that S is a finite centralizing extension of R. Then R is simple if and only if S is simple.*

Lemma 22. *Let $R \subseteq S$ be a chain of rings and suppose that S is a finite normalizing extension of R. If S satisfies (*), then R satisfies (*).*

Theorem 23. *Let S be a finite centralizing extension of a ring R. Then the following statements are equivalent:*

(1) *R satisfies (*);*
(2) *S satisfies (*).*

We denote the n by n full matrix ring over a ring R by $Mat_n(R)$ for any positive integer n.

Corollary 24. *For a ring R and any positive integer n, the following statements are equivalent:*

(1) *R satisfies (*);*
(2) *$Mat_n(R)$ satisfies (*).*

Now we prove that the condition (*) is Morita invariant.

Corollary 25. *Suppose that a ring R satisfies (*). Then for every finitely generated projective right R-module P, $End_R(P)$ satisfies (*) too; especially the condition (*) is a Morita invariant property, where $End_R(P)$ is the endomorphism ring of P over R.*

Note that a finite group ring over a ring R is a finite centralizing extension of R, and so we have the following result by Theorem 23.

Corollary 26. *Let R be a ring and S be a finite group ring over R. Then R satisfies (*) if and only if S satisfies (*).*

In the following Remark, we consider another extension by tensor product, and check whether it satisfies (*) when ground rings satisfy (*).

Remark. Regev [10, Theorem 6.1.1] proved that the tensor product of any two PI-rings is a PI-ring. By Corollary 2, PI-rings satisfy (*) and hence we may conjecture that the tensor product of any two rings which satisfy (*) also satisfies (*). But it fails in general to be true by Example 14. Notice that $R[t] \cong R \otimes_{\mathbb{Q}} \mathbb{Q}[t]$ by [11, Corollary 1.7.20]. Both R and $\mathbb{Q}[t]$ satisfy (*), but $R[t]$ does not satisfy (*).

Next we obtain a similar result to Theorem 16.

Proposition 27. *Let R be a field and $R[x;\alpha]$ be the Ore extension of R by an endomorphism α of R, subject to $rx = x\alpha(r)$ for $r \in R$. If α is not onto, then $R[x;\alpha]$ does not satisfy (*).*

Recall that polynomial rings over PI-rings are also PI-rings. Hence we may conjecture that Ore extensions of endomorphism type over PI-rings are also PI-rings. As a byproduct of Proposition 27, we get a negative answer as in the following example.

Example 28 Let $R = \mathbb{Q}(x)$ be the quotient field of the polynomial ring $\mathbb{Q}[x]$ over $\mathbb{Q}$, where $\mathbb{Q}$ is the rationals. Define an endomorphism $\alpha : R \to R$ by $\alpha(f(x)) = f(x^2)$ for $f(x) \in R$. Next consider the Ore extension $S = R[t;\alpha]$ subject to $at = t\alpha(a)$ for $a \in R$.

Then since α is not onto, S does not satisfy (*) by Proposition 26. Assume that S is a PI-ring. Then S satisfies (*) by Corollary 2, which is a contradiction.

Acknowledgement. The author would like to express his gratitude to Professor Masahisa Sato for his giving the opportunity to present this material on the Japan-Korea Ring Theory and Representation Theory Seminar joined with the 31st Japan Ring Theory and Representation Theory Symposium.

References

1. F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, Springer-Verlag New York, Inc. (1992).
2. H. Bass, *Finitistic dimension and a generalization of semiprimary rings*, Trans. Amer. Math. Soc. 95 (1960), 466–488.
3. G. F. Birkenmeier, J. Y. Kim and J. K. Park, *Regularity conditions and the simplicity of prime factor rings*, J. Pure Appl. Algebra 115 (1997), 213–230.
4. J. Cozzens and C. Faith, *Simple Noetherian rings*, Cambridge University Press (1975).
5. N. Jacobson, *Structure of rings*, American Mathematical Society (1956).
6. I. Kaplansky, *Rings with polynomial identity*, Bull. Amer. Math. Soc. 54 (1948), 575–580.
7. Y. Lee and C. Huh, *Some results on quasi-duo rings*, submitted.
8. J. C. McConnell and J. C. Robson, *Noncommutative Noetherian rings*, John Wiley & Sons Ltd. (1987).
9. J. C. Robson and L. W. Small, *Liberal Extensions*, Proc. London Math. Soc. (3) 42 (1981), 87–103.
10. L. H. Rowen, *Polynomial Identities in Ring Theory*, Academic Press, New York (1980).
11. L. H. Rowen, *Ring Theory*, Academic Press, Inc., San Diego (1991).
12. H. P. Yu, *On quasi-duo rings*, Glasgow Math. J. 37 (1995), 21–31.

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Quadratic bimodule problem *

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Abstract

It is shown that a kind of bimodule problem, called quadratic bimodule problem, can be always solved. As applications, we will show that a kind of orders which form infinite sequences are of tame representation type, and a kind of algebras which generalize string algebras and clannish algebras are also of tame representation type.

周知の通り、代数閉体上の有限次多元環は二つの類、即ち“tame 表現型”及び“wild 表現型”に分けられる。これが有名な Drozd の tame-wild Theorem である。

“tame 表現型”とはその加群の圏が一変数多項式環上の加群の圏によって近似可能という事であり、“wild 表現型”とはその加群の圏が非可換二変数多項式環上の加群の圏よりも複雑という事である。正確な定義及びその証明は [CB5] 参照。

この定理は、有限次多元環よりも広い“BOCS”の領域において証明される。領域を拡げたために reduction という操作が可能となった事が証明の成功の一つの要因であり、またその恩恵として種々の問題を統一的に取り扱う事が可能となる。

本文では BOCS の dual を取る事により生じる“bimodule problem”（略して BP）なる対象を取り扱う。

1. 目的（詳しくは [I] を参照）

(I) ある種の bimodule problem が常に解ける事を示す。この種の bimodule problem においては、二次拡大が特徴的に現れる事より“quadratic bimodule problem”と呼ぶ事にする。(5.2, 5.5)

*The detailed version of this paper will be submitted for publication elsewhere.

(II) (I) の応用として、完備離散付値環 R 上のある種の order の直既約格子を全て求めることが可能であることを示す (7.1.3)。この種の order においても、(I) と同様に二次拡大が特徴的に現れる事より "quadratic order" と呼ぶ事にする (2.4)。

(III) (I) の他の応用として、ある種の artin 多元環の直既約加群を全て求めることが可能であることを示す。これは special biserial algebra および clannish algebra として知られているものを完全に含んでいる。(7.2.1)

(IV) (III) の artin 多元環は全て特殊な quadratic order から得られる事を注意する。これは (III) の artin 多元環が常に射影極限として order を持つという興味深い現象を示している。(2.5)

1.1. 注意

(1) 講演においては、(II) に "restricted" なる技術上の仮定を付けていたが、それは全く不要である事がその後の進展により分かった。詳しくは (6) 参照。

(2) 以下断らない限り、完備離散付値環 R 及び体 k にはこれ以上の仮定 (例えば、代数的閉体、分離的、剰余体の有限性 etc.) は課さない。

(3) (II) において R の剰余体が代数的閉体と仮定した場合、並びに (III) において基礎環が代数的閉体と仮定した場合はそれぞれ tame 表現型である事が示される。

(4) 一般に (I) の表現の圏は、skew polynomial ring 及び $\widetilde{A_{12}}$, $\widetilde{A_{11}}$, $\widetilde{B}_2$, $\widetilde{C}_2$, $\widetilde{A}_3$ 型の extended Dynkin diagram に対応する hereditary algebra 上の加群の圏によって近似可能である事が分かる。

(5) 講演でも少し触れたが、代数閉体上でない多元環に対しては "tame 表現型" 及び "wild 表現型" の定義は、dichotomy (即ち Drozd の tame-wild thorem の一般化) が成立しかつその多元環自身の表現の性質を明確に規定するという要請を満たす様な形では、今の所成されていない。しかし (4) の事実はその方向における "tame 表現型" の定義に対し、一つの指針を与えていると思われる。

一方、別方向での定義として generically tame 及び generically wild ([CB3] 参照) があげられる。これらの概念は全く一般の環に対してさえ適用可能な反面、表現の性質との関係がそれほど明確ではない様に思われる。さらに代数閉体上の多元環以外では dichotomy も今だ確立されていない。

(6) 以前、環論メーリングリストで質問した内容に関連する。

(I) の証明の過程で、$\widetilde{A_{12}}$, $\widetilde{A_{11}}$, $\widetilde{B}_2$, $\widetilde{C}_2$, $\widetilde{A}_3$ 型の hereditary algebra 上の加群の圏において、ある性質 (5.3) が成立する事を示す必要が生じる。当初は加群の完全決定によりこの性質を示そうと試みていたが、そもそも基礎体に制

限を付けない限り完全決定はほぼ無理であろうと現在は考えている（この方向の研究は [CB2] 参照）。講演の時点ではその点がギャップとなり "restricted" なる制限を課していたが、その後別方向で上記性質が成立する事の証明に成功した。（6 節参照）

2. Quadratic extension 及び Quadratic order

以下、環 A に対し、A- 加群と言えば左 A- 加群を意味するものとし、$\mathrm{mod}\,A$ で有限生成 A- 加群の成す圏を表わすものとする。また J_A で A の Jacobson radical を表わすものとする。

まず、本文で考察する問題において特徴的に現れてくる "quadratic extension" の定義をする。この定義は簡単すぎて逆に人為的に見えるかも知れないが、例えばその分類が容易に得られる (2.3) 事からも分かるように、非常に良い性質である。

2.1. 定義 K を体とする。有限次 K- 多元環の間の単射 $B \longrightarrow A$ が **quadratic extension** であるとは、A が半単純であり、任意の B の巾等元 e に対して $2\dim_K eBe \geq \dim_K eAe$ が成立する事である。

2.2. 例

(1) $f : B \longrightarrow A$ を quadratic extension 、$\mathcal{E}$ を B の直交巾等元よりなる集合とする。この時 f から導かれる $f' : B' := \bigoplus_{i,j} e_iBe_j \longrightarrow \bigoplus_{i,j} f(e_i)Af(e_j) =: A'$ $(e_i \in \mathcal{E})$ も quadratic extension となる。この形の f' を、f から得られる **quadratic extension** と呼ぶ事にする。

(2) A が半単純で $1 = \sum_{i=1}^{n} e_i$ を直交巾等元への分解とする。この時 $B := \bigoplus_{1 \leq i \leq j \leq n} e_iAe_j \longrightarrow A$ は quadratic extension である。この形の quadratic extension を **A-extension** と呼ぶことにする。

(3) 以下の形の quadratic extension をそれぞれ **type (I),..., type (V)** と呼ぶことにする。

(I) $B = A$ は斜体。

(II) 斜体の間の二次拡大 $D \subset B$ が存在して、$B \longrightarrow M_2(D) =: A$ は正則表現。

(III) B は斜体であり $B \longrightarrow B \prod B =: A$ は対角埋め込み。

(IV) $B \longrightarrow A$ は斜体の間の二次拡大。

(V) 斜体 D が存在して $B := D \prod D \longrightarrow M_2(D) =: A$ は $(x, y) \mapsto \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$。

2.2.1. 注意 Bass order が quadratic extension から「生じる」事はよく

知られている。(3) における type (I),..., type (V) なる名称は [HN] における Bass order の分類に若干の変更を施したものである。正確には、ここでの type (I) は Bass order には現れず、type (II) は (I) 型及び (II) 型 Bass order に対応する。その他はそのまま。

2.3. 命題（構造定理） $f: B \longrightarrow A$ を有限次 K- 多元環の間の単射、A を半単純とする時、f が quadratic extension である必要十分条件は、ある多元環 C で以下の条件を満たすものが存在する事。

(1) $J_C = J_B \subset B \subseteq C \subseteq A$。

(2) $C \longrightarrow A$ は $\mathbb{A}$-extension。

(3) $B/J_B \longrightarrow C/J_C$ は type (I) から (V) の形の写像の直積から得られる quadratic extension。

2.4. 定義 （詳しくは [CR] 参照）

R を完備離散付値環、K を R の商体、k を R の剰余体とする。

(1) R- 多元環 Λ が R-order であるとは、R- 加群として有限生成射影加群である事。この時 Λ- 加群 L が Λ- 格子 であるとは、R- 加群として有限生成射影加群である事。$\mathrm{lat}\,\Lambda$ で Λ- 格子の成す圏を表わす事にする。

以下 R-order Λ、K- 多元環 A および R- 多元環の間の単射 $\Lambda \longrightarrow A$ が与えられているとする。

(2) Λ が A の full order であるとは、$K \otimes_R \Lambda = A$ が成立する事。

(3) $\mathrm{lat}(\Lambda, A)$ で順極限 $\varinjlim_{\Delta} \mathrm{lat}\,\Delta$ を表わすものとする。ここで Δ は Λ を含む A の full order を動くものとする。例えばもし Λ 自身が A の full order ならば、$\mathrm{lat}(\Lambda, A)$ は $\mathrm{lat}\,\Lambda$ に一致する。

(4) Λ が A の quadratic order であるとは、quadratic extension $B \longrightarrow A$ 及び B の hereditary full order Δ で $J_\Delta = J_\Lambda \subset \Lambda \subseteq \Delta$ なるものが存在して $\Lambda/J_\Lambda \longrightarrow \Delta/J_\Delta$ が quadratic extension となる事。

(5) (4) において $B = A$ が成立する時、Λ を A の quadratic Bäckström order と呼ぶ事にする。

2.5. k 上の special biserial algebra 及びより一般に clannish algebra は quadratic Bäckström order から得られる事が分かる。special biserial algebra の定義は [WW]、clannish algebra の定義は [CB4] 参照。

命題 (1) T を k 上の clannish algebra とする。この時、$k[[x]]$ 上の quadratic Bäckström order Λ およびその両側 ideal I が存在して、$T = \Lambda/I$。

(2) 逆に Λ を $k[[x]]$ 上の quadratic Bäckström order、Δ を hereditary

order で $J_\Delta = J_\Lambda \subset \Lambda \subsetneq \Delta$ なるものとすると、任意の Λ の両側 ideal I で $\dim_k \Lambda/I < \infty$ なるものに対し以下が成立する。

(i) $\Lambda/J_\Lambda \longrightarrow \Delta/J_\Delta$ が type (I)(III) の形の写像の直積から得られるならば Λ/I は k 上の special biserial algebra。

(ii) $\Lambda/J_\Lambda \longrightarrow \Delta/J_\Delta$ が type (I)(III)(V) の形の写像の直積から得られるならば Λ/I は k 上の clannish algebra。

証明は sink を持たない $\tilde{A}_n$ 型の quiver から定義される無限次元 k- 多元環が $k[x]$ 上の hereditary order になる事より容易に得られる。

以下の章で、 quadratic order Λ に対する $\mathrm{lat}(\Lambda, A)$ 及び quadratic Bäckström order Λ に対する $\mathrm{mod}\,\Lambda$ を決定する事が可能となる（7.1.3, 7.2.1）。特に clannish algebra T に対する $\mathrm{mod}\,T$ も決定される。

3. bimodule problem

bimodule problem は BOCS 同様、表現論における種々の問題を統一的に記述し取り扱う事を目的に導入された概念である。多元環の表現をこれらの言葉に翻訳する際の距離において、 bimodule problem は BOCS よりも優れている。

以下、「圏」は常に skeletally small な加法圏、「関手」は加法関手を意味するものとする。圏 $\mathcal{C}$ に対し $\mathcal{O}b(\mathcal{C})$ で $\mathcal{C}$ の対象の全体、$\mathcal{C}(X,Y)$ で X から Y への射の全体を表わし、$f \in \mathcal{C}(X,Y)$ と $g \in \mathcal{C}(Y,Z)$ の合成を fg で表わす。また $\mathcal{J}_\mathcal{C}$ で $\mathcal{C}$ の Jacobson radical を表わすことにする。

まず、 bimodule problem の定義をする。

3.1. 定義 [CB] $\mathcal{C}, \mathcal{D}$ 等を圏とする。

(1) $\mathcal{C}$- 加群 (resp. $(\mathcal{C},\mathcal{D})$- 加群) とは反変関手 $F : \mathcal{C} \longrightarrow \mathcal{A}b$ (resp. $F : \mathcal{C} \amalg \mathcal{D}^{op} \longrightarrow \mathcal{A}b$) の事を意味する。$\mathcal{C}$- 加群 (resp. $(\mathcal{C},\mathcal{D})$- 加群) 全体は、関手間の自然変換を射と定める事により Abel 圏を成す。

(2) **bimodule problem** （略して **BP**）とは、対 $(\mathcal{C}, M)$ で、$\mathcal{C}$ は圏、M は $(\mathcal{C},\mathcal{C})$- 加群 であるようなものとする。

(3) BP $(\mathcal{C}, M)$ に対し、圏 $\mathrm{Mat}(\mathcal{C}, M)$ を次のように定義する。 $\mathrm{Mat}(\mathcal{C}, M)$ の対象は、対 Xm で $X \in \mathcal{O}b(\mathcal{C})$ と $m \in M(X,X)$ より成るものとする。射は $\mathrm{Mat}(\mathcal{C}, M)(Xm, X'm') = \{f \in \mathcal{C}(X,X') \mid mf = fm'\}$ とし、射の合成は $\mathcal{C}$ における合成をそのまま用いて定義する。

(4) 忘却関手 $\mathrm{Mat}(\mathcal{C}, M) \longrightarrow \mathcal{C}$ が自然に定義される。これによる $\mathcal{J}_\mathcal{C}$ の引き戻しを $(\mathcal{J}_\mathcal{C}, M)$ とおくとこれは $\mathrm{Mat}(\mathcal{C}, M)$ の ideal となり、それによる商

圏を $\overline{\mathrm{Mat}}(\mathcal{C},M) := \mathrm{Mat}(\mathcal{C},M)/(\mathcal{J}_{\mathcal{C}},M)$ と表わす事にする。

(5) $X \in \mathcal{O}b(\mathcal{C})$ に $X0 \in \mathcal{O}b(\mathrm{Mat}(\mathcal{C},M))$ を対応させる事により、$\mathcal{C}$ を $\mathrm{Mat}(\mathcal{C},M)$ の充満部分圏と見なす事ができる。

3.2. 注意

(1) 本来 BP は $(\mathcal{C},M)$ の他に derivation $d:\mathcal{C} \longrightarrow M$ を加えた $(\mathcal{C},M,d)$ として定義される。上の定義は $d=0$ とした時のものに一致する。

(2) $d \neq 0$ の BP を考察する事による最大の恩恵は（単なる拡張という事では無く） reduction という操作を施す事が可能となる点にある。即ち BP は 0 でない derivation を許す事により、はじめて reduction において閉じるのである。それは Drozd の tame-wild Theorem が多元環の枠組みを拡げた BOCS の枠組みにおいて証明された事に類似する。

(3) にも関わらず以下では $d=0$ の場合のみ扱う。これは我々が取り扱う QBP （5.2 で定義）においては、$d=0$ のまま reduction を施す事が可能であるためである。即ち（$d=0$ の） QBP は reduction により閉じているのである。ただしそのために通常のものとは異なる別の reduction を定義する必要がある（4.4）。

3.3. 用語

(1) 圏 $\mathcal{C}$ が Krull-Schmidt であるとは、任意の対象が、自己準同型環が局所環であるような対象の有限個の直和に同型である事を意味する。

(2) $\mathrm{ind}(\mathcal{C})$ で $\mathcal{C}$ の直既約対象の同型類の全体を表わす事にする。以下便宜的に、$X \in \mathcal{O}b(\mathcal{C})$ に対し $X \in \mathrm{ind}(\mathcal{C})$ で X が直既約である事を表わす。

(3) 関手 $\mathbb{F}:\mathcal{C} \longrightarrow \mathcal{D}$ が稠密であるとは、任意の $Y \in \mathcal{O}b(\mathcal{D})$ に対し $\mathbb{F}(X)$ が Y と同型である様な $X \in \mathcal{O}b(\mathcal{C})$ が存在する事を意味する。

(4) 関手 $\mathbb{F}:\mathcal{C} \longrightarrow \mathcal{D}$ が表現同値であるとは、稠密かつ充満であり、任意の $0 \neq X \in \mathcal{O}b(\mathcal{C})$ は $\mathbb{F}(X) \neq 0$ を満たす事を意味するものとする。（同値関係でないのであまり良い用語ではない）

(5) 加群が局所であるとは、唯一つの極大部分加群を持つ事。加群が単列であるとは、唯一つの組成列を持つ事。

3.4. $\mathrm{Mat}(\mathcal{C},M)$ の対象の同型類を完全に決定する事を BP を解くと呼ぶことにすると、表現論の多くの問題は BP を解く事に帰着する事が出来る。以下の例は最も基本的な帰着方法を示す。3.4.1 により Noether 環 Λ 上の $\mathrm{mod}\,\Lambda$ を求める問題が BP を解く事に帰着される。一方 3.4.2 によりある種の order Δ' 上の $\mathrm{lat}(\Delta',A)$ を求める問題が BP を解く事に帰着される。

3.4.1. 例 [CB] Λ を Noether 環とし $\mathcal{P}$ で有限生成射影 Λ- 加群の成す圏を表す。すると $\mathcal{J}_{\mathcal{P}}$ は自然に $(\mathcal{P},\mathcal{P})$- 加群 と見なされる。$\mathrm{Mat}(\mathcal{P}\prod\mathcal{P},\mathcal{J}_{\mathcal{P}})$ から $\mathrm{mod}\,\Lambda$ への稠密充満関手 $\mathbb{F}$ が次の様に定義される。

$(P\prod Q)f\in\mathcal{O}b(\mathrm{Mat}(\mathcal{P}\prod\mathcal{P},\mathcal{J}_{\mathcal{P}}))$ に対し、$f\in\mathrm{Hom}_{\Lambda}(P,Q)$ であるので $\mathbb{F}((P\prod Q)f):=\mathrm{Cok}(f)\in\mathcal{O}b(\mathrm{mod}\,\Lambda)$ とおく。射についても同様。

3.4.2. R を完備離散付値環、K を R の商体、$A\supseteq B$ を有限次 K- 多元環、$\Delta\supseteq\Lambda$ をともに B の full R-order で $I:=J_{\Lambda}$ が Δ の両側 ideal である様なものとする。

$\overline{\Delta}:=\Delta/I,\overline{\Lambda}:=\Lambda/I$ とおき、$\mathcal{I}$ を $\mathcal{I}(L,L'):=\{f\in\mathrm{lat}(\Lambda,A)(L,L')\mid(L)f\subseteq IL'\}$, で定義される $\mathrm{lat}(\Lambda,A)$ の ideal とする。

関手 $\mathrm{mod}\,\overline{\Lambda}\longrightarrow\mathrm{mod}\,\overline{\Delta}$, $X\mapsto\overline{\Delta}\otimes_{\overline{\Lambda}}X$ 及び $\mathrm{lat}(\Delta,A)\longrightarrow\mathrm{mod}\,\overline{\Delta}$, $Y\mapsto Y/IY$ の像をそれぞれ $\mathcal{K},\mathcal{L}$ とおき、$\mathrm{mod}\,\overline{\Delta}$ を $(\mathcal{K},\mathcal{L})$- 加群と見なしたものを N とする。

命題 $\mathrm{Mat}_0(\mathcal{K}\prod\mathcal{L},N)$ を $f\in\mathrm{Hom}_{\overline{\Delta}}(\overline{\Delta}\otimes_{\overline{\Lambda}}X,Y/IY)$ が同型となるような対象 $(X\prod Y)f$ よりなる $\mathrm{Mat}(\mathcal{K}\prod\mathcal{L},N)$ の充満部分圏とする時、圏の同値 $\mathbb{F}:\mathrm{lat}(\Lambda,A)/\mathcal{I}\longrightarrow\mathrm{Mat}_0(\mathcal{K}\prod\mathcal{L},N)$ が次の様に定義される。

対象 $L\in\mathcal{O}b(\mathrm{lat}(\Lambda,A))$ に対し $f_L:\overline{\Delta}\otimes_{\overline{\Lambda}}(L/IL)\longrightarrow\Delta L/IL$ を自然な同一視として $\mathbb{F}L:=(L/IL\prod\Delta L)f_L$ とおく。射についても同様に定める。

4. 可約な bimodule problem 及びその Reduction Lemma

一般の BP に対して [CB] は Reduction Lemma と呼ばれる計算方法を提案した。それ自体は殆んど自明なものであり、あらゆる BP に適用可能な反面精度が悪い。ここで述べる Reduction Lemma (4.4) は、4.1 で定義する特殊な BP に対してしか適用できない代わりに、精密な取り扱いが可能となっている。

4.1. 定義 $(\mathcal{C},\mathcal{C}_l,\mathcal{C}_r,M,M')$ が可約な BP であるとは、以下の条件が成立すること。

(1) $\mathcal{C}$, $\mathcal{C}_l$, $\mathcal{C}_r$ は Krull-Schmidt 圏、$M\supset M'$ はともに $(\mathcal{C}_l,\mathcal{C}_r)$- 加群。$\overline{\mathcal{C}_l}:=\mathcal{C}_l\,/\,\mathcal{J}_{\mathcal{C}_l}$, $\overline{\mathcal{C}_r}:=\mathcal{C}_r\,/\,\mathcal{J}_{\mathcal{C}_r}$, $\overline{M}:=M/M'$ とおく。

(2) $\mathcal{J}_{\mathcal{C}_l}M+M\,\mathcal{J}_{\mathcal{C}_r}\subseteq M'$。ゆえに $\overline{M}$ は $(\overline{\mathcal{C}_l},\overline{\mathcal{C}_r})$- 加群 になる。

(3) $X\in\mathrm{ind}(\mathcal{C}_l)$ が $\overline{M}(X,\ )\neq 0$ を満たせば、$M(X,\ )$ は局所 $\mathcal{C}_r^{op}$- 加群。

(4) $X\in\mathrm{ind}(\mathcal{C}_r)$ が $\overline{M}(\ ,X)\neq 0$ を満たせば、$M(\ ,X)$ は局所 $\mathcal{C}_l$- 加群。

(5) $\mathbb{P}_l\prod\mathbb{P}_r:\mathcal{C}\longrightarrow\mathcal{C}_l\prod\mathcal{C}_r$ は忠実関手、$\mathcal{C}\supset\mathcal{J}_{\mathcal{C}_l\prod\mathcal{C}_r}$ (i.e. 任意の $X\in\mathcal{O}b(\mathcal{C})$ に対し $\mathcal{C}(X,X)\supset\mathcal{J}_{\mathcal{C}_l}(\mathbb{P}_lX,\mathbb{P}_lX)\prod\mathcal{J}_{\mathcal{C}_r}(\mathbb{P}_rX,\mathbb{P}_rX)$) 。$\overline{\mathcal{C}}:=\mathcal{C}\,/\,\mathcal{J}_{\mathcal{C}_l\prod\mathcal{C}_r}$

とおく。

以下この節では可約な BP $(\mathcal{C},\mathcal{C}_l,\mathcal{C}_r,M,M')$ を固定する。

4.2. 補題 任意の $m\in M(X,Y)$ $(X\in Ob(\mathcal{C}_l), Y\in Ob(\mathcal{C}_r))$ に対し $W\in Ob(\mathcal{C}_l)$, $k\in\mathcal{C}_l(W,X)$, $p\in\mathcal{C}_l(X,W)$, $Z\in Ob(\mathcal{C}_r)$, $c\in\mathcal{C}_r(Y,Z)$, $i\in\mathcal{C}_r(Z,Y)$ で (1)(2)(3) を満たすものが存在する。

(1) 次は完全。(この時便宜的に $0\longrightarrow W\xrightarrow{\bar{k}}X\xrightarrow{\bar{m}}Y\xrightarrow{\bar{c}}Z\longrightarrow 0$ が完全であると呼ぶ事にする。)

$$0\longrightarrow\overline{\mathcal{C}_l}(\ ,W)\xrightarrow{\cdot\bar{k}}\overline{\mathcal{C}_l}(\ ,X)\xrightarrow{\cdot\bar{m}}\overline{M}(\ ,Y)\xrightarrow{\cdot\bar{c}}\overline{M}(\ ,Z)\longrightarrow 0$$
$$0\longrightarrow\overline{\mathcal{C}_r}(Z,\ )\xrightarrow{\bar{c}\cdot}\overline{\mathcal{C}_r}(Y,\ )\xrightarrow{\bar{m}\cdot}\overline{M}(X,\ )\xrightarrow{\bar{k}\cdot}\overline{M}(W,\ )\longrightarrow 0$$

(2) $kp=1_W$, $ic=1_Z$.
(3) $mci=pkm$.
この時任意に $W_1\in Ob(\mathcal{C}_l)$, $p_1\in\mathcal{C}_l(W_1,X)$, $k_1\in\mathcal{C}_l(X,W_1)$, $Z_1\in Ob(\mathcal{C}_r)$, $i_1\in\mathcal{C}_r(Z_1,Y)$, $c_1\in\mathcal{C}_r(Y,Z_1)$ で $k_1p_1=1_{W_1}$, $i_1c_1=1_{Z_1}$, $1_X=pk+p_1k_1$ 及び $1_Y=ci+c_1i_1$ を満たすようなものを取る. すると (4)(5) が成立する。

$$\begin{array}{ccccc}
W \underset{k}{\overset{p}{\rightleftarrows}} & X & \xrightarrow{m} & Y & \underset{c}{\overset{i}{\leftrightarrows}} Z \\
 & \downarrow^{p_1}\uparrow^{k_1} & & \downarrow^{c_1}\uparrow^{i_1} & \\
 & W_1 & & Z_1 &
\end{array}$$

(4) $(\cdot k_1m)$ が導く $\mathcal{C}_l(\ ,W_1)\longrightarrow M(\ ,Y)c_1i_1$ 及び $\mathcal{J}_{\mathcal{C}_l}(\ ,W_1)\longrightarrow M'(\ ,Y)c_1i_1$ は全射。
(5) $(mc_1\cdot)$ が導く $\mathcal{C}_r(Z_1,\ )\longrightarrow p_1k_1M(X,\ )$ 及び $\mathcal{J}_{\mathcal{C}_r}(Z_1,\ )\longrightarrow p_1k_1M'(X,\ )$ は全射。

4.3. 定義 (1) 関手 $\{\ \}:=\langle\ \rangle\amalg[\ ]:\mathrm{Mat}(\overline{\mathcal{C}},\overline{M})\longrightarrow\overline{\mathcal{C}_l}\amalg\overline{\mathcal{C}_r}$ を以下のように定義する。
対象 $X\overline{m}\in\mathrm{Mat}(\overline{\mathcal{C}},\overline{M})$ に対して、次の列が 4.2 の意味で完全になるように $\langle X\overline{m}\rangle\in Ob(\mathcal{C}_l)$, $[X\overline{m}]\in Ob(\mathcal{C}_r)$ を定義する。

$$0\longrightarrow\langle X\overline{m}\rangle\xrightarrow{\bar{k}}\mathbb{P}_lX\xrightarrow{\overline{m}}\mathbb{P}_rX\xrightarrow{\bar{c}}[X\overline{m}]\longrightarrow 0$$

射 $\overline{f} \in \mathrm{Mat}(\overline{\mathcal{C}}, \overline{M})(X\overline{m}, X'\overline{m'})$ に対して、5.2 (2) より $\langle f\rangle \in \overline{\mathcal{C}_l}(\langle X\overline{m}\rangle, \langle X'\overline{m'}\rangle)$ 及び $[f] \in \overline{\mathcal{C}_r}([X\overline{m}], [X'\overline{m'}])$ を以下が可換になるようなものとして定義する事ができる。

$$\begin{array}{ccccccccc} 0 \longrightarrow & \langle X\overline{m}\rangle & \xrightarrow{\overline{k}} & \mathbb{P}_l X & \xrightarrow{\overline{m}} & \mathbb{P}_r X & \xrightarrow{\overline{c}} & [X\overline{m}] & \longrightarrow 0 \\ & \downarrow{\scriptstyle \langle\overline{f}\rangle} & & \downarrow{\scriptstyle \mathbb{P}_l\overline{f}} & & \downarrow{\scriptstyle \mathbb{P}_r\overline{f}} & & \downarrow{\scriptstyle [\overline{f}]} & \\ 0 \longrightarrow & \langle X'\overline{m'}\rangle & \xrightarrow{\overline{k'}} & \mathbb{P}_l X' & \xrightarrow{\overline{m'}} & \mathbb{P}_r X' & \xrightarrow{\overline{c'}} & [X'\overline{m'}] & \longrightarrow 0 \end{array}$$

$\langle\ \rangle$, $[\ ]$ 及び $\{\ \}$ が well defined である事は容易に分かる。

$\{A\} = 0$ なる対象 A より成る $\mathrm{Mat}(\overline{\mathcal{C}}, \overline{M})$ の充満部分圏を band 圏、関手 $\{\ \}$ の像を string 圏と呼ぶ事にする。

(2) $\mathrm{Mat}(\mathcal{C}, M, M')$ を以下の様に定義する。

$$\begin{aligned} \mathcal{O}b(\mathrm{Mat}(\mathcal{C}, M, M')) &:= \mathcal{O}b(\mathrm{Mat}(\mathcal{C}, M)) \\ \mathrm{Mat}(\mathcal{C}, M, M')(Xm, X'm') &:= \{f \in \mathcal{C}(X, X') \mid mf - fm' \in M'\} \end{aligned}$$

$Xm \mapsto X\overline{m}$ より関手 $\mathrm{Mat}(\mathcal{C}, M, M') \longrightarrow \mathrm{Mat}(\overline{\mathcal{C}}, \overline{M})$ を得る。これと $\{\ \} = \langle\ \rangle \amalg [\ ]$ を合成する事により、$\{\ \} = \langle\ \rangle \amalg [\ ] : \mathrm{Mat}(\mathcal{C}, M, M') \longrightarrow \overline{\mathcal{C}_l} \amalg \overline{\mathcal{C}_r}$ を得る。

(3) 圏 $\mathcal{C}'$ を以下の様に定義する。

$$\begin{aligned} \mathcal{O}b(\mathcal{C}') &:= \mathcal{O}b(\mathrm{Mat}(\mathcal{C}, M, M')) \\ \mathcal{C}'(Xm, X'm') &:= \{f \textstyle\coprod g \in \mathcal{C}_l(\langle Xm\rangle, \langle X'm'\rangle) \coprod \mathcal{C}_r([Xm], [X'm']) \mid \\ &\qquad \overline{f} \textstyle\coprod \overline{g} = \{h\} \text{ なる } h \in \mathrm{Mat}(\mathcal{C}, M, M')(Xm, X'm') \text{ が存在}\} \end{aligned}$$

関手 $\mathbb{P}_l \amalg \mathbb{P}_r : \mathcal{C}' \longrightarrow \mathcal{C}_l \amalg \mathcal{C}_r$ が次の様に定義される。対象 $Xm \in \mathcal{O}b(\mathcal{C}') = \mathcal{O}b(\mathrm{Mat}(\mathcal{C}, M, M'))$ に対して $\mathbb{P}_l \amalg \mathbb{P}_r(Xm) := \langle Xm\rangle \amalg [Xm]$。射 $f \amalg g \in \mathcal{C}'(Xm, X'm') \subseteq \mathcal{C}_l(\langle Xm\rangle, \langle X'm'\rangle) \amalg \mathcal{C}_r([Xm], [X'm'])$ に対して $\mathbb{P}_l \amalg \mathbb{P}_r(f \amalg g) := f \amalg g$。

$\mathbb{P}_l \amalg \mathbb{P}_r$ により M 及び M' は $\mathcal{C}'$-加群とみなされる。

$$M(Xm, X'm') := M(\mathbb{P}_l(Xm), \mathbb{P}_r(X'm')) = M(\langle Xm\rangle, [X'm'])$$

4.4. 定理 (Reduction Lemma) $(\mathcal{C}, \mathcal{C}_l, \mathcal{C}_r, M, M')$ を可約な BP とする。この時、稠密充満関手 $\mathbb{F} : \overline{\mathrm{Mat}}(\mathcal{C}, M) \longrightarrow \overline{\mathrm{Mat}}(\mathcal{C}', M')$ が存在して次が成立する。

(1) $\mathbb{F}(A) = 0$ なる対象 A より成る $\overline{\mathrm{Mat}}(\mathcal{C}, M)$ の充満部分圏は、$\mathrm{Mat}(\overline{\mathcal{C}}, \overline{M})$ の band 圏と同値。

(2) $\{\ \}$ は $\mathrm{Mat}(\overline{\mathcal{C}},\overline{M})$ の string 圏と $\mathcal{C}'/\mathcal{J}_{\mathcal{C}_l\coprod\mathcal{C}_r}$ の同値を与える。

$\mathbb{F}$ を還元関手と呼ぶ事にする。$\mathbb{F}$ により $\overline{\mathrm{Mat}}(\mathcal{C},M)$ から、$\mathrm{Mat}(\overline{\mathcal{C}},\overline{M})$ の band 圏に同値な部分が落ちた訳である。

以下 $\mathbb{F}$ の構成をする。そのためには $\mathrm{Mat}(\mathcal{C},M)$ 及び $\mathrm{Mat}(\mathcal{C}',M')$ に若干の構造を付け加えた $\widehat{\mathrm{Mat}}(\mathcal{C},M)$ 及び $\widetilde{\mathrm{Mat}}(\mathcal{C}',M')$ を用いるのが分かりやすい。

4.5. 定義 圏 $\widehat{\mathrm{Mat}}(\mathcal{C},M)$ を以下の様に定義する。

対象は組 $(Xm,k,p,c,i)\in\mathcal{O}b(\mathrm{Mat}(\mathcal{C},M))\times\mathcal{C}_l(\langle Xm\rangle,\mathbb{P}_lX)\times\mathcal{C}_l(\mathbb{P}_lX,\langle Xm\rangle)\times\mathcal{C}_r(\mathbb{P}_rX,[Xm])\times\mathcal{C}_r([Xm],\mathbb{P}_rX)$ で以下の条件を満たすもの。$kp=1_{\langle Xm\rangle}$, $ic=1_{[Xm]}$, $pkm=mci$ 及び次が 4.2 の意味で完全。

$$0\longrightarrow\langle Xm\rangle\xrightarrow{\overline{k}}\mathbb{P}_lX\xrightarrow{\overline{m}}\mathbb{P}_rX\xrightarrow{\overline{c}}[Xm]\longrightarrow 0$$

射は $\widehat{\mathrm{Mat}}(\mathcal{C},M)((Xm,k,p,c,i),(X'm',k',p',c',i')):=\mathrm{Mat}(\mathcal{C},M)(Xm,X'm')$ で定義する。4.2 より $\widehat{\mathrm{Mat}}(\mathcal{C},M)$ は $\mathrm{Mat}(\mathcal{C},M)$ と同値である。

4.6. 定義 圏 $\widetilde{\mathrm{Mat}}(\mathcal{C}',M')$ を以下の様に定義する。

$\mathrm{Mat}(\mathcal{C}',M')$ の対象は $(Ym)n$ $(Ym\in\mathcal{O}b(\mathcal{C}')=\mathcal{O}b(\mathrm{Mat}(\mathcal{C},M,M'))$, $n\in M'(Ym,Ym)=M'(\langle Ym\rangle,[Ym]))$ の形をしている事に注意。$\widetilde{\mathrm{Mat}}(\mathcal{C}',M')$ の対象は組 $((Ym)n,k,p,c,i)\in\mathcal{O}b(\mathrm{Mat}(\mathcal{C},M,M'))\times\mathcal{C}_l(\langle Ym\rangle,\mathbb{P}_lY)\times\mathcal{C}_l(\mathbb{P}_lY,\langle Ym\rangle)\times\mathcal{C}_r(\mathbb{P}_rY,[Ym])\times\mathcal{C}_r([Ym],\mathbb{P}_rY)$ で以下の条件を満たすもの。$kp=1_{\langle Ym\rangle}$, $ic=1_{[Ym]}$, $pkm=mci$ 及び次が 4.2 の意味で完全。

$$0\longrightarrow\langle Ym\rangle\xrightarrow{\overline{k}}\mathbb{P}_lY\xrightarrow{\overline{m}}\mathbb{P}_rY\xrightarrow{\overline{c}}[Ym]\longrightarrow 0$$

射は $\widetilde{\mathrm{Mat}}(\mathcal{C}',M')(((Ym)n,k,p,c,i),((Y'm')n',k',p',c',i')):=\mathrm{Mat}(\mathcal{C}',M')((Ym)n,(Y'm')n')$ とおく。4.2 より $\widetilde{\mathrm{Mat}}(\mathcal{C}',M')$ は $\mathrm{Mat}(\mathcal{C}',M')$ に同値である。

4.7. 定義 $\mathcal{K}$ を関手 $\widehat{\mathrm{Mat}}(\mathcal{C},M)\longrightarrow\mathrm{Mat}(\overline{\mathcal{C}},\overline{M})\xrightarrow{\{\ \}}\overline{\mathcal{C}_l}\coprod\overline{\mathcal{C}_r}$ の核とし、$\mathcal{Q}$ を関手 $\widetilde{\mathrm{Mat}}(\mathcal{C}',M')\longrightarrow\mathcal{C}'\longrightarrow\mathcal{C}'/\mathcal{J}_{\mathcal{C}_l\coprod\mathcal{C}_r}$ の核とする。この時、関手 $\mathbb{F}:\widehat{\mathrm{Mat}}(\mathcal{C},M)/\mathcal{K}\longrightarrow\widetilde{\mathrm{Mat}}(\mathcal{C}',M')/\mathcal{Q}$ を以下の様に定義する。

対象 $\mathbf{X}=(Xm,k,p,c,i)\in\mathcal{O}b(\widehat{\mathrm{Mat}}(\mathcal{C},M))$ に対し $\overline{kmc}=0$ なので $kmc\in M'(\langle Xm\rangle,[Xm])=M'(Xm,Xm)$ である。一方 $Xm\in\mathcal{O}b(\mathcal{C}')$ であるので $(Xm)(kmc)\in\mathcal{O}b(\mathrm{Mat}(\mathcal{C}',M'))$ である。$\mathbb{F}(\mathbf{X}):=((Xm)(kmc),k,p,c,i)$ とおく。

射 $f \in \widehat{\mathrm{Mat}}(\mathcal{C},M)(\mathbf{X},\mathbf{X}')$ $(\mathbf{X}' = (X'm',k',p',c',i'))$ に対し次の図式を考える。

$$
\begin{array}{ccccccccc}
0 \longrightarrow & \langle Xm\rangle & \underset{\overline{k}}{\overset{\overline{p}}{\rightleftarrows}} & \mathbb{P}_l X & \xrightarrow{\overline{m}} & \mathbb{P}_r X & \underset{\overline{c}}{\overset{\overline{i}}{\rightleftarrows}} & [Xm] & \longrightarrow 0 \\
& \downarrow{\scriptstyle \langle f\rangle} & & \downarrow{\scriptstyle \overline{\mathbb{P}_l f}} & & \downarrow{\scriptstyle \overline{\mathbb{P}_r f}} & & \downarrow{\scriptstyle [f]} & \\
0 \longrightarrow & \langle X'm'\rangle & \underset{\overline{k'}}{\overset{\overline{p'}}{\rightleftarrows}} & \mathbb{P}_l X' & \xrightarrow{\overline{m'}} & \mathbb{P}_r X' & \underset{\overline{c'}}{\overset{\overline{i'}}{\rightleftarrows}} & [X'm'] & \longrightarrow 0
\end{array}
$$

この時 $\mathbb{F}f := k(\mathbb{P}_l f)p' \prod i(\mathbb{P}_r f)c'$ とおく。すると上図の可換性より $\overline{k(\mathbb{P}_l f)p'} = \langle f\rangle$ 及び $\overline{i(\mathbb{P}_r f)c'} = [f]$ であるので $\mathcal{C}'$ の定義より $\mathbb{F}f \in \mathcal{C}'(Xm, X'm')$ である。一方、$kmci = kpkm = km$ 及び $p'k'm'c' = m'c'i'c' = m'c'$ であるので次式より $\mathbb{F}f$ が $\mathrm{Mat}(\mathcal{C}',M')((Xm)(kmc),(X'm')(k'm'c'))$ の射を与える事が分かる。

$$(kmc)(i(\mathbb{P}_r f)c') = km(\mathbb{P}_r f)c' = k(\mathbb{P}_l f)m'c' = (k(\mathbb{P}_l f)p')(k'm'c')$$

(1) $\overline{\mathbb{F}f} = \{f\}$ が成立。
(2) $\mathbb{F}(\mathcal{K}) \subseteq \mathcal{Q}$ が成立。
(3) 任意の $f, f' \in \widehat{\mathrm{Mat}}(\mathcal{C},M)(\mathbf{X},\mathbf{X}')$ に対し $\mathbb{F}(f)+\mathbb{F}(f') = \mathbb{F}(f+f')$ が成立。
(4) 任意の $f \in \widehat{\mathrm{Mat}}(\mathcal{C},M)(\mathbf{X},\mathbf{X}')$ 及び $f' \in \widehat{\mathrm{Mat}}(\mathcal{C},M)(\mathbf{X}',\mathbf{X}'')$ に対し、$\mathbb{F}(f)\mathbb{F}(f') - \mathbb{F}(ff') \in \mathcal{Q}$ が成立。
(5) $\mathbb{F}$ は忠実関手 $\mathbb{F} : \widehat{\mathrm{Mat}}(\mathcal{C},M)/\mathcal{K} \longrightarrow \widetilde{\mathrm{Mat}}(\mathcal{C}',M')/\mathcal{Q}$ を導く。

証明 (1) $\overline{\mathbb{F}f} = \overline{k(\mathbb{P}_l f)p'} \prod \overline{i(\mathbb{P}_r f)c'} = \langle f\rangle \prod [f] = \{f\}$。
(2) (1) よりすぐ。(3) 定義より。
(4) $\overline{\mathbb{F}(f)\mathbb{F}(f')} - \overline{\mathbb{F}(ff')} = \{f\}\{f'\} - \{ff'\} = 0$。
(5) $\mathbb{F}$ が忠実であることのみ示せば良い。$\mathbb{F}(f) \in \mathcal{Q}$ ならば、(1) より $\{f\} = 0$。ゆえに $f \in \mathcal{K}$。∎

4.8. 定義 写像 $\mathbb{G} : Ob(\widetilde{\mathrm{Mat}}(\mathcal{C}',M')) \longrightarrow Ob(\widehat{\mathrm{Mat}}(\mathcal{C},M))$ 及び任意の $\mathbf{Y},\mathbf{Y}' \in Ob(\widetilde{\mathrm{Mat}}(\mathcal{C}',M'))$ に対し写像 $\mathbb{G} : \widetilde{\mathrm{Mat}}(\mathcal{C}',M')(\mathbf{Y},\mathbf{Y}') \longrightarrow \widehat{\mathrm{Mat}}(\mathcal{C},M)(\mathbb{G}(\mathbf{Y}),\mathbb{G}(\mathbf{Y}'))$ を以下の様に定義する。$\mathbb{G}$ は関手ではないことに注意。

対象 $\mathbf{Y} = ((Ym)n,k,p,c,i) \in Ob(\widetilde{\mathrm{Mat}}(\mathcal{C}',M'))$ に対し $pk(pni+m-pkmci) = pni = (pni+m-pkmci)ci$ であるので、$(Y(pni+m-pkmci),k,p,c,i) \in \widehat{\mathrm{Mat}}(\mathcal{C},M)$ を得る。$\mathbb{G}(\mathbf{Y}) := (Y(pni+m-pkmci),k,p,c,i)$ とおく。

射 $f\coprod g \in \widetilde{\mathrm{Mat}}(\mathcal{C}', M')(\mathbf{Y}, \mathbf{Y}')$ $(\mathbf{Y}' = ((Y'm')n', k', p', c', i')))$ に対し、$f\coprod g \in \mathcal{C}'(Ym, Y'm')$ であるので、$\mathcal{C}'$ の定義より $h \in \mathrm{Mat}(\mathcal{C}, M, M')(Ym, Y'm')$ で $\{h\} = \overline{f}\coprod\overline{g}$ を満たすものがとれる。

$$
\begin{array}{ccccccccccc}
 & & & \xleftarrow{\overline{p}} & & & & \xleftarrow{\overline{i}} & & & \\
0 & \longrightarrow & \langle Ym\rangle & \xrightarrow{\overline{k}} & \mathbb{P}_lY & \xrightarrow{\overline{m}} & \mathbb{P}_rY & \xrightarrow{\overline{c}} & [Ym] & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \overline{f}=\langle h\rangle} & & \downarrow{\scriptstyle \overline{\mathbb{P}_lh}} & & \downarrow{\scriptstyle \overline{\mathbb{P}_rh}} & & \downarrow{\scriptstyle \overline{g}=[h]} & & \\
 & & & \xleftarrow{\overline{p'}} & & & & \xleftarrow{\overline{i'}} & & & \\
0 & \longrightarrow & \langle Y'm'\rangle & \xrightarrow{\overline{k'}} & \mathbb{P}_lY' & \xrightarrow{\overline{m'}} & \mathbb{P}_rY' & \xrightarrow{\overline{c'}} & [Y'm'] & \longrightarrow & 0
\end{array}
$$

$W_1' \in \mathcal{O}b(\mathcal{C}_l)$, $p_1' \in \mathcal{C}_l(W_1', \mathbb{P}_lY')$, $k_1' \in \mathcal{C}_l(\mathbb{P}_lY', W_1')$ で $k_1'p_1' = 1_{W_1'}$, $1_{\mathbb{P}_lY'} = p'k' + p_1'k_1'$ を満たすもの及び $Z_1 \in \mathcal{O}b(\mathcal{C}_r)$, $i_1 \in \mathcal{C}_r(Z_1, \mathbb{P}_rY)$ $c_1 \in \mathcal{C}_r(\mathbb{P}_rY, Z_1)$ で $i_1c_1 = 1_{Z_1}$, $1_{\mathbb{P}_rY} = ci + c_1i_1$ を満たすものを任意にとる。

次式と 4.2(5) より $\gamma \in \mathcal{J}_{\mathcal{C}_r}(Z_1, \mathbb{P}_rY')$ で $(1_{\mathbb{P}_lY} - pk)(\mathbb{P}_lh)(p'n'i' + m' - p'k'm'c'i') - (m - pkmci)(\mathbb{P}_rh)(1 - c'i') = mc_1\gamma$ を満たすものがとれる。同様に 4.2(4) より $\delta \in \mathcal{J}_{\mathcal{C}_l}(\mathbb{P}_lY, W_1')$ で $pni(\mathbb{P}_rh)(1_{\mathbb{P}_rY'} - c'i') = \delta k_1'm'$ を満たすものがとれる。

$$
\begin{aligned}
&\overline{(1_{\mathbb{P}_lY} - pk)(\mathbb{P}_lh)(p'n'i' + m' - p'k'm'c'i') - (m - pkmci)(\mathbb{P}_rh)(1 - c'i')} \\
&= \overline{(1_{\mathbb{P}_lY} - pk)(\mathbb{P}_lh)m' - m(\mathbb{P}_rh)(1 - c'i')} \\
&= \overline{-pk(\mathbb{P}_lh)m' + (\mathbb{P}_lh)m' - m(\mathbb{P}_rh) + m(\mathbb{P}_rh)c'i'} = 0 \\
&\overline{pni(\mathbb{P}_rh)(1_{\mathbb{P}_rY'} - c'i')} = 0
\end{aligned}
$$

$\mathbb{G}(f\coprod g) := (pfk' + (1 - pk)(\mathbb{P}_lh) + \delta k_1')\coprod(cgi' + (\mathbb{P}_rh)(1 - c'i') + c_1\gamma)$ とおく。すると $\overline{\mathbb{G}(f\coprod g)} = \overline{pfk' + (1 - pk)(\mathbb{P}_lh)}\coprod\overline{cgi' + (\mathbb{P}_rh)(1 - c'i')} = \overline{\mathbb{P}_lh}\coprod\overline{\mathbb{P}_rh} \in \overline{\mathcal{C}}(Y, Y)$ より $\mathbb{G}(f\coprod g) \in \mathcal{C}(Y, Y)$ が成立し、次式より $\mathbb{G}(f\coprod g) \in \widetilde{\mathrm{Mat}}(\mathcal{C}, M)(\mathbb{G}(\mathbf{Y}), \mathbb{G}(\mathbf{Y}'))$ が分かる。

$$
\begin{aligned}
&(pfk')(p'n'i' + m' - p'k'm'c'i') = pfk'p'n'i' = pfn'i' = pngi' = pnicgi' \\
&= (pni + m - pkmci)(cgi') \\
&((1 - pk)(\mathbb{P}_lh) + \delta k_1')(p'n'i' + m' - p'k'm'c'i') \\
&= (m - pkmci)(\mathbb{P}_rh)(1 - c'i') + mc_1\gamma + \delta k_1'(p'n'i' + m' - p'k'm'c'i') \\
&= (pni + m - pkmci)((\mathbb{P}_rh)(1 - c'i') + c_1\gamma)
\end{aligned}
$$

4.9. 次の補題より 4.4 は容易に得られる。

補題 $(\mathcal{C},\mathcal{C}_l,\mathcal{C}_r,M,M')$ を可約な BP、$\mathcal{K}$ 及び $\mathcal{Q}$ を 4.7 で定義した ideal とする。

この時 $\mathbb{F}$ 及び $\mathbb{G}$ は互いに逆な同値 $\mathbb{F}:\widehat{\mathrm{Mat}}(\mathcal{C},M)/\mathcal{K}\longrightarrow\widetilde{\mathrm{Mat}}(\mathcal{C}',M')/\mathcal{Q}$ 及び $\mathbb{G}:\widetilde{\mathrm{Mat}}(\mathcal{C}',M')/\mathcal{Q}\longrightarrow\widehat{\mathrm{Mat}}(\mathcal{C},M)/\mathcal{K}$ を導く。

証明 4.7(5) より $\mathbb{F}$ は忠実。ゆえに $\mathbb{F}$ が同値である事を示すためには次の (i)(ii) を示せば十分。

(i) $\mathbb{F}\circ\mathbb{G}$ は $\widetilde{\mathrm{Mat}}(\mathcal{C}',M')/\mathcal{Q}$ の恒等関手に同型。

(ii) $\mathbb{G}\circ\mathbb{F}$ は $\mathcal{O}b(\widehat{\mathrm{Mat}}(\mathcal{C},M)/\mathcal{K})$ 上恒等。

(i) を示す。任意の $\mathbf{Y}=((Ym)n,k,p,c,i)\in\mathcal{O}b(\widetilde{\mathrm{Mat}}(\mathcal{C}',M'))$ に対し次式は $\mathbb{F}\circ\mathbb{G}(\mathbf{Y})\simeq\mathbf{Y}$ を示す。

$$\begin{aligned}&\mathbb{F}\circ\mathbb{G}(\mathbf{Y})=\mathbb{F}(Y(pni+m-pkmci),k,p,c,i)\\&=((Y(pni+m-pkmci))(k(pni+m-pkmci)c),k,p,c,i)\\&=((Y(pni+m-pkmci))n,k,p,c,i)\simeq\mathbf{Y}\end{aligned}$$

任意の $f\prod g\in\widetilde{\mathrm{Mat}}(\mathcal{C}',M')(\mathbf{Y},\mathbf{Y}')$ $(\mathbf{Y}'=((Y'm')n',p',i'))$ に対し次式は $\mathbb{F}\circ\mathbb{G}(f\prod g)=f\prod g$ を示す。

$$\begin{aligned}&\mathbb{F}\circ\mathbb{G}(f\textstyle\prod g)=\mathbb{F}((pfk'+(1-pk)(\mathbb{P}_lh)+\delta k_1')\prod(cgi'+(\mathbb{P}_rh)(1-c'i')+c_1\gamma))\\&=k(pfk'+(1-pk)(\mathbb{P}_lh)+\delta k_1')p'\textstyle\prod i(cgi'+(\mathbb{P}_rh)(1-c'i')+c_1\gamma)c'=f\prod g\end{aligned}$$

(ii) を示す。任意の $\mathbf{X}=(Xm,k,p,c,i)\in\mathcal{O}b(\widehat{\mathrm{Mat}}(\mathcal{C},M))$ に対し次式より $\mathbb{G}\circ\mathbb{F}(\mathbf{X})=\mathbf{X}$。

$$\begin{aligned}&\mathbb{G}\circ\mathbb{F}(\mathbf{X})=\mathbb{G}((Xm)(kmc),k,p,c,i)\\&=(X(pkmci+m-pkmci),k,p,c,i)=(Xm,k,p,c,i)=\mathbf{X}\blacksquare\end{aligned}$$

5. Quadratic bimodule problem

5.1. 定義 (1) 圏 $\mathcal{C}$ が半単純であるとは、任意の $X\in\mathcal{O}b(\mathcal{C})$ に対し $\mathcal{C}(X,X)$ が半単純である事を意味する。

(2) 忠実関手 $\mathbb{F}:\mathcal{C}\longrightarrow\mathcal{D}$ が quadratic extension であるとは、$\mathcal{D}$ が半単純であり任意の $X\in\mathcal{O}b(\mathcal{C})$ に対し $\mathcal{C}(X,X)\longrightarrow\mathcal{D}(\mathbb{F}X,\mathbb{F}X)$ が quadratic extension (2.1) である事を意味する。

5.2. 定義 $(\mathcal{C},M)$ （正確には $(\mathcal{C},\mathcal{C}_l,\mathcal{C}_r,M)$）が quadratic BP（略して QBP）であるとは (1)-(4) が成立する事であり、正規化された QBP であるとは (1)-(5) が成立する事。

(1) $\mathcal{C}$, $\mathcal{C}_l$, $\mathcal{C}_r$ はある可換局所環 R 上の Krull-Schmidt 圏、M は $(\mathcal{C}_l, \mathcal{C}_r)$-加群で R が中心的に作用する。$\overline{\mathcal{C}_l} := \mathcal{C}_l / \mathcal{J}_{\mathcal{C}_l}$, $\overline{\mathcal{C}_r} := \mathcal{C}_r / \mathcal{J}_{\mathcal{C}_r}$ とおくと、これらは対象毎に k 上有限次（$k := R/J_R$）。

(2) $\#\mathrm{ind}(\mathcal{C}_{r0}) < \infty$ なる $\mathcal{C}_r$ の充満部分圏 $\mathcal{C}_{r0}$ 及び $X \in \mathrm{ind}(\mathcal{C}_l)$ に対し N を $M(X, \ )$ の $\mathcal{C}_{r0}$ への制限とすると、N は単列 $\mathcal{C}_{r0}^{op}$- 加群で $\bigcap_{n \geq 0} N \mathcal{J}_{\mathcal{C}_{r0}}^n = 0$。

(3) $\#\mathrm{ind}(\mathcal{C}_{l0}) < \infty$ なる $\mathcal{C}_r$ の充満部分圏 $\mathcal{C}_{l0}$ 及び $X \in \mathrm{ind}(\mathcal{C}_r)$ に対し N を $M(\ ,X)$ の $\mathcal{C}_{l0}$ への制限とすると、N は単列 $\mathcal{C}_{l0}$- 加群で $\bigcap_{n \geq 0} \mathcal{J}_{\mathcal{C}_{l0}}^n N = 0$。

(4) $\mathbb{P}_l \coprod \mathbb{P}_r : \mathcal{C} \longrightarrow \mathcal{C}_l \coprod \mathcal{C}_r$ は忠実関手、$\mathcal{C} \supset \mathcal{J}_{\mathcal{C}_l \coprod \mathcal{C}_r}$（i.e. 任意の $X \in \mathcal{O}b(\mathcal{C})$ に対し $\mathcal{C}(X,X) \supset \mathcal{J}_{\mathcal{C}_l}(\mathbb{P}_l X, \mathbb{P}_l X) \coprod \mathcal{J}_{\mathcal{C}_r}(\mathbb{P}_r X, \mathbb{P}_r X)$）であり、$\overline{\mathcal{C}} := \mathcal{C} / \mathcal{J}_{\mathcal{C}_l \coprod \mathcal{C}_r}$ に対し $\overline{\mathcal{C}} \longrightarrow \overline{\mathcal{C}_l} \coprod \overline{\mathcal{C}_r}$ は quadratic extension。

(5) $\mathcal{J}_{\mathcal{C}} = \mathcal{J}_{\mathcal{C}_l \coprod \mathcal{C}_r}$。

5.2.1. 注意 (1) QBP $(\mathcal{C}, M)$ が $\#\mathrm{ind}(\mathcal{C}) < \infty$ を満たせば、$\mathcal{C}_l \coprod \mathcal{C}_r$ を $(\mathcal{C}, \mathcal{C}_l, \mathcal{C}_r, M)$ が正規化された QBP である様に取り直す事が可能である。

(2) 具体例は 7.1.3, 7.2.1 参照。

5.3. 補題 $(\mathcal{C}, \mathcal{C}_l, \mathcal{C}_r, M)$ を正規化された QBP とし、M' を M の極大 $(\mathcal{C}_l, \mathcal{C}_r)$- 部分加群とする時、以下が成立する。

(1) $\mathrm{Mat}(\overline{\mathcal{C}}, \overline{M})$ は $\mathrm{mod}\, A$ と半単純な圏の直積に同値になる。但し A は有限次 k- 多元環で次のいずれか。

(a) D は斜体、σ は D の環自己同型、$A := D[t; \sigma]$ は skew polynomial ring（$d \in D$ に対し $td^\sigma = dt$）。

(b) F は斜体、$\iota_D : D \longrightarrow M_a(F)$ 及び $\iota_E : E \longrightarrow M_b(F)$ $(a, b = 1, 2)$ は type (I)(II)(IV)(V) の quadratic extension, $M := \mathrm{Hom}_k(M_{a,b}(F), k)$ は (E, D)- 加群で $A := \begin{pmatrix} D & 0 \\ M & E \end{pmatrix}$。

(2) $(\mathcal{C}, \mathcal{C}_l, \mathcal{C}_r, M, M')$ は可約な BP であり $\mathcal{C}'$ を 4.3 の様に定義すると、$\mathbb{P}_l \coprod \mathbb{P}_r$ より導かれる $\mathcal{C}' / \mathcal{J}_{\mathcal{C}_l \coprod \mathcal{C}_r} \longrightarrow \overline{\mathcal{C}_l} \coprod \overline{\mathcal{C}_r}$ は quadratic extension となる。特に、$(\mathcal{C}', \mathcal{C}_l, \mathcal{C}_r, M')$ は再び（正規化されていると限らない） QBP となる。

証明 (1) は容易。(2) は 6.1.1, 6.2.1, 6.2.2, 6.2.3, 6.2.4 より分かる。■

5.3.1. 定義 一般に (1) の形の A から関手 $\{ \ \} : \mathrm{mod}\, A \longrightarrow \mathrm{Mat}(\overline{\mathcal{C}}, \overline{M}) \xrightarrow{\{\ \}} \overline{\mathcal{C}_l} \coprod \overline{\mathcal{C}_r}$ を定める。$\{ \ \}$ の像を string 圏、$\{A\} = 0$ なる A よりなる $\mathrm{mod}\, A$ の充満部分圏を band 圏と呼ぶ事にする。

5.4. 定義 $(\mathcal{C}, \mathcal{C}_l, \mathcal{C}_r, M)$ を QBP とする。

(1) 関手の有限列 $\mathbf{F}=\{\mathbb{F}_j \mid 0<j\leq n=n_{\mathbf{F}}\}$ が $(\mathcal{C},M)$ の還元列 であるとは、$\mathbf{F}$ が以下の方法で得られる事を意味する。

$$\overline{\mathrm{Mat}}(\mathcal{C}'_{j-1},M'_{j-1})\supseteq\overline{\mathrm{Mat}}(\mathcal{C}_j,M_j)\xrightarrow{\mathbb{F}_j}\overline{\mathrm{Mat}}(\mathcal{C}'_j,M'_j)$$

$(\mathcal{C}'_0,\mathcal{C}_{l0},\mathcal{C}_{r0},M'_0):=(\mathcal{C},\mathcal{C}_l,\mathcal{C}_r,M)$ とおく。$0<j\leq n_{\mathbf{F}}$ に対し $(\mathcal{C}'_{j-1},\mathcal{C}_{l,j-1},\mathcal{C}_{r,j-1},M'_{j-1})$ が QBP であると仮定する。$\mathcal{C}_j$ を $\mathcal{C}'_{j-1}$ の充満部分圏で $\#\mathrm{ind}(\mathcal{C}_j)<\infty$ なるものとし、M_j を M'_{j-1} の $\mathcal{C}_j$ への制限とする。5.2.1 より $\mathcal{C}_{l,j-1}\coprod\mathcal{C}_{r,j-1}$ の充満部分圏 $\mathcal{C}_{lj}\coprod\mathcal{C}_{rj}$ を $(\mathcal{C}_j,\mathcal{C}_{lj},\mathcal{C}_{rj},M_j)$ が正規化された QBP である様に取る。M_j の任意の極大 $(\mathcal{C}_{lj},\mathcal{C}_{rj})$- 部分加群 M'_j を取り、$\mathbb{F}_j:\overline{\mathrm{Mat}}(\mathcal{C}_j,M_j)\longrightarrow\overline{\mathrm{Mat}}(\mathcal{C}'_j,M'_j)$ を還元関手とする。すると $(\mathcal{C}'_j,\mathcal{C}_{lj},\mathcal{C}_{rj},M'_j)$ は 5.3 より再び QBP である。以下同様。

(2) $(\mathcal{C},M)$ の還元列 $\mathbf{F}=\{\mathbb{F}_j \mid 0<j\leq n=n_{\mathbf{F}}\}$ を固定する。$\mathcal{C}_{\mathbf{F}}$ を $\overline{\mathrm{Mat}}(\mathcal{C},M)$ の充満部分圏で $\mathbb{F}_n\circ\ldots\circ\mathbb{F}_1(A)$ が定義される様な対象 A より成るものとする。関手 $\mathbb{E}_{\mathbf{F}}$ 及び $\mathbb{E}'_{\mathbf{F}}$ を次で定義する。

$$\begin{aligned}\mathbb{E}'_{\mathbf{F}}:=\mathbb{F}_{n-1}\circ\ldots\circ\mathbb{F}_1 &: \mathcal{C}_{\mathbf{F}}\longrightarrow\overline{\mathrm{Mat}}(\mathcal{C}_n,M_n)\subseteq\overline{\mathrm{Mat}}(\mathcal{C}'_{n-1},M'_{n-1})\\ \mathbb{E}_{\mathbf{F}}:=\mathbb{F}_n\circ\mathbb{E}'_{\mathbf{F}} &: \mathcal{C}_{\mathbf{F}}\longrightarrow\overline{\mathrm{Mat}}(\mathcal{C}'_n,M'_n)\end{aligned}$$

$\mathcal{B}_{\mathbf{F}}$ を $\mathcal{C}_{\mathbf{F}}$ の充満部分圏で、任意の 0 でない直和因子 B が $\mathbb{E}'_{\mathbf{F}}(B)\neq 0$ かつ $\mathbb{E}_{\mathbf{F}}(B)=0$ を満たす様な対象 A より成るものとする。

$\mathcal{C}'_n$ 及び $\mathcal{C}_n$ をそれぞれ $\mathrm{Mat}(\mathcal{C}'_n,M'_n)$ 及び $\mathrm{Mat}(\mathcal{C}_n,M_n)$ の充満部分圏とみなし、$\mathcal{S}_{\mathbf{F}}$ を $\mathcal{C}_{\mathbf{F}}$ の充満部分圏で、任意の 0 でない直和因子 B が $\mathbb{E}'_{\mathbf{F}}(B)\notin\mathcal{O}b(\mathcal{C}_n)$ かつ $\mathbb{E}_{\mathbf{F}}(B)\in\mathcal{O}b(\mathcal{C}'_n)$ を満たす様な対象 A より成るものとする。

5.5. 主定理 $(\mathcal{C},M)$ を QBP とし I を $(\mathcal{C},M)$ を還元列全体とする。この時以下が成立する。

(1) $\mathrm{ind}(\mathrm{Mat}(\mathcal{C},M))=\bigcup_{\mathbf{F}\in I}(\mathrm{ind}(\mathcal{B}_{\mathbf{F}})\cup\mathrm{ind}(\mathcal{S}_{\mathbf{F}}))$。

(2) $\mathcal{B}_{\mathbf{F}}$ からある band 圏への表現同値が存在する。

(3) $\mathcal{S}_{\mathbf{F}}$ からある string 圏への表現同値が存在する。

この定理は、QBP は常に解く事が可能である事を主張する。

6 節における Im{ } の記述を用いる事により、詳細な $\mathrm{ind}(\mathrm{Mat}(\mathcal{C},M))$ の記述が可能となるが、ここでは省略する。詳しくは [I] 参照の事。

以下、5.5 を証明するために二種類の関数を定義する。

5.6. 定義 圏 $\mathcal{C}$ に対し、$\|\ \|:\mathcal{O}b(\mathcal{C})\longrightarrow\mathrm{N}_{\geq 0}$ が norm function であるとは任意の $X,Y,Z\in\mathcal{O}b(\mathcal{C})$ に対し (a)(b) が成立する事。

(a) $||X|| = 0$ と $X = 0$ は同値。
(b) X が $Y \oplus Z$ に同型ならば $l(X) = l(Y) + l(Z)$ が成立する。

5.6.1. $|| \ || : \mathcal{O}b(\mathcal{C}_l \coprod \mathcal{C}_r) \longrightarrow \mathbb{N}_{\geq 0}$ を norm function、$\mathbb{F} : \overline{\mathrm{Mat}}(\mathcal{C}, M) \longrightarrow \overline{\mathrm{Mat}}(\mathcal{C}', M)$ を還元関手とする。$|| \ || : \mathcal{O}b(\mathrm{Mat}(\mathcal{C}, M)) \longrightarrow \mathbb{N}_{\geq 0}$ 及び $|| \ || : \mathcal{O}b(\mathrm{Mat}(\mathcal{C}', M')) \longrightarrow \mathbb{N}_{\geq 0}$ をそれぞれ忘却関手と $\mathbb{P}_l \coprod \mathbb{P}_r : \mathcal{C} \longrightarrow \mathcal{C}_l \coprod \mathcal{C}_r$ 乃至 $\mathbb{P}_l \coprod \mathbb{P}_r : \mathcal{C}' \longrightarrow \mathcal{C}_l \coprod \mathcal{C}_r$ を合成して定める。すると任意の $A \in \mathcal{O}b(\mathrm{Mat}(\mathcal{C}, M))$ に対して $||A|| \geq ||\mathbb{F}(A)||$ が成立。

5.7. 定義 $(\mathcal{C}, M)$ を QBP とする時、以下の様に定義される $d : \mathrm{ind}(\mathrm{Mat}(\mathcal{C}, M)) \longrightarrow \mathbb{N}_{\geq 0} \cup \{\infty\}$ を **depth function** と呼ぶ。

$Xm \in \mathrm{ind}(\mathrm{Mat}(\mathcal{C}, M))$ を固定する。X の直和の直和因子に同型となる対象より成る $\mathcal{C}$ の充満部分圏を $\mathcal{C}_X$ とし、M_X を M の $\mathcal{C}_X$ への制限とする。5.2.1 より $\mathcal{C}_l \coprod \mathcal{C}_r$ を $(\mathcal{C}_X, \mathcal{C}_l, \mathcal{C}_r, M_X)$ が正規化された QBP である様に取る事ができる。$m = 0$ ならば $d(Xm) := \infty$ とおき、そうでなければ $d(Xm)$ を $(\mathcal{C}_l, \mathcal{C}_r)$- 加群 $M_X / \mathcal{C}_l m \mathcal{C}_r$ の長さとする。

(1) $\mathcal{C}$ を $\mathrm{Mat}(\mathcal{C}, M)$ の充満部分圏とみなせば、$A \in \mathrm{ind}(\mathrm{Mat}(\mathcal{C}, M))$ に対して $d(A) = \infty$ と $A \in \mathcal{O}b(\mathcal{C})$ は同値。

(2) $\mathbb{F} : \overline{\mathrm{Mat}}(\mathcal{C}, M) \longrightarrow \overline{\mathrm{Mat}}(\mathcal{C}', M')$ を還元関手、$d : \mathrm{ind}(\mathrm{Mat}(\mathcal{C}', M')) \longrightarrow \mathbb{N}_{\geq 0} \cup \{\infty\}$ を QBP $(\mathcal{C}', M')$ の depth function とする時、$A \in \mathrm{ind}(\mathrm{Mat}(\mathcal{C}, M))$ が $||A|| = ||\mathbb{F}(A)||$ かつ $d(A) < \infty$ を満たせば $d(A) > d(\mathbb{F}(A))$。

5.5 の証明 (2)(3) は 4.4, 5.3 よりすぐ。(1) を示す。$A \in \mathrm{ind}(\mathrm{Mat}(\mathcal{C}, M))$ を固定し、$a := \min\{||\mathbb{E}_{\mathbf{F}}(A)|| \mid \mathbf{F} \in I, A \in \mathcal{O}b(\mathcal{C}_{\mathbf{F}})\}$ とおく。

(i) $a = 0$ と仮定する。

$\mathbf{F} = \{\mathbb{F}_j \mid 0 < j \leq n = n_{\mathbf{F}}\} \in I$ を $||\mathbb{E}_{\mathbf{F}}(A)|| = 0$ なるよう取る。すると $0 < k \leq n = n_{\mathbf{F}}$ で $\mathbb{F}_k \circ \ldots \circ \mathbb{F}_1(A) = 0$ かつ $\mathbb{F}_{k-1} \circ \ldots \circ \mathbb{F}_1(A) \neq 0$ なるものが存在する。$\mathbf{G} := \{\mathbb{F}_j \mid 0 < j \leq k\}$ とおくと $A \in \mathcal{O}b(\mathcal{B}_{\mathbf{G}})$ が成立する。

(ii) $a > 0$ と仮定し、$b := \min\{d(\mathbb{E}_{\mathbf{F}}(A)) \mid \mathbf{F} \in I, A \in \mathcal{O}b(\mathcal{C}_{\mathbf{F}}), ||\mathbb{E}_{\mathbf{F}}(A)|| = a\}$ とおき、$\mathbf{F} = \{\mathbb{F}_j \mid 0 < j \leq n = n_{\mathbf{F}}\} \in I$ を $||\mathbb{E}_{\mathbf{F}}(A)|| = a$ かつ $d(\mathbb{E}_{\mathbf{F}}(A)) = b$ なるよう取る。

$b = \infty$ と仮定する。すると $0 \leq k \leq n$ で $d(\mathbb{F}_k \circ \ldots \circ \mathbb{F}_1(A)) = \infty$ かつ $d(\mathbb{F}_{k-1} \circ \ldots \circ \mathbb{F}_1(A)) \neq \infty$ なるものが存在する。$\mathbf{G} := \{\mathbb{F}_j \mid 0 < j \leq k\}$ とおくと $A \in \mathcal{O}b(\mathcal{S}_{\mathbf{G}})$ が成立する。

$b \neq \infty$ と仮定する。$\mathbb{E}_{\mathbf{F}}(A) = Xm$, $X \in \mathcal{O}b(\mathcal{C}'_n)$, $m \in M'_n$ とおく。X の直和の直和因子に同型となる対象より成る $\mathcal{C}'_n$ の充満部分圏を $\mathcal{C}_{n+1}$ とし、M_{n+1} を M'_n の $\mathcal{C}_{n+1}$ への制限とする。$\#\mathrm{ind}(\mathcal{C}_{n+1}) < \infty$ より、5.2.1 よ

り $C_{l,n+1}\prod C_{r,n+1}$ で $(C_{n+1}, C_{l,n+1}, C_{r,n+1}, M_{n+1})$ が正規化された QBP である様なものが存在し、更に M_{n+1} の極大 $(C_{l,n+1}, C_{r,n+1})$- 部分加群 M'_{n+1} が存在する。$\mathbb{F}_{n+1} : \overline{\mathrm{Mat}}(C_{n+1}, M_{n+1}) \longrightarrow \overline{\mathrm{Mat}}(C'_{n+1}, M'_{n+1})$ を還元関手とし、$\mathbf{G} := \{\mathbb{F}_j \mid 0 < j \leq n+1\}$ とおく。すると $||\mathbb{E}_{\mathbf{G}}(A)|| = ||\mathbb{E}_{\mathbf{F}}(A)||$ が a の最小性及び 5.6.1 より成立する。ゆえに $d(\mathbb{E}_{\mathbf{F}}(A)) > d(\mathbb{E}_{\mathbf{G}}(A))$ が $d(\mathbb{E}_{\mathbf{F}}(A)) < \infty$ 及び 5.7(2) より成立する。これは矛盾。∎

6. 特別な Quadratic bimodule problem

5.3 は Reduction Theorem を QBP に適用して得られる新しい圏が再び QBP になる事を保証する。この節ではその証明に不可欠な補題を挙げる。

これは特に skew polynomial ring 及び $\widetilde{A}_{12}$, $\widetilde{A}_{11}$, $\widetilde{B}_2$, $\widetilde{C}_2$, $\widetilde{A}_3$ 型の hereditary algebra 上の加群の圏においてある興味深い性質が成立する事を意味する。hereditary algebra の基礎事項は [DR] 参照。

6.1. Skew polynomial ring の場合 D を斜体、σ を D の環自己同型、$A := D[t;\sigma]$ を skew polynomial ring ($td^{\sigma} = dt$ for any $d \in D$) とし、$\mathrm{mod}\,A$ で D 上有限次な左 A- 加群の成す圏を表わす。

(1) $\mathrm{mod}\,A$ は以下の様に定義される圏と同値。対象は組 $X = (U, \phi)$ で、$U \in \mathcal{O}b(\mathrm{mod}\,D)$ 及び $\phi \in \mathrm{Hom}_{\mathbb{Z}}(U, U)$ で $d\phi(u) = \phi(d^{\sigma}u)$ $(u \in U,\, d \in D)$ を満たすもの。$\mathrm{Hom}_A((U,\phi),(U',\phi')) := \{f \in \mathrm{Hom}_D(U,U') \mid f\phi' = \phi f\}$。

(2) 関手 $\langle\ \rangle, [\ ] : \mathrm{mod}\,A \longrightarrow \mathrm{mod}\,D$ が次の様に定義される。$X = (U,\phi) \in \mathcal{O}b(\mathrm{mod}\,A)$ に対し、$\langle X \rangle := \mathrm{Ker}\,\phi$, $[X] := \mathrm{Cok}\,\phi$ とおく。$\mathrm{Hom}_A(X,Y) \longrightarrow \mathrm{Hom}_D(\langle X \rangle, \langle Y \rangle)$, $f \mapsto \langle f \rangle$ 及び $\mathrm{Hom}_A(X,Y) \longrightarrow \mathrm{Hom}_D([X],[Y])$, $f \mapsto [f]$ は f より導かれるものとする。

6.1.1. 定理 $\langle\ \rangle$ 及び $[\ ]$ を 6.1 で定義した関手とし、$\{\ \} := \langle\ \rangle \prod [\ ] : \mathrm{mod}\,A \longrightarrow \mathrm{mod}(D \prod D)$ とおく。この時 $S_n \in \mathrm{ind}(\mathrm{mod}\,A)$ $(n \geq 0)$ が存在し、$\mathrm{Im}\{\ \}$ は以下で与えられる。

$\mathrm{ind}(\mathrm{Im}\{\ \})$ は S_n $(n \geq 0)$ より成り、$\langle S_n \rangle = D$, $[S_n] = D$ が成立する。

$$\{\mathrm{Hom}_A(S_n, S_m)\} = \begin{cases} \mathrm{Hom}_D(\langle S_n \rangle, \langle S_m \rangle) \prod 0 & (n \leq m) \\ D & (n = m) \\ 0 \prod \mathrm{Hom}_D([S_n], [S_m]) & (n \geq m) \end{cases}$$

但し $n = m$ の場合の D は対角に埋め込まれた $\mathrm{End}_D(\langle S_n \rangle) \prod \mathrm{End}_D([S_n])$ の部分体を表わす。

6.2. Hereditary algebra の場合 この節では環は全て体 k 上有限次であるとする。

F を斜体、$\iota_D : D \longrightarrow M_a(F)$ 及び $\iota_E : E \longrightarrow M_b(F)$ $(a, b = 1, 2)$ を type (I)(II)(IV)(V) の quadratic extension、$A := \begin{pmatrix} D & 0 \\ M_{b,a}(F) & E \end{pmatrix}$ なる hereditary algebra を考察する。

(1) mod A は以下の様に定義される圏と同値。対象は組 $X = (U, V, \phi)$ で、$U \in \mathcal{O}b(\mathrm{mod}\, D)$, $V \in \mathcal{O}b(\mathrm{mod}\, F)$ 及び $\phi \in \mathrm{Hom}_F(\overline{U}(X), \overline{V}(X))$ $(\overline{U}(X) := M_{1,a}(F) \otimes_D U, \overline{V}(X) := M_{1,b}(F) \otimes_E V)$ より成るもの。$\mathrm{Hom}_A((U, V, \phi), (U', V', \phi'))$ は次で与えられる。

$$\{(f, g) \in \mathrm{Hom}_D(U, U') \times \mathrm{Hom}_E(V, V') \mid (1_{M_{1,a}(F)} \otimes f)\phi' = \phi(1_{M_{1,b}(F)} \otimes g)\}$$

(2) 関手 $\langle\ \rangle$ 及び $[\ ]$ が次の様に定義される。$X = (U, V, \phi) \in \mathcal{O}b(\mathcal{L})$ に対し、$\langle X \rangle := \mathrm{Ker}\, \phi$, $[X] := \mathrm{Cok}\, \phi$ とおく。$\mathrm{Hom}_A(X, Y) \longrightarrow \mathrm{Hom}_F(\langle X \rangle, \langle Y \rangle)$, $f \mapsto \langle f \rangle$ 及び $\mathrm{Hom}_A(X, Y) \longrightarrow \mathrm{Hom}_F([X], [Y])$, $f \mapsto [f]$ は f より導かれるものとする。

(3) $\mathcal{I}_A$, $\mathcal{P}_A$, $\mathcal{R}_A$ でそれぞれ preinjective A- 加群、preprojective A- 加群、及び regular A- 加群の成す圏を表わすものとする。

6.2.1. 定理 ι_D 及び ι_E を type (II)(IV)(V)、$\langle\ \rangle, [\ ]$ を 6.2 で定義した関手とし、$\{\ \} := \langle\ \rangle \coprod [\ ] : \mathrm{mod}\, A \longrightarrow \mathrm{mod}(F \coprod F)$ とおく。この時 $S_n \in \mathrm{ind}(\mathcal{R}_A)$ $(n \geq 0)$ が存在し、$\mathrm{Im}\{\ \}$ は以下で与えられる。

(1) $\mathrm{ind}(\mathrm{Im}\langle\ \rangle)$ は $\mathrm{ind}(\mathcal{I}_A)$ 及び S_n $(n \geq 0)$ より成っており、$\langle S_n \rangle = F$, $\langle X \rangle = F^{d_X}$ $(X \in \mathrm{ind}(\mathcal{I}_A), d_X = 1, 2)$ である。$\mathrm{ind}(\mathrm{Im}\langle\ \rangle)$ 上の半順序 $\leq$ を $n < m$ 及び $X, Y \in \mathrm{ind}(\mathcal{I}_A)$ で $\mathrm{Hom}_A(X, Y) \neq 0$ なるものに対し、$S_n < S_m < X < Y$ とおく事により定める。この時、

$$\langle \mathrm{Hom}_A(X, Y) \rangle = \begin{cases} \mathrm{Hom}_F(\langle X \rangle, \langle Y \rangle) & (X < Y) \\ \mathrm{End}_F(\langle X \rangle) & (X = Y = S_n) \\ \mathrm{End}_A(X) & (X = Y \in \mathrm{ind}(\mathcal{I}_A)) \\ 0 & (X > Y) \end{cases}$$

但し $X = Y \in \mathrm{ind}(\mathcal{I}_A)$ の場合の $\mathrm{End}_A(X)$ は、それに同型な $\mathrm{End}_F(\langle X \rangle)$ の部分体を表わす。

(2) $\mathrm{ind}(\mathrm{Im}[\ ])$ は $\mathrm{ind}(\mathcal{P}_A)$ 及び S_n $(n \geq 0)$ より成っており、$[S_n] = F$ and $[X] = F^{d_X}$ $(X \in \mathrm{ind}(\mathcal{P}_A), d_X = 1, 2)$ である。$\mathrm{ind}(\mathrm{Im}[\ ])$ 上の半順序 $\preceq$ を $n < m$ 及び $X, Y \in \mathrm{ind}(\mathcal{P}_A)$ で $\mathrm{Hom}_A(Y, X) \neq 0$ なるものに対し、

$S_n \succ S_m \succ X \succ Y$ とおく事により定める。この時、

$$[\mathrm{Hom}_A(X,Y)] = \begin{cases} \mathrm{Hom}_F([X],[Y]) & (X \prec Y) \\ \mathrm{End}_A(X) & (X = Y \in \mathrm{ind}(\mathcal{P}_A)) \\ \mathrm{End}_F([X]) & (X = Y = S_n) \\ 0 & (X \succ Y) \end{cases}$$

但し $X = Y \in \mathrm{ind}(\mathcal{P}_A)$ の場合の $\mathrm{End}_A(X)$ は、それに同型な $\mathrm{End}_F([X])$ の部分体を表わす。

(3) $\mathrm{ind}(\mathrm{Im}\{\ \})$ は $\mathrm{ind}(\mathcal{I}_A)$, $\mathrm{ind}(\mathcal{P}_A)$ 及び S_n $(n \geq 0)$ より成る。$X, Y \in \mathrm{ind}(\mathrm{Im}\{\ \})$ に対し、

$$\{\mathrm{Hom}_A(X,Y)\} = \begin{cases} F & (X = Y = S_n) \\ \langle \mathrm{Hom}_A(X,Y)\rangle \prod [\mathrm{Hom}_A(X,Y)] & (\text{その他}) \end{cases}$$

但し $X = Y = S_n$ の場合の F は対角に埋め込まれた $\mathrm{End}_F(\langle S_n\rangle) \prod \mathrm{End}_F([S_n])$ の部分体を表わす。

(4) 特に、$\mathrm{Im}\{\ \} \longrightarrow \mathrm{mod}(F \prod F)$ は quadratic extension。

6.2.2. ι_D が type (I) の時、$\{\ \} := \langle\ \rangle \prod \overline{U} \prod [\ ] : \mathrm{mod}\, A \longrightarrow \mathrm{mod}(F \prod F \prod F)$ の像は以下で与えられる。(a)(b)(c) の3通りに分ける。

(a) ι_E が type (II) の時。$\mathrm{ind}(\mathrm{mod}\, A) = \mathrm{proj}\, A \cup \mathrm{inj}\, A = \{P, Q\} \cup \{I, J\}$ 上の全順序を $P < Q < I < J$ と定める。

(b) ι_E が type (V) の時。$\mathrm{ind}(\mathrm{mod}\, A) = \mathrm{proj}\, A \cup \mathrm{inj}\, A = \{P, Q, R\} \cup \{I, J, K\}$ 上の半順序を $P, Q < R < I, J < K$ と定める。

(c) ι_E が type (IV) の時。$\mathrm{ind}(\mathrm{mod}\, A) = \mathrm{proj}\, A \cup \mathrm{inj}\, A = \{P, Q\} \cup \{I, J\}$ 上の全順序を $P < Q < I < J$ と定める。

(1)(a) $\{P\} = 0 \prod 0 \prod F^2$, $\{Q\} = 0 \prod F \prod F$, $\{I\} = 0 \prod F^2 \prod 0$ 及び $\{J\} = F \prod F \prod 0$。

(b) $\{P\} = 0 \prod 0 \prod F$, $\{Q\} = 0 \prod 0 \prod F$, $\{R\} = 0 \prod F \prod F$, $\{I\} = 0 \prod F \prod 0$, $\{J\} = 0 \prod F \prod 0$ 及び $\{K\} = F \prod F \prod 0$。

(c) $\{P\} = 0 \prod 0 \prod F$, $\{Q\} = 0 \prod F \prod F$, $\{I\} = 0 \prod F \prod 0$ 及び $\{J\} = F \prod F \prod 0$。

(2) (F) で、対角に埋め込まれた F と同型な $F \prod F$ の部分体を表わす時、

(a) $\{\mathrm{End}_A(P)\} = 0 \prod 0 \prod E$, $\{\mathrm{End}_A(Q)\} = 0 \prod (F)$, $\{\mathrm{End}_A(I)\} = 0 \prod E \prod 0$ 及び $\{\mathrm{End}_A(J)\} = (F) \prod 0$。

(b) $\{\mathrm{End}_A(P)\} = 0 \prod 0 \prod F$, $\{\mathrm{End}_A(Q)\} = 0 \prod 0 \prod F$, $\{\mathrm{End}_A(R)\} = 0 \prod (F)$, $\{\mathrm{End}_A(I)\} = 0 \prod F \prod 0$, $\{\mathrm{End}_A(J)\} = 0 \prod F \prod 0$ 及び $\{\mathrm{End}_A(K)\} = (F) \prod 0$。

(c) $\{\mathrm{End}_A(P)\} = 0 \amalg 0 \amalg E$, $\{\mathrm{End}_A(Q)\} = 0 \amalg (F)$, $\{\mathrm{End}_A(I)\} = 0 \amalg E \amalg 0$ 及び $\{\mathrm{End}_A(J)\} = (F) \amalg 0$。

(3) $X < Y$ なら $\{\mathrm{Hom}_A(X,Y)\} = \mathrm{Hom}_{(F \amalg F \amalg F)}(\{X\},\{Y\})$、$X > Y$ なら $= 0$。

(4) 特に、$\mathrm{Im}\{\ \} \longrightarrow \mathrm{mod}(F \amalg F \amalg F)$ は quadratic extension。

6.2.3. ι_E が type (I) の時、$\{\ \} := \langle\ \rangle \amalg \overline{V} \amalg [\ ] : \mathrm{mod}\, A \longrightarrow \mathrm{mod}(F \amalg F \amalg F)$ の像は以下で与えられる。(a)(b)(c) の 3 通りに分ける。

(a) ι_D が type (II) の時。$\mathrm{ind}(\mathrm{mod}\, A) = \mathrm{proj}\, A \cup \mathrm{inj}\, A = \{P, Q\} \cup \{I, J\}$ 上の全順序を $P < Q < I < J$ と定める。

(b) ι_D が type (V) の時。$\mathrm{ind}(\mathrm{mod}\, A) = \mathrm{proj}\, A \cup \mathrm{inj}\, A = \{P, Q, R\} \cup \{I, J, K\}$ 上の半順序を $P < Q, R < I < J, K$ と定める。

(c) ι_D が type (IV) の時。$\mathrm{ind}(\mathrm{mod}\, A) = \mathrm{proj}\, A \cup \mathrm{inj}\, A = \{P, Q\} \cup \{I, J\}$ 上の全順序を $P < Q < I < J$ と定める。

(1)(a) $\{P\} = 0 \amalg F \amalg F$, $\{Q\} = 0 \amalg F^2 \amalg 0$, $\{I\} = F \amalg F \amalg 0$ 及び $\{J\} = F^2 \amalg 0 \amalg 0$。

(b) $\{P\} = 0 \amalg F \amalg F$, $\{Q\} = 0 \amalg F \amalg 0$, $\{R\} = 0 \amalg F \amalg 0$, $\{I\} = F \amalg F \amalg 0$, $\{J\} = F \amalg 0 \amalg 0$ 及び $\{K\} = F \amalg 0 \amalg 0$。

(c) $\{P\} = 0 \amalg F \amalg F$, $\{Q\} = 0 \amalg F \amalg 0$, $\{I\} = F \amalg F \amalg 0$ 及び $\{J\} = F \amalg 0 \amalg 0$。

(2) (F) で、対角に埋め込まれた F と同型な $F \amalg F$ の部分体を表わす時、

(a) $\{\mathrm{End}_A(P)\} = 0 \amalg (F)$, $\{\mathrm{End}_A(Q)\} = 0 \amalg D \amalg 0$, $\{\mathrm{End}_A(I)\} = (F) \amalg 0$ 及び $\{\mathrm{End}_A(J)\} = D \amalg 0 \amalg 0$。

(b) $\{\mathrm{End}_A(P)\} = 0 \amalg (F)$, $\{\mathrm{End}_A(Q)\} = 0 \amalg F \amalg 0$, $\{\mathrm{End}_A(R)\} = 0 \amalg F \amalg 0$, $\{\mathrm{End}_A(I)\} = (F) \amalg 0$, $\{\mathrm{End}_A(J)\} = F \amalg 0 \amalg 0$ 及び $\{\mathrm{End}_A(K)\} = F \amalg 0 \amalg 0$。

(c) $\{\mathrm{End}_A(P)\} = 0 \amalg (F)$, $\{\mathrm{End}_A(Q)\} = 0 \amalg D \amalg 0$, $\{\mathrm{End}_A(I)\} = (F) \amalg 0$ 及び $\{\mathrm{End}_A(J)\} = D \amalg 0 \amalg 0$。

(3) $X < Y$ なら $\{\mathrm{Hom}_A(X,Y)\} = \mathrm{Hom}_{(F \amalg F \amalg F)}(\{X\},\{Y\})$、$X > Y$ なら $= 0$。

(4) 特に、$\mathrm{Im}\{\ \} \longrightarrow \mathrm{mod}(F \amalg F \amalg F)$ は quadratic extension。

6.2.4. ι_D 及び ι_E を type (I) とし $\mathrm{ind}(\mathrm{mod}\, A) = \{P, B, J\}$ 上の全順序を $P < B < J$ と定める。$\{\ \} := \langle\ \rangle \amalg \overline{U} \amalg \overline{V} \amalg [\ ] : \mathrm{mod}\, A \longrightarrow \mathrm{mod}(F \amalg F \amalg F \amalg F)$ の像は以下で与えられる。

(1) $\{P\} = 0 \amalg 0 \amalg F \amalg F$, $\{B\} = 0 \amalg F \amalg F \amalg 0$ 及び $\{I\} = F \amalg F \amalg 0 \amalg 0$。

(2) $\{\mathrm{End}_A(P)\} = 0 \prod 0 \prod (F)$, $\{\mathrm{End}_A(B)\} = 0 \prod (F) \prod 0$ 及び $\{\mathrm{End}_A(I)\} = (F) \prod 0 \prod 0$。但し (F) は、対角に埋め込まれた F と同型な $F \prod F$ の部分体を表わす。

(3) $X < Y$ なら $\{\mathrm{Hom}_A(X,Y)\} = \mathrm{Hom}_{(F \prod F \prod F \prod F)}(\{X\},\{Y\})$、$X > Y$ なら $= 0$。

(4) 特に、$\mathrm{Im}\{\ \} \longrightarrow \mathrm{mod}(F \prod F \prod F \prod F)$ は quadratic extension。

7.1. Quadratic order への応用

この節では R は完備離散付値環、K は R の商体とする。

7.1.1. 定理 $B \longrightarrow A$ を有限次 K-多元環の quadratic extension とし、Δ を B の hereditary full order とする。関手 $\mathrm{lat}(\Delta, A) \longrightarrow \mathrm{mod}\,\Delta/J_\Delta$, $Y \mapsto Y/J_\Delta Y$ の像を $\mathcal{L}$ とおく時、$\mathcal{L} \longrightarrow \mathrm{mod}\,\Delta/J_\Delta$ は quadratic extension となる。

7.1.2. 注意 この定理の証明は省略するが、almost Bass order との関係を注意しておく。

[H] において、almost Bass order の概念が導入された。almost Bass order は分類が可能であり、大まかに言うと次の二種類が存在する。

(1) 7.1.1 の Δ を含むような A の full order。

(2) "最小" の quadratic Bačkstrom̈ order。

逆に、無限系列を成す almost Bass order は全て (1) の形で得られる。

7.1.3. 系 Λ を A の quadratic order とし、hereditary order Δ で $J_\Delta = J_\Lambda \subset \Lambda \subsetneq \Delta$ なるものを取る。$\mathcal{I}$ を $\mathrm{lat}(\Lambda, A)$ の ideal で $\mathcal{I}(L,L') := \{f \in \mathrm{lat}(\Lambda,A)(L,L') \mid (L)f \subseteq J_\Lambda L'\}$ で定義されるものとすると、QBP $(\mathcal{C}, M)$ で $\mathrm{lat}(\Lambda,A)/\mathcal{I}$ が $\mathrm{Mat}(\mathcal{C},M)$ の充満部分圏に同値となるものが存在する。

特に $\mathrm{lat}(\Lambda, A)$ の直既約対象は全て 5.5 により求める事が可能である。

証明 3.4.2 の記号で $(\mathcal{K} \prod \mathcal{L}, \mathrm{mod}\,\overline{\Delta}, \mathrm{mod}\,\overline{\Delta}, N)$ が条件を満たす事を示す。

$\overline{\Lambda} \longrightarrow \overline{\Delta}$ は quadratic extension なので、$\mathcal{K} \longrightarrow \mathrm{mod}\,\overline{\Delta}$ も quadratic extension である。一方 7.1.1 より $\mathcal{L} \longrightarrow \mathrm{mod}\,\overline{\Delta}$ も quadratic extension である。ゆえに命題 3.4.2 より主張を得る。∎

7.2. String algebra 等への応用

この節では R は完備離散付値環、k を R の剰余体とする。

7.2.1. 系 Λ を quadratic Bačkstrom̈ order とし、hereditary order Δ で

$J_\Delta = J_\Lambda \subset \Lambda \subseteq \Delta$ なるものを取る。すると QBP $(\mathcal{C}, M)$ 及び $\mathrm{Mat}(\mathcal{C}, M)$ から $\mathrm{mod}\,\Lambda$ への稠密充満関手が存在する。

特に R 上の quadratic Bačkstroṁ order、 k 上の string algebra 及び clannish algebra についてはいずれもその直既約加群を 5.5 により求める事が可能である。

証明 $\mathcal{P}$ を有限生成射影 Λ- 加群の圏、$\mathcal{P}'$ を有限生成射影 Δ- 加群の圏とする。前半は 3.4.1 より $(\mathcal{P} \coprod \mathcal{P}, \mathcal{P}', \mathcal{P}', \mathcal{J}_{\mathcal{P}})$ が条件を満たす事よりすぐ。

後半は 2.5 より分かる。■

[CB] W. W. Crawley-Boevey: Matrix problems and Drozd's theorem, Topics in algebra, Banach center publications, vol. 26. part 1, 199-222 (1990).

[CB2] W. W. Crawley-Boevey: Regular modules for tame hereditary algebras, Proc. London Math. Soc. (3) **62** 490-508 (1991).

[CB3] W. W. Crawley-Boevey: Modules of finite length over their endomorphism rings, London Math. Soc. Lect. Notes, **168** 127-184 (1992).

[CB4] W. W. Crawley-Boevey: Functorial filtrations II: clans and the Gelfand problem, J. London Math. (2) **40** 9-30 (1989).

[CB5] W. W. Crawley-Boevey: On tame algebras and bocses, Proc. London Math. Soc. (3) **56** 451-483 (1988).

[CR] C.W. Curtis, I. Reiner: Methods of representation theory with application to finite groups and orders, vol.1, Interscience, 1981.

[DR] V. Dlab, C. M. Ringel: Representation of Graphs and Algebras, Mem. Amer. Math. Soc. **173** (1976).

[H] H. Hijikata: Bass orders in non semisimple algebras, Report of the 4-th Symposium on Representation of Algebras, at Shizuoka, 95-118 (1993).

[HN] H. Hijikata, K. Nishida: Classification of Bass orders, J. reine angew. Math. **431** 191-220 (1992).

[I] O. Iyama: Quadratic bimodule problem, Preprint.

[WW] B. Wald, J. Waschbüsch: Tame biserial algebras, J. Algebra, **95** 480-500 (1985).

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QUASI-DUO RINGS AND 2-PRIMAL RINGS

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1. Introduction

Throughout this paper, all rings are associative with identity. Observing the properties of quasi-duo rings was initiated by Yu in [14], related to the Bass' conjecture in [1]. Given a ring R the polynomial ring over R and the formal power series ring over R are denoted by $R[x]$ and $R[[x]]$, respectively. In [9], if $R[x]$ is right quasi-duo then R is right quasi-duo but the converse is not true in general: similarly R is 2-primal if and only if $R[x]$ is 2-primal by [2, Proposition 2.6] and [7, Proposition 4]. R is right quasi-duo if and only if $R[[x]]$ is right quasi-duo [9, Proposition 6]: similarly by [7, Proposition 12] if $R[[x]]$ is 2-primal then R is 2-primal, but the converse is not true in general by [10, Example 1.1]. Considering the preceding results, we observe the similarities or differences between right quasi-duo rings and 2-primal rings.

A ring R is called *right (left) duo* if every right (left) ideal of R is two-sided. A ring R is called *right (left) quasi-duo* if every maximal right (left) ideal of R is two-sided. Commutative rings are clearly right and left duo; right (left) duo rings are right (left) quasi-duo obviously. The n by n upper triangular matrix rings over right quasi-duo rings are also right quasi-duo by [14, Proposition 2.1]. But the n by n full matrix rings over right quasi-duo rings are not right quasi-duo.

As another generalization of commutative rings, there are 2-primal rings. The term *2-primal* was come upon originally by Birkenmeier-Heatherly-Lee [2] in the context of left near rings. Shin [13] proved that a ring R is 2-primal if and only if every minimal prime ideal of R is completely prime, which was one of the earliest results known to us about 2-primal rings (although not so called at the time.) A ring R is called *2-primal* if $P(R) = N(R)$, where $P(R)$ is the prime radical of R and $N(R)$ is the set of all nilpotent elements in R. It is straightforward to check that a ring R is 2-primal if and only if $R/P(R)$ is a reduced ring (i.e., a ring without nonzero nilpotent elements). Commutative rings and reduced rings are 2-primal obviously, and the n by n upper triangular matrix rings over 2-primal rings are also 2-primal by [2, Proposition 2.5]. But the n by n full matrix rings over 2-primal rings are not 2-primal.

2. Counterexamples and related results

This is a part of forthcoming joint paper "A study on quasi-duo rings, to appear in Bull. Korean Math. Soc." with Hong Kee Kim and Sung Hee Jang

An ideal I of a ring R is called *completely prime* if R/I is a domain.

Proposition 1. *Right (or left) duo rings are 2-primal.*

By Proposition 1 and the fact that right duo rings are right quasi-duo, we may raise the following question:

Question (1). Are right quasi-duo rings 2-primal?

But the answer is negative by the following Example 2.

Example 2. *There exists a right quasi-duo ring but not 2-primal.*

Proof. We take the ring R in [10, Example 1.1]. Let F be a field and let V be a infinite dimensional left vector space over F with $\{v_1, v_2, \dots\}$ a basis. For the endomorphism ring $A = End_F(V)$, define

$$A_1 = \{f \in A \mid \mathrm{rank}(f) < \infty \text{ and } f(v_i) = a_1v_1 + \cdots + a_iv_i \text{ for } i = 1, 2, \dots \text{ with } a_j \in F\}$$

and let R be the F-subalgebra of A generated by A_1 and 1_A. Let M be a maximal right ideal of R. Then M is of the form

$$M = \{r \in R \mid (i,i)-\text{entry of } r \text{ is } zero\}$$

for some $i \in \{1, 2, \dots\}$. But M is also a 2-sided ideal of R and so R is right quasi-duo. By [9, Proposition 6] the formal power series ring $R[[x]]$ over R is also right quasi-duo. However $R[[x]]$ is not 2-primal by the argument in [10, Example 1.1].

The ring $R[[x]]$ in Example 2 is not semiprimitive. We have an affirmative answer to Question (1) when given a ring is semiprimitive.

Proposition 3. *Semiprimitive right (or left) quasi-duo rings are reduced (hence 2-primal).*

Remark. The converse of Proposition 3 (i.e., for a semiprimitive ring R, is R right quasi-duo if R is reduced?) is not true in general by the following Example 9.

There is another condition under which the answer to Question (1) is affirmative. A subset of a ring is said to be *nil* if each element of it is nilpotent. The *index of nilpotency* of a nilpotent element x in a ring R is the least positive integer n such that $x^n = 0$. The *index of nilpotency* of a two-sided ideal I of R is the supremum of the indices of nilpotency of all nilpotent elements in I. If such a supremum is finite, then I is said to be *of bounded index of nilpotency*. By [6, Theorem 6], if R is a right quasi-duo ring of bounded index of nilpotency and the Jacobson radical of R is nil then R is 2-primal. Notice that for the ring $R[[x]]$ in Example 2, the preceding conditions are not satisfied.

A ring R is called *weakly right (left) duo* if for each a in R there exists a positive integer $n = n(a)$, depending on a, such that a^nR (Ra^n) is a two-sided ideal of R. Weakly right duo rings are abelian right quasi-duo rings by [14, Proposition 2.2].

Lemma 4. *Let R be a ring. Then we have the following statements:*

(1) *$R[[x]]$ is right quasi-duo if and only if R is right quasi-duo.*
(2) *If $R[[x]]$ is right duo then R is right duo.*
(3) *If $R[[x]]$ is weakly right duo then R is weakly right duo.*

Remark. We may obtain the same results for the left cases by replacing "right" by "left" in the preceding lemma and its proof.

The converse of (2) in Lemma 4 is not true in general by [5, Example 4]. So we raise following as the converse of (3).

Question (2). Are formal power series rings over weakly right duo rings also weakly right duo?

But the answer is negative by the following Example 5. Recall that a *right Ore domain* is a domain R in which every two nonzero elements have a nonzero common right multiple, i.e., for each nonzero $x, y \in R$ there exist $r, s \in R$ such that $xr = ys \neq 0$. Right Noetherian domains are right Ore by [4, Corollary 5.16].

Example 5. *There exists a weakly right duo ring such that the formal power series ring over it is not weakly right duo.*

Proof. We hire the method in [5, Example 4]. Let F be a field of characteristic zero, $A = F[y]$ be the polynomial ring over F. Define $\sigma : A \to A$ with $\sigma(y) = 1+y$, then σ is an automorphism of A. Let $B = A[z;\sigma]$ be the skew polynomial ring over A, subject to $az = z\sigma(a)$ for all $a \in A$. Then since A is right Noetherian and σ is an automorphism, B is a right Noetherian domain by [11, Theorem 1.2.9]. Hence B is also right Ore by the preceding argument and thus B is a right order in a division ring by [4, Theorem 5.17], say D is the division ring. Then D is a noncommutative division ring.

For each $i = 1, 2, \ldots$, let $D_i = D$ and $S = \prod D_i$ be the direct product of D_i's. Now define

$$R = \{(d_i) \in S \mid \text{ there exists } n \text{ such that } d_i = d_n \text{ for all } i \geq n\}.$$

Then R is a strongly regular ring and so it is right duo (hence weakly right duo). Let $e_j = (d_i) \in R$ such that $d_j = 1$ and $d_i = 0$ for $i \neq j$, and consider $f = ze_1 + z^2e_2x + z^3e_3x^2 + \cdots \in R[[x]]$. Then for any positive integer k, $f^k = z^ke_1 + z^{2k}e_2x^k + z^{3k}e_3x^{2k} + \cdots$. Assume that $R[[x]]$ is weakly right duo. Then for some positive integer k, $f^kR[[x]]$ is a 2-sided ideal of $R[[x]]$; hence for $g = (y, y, y, \ldots) \in R$ there exists $h \in R[[x]]$ such that $gf^k = f^kh$. Then

$$ye_nz^{nk}x^{(n-1)k} = e_ngf^k = e_nf^kh = z^{nk}e_nx^{(n-1)k}(e_nh)$$

and so $e_n h = e_n z^{-nk} y z^{nk} = e_n(nk+y)$. This implies that $h = (nk+y) \in S\backslash R$ because the characteristic of F is zero, a contradiction. Therefore $R[[x]]$ is not weakly right duo.

From [5, Theorem 4] we obtain conditions under which the converses in Lemma 4 may be true as in the following.

Theorem 6. *Suppose that R is a right self-injective von Neumann regular ring. Then the following statements are equivalent:*

(1) *R is right (left) duo.*
(2) *R is weakly right (left) duo.*
(3) *R is right (left) quasi-duo.*
(4) *R is a reduced ring.*
(5) *R is a 2-primal ring.*
(6) *$R[[x]]$ is right (left) duo.*
(7) *$R[[x]]$ is weakly right (left) duo.*
(8) *$R[[x]]$ is right (left) quasi-duo.*
(9) *$R[[x]]$ is a reduced ring.*
(10) *$R[[x]]$ is a 2-primal ring.*

Remark. The equivalences of (1), (2), (3), (4) and (5) in Theorem 6 hold only when R is von Neumann regular.

Now we consider a characterization of right quasi-duo rings, obtained from [9, Proposition 1] and the proof of [14, Proposition 2.1].

Lemma 7. *For a ring R, the followings are equivalent:*

(1) *R is right quasi-duo.*
(2) *Every right primitive factor ring of R is a division ring.*
(3) *$R/J(R)$ is right quasi-duo ($J(R)$ is the Jacobson radical of R).*

By Lemma 7, every right primitive factor ring of a right quasi-duo ring is Artinian. Whence we have the following useful result if given a ring is both 2-primal and right quasi-duo.

Proposition 8. *Suppose that R is a 2-primal right quasi-duo ring. Then the following statements are equivalent:*

(1) *R is strongly π-regular.*
(2) *R is π-regular.*
(3) *R is weakly π-regular.*
(4) *R is right weakly π-regular.*
(5) *$R/J(R)$ is right weakly π-regular and $J(R)$ is nil where $J(R)$ is the Jacobson radical of R.*
(6) *Every prime ideal of R is maximal.*

By this result, Question (1) and the following Question (3) may be meaningful. Recall that a ring R is called a *PI-ring* if R satisfies a polynomial identity with coefficients in the ring of integers. PI-rings are another generalization of commutative rings. As the converse of Question (1) one may ask the following.

Question (3). Are 2-primal rings right quasi-duo?

But the answer is also negative by the following example, although it is a semiprimitive PI-ring.

Example 9. *There exists a ring S such that*

(1) *S is a domain (hence reduced and so 2-primal),*
(2) *S is a semiprimitive PI-ring and*
(3) *S is not right quasi-duo.*

Proof. We take the ring $R[x]$ in [9, Example 9]. Let R be the Hamilton quaternion over the field of real numbers and $S = R[x]$ be the polynomial ring over R. Then clearly $R[x]$ is a domain and so it is semiprimitive by [12, Reproof of Amitsur's Theorem (2.5.23) after Lemma 2.3.41]. $R[x]$ is also a PI-ring becaues R is a PI-ring by [4, Prologue]. But $R[x]$ is not right quasi-duo by [9, Lemma 8].

As we see in the proof of Theorem 6, R is 2-primal if and only if R is right (left) duo if and only if R is weakly right (left) duo if and only if R is right (left) quasi-duo if and only if R is reduced, when R is a von Neumann regular ring.

Note that a semiprimitive right (or left) quasi-duo ring is a subdirect product of division rings, and if the polynomial ring over a ring R is right quasi-duo then $R/J(R)$ is commutative [9, Theorem 12] where $J(R)$ is the Jacobson radical of R; hence one may suspect that semiprimitive right quasi-duo rings are PI-rings. However the answer is negative because there is a division ring which is not a PI-ring. Recall that a ring R is right quasi-duo if the polynomial ring $R[x]$ over R is right quasi-duo. The preceding suspicion is affirmative when $R[x]$ is right quasi-duo by [9, Corollary 14]. However this argument does not hold in general for the formal power series rings. Let D be a noncommutative division ring, then D is semiprimitive right quasi-duo obviously and $D[[x]]$ is right quasi-duo by [9, Proposition 6]; but $D[[x]]$ is noncommutative.

Lastly we obtain similar results to [14, Proposition 2.1].

Proposition 10. *Let R, S be rings and ${}_RM_S$ be a (R,S)-bimodule. Let $E = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$. Then E is a right quasi-duo ring if and only if R and S are right quasi-duo rings.*

For a ring R, let $T(R,R) = \{(a,x) \mid a,x \in R\}$ with the addition componentwise and the multiplication defined by $(a_1,x_1)(a_2,x_2) = (a_1a_2, a_1x_2 + x_1a_2)$. Then $T(R,R)$ is a ring which is called the *trivial extension* of R by R. $T(R,R)$ is isomorphic to the ring of matrices $\begin{pmatrix} a & x \\ 0 & a \end{pmatrix}$ with $a, x \in R$.

Proposition 11. *For a ring R, $T(R, R)$ is a right quasi-duo ring if and only if R is a right quasi-duo ring.*

Acknowledgement. The author would like to express his gratitude to Professor Masahisa Sato for his giving the opportunity to present this material on Japan-Korea Ring Theory and Representation Theory Seminar joined with the 31st Japan Ring Theory and Representation Theory Symposium.

REFERENCES

1. H. Bass, *Finitistic dimension and a generalization of semiprimary rings*, Trans. Amer. Math. Soc. **95** (1960), 466–488.

2. G. F. Birkenmeier, H. E. Heatherly and E. K. Lee, *Completely prime ideals and associated radicals*, Proc. Biennial Ohio State-Denison Conference 1992, edited by S. K. Jain and S. T. Rizvi, World Scientific, Singapore-New Jersey-London-Hong Kong (1993), 102–129.

3. K. R. Goodearl, *Von Neumann Regular Rings*, Pitman, London (1979).

4. K. R. Goodearl and R. B. Warfield, Jr., *An introduction to noncommutative Noetherian rings*, Cambridge University Press, Cambridge (1989).

5. Y. Hirano, C. Y. Hong, J. Y. Kim and J. K. Park, *On strongly bounded rings and duo rings*, Comm. Algebra **23(6)** (1995), 2199–2214.

6. C. Y. Hong, N. K. Kim, T. K. Kwak and Y. Lee, *On weak π-regularity of rings whose prime ideals are maximal*, To appear in J. Pure Appl. Algebra.

7. N. K. Kim, T. K. Kwak and Y. Lee, *On extensions of 2-primal rings*, Preprint.

8. Y. Lee and C. Huh, *On rings in which every maximal one-sided ideal contains a maximal ideal*, To appear in Comm. Algebra.

9. Y. Lee and C. Huh, *Some results on quasi-duo rings*, Submitted.

10. Y. Lee, C. Huh and H. K. Kim, *Questions on 2-primal rings*, Comm. Algebra **26(2)** (1998), 595–600.

11. J. C. McConnell and J. C. Robson, *Noncommutative Noetherian rings*, John Wiley & Sons Ltd. (1987).

12. L. H. Rowen, *Ring Theory*, Academic Press, Inc. (1991).

13. G. Shin, *Prime ideals and sheaf representation of a pseudo symmetric rings*, Trans. Amer. Math. Soc. **184** (1973), 43–60.

14. H.-P. Yu, *On quasi-duo rings*, Glasgow Math. J. **37** (1995), 21–31.

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SOME RESULTS ON SKEW POLYNOMIAL RINGS OVER A REDUCED RING

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Throughout this paper, all rings are associative with unity. A ring R is called *(quasi-) Baer* if the right annihilator of every ((right) ideal)nonempty subset of R is generated by an idempotent. In [3], a ring R is called a *right(resp. left)principally quasi-Baer* (or simply *right(resp. left) p.q.-Baer*) if the right(resp. left) annihilator of a principal right(resp. left) ideal is generated by an idempotent. A ring R is called a *p.q.-Baer* ring if it is both right and left p.q.-Baer. Another generalization of Baer ring is the p.p.-ring. A ring R is called a *right(resp. left) p.p.*- ring if the right(resp. left) annihilator of an element of R is generated by an idempotent. Also, a ring R is called a *p.p.*-ring if it is both right and left p.p.

In [3], the following fact was proved;

Proposition 1. *The following are equivalent;*

(1) *R is a right p.q.-Baer ring.*
(2) *The right annihilator of any finitely generated right ideal is generated (as a right ideal) by an idempotent.*
(3) *The right annihilator of every principal right ideal is generated (as a right ideal) by an idempotent.*
(4) *The right annihilator of every finitely generated ideal is generated (as a right ideal) by an idempotent.*

Note that this statement is true if "right" is replaced by "left" throughout.

In [3], they also have shown the following results;

Theorem A. *R is a right(resp. left) p.q.-Baer ring if and only if the polynomial ring $R[x]$ is a right(resp. left) p.q.- Baer ring.*

Theorem B. *For a ring R, the following are equivalent;*

(1) *R is a quasi-Baer ring;*
(2) *the polynomial ring $R[x]$ over R is a quasi-Baer ring;*
(3) *the formal power series ring $R[[x]]$ over R is a quasi-Baer ring.*

This is a part of the paper to submit in Comm. in Algebra with Jun Cheol Han and Yasuyuki Hirano

Now we try to apply those results for Ore extension and so we recalled the following :

Let α be an endomorphism of a ring R. An $\alpha-$*derivation* of R is an additive map $\delta : R \longmapsto R$ such that $\delta(ab) = \alpha(a)\delta(b) + \delta(a)b$ for all $a, b \in R$. The Ore extension $R[x;\alpha,\delta]$ is the ring of polynomials in x over R with the usual addition and with new multiplication by $xa = \alpha(a)x + \delta(a)$ for each $a \in R$. If $\delta = 0$, we write $R[x;\alpha]$ for $R[x;\alpha,\delta]$ and is called an *Ore extension of endomorphism type*(also called a *skew polynomial* ring). While if $\alpha = 1$, we write $R[x;\delta]$ for $R[x;1,\delta]$ and is called an *Ore extension of derivation type*(also called a *differential polynomial* ring).

Also, we recalled that R is called a *reduced* ring if it has no nonzero nilpotent elements and R is called an *abelian* ring if every idempotent of R is central. We can observe easily that every reduced ring is abelian and in a reduced ring R left and right annihilaters coincide for any subset U of R, where a left (right) annihilator of U is denoted by $l_R(U) = \{a \in R \mid aU = 0\}(r_R(U) = \{a \in R \mid Ua = 0\})$.

Of course, for a reduced ring R, the following are equivalent clearly;

(1) R is a right p.p.-ring.
(2) R is a left p.p.-ring.
(3) R is a right p.q.-Baer ring.
(4) R is a left p.q.-Baer ring.

We have well- known fact:

Theorem 2. *Let R be an integral domain with a monomorphism α. Then the skew polynomial ring $R[x;\alpha]$ is an integral domain.*

In this case, the skew polynomial ring $R[x;\alpha]$ is a Baer ring. Also, we have the following well-known fact:

Theorem 3. *Let α be an inner automorphism of a ring R induced by an invertible element c(i.e. $\alpha(r) = c^{-1}rc$ for all $r \in R$) and $R[x;\alpha]$ the Ore extension of automorphism type.*

Then the polynomial ring $R[x]$ is isomorphic to $R[x;\alpha]$.

In this case, the skew polynomial ring $R[x;\alpha]$ is a Baer ring by Corollary 2.7[3].

Example 4.

(1) *The ring $R = Z_2[x]/(x^2)$ is not a quasi-Baer, where Z_2 is the field of two elements and (x^2) is the ideal of the ring $Z_2[x^2]$ generated by x^2. In fact, $l_R(R(x+(x^2))$ is not generated by an idempotent of R.*

But since $R[y;\delta] \simeq Mat_2(Z_2[y^2])$, where a derivation δ is defined by $\delta(x + (x^2)) = 1 + (x^2)$, $R[y;\delta]$ is a quasi-Baer because $Z_2[y^2]$ is a quasi-Baer and so $Mat_2(Z_2[y^2])$ is also a quasi-Baer.

(2) *Let F be a field and $R = F[t]$ a polynomial ring over F with the endomorphism α given by $\alpha(f(t)) = f(0)$ for all $f(t) \in R$. Then R is a principal ideal domain but the skew polynomial ring $R[x;\alpha]$ is not an integral domain because $xt = \alpha(t)x = 0$. Also, the skew polynomial ring $R[x;\alpha]$ is neither a right p.q.-Baer nor a right p.p.-ring.*

In [1], Armendariz proved that if R is a reduced ring, then R is a p.p.(resp. Baer)-ring if and only if the polynomial ring $R[x]$ is a p.p.(resp. Baer)-ring. We will generalize this result by showing that if R is a reduced ring with a monomorpism α and $\alpha(P) \subseteq P$ for any minimal prime ideal P in R, then R is a p.p.(resp. Baer)-ring if and only if the skew polynomial ring $R[x;\alpha]$ is a p.p.(resp. Baer)-ring. Based on these facts, we have the following;

Lemma 5. *Let R be a reduced ring. Then for all a, b,c, and $d \in R$,*

(1) *$ab = 0$ if and only if $ba = 0$;*
(2) *If $ab = 0$ and $cb + ad = 0$, then $cb = ad = 0$;*

Proposition 6. *If S is a multiplicative subset(i.e. $a,b \in S$ implies $ab \in S$) of a ring R which is disjoint from an ideal K of R, then there exists an ideal P which is maximal in the set of all ideals of R disjoint from S and containing K. Futhermore any such ideal P is a prime ideal.*

Lemma 7. *Let R be a reduced ring with a monomorpism α. Assume that $\alpha(P) \subseteq P$ for any minimal prime ideal P in R. Then, for each $a,b \in R$, $ab = 0$ if and only if $a\alpha^k(b) = 0$ for $k = 1,2,\ldots$.*

Proposition 8. *Let R be a reduced ring with a monomorpism α and let f and $g \in R[x;\alpha]$ with $f = \sum_{i=0}^{n} a_i x^i, g = \sum_{i=0}^{m} b_i x^i$. Assume that $\alpha(P) \subseteq P$ for any minimal prime ideal P in R. Then $fg = 0$ if and only if $a_i b_j = 0$ for all i and j $(0 \le i \le n, 0 \le j \le m)$.*

Corollary 9. *Let R be a reduced ring with a monomorpism α. Assume that $\alpha(P) \subseteq P$ for any minimal prime ideal P in R. Then R is a reduced ring if and only if $R[x;\alpha]$ is a reduced ring.*

Corollary 10. *Let R be a reduced ring with a monomorpism α. Assume that $\alpha(P) \subseteq P$ for any minimal prime ideal P in R. If $f \in R[x;\alpha]$ is an idempotent, then $f \in R$, that is, every idempotent of $R[x;\alpha]$ is an idempotent of R.*

Corollary 11. *Let R be a reduced ring with a monomorpism α. Assume that $\alpha(P) \subseteq P$ for any minimal prime ideal P in R. If $T \subseteq R[X;\alpha]$ and $S_f = \{a_0, a_1, \ldots, a_n\}$, where $f = a_0 + a_1x + \cdots + a_nx^n \in T$, then $r_{R[x;\alpha]}(T) = r_R(S_T)[x;\alpha]$, where $S_T = \cup_{f \in T} S_f$.*

Theorem 12. *Let R be a reduced ring with a monomorpism α. Assume that $\alpha(P) \subseteq P$ for any minimal prime ideal P in R. Then $R[x;\alpha]$ is a p.p.-ring if and only if R is a p.p.-ring,* Since a p.p.-ring is equivalent to a p.q.-Baer ring for a reduced ring, we have the following;

Corollary 13. *Let R be a reduced ring with a monomorpism α. Assume that $\alpha(P) \subseteq P$ for any minimal prime ideal P in R. Then $R[x;\alpha]$ is a p.q.-Baer ring if and only if R is a p.q.-Baer ring.*

Similarly we can also have the following Theorem:

Theorem 14. *Let R be a reduced ring with a monomorpism α. Assume that $\alpha(P) \subseteq P$ for any minimal prime ideal P in R. Then $R[x;\alpha]$ is a Baer ring if and only if R is a Baer ring.*

Theorem 12 and 14 extend Armendariz's results[1, Theorem A and B] if α is the identity. Also, for a reduced ring R, the following are equivalent clearly;

(1) R is a Baer ring.
(2) R is a quasi-Baer ring.

Hence we have the following;

Corollary 15. *Let R be a reduced ring with a monomorpism α. Assume that $\alpha(P) \subseteq P$ for any minimal prime ideal P in R. Then $R[x;\alpha]$ is a quasi-Baer ring if and only if R is a quasi-Baer ring.*

All results in this paper does not hold if the endomorphism α of a reduced ring R is not a monomorphism even though $\alpha(P) \subseteq P$ for any minimal prime ideal P in R.

For an example, let F be a field and $R = F[[t]]$ the formal power series ring over F with the endomorphism α given by $\alpha(f(t)) = f(0)$ for all $f(t) \in R$. In this case, $R = F[[t]]$ is a P. I. D. and so R is a Baer ring and also (0) is a uniqe minimal prime ideal. Since $\alpha(0) = (0)$, the assumption that $\alpha(P) \subseteq P$ for any minimal prime ideal P in R is satisfied. But $R[x;\alpha]$ is neither a right p.q.-Baer nor a right p.p.-ring.

Furthermore, there exists an example such that if R is a reduced ring with an automorpism α and R is a Baer ring, then $R[x;\alpha]$ is not a p.p.-ring.

Example 16. *Let $R = Z_{10} \times Z_{10}$ with an automorphism α given by $\alpha(a,b) = (b,a)$ for all $(a,b) \in R$. Then R is a reduced and Baer ring. But the skew polynomial ring $R[x;\alpha]$ is not a p.p.-ring.*

References

[1] E. P. Armendariz, *A note on extensions of Baer and p.p. rings*, Australian Math. Soc. **18** (1974), 470 – 473.

[2] E. P. Armendariz, H. K. Koo and J. K. Park, *Isomorphic Ore extensions*, Comm. in Algebra 15(12) (1987), 2633 – 2652.

[3] G. F. Birkenmeier, J. Y. Kim and J. K. Park, *On extensions of quasi-Baer and principally quasi-Baer rings, preprint.*

[4] K. R. Goodearl, and R. B. Warfield, JR., *An Introduction to Noncommutative Noetherian Rings*, Cambridge University Press, Cambridge. 1989.

[5] J. C. McConnell, and J. C. Robson, *Noncommutative Noetherian rings*, John Wiley & Sons Ltd. 1987.

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ON EXCHANGE RINGS

NAM KYUN KIM

Throughout this paper, R denotes an associative ring with identity and all modules are unitary. Let M_R be a right R-module. Following Crawley and Jónsson [4], M_R is said to have the *exchange property* if for every module A_R and any two decompositions of A_R

$$A_R = M'_R \oplus N_R = \bigoplus_{i \in I} A_i$$

with $M'_R \cong M_R$, there exists submodules $A'_i \subseteq A_i$ such that

$$A_R = M' \oplus (\bigoplus_{i \in I} A'_i).$$

M_R is said to have the *finite exchange property* if the above condition is satisfied whenever the index set I is finite.

Warfield [10] introduced the class of exchange rings. He called a ring R an *exchange ring* if R_R has the exchange property.

Nicholson [9, Theorem 2.1] proved that R is an exchange ring if and only if the n-by-n full matrix ring $\mathrm{Mat}_n(R)$ over R is an exchange ring. But, in general a subring of an gexchange ring is not an exchange ring. For example, $\mathbb{Q}$, the field of all rational numbers, is an exchange ring but the subring $\mathbb{Z}$, the integer of integers, is not an exchange ring. So we may suspect that the n-by-n upper (or lower) triangular matrix ring over R is an exchange ring.

Proposition 1. The following statements are equivalent:

(1) R is an exchange ring.

(2) Every upper triangular matrix ring (finite or infinite) over R is an exchange ring.

(3) Every lower triangular matrix ring (finite or infinite) over R is an exchange ring.

Recall that a ring R is called to be *strongly π-regular* if for any $a \in R$, there exists a positive integer n, depending on a, such that $a^n = a^{n+1}x$ for some $x \in R$. In case $n = 1$, a ring R is called to be *strongly regular*. A ring R is called to be *π-regular* if for

This is a part of the joint paper "A note on exchange rings" with Chan Yong Hong, Tai Keun Kwak and Yang Lee

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

any $a \in R$, there exists a positive integer n, depending on a, such that $a^n = a^n x a^n$ for some $x \in R$. In case $n = 1$, a ring R is called to be *von Neumann regular*.

The following first example shows that there exists an exchange ring whose primitive factor rings are Artinian but not π-regular and hence it is not von Neumann regular. The second example shows that there exists a semiprimitive exchange ring whose primitive factor rings are Artinian but not von Neumann regular.

Example 2. (1)[12, Example 3.3] Let p be a prime number and let $R = \mathbb{Z}_{(p)}$, the localization of integers at (p). Then R is a commutative (hence all primitive factor rings are Artinian) exchange ring since R is a local ring. If R is π-regular, then $J(R)$ is nil. However $J(\mathbb{Z}_{(p)})$ is not nil. Therefore R is not π-regular.

(2)[9, Example 1.7] Let $\mathbb{Q}$ be the field of all rationals and S the ring of all rationals with odd denominators. Let

$$R = \{< a_i > \in \prod_{i=1}^{\infty} \mathbb{Q}_i \mid a_i \text{ is eventually in } S\},$$

where $\mathbb{Q}_i = \mathbb{Q}$ for all i. Since $S/J(S)$ is a division ring, it follows that R is a commutative semiprimitive ring. Moreover R is an exchange ring. But, since $J(S) \neq 0$, S is not von Neumann regular. Note that S is a homomorphic image of R. Therefore R is not von Neumann regular.

(3) There exists a semiprimitive ring whose primitive factor rings are von Neumann regular but not exchange. For example, in $\mathbb{Z}$, (0) is not a primitive ideal. But every prime ideal of $\mathbb{Z}$ is primitive. So every primitive factor ring of $\mathbb{Z}$ is a field and hence von Neumann regular. Moreover $J(\mathbb{Z}) = 0$. However $\mathbb{Z}$ is not exchange.

A ring R is called *homomorphically semiprimitive* if every ring homomorphic image (including R) of R has zero Jacobson radical. Von Neumann regular rings are clearly homomorphically semiprimitive. But the converse is not true in general, for example, let $R = W_1[F]$ be the first Weyl algebra over a field F of characteristic zero, then R is simple domain which is not division. So R is homomorphically semiprimtive. But R is not von Neumann regular. Recently, Yu proved [12, Theorem 3.6] the following: Let R be an exchange ring whose primitive factor rings are Artinian. If $R/J(R)$ is homomorphically semiprimitive, then $R/J(R)$ is strongly π-regular. But we have the following theorem which generalizes Yu's result and [5, Theorem 1].

Theorem 3. Suppose that R is an exchange ring whose primitive factor rings are Artinian. If R is homomorphically semiprimitive, then R is von Neumann regular. In particular, R is unit-regular.

In Theorem 3, the condition "homomophically semiprimitive" is not superfluous by Example 2(2). Also the condition "primitive factor rings are Artinian" is not superfluous. Let V be an infinite vector space of a field F. Then $R = \text{End}(V)$ is von Neumann regular and so is homomorphically semiprimitive exchange. While R is primitive but not Artinian. In fact, R is unit-regular if and only if V is finite dimensional.

Corollary 4.[5, Theorem 1] Suppose that R is a von Neumann regular ring whose primitive factor rings are Artinian. Then R is unit-regular.

Recall that a ring R is called *right quasi-duo* if every maximal right ideal of R is an ideal. Right primitive right quasi-duo rings are division rings. This implies that all primitive factor rings of a right quasi-duo ring are Artinian. So we have the following which generalize results [2, Proposition 3.2] and [12, Theorem 3.8]. A ring R is called to be *biregular* if every principal two-sided ideal of R is generated by a nonzero central idempotent.

Proposition 5. Suppose that R is a right quasi-duo exchange ring. Then the following statements are equivalent:

(1) R is strongly regular.

(2) R is von Neumann regular.

(3) R is biregular.

(4) R is homomorphically semiprimitive.

The following result extends a result [6, Theorem 3.2].

Proposition 6. Suppose that R is a homomorphically semiprimitive exchange ring. Then the following statements are equivalent:

(1) R is abelian

(2) R is reduced.

(3) R/P is a division ring for all prime ideals P of R.

(4) R is strongly regular.

(5) Every nonzero right ideal of R contains a nonzero central idempotent.

Recall that a ring R is called to be *right (left)weakly π-regular* if for any $a \in R$, there exists a positive integer n, depending on a, such that $a^nR = a^nRa^nR$. In case $n = 1$, a ring R is called to be *right (left) weakly regular*.

Proposition 7. Let R be an abelian exchange ring. The following statements are equivalent.

(1) R is strongly π-regular.

(2) R is π-regular.

(3) R is right weakly π-regular.

(4) $R/J(R)$ is right weakly π-regular and $J(R)$ is nil.

Recall that for a ring R with a ring endomorphism $\alpha : R \to R$, the *skew polynomial ring* $R[x;\alpha]$ of R is the ring obtained by giving the polynomial ring over R with the new multiplication $xr = \alpha(r)x$ for all $r \in R$. While $R[[x;\alpha]]$ is called the *skew power series ring.*

Proposition 8. If the skew polynomial ring $R[x;\alpha]$ of R is an exchange ring, then R is an exchange ring.

As a corollary of Proposition 8, we have the following.

Corollary 9. If the polynomial ring $R[x]$ of R is an exchange ring, then R is an exchange ring.

The converse of Corollary 9 is not true in general. For example, let F be a field. Then clearly F is an exchange ring. For $x \in F[x]$, $xF[x]$ contains the only idempotent 0. If $F[x]$ is exchange, then $1 \in (1-x)F[x]$ and so $1-x$ is invertible in $F[x]$, which is a contradiction. Therefore $F[x]$ is not exchange.

Recall that a two-sided ideal I of a ring R is called an exchange ring without identity if there exists $e^2 = e \in Ix$ such that $1-e \in (1-x)R$ for any $x \in I$.

Proposition 10. Let R be an exchange ring. Then $R[x]/(x^{n+1})$ is an exchange ring for any $n \geq 1$.

However, for formal (skew) power series ring over an exchange ring, we have the following result.

Theorem 11. For a ring R, the following statements are equivalent:

(1) R is an exchange ring.

(2) The formal power series ring $R[[x]]$ of R is an exchange ring.

(3) The skew power series ring $R[[x;\alpha]]$ of R is an exchange ring.

The following examples show that the center $C(R)$ of R is exchange but R is not exchange.

Example 12. (1) Let $F(x)$ be the rational functions over a field F,

$$K = \begin{bmatrix} F(x) & F(x) \\ F(x) & F(x) \end{bmatrix} \text{ and } L = \begin{bmatrix} F[x] & F(x) \\ 0 & F \end{bmatrix}.$$

Consider the ring

$$R = \{< a_n >_{i=1}^{\infty} \mid a_n \in K \text{ and } a_n \text{ is eventually constant in } L\}.$$

Noet that R is a semiprime PI-ring with von Neumann regular (hence exchange) center. Also note that R is not exchange.

(2) Let D be a division ring and $S = \prod_{i=1}^{\infty} D_i$, where $D_i = D$ for all i. Define $\alpha : S \to S$ by $\alpha(a_1, a_2, \dots) = (a_1, a_1, a_2, \dots)$. Then α is injective but not onto. Let $R = S[x; \alpha]$ be a skew polynomial ring. Then $C(R) = \{(a, a, \dots) \mid a \in D\} \cong D$ is exchange. However 0 is the only idempotent contained in xR and $1 \notin (1 - x)R$. Therefore R is not exchange.

(3) Let $W[\mathbb{Q}]$ be the 1st Weyl Algebra over $\mathbb{Q}$. Consider $R = xW_1[\mathbb{Q}] * \mathbb{Q}$. with $(f, r)(g, s) = (fg + sf + rg, rs)$. Then R is a domain and $C(R) \cong \mathbb{Q}$ is exchange. But 0 is the only idempotent contained in $(x, 0)R$ and $1 \notin (1_R - (x, 0))R = (-x, 1)R$. Therefore R is not exchange.

Proposition 13. Let R be a reduced PI-ring. If $C(R)$ is semiprimitive exchange, then R is a semiprimitive I-ring.

Acknowledgement. The author would like to express his gratitude to Professor Masahisa Sato for his giving the opportunity to present this material on the Japan-Korea Ring Theory and Representation Theory Seminar jointed with the 31st Japan Ring Theory and Representation Theory Symposium.

References

1. P. Ara, *Extensions of exchange rings*, J. Algebra **197** (1997), 409–423.
2. W.D. Burgess and W. Stephenson, *Rings all of whose Pierce stalks are local*, Canad. Math. Bull. **22(2)** (1979), 159–164.
3. V.P. Camillo and H.-P. Yu, *Exchange rings, units and idempotents*, Comm. Algebra **22(12)** (1994), 4737–4749.
4. P. Crawley and B. Jónsson, *Refinements for infinite direct decompositions of algebraic systems*, Pacific J. math. **14** (1964), 797–855.
5. J.W. Fisher and R.L. Snider, *Rings generated by their units*, J. Algebra **42** (1976), 363–368.
6. K.R. Goodearl, *On von Neumann regular rings*, Pitman Publishing Limited, New York, 1979..
7. C. Y. Hong, N. K. Kim, T. K. Kwak and Y. Lee, *On weak π-regularity of rings whose prime ideals are maximal*, J. Pure and Applied Algebra, to appear.
8. P. Menal, *On π-regular rings whose primitive factor rings are Artinian*, J. Pure and Applied Algebra **20** (1981), 71–78.
9. W.K. Nicholson, *Lifting idempotents and exchange rings*, Trans. Amer. math. Soc. **229** (1977), 269–278.
10. R.B. Warfield, Jr., *Exchange rings and decompositions of modules*, Math. Ann. **199** (1972), 31–36.
11. H.-P. Yu, *On quasi-duo rings*, Glasgow Math. J. **37** (1995), 21–31.
12. H.-P. Yu, *On the structure of exchange rings*, Comm. Algebra **25(2)** (1997), 661–670.

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SOME CHARACTERIZATIONS OF 2-PRIMAL RINGS

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Throughout this note R denotes an associative ring with identity and all prime ideals of R are assumed to be proper.

Birkenmeier, Heatherly and Lee [1] called a ring R *2-primal* if its prime radical $\mathbf{P}(R)$ coincides with the set $\mathrm{N}(R)$ of all nilpotent elements of R. Note that commutative rings and reduced rings (i.e., *rings without nilpotent elements*) is a 2-primal ring.

Historically, some of the earliest results known to us about 2-primal rings (although not so called at the time) and prime ideals were due to Shin [10]. He proved that a ring R is 2-primal if and only if every minimal prime ideal of R is completely prime. Hirano [5] considered the 2-primal condition in the context of strongly π-regular rings. He used the term *N-ring* for what we call a 2-primal ring. The 2-primal condition was taken up independently by Sun [11], where in the setting of rings with identity he introduced a condition to be called *weakly symmetric*, which is equivalent to the 2-primal condition for rings. Sun [11] showed that if R is weakly symmetric, then each minimal prime ideal of R is a completely prime ideal, and that the ring of n-by n upper triangular matrices over R inherits the weakly symmetric condition. The name 2-primal rings originally came from the context of left near rings by Birkenmeier, Heatherly and Lee [1].

Following Birkenmeier, Kim and Park [2], Koh [7] and Shin [10], for a prime ideal P of a ring R, we put

$$\begin{aligned} O(P) &= \{a \in R \mid aRb = 0 \text{ for some } b \in R\backslash P\}, \\ O_P &= \{a \in R \mid ab = 0 \text{ for some } b \in R\backslash P\}, \\ \overline{O}_P &= \{a \in R \mid a^m b = 0 \text{ for some positive integer } m \text{ and some } b \in R\backslash P\}, \\ N(P) &= \{a \in R \mid aRb \subseteq \mathbf{P}(R) \text{ for some } b \in R\backslash P\}, \text{ and} \\ N_P &= \{a \in R \mid ab \in \mathbf{P}(R) \text{ for some } b \in R\backslash P\}. \end{aligned}$$

It can be easily checked that for each prime ideal P of R, $O(P)$ and $N(P)$ are two-sided ideals, but $O_P, \overline{O}_P$, and N_P are not one-sided ideals of R. Also $O(P)$ and $N(P)$ are subsets of P, $O(P) \subseteq O_P \subseteq \overline{O}_P$ and $N(P) \subseteq N_P$.

Recall that a two-sided ideal P of R is *completely prime* (*completely semiprime*) if $ab \in P$ implies $a \in P$ or $b \in P$ (if $a^2 \in P$ implies $a \in P$) for $a, b \in R$. Note that if P is a completely prime ideal of R, then $\overline{O}_P$ is a subset of P.

We use $\mathbf{P}(R)$, $\mathrm{N}(R)$, and (m)Spec(R) to represent the prime radical, the set of all nilpotent elements, and the set of all (minimal) prime ideals of R, respectively.

1991 Mathematics Subject Classification: 16D30, 16S70.
Key words and phrases: 2-primal rings, minimal prime ideals, insertion of factors property.

Koh [7] proved the following: Let R be a reduced ring.

(1) [Proposition 2.3] If $P \in \mathrm{Spec}(R)$, then O_P is a ideal and $O_P \subseteq P$.
(2) [Theorem 2.4] $P \in \mathrm{mSpec}(R)$ if and only if $P = O_P$ and in this case, P is a completely prime ideal.

Recall from Lambek [8], a one-sided ideal I of R is called *right* (*left*) *symmetric* if $xyz \in I$ implies $xzy \in I$ ($yxz \in I$). We shall say that R is *symmetric* whenever (0) has the corresponding property.

Shin [10] called a ring R *almost symmetric* if it satisfies;

(S I) for each $a \in R$, $r(a)$ is a two-sided ideal of R, where $r(a) = \{b \in R \mid ab = 0\}$, and
(S II) for $a, b, c \in R$, if $a(bc)^n = 0$ for a positive integer n, then $ab^m c^m = 0$ for some positive integer m.

Also, he showed the following:

(1) [Lemma 1.1] Any reduced ring is symmetric.
(2) [Lemma 1.2] R satisfies (S I) if and only if $ab = 0$ implies $aRb = 0$ for any $a, b \in R$.
(3) [Proposition 1.4] Any symmetric ring is almost symmetric.
(4) [Theorem 1.5] If R satisfies (S I) then R is 2-primal.
(5) [Corollary 1.10] If R is a 2-primal ring, then $P \in \mathrm{mSpec}(R)$ if and only if $P = N(P)$ if and only if for any $a \in P$, ab is nilpotent for some $b \in R \backslash P$.
(6) [Proposition 1.11] R is a 2-primal ring if and only if every minimal prime ideal is completely prime.

The converse of [10, Theorem 1.5] does not hold by the following example.

Example 1. Let $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$, where F is a field. Then $\mathrm{P}(R) = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix} = \mathrm{N}(R)$, so R is 2-primal, but it does not satisfy (S I).

Now, as the parallel definition to $\overline{O}_P$, we define the following.

Definition 2. $\overline{N}_P = \{a \in R \mid a^m b \in \mathrm{P}(R)$ for some positive integer m and some $b \in R \backslash P\}$ for $P \in \mathrm{mSpec}(R)$.

Note that for $P \in \mathrm{mSpec}(R)$, $\overline{N}_P$ is not a one-sided ideal of R and $\overline{O}_P, N_P \subseteq \overline{N}_P$. Moreover, if P is a completely prime ideal of R, then $\overline{N}_P$ is a subset of P.

With the above facts, we characterize 2-primal rings. To characterize a 2-primal ring, we consider a useful property which has been profitable in the study of near-rings.

Recall from Mason [9], a one-sided ideal I of R has the *insertion of factors property* (or simply, IFP) if $xy \in I$ implies $xRy \subseteq I$ for $x, y \in R$. Observe that every completely semiprime ideal of R has the IFP.

The following results might be helpful for the criterion for a certain class of rings to be 2-primal.

Theorem 3. *The following statements are equivalent:*

(a) *R is a 2-primal ring.*

(b) $\mathbf{P}(R)$ *has the IFP.*
(c) $N(P)$ *has the IFP for each* $P \in mSpec(R)$.
(d) $N(P) = \overline{N}_P$ *for each* $P \in mSpec(R)$.
(e) $N(P) = N_P$ *for each* $P \in mSpec(R)$.
(f) $N_P \subseteq P$ *for each* $P \in mSpec(R)$.
(g) $N_{P/\mathbf{P}(R)} \subseteq P/\mathbf{P}(R)$ *for each* $P \in mSpec(R)$.

In Theorem 3, "$P \in$ mSpec(R)" can be replaced by "$P \in$ Spec(R)".

Note that if R satisfies (S I), then $O_P = O(P)$ has the IFP for each $P \in$ Spec(R). On the other hand, we have the following result by using Theorem 3.

Proposition 4. *We have the following:*

(a) *If* $O(P)$ *has the IFP for each* $P \in mSpec(R)$, *then* R *is a 2-primal ring.*
(b) *If* R *is a 2-primal ring and* $O_P = P$ *for some* $P \in Spec(R)$, *then* P *is a completely prime ideal of* R, *in particular,* O_P *has the IFP.*

The converse of Proposition 4(a) does not hold. Moreover, the condition "$O_P = P$" in Proposition 4(b) is not superfluous by the following example.

Example 5. In Example 1, the ring $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ is 2-primal and $P = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}$ is a minimal prime ideal of R. Then $O(P) = 0$ and $O(P)$ does not have the IFP. Furthermore, $O_P \neq P$ and O_P is not a one-sided ideal of R.

In [6], Hirano, Huynh and Park showed the following: Let R be a ring, then its prime radical $\mathbf{P}(R)$ contains all nilpotent elements of index two if and only if for any $a, b \in R$ with $xy = 0$, it holds $yRx \subseteq \mathbf{P}(R)$. Observe that if R is a 2-primal ring, then $\mathbf{P}(R)$ contains all nilpotent elements of index two. However, in general, the converse is not true.

Proposition 6. *The following statements are equivalent:*

(a) R *is a 2-primal ring.*
(b) $\mathbf{P}(R)$ *is a completely semiprime ideal of* R.
(c) $\mathbf{P}(R)$ *is a left and right symmetric ideal of* R.
(d) $xy \in \mathbf{P}(R)$ *implies* $yRx \subseteq \mathbf{P}(R)$ *for* $x, y \in R$.
(e) $N(P)$ *is a completely semiprime ideal of* R *for each* $P \in mSpec(R)$.
(f) $N(P)$ *is a left and right symmetric ideal of* R *for each* $P \in mSpec(R)$.
(g) $xy \in N(P)$ *implies* $yRx \subseteq N(P)$ *for* $x, y \in R$ *and for each* $P \in mSpec(R)$.

Recently, Birkenmeier, Kim and Park [3] showed the following:
(1) [Proposition 1.2] Let R be a 2-primal ring and P a prime ideal of R. Then
(i) $\overline{O}_P \subseteq N_P \subseteq P$;
(ii) $\mathbf{N}(R) = \bigcap_{P \in \mathrm{Spec}(R)} \overline{O}_P = \mathbf{P}(R)$.

(2) [Definition 1.3] Let $x, y \in R$ and n a positive integer. We say R *satisfies* the condition (CZ 1) if whenever $(xy)^n = 0$ then $x^m y^m = 0$ for some positive integer m. Observe that if R satisfies (S II), then R satisfies the condition (CZ 1).

(3) [Lemma 2.1] Let P be a prime ideal of R.

(i) If $P = \overline{O}_P$, then P is completely prime. In particular, If $P = \overline{O}_P$ for every $P \in \mathrm{mSpec}(R)$ then R is a 2-primal ring.

(ii) If $\overline{O}_P = P$ and R is a 2-primal ring, then P is a minimal prime ideal of R which is completely prime.

(4) [Theorem 2.3] Let R be a 2-primal ring and P a prime ideal of R. If R satisfies (CZ 1), then P is a minimal prime ideal of R if and only if $P = \overline{O}_P$.

However, we have the following results for 2-primal rings.

Theorem 7. *The following statements are equivalent:*

(a) *R is a 2-primal ring.*

(b) *$\overline{O}_P \subseteq P$ for each $P \in mSpec(R)$.*

(c) $\mathbf{N}(R) = \bigcap_{P \in \mathrm{mSpec}(R)} \overline{O}_P = \mathbf{P}(R)$.

Recall that R is a 2-primal ring if and only if every minimal prime ideal of R is completely prime.

Corollary 8. *Assume that $\overline{O}_P = P$ for each $P \in Spec(R)$. Then we have the following:*

(a) *R is a 2-primal ring.*

(b) *$\overline{O}_P = N(P)$ for each $P \in Spec(R)$.*

(c) *Every prime ideal of R is minimal and completely prime.*

In Corollary 8, even if we replace "$P \in \mathrm{Spec}(R)$" by "$P \in \mathrm{mSpec}(R)$", we have the same results.

In [4], Camillo and Xiao showed the following: There exists a simple domain which is not right Ore. Related to [3, Theorem 2.3], we shall show that there exists a 2-primal ring R with $\overline{O}_P \neq P$ for some minimal prime ideal P of R, even if every prime ideal of R is maximal. Moreover, R does not satisfy (CZ 1).

Example 9. By [4, Theorem 17], there exists a simple domain D which is not right Ore. Hence we let $R = \begin{bmatrix} D & D \\ 0 & D \end{bmatrix}$. Then R is a 2-primal ring which does not satisfy (CZ 1). Furthermore, every prime ideal of R is maximal. Now, we consider $P = \begin{bmatrix} D & D \\ 0 & 0 \end{bmatrix}$. Then $\overline{O}_P \neq P$.

Acknowledgement. The authors would like to express their gratitude to Professor Masahisa Sato and Professor Jae Keol Park for their giving the opportunity to present this material on the Japan-Korea Ring Theory and Representation Theory Seminar jointed with the 31st Japan Ring Theory and Representation Theory Symposium.

References

1. G.F. Birkenmeier, H.E. Heatherly and E.K. Lee, *Completely prime ideals and associated radicals*, Proc. Biennial Ohio State-Denison Conference 1992, edited by S.K. Jain and S.T. Rizvi, World Scientific, New Jersey (1993), 102-129.
2. G.F. Birkenmeier, J.Y. Kim and J.K. Park, *Regularity conditions and the simplicity of prime factor rings*, J. Pure and Appl. Algebra **115** (1997), 213-230.
3. G.F. Birkenmeier, J.Y. Kim and J.K. Park, *A characterization of minimal prime ideals*, Glassgow Math. J. **40** (1998), 223-236.
4. V. Camillo and Y. Xiao, *Weakly regular rings*, Comm. Algebra **22**(10) (1994), 4095-4112.
5. Y. Hirano, *Some studies on strongly π-regular rings*, Math. J. Okayama Univ. **20** (1978), 141-149.
6. Y. Hirano, D.V. Huynh and J.K. Park, *On rings whose prime radical contains all nilpotent elements of index two*, Arch. Math. **66** (1996), 360-365.
7. K. Koh, *On functional representation of a ring without nilpotent elements*, Canad. Math. Bull. **14** (1971), 349-352.
8. J. Lambek, *On the representation of modules by sheaves of factor modules*, Canad. Math. Bull. **14** (1971), 359-368.
9. G. Mason, *Reflexive ideals*, Comm. Algebra **9** (1981), 1709-1724.
10. G.Y. Shin, *Prime ideals and sheaf representation of a pseudo symmetric ring*, Trans. Amer. Math. Soc. **184** (1973), 43-60.
11. S.H. Sun, *Noncommutative rings in which every prime ideal is contained in a unique maximal ideal*, J. Pure and Appl. Algebra **76** (1991), 179-192.

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Noncommutative Poisson algebras and the Gerstenhaber's deformation theory

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Abstract. Noncommutative Poisson algebras are the alggebras having an associative algebra structure and a Lie algebra structure together with the Leibniz law. A standard such algebra is an associative algebra with its Lie product a scalar multiple of the ordinary associative commutator. In this article we present several noncommutative Pisson algebras which are allowed to have only standard ones. In the second part we take a look at the origin of the Gerstenhaber's deformation theory and then go into the deformation theory of noncmmutative Poisson algebras.

I Noncommutative Poisson algebras

1 Noncommutative Poisson algebras

If a vector space A over a field k of the characteristic zero has an associative algebra product, being denoted by ab the assiciative algebra product of a, b in A and the Lie algebra bracket $\{-,-\}$ satisfying the Leibniz law

$$\{a, bc\} = \{a, b\}c + b\{a, c\} \quad (a, b, c \in A),$$

we call such an A a *noncommutative Poisson algebra.*

Let A be an associative algebra and denote the ordinary associative commutator by

$$[a, b] = ab - ba \quad (a, b \in A).$$

We always have a noncommutative Poisson algebra structure on A whose Lie bracket is given by

$$\{-,-\} = \lambda[-,-] \quad (\lambda \in k).$$

Let us call this a *standard structure*, and denote an algebra A with such a structure by

$$A^\lambda.$$

If we take a λ in an extension K of k, we say it *K-standard*.

A deformation quantization of a commutative Poisson algebra brings about noncommutative Poisson algebras such the Weyl algebra after deforming the algebra of polynomials. In the nect section we will find a work of Farkas and Letzter [FL] and see that the Weyl algebra is allowed to have only a standard noncommutative Poisson algebra structure.

In another stream the author has asked the question of what possibilities are allowed when A is finite dimensional over k. Some of answers to this question will be also found in the next section. Before going next let us state the foundamental results.

Theorem ([K1]) *The noncommutative Poisson algebra structures on either the full algebra $M_n(k)$ of $n \times n$ marices must be standard.*

We close this section with basic tools of our investigation.

Theorem ([K3]) *Let L be a finite dimensional Lie algebra, A a finite dimensional associative algebra and simultaniously an L-module such that an L-action $[l, -]$ on A satisfies*

$$[l, ab] = [l, a]b + a[l, b] \quad (l \in L, a, b \in A).$$

If A contains no proper ideal which is simultaneously an L-submudule, then
(1) *$A = M_n(k)$ for some n, or*
(2) *A is a simple L-mudule and $AA = 0$.*

Sketch of the proof: The Jacobson radical $\mathrm{rad}(A)$ of A is invariant under all associative derivation of A, one has $\mathrm{rad}(A) \triangleleft A$ and $[L, \mathrm{rad}(A)] \subseteq \mathrm{rad}(A)$, and hence, $\mathrm{rad}(A) = A$ or $\mathrm{rad}(A) = 0$ by our hypothesis. When $\mathrm{rad}(A) = A$, we arrive at the assertion (2). We need a little consideration when $\mathrm{rad}(A) = 0$, that is, in the case that A is a semisimple associative algebra. Let $1 = e_1 + \cdots + e_r$, $e_i^2 = e_i$ be a central decomposition of the identity 1 and $\exp t\delta$ $(t \in k)$ be an automorphism that exponentioates an associative derivation δ of A. Considering the equation

$$\exp t\delta(e_i) = a_1(t)e_1 + \cdots + a_r(t)e_r,$$

one gets $a_i(t) = e_i$ and $a_j(t) = 0 \quad (j \neq i)$ by the continuity on t and observing the starting point $a_i(0) = e_i$, $a_j(0) = 0 \quad (j \neq i)$. This leads us that $\exp t\delta(e_i) = e_i$, so that $\delta(e_i) = 0$. Hence $Ae_i \lhd A$ and $[L, Ae_i] \subseteq Ae_i$, which imply $A = Ae_i$ for some i. □

Let us say a noncommutative Poisson algebra A to be *simple* if A has no proper ideal which is simultaneously a Lie ideal.

Corollary *If A is a finite dimensional simple noncommutative Poisson algebra then*

(1) $A = M_n(k)$ *for some n, or*

(2) *A is a simple Lie algebra with the associative algebra multiplication* $AA = 0$.

2 Standard noncommutative Poisson algebra structures

2.1 Poset algebras

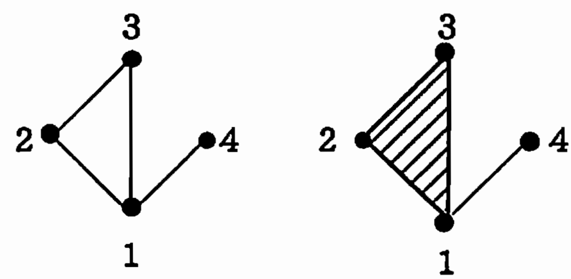
When a subalgebra A of $M_n(k)$ contains all the diagonal matrices, hence, in particular, all e_{ii}, $i = 1, \ldots, n$, A is spanned by those e_{ij} which it contains (e_{ij} is the matrix with 1 in the (i,j)-th place and 0 elsewhere). For if an a in A has the form $a = \sum \lambda_{ij} e_{ij}$ then $e_{ii} a e_{jj} = \lambda_{ij} e_{ij}$, so if $\lambda_{ij} \neq 0$ then $e_{ij} \in A$. Such algebras A are called *poset algebras* ([GS 2]). Now define a poset $I = I(A)$ by setting $i \prec j$ if $e_{ij} \in A$, and let $\overline{I}$ be the poset (without loops) determined by reducing I modulo the equivalence relation defined by the loops, i.e., by identifying to a single element any i and j for which both $i \prec j$ and $j \prec i$ (hence identifying any $i_1, i_2, \ldots, i_r$ whenever $i_1 \prec i_2 \prec \ldots \prec i_r$.) Let $\Sigma = \Sigma(A)$ be the nerve of $\overline{I}(A)$. This is a finite simplicial complex.

Example : We express the poset I by their Hasse diagrams. The following examples consist of the poset algebra A pictured as on the left in the figure 1 (the entries in position marked by $*$ may be chosen arbitrarily from k), the Hasse diagrams on the center and the geometric realization of I (the nerve $\Sigma(A)$ of $\overline{I}$) are given on the left in the figure 1.

We now state the following foundamental result.

Theorem ([K1]) *Let A be a poset algebra in $M_n(k)$ and $\Sigma(A)$ the simplical complex associated to A. Suppose that $\Sigma(A)$ is connected and has the property that for any pair of 1-faces as its edges. Then any noncommutative Poisson algebra structure on A must be standard.*

(1) $\begin{pmatrix} * & * & * & * \\ 0 & * & * & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}$



(2) $\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}$

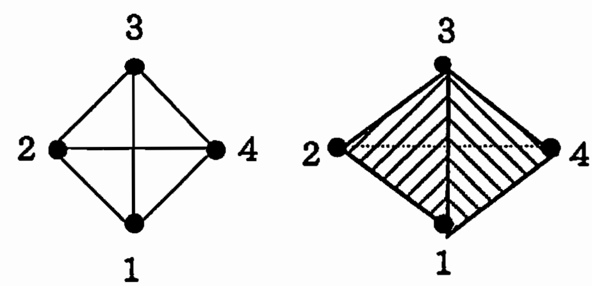


Figure 1

Note here that since the simplical complex associated to $M_n(k)$ or $T_n(k)$ is just the $(n-1)$-simplex as an example (2) above, noncommutative Poisson algebra structures on these algebras are allowed to be only standard ones. The algebra given in the example (1) has a noncommutative Poisson algebra structure which is not standard ([K2]).

We have some informations about the infinite dimensional cases in [K4].

(1) Every noncommutative Poisson algebra structure on $M_\infty(k)$ of all k-endomorphisms of a countable dimensional k-vector space must be standard.

(2) If A is a poset algebra in $M_\infty(k)$ has the same property as in the finite dimensional case, then A has only the standard structures.

(3) The Kac-Moody algebra L of affine type have only an almost trivial structure, that is, $[L, L][L, L] = 0$.

2.2 Farkas and Letzter's Theorem

Following to Farkas and Letzter [FL] we take a look at the noncommutative Poisson algebra structures on the prime algebras. Before going into their theorem we recall the Martindale ring of quotients and the extended centroid.

The Martindale ring of quotient Q of a prime algebra A consists of all the pairs (U, f) of a nonzero ideal U of A and a right A-module homomorphism $f : U \to A$ by reducing modulo the equivalence relation identifying to a single element any (U, f) and (V, g) for which $f = g$ on some nonzero ideal W of A with $W \subseteq U \cap V$. Its ring structures are given by $\bar{f} + \bar{g} :=$ "a class of $(U \cap V, f + g)$", $\bar{f}\bar{g} :=$ "a class of $(VU, f \circ g)$" for $\bar{f} =$ "a class of (U, f)" and $\bar{g} =$ "a class of (V, g)". Then the extended centroid $C^+(A)$ of A is defined to be the center of Q.

By using the corollary in the book [He] that if A is a prime ring and $0 \neq a, b \in A$ satisfy the condition $axb = bxa$ for any $x \in A$ then there exists a $\lambda \in C^+(A)$ such that $b = \lambda a$, Farkas and Letzter prove the following theorem.

Theorem *If A is a prime not-commutative Poisson algebra with a Lie product $\{-,-\}$ then there exists a $\lambda \in C^+(A)$ such that $\{c, d\} = \lambda[c, d]$ for all $c, d \in A$.*

Hence a prime not-commutative Poisson algebra is allowed to be only the $C^+(A)$-standard structures in our context.

This theorem is applied to the Weyl algebras and $M_n(k)$ observing that they are simple and $C^+(A) = k$. Then it follows that every noncommutative Poisson algebra structure on them must be standard.

II A brief observation of the Gerstenhaber's deformation theorey

3 Deformation theorey of associative algebras

The deformation theory of algebras was introduced by Gerstenhaber in a series of papers [G1-G5]. It has subsequently been extended by Gerstenhaber and Schack to covariant functors from a small category to algebras [GS 2] and to algebraic systems, bialgebras, Hopf algebras [GS 3], Leibniz pairs and Poisson algebras [FGV], etc. We will discuss a little about the deformation theory of noncommutative Poisson algebras in the next section. In this section we go back to the origin of the deformation theory of algebras by going through the chapters I and II of the Gerstenhaber's paper [G2]. We also recommend the readers to take a look at an essay of Hazenwinkel [Ha] stated in the opening of the big volume which is edeted by him and M.Gerstenhaber.

3.1 Aspects of Deformation Theory

To explore our subject let us keep the basic key words used here in mind. These are found in the following aspects of a deformation theory (We pick up here some of the aspects given by Gerstenhaber).

- *Aspects of deformation theory*

(1) A definition of the class of objects within which deformation takes place and identification of the infinitesimal deformations of a given object with the elements of a suitable cohomology group.

(2) A theory of the obstructions to the integration of an infinitesimal deformation.

(3) A parameterization of the set of objects.

(4) A determination of the natural automorphisms of the parameter space and the determination of the rigid objects.

3.2 Basic concepts

- *Deformation*

Let A be an algebra over a commutative associative unital ring k with multiplication $\alpha : A \times A \to A$. A *deformation* of A is a formal power series $\alpha_t : A[[t]] \times A[[t]] \to A[[t]]$ (t: deformation parameter) of the form

$$\alpha_t := \alpha + t\alpha_1 + t^2\alpha_2 + \cdots$$

where each $\alpha_i : A \times A \to A$ is a k-bilinear map and extended to that of $k[[t]]$. Then α_t define an algebra structure on $A[[t]]$ and write such an algebra by

$$A_t := (A[[t]], \alpha_t).$$

If A is an associative algebra, A_t is required to be the same kind as α, that is, an associative algebra following to the aspect (1) in 3.1.

When $\alpha_1 = \alpha_2 = \cdots = 0$, we say that α_t is a *null deformation* and write A_0, hence,

$$A_0 = A \otimes_k k[[t]]$$

is a just extension of the coefficients of A.

- *Equivalence*

Let A_t, A'_t be the deformations of an algebra A with $\alpha'_t := \alpha + t\alpha'_1 + t^2\alpha'_2 + \cdots$. A deformation A'_t is *equivalent* to A_t, denoted by $A'_t \sim A_t$, if there exists a $k[[t]]$-isomorphism $f_t : A'_t \to A_t$ of the form

$$f_t = 1_A + tf_1 + t^2 f_2 + \cdots,$$

where $f_i \in \mathrm{Hom}_k(A, A)$ is extended to $k[[t]]$, such that

$$\begin{aligned} \alpha'_t(a,b) &= f_t^{-1}\alpha_t(f_t(a), f_t(b)) \\ &= f_t^{-1}\alpha_t(f_t, f_t)\}(a,b) \quad (a, b \in A). \end{aligned}$$

If we write $a * b = \alpha'_t(a,b)$, $a \cdot b = \alpha_t(a,b)$ then the above equality means $f_t(a * b) = f_t(a) \cdot f_t(b)$, i.e., f_t is an algebra isomorphism. A deformation A_t is said to be a *trivial deformation* when $A_t \sim A_0$.

- *Comomology*

Let A be an associative algebra over k. In the deformation theory of the associative algebras we require A_t to be an associative algebra, that is,

$$\alpha_t(a, \alpha_t(b,c)) = \alpha_t(\alpha_t(a,b), c).$$

This leads the *deformation equation*

$$\sum_{\substack{p+q=n \\ p>0, q>0}} \{\alpha_p(\dot{\alpha}_q(a,b),c) - \alpha_p(a,\alpha_q(b,c))\} = (\delta\alpha_n)(a,b,c) \qquad (*)$$

here, δ is a Hochschild coboundary operator, namely, $\delta^n : C^n(A,A) := \mathrm{Hom}_k(A^n, A) \to C^{n+1}(A,A)$ defined by

$$\begin{aligned}\delta^n f(a_1,\ldots,a_{n+1}) &= a_1 f(a_2,\ldots,a_n) + \textstyle\sum_{i=1}^n (-1)^i f(\ldots,a_i a_{i+1},\ldots) \\ &\quad +(-1)^{n+1} f(a_1,\ldots,a_n)a_{n+1}.\end{aligned}$$

One can see the easy consequences of the equation $(*)$:

$$\begin{aligned}\text{Substitute } n=1 \text{ in } (*) &\implies \delta\alpha_1 = 0, \text{ i.e., } \alpha_1 \in Z^2(A,A) \\ \alpha'_t \sim \alpha_t &\iff \alpha'_1 - \alpha_1 \in B^2(A,A).\end{aligned}$$

In fact, assume $\alpha'_t \sim \alpha_t$ and write $\alpha'_t - \alpha_t = \delta f_1$. Then the isomorphism $f_t : A_t \to A'_t$ is given by $f_t = 1_A + t f_1$.

- *Rigidity*

A k-algebra A is *analytically rigid* if every deformation A_t is equivalent to the null deformation A_0 ($A_t \sim A_0$). Here is a foundamental theorem.

Theorem ([G 2]) *If A is an associative algebra with $H^2(A,A) = 0$ then A is analytically rigid. In particular, the separable algebras are analytically rigid.*

Proof. Let α_t is a deformation and write $\alpha_t = \alpha + t\alpha_1 + t^2\alpha_2 + \cdots$. By the assumption that $\alpha_1 \in Z^2(A,A) = B^2(A,A)$, one can find $f^{(1)} \in C^1(A,A)$ such that $\delta f^{(1)} = \alpha_1$. Then we have

$$\begin{aligned}\alpha_t^{(1)} &:= (1_A - tf^{(1)})^{-1}\alpha_t(1_A - tf^{(1)}, 1_A - tf^{(1)}) \\ &= \alpha + t^2\alpha_2 + \cdots,\end{aligned}$$

which is equivalent to α_t. Repeating this procedure, one gets

$$\prod_{i=1}^{\infty}(1_A - tf^{(i)})^{-1}\alpha_t \prod_{i=1}^{\infty}(1_A - tf^{(i)}, 1_A - tf^{(i)}) = \alpha_0.$$

□

An associative algebra A is said to be *infinitesimal rigid* if A satisfies $H^2(A, A) = 0$.

- *Integrable Problem*

A 2-cocycle $\alpha_1 \in Z^2(A, A)$ is said to be *integrable* if there exists a one parameter family α_t whose first coefficient is α_1, so that,

$$\alpha_t = \alpha + t\underline{\alpha_1} + t^2\alpha_2 + \cdots.$$

Now go back to the deformation equation $(*)$ in the part of 'ohomology" . Suppose that we can find $\alpha_1, \ldots, \alpha_{n-1}$ satisfying $(*)$. We want to construct α_n from $\alpha_1, \ldots, \alpha_{n-1}$ satisfying $(*)$. But there is an *obstruction* to do so. If we put

$$f\bar{\circ}g(a, b, c) = f(g(a, b), c) - f(a, g(b, c)),$$

the left hand side of $(*)$ can be written in the form $(\alpha_1\bar{\circ}\alpha_{n-1} + \cdots + \alpha_{n-1}\bar{\circ}\alpha_1)$ (a, b, c), and then the deformation equation $(*)$ is of the form

$$\alpha_1\bar{\circ}\alpha_{n-1} + \cdots + \alpha_{n-1}\bar{\circ}\alpha_1 = \delta\alpha_n.$$

The left hand side of this equation define the 3-cocycle and is called an obstruction cocycle. Hence we can now easily understand

Theorem ([G 2]) *If A is an associative algebra with $H^3(A, A) = 0$, then every 2-cocycle is integrable.*

Example of such an associative algebra A. We consider the poset algebra A given in the figure 2. Then one has

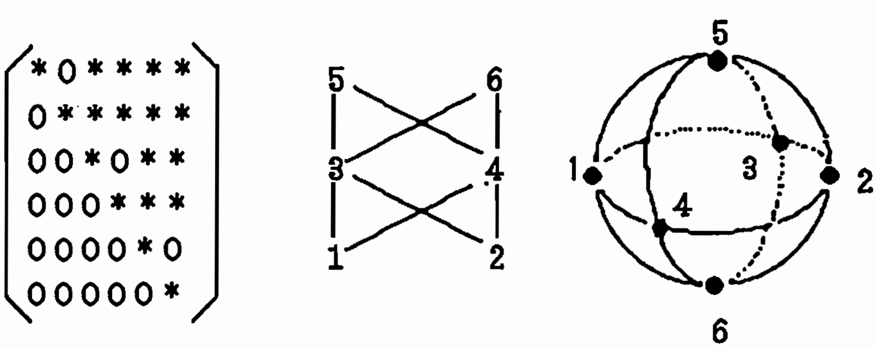


Figure 2

$$\begin{aligned} H^2(A,A) &\cong H^2(\Sigma(A),k) &\cong H_2(\Sigma(A),k) &= k, \\ H^3(A,A) &\cong H^3(\Sigma(A),k) &\cong H_3(\Sigma(A),k) &= 0, \end{aligned}$$

by using the Gerstenhaber and Schack's theorem $H^\bullet(A,A) \cong H^\bullet(\Sigma(A),k)$ ([GS 2], p.138) and the famous Künneth theorem in algebraic topology.

3.3 Parameter space

Let k be a field and Ω an universal domain with $k \subseteq \Omega$. Let C be an n-dimensional vector space over k with a basis $\{x_1, \ldots, x_n\}$. For each point $P = (c_{ijk})$ in Ω^{n^3} we have an algebra whose structure constants are (c_{ijk}), that is, $x_i x_j = \sum_k c_{ijk} x_k$. Let A be an assosciative algebra over k and fix a basis $\{a_1, \ldots, a_n\}$ of A. Since the condition that an n-dimensional algebra defined by a set of structure constants (c_{ijk}) be associative is expressible by vanishing of certain quadratic polynomials in these constants, such points (c_{ijk}) form an algebraic set C of the affine n^3-space. We have now a *parameter space* C associated to the class of the associative algebras of dimension n.

There are several correspondence between the concepts of algebras and those of geometric objects. Let $G = GL(n, \Omega)$ be the general linear group. Then G acts on C.

(1) Let $P \in$ C represent an algebra A. Then a point gP $(g \in G)$ represents an algebra which is isomorphic to A over some common extension of their coefficient field.

$$\begin{array}{ccccc} P \in V & \longleftrightarrow & A & \longrightarrow & A \otimes_k L \\ \downarrow & & & & \wr\| \\ gP \in V & \longleftrightarrow & A' & \longrightarrow & A' \otimes_k L \end{array}$$

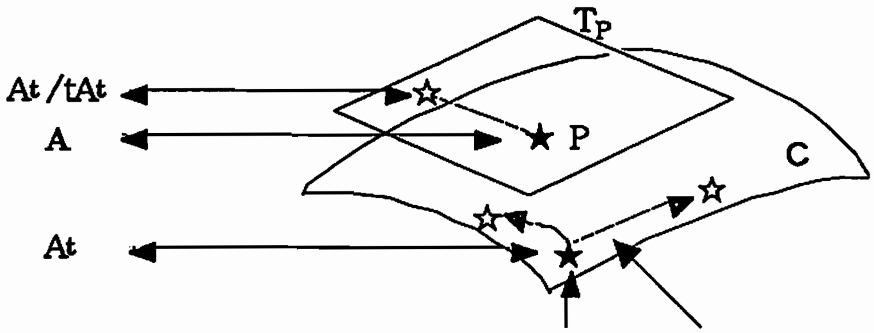
(2) Let $P = (c_{ijk})$ be a point in V corresponding to A. An associative algebra A is *geometrically rigid* if the orbit GP contains a Zarisky open subset of P on C, equivalently a component of C containing P has a generic point which represents an algebra isomorphic to A over some extension of k.

(3) An element of $Z^2(A_\Omega, A_\Omega)$ corresponds to a tangent vector of C at P. Let $P = (c_{ijk})$ be a pont in C corresponding to A and $P' = (c_{ijk} + tc'_{ijk})$ a

point in the tangent space of C at P. Then we have an algebra A_t/t^2A_t over $k[[t]]/t^2k[[t]]$ with a multiplication

$$a_ia_j = \sum_k (c_{ijk} + tc'_{ijk})a_k,$$

called *infinitesimal deformation.*



We close this section with some results on rigidity.

Theorem ([GS 1]) *Let k be a field of characteristic zero and A a finite dimensional k-algebra in any equationally defined category. Then A is geometrically rigid if and only if it is analytically rigid.*

Note It is still open problem (35years old) to find analytically, but not infinitesimally, rigid associative algebras in characteristic zero.

infinitesimally rigid	$\Rightarrow$	analytically rigid	$\Rightarrow$	geometrically rigid
$(H^2(A,A)=0)$	$\not\Leftarrow$	$(A_t \sim A_0)$	$\Leftarrow$	(GP is open)

4 Deformation theory of noncommutative Poisson algebras

We have seen the aspects of the deformation theory in the previous subsections. The deformation theory of any other algebraic systems seems to be developped based on these aspects. In this section we will concern such a model, a deformation theory of noncommutative Poisson algebras, introduced by Flato, Gerstenhaber and Voronov [FGV].

- *Deformation of noncommutative Poisson algebras*
  Let a noncommutative Poisson algebra (A, α, λ) be given by

$$\begin{array}{rcl} \text{an associative product } \alpha & : & A \times A \to A \\ \text{a Lie product } \lambda & : & A \times A \to A \end{array}$$

with the Leibniz law $\lambda(a, \alpha(b, c)) = \alpha(a, \lambda(b, c)) + \alpha(\lambda(a, b), c)$. A *deformation* $(A[[t]], \alpha_t, \lambda_t)$ of (A, α, λ) is given by

$$\begin{array}{rcl} \text{an associative product } \alpha_t & : & A[[t]] \times A[[t]] \to A[[t]] \\ \text{a Lie product } \lambda_t & : & A[[t]] \times A[[t]] \to A[[t]] \end{array}$$

of the form

$$\alpha_t = \alpha + t\alpha_1 + t^2\alpha_2 + \cdots, \quad \lambda_t = \lambda + t\lambda_1 + t^2\lambda_2 + \cdots$$

with the Leibniz law $\lambda_t(a, \alpha_t(b, c)) = \alpha_t(a, \lambda_t(b, c)) + \alpha_t(\lambda_t(a, b), c)$.

- *Cohomology for noncommutative Poisson algebras*
  A cohomology group controling a deformation theory of noncommutative Poisson algebras is proposed to choose a chain complex $(C^{\bullet}_{tot}(A, A), \delta_{tot})$ defined by

$$C^n_{tot}(A, A) = \oplus_{p+q=n} C^{p,q}(A, A), \ \delta_{tot} = \left\{ \begin{array}{rcl} \delta_{tot}|_{C^{1,q}} & = & \delta_P + \delta_{CE} \\ \delta_{tot}|_{C^{p,q}} & = & \delta_H + (-1)^q \delta_{CE} \ (p \geq 2) \end{array} \right. ,$$

where

$$C^{p,q}(A, A) = \left\{ \begin{array}{rcl} C^{1,q} & = & \mathrm{Hom}_k(\wedge^{q+1} A, A) \\ C^{p,q} & = & \mathrm{Hom}_k(A^p \otimes (\wedge^q A), A) \ (p \geq 2) \end{array} \right. ,$$

δ_H is a Hochschild coboundary, and δ_{CE} is a Chevalley-Eilenbrg coboundary operator defined by, denoting by $[-, -] := \lambda(-, -)$,

$$\begin{array}{rcl} (\delta_{CE} f)(a_1, \ldots, a_n) & = & \sum_{i=1}^n (-1)^{i+1} [a_i, f(a_1, \ldots, \hat{a}_i, \ldots) \\ & & + \sum_{1 \leq i < j \leq n} (-1)^{i+j} f([a_i, a_j], \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots), \end{array}$$

and $\delta_P : C^{1,q} = \mathrm{Hom}_k(\wedge^{q+1} A, A) \to C^{2,q} = \mathrm{Hom}_k(A^2 \otimes (\wedge^q A), A)$ is a composition $\mathrm{Hom}_k(\wedge^{q+1} A, A) \xrightarrow{\varepsilon^*} \mathrm{Hom}_k(A \otimes (\wedge^q A), A) \xrightarrow{\delta_H} \mathrm{Hom}_k(A^2 \otimes (\wedge^q A), A)$ $(\varepsilon^* f(a_1 \otimes (a_2 \wedge \ldots \wedge a_q)) := f(a_1 \wedge a_2 \wedge \ldots \wedge a_q))$.

$$
\begin{array}{ccccccc}
\uparrow \delta_H & & \uparrow \delta_H & & \uparrow \delta_H & & \\
\mathrm{Hom}_k(A^3, A) & \xrightarrow{\delta_{CE}} & \mathrm{Hom}_k(A^3 \otimes A, A) & \xrightarrow{\delta_{CE}} & \mathrm{Hom}_k(A^3 \otimes \wedge^2 A, A) & \xrightarrow{\delta_{CE}} & \\
\uparrow \delta_H & & \uparrow \delta_H & & \uparrow \delta_H & & \\
\mathrm{Hom}_k(A^2, A) & \xrightarrow{\delta_{CE}} & \mathrm{Hom}_k(A^2 \otimes A, A) & \xrightarrow{\delta_{CE}} & \mathrm{Hom}_k(A^2 \otimes \wedge^2 A, A) & \xrightarrow{\delta_{CE}} & \\
\uparrow \delta_P = \delta_H & & \uparrow \delta_P & & \uparrow \delta_P & & \\
\mathrm{Hom}_k(A, A) & \xrightarrow{\delta_{CE}} & \mathrm{Hom}_k(\wedge^2 A, A) & \xrightarrow{\delta_{CE}} & \mathrm{Hom}_k(\wedge^3 A, A) & \xrightarrow{\delta_{CE}} &
\end{array}
$$

Let us find out a reason why this chain complex works well in our theory. We denote by $B^n_{tot}(A, A)$, $Z^n_{tot}(A, A)$, $H^n_{tot}(A, A)$ the k-modules of n-cocycles, n-coboundaries and n-th cohomology group respectively. We have three deformation equations for an associative algebra, a Lie algebra and the Leibniz law. By looking at the first coefficient of the Leibniz law $\lambda_t(a, \alpha_t(b, c)) = \alpha_t(a, \lambda_t(b, c)) + \alpha_t(\lambda_t(a, b), c)$, one has

$$[x, \alpha_1(a, b)] + \lambda_1(x, ab) = a\lambda_1(x, b) + \alpha_1(a, [x, b]) + \lambda_1(x, a)b + \alpha_1([x, a], b),$$

being denoted by $[-, -] = \lambda(-, -)$, $ab = \alpha(a, b)$. On the other hand, by identifying $\mathrm{Hom}_k(A^2 \otimes A, A)$ with $\mathrm{Hom}_k(A, \mathrm{Hom}_k(A^2, A))$, one gets

$$
\begin{aligned}
(\delta_{CE}\alpha_1)(x)(a \otimes b) &= [x, \alpha_1(a \otimes b)] - \alpha_1([x, a] \otimes b) - \alpha_1(a \otimes [x, b]) \\
(\delta_P \lambda_1)(x)(a \otimes b) &= a\varepsilon^*(\lambda_1)(b \otimes x) - \varepsilon^*(\lambda_1)(ab \otimes x) + \varepsilon^*(\lambda_1)(a \otimes x)b \\
&= a\lambda_1(b \wedge x) - \lambda_1(ab \wedge x) + \lambda_1(a \wedge x)b.
\end{aligned}
$$

Hence the Leibniz law holds modulo $t^2 A_t$ if and only if

$$\delta_P \lambda_1 + \delta_{CE} = 0, \text{or }, \ \lambda_1 + \alpha_1 \in Z^2_{tot}(A, A).$$

- *Rigidity of noncommutative Poisson algebras*

Now assume that A is infinitesimally rigid, that is, $H^2_{tot}(A, A) = 0$. Let $(A_t, \alpha_t, \lambda_t)$ be a deformation of a noncommutative Poisson algebra (A, α, λ).

Since $\lambda_1 + \alpha_1 \in Z^2_{tot}(A,A) = B^2_{tot}(A,A)$, one can find an $f \in C^1_{tot}(A,A) = C^{1,0}(A,A) = \mathrm{Hom}_k(A,A)$ such that $\delta^1_{tot}(f) = \lambda_1 + \alpha_1$, hence

$$\alpha_1 = \delta_P(f), \quad \lambda_1 = \delta_{CE}(f).$$

Then consider the $k[[t]]$-automorphism $\Phi_t = 1_A + tf$ of A_t. As in the proof stated in the part of 'ohomology" in the section 3, one has

$$\Phi_t^{-1}\alpha_t(\Phi_t, \Phi_t) = \alpha + t^2\alpha_2' + \cdots, \quad \Phi_t^{-1}\lambda_t(\Phi_t, \Phi_t) = \lambda + t^2\lambda_2' + \cdots.$$

Repeating this procedure we have

Theorem *If A is a noncommutative Poisson algebra with $H^2_{tot}(A,A) = 0$ then A is analytically rigid in the category of noncommutative Poisson algebras.*

The analogous theorems to those in the section 3 are expected for noncommutative Poisson algebras, but we can not find them in the paper [FGV] (The authors might think that such theorems are obviously derived). It is a good exercise to verify these theorems.

References

[FGV] M.Flato,M.Gerstenhaber and A.A.Voronov, *Cohomology and deformation of Leibniz pairs*, Lett.Math.Phys. 34(1995) 77-90.

[F L] D.R.Farkas and G.Letzter, *Ring theory from symplectic geomery*, J.Pure Appl. Algebra 125(1998)155-190.

[G 1] M.Gerstenhaber, *On the cohomology structure of an associative ring*, Ann. of Math. 78 (1963) 59-103.

] M.Gerstenhaber, *On the deformation of rings and algebras*, Ann.of Math. 79(1964) 267-288.

[G 3] M.Gerstenhaber, *On the deformation of rings and algebras II*, Ann.of Math. 84 (1966) 1-19.

[G 4] M.Gerstenhaber, *On the deformation of rings and algebras III*, Ann.of Math. 88 (1968) 1-34.

[G 5] M.Gerstenhaber, *On the deformation of rings and algebras IV*, Ann.of Math. 99 (1974) 257-276.

[GS 1] M.Gerstenhaber and S.D.Schack, *Relative Hochschild cohomology, rigid algebras, and the Bockstein*, J.Pure Appl. Algebra 43(1986) 53-74.

[GS 2] M.Gerstenhaber and S.D.Schack, *Algebraic cohomology and deformation theory*, in: M.Hazewinkel and M.Gerstenhaber, eds., Deformation Theory of Algebras and Structures and Applications (Kluwer Academic Publishers, Dordrecht,1988) 11-264.

[GS 3] M.Gerstenhaber and S.D.Schack, *Algebras, bialgebras, quantum groups, and algebraic deformations*, in: M.Gerstenhaber and J.Stasheff, eds., Deformation Theory and Quantum Groups with Applications to Mathematical Physics , Contemporary Math., Vol. 134 (Amer. Math. Soc., Providence, RI, 1992) 51-92.

[H a] M.Hazewinkel, *The philosophy of deformations : introductory remarks and a guide to this volume*, in: M.Hazewinkel and M.Gerstenhaber, eds., Deformation Theory of Algebras and Structures and Applications (Kluwer Academic Publishers, Dordrecht,1988) 1-7.

[H e] I.N.Herstein, Rings with Involution, Chicago Lect. in Math. (Univ.Chicago Press, Chicago, 1976)

[K 1] F.Kubo, *Finite-dimensional non-commutative Poisson algebras*, J.Pure Appl.Algebra 113(1996) 307-314.

[K 2] F.Kubo, *Noncommutative Poisson algebra structures on poset algebras and morphisms of Leibniz pairs*, Bull.Kyushu Inst.Tech. (Math.Natur.Sci.) 44 (1997) 1-5.

[K 3] F.Kubo, *Finite-dimensional simple Leibniz pairs and simple Poisson modules*, Lett.Math.Phys. 43 (1998) 21-29.

[K 4] F.Kubo, *Non-commutative Poisson algebra structure on Kac-Moody algebras*, J.Pure Appl.Algebra 126(1998) 267-286.

ON RELATIVE PROJECTIVITY FOR FINITE GROUP ALGEBRAS

NAOKO KUNUGI

§1. Introduction

Let G be a finite group. Let k be an algebraically closed field of characteristic $p > 0$. In modular representation theory of finite groups, it is important to investigate relations between representations of G and representations of $N_G(P)$ where P is a p-subgroup of G. There is a well-known conjecture due to Broué.

Conjecture(Broué [1, 4.9 Conjecture]) Let G be a finite group with abelian Sylow p-subgroup P. Then the principal block of kG and the principal block of $kN_G(P)$ are derived equivalent.

T. Okuyama showed that the Broué conjecture holds in some cases(see [5]). The Green correspondents of the simple modules play an important role, there.

In this paper, we state results which may be useful for calculating the Green correspondents. Moreover we apply these results to $G = PSL(3, q)$ in characteristic 3.

§2. Relative projectivity

Let G be a finite group and let $\mathcal{H}$ be a family of p-subgroups of G. A kG-module U is said to be $\mathcal{H}$-projective if each direct summand U_i of U is H_i-projective for some $H_i \in \mathcal{H}$. We recall the definition of relative projective covers which was introduced by Knörr.

Definition 1 ([2, 1.Definition, 2.Proposition]) A short exact sequence

$$0 \longrightarrow U \longrightarrow V \longrightarrow M \longrightarrow 0 \qquad (1)$$

of kG-modules is called an *$\mathcal{H}$-projective resolution of M* if

(i) V is $\mathcal{H}$-projective

(ii) the sequence (1) is $\mathcal{H}$-split.

Moreover if U has no non-zero $\mathcal{H}$-projective direct summand then (1) is called an *$\mathcal{H}$-projective cover of M*, and we write $P_{\mathcal{H}}(M)$ for V and $\Omega_{\mathcal{H}}(M)$ for U.

An $\mathcal{H}$-projective cover of any kG-module exists and uniquely determined up to isomorphism. The module $P_{\mathcal{H}}(M)$ is also called the $\mathcal{H}$-projective cover of M.

Dually we can define $\mathcal{H}$-injective hulls. The $\mathcal{H}$-projective resolution (1) of M in Definition 1 is also called an $\mathcal{H}$-injective resolution of U. If M has no non-zero $\mathcal{H}$-injective

The detailed version of this paper will be submitted for publication elsewhere.

summand then (1) is called an $\mathcal{H}$-injective hulls, and we write $I_{\mathcal{H}}(U)$ for V and $\Omega_{\mathcal{H}}^{-1}(U)$ for M.

Next we consider the relative projective cover of the trivial kG-module k_G. Let H be a subgroup of G. The induced module $k_H\uparrow^G$ has a unique indecomposable direct summand containing the trivial module k_G in the head. This summand is called the Scott module and is denoted by $S_G(H)$. The following properties of Scott modules are important.

Proposition 2.1 (Scott, Alperin)(see [4, Chapter 4 Theorem 8.4, Corollary 8.5]) (i) *$S_G(H)$ is self-dual.*

(ii) $\dim_k \mathrm{Hom}_{kG}(k_G, S_G(H)) = \dim_k \mathrm{Hom}_{kG}(S_G(H), k_G) = 1$

(iii) *Let Q be a p-subgroup of G. Then, $S_G(H) \cong S_G(Q)$ if and only if Q is conjugate to a Sylow p-subgroup of H in G.*

Proposition 2.2 *The Scott module $S_G(Q)$ is the Q-projective cover of k_G, and is the Q-injective hull of k_G.*

Lemma 2.3 *Let Q be a proper subgroup of a Sylow p-subgroup of G. Then the inclusion map $\phi : k_G \to \Omega_Q(k_G)$ is not a Q-projective homomorphism.*

§3. Green correspondence

In this section we state two lemmas which may be useful to calculate the Green correspondents of some modules.

Let P be a p-subgroup of G and let $H = N_G(P)$ be the normalizer of P in G. We set

$$\begin{aligned} \mathfrak{X} &= \{sPs^{-1} \cap P \mid s \in G,\ s \notin H\} \\ \mathfrak{Y} &= \{sPs^{-1} \cap H \mid s \in G,\ s \notin H\} \\ \mathfrak{Z} &= \{Q \subseteq P \mid Q \not\subseteq_G \mathfrak{X}\}. \end{aligned}$$

Then, the Green correspondence $f = f(G, P, H)$ with respect to (G, P, H) is a one-to-one correspondence between isomorphism classes of indecomposable kG-modules with vertex in $\mathfrak{Z}$ and isomorphism classes of indecomposable kH-modules with vertex in $\mathfrak{Z}$. For such a kG-module U, we write $f(U)$ for the corresponding kH-module. Then, U and $f(U)$ have the same vertex and

$$\begin{aligned} U\downarrow_H &= f(U) \oplus \mathfrak{Y}\text{-proj} \\ f(U)\uparrow^G &= U \oplus \mathfrak{X}\text{-proj} \end{aligned}$$

(see [4]). It is difficult to determine the Green correspondent of a module by direct calculations, in general. However, we can give the following correspondence using the relative projective cover of the trivial module.

Lemma 3.1 *Let G be a finite group with abelian Sylow p-subgroup P, and $H = N_G(P)$. For a proper subgroup of P,*

$$(\Omega_Q(k_G)/k_G)\downarrow_H \cong \Omega_Q(k_H)/k_H \oplus (Q\text{-proj}).$$

Remark 1 In general, the restriction to H of a Q-projective kG-module is a $\{sQs^{-1} \cap H \mid s \in G\}$-projective kH-module. In the above lemma, by the assumption on P, this family is conjugate to $\{Q\}$. Therefore restriction of a Q-projective kG-module is a Q-projective kH-module.

Therefore we can determine the Green correspondent of $\Omega_Q(k_G)/k_G$ by calculating the Scott module $S_H(Q)$. Similarly we consider tensoring by a simple module instead of restriction to H.

Lemma 3.2 *Let G be a finite group and let Q be a proper subgroup of a Sylow p-subgroup of G. Let S be a simple module such that $p \nmid \dim_k S$. Assume that $P_Q(S) \cong I_Q(S)$. Then*

$$S \otimes (\Omega_Q(k_G)/k_G) \cong \Omega_Q(S)/S' \oplus (Q\text{-proj}),$$

where S' is a submodule of $\Omega_Q(S)$ isomorphic to S.

The condition $p \nmid |G : H|$ in Lemma 3.1 corresponds to the condition $p \nmid \dim_k S$.

§4. Application

Let k be an algebraically closed field of characteristic 3, and let $G = PSL(3,q)$ be the 3-dimensional projective special linear group, where q is a power of a prime satisfying the condition $q \equiv 4$ or $7 \pmod 9$. In this section, we calculate the Green correspondents of the simple modules in the principal block of kG by applying the results of section 3 to G.

Assumption. Let k be an algebraically closed field of characteristic 3, and $G = PSL(3,q)$ where $q \equiv 4$ or $7 \pmod 9$. Let P be a Sylow p-subgroup of G. Then $P \cong C_3 \times C_3$. We set $H = N_G(P)$. Then $H \cong (C_3 \times C_3) \rtimes Q_8$ where Q_8 is a quaternion group of order 8. We denote by A the principal block of kG and B by the principal block of kH. In particular, we denote by A_1 the principal block of $kPSL(3,4)$.

We consider the Green correspondence f with respect to (G, P, H). The principal block B of kH has five simple modules k_H, 1_1, 1_2, 1_3 and 2. The principal block A of kG has five simple modules k_G, S, T_1, T_2 and T_3, where $\dim_k S = q^2 + q - 1$. Since P is abelian, all simple modules in A and B have P as their vertex. Therefore the Green correspondence of the simple modules is defined.

Let Q be a subgroup of P of order 3. The group G has a maximal subgroup L of index $q^2 + q + 1$ and H has a subgroup K of index 12. Induced modules $k_L{\uparrow}^G$ and $k_K{\uparrow}^H$ are both indecomposable. We have the following.

Lemma 4.1

$$S_G(Q) \cong \begin{pmatrix} k_G \\ S \\ k_G \end{pmatrix}, \; S_H(Q) \cong \begin{pmatrix} k_H \; 1_1 \; 1_2 \; 1_3 \\ 2 \; 2 \\ k_H \; 1_1 \; 1_2 \; 1_3 \end{pmatrix}.$$

We get the following by Lemma 3.1.

Lemma 4.2 *The Loewy series of the Green correspondent of the simple module S is*

$$\begin{pmatrix} 1_1\ 1_2\ 1_3 \\ 2\ 2 \\ 1_1\ 1_2\ 1_3 \end{pmatrix}.$$

In general, to calculate the relative projective cover of a module is difficult. However, we can determine the Q-projective cover of the simple module S.

Lemma 4.3

$$P_Q(S) = I_Q(S) = \begin{pmatrix} S \\ k_G T_1 T_2 T_3 \\ S \end{pmatrix}.$$

Since $3 \nmid \dim_k S$, we have the following lemma by applying Lemma 3.2.

Lemma 4.4

$$S \otimes S \cong k_G \oplus T_1 \oplus T_2 \oplus T_3 \oplus (Q\text{-proj}).$$

Therefore we have

$$f(S) \otimes f(S) \cong k_H \oplus f(T_1) \oplus f(T_2) \oplus f(T_3) \oplus (Q\text{-proj}).$$

Note that the Green correspondent of S dose not depend on q. Thus the Green correspondence of T_i dose not depend on q. Schneider determined the Green correspondents in the case $q = 4$ (see [6, Theorem]). Hence we get the Green correspondents of all simple modules.

Lemma 4.5 *The Green correspondents of the simple kG-modules in A are determined and they do not depend on q. The Loewy series are*

$$f(k_G) = k_H, \quad f(S) = \begin{pmatrix} 1_1\ 1_2\ 1_3 \\ 2\ 2 \\ 1_1\ 1_2\ 1_3 \end{pmatrix}, \quad f(T_i) = \begin{pmatrix} 2 \\ k_H\ 1_i \\ 2 \end{pmatrix}$$

for $i = 1, 2, 3$.

Theorem 4.6 *The principal block A of kG is Morita equivalent to the principal block A_1 of $kPSL(3,4)$*

Proof. We write ${}_AA_B$ if we regard A as an (A, B)-bimodule and ${}_BA_A$ if we regard A as a (B, A)-bimodule, where $B = B_0(kH) = kH$. Then the bimodules ${}_BA_A$ and ${}_AA_B$ induce a stable equivalence of Morita type between A and B by a theorem of Broué([1, 6.3 Theorem]). Therefore the bimodules $A_1 \otimes_B A$ and $A \otimes_B A_1$ induce a stable equivalence of Morita type between A and A_1. By Lemma 4.5, for any simple A-module U, we have that $A_1 \otimes_B A \otimes_A U = (\text{simple}) \oplus (\text{proj})$. Therefore we have that A and A_1 are Morita equivalent by a theorem of Linckelmann([3, Theorem 2.1]). □

The following theorem is showed by T. Okuyama (see [5]).

Theorem 4.7 (Okuyama) *A_1 and B are derived equivalent.*

Therefore we have the following as a corollary to Theorems 4.6 and 4.7.

Corollary 4.8 *A and B are derived equivalent, that is, the Broué conjecture holds for the cases $G = PSL(3,q)$, $q \equiv 4 \,\text{or}\, 7 \pmod 9$.*

References

[1] M. Broué *Equivalences of blocks of group algebras*, in "Finite dimensional algebras and related topics", pp.1–26, edited by V. Dlab and L.L. Scott, Kluwer Acad. Pub. Dordrecht, 1994.

[2] R. Knörr, *Relative projective covers*, in "Proceedings of the Symposium on Modular Representations of Finite Groups", Aarhas Univ. (1978). pp.28–32.

[3] M. Linckelmann, *Stable equivalences of Morita type for self-injective algebras and p-groups*, Math. Z. **223** (1996),87–100.

[4] H. Nagao and Y. Tsushima, *"Representations of Finite Groups"* Academic Press, New York, 1989.

[5] T. Okuyama, *Some examples of derived equivalent blocks of finite groups*, in "Proceedings of the 6th Symposium on Representation Theory of Algebras" pp.108–122(Tateyama, Chiba Prefecture, Japan) edited by S. Koshitani and M. Sato (in Japanese),1996

[6] G.J.A. Schneider, *$PSL(3,4)$ in characteristic* 3, Commun. Algebra **15** (1987), 1543–1547.

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POLYNOMIAL RINGS WHICH ARE QUASI-DUO

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Throughout this paper, all rings are associative with identity. Given a ring R, the Jacobson radical of R, the polynomial ring over R and the formal power series ring over R are denoted by $J(R)$, $R[x]$ and $R[[x]]$, respectively. In this paper we continue the study of quasi-duo rings that was initiated by Yu in [12], related to the Bass' conjecture in [2]. In section 1 we observe some properties of right quasi-duo rings, and investigate whether right quasi-duo rings (or thier factor rings) are commutative if the polynomial rings over them are right quasi-duo. In section 2 we study the connections between right quasi-duo rings and weakly right duo rings. A ring R is *right (left) duo* if every right (left) ideal of R is two-sided. A ring R is called *weakly right (left) duo* if for each a in R there exists a positive integer $n = n(a)$, depending on a, such that $a^n R$ (Ra^n) is two-sided. A ring R is called *right (left) quasi-duo* if every maximal right (left) ideal of R is two-sided. Commutative rings are clearly right and left duo. Right duo rings are obviously weakly right duo, and weakly right duo rings are right quasi-duo by [12, Proposition 2.2].

1. Polynomial rings which are right quasi-duo

In this section we study whether a ring R (or its factor ring) is commutative if $R[x]$ is right quasi-duo. We first observe some properties of right quasi-duo rings.

Proposition 1. *For a ring R, the followings are equivalent:*

(1) *R is right quasi-duo.*
(2) *Every right primitive factor ring of R is a division ring.*

Remark. We may obtain the same result for left quasi-duo rings by replacing "right" by "left" in the preceding proposition.

Recall that a ring R is called *reduced* if R has no nonzero nilpotent elements. As in the next corollary we obtain the result [12, Lemma 2.3] (i.e., For a left or right quasi-duo ring R, all nilpotent elements are in $J(R)$), using a different method of proof.

Corollary 2. *[12, Lemma 2.3] For a right (or left) quasi-duo ring R, $R/J(R)$ is reduced.*

This is a part of forthcoming joint paper "Some results on quasi-duo rings" with Chan Huh

As an elementary fact, a commutative semiprimitive ring is a subdirect product of fields. In the next corollary, we may generalize this result to right or left quasi-duo rings.

Corollary 3. *(1) A right (left) quasi-duo semiprimitive ring is a subdirect product of division rings. (2) Every homomorphic image of a right (left) quasi-duo ring is also right (left) quasi-duo.*

As another corollary, we obtain [12, Corollary 2.4] (i.e., A semiprimitive right (or left) quasi-duo ring is reduced) from Corollary 3. Following Theorem 5 is one of our main results of this paper.

Lemma 4. *Let R be a ring and $0 \neq e^2 = e \in R$. If I is a maximal ideal of R then either $eIe = eRe$ or eIe is a maximal ideal of eRe.*

Remark. In the preceding lemma, if $ReR = R$ then $eJe \subsetneq eRe$ for every proper ideal J of R.

Theorem 5. *Let R be a ring and $0 \neq e^2 = e \in R$. If R is right quasi-duo then so is eRe.*

Remark. (1) In spite of Theorem 5, the quasi-duo condition is not a Morita invariant property by the n by n full matrix ring over a division ring which is neither right nor left quasi-duo, where n is any positive integer. (2) The converse of Theorem 5 does not hold in general. Let R be the n by n full matrix ring over a division ring D and $e \in R$ be the nonzero idempotent for which $(1,1)$-entry is 1_D and other entries are 0_D. Then $ReR = R$ and $eRe(\cong D)$ is right quasi-duo; but R is not right quasi-duo.

Moreover we observe some properties of polynomial rings and formal power series rings which are right quasi-suo.

Proposition 6. *For a ring R, the followings are equivalent:*

(1) *R is right quasi-duo.*
(2) *$R[[x;\theta]]$ is right quasi-duo for every endomorphism $\theta : R \to R$.*
(3) *$R[[x;\theta]]$ is right quasi-duo for some endomorphism $\theta : R \to R$.*
(4) *$R[[x]]$ is right quasi-duo,*

where $R[[x;\theta]]$ is the skew power series ring over R by θ, every element of which is of the form $\sum_{n=1}^{\infty} a_n x^n$, only subject to $xa = \theta(a)x$ for each $a \in R$.

Similarly we obtain the following result.

Lemma 7. *For a ring R, if $R[x]$ is right quasi-duo then R is right quasi-duo.*

Based on Proposition 6 and Lemma 7, we may raise the following question.

Question (1). Is $R[x]$ right quasi-duo if R is right quasi-duo?

Answer. Negative by the following Example 9.

Lemma 8. *Let R be a right primitive ring. Then $R[x]$ is right quasi-duo if and only if R is a field.*

Recall that a ring R is called a *PI-ring* if R satisfies a polynomial identity with coefficients in the ring of integers. We may conjecture that the answer of Question (1) with PI-condition is affirmative. However followings, although they are PI-rings, are counterexamples to the question.

Example 9. (1) Consider the field $F = \{0, 1, u, 1+u\}$ with $u^2 = 1+u$ and the automorphism $\theta : F \to F$ given by $\theta(\alpha) = \alpha^2$ for each $\alpha \in F$. Let $R = F[[t;\theta]]$ be the skew power series ring over R by θ with t its indeterminate, subject to $t\alpha = \theta(\alpha)t$ for each $\alpha \in F$. By Proposition 6, R is right quasi-duo. Since t^2 is central and $\alpha^2 + \alpha \in \mathbb{Z}_2$ for all $\alpha \in F$, R satisfies the polynomial identity

$$x_1(x_2x_3 - x_3x_2)^2 - (x_2x_3 - x_3x_2)^2 x_1.$$

Let $I = (1+tx)R[x]$ then since I is a proper right ideal of $R[x]$ there exists a maximal right ideal M of R such that $I \subseteq M$. Now assume that $R[x]$ is right quasi-duo, then M is two-sided and so

$$1 = u + u^2 = u(1+tx) + (1+tx)u^2 \in M,$$

a contradiction. Thus $R[x]$ cannot be right quasi-duo.

(2) Let R be the Hamilton quaternion over the field of real numbers. Then clearly R is right quasi-duo and right primitive; but by Lemma 8, $R[x]$ cannot be right quasi-duo.

The following fact, although the proof is simple, may give a motivation to our goal in this section.

Proposition 10. *[4, Lemma 3] For a ring R, if $R[x]$ is right duo then R is commutative.*

Note that right duo rings are right quasi-duo. So from Lemma 8 and Proposition 10, we ask the following question:

Question (2). Is R commutative if $R[x]$ is right quasi-duo for a ring R?

Answer. Negative by the following Example 11.

Example 11. Let F be a field and $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. Then R is right quasi-duo and

$$R[x] = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}[x] \cong \begin{pmatrix} F[x] & F[x] \\ 0 & F[x] \end{pmatrix}$$

is also right quasi-duo. But R is noncommutative.

Commutative rings are clearly PI-rings; hence we may consider another question:

Question (3). Let R be a PI-ring and $R[x]$ right quasi-duo. Is R then commutative?

Answer. Negative by Example 11. Note that the ring R is a PI-ring.

We will find a condition under which Question (2) may be true. For doing it, we need to prove the following that is one of our main results of this paper.

Theorem 12. *For a ring R, if $R[x]$ is right quasi-duo then $R/J(R)$ is commutative.*

Remark. As the converse of Theorem 12, one may conjecture that for a right quasi-duo ring R if $R/J(R)$ is commutative then $R[x]$ is right quasi-duo. However it fails in general by the following example.

Example 13. Let $\mathbb{C}$ be the field of complex numbers. Define a field isomorphism $\sigma : \mathbb{C} \longrightarrow \mathbb{C}$ by $\sigma(a+bi) = a - bi$ where a, b are real numbers. Next consider the skew power series ring $R = \mathbb{C}[[t;\sigma]]$, every element is of the form $\sum_{n=1}^{\infty} a_n t^n$ over $\mathbb{C}$, only subject to $t(a+bi) = (\sigma(a+bi))t$, where t is the indeterminate of R. Note that R is a right quasi-duo local domain and $R/J(R) \cong \mathbb{C}$; hence $R/J(R)$ is commutative. Next consider a maximal right ideal M of the polynomial ring $R[x]$ over R such that $1+(it)x \in M$. Assume that $R[x]$ is right quasi-duo. Then M is 2-sided and so

$$2i = i(1+(it)x) + (1+(it)x)i \in M,$$

which is a contradiction. Thus $R[x]$ is not right quasi-duo.

We now have a condition under which Question (2) may be true, from Theorem 12.

Corollary 14. *Let R be a semiprimitive ring. Then $R[x]$ is right quasi-duo if and only if R is commutative.*

In Corollary 14, the ring R is semiprimitive and right quasi-duo so it is reduced by [12, Corollary 2.4]; hence as a generalization of Corollary 14 we may ask whether R is commutative if R is reduced and $R[x]$ is right quasi-duo. But the following proposition shows that this question is equivalent to the corresponding question in the case where R is a domain. We raise it at the end of this paper.

Proposition 15. *For a ring R, the following statements are equivalent:*

(1) *If R is reduced and $R[x]$ is right quasi-duo then R is commutative.*
(2) *If R is a domain and $R[x]$ is right quasi-duo then R is commutative.*

Given a ring R we denote the prime radical of R, the set of all nilpotent elements of R and the Jacobson radical of $R[x]$ by $P(R)$, $N(R)$ and $J(R[x])$ respectively. In the followings we obtain some facts about $J(R[x])$ when $R[x]$ is right quasi-duo, related to $N(R)$ and $P(R)$.

Proposition 16. *Let R be a ring and suppose that $R[x]$ is right quasi-duo. Then $N(R)$ is an ideal of R and $J(R[x]) = N(R)[x]$.*

Corollary 17. *Let R be a ring with right Krull dimension (in the sense of Gabriel and Rentschler, see [6] for more detail). If $R[x]$ is right quasi-duo then $N(R) = P(R)$ and $J(R[x]) = P(R)[x]$.*

Corollary 18. *Let R be a ring which is right Goldie or satisfies ascending chain condition on both right and left annihilators. If $R[x]$ is right quasi-duo then $N(R) = P(R)$ and $J(R[x]) = P(R)[x]$.*

2. Connections between right quasi-duo rings and weakly right duo rings

In this section we study the connections between right quasi-duo rings and weakly right duo rings. A ring R is called *abelian* if every idempotent is central. First we recall the following fact.

Proposition 19. *[12, Proposition 2.2] Weakly right duo rings are abelian right quasi-duo rings.*

It is natural to ask the following question:

Question (4). Is an abelian right quasi-duo ring weakly right duo?

Answer. Negative by the following.

Example 20. Let F be a field and $S = F[t]$ be the polynomial ring over F with t its indeterminate. Define a ring homomorphism $\sigma : S \longrightarrow S$ by $\sigma(f(t)) = f(t^2)$. Next consider the skew power series ring $R = S[[x;\sigma]]$ over S by σ, every element of which is of the form $\sum_{n=1}^{\infty} a_n x^n$, only subject to $xa = \sigma(a)x$ for each $a \in S$. Note that R is an integral domain and so $0, 1$ are the only idempotents of R; hence R is abelian. Since S is clearly right quasi-duo, R is also right quasi-duo by Proposition 6. Next, notice that all coefficients of the elements of $x^n R$ are of the form $f(t^{2n}) \in S$ by the definition of σ and the property $xa = \sigma(a)x$ for $a \in S$, where n is any positive integer. So $x^n R$ cannot contain Rx^n and hence R is not weakly right duo.

Every factor ring of a given weakly right duo ring is also weakly right duo. So we may raise the following question.

Question (5). Is an abelian right quasi-duo ring R, such that $R/J(R)$ is weakly right duo, weakly right duo?

Answer. Negative by Example 20. For, since S is semiprimitive, $J(R) = Rx$ and so $R/J(R)$ is isomorphic to S; hence $R/J(R)$ is commutative.

Based on this answer we will consider a stronger condition than the condition that $R/J(R)$ is weakly right duo. Recall that a ring R is called *von Neumann regular* if for each a in R there exists an x in R such that $a = axa$; a ring R is called *strongly regular* if for each a in R there exists x in R such that $a = a^2x$; a ring R is called *π-regular* if for each a in R there exists an x in R and a positive integer $n = n(a)$, depending on a, such that $a^n = a^nxa^n$; a ring R is called *right weakly π-regular* if for each a in R there exists a positive integer $n = n(a)$, depending on a, such that $a^n \in a^nRa^nR$. A prime ideal P of a ring R is called *completely prime* if R/P is a domain.

Lemma 21. *For a right quasi-duo ring R the followings are equivalent:*

(1) *$R/J(R)$ is right weakly π-regular.*
(2) *$R/J(R)$ is strongly regular.*
(3) *$R/J(R)$ is von Neumann regular.*
(4) *$R/J(R)$ is π-regular.*

Now we obtain one of our main results of this paper from Lemma 21.

Theorem 22. *For a ring R suppose that $R/J(R)$ is right weakly π-regular and $J(R)$ is nil. Then the following are equivalent:*

(1) *R is weakly right duo.*
(2) *R is abelian and right quasi-duo.*

In Theorem 22, the condition "$J(R)$ is nil" is not superfluous by the following example.

Example 23. Let S be the quotient field of the polynomial ring $F[t]$ over a field F with t its indeterminate and define a ring homomorphism $\sigma : S \longrightarrow S$ by

$$\sigma\left(\frac{f(t)}{g(t)}\right) = \frac{f(t^2)}{g(t^2)}.$$

Next consider the skew power series ring $R = S[[x;\sigma]]$ over S by σ, every element of which is of the form $\sum_{n=1}^{\infty} a_nx^n$, only subject to $xa = \sigma(a)x$ for each $a \in S$. Note that R is an integral domain and so $0, 1$ are the only idempotents of R; hence R is abelian. Also note that R is local with $J(R) = Rx$, so it is right quasi-duo. Next, notice that

$$\text{all coefficients of the elements of } x^nR \text{ are of the form } \frac{f(t^{2n})}{g(t^{2n})}$$

by the definition of σ and the property $xa = \sigma(a)x$ for $a \in S$, where n is any positive integer. So x^nR cannot contain Rx^n and hence R is not weakly right duo. And $R/J(R)$ is isomorphic to S and hence it is right weakly π-regular. However $J(R)$ is not nil.

From Lemma 21 and Theorem 22, it is natural to ask the following question:

Question (6). Assume that R is abelian and right quasi-duo so that $J(R)$ is nil and $R/J(R)$ weakly right duo. Is then R weakly right duo?

Answer. Negative by the following example.

Example 24. Let F be a field and $S = F[t]$ be the polynomial ring over F with t its indeterminate. Define a ring homomorphism $\sigma : S \longrightarrow S$ by $\sigma(f(t)) = f(t^2)$. Consider the skew power series ring $T = S[[y;\sigma]]$ over S by σ, every element of which is of the form $\sum_{n=0}^{\infty} y^n a_n$, only subject to $ay = y\sigma(a)$ for each $a \in S$, . Next define

$$R = \{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in Mat_2(T) \mid a \in S,\ b \in T\}.$$

Since S is a domain, $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are the only idempotents in R; hence R is abelian. Every maximal right ideal of R is of the form

$$\{\begin{pmatrix} m & b \\ 0 & m \end{pmatrix} \in R \mid m \in M,\ b \in T\}$$

where M is a maximal ideal of S; but it is two-sided so R is right quasi-duo. Note that S is semiprimitive whence

$$J(R) = \{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in R \mid b \in T\}.$$

Thus $J(R)$ is nil and $R/J(R)$ is isomorphic to S so it is clearly weakly right duo. However

the right ideal $f^n R$ of R, generated by f^n, cannot contain the left ideal Rf^n of R, generated by f^n, using a computation similar to one in Example 20, where $f = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}$ and n is any positive integer. It then follows that R is not weakly right duo. Notice that $R/J(R)$ is not right weakly π-regular.

We end this paper with raising the following questions.

Questions.

(1). Is the quasi-duo condition left-right symmetric?

(2). Suppose that R is a domain. Then $R[x]$ is right quasi-duo if and only if R is commutative?

Acknowledgement. The author would like to express his gratitude to Professor Masahisa Sato for his giving the opportunity to present this material on Japan-Korea Ring Theory and Representation Theory Seminar joined with the 31st Japan Ring Theory and Representation Theory Symposium.

References

1. F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, Springer-Verlag New York, Inc. (1992).
2. H. Bass, *Finitistic dimension and a generalization of semiprimary rings*, Trans. Amer. Math. Soc. **95** (1960), 466–488.
3. G. F. Birkenmeier, J. Y. Kim and J. K. Park, *A connection between weak regularity and the simplicity of prime factor rings*, Proc. Amer. Math. Soc. **122** (1994), 53–58.
4. A. W. Chatters and C. R. Hajarnavis, *Rings with chain conditions*, Pitman Advanced Publishing Program (1980).
5. K. R. Goodearl, *Von Neumann regular rings*, Pitman, London (1979).
6. R. Gordon and J. C. Robson, *Krull dimension*, Memoirs Amer. Math. Soc. **133** (1973).
7. Y. Hirano, C. Y. Hong, J. Y. Kim and J. K. Park, *On strongly bounded rings and duo rings*, Comm. Algebra **23(6)** (1995), 2199–2214.
8. J. Lambek, *Lectures on rings and modules*, Blaisdell, Waltham (1966).
9. C. Lanski, *Nil subrings of Goldie rings are nilpotent*, Canad. J. Math. **21** (1969), 904–907.
10. T. H. Lenagan, *Nil ideals in rings with finite Krull dimension*, J. Algebra **29** (1974), 77–87.
11. L. H. Rowen, *Ring Theory*, Academic Press, Inc. (1991).
12. H. P. Yu, *On quasi-duo rings*, Glasgow Math. J. **37** (1995), 21–31.

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ホップ代数，リー双代数の拡大とコホモロジー

Extensions and Cohomology of Hopf Algebras, Lie Bialgebras

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ABSTRACT

The theory of group extensions was extended to Hopf algebra extensions (Singer, Hofstetter), enriched with a beautiful selfduality. Previously, special extensions $(kG)^* \rightarrowtail ? \twoheadrightarrow kF$ of a finite group algebra by the dual of such an algebra had been investigated by G.I. Kac, who showed an interesting, exact cohomology sequence involving the extension group. To obtain a variation of this exact sequence due to Kac with F, G replaced by finite-dimensional Lie algebras $\mathfrak{f}$, $\mathfrak{g}$ in characteristic zero, we shall discuss a correspondence between Hopf algebra extensions $(U\mathfrak{g})^\circ \rightarrowtail ? \twoheadrightarrow U\mathfrak{f}$ of the universal envelope $U\mathfrak{f}$ by the Hopf dual $(U\mathfrak{g})^\circ$ (or its appropriate Hopf subalgebra) and Lie bialgebra extensions $\mathfrak{g}^* \rightarrowtail ? \twoheadrightarrow \mathfrak{f}$ of $\mathfrak{f}$ (with zero co-bracket) by $\mathfrak{g}^*$ (with zero bracket).

The text of this report is written in Japanese. The survey article [4] written in English treats the same subject, which is available from the author on request.

当該研究のために文部省より科研費補助金の交付を得ています：
課題番号 10740012.

弁明：集会の会場に韓国からの方々をお見受けしたので，英語で発表しましたが，その評判は決して芳しくありませんでした．せめて日本の方々への埋め合わせにこの報告は日本語で行い，興味を持たれた日本語圏外の方には（よりフォーマルな）[4] を参照していただくことにします．集会の責任者，浅芝・西田両先生に申し上げたいのは，お世話になったお礼と，これに懲りずに今後もよろしくお願いします．

はじめに

ホップ代数学とは何だろうか．大ざっぱに言って，群（Γで表す）と可換ホップ代数（A で表す）とが対応する．この対応は反変，つまり矢印を逆にするものだから，両者の構造射を対応させて書けば次のようになる．

$$\begin{array}{ccc} \Gamma\times\Gamma & \{1\} & \Gamma \\ \downarrow \text{積} & \downarrow \text{単位元} & \downarrow \text{逆元} \\ \Gamma & \Gamma & \Gamma \end{array} \qquad\qquad \begin{array}{ccc} A\otimes A & k & A \\ \uparrow \Delta & \uparrow \varepsilon & \uparrow S \\ A & A & A \end{array}$$

ここにホップ代数の可換性を取り去って，

（非可換）ホップ代数 ＝ 群の概念の非可換化

という等式が成り立つ．この右辺は，近年 Drinfeld に従い「量子群」と呼ばれるが，この等式自体，35年余り前にホップ代数の研究を始めた Chase と Sweedler の念頭に存在していたのであり[1]，それ以来スローガン

「群論の結果をホップ代数に拡張しよう」

のもと，さまざまな分野の専門家を交え[2]研究が続けられてきた．日本でも

1) この辺りの歴史は竹内光弘先生に教わりました（数理研講究録 877, pp. 101-104 を参照のこと）．

2) ここに土井幸雄先生のことば「ホップ代数はみんなのもの」を引用したい．

故服部昭先生がこの研究対象を紹介されて以来，諸先輩によって誠実に研究が続けられてきたこと，数多くの優れた成果が生まれたことを誇りとしたい．

ところでホップ代数は，その双対が再びホップ代数をなすことからもわかるように，双対性を，言わば生まれながらにして持っている．それ故上記の「拡張」は双対性の美をもたらすが，反面，計算のむつかしさという試練も与える．筆者はこの分野で具体的な計算をしたい（この際コンピュータを使ってでも）．

以下，ある種典型的なホップ代数拡大を自由に計算しようとする筆者の最近の試みを，その動機となった G.I. Kac による約30年前の仕事の視点から，ざっくばらんに述べたい．

1. 群拡大について復習

Γ 群，M アーベル群 とする．

Γ のMによる（群）拡大とは，Γ を余核，M を核にもつ群の短完全列

(E) = $M \rightarrowtail E \overset{\pi}{\twoheadrightarrow} \Gamma$ をさす．2つの拡大 (E), (E') が同値であるとは，準同形（必然的に，同形）$f : E \longrightarrow E'$ が存在して図形

$$\begin{array}{ccccc} M & \rightarrowtail & E & \twoheadrightarrow & \Gamma \\ \| & & \downarrow f & & \| \\ M & \rightarrowtail & E' & \twoheadrightarrow & \Gamma \end{array}$$

を可換にすることをいう．

各拡大 (E) はMを左 Γ 加群にする作用 $\rightharpoonup : \Gamma \times M \longrightarrow M$ を次の式で決める．

$$\gamma \rightharpoonup m = \hat{\gamma} m \hat{\gamma}^{-1} \quad (\text{右辺で} M \subset E \text{と見る})$$

ここに $\gamma \in \Gamma$, $m \in M$; $\hat{\gamma}$ は γ の $\pi : E \longrightarrow \Gamma$ による引き戻し，つまり $\pi(\hat{\gamma}) = \gamma$ を満たすEの元であり，これの選び方に $\rightharpoonup$ は依らない．従って

立場を変えて，(E) は $(\Gamma, M, \rightharpoonup)$ に属す，と言い表してよい．$\rightharpoonup$を固定するとき，MacLane に従い

$$\mathrm{Opext}(\Gamma, M, \rightharpoonup) = (\Gamma, M, \rightharpoonup) \text{ に属す拡大の同値類全体}$$

と書く（Opは operation の意）．これは Baer 積によりアーベル群をなし，次のように群コホモロジーで記述される．

$$H^2(\Gamma, M) = \mathrm{Opext}(\Gamma, M, \rightharpoonup)$$

Γ のMによる拡大の同値類全体 $\mathrm{Ext}(\Gamma, M)$ は，すべての $\mathrm{Opext}(\Gamma, M, \rightharpoonup)$ の$\rightharpoonup$に関する disjoint union に等しいから， $\mathrm{Ext}(\Gamma, M)$ を捉えるには

①まずMを左Γ加群にする作用 $\rightharpoonup : \Gamma \times M \longrightarrow M$ をすべて求め，

②ついで各群 $\mathrm{Opext}(\Gamma, M, \rightharpoonup)$ を求めれば良い．

想像するに，ホップ代数 H，K が与えられたときホップ代数拡大 $K \rightarrowtail A \twoheadrightarrow H$ のすべてを捉えるには

①まず拡大の引き起こし得る作用 $\rightharpoonup : H \otimes K \longrightarrow K$ と余作用 $\rho : H \longrightarrow H \otimes K$ をすべて求め，

②ついで各 $\mathrm{Opext}(H, K, \rightharpoonup, \rho)$（アーベル群をなすべき）を求めれば良いだろう．

実際，H が余可換かつK が可換[3)]ならばこれは正しく，我々の考察はこの場合に限る．それでも，A として可換でも余可換でもない興味深いホップ代数が得られることを注意したい．

2. George I. Kac の仕事（1960年代後半）

以下，基礎体kを固定し，代数，ホップ代数，テンソル積等，すべて k 上のものとする．

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3)ホップ代数が可換とは代数として可換なこと，余可換とは余代数として可換，つまり合成 $H \xrightarrow{\Delta} H \otimes H \longrightarrow H \otimes H$ がΔに等しいことをいう．
$x \otimes y \longmapsto y \otimes x$

いま

$$F,\ G\ \text{有限群}$$

から２つの初等的なホップ代数を構成する．

ⓐ群環 kF: これはFの各元xを群様元にとる，つまり

$$\Delta(x) = x \otimes x,\quad \varepsilon(x) = 1,\quad S(x) = x^{-1}$$

と定めることにより余可換ホップ代数をなす．

ⓑ群環の線形双対 $k^G = (kG)^*$ (= 関数 $G \longrightarrow k$ 全体): これは $(s)_{s \in G}$ の双対基底 $(e_s)_{s \in G}$ を直交巾等元にもつ可換ホップ代数で，

$$\Delta(e_s) = \sum_{tu=s} e_t \otimes e_u,\quad \varepsilon(e_s) = \delta_{1s},\quad S(e_s) = e_{s^{-1}}$$

を構造射とする．

さて，
$$(A) = k^G \rightarrowtail A \twoheadrightarrow kF$$
の形のホップ代数拡大を考える．この双対 $(A^*) = kG \twoheadleftarrow A^* \leftarrowtail k^F$ も同じ形の拡大であることに注意する．ホップ代数拡大の定義は述べないが，群の場合と同様にホップ代数の短完全列が定義でき，上記(A)が短完全列のときこれを kF の k^G による拡大と呼ぶ．ここで問題は

①拡大(A)の引き起こし得る作用，余作用はどんな条件を満たすか

②各 Opext(kF, k^G,....) は（コホモロジカルに）どう記述されるか

であることを思い出そう．

これらに答えるため，まず群の接合積につき復習する．可換代数R上の<u>左F接合積</u>とは，Fの各元で添字付けられた $\{u_x\}_{x \in F}$ を基底にもつ自由左R加群

$$R \mathop{*}_{\rightharpoonup, \sigma} F = \bigoplus_x Ru_x$$

であって次で決まる積を持つ代数をいう．

$$(au_x)(bu_y) = a(x \rightharpoonup b)\,\sigma(x, y)u_{xy} \quad (a, b \in R;\ x, y \in F)$$

ここに，積が結合律を満たすべく

$$\rightharpoonup : F \times R \longrightarrow R$$

はRの自己同形を与える作用，

$$\sigma : F \times F \longrightarrow R^{\times}$$

は単数群$R^{\times}$（$\rightharpoonup$の制限によりF加群）に関する2コサイクルである．

Fの作用と両立する代数射 $\varepsilon : R \longrightarrow k$ が存在すれば，この接合積は一種の拡大

$$R \rightarrowtail R \underset{\rightharpoonup,\sigma}{*} F \twoheadrightarrow kF$$

と思える．但し，この全射は $au_x \longmapsto \varepsilon(a)x$ で与えられ，弱い意味のセクション$u_x \longleftarrow\!\mid x$ をもつのが特徴である．次の命題（H.-J. Schneider の結果のごく特別な場合）は，先の拡大(A) 及びその双対（A^*）がこの種の拡大であることをいうものである．

<u>命題</u>(Schneider)：代数Aは左F接合積に等しく，代数 A^* は右G接合積に等しい．より正確に，あるデータ

$$\rightharpoonup : F \times k^G \longrightarrow k^G, \qquad \sigma : F \times F \longrightarrow (k^G)^{\times},$$

$$\leftharpoonup : k^F \times G \longrightarrow k^F, \qquad \tau : G \times G \longrightarrow (k^F)^{\times}$$

を以て $\qquad A = k^G \underset{\rightharpoonup,\sigma}{*} F, \quad A^* = G \underset{\leftharpoonup,\tau}{*} k^F.$

ここで$\rightharpoonup$と$\leftharpoonup$は拡大(A) に固有で基底のとり方に依らない．しかし，無限次元も含んだ一般のホップ代数拡大を考えるには，$\rightharpoonup$の線形化

$$\rightharpoonup : kF \otimes k^G \longrightarrow k^G$$

と$\leftharpoonup$の双対

$$\rho = (\leftharpoonup)^* : kF \longrightarrow kF \otimes k^G$$

を(A) が引き起こす構造と思うのが良い．これらが満たすべき条件は，さらに

$$x \rightharpoonup e_s = e_{s \triangleleft x^{-1}}, \quad e_x \leftharpoonup s = e_{s^{-1} \triangleright x} \quad (x \in F,\ s \in G)$$

で決まる作用（集合 G, F の置換を引き起こす）

$$G \overset{\triangleleft}{\longleftarrow} G \times F \overset{\triangleright}{\longrightarrow} F$$

（つまり⇀と ◁，↼と ▷ が互いに転置）を用いると，次の命題（竹内の結果のごく特別な場合）にいうようにうまく表せる．これが前述の問題①に対する答．

<u>命題</u>（竹内）：作用 ▷, ◁ の満たすべき条件は，(F, G, ▷, ◁)が<u>整合ペア</u>[4)] (<u>matched</u> <u>pair</u>)なること，つまり直積集合 F × G が

$$(x, s)(y, t) = (x(s \triangleright y), (s \triangleleft y)t) \quad (x, y \in F;\ s, t \in G)$$

を積として (1, 1) を単位元にもつ群をなすこと．この群を $F \bowtie G$ とかく．

F と G を部分群として含む群 E に対し，分解

$$F \times G = E, \quad (x, s) \longleftrightarrow xs$$

が成り立てば，(F, G)は $F \bowtie G = E$ となるような唯一の方法で整合ペアをなす．

例：対称群 S_n $(n > 2)$ は次に定める2つの部分群を含む．

$C_n = \langle a \rangle =$ 巡回置換 $a = (1, \ldots, n)$ の生成する部分群，

$S_{n-1} = \{s \in S_n \mid s(n) = n\}$.

分解 $C_n \times S_{n-1} = S_n$ が成り立つから (C_n, S_{n-1}) は整合ペアをなし，$C_n \bowtie S_{n-1} = S_n$．一方の作用 $\triangleright : S_{n-1} \times C_n \longrightarrow C_n$ は

$$s \triangleright 1 = 1, \quad s \triangleright a^i = a^{s(i)} \quad (0 < i < n)$$

と表せるが，もう一方の $\triangleleft : S_{n-1} \times C_n \longrightarrow S_{n-1}$ を書き下すのは難しい（トリビアルな n = 3 のときを除いては）．

4) ▷, ◁ を構造と考えて (F, G) = (F, G, ▷, ◁) をペアと呼ぶ．

次の Kac [1] の結果が前述の問題②に答える．

<u>命題</u>(Kac) ：固定した$(kF,\ k^G,\ \rightharpoonup,\ \rho)$に付随する拡大の同値類全体，それを

$$\mathrm{Opext}(kF,\ k^G) = \mathrm{Opext}(kF,\ k^G,\ \rightharpoonup,\ \rho)$$

で表す，は自然にアーベル群をなし，次に挙げる双複体 $C_H^{\cdot\cdot}$（H は Hopf の意）の全１コホモロジー $H^1\mathrm{Tot}(C_H^{\cdot\cdot})$ に同形である．

$$\begin{array}{ccccc} & & \vdots & & \vdots \\ & & \uparrow & & \uparrow \\ C_H^{\cdot\cdot} = & & \mathrm{Map}(G^2 \times F,\ k^\times) & \longrightarrow & \mathrm{Map}(G^2 \times F^2,\ k^\times) \longrightarrow \ \cdots\cdots \\ & & \uparrow & & \uparrow \\ & & \mathrm{Map}(G \times F,\ k^\times) & \longrightarrow & \mathrm{Map}(G \times F^2,\ k^\times) \longrightarrow \ \cdots\cdots \end{array}$$

おおざっぱに，この双複体は k のトリビアルな左F加群また右G加群としての，２つの標準的分解から引き起こされる．

Opext と H^1 の間の同形は次のように与えられる．命題(Schneider) において，A に付随するσを写像$G \times F \times F \longrightarrow k^\times$と，$A^*$に付随する$\tau$を写像 $G \times G \times F \longrightarrow k^\times$と見なせば，$\sigma$は $C_H^{\cdot\cdot}$の最下行の１コサイクルに，τは最左列の１コサイクルに，さらに（A の積と余積の両立条件のために）$(\sigma,\ \tau)$ は全１コサイクルになる．そのコホモロジー類は，A や A^* の基底の選び方に依らずに決まり，(A) の同値類にそのコホモロジー類を対応させて同形が得られる．

Kac はさらに次の興味深い完全列を示した．

<u>定理</u>(Kac) ：次の完全列が成り立つ．

$$H^2(F \bowtie G,\ k^\times) \longrightarrow H^2(F,\ k^\times) \oplus H^2(G,\ k^\times) \longrightarrow$$

$$\mathrm{Opext}(kF,\ k^G) \longrightarrow H^3(F \bowtie G,\ k^\times) \longrightarrow H^3(F,\ k^\times) \oplus H^3(G,\ k^\times).$$

ここに $H^\cdot$ はトリビアル加群 $k^\times$ に係数をもつ群コホモロジーを表す．

Kac の仕事はこれに留まらないが，少なくともホップ代数びとの間では長いこと忘れられていたようである（かつては知られていた痕跡が，例えば Kaplansky のシカゴ講義録に見られる）．筆者が Kac の仕事を関根義弘氏（静岡大）から教えて頂いた時[5]，解析の分野と思われる人が，30年も前にこれだけのことをしていたことに，感心すると同時に少なからずショックを受けた．そこから立ち直るため，また Kac 完全列の実用性を示すため，これを応用して次の計算をした．

例：前の例から得られるペア $(kC_n, k^{S_{n-1}})$ に対し，

$$\mathrm{Opext}(kC_n, k^{S_{n-1}}) = \begin{cases} k^\times/(k^\times)^n & (n \neq 4) \\ k^\times/(k^\times)^8 & (n = 4). \end{cases}$$

3. 一般のホップ代数拡大

Kac 完全列を除く先の結果は，kF, k^G をそれぞれ

H 余可換ホップ代数， K 可換ホップ代数

に替えて成り立つ（Singer, Hofstetterらに負う．但し彼らも Kac の仕事は知らなかったと思う）．

作用 $\rightharpoonup : H \otimes K \longrightarrow K$ と余作用 $\rho : H \longrightarrow H \otimes K$ が然るべき条件を満たすとき $(H, K, \rightharpoonup, \rho)$ を <u>Singer ペア</u>[6]（$\rightharpoonup$, ρ を構造と考えて）と呼べば，このペアに付随する（正確には，クレフト）ホップ代数拡大 $K \rightarrowtail A \twoheadrightarrow H$ の同値類全体

$$\mathrm{Opext}(H, K) = \mathrm{Opext}(H, K, \rightharpoonup, \rho)$$

が双テンサー積 $\otimes_K^H$ に関しアーベル群をなし，ある双複体

[5] 1994年頃だったと思う．改めて関根氏に感謝します．

[6] これは通常アーベル整合ペア (abelian matched pair)と呼ばれるが，前出の「整合ペア」と区別するため，W. Singer に敬意を表しこの用語を提案する．

$$
\begin{array}{ccccc}
 & \vdots & & \vdots & \\
 & \uparrow & & \uparrow & \\
C_H^{\cdot\cdot} = & \mathrm{Reg}(H,\ K^{\otimes 2}) & \longrightarrow & \mathrm{Reg}(H^{\otimes 2},\ K^{\otimes 2}) & \longrightarrow \ldots\ldots \\
 & \uparrow & & \uparrow & \\
 & \mathrm{Reg}(H,\ K) & \longrightarrow & \mathrm{Reg}(H^{\otimes 2},\ K) & \longrightarrow \ldots\ldots
\end{array}
$$

の全１コホモロジー $H^1\mathrm{Tot}(C_H^{\cdot\cdot})$ に同形である．Reg はたたみ込積に関し可逆な線形写像のなすアーベル群を表す．

特に $H = kF,\ K = k^G$ のときには，各 Singer ペア $(kF,\ k^G)$ は一意に決まる整合ペア $(F,\ G)$ からくる．

$\mathfrak{f}$, $\mathfrak{g}$ を有限次元リー代数とするとき

$H = U\mathfrak{f}$（普遍包絡環）， $K = U\mathfrak{g}$ の双対

に対するKac 完全列を得たい．そのために，まずKac の結果のリー双代数版を考える．

4. リー双代数拡大

有限次元ベクトル空間 $\mathfrak{l}$ が $\delta : \mathfrak{l} \longrightarrow \mathfrak{l}\otimes\mathfrak{l}$ を余ブラケットにもつリー余代数であるとは，線形双対 $\mathfrak{l}^*$ が $\delta^* : \mathfrak{l}^* \otimes \mathfrak{l}^* = (\mathfrak{l}\otimes\mathfrak{l})^* \longrightarrow \mathfrak{l}^*$ をブラケットにもつリー代数であること．リー双代数とはリー代数かつリー余代数 $\mathfrak{l}$ で両立条件

$$\delta[a,\ b] = a\,\delta(b) + \delta(a)b \qquad (a,\ b \in \mathfrak{l})$$

を満たすものをいう (Drinfeld)．この右辺において，$\mathfrak{l}$ は $\mathfrak{l}\otimes\mathfrak{l}$ に左及び右から対角的に随伴 (adjoint) により作用する．

以下を通じて

$\mathfrak{f}$, $\mathfrak{g}$ 有限次元リー代数

とし，$\mathfrak{f}$ をゼロ余ブラケットをもつ（いわば余可換）リー双代数と見る．

$\mathfrak{g}^*$は自然にリー余代数，これをゼロブラケットをもつ（可換）リー双代数と見る．

$\mathfrak{f}$の $\mathfrak{g}^*$ によるリー双代数拡大とは，リー双代数の列 $(\ell) = \mathfrak{g}^* \rightarrowtail \ell \twoheadrightarrow \mathfrak{f}$ でベクトル空間の列と見て短完全なるものをいう．そのような拡大 (ℓ) は（リー双代数の）Singerペアと呼ぶべき（$\mathfrak{f}$, $\mathfrak{g}^*$, $\rightharpoonup$, ρ）に付随する．この作用 $\rightharpoonup: \mathfrak{f} \otimes \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$ と余作用 $\rho: \mathfrak{f} \longrightarrow \mathfrak{f} \otimes \mathfrak{g}^*$ の満たすべき条件は，$\rightharpoonup$の転置 $\triangleleft: \mathfrak{g} \otimes \mathfrak{f} \longrightarrow \mathfrak{g}$ と ρ^* の転置 $\triangleright: \mathfrak{g} \otimes \mathfrak{f} \longrightarrow \mathfrak{f}$（共にリー加群の作用）を用いると，次のようにうまく表せる．

命題：$\rightharpoonup$, ρ が満たすべき条件は（$\mathfrak{f}$, $\mathfrak{g}$, $\triangleright$, $\triangleleft$）が（リー代数の）整合ペアなること，つまり直和空間 $\mathfrak{f} \oplus \mathfrak{g}$ が

$$[x \oplus s, y \oplus t] = ([x, y] + s \triangleright y - t \triangleright x) \oplus (s \triangleleft y - t \triangleleft x + [s, t])$$

をブラケットとしてリー代数をなすこと．このリー代数を $\mathfrak{f} \bowtie \mathfrak{g}$ とかく．

群 vs リー：全く平行している．

ホップ代数拡大 $k^G \rightarrowtail A \twoheadrightarrow kF$ はある（ホップ代数の）Singerペア（kF, k^G, $\rightharpoonup$, ρ)に付随する．そのようなペアは群の整合ペア（F, G, $\triangleright$, $\triangleleft$）と対応し，さらにそれはより大きな群Eの分解 $F \times G = E$ からくる．

リー双代数拡大 $\mathfrak{g}^* \rightarrowtail \ell \twoheadrightarrow \mathfrak{f}$ はある（リー双代数の）Singerペア（$\mathfrak{f}$, $\mathfrak{g}^*$, $\rightharpoonup$, ρ)に付随する．そのようなペアはリー代数の整合ペア（$\mathfrak{f}$, $\mathfrak{g}$, $\triangleright$, $\triangleleft$）と対応し，さらにそれはより大きなリー代数 $\mathfrak{e}$の分解 $\mathfrak{f} \oplus \mathfrak{g} = \mathfrak{e}$ からくる．

例：ξ, η をkの任意の元とする．2次元ベクトル空間 $\mathfrak{e} = kx \oplus ks$ は

$$[s, x] = \xi x \oplus \eta s$$

で決まるブラケットによりリー代数をなす．分解 $kx \oplus ks = \mathfrak{e}$ は1次元

リー代数からなる (kx, ks) を

$$s \triangleright x = \xi x, \quad s \triangleleft x = \eta s$$

で決まる作用を伴う整合ペアとする．

Kac の結果のリー双代数版が下に得られる．

<u>命題</u>：固定した Singer ペア $(\mathfrak{s}, \mathfrak{g}^*, \rightharpoonup, \rho)$ に付随するリー双代数拡大の同値類全体 $\mathrm{Opext}(\mathfrak{s}, \mathfrak{g}^*, \rightharpoonup, \rho)$ は自然にアーベル群をなし，ある双複体

$$\begin{array}{cccccc}
 & \vdots & & \vdots & & \\
 & \uparrow & & \uparrow & & \\
C_L^{\cdot\cdot} = & \mathrm{Hom}(\Lambda^2\mathfrak{g} \otimes \mathfrak{s}, k) & \longrightarrow & \mathrm{Hom}(\Lambda^2\mathfrak{g} \otimes \Lambda^2\mathfrak{s}, k) & \longrightarrow & \cdots\cdot \\
 & \uparrow & & \uparrow & & \\
 & \mathrm{Hom}(\mathfrak{g} \otimes \mathfrak{s}, k) & \longrightarrow & \mathrm{Hom}(\mathfrak{g} \otimes \Lambda^2\mathfrak{s}, k) & \longrightarrow & \cdots\cdot
\end{array}$$

の全 1 コホモロジー $H^1\mathrm{Tot}(C_L^{\cdot\cdot})$ に同形である（下添 L は Lie の意）．

おおざっぱに，この双複体は k のトリビアルな左 $\mathfrak{s}$ 加群及び右 $\mathfrak{g}$ 加群としての自由分解を与える，2 つの Chevalley-Eilenberg 複体からくる．

例：$\mathfrak{s}$, $\mathfrak{g}$ が共に 1 次元ならば，$\mathrm{Hom}(\mathfrak{g} \otimes \mathfrak{s}, k)$を除き $C_L^{\cdot\cdot}$ の各項はゼロだから，どんな $\rightharpoonup, \rho$に対しても

$$\mathrm{Opext}(kx, (ks)^*, \rightharpoonup, \rho) = 0.$$

<u>定理</u>：次の完全列が成り立つ．

$$H^2(\mathfrak{s} \bowtie \mathfrak{g}, k) \longrightarrow H^2(\mathfrak{s}, k) \oplus H^2(\mathfrak{g}, k) \longrightarrow$$
$$\mathrm{Opext}(\mathfrak{s}, \mathfrak{g}^*) \longrightarrow H^3(\mathfrak{s} \bowtie \mathfrak{g}, k) \longrightarrow H^3(\mathfrak{s}, k) \oplus H^3(\mathfrak{g}, k).$$

ここに $H^\cdot$ はトリビアル加群 k に係数を持つリー代数コホモロジーを表す．

コホモロジーに関して有限群とリー代数の相違は，リー代数に半単純の概念があり，その種のリー代数の低次コホモロジーが消滅する(Whitehead Lemma)点にある．これを利用し次を得る．

系：基礎体kの標数＝ゼロと仮定する．Singerペア($\mathfrak{g}$, $\mathfrak{g}^*$, $\rightharpoonup$, ρ)において

1) $\mathfrak{g}$が半単純かつ$\rightharpoonup$がゼロ写像　又は　2) $\mathfrak{g}$が半単純かつρがゼロ写像ならば

$$\mathrm{Opext}(\mathfrak{g}, \mathfrak{g}^*, \rightharpoonup, \rho) = 0.$$

以下を通じて，基礎体kの標数＝ゼロと仮定する．

5. ホップ代数拡大からリー双代数拡大へ
ケースI－核が最大双対 $(U\mathfrak{g})^\circ$ の場合

$\mathfrak{g}$の普遍包絡環$U\mathfrak{g}$は$\mathfrak{g}$の各元xを原始元，つまり$\Delta(x) = 1 \otimes x + x \otimes 1$として余可換ホップ代数をなす．

一般に，無限次元ホップ代数Aに対し，その線形双対A^*はホップ代数をなさない．A^*の元のうち，ある有限余次元のイデアル$I \subset A$を消すもの全体，つまり

$$A^\circ = \bigcup_I I^\perp, \quad I^\perp = \{p \in A^* | p(I) = 0\}$$

はA^*に含まれる最大のホップ代数をなす（これは双対代数A^*の部分代数，余代数として有限次元余代数$(A/I)^*(= I^\perp)$の有向ユニオン）．

我々は普遍包絡環$U\mathfrak{g}$の"最大双対"$(U\mathfrak{g})^\circ$を考える．このホップ代数は可換で，Hochschildによれば，これが有限生成であるための必要十分条件は$\mathfrak{g}$が完全なること，つまり$\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$を満たすことである．

$U\mathfrak{g}$の$(U\mathfrak{g})^\circ$によるホップ代数拡大$(U\mathfrak{g})^\circ \rightarrowtail A \twoheadrightarrow U\mathfrak{g}$を知るために，これらを$\mathfrak{g}$の$\mathfrak{g}^*$によるリー双代数拡大$\mathfrak{g}^* \rightarrowtail \mathfrak{l} \twoheadrightarrow \mathfrak{g}$と結びつける．

まず双方の Singer ペアの間の関係を見よう．

<u>命題</u>： $(\mathfrak{f}, \mathfrak{g}^*)$ をリー双代数の Singer ペアにする構造 $(\rightharpoonup, \rho)$ 全体から $(U\mathfrak{f}, (U\mathfrak{g})^\circ)$ をホップ代数の Singer ペアにする構造 $(\rightharpoonup', \rho')$ 全体への自然な単射が存在する．$\mathfrak{g}$ が完全ならばこれは全単射である．

ここにいう対応は，第７節でもう少し一般的な状況のもと記述される．次にいう２つの Opext 群の間の同形が第１の主結果である．

<u>定理</u>：$(\mathfrak{f}, \mathfrak{g}^*, \rightharpoonup, \rho)$ をリー双代数の Singer ペアとすると，上にいうようにホップ代数の Singer ペア $(U\mathfrak{f}, (U\mathfrak{g})^\circ, \rightharpoonup', \rho')$ が引き起こされる．このとき自然な同形

$$\mathrm{Opext}(U\mathfrak{f}, (U\mathfrak{g})^\circ, \rightharpoonup', \rho') \cong \mathrm{Opext}(\mathfrak{f}, \mathfrak{g}^*, \rightharpoonup, \rho)$$

が存在する．

この同形がどう与えられるかは少々面倒なので述べないが，わかっているのは左辺から右辺への対応で，逆の対応はまだわかっていないことを注意する（この事実が本節のタイトルにつながった）．

<u>系１</u>：ホップ代数の Singer ペア $(U\mathfrak{f}, (U\mathfrak{g})^\circ)$ がリー双代数の Singer ペア $(\mathfrak{f}, \mathfrak{g}^*)$ からくるものであれば，次の完全列が成り立つ．

$$H^2(\mathfrak{f} \bowtie \mathfrak{g}, k) \longrightarrow H^2(\mathfrak{f}, k) \oplus H^2(\mathfrak{g}, k) \longrightarrow$$
$$\mathrm{Opext}(U\mathfrak{f}, (U\mathfrak{g})^\circ) \longrightarrow H^3(\mathfrak{f} \bowtie \mathfrak{g}, k) \longrightarrow H^3(\mathfrak{f}, k) \oplus H^3(\mathfrak{g}, k).$$

<u>系２</u>：ホップ代数の Singer ペア $(U\mathfrak{f}, (U\mathfrak{g})^\circ, \rightharpoonup, \rho)$ において 1) $\mathfrak{f}$ が半単純かつ $\rightharpoonup$ がトリビアル　又は　2) $\mathfrak{g}$ が半単純かつ ρ がトリビアルならば

$$\mathrm{Opext}(U\mathfrak{f}, (U\mathfrak{g})^\circ, \rightharpoonup, \rho) = 0.$$

系３：$\mathfrak{g}$が完全ならば，双方の拡大の同値類全体の間に自然な１対１対応が存在する．つまり拡大の同値類全体をExt で表せば，

$$\mathrm{Ext}(U\mathfrak{h},\ (U\mathfrak{g})^{\circ}) = \mathrm{Ext}(\mathfrak{h},\ \mathfrak{g}^{*}).$$

（一般には包含 $\supset$ が成り立つのみ．）

前定理と前節の定理，前節の系，前命題の最後の主張をそれぞれ組み合わせることにより，系１，系２，系３が従う．

6．ホップ代数拡大からリー双代数拡大へ
ケースII－核が既約双対 $(U\mathfrak{g})'$ の場合

この節を通し，$\mathfrak{g}$はベキ零リー代数と仮定し，$(U\mathfrak{g})^{\circ}$の中で１を含む最大既約部分余代数，必然的に部分ホップ代数，を $(U\mathfrak{g})'$ とかく．余代数が既約とは，その余根基が１次元なることをいう．

再び Hochschild によれば，$(U\mathfrak{g})'$は有限生成可換既約ホップ代数であり，逆にこの種のホップ代数はこの形をしている．

前節の $(U\mathfrak{g})^{\circ}$ を $(U\mathfrak{g})'$ に替えてパラレルな結果が得られる．

命題：（$\mathfrak{h}$, $\mathfrak{g}^{*}$）をリー双代数の Singer ペアにする構造（$\rightharpoonup$, ρ）のうち，ρの随伴たる作用 $\triangleright$: $\mathfrak{g}\otimes\mathfrak{h}\longrightarrow\mathfrak{h}$ がベキ零なるもの全体と （$U\mathfrak{h}$, $(U\mathfrak{g})'$）をホップ代数のSingerペアにする構造（$\rightharpoonup'$, ρ'）全体との間に自然な１対１対応が存在する．

定理：上において互いに対応する構造 （$\rightharpoonup$, ρ), （$\rightharpoonup'$, ρ'）をとれば，自然な同形

$$\mathrm{Opext}(U\mathfrak{h},\ (U\mathfrak{g})',\ \rightharpoonup',\ \rho') = \mathrm{Opext}(\mathfrak{h},\ \mathfrak{g}^{*},\ \rightharpoonup,\ \rho)$$

が存在する．（前命題と合わせて $\mathrm{Ext}(U\mathfrak{h},\ (U\mathfrak{g})') \subset \mathrm{Ext}(\mathfrak{h},\ \mathfrak{g}^{*})$.）

例：多項式ホップ代数 k[x], k[p] (x, p は原始元）に対し，Ext(k[x], k[p]) を決定しよう．$\mathfrak{g} = kx$, $\mathfrak{a} = ks$ （1次元リー代数）にとれば

$$U\mathfrak{g} = k[x], \quad U\mathfrak{a} = k[s], \quad (U\mathfrak{a})' = k[p].$$

ここに $p \in (U\mathfrak{a})^*$ は $p(s^n) = \delta_{1n}$ で決まる．第4節最初の例で見たように，$(\mathfrak{g}, \mathfrak{a})$ を整合ペアにする作用は k の勝手な元 ξ, η を以て

$$s \triangleright x = \xi x, \quad s \triangleleft x = \eta s$$

で与えられる．作用 $\triangleright$ がベキ零となる条件は $\xi = 0$ だから，応じてトリビアルな余作用 $\rho' : k[x] \longrightarrow k[x] \otimes k[p]$, $\rho'(a) = a \otimes 1$ と $\triangleleft$ に対応する作用

$$\rightharpoonup' : k[x] \otimes k[p] \longrightarrow k[p], \quad x \rightharpoonup' p^n = n\eta p^n \quad (n = 0, 1, \ldots)$$

が (k[x], k[p]) をホップ代数の Singer ペアにする構造を尽くす．このような各ペア (k[x], k[p]) に対し，前定理と第4節第2の例から

$$\mathrm{Opext}(k[x], k[p]) = \mathrm{Opext}(kx, (ks)^*) = 0.$$

こうして，k[x] の k[p] による拡大は η のみで決まることがわかり，

$$\mathrm{Ext}(k[x], k[p]) = k.$$

7. 2つのケースの統一

この目的のため，$U\mathfrak{a}$ の有限余次元イデアルからなる集合 $\mathfrak{I}$ で，次の条件を満たすものをとる．

① $I_1, I_2 \in \mathfrak{I}$ ならば $J \in \mathfrak{I}$ が存在して $J \subset I_1 I_2$;
② $I \in \mathfrak{I}$ ならば $J \in \mathfrak{I}$ が存在して $\Delta(J) \subset U\mathfrak{a} \otimes I + I \otimes U\mathfrak{a}$ かつ $S(J) \subset I$;
③ $\mathfrak{a}$ の生成するイデアル $I_{\mathfrak{a}}$，等しく $\varepsilon : U\mathfrak{a} \longrightarrow k$ の核，が $\mathfrak{I}$ に含まれる．

これらの条件により，

$$(U\mathfrak{g})^\circ_{\mathfrak{I}} := \bigcup_{I\in\mathfrak{I}} I^\perp, \quad I^\perp = \{p \in (U\mathfrak{g})^* \mid p(I) = 0\}$$

が $(U\mathfrak{g})^\circ$ の部分ホップ代数をなすことが保証される．

さらに次のような位相を導入する．

ⓐ $U\mathfrak{h}$ はディスクリート空間；

ⓑ $U\mathfrak{g}$ は $\mathfrak{I}$ を 0 の基本近傍系にもつ [5] の意味の位相ベクトル空間（すると $(U\mathfrak{g})^\circ_{\mathfrak{I}}$ は $U\mathfrak{g}$ の連続双対に等しい）；

ⓒ $\{a_\lambda\}$ を $U\mathfrak{h}$ の基底とするとき，テンソル積 $U\mathfrak{g}\otimes U\mathfrak{h}$ はすべての $\sum_\lambda I_\lambda\otimes ka_\lambda$ $(I_\lambda\in\mathfrak{I})$ を 0 の基本近傍系にもつ位相ベクトル空間．

いま，リー双代数の Singer ペア $(\mathfrak{h}, \mathfrak{g}^*, \rightharpoonup, \rho)$ を固定する．対応するリー代数の整合ペアの作用 $\mathfrak{g}\xleftarrow{\triangleleft}\mathfrak{g}\otimes\mathfrak{h}\xrightarrow{\triangleright}\mathfrak{h}$ を一意的に拡張して，$(U\mathfrak{h}, U\mathfrak{g})$ をホップ代数の整合ペア（定義は述べない）とする作用

$$U\mathfrak{g}\xleftarrow{\triangleleft'} U\mathfrak{g}\otimes U\mathfrak{h}\xrightarrow{\triangleright'} U\mathfrak{h}$$

が得られる．

<u>定理</u>：これらの作用が共に連続であると仮定する．すると，$\triangleleft'$ の転置として作用 $\rightharpoonup' : U\mathfrak{h}\otimes(U\mathfrak{g})^\circ_{\mathfrak{I}}\longrightarrow(U\mathfrak{g})^\circ_{\mathfrak{I}}$ が，$\triangleright'$ の随伴として余作用 $\rho' : U\mathfrak{h}\longrightarrow U\mathfrak{h}\otimes(U\mathfrak{g})^\circ_{\mathfrak{I}}$ が引き起こされ，$(U\mathfrak{h}, (U\mathfrak{g})^\circ_{\mathfrak{I}}, \rightharpoonup', \rho')$ はホップ代数の Singer ペアをなす．さらに自然な群準同形

$$\mathrm{Opext}(U\mathfrak{h}, (U\mathfrak{g})^\circ_{\mathfrak{I}}, \rightharpoonup', \rho')\longrightarrow\mathrm{Opext}(\mathfrak{h}, \mathfrak{g}^*, \rightharpoonup, \rho)$$

が存在し，もし

$$H^2(\mathfrak{g}, (U\mathfrak{g})^\circ_{\mathfrak{I}}) = 0$$

が成り立てば，これは同形となる．但し，係数域 $(U\mathfrak{g})^\circ_{\mathfrak{I}}$ を左または右 $\mathfrak{g}$ 加群と見る方法は，$\mathfrak{g}$ の $U\mathfrak{g}$ への右または左乗法の転置による．

この定理が前２節の定理を統一することが，次のようにわかる．

まず，$\mathcal{J}$ = {すべての有限余次元イデアル} にとれば $(U\mathfrak{g})^{\circ}_{\mathcal{J}} = (U\mathfrak{g})^{\circ}$．H.-J. Schneider による最近の結果

$$H^2(\mathfrak{g},\ (U\mathfrak{g})^{\circ}) = 0$$

と合わせて第５節の定理が従う．

次に，$\mathcal{J} = \{I_{\mathfrak{g}}^{\,n} \mid n = 1,\ 2, \ldots\}$ にとれば $(U\mathfrak{g})^{\circ}_{\mathcal{J}} = (U\mathfrak{g})'$． Koszul による古い結果

$$H^n(\mathfrak{g},\ (U\mathfrak{g})') = 0 \quad (n > 0,\ \mathfrak{g}\text{ベキ零})$$

と合わせて第６節の定理が従う．

定理にいう同形の証明には，双方の群が「導来関手のようなもの」であることに注意し，お馴染みの単射分解の一意性を用いる．ただ位相を持つ加群からなる非アーベル圏で考える必要があり，多少デリケートな議論をする（詳細は [2] の第６節）．

そこで用いられる位相（[5] に依る）は基礎体を（たとえ実数体や複素数体であろうとも）ディスクリートとするもので，一見奇妙に見える．しかし，少なくとも代数的には自然なもので，実際ガロア理論に見られる位相に近い[7]．

第４，５節の詳細は [2] にあり，第６，７節の詳細は [3] に含まれる予定．英文のサーベイ [4] も加え，ご希望により喜んでお送りします．

文　献

[1] G.I. Kac, Extensions of groups to ring groups, Math. USSR Sbornik 5(1968), 451-474.

[2] A. Masuoka, Extensions of Hopf algebras and Lie bialgebras, Trans. Amer. Math. Soc., to appear.

[7] この点，日本の環論の伝統を踏襲していると勝手に思っている．

[3] A. Masuoka, Extensions of Hopf algebras and Lie bialgebras II, in preparation.

[4] A. Masuoka, From Hopf algebra extensions to Lie bialgebra extensions, preprint.

[5] M. Takeuchi, Topological coalgebras, J. Algebra 87(1985), 505-539.

$DT\tau$-VARIANT MODULES OVER WILD ALGEBRAS

Hiroshi Nagase

Introduction

Throughout this paper k denotes an algebraically closed field and all algebras are connected, basic, associative k-algebras with identity. We assume that all algebras are finite-dimensional unless otherwise stated. An algebla A is called *wild* if there is an A-$k\langle x,y\rangle$-bimodule M which is a free right $k\langle x,y\rangle$-module of finite rank such that the functor $M\otimes_{k\langle x,y\rangle} - : \operatorname{mod} k\langle x,y\rangle \to \operatorname{mod} A$ preserves indecomposability and isoclasses. In this case we simply say that the pair (A,F) is *wild*. Note that the functor above is faithful and exact by construction. An algebla A is called *strictly wild* if there exists a wild pair (A,F) such that the functor F is full.

In [4] Crawley-Boevey showed that if for an algebra A there exists a natural number d such that infinitely many nonisomorphic d-dimensional indecomposable A-modules are τ-variant, then A is wild, where we call an A-module X *τ-variant* if $\tau X \not\cong X$. Therefore in above case we simply call the algebra A *τ-wild*. In the same paper he conjectured that the converse is true, i.e., wild algebras are τ-wild. In this paper we will consider the conjecture.

In [14] de la Peña proved that if A is strictly wild, then the conjecture is true. This fact immediately follows from a fact (we will see later) and the following our result in the case that $t=1$: Let (A,F) be wild pair and C component of Γ_A of the form $\mathbb{Z}A_\infty/\langle\tau^t\rangle$ with a natural number t.

$$\text{If } F \text{ is full, then } \#\{(\operatorname{Im} F)_0 \cap C_0\} \le t/2.$$

Hence this inequality gives an alternative proof of de la Peña's one. The proof uses the non-existence of irreducible maps in $\operatorname{mod} k\langle x,y\rangle$. But, in the case that $t=1$, we will prove the inequality without using this fact.

In the next step, we will consider the conjecture in the case that A is not strictly wild algebras. It seems, however, difficult to deal with such algebras in general. Therefore we will restrict the case of wild local algebras and some two-point algebras, namely algebras with two isoclasses of simple modules. Our second theorem will show that these algebras are not strictly wild and τ-wild, i.e., the conjecture is ture in the case of some not strictly wild algebras. Our proof uses the facts that wild concealed algebras are strongly τ-wild and Galois covering functors with free Galois group preserve strongly τ-wildness.

The detailed version of this paper will be submitted for publication elsewhere.

1. Preliminaries

For an algebra A (not necessarily finite-dimensional), we denote by $\operatorname{mod} A$ the category of finite-dimensional left A-modules. For a subcategory $\mathcal{A}$ of $\operatorname{mod} A$, define $\mathcal{A}_0 := \{[X] \mid X \text{ is a module in } \mathcal{A}\}$ where $[X]$ denotes the isoclass of X. We denote by rad_A the *radical* of $\operatorname{mod} A$, namely rad_A is an ideal of $\operatorname{mod} A$ defined as follows : for each X, $Y \in \operatorname{mod} A$, $\operatorname{rad}_A(X, Y)$ is the subset of $\operatorname{Hom}_A(X, Y)$ consisting of those f such that hfg is not an isomorphism for each $g \in \operatorname{Hom}_A(U, X)$ and $h \in \operatorname{Hom}_A(Y, V)$ with U, $V \in \operatorname{mod} A$. Then we can consider the powers rad_A^n for all natural numbers n. We set $\operatorname{rad}_A^0 := \operatorname{mod} A$. For an algebla A, Γ_A denote the Auslander-Reiten quiver of A and for a component $\mathcal{C}$ of Γ_A, $\mathcal{C}_0$ denotes the set of vertices in $\mathcal{C}$. By τ we denotes the Auslander-Reiten translation DTr. A component of Γ_A of the form $\mathbb{Z}A_\infty/\langle\tau\rangle$ is called *homogeneous tube*. By $k\langle x, y\rangle$ we denote a free associative algebra with two indeterminates. For any set S, by $\#S$ we denote the cardinality of S.

2. The first main theorem

In [14] de la Peña proved that if A is strictly wild, then the conjecture is true. This fact immediately follows from the fact that any module which dose not lie in any homogeneous tube is τ-variant if it is a module over an algebra of infinite representation type (see Hoshino [9]) and the following theorem in the case that $t = 1$:

Theorem 2.1. *Let (A, F) be wild pair and $\mathcal{C}$ component of Γ_A of the form $\mathbb{Z}A_\infty/\langle\tau^t\rangle$ with a natural number t.*

$$\text{If } F \text{ is full, then } \#\{(\operatorname{Im} F)_0 \cap \mathcal{C}_0\} \leq t/2.$$

The proof of this iniequality uses the following two lemmas. But, in the case that $t = 1$, we will prove the inequality without using the first lemma as the following propsition.

Lemma 2.1. $\operatorname{rad}_{k\langle x,y\rangle} = \operatorname{rad}^2_{k\langle x,y\rangle}$.

Lemma 2.2. $\operatorname{Ext}^1{}_{k\langle x,y\rangle}(X, Y) \neq 0$ *for non-zero finite-dimensional $k\langle x, y\rangle$-modules X and Y.*

Proposition 2.1. *Let (A, F) be wild pair and $\mathcal{C}$ a homogeneous tube, namely a component of Γ_A of the form $\mathbb{Z}A_\infty/\langle\tau\rangle$.*

$$\text{If } F \text{ is full, then } \#\{(\operatorname{Im} F)_0 \cap \mathcal{C}_0\} = 0.$$

Proof. Assume that there exists a module X in $\operatorname{mod} k\langle x, y\rangle$ with $[FX]$ in $(\operatorname{Im} F)_0 \cap \mathcal{C}_0$. Then, for any non-zero module Y in $\operatorname{mod} k\langle x, y\rangle$, we have $\underline{\operatorname{Hom}}_A(FY, FX) \cong D \operatorname{Ext}^1{}_A(FX, FY) \neq 0$ by lemma above and the assumption that F is a fully faithful exact functor. But, for any module Z in $\operatorname{mod} k\langle x, y\rangle$, there exists a module W in $\operatorname{mod} k\langle x, y\rangle$ such that $\operatorname{Hom}_A(FZ, FW) = 0$ because F is a fully faithful functor and $\operatorname{mod} k\langle x, y\rangle$ has infinitely many non-isomorphic simple modules. This gives a contradiction. □

3. Not strictly wild algebras

We want to consider the conjecture in the case of not strictly wild algebras. It seems, however, difficult to deal with such algebras in general and we know only a few examples of not strictly wild algebras. Hence in this section we make some examples of not strictly wild algebras.

Proposition 3.1. *Let B be tame and $A \to B$ an algebra homomorphism with surjection. If all bricks over A are B-modules as canonical way, then A is not strictly wild.*

Proof. Assume that A is strictly wild. Then A has infinitely many non-isomorphic bricks $\{X_i\}$ such that $\mathrm{Ext}^1{}_A(X_i, X_j) \neq 0$ and $\mathrm{Hom}_A(X_i, X_j) = 0$ if $i \neq j$ by Lemma 2.2 and which have same dimension. Since B is tame, for any natural number d, there exist only finitely many τ-variant d-dimensional B-modules by [4]. Therefore we can choose X_1 such that $X_1 \cong \tau_B X_1$. Since $\mathrm{Hom}_A(X_i, X_1) = \mathrm{Hom}_B(X_i, X_1) \to \mathrm{Ext}^1{}_B(\tau_B X_1, X_i) \to 0$, we have that $\mathrm{Ext}^1{}_B(X_1, X_i) = 0$ for $i \neq 1$. On the other hand, since $\mathrm{Ext}^1{}_A(X_1, X_i) \neq 0$, we have non-split short exact sequence $\delta : 0 \to X_i \to Y \to X_1 \to 0$ such that Y is brick in $\mathrm{mod}\, A$, because X_1 and X_i are bricks such that $\mathrm{Hom}_A(X_i, X_1) = 0 = \mathrm{Hom}_A(X_i, X_1)$ for $i \neq 1$. Hence δ is non-split short exact sequence in $\mathrm{mod}\, B$, a contradiction. □

Example 1. (1) Wild local algebras are not strictly wild. Let A be wild local. Then the algebra homomorphism $A \to k$ satisfies the conditions of the proposition above.

(2) Wild two-point algebras in the list of [10] except one example numbered (0) are not strictly wild. For example, the following natural homomorphism satisfies the condition.

$$\begin{pmatrix} C & k^2 \\ 0 & k \end{pmatrix} \longrightarrow \begin{pmatrix} k & k^2 \\ 0 & k \end{pmatrix}, C = k[x]/x^2.$$

(3) For any tame algebra B, we can make an algebra homomorphism $A \to B$ which satisfies the condition and such that A is wild.

4. The second main theorem

From now on, we consider the conjecture in the case of wild local algebras and wild two-point algebras in the list of Hoshino and Miyachi [10].

A module N is called *τ-variant* (resp.*strongly τ-variant*) if τN is not isomorphic to N (resp. $\dim_k \tau N > \dim_k N$). An algebra A is called *τ-wild* (resp. *strongly τ-wild*) if for some natural number d there exist infinitely many non-isomorphic d-dimensional indecomposable τ-variant (resp. strongly τ-variant) modules. For a module N, we denote by $\underline{\dim}\, N$ the dimension-vector of N. For two vectors $\mathbf{x}$ and $\mathbf{y}$, we write $\mathbf{x} \gg \mathbf{y}$ (resp. $\mathbf{x} \geq \mathbf{y}$) if $\mathbf{x}_i > \mathbf{y}_i$ (resp. $\mathbf{x}_i \geq \mathbf{y}_i$) for all entries.

Theorem 4.1. *(1) Wild local algebras are strongly τ-wild;*

(2) Wild two-point algebras in the list of [10] *except one example numbered (0) are strongly τ-wild.*

For the proof of the theorem, we prepare the following lemmas.

An algebra A is called *wild concealed* if there exists a wild hereditary algebra B and preprojective tilting B-module T such that $A = \mathrm{End}_B(T)$. Here note that wild hereditary algebras are wild concealed.

Lemma 4.1. *Wild concealed algebras are strongly τ-wild.*

Proof. The proof uses the fact that for any wild hereditary algebra B and indecomposable regular B-module X, there is a natural number N and for any $n \geq N$, we have that $\underline{\dim}\,\tau^{n+1}X \gg \underline{\dim}\,\tau^n X$ (see de la peña [15]). □

From now on we regard each algebra A as a locally bounded category as follows:

(1) Objects consist of primitive orthogonal idempotents $e_1, \dots, e_n$ with $e_1 + \cdots + e_n = 1$ and null object;
(2) $A(e_i, e_j) := e_j A e_i$;
(3) the composition is given by the multiplication of A

We call locally bounded spectoroid A (see [7]) *strongly τ-wild* if for some dimension vector d there exist infinitely many non-isomorphic finite dimensional indecomposable strongly τ-variant modules whose dimension vectors are d.

Let A and B be locally bounded spectroids and $F : A \to B$ functor. The *pull-up* functor $F^{\cdot} : \mathrm{Mod}\, B \to \mathrm{Mod}\, A$ is defined as $F^{\cdot}M = M \circ F$ for any module $M \in \mathrm{Mod}\, B$. The pull-up functor $F^{\cdot}$ admits a left adjoint called *push-down* functor $F_{\cdot} : \mathrm{Mod}\, A \to \mathrm{Mod}\, B$ which is uniquely defined (up to isomorphism) by the following requirements:

(1) $F_{\cdot}A(x, -) = B(Fx, -)$;
(2) $F_{\cdot}$ commutes with direct limits.

Keeping the notation above, we prepare the following two lemmma.

Lemma 4.2. *Let F be a Galois covering functor with Galois group G which acts freely on* $\mathrm{ind}\, A$. *Then F preserves strongly τ-wildness.*

Lemma 4.3. *If F be full and dense, then F preserves both τ-wildness and strongly τ-wildness.*

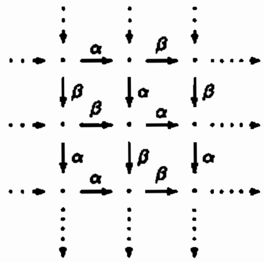
We show ideas of the proof of the theorem.

Proof. (1) By Ringel [16], for any wild local algebra L there exists one of the following wild local algebras as a factor algebra of L.

(1) $L_1 = k\langle x, y\rangle/(x^2, xy, y^2x, y^3)$;
(2) L_1^{op};
(3) $L_2^c = k\langle x, y\rangle/(x^2, y^2x, y^3, xy - cyx)$ $(0 \neq c \in k)$;
(4) $L_3 = k\langle x, y\rangle/(x^2 - y^2, yx)$;
(5) $L_4 = k\langle x, y, z\rangle/(x^2, y^2, z^2, xy, yx, xz, zx, yz, zy)$.

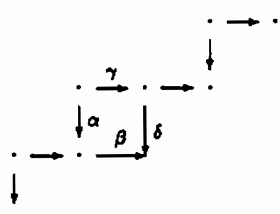
Hence, by Lemma above, it is enough to show that $L_1, \dots, L_4$ are strongly τ-wild. We show only the case of L_3.

L_3 has a Galois covering from the following locally bounded spectroid A with Galois group $\mathbb{Z} \times \mathbb{Z}$:



with $\alpha^2 - \beta^2 = 0$ and $\beta\alpha = 0$.

Hence L_3 is τ-wild because there exists a full dense functor from A to the following algebra which is wild concealed (see [18]):



with $\beta\alpha - \delta\gamma = 0$.

(2) We can find the coverings of the two-point algebras in [10]. □

References

[1] Auslander, M., Reiten, I. and Smalø, O.: *Representation Theory of Artin Algebras*, Cambridge studies in advanced mathematics **36**, Cambridge Univ. Press, 1995.

[2] Auslander, M. and Reiten, I.: *DTr-periodic modules and functors*, C. M. S. Conference Proceedings **18**, 1996, 39–50.

[3] Bongaltz, K.: *Tilted Algebras*, Springer Lect. Notes Math., 903, 1980, 26–38.

[4] Crawley-Boevey, W. W.: *On tame algebras and bocses*, Proc. London Math. Soc. (3) **56**, 1988, 451–483.

[5] Crawley-Boevey, W. W.: *Modules of finite length over their endmorphism rings*, London Math. Soc. Lect. Note Series **168**, Cambridge Univ. Press, 1992, 127–184.

[6] Gabriel, P.: *The universal cover of a representation-fnite algebra*, Springer Lect. Notes Math., 903, 1980, 68–105.

[7] Gabriel, P. and Roiter, A.V.: *Representations of finite-dimensional algeblas*, Encyclopaedia of Math. Sci. Vol. **73**, Springer, 1992.

[8] Happle, D., Preiser, U. and Ringel, C.M.: *Vinberg's characterization of Dynkin diagrams using subadditive functions with application to DTr-periodic modules*, Springer Lect. Notes Math., 832, 1980, 280–294.

[9] Hshino, M.: *DTr-Invariant modules*, Tsukuba J. Math **7**, 1983, 205–214.

[10] Hshino, M. and Miyachi, J.: *Tame two-point algebras*, Tsukuba J. Math **12**, 1988, 65–96.
[11] Kerner, O. and Lukas, F.: *Regular modules over wild hereditary algebras*, C. M. S. Conference Proceedings **11**, 1991, 191–208.
[12] Krause, H.: *Constructing large modules over artin algebras*, Ergänzungsreihe Sonderforschungsbereich 343 "Diskrete Strukturen in der Mathematik", **95-029**, the University of Bielefeld, 1995.
[13] Liu, S.: *Degrees of irreducible maps and the shapes of Auslander-Reiten quivers*, J. London Math. Soc. (2) **45**, 1992, 32–54.
[14] de la Peña, J.A.: *On the dimension of the module-varieties of tame and wild algeblas*, Comm. in Alg. **19(6)**, 1991, 1795–1807.
[15] de la peña J.A.: *Coxeter transformations and the representation theory of algebras*, in Finite Dimensional Algebras and Related topics, Kluwer Academic Publishers, 1994, 223–253.
[16] Ringel, C.M.: *The representation type of local algebras*, Springer Lect. Notes Math., 488, 1974, 282–305.
[17] Ringel, C.M.: *Finite dimensional hereditary algeblas of wild representation type*, Math. Z. **161**, 1978, 235–255.
[18] Ringel, C.M.: *Tame algebras*, Springer Lect. Notes Math., 831, 1980, 137–297.
[19] Yingbo, Z., Tiangang, L. and Raymundo, B.: *The representation category of a bocs. II. The almost split maps*, Beijing Shifan Daxue Xuebao **31**, 1995, no. 4, 440–445.

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The Symmetricity of Hochschild extension algebra given by a 2-cocycle

YOSUKE OHNUKI AND KAORU TAKEDA

Abstract

Let L be a finite extension field of a basic field K and Q a finite quiver without an oriented cycle. We study the simmetricity of the Hochschild extension algebra of a K-algebra LQ corresponding to a 2-cocycle $LQ \times LQ \to \mathrm{Hom}_K(LQ, K)$. In this paper, we construct the sufficient condition for a Hochschild extension algebra to be symmetric and we introduce some examples of non-symmetric Hochschild extension algebras.

K を体, A を有限次元 K-多元環とする. (但し, 多元環は単位元を持つ有限次元多元環で結合的とする). 本文では Hochschild 拡大環や多元環が対称的である事などの定義やそれらの性質については省略してある. 詳細は [9] を参照してほしい. また多元環のクイバーについては [1] を参照.

Happel [2] は多元環が derived equivalent を引き起こすならば, その多元環が stable equivalent を引き起こす事を示した. 本文では, この逆が成立するかという問題を考える. ここで多元環 A, B が stable equivalent, derived equivalent を引き起こすとは, それぞれ $\underline{\mathrm{mod}}\, A \simeq \underline{\mathrm{mod}}\, B$, $D^b(\mathrm{mod}\, A) \simeq D^b(\mathrm{mod}\, B)$ の事を意味する. また, Skowroński-山形 [7] ,[8] により多元環 A のクイバーが有向サイクルを含まないとき, A の $\mathrm{Hom}_K(A, K)$ による Hochschild 拡大環と A の $\mathrm{Hom}_K(A, K)$ による自明な多元環が stable equivalent を引き起こす事が示されている. そこで次の問題が考えられる.

問題 1 多元環 A がそのクイバー内に有向サイクルを含まないとき A の $\mathrm{Hom}_K(A, K)$ による *Hochschild* 拡大環と自明な拡大環が *derived equivalent* を引き起こすか?

また次の定理が Rickard [4], [5], [6] により示されている.

The detail paper of this one has been submitted for publication elsewhere.

定理 0.1 K 上対称的な多元環に *derived equivalence* な多元環もまた K 上対称的である.

自明な拡大環が対称的なので, 問題 1 が成立するならば, Hochschild 拡大環は全て対称的でなければならないが, これは一般には示されていない.

問題 2 クイバー内に有向サイクルを含まない多元環 A の $\mathrm{Hom}_K(A,K)$ による *Hochschild* 拡大環がすべて対称的になるか?

本文では 問題 2 の反例を与えることができた. よって, 多元環が derived equivalent を引き起こすならば, その多元環が stable equivalent を引き起こす事の逆は一般には成立しないことが分かった. この反例を与える準備としていくつかの条件の下での対称性を調べるための性質を与える. ここでは対称性を調べるための性質を得るために, Hochschild 拡大環と 2-コサイクルとの関係を用いる. ここで Hochschild 拡大環と 2-コサイクルの間にどのような関係があるかについて触れておく.

A を多元環, X を両側 A 加群とする. 2 次の Hochschild コホモロジー群 $\mathrm{H}^2(A,X)$ の各元は 2-コサイクル $A\times A\to X$ を 任意の K-線形写像 $A\to X$ のコバウンダリーで同一視したものと考えられる. ここで$\alpha: A\times A\to X$ が 2-コサイクルであるとは A の元 a,b,c に対し, $\alpha(a,bc)+a\alpha(b,c)=\alpha(ab,c)+\alpha(a,b)c$ を満たす K-双線形写像とし, K-線形写像 $f: A\to X$ のコバウンダリーとは $\alpha_f(a,b)=af(b)-f(ab)+f(a)b$ により定義される $\alpha_f: A\times A\to X$ の事とする.

また 2-コサイクル $\alpha: A\times A\to X$ が与えられたとき, α に対応する Hochschild 拡大環 $T(A,X,\alpha)$ を K-線形空間 $A\oplus X$ に $a,b\in A$, $x,y\in X$ について

$$(a,x)(b,y)=(ab,ax+yb+\alpha(a,b))$$

で積を定義して得られる K-多元環とする.

すると, 任意に A の X による Hochschild 拡大環 T を与えると, $T\simeq T(A,X,\alpha)$ を満たす 2-コサイクル $\alpha: A\times A\to X$ が存在する. よって, 与えられた Hochschild 拡大環 T が対称的であるかどうかを判定するためには, ある 2-コサイクル α に対する, $T(A,X,\alpha)$ の対称性を調べると十分である. 最後に, 自明な拡大環とは $T(A,X,0)$ の事である事に注意しておく.

§1 双対性加群による Hochschild 拡大環

多元環 A について, 両側 A 加群 DA により定まる反変関手 $\mathrm{Hom}_A(-,DA)$: $\mathrm{mod}\,A \rightleftarrows \mathrm{mod}\,A^{op}$ が $\mathrm{Hom}_A(\mathrm{Hom}_A(-,DA),DA) \simeq id$ を満たすとき DA を A の双対性加群とする. 特に, $\mathrm{Hom}_K(A,K)$ は双対性加群である.

以下, この節では A を K-多元環とし, DA によりその双対性加群を表すとする.

A の DA による Hochschild 拡大環は全て自己入射多元環である事. また A の DA による対称的な Hochschild 拡大環が存在する事は DA が $\mathrm{Hom}_K(A,K)$ に 両側 A 加群として同型である事に同値である. この 2 つの事実が山形 [9] により示されている.

これらを踏まえて問題 2 の条件は設定されている. また問題 2 とは少し話がそれるが次の命題も Happel, Skowroński-山形 [3] , [7] により示されている.

命題 1.1 体 K が代数的閉体で, K-多元環 A のクイバーが有向サイクルを含まないとき, $\mathrm{H}^2(A,DA)=0$ が成り立つ.

よって, この命題の条件の下では A の DA による Hochschild 拡大環は自明な拡大環 $T(A,DA,0)$ に同値 (つまり, 同型) で, 対称的である. この命題を一般化した次の定理を示すことができた.

定理 1.2 体 K を代数的閉体とし, K-多元環 $A=KQ/I$ をとる. 但し, Q は有限クイバー, I は *admissible* イデアルとする. また $\alpha: A\times A\to \mathrm{Hom}_K(A,K)$ をとる. この時, Q 内の *path* の組 p,q で結合 pq が有向サイクルであるものについて, これらの組が

$$\alpha(p,q)(e_{s(q)})=\alpha(q,p)(e_{s(p)})$$

を満たすとする. 但し, $e_{s(p)}, e_{s(q)}$ はそれぞれ *path* p,q にの始点に対応するベキ等元とする. このとき, $T(A,\mathrm{Hom}_K(A,K),\alpha)$ は K 上対称的である.

§2 example を構成するための準備

L を体 K の有限次拡大体とし, Q を有向サイクルを含まない有限クイバーとする. さらに, Q_0, Q_1, Q_+ によりそれぞれ Q 内の vertex, arrow, 長さ 1 以上の path の集合を表すとする. この節ではパス多元環 $A=LQ$ を K-多元環と見なし, この多元環 A の $\mathrm{Hom}_K(A,K)$ による Hochschild 拡大環

について考える．特に，双対性加群 $\mathrm{Hom}_K(A,K)$ による Hochschild 拡大環 $T(A,\mathrm{Hom}_K(A,K),\alpha)$ を単に $T(A,\alpha)$ と表す事にする．

注意 1 両側 A 加群として，$\mathrm{Hom}_K(A,K)$ と $\mathrm{Hom}_L(A,L)$ は同型である．この事から $\mathrm{Hom}_K(A,K)$ の L-基底として A の L-基底 $\{e_i\}_{i\in Q_0}\sqcup Q_+$ に関する双対基底をとることができる．但し，$1_A=\sum_{i\in Q_0}e_i$ を互いに直交する原始ベキ等元への分解で各 e_i は *vertex* i に対応しているとする．

簡単のために，次の記号を用いる．
2-コサイクル $\alpha: A\times A\to \mathrm{Hom}_K(A,K)$ に対して，

$$[x,y]_\alpha := \alpha(x,y)-\alpha(y,x)$$
$$\langle x,y\rangle_\alpha := [x,y]_\alpha(1_A)$$

と定める．また $\alpha_i: L\times L\to L((x,y)\mapsto \alpha(xe_i,ye_i)(e_i))$ により 2-コサイクル α_i が得られる．ここで次の 2 つの定理が得られた．

定理 2.1 $T(A,\alpha)$ が対称的ならば，$\sum_{i\in Q_0}[L,L]_{\alpha_i}\neq L$

定理 2.2 $\sum_{i\in Q_0}\Big([L,L]_{\alpha_i}+\langle e_i\,\mathrm{rad}\,A,e_i\rangle_\alpha\Big)\neq L$ ならば，$T(A,\alpha)$ は対称的である．

この 2 つの定理により次の系 2.3 が 得られる．

系 2.3 $T(L,\alpha_i)$ が対称的 $\Longleftrightarrow [L,L]_{\alpha_i}\neq L$

$T(A,\alpha)$ が対称的 $\Longrightarrow$ 各 $i\in Q_0$ に対し $T(L,\alpha_i)$ が対称的

拡大体 L の拡大次数が 2 次以下のときは，任意の 2-コサイクル$\alpha: L\times L\to L$ に対し，$[L,L]_\alpha\neq L$ が成り立つ．このため，$T(L,\alpha)$ が対称的になる．しかし，全ての拡大体 L に対して $T(L,\alpha)$ が対称的になるのではない．次節で実際に $T(L,\alpha)$ が対称的でない例を挙げる．

§3 反例の構成

標数 2 の有理関数体 $K=\mathbf{Z}_2(x,y,z)$ とその拡大体 $L=K[a,b,c]/(a^2-x,b^2-y,c^2-z)$ に対し双 K 線形写像 $\alpha: L\times L\to L$ を L の K 基底の元 $\overline{a^lb^mc^n},\overline{a^{l'}b^{m'}c^{n'}}$ (l,m,n,l',m',n' は 0 または 1) に対し，

$$\alpha(\overline{a^l b^m c^n}, \overline{a^{l'} b^{m'} c^{n'}}) = \overline{a^{l+l'-1} b^{m+m'-1} c^{n+n'-1}}(lm'\overline{c} + mn'\overline{ab})$$

により与えると α は 2-コサイクルになる.

命題 3.1 $\alpha : L \times L \to L$ に対応する *Hochschild* 拡大環 $T(L, L, \alpha) \simeq T(L, \alpha)$ は対称的でない.

この α を用いて任意の有向サイクルを含まない有限クイバー Q について, これをクイバーに持つ多元環 A で A の $\mathrm{Hom}_K(A, K)$ による Hochschild 拡大環が対称的でないものが次のようにして構成できる.

任意の有向サイクルを含まない有限クイバー Q をとり, K-多元環 $A = LQ$ を考える. 2-コサイクル $\beta : L \times L \to L$ に対して, $\hat{\beta} : A \times A \to \mathrm{Hom}_K(A, K) \simeq \mathrm{Hom}_L(A, L)$ を

$$\hat{\beta}((a, b)) = \sum_{i \in Q_0} \beta(a_i e_i, b_i e_i) e_i^*$$

により与える. 但し, $a = \sum_{i \in Q_0} a_i e_i + \sum_{p \in Q_+} a_p p$, $b = \sum_{i \in Q_0} b_i e_i + \sum_{p \in Q_+} b_p p$ $(a_i, a_p, b_i, b_p \in L)$, 各 e_i^* は注意 1 のようにして得られる双対基底の元とする. すると $\hat{\beta}$ も 2-コサイクルとなり, 定理 2.1, 2.2 から次の命題が得られる.

命題 3.2 次は同値である.

i) $[L, L] \neq L$

ii) $T(L, \beta)$ は対称的

iii) 有向サイクルを含まないある有限クイバー Q について
$T(LQ, \hat{\beta})$ は対称的

iv) 有向サイクルを含まない任意の有限クイバー Q について
$T(LQ, \hat{\beta})$ は対称的.

この命題により $K, L, \beta = \alpha$ をこの節の初めのようにとると, 任意の有向サイクルを含まない有限クイバー Q に対して $\hat{\alpha}$ に対応する LQ の $\mathrm{Hom}_K(LQ, K)$ による対称的でない Hochschild 拡大環が得られる.

ここで挙げた例は標数 2 の体上の多元環であるが, 同様に各素数 p を標数に持つ体上の多元環で問題 2 の反例も構成できる. 補足ではあるが基礎体 K の標数が 0 のときは, 拡大 L/K が分離拡大となるため拡大次数は 1 以下である. よって 2-コサイクル α に対し, $T(L, \alpha)$ は対称的である. (特に, $T(L, \alpha)$ は可換である).

References

[1] Yu. A. Drozd and V. V. Kirichenko, *Finite dimensional algebras*, J. London Math. Soc. **39** (1989), 436–456.

[2] D. Happel, *Triangulated Categories in the representation theory of finite-dimensional algebras*, London Math. Soc. Lecture Notes Series **119** (Cambridge University Press, Cambridge , 1988).

[3] D. Happel, *Hochschild cohomology of finite dimensional algebras*, Lecture Notes in Mathematics **1404** (Springer, Berlin, 1989), 108–126.

[4] J. Rickard, *Morita theory for derived categories*, Springer-Verlag, (Berlin Heidelberg, 1994).

[5] J. Rickard, *Derived categories and stable equivalence*, J. Pure Appl. Algebra **61** (1989), 303–317.

[6] J. Rickard, *Derived equivalence as derived functors*, J. London Math. Soc. **43** (1991), 37–48.

[7] A. Skowroński and K. Yamagata, *Socle deformations of selfinjective algebras*, Proc. London Math. Soc. **72** (1996), 545–566.

[8] K. Yamagata, *Representations of non-splittable extension algebras*, J. Algebra **115** (1988), 32–45.

[9] K. Yamagata, *Frobenius algebras*, Handbook of Algebra, Vol.1, (Elsevier, 1996), 841–887.

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On QF-serial rings

Kiyoichi Oshiro

Abstract

In this note, we discuss the relationship betuween author [2] and Kupisch [3] on serial rings.

1939 年に Nakayama によって考察された (artinian) serial ring は、半世紀以上経過した現在、その構造はどこまで解明されてきたのであろうか。少なくとも、筆者の研究 [2] と Kupisch の研究 [3] では、この環の本質は local serial ring 上の skew-matrix ring であると主張している。その状況、及び両研究の整合性について説明するのがこの稿の目的である。

筆者の研究

第 17 回環論セミナー報告集 [2] で筆者は次の定理を示した。

Theorem A. すべての serial 環は QF-serial 環を自然に拡大し、その適当な factor ring をとることによって構成される。

例えば、Division ring D 上の triangular matrix ring

$$R = \begin{pmatrix} D & \cdots & D \\ & \ddots & \vdots \\ 0 & & D \end{pmatrix}$$

を考えてみると、この環は D を自然に拡大して得られた環である。

もう少し具体的な場合にこの定理の意味を説明するために、Q を local serial ring、J, S をそれぞれ Q の Jacobson radical 及び socle とする。

$$\begin{pmatrix} Q & \cdots & Q \\ & \ddots & \vdots \\ J & & Q \end{pmatrix}$$

$$\begin{pmatrix} Q & \cdots & Q \\ & \ddots & \vdots \\ J & & Q \end{pmatrix} \Big/ \begin{pmatrix} 0 & & S \\ & & \\ 0 & & 0 \end{pmatrix}$$

などは、serial 環である。これらの serial 環は、Q を自然に拡大し、その factor ring をとることにより構成されていると見れる。

次に、もっと一般的に $R = eR \oplus fR$ を QF-serial 環、$\{e, f\}$ を直交原始冪等元とし、$eR/eJ \simeq S(fR)$、$fR/fJ \simeq S(eR)$ が成り立つとする. ここで

$$Q = eRe \qquad A = eRf$$
$$B = fRe \qquad T = fRf$$

とおくと $R = \begin{pmatrix} Q & A \\ B & T \end{pmatrix}$ と表現される. このとき、自然な拡大環

$$\left(\begin{array}{ccc|ccc} Q & \cdots & Q & A & \cdots & A \\ & \ddots & \vdots & & \cdots & \\ J(Q) & & Q & A & \cdots & A \\ \hline B & \cdots & B & T & \cdots & T \\ & \cdots & & & \ddots & \vdots \\ B & \cdots & B & J(T) & & T \end{array}\right)$$

やその factor ring

$$\left(\begin{array}{ccc|ccc} Q & \cdots & Q & A & \cdots & A \\ & \ddots & \vdots & & \cdots & \\ J(Q) & & Q & A & \cdots & A \\ \hline B & \cdots & B & T & \cdots & T \\ & \cdots & & & \ddots & \vdots \\ B & \cdots & B & J(T) & & T \end{array}\right) \Bigg/ \left(\begin{array}{c|c} & S(A) \\ 0 & \\ & 0 \\ \hline S(B) & \\ & 0 \\ 0 & \end{array}\right)$$

も、また serial ring である。ここで、$J(\ ), S(\ )$ はそれぞれ Jacobson radical と socle を意味する。

このように QF-serial ring から自然な拡大環を作り、その fuctor ring を考えることにより、多くの serial ring が作れる。Theorem A はその状況を説明した定理である。かくして、全ての serial ring が QF-serial ring から構成されることになり、serial ring の研究は QF-serial ring の研究に帰着される。更に、筆者は QF-serial ring を分析し、次の定理を示した。

Theorem B. Nakayama 置換が identity でない、つまり weakly symmetric でない QF-serial ring は、local serial ring 上の skew matrix ring を自然に拡大し、その factor ring をとることにより構成される。更に、Nakayama 置換が identity なる環 R については、$R/S(R)$ が local serial 環上の skew matrix ring として表現される。

かくして、local serial ring 上の skew matrix ring が serial ring の本質である。この定理の状況を説明するために、skew matrix ring の定義及び QF-ring に関する Nakayama 置換、Nakayama 自己同型写像、serial ring に関する Kupisch series なるものの定義を思い起こそう。

Skew matrix ring

Q を環とし、$c \in Q$ と $\sigma \in \mathrm{End}(Q)$ を考え、$\sigma(c) = c, \sigma(q)c = cq \ \forall q \in Q$ が成り立つとする。Q の元を成分とする n 次の行列の全体

$$R = \begin{pmatrix} Q & \cdots & Q \\ & \cdots & \\ Q & \cdots & Q \end{pmatrix}$$

に対して、積を次のように定義する:
$(x_{ik}), (y_{ik}) \in R$ に対して

$$(x_{ik})(y_{ik}) = (z_{ik})$$

ここで、$k \geq i$ のとき

$$z_{ik} = \sum_{j<i} x_{ij}\sigma(y_{ik})c + \sum_{i \leq j \leq k} x_{ij}y_{jk} + \sum_{k<j} x_{ij}y_{jk}c$$

$k < i$ のとき

$$z_{ik} = \sum_{i \leq j \leq k} x_{ij}\sigma(y_{jk}) + \sum_{k<j<i} x_{ij}\sigma(y_{jk})c + \sum_{i \leq j} x_{ij}y_{jk}$$

この積と普通の行列の和で R は環になる。これを Q 上の σ, c に関する n 次の skew-matrix ring と云う。

Nakayama 置換と Nakayama 自己同型写像

R を Artin 環でいわゆる basic とし、J を R の Jacobson radical、$E = \{e_1, \ldots, e_n\}$ を完全直交原始冪等元とする。

R が QF-ring のとき、各 e_i に対して $f_i \in E$ が一意的に定まり e_iR の socle$S(e_iR)$ の projective cover が f_iR となる。従って $\begin{pmatrix} e_1 & \cdots & e_n \\ f_1 & \cdots & f_n \end{pmatrix}$ は一つの置換である。これを R の Nakayama 置換という。R の環同型写像 φ が $\varphi(e_i) = f_i \ \forall i$ をみたすとき、φ を Nakayama 自己同型写像と云う。この写像が存在するか否かは環論における基本的な問題と思うがここではこれについては言及しない。

さて、R を serial ring とし、ここで e_iJ の projective cover が $e_{i+1}R$ として $\{e_1R, \ldots, e_nR\}$ を並べた $\{e_nR, \ldots, e_1R\}$ を Kupisch series という。

QF-serial ring R に関して次が云える:

Proposition: もし、$\{e_nR, \ldots, e_1R\}$ が Kupisch series とすると、$\exists s$:

$$\begin{pmatrix} e_1 & e_2 & \cdots\cdots\cdots\cdots\cdots\cdots & & & e_n \\ e_s & e_{s+1} & \cdots & e_n & e_1 & \cdots & e_{n-1} \end{pmatrix}$$

が Nakayama 置換となる。

この s について、次の4つの場合を考え、R を分析する。

(I) $s = n$

(II) $s > n - s, s \neq n$

(III) $n = sq$

(IV) $n = sq + r, 0 < r < s$

Case (I): これは、次を意味する: (*) もし $\{e_nR, e_{n-1}R, \ldots, e_1R\}$ が Kupisch series ならば

$$\begin{pmatrix} e_1 & e_2 & e_3 & \ldots & e_n \\ e_n & e_1 & e_2 & \ldots & e_{n-1} \end{pmatrix}$$

は R の Nakayama 置換。この場合に関しては、次の一般的なことが成り立つ。

Theorem C. R が basic indecomposable serial ring で (*) をみたすならば、ある local serial ring $Q, c \in J(Q), \sigma \in \mathrm{Aut}(Q)$ があって、

$$\sigma(c) = c$$
$$\sigma(q)c = cq \qquad \forall q \in Q$$
$$J(Q) = cQ$$

が成り立ち、R は Q 上の σ, c に関する skew matrix ring として表現される。

Case (II): $l = n - s + 1, w = l - 1, t = n - w + 1$ とおくと R は次のように表現される。

$$R = \begin{pmatrix} A_{11} & \ldots & A_{1w} & A_{1l} & & & & A_{1n} \\ & \ldots & & & \ddots & & & \\ A_{w1} & \ldots & A_{ww} & A_{wl} & & \ddots & & A_{wn} \\ A_{l1} & \ldots & A_{lw} & A_{ll} & & & \ddots & A_{ln} \\ & \ldots & & & & & & \\ A_{t-1,1} & & \ldots\ldots & & A_{t-1,t} & \ldots & & A_{t-1,n} \\ A_{t1} & & \ldots\ldots & & A_{tt} & \ldots & & A_{tn} \\ & \ddots & & & & & & \\ A_{n1} & \ldots & A_{nw} & A_{nl} & & \ldots\ldots & & A_{nn} \end{pmatrix}$$

$$R_{11} = \begin{pmatrix} A_{11} & \ldots & A_{1w} \\ & \ldots & \\ A_{1w} & \ldots & A_{ww} \end{pmatrix} \quad R_{12} = \begin{pmatrix} A_{1l} & \ldots & A_{1n} \\ & \ldots & \\ A_{wl} & \ldots & A_{wn} \end{pmatrix}$$

$$R_{21} = \begin{pmatrix} A_{l1} & \ldots & A_{lw} \\ & \ldots & \\ A_{t1} & & \\ \vdots & \ddots & \\ A_{n1} & \ldots & A_{nw} \end{pmatrix} \quad R_{22} = \begin{pmatrix} A_{ll} & \ldots & A_{ln} \\ & & \\ & \ldots & \\ & & \\ A_{nl} & \ldots & A_{nn} \end{pmatrix}$$

とおくと

$$R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}$$

Pを

$$P = \begin{pmatrix} A_{11} & A_{1l} \\ A_{t1} & A_{ll} \end{pmatrix}$$

とおくと P に自然な積が定義され、P は (*) をみたす basic indecomposable QF serial ring になる。従って、P は 2 次の skew matrix ring として表現される。この環を次のように自然に拡大する。

各 R_{ij} に対して同じ型の行列 P_{ij} を次のように定義する:

$$P_{11} = \begin{pmatrix} A_{11} & \dots & A_{11} \\ & \ddots & \vdots \\ J(A_{11}) & & A_{11} \end{pmatrix} \quad P_{12} = \begin{pmatrix} A_{1l} & \dots & A_{1l} \\ & \dots & \\ A_{1l} & \dots & A_{1l} \end{pmatrix}$$

$$P_{21} = \begin{pmatrix} A_{t1} & \dots & A_{t1} \\ & \dots & \\ A_{t1} & & \\ & \ddots & \\ A_{t1} & \dots & A_{t1} \end{pmatrix} \quad P_{22} = \begin{pmatrix} A_{ll} & \dots & A_{ll} \\ & & \\ & \ddots & \vdots \\ & & \\ J(A_{ll}) & & A_{ll} \end{pmatrix}$$

ここで

$$Z = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$$

とおくと、この Z は P を自然に拡大した環である。これが求める serial ring であり、Z から R へ自然な上への環準同型写像があることが示される。

Case (III): $n = sq, n \neq s$。このとき次の分割を考える

$$\{1, 2, \dots, s\} \cup \{s+1, \dots, 2s\} \cup \dots \cup \{(q-1)s+1, \dots, n\}$$

$t_1 = 1, s_1 = s, t_2 = s+1, s_2 = 2s, \dots, t_q = (q-1)s+1, s_q = n$ とおき、R の subring

$$P = \begin{pmatrix} A_{t_1t_1} & A_{t_1t_2} & \dots & A_{t_1t_q} \\ A_{t_2t_1} & A_{t_2t_2} & \dots & A_{t_2t_q} \\ & & \dots & \\ A_{t_qt_1} & A_{t_qt_2} & \dots & A_{t_qt_q} \end{pmatrix}$$

を考える。このとき P は (*) をみたす basic indecomposable QF serial ring である。

ここで次のように (s,s)matrix $Q_{i,j} (1 \leq i \leq q, 1 \leq j \leq q)$ を定義する:

$$Q_{ii} = \begin{pmatrix} A_{t_it_i} & \dots & A_{t_it_i} \\ & \ddots & \vdots \\ J(A_{t_it_i}) & \dots & A_{t_it_i} \end{pmatrix}$$

$$Q_{ij} = \begin{pmatrix} A_{t_it_j} & \cdots & A_{t_it_j} \\ & \cdots & \\ A_{t_it_j} & \cdots & A_{t_it_j} \end{pmatrix} (i \neq j)$$

そして

$$Z = \begin{pmatrix} Q_{11} & \cdots & Q_{1q} \\ & \cdots & \\ Q_{q1} & \cdots & Q_{qq} \end{pmatrix}$$

とおくと Z から R の上へ環準同型写像があることが示され、R は (*) をみたす basic indecomposable QF serial ring P 自然に拡張し、その factor ring をとることにより表現される。

Case (IV): $n = sq + r, 0 < r < s$. この場合、次の分割を考える。

$$\{1, \ldots, s\} \cup \{s+1, \ldots, 2s\} \cup \ldots \cup \{(q-2)s+1, \ldots, (q-1)s\} \cup$$

$$\{(q-1)s+1, \ldots, (q-1)s+r\} \cup \{(q-1)s+r+1, \ldots, n\}.$$

ここで

$$t_1 = 1, s_1 = s, t_2 = s+1, s_2 = 2s, \ldots, t_{q-2} = (q-3)s+1,$$

$$s_{q-2} = (q-2)s, t_{q-1} = (q-2)s+1, s_{q-1} = (q-2)s+r,$$

$$t_q = (q-2)s+r+1, s_q = n.$$

とおき

$$R_{ij} = \begin{pmatrix} A_{t_it_j} & A_{t_it_{j+1}} & \cdots & A_{t_is_j} \\ A_{t_i+1,t_j} & A_{t_i+1,t_j+1} & \cdots & A_{t_i+1,s_j} \\ & & \cdots & \\ A_{s_i t_j} & A_{s_i,t_j+1} & \cdots & A_{s_is_j} \end{pmatrix}$$

とおくと、

$$R = \begin{pmatrix} R_{11} & \cdots & R_{1,q+1} \\ & \cdots & \\ R_{q+1,1} & \cdots & R_{q+1,q+1} \end{pmatrix}$$

特に

$$\begin{pmatrix} R_{q,q-1} & R_{q,q} & R_{q,q+1} \\ R_{q+1,q-1} & R_{q+1,q} & R_{q+1,q+1} \end{pmatrix}$$

は次の形をしている:

$$\left(\begin{array}{c|c|cc} A_{t_q,t_{q-1}} & A_{t_q,t_q} & A_{t_q,t_{q+1}} & A_{t_q,t_{q+1}} \\ \ddots & \ddots & A_{t_{q+1},t_{q+1}} & \\ & A_{x,t_{q+1}} & & \ddots \\ A_{t_{q+1},t_{q-1}} & \ddots & A_{t_{q+1},t_{q+1}} & A_{t_{q+1},t_{q+1}} \end{array}\right)$$

ここで P を次のようにおく:

$$P = \begin{pmatrix} A_{t_1t_1} & A_{t_1t_2} & \cdots & A_{t_1t_{q-1}} & A_{t_1t_q} & A_{t_1t_{q+1}} \\ A_{t_2t_1} & A_{t_2t_2} & \cdots & A_{t_2t_{q-1}} & A_{t_2t_q} & A_{t_2t_{q+1}} \\ & & \cdots & & & \\ A_{t_qt_1} & A_{t_qt_2} & \cdots & A_{t_qt_{q-1}} & A_{t_qt_q} & A_{t_qt_{q+1}} \\ A_{t_{q+1}t_1} & & \cdots & A_{t_{q+1}t_{q-1}} & A_{xt_q} & A_{t_{q+1}t_{q+1}} \end{pmatrix}$$

このとき P に自然に積が定義され P は (*) をみたす basic QF-serial ring になる。ここで、R_{ij} に対応して、同じ型の次のような Q_{ij} を作る。

$$Q_{ii} = \begin{pmatrix} A_{t_it_i} & \cdots & A_{t_it_i} \\ & & \vdots \\ J(A_{t_it_i}) & \cdots & A_{t_it_i} \end{pmatrix}$$

$$Q_{q+1,q} = \begin{pmatrix} A_{xt_q} & \cdots & A_{xt_q} \\ & \cdots & \\ A_{xt_q} & \cdots & A_{xt_q} \end{pmatrix}$$

$$Q_{ij} = \begin{pmatrix} A_{t_it_j} & \cdots & A_{t_it_j} \\ & \cdots & \\ A_{t_it_j} & \cdots & A_{t_it_j} \end{pmatrix}$$

そして

$$Z = \begin{pmatrix} Q_{11} & \cdots & Q_{1,q+1} \\ & \cdots & \\ Q_{q+1,1} & \cdots & Q_{q+1,q+1} \end{pmatrix}$$

とおくと、Z は P の自然な拡大環になり、この Z から R の上への環準同型写像があることが示され、R は Z の factor ring として表現される。

Theorem B の前半の状況は以上のとおりである。後半に関しては、つまり、R の Nakayama 置換が identity の場合は $R/S(R)$ が (*) を満たす QF-serial ring になることが容易にわかり、$R/S(R)$ は local serial ring 上の skew matrix ring として表現されることになる。

Kupisch の研究

ここでも、R を indecomposable basic serial ring とし、$E = \{e_1, \ldots, e_n\}$ を完全直交原始冪等元とする。e_iR の組成列の長さを d_i とおくとき、Kupisch は次の定理を示している。

Theorem D. もし $d_i \not\equiv 1 \mod(n)$ $\forall i$ ならば R は local serial ring 上の skew matrix ring の factor ring として表現される。従って、特に、Nakayama 置換が identity でない QF serial ring はこのように表現される。

さて、筆者の Theorem B では、QF-serial ring を四つの場合に分けて考察し、それぞれが local serial ring 上の skew-matrix ring を自然に拡張し、その factor をとることにより構成されることを示している。

一方、Kupisch は直接 QF-serial ring が local serial ring 上の skew-matrix ring の factor ring として構成されることを示している。従って、筆者のそれぞれの場合もそのように、skew-matrix ring の factor ring になっている筈である。

この整合性を検証してみると、確かに簡単にそのようになっていることがわかる。次の具体的な場合をみれば一般の場合の状況も了解できよう。

Q を local serial ring とし、$J = J(Q)$ とおく。$\sigma \in \mathrm{Aut}(Q), c \in J$ は次をみたすとする:

$$
\begin{aligned}
J &= cQ \\
\sigma(c) &= c \\
\sigma(q)c &= cq \qquad \forall q \in Q
\end{aligned}
$$

このとき

$$P = \begin{pmatrix} Q & Q \\ Q & Q \end{pmatrix}_{\sigma,c}$$

とおき、$e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ とおくと両側 Q 加群 $A, B, \alpha \in A, \beta \in B$ があって

$$P = \begin{pmatrix} Q & A \\ B & Q \end{pmatrix}$$

$$
\begin{aligned}
A &= \alpha Q = Q\alpha \\
B &= \beta Q = Q\beta \\
\alpha\beta &= \beta\alpha = c \\
\alpha q &= q\alpha \qquad \forall q \in Q \\
\beta q &= \sigma(q)\beta \qquad \forall q \in Q
\end{aligned}
$$

となる。$P = \begin{pmatrix} Q & A \\ B & Q \end{pmatrix}$ を自然に拡張した環

$$Z = \left(\begin{array}{cccc|c} Q & - & - & Q & A \\ J & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & \vdots & \vdots \\ J & \cdots & J & Q & A \\ \hline B & - & - & B & Q \end{array}\right) \Bigg\} k$$

を考えてみると、この環は次の対応で skew matrix ring

$$\begin{pmatrix} Q & \cdots & Q \\ & \cdots & \\ Q & \cdots & Q \end{pmatrix}_{\sigma,c,k+1}$$

の factor ring になっていることがわかる:

$$\begin{pmatrix} x_{11} & \cdots & x_{1,k+1} \\ & \cdots & \\ x_{k+1,1} & \cdots & x_{k+1,k+1} \end{pmatrix} \longrightarrow \begin{pmatrix} x_{11} & \cdots & x_{1k} & x_{1k+1}\alpha \\ x_{21}c & \ddots & \vdots & \vdots \\ \vdots & \ddots & \vdots & \vdots \\ x_{k1}c & \cdots \; x_{k,k-1}c & x_{kk} & x_{1,k+1}\alpha \\ x_{k+1,1}\beta & \cdots & x_{k+1,k}\beta & x_{k+1,k+1} \end{pmatrix}$$

Theorem B では与えられた QF-serial ring R から (*) をみたす QF-serial ring P をつくりその自然な拡大環 Z の factor ring として R をとらえた。しかし、この Z が実は今みたような方法で local serial ring Q 上の skew-matrix ring の factor ring になっていて、その結果、R もその skew-matrix ring の factor ring になる訳である。

Kupisch の論文では Theorem A に相当する部分がなく主定理が上記の Theorem D である。これを見て、serial ring がどこまで解明されていると解釈すべきか筆者にはわからない。

尚、付言すれば、Theorem A は筆者の Harada 環に関する理論の一つの応用としてできた定理である。Harada 環は serial ring のもつ自己双対的な性質を抽出して考察された環であり、いわば serial ring の骨格のようなものである。

最後に serial ring の自己双対性が skew-matrix ring のどの部分に関わっているかを述べよう。

local serial ring$Q$ 上の skew-matrix ring

$$R = \begin{pmatrix} Q & \cdots & Q \\ & \cdots & \\ Q & \cdots & Q \end{pmatrix}_{\sigma,c,n} \quad (\sigma \in \mathrm{Aut}(Q), cR = J)$$

において写像

$$\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ & & \cdots & \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix} \longrightarrow \begin{pmatrix} x_{nn} & x_{n1} & \cdots & x_{n,n-1} \\ \sigma(x_{1n}) & \sigma(x_{11}) & \cdots & \sigma(x_{1,n-1}) \\ & & \cdots & \\ \sigma(x_{n-1,n}) & \sigma(x_{n-1,1}) & \cdots & \sigma(x_{n-1,n-1}) \end{pmatrix}$$

は R の Nakayama 自己同型写像になり、これが存在するために、すべての serial ring が自己双対的になるのである。(詳細は [1] を参照)

参考文献

[1] J. Kado and K. Oshiro: Self-Duality and Harada Rings To appear in J.Algebra

[2] K. Oshiro: Proceedings of The 20th Simposium on Ring Theory. 109-133(1987)

[3] H. Kupisch: Über ein Klasse von Ringen mit Minimalbedingung II.Arch. Math.26,23-35(1975).

On semisimple extensions of serial rings

Kazuhiko Hirata and Kozo Sugano

Throughout this report A will always be a ring with identity 1, and B a subring of A containing 1. In their previous paper [4] the authors introduced the notion of semisimple extensions of a ring. A ring A is said to be a left semisimple extension of B in the case where every left A-module M is (A,B)-projective, that is, the map π of $A\otimes_B M$ to M, defined by $\pi(a\otimes m) = am$ for any $a \in A$ and $m \in M$, splits as left A-homomorphism, or equivalently, for every left A-module M, every A-submosule which is a B-direct summand of M is always an A-direct summand. (See Theorem 1.1 [4]). The right semisimple extension is similarly defined, and the both left and right semisimple extension is called semisimple extension. Till now some tipical examples of the semisimple extension are known, for example, each semisimple ring is a semisimple extension of each subring of it, and each separable extension is a semisimple extension. However, since the semisimplicity is a quite abstract condition, it is very difficult to research the structure of the semisimple extension or find proper examples of it.

In this report we will give some structure theorem of semisimple extensions of (two-sided) uniserial local rings. A ring R is said to be left serial in the case where R is left artinian and Re has the unique composition series for each primitive idempotent e of R. In the case where R is a direct sum of finite primary left serial rings, R is said to be a left uniserial ring. It is a well known fact that R is primary left uniserial if and only if R is a full matrix ring over a local left serial ring. Right (uni) serial ring is defined similarly, and a both left and right (uni) serial ring is called (uni) serial ring. It is also a well known fact that, if R is serial, R satisfies the following two conditions;

(1) Each left R-module is a direct sum of indecomposable submodules

(2) A left R-module is indecomposable if and only if it is a homomorphic image of some Re, where e is a primitive idempotent of R.

In the case where R satisfies the condition (1), the indecomposable decomposition of each module compliments direct summands by Corollary 2 to Theorem A [7]. Therefore in this case we see that the indecomposable decomposition of each module is unique up to isomorphism by Theorem 12. 4 [1], and that R is left artinian by Corollary 28. 15 [1]. Consequently each left R-module has the projective cover. In addition it can be easily proved that, under the condition (2), for each primitive idempotent e of R Re has the unique maximal left subideal, and each epimorphism of Re to M is a projective cover of a indecomposable left R-module M. Under these preparations we have;

Theorem 1. Let both A and B satisfy the above conditions (1) and (2), and suppose that A is a left semisimple extension of B. Then for each left ideal L of A and each primitive idempotent e of A, there exist a left ideal I of B and a primitive idempotent e' of A such that there is an A-isomorphism of Ae to Ae' whose restricion on Le is an isomorphism of Le to AIe'.

The detailed version of this report will appear in Hokkaido Mathematical Journal.

We will apply Theorem 1 to two cases. One is the case where B is a commutative local serial ring and A is a B-algebra, the other is the case where both A and B are local serial rings. In either case the converse of Theorem 1 is true.

Proposition 1. Let B be a commutative ring and A a B-algebra, and suppose that both A and B satisfy the conditions (1) and (2). Then A is a left semisimple extension of B if and only if, for each left ideal L and primitive idempotent e of A, there exist an ideal I of B and a primitive idempotent e' of A such that there is an A-isomorphism of Ae to Ae' whose restriction on Le is an isomorphism of Le to AIe'.

Theorem 2. Let B be a commutative local serial ring, and A a B-algebra. Then if A is a left semisimple extension of B, A is a uniserial ring. In addition if A is indecomposable as a ring, the length of the composition series of Ae coincides with that of B for each primitive idempotent e of A.

In what follows we will always denote the radicals of A and B by N and J, respectively. The above theorem can be described more generally as follows;

Theorem 3. Let B be a commutative local serial ring and A a semiprimary B-algebra such that N = AJ. Then A is a uniserial ring. If furthermore A is indecomposable as a ring, the length of Ae is equal to the length of B for each primitive idempotent e of A.

By Theorem 1 and a part of the proof of Theorem 2 we have

Theorem 4. Let B be a commutative local serial ring and assume that A is a serial B-algrbra. Then A is a left semisimple extension of B if and only if N = JA

Nextly we will consider the case where B is not necessarily commutative, and we have

Theorem 5. Let A and B satisfy the conditions (1) and (2), and assume A has no idempotent except for 1 and 0. Then A is a left semisimple extension of B if and only if for each left ideal L of A there exist a left ideal I of B and a unit u of A such that Lu = AI.

Applying the above theorem we see that the same results as Theorems 2 and 4 hold in the case of local serial rings as follows;

Theorem 6. Let A and B be local serial rings. Then the following conditions are equivalent;

(i) A is a left semisimple extension of B

(ii) N = AJ

(iii) The lengths of the composition serieses of the left A-module A and the left B-module B are same.

(vi) A is a right semisimple extension of B

Theorem 7. Let B be a local serial ring and A a local ring, and assume that A is finitely generated as a left B-module. Then if A is a left semisimple extension of B, A is a left serial ring.

Finally we will give examples of ring extensions which satisfy the conditions of Theorem 6. Let D be a division ring with a discrete valuation v. Proposition 17.6 [6] shows that such division rings really exist. An uniformizer at v is an element z of D such that $v(z) < 1$ and $v(z)$ generates the cyclic group $v(D - \{0\})$. As usual we write

$$O(D) = O(D, v) = \{x \in D \mid v(x) \leq 1\}, \quad P(D) = P(D, v) = \{x \in D \mid v(x) < 1\}$$

It is well known that $O(D)$ is a local ring with the radical $P(D)$, and $P(D) = O(D)z = zO(D)$ for each uniformizer z at v. It is also obvious that $O(D)/P(D)^n$ is a local serial ring with the length of the composition series n for each natural number n. Now the next proposition gives examples one of which satisfies the conditions of Theorem 6 and some other do not.

Proposition 2. Let D be a division ring with a discrete valuation v and E a division subring of D. Then $O(D)/P(D)^n$ is a semisimple extension of $O(E)/P(E)^n$ for each natural number n if and only if E contains a uniformizer at v, that is, if and only if $v(D) = v(E)$.

References

[1] F. Anderson and K. Fuller, Rings and Categories of Modules, Springer G.T.M., 13 (1992).

[2] K. Asano, Theory of Rings and Ideals (in Japanese), Kyoritsu Press, Tokyo (1949).

[3] A. Hattori, Semisimple algebras over a commutative ring, J. Math. Soc Japan, 15 (1963), 404-419.

[4] K. Hirata and K. Sugano, On semisimple extensions and separable extensions over non commutative rings, J. Math. Soc. Japan, 18 (1966), 360-373.

[5] R. Knorr, Blocks, Vertices and Normal Subgroups, Math. Z., 148 (1976), 53-60.

[6] R. Pierce, Associative Algebras, Springer G. T. M., 88 (1980).

[7] B. Zimmermann-Huisgen, Rings whose right modules are direct sum of indecomposable modules, Proc. Amer. Math. Soc., 77 (1979), 191-197.

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THE FUNDAMENTAL GROUPS OF HAKEN MANIFOLDS

Shinsuke Takashima

Abstract

The main result of this paper is the existence of recursive algorisms of finding the sequence called "Hierarchy" of "Haken manifolds".

1 Introduction

The classification theory of 2-dimensional bounded PL-manifolds is already completed and it has been well known the existence of recursive algorisms for compareing given two such a spaces. But it is not found the same theories for 3-dimensional version. For this, we have to study for fundamental groups of such a spaces.

2 Preliminals

In this paper, all of the topologies are considered in PL-categories, and we'll omit the word "PL-". When we want to consider in general topological category we put the word "topological-" above the words, for example, "topological-continuous map" or " topological-manifolds" and so on. And all manifolds are admited "bound". We call n-dimensional manifold to n-manifold. We use following notations throughout this paper;

$I := [0,1]$

$B^n := I^n$

$\partial X :=$ (the boundary of X)

$\text{int}X :=$ (the topological-interior of X)

$S^n := \partial B^{n+1}$

The detail version of this paper will be submitted for publication elsewhere

A sub 2-manifold F of 3-manifold M is called incompressible if there exists a some $x \in F$ such that the cannocical inclusion map $F \hookrightarrow M$ induces the injection $\pi_1(F,x) \to \pi_1(M,x)$. A 3-manifold M is called irreducible if any sub 2-manifold F homeomorphic to S^2 is boundary of some sub 3-manifold homeomorphic to B^3. An irreducible orientable 3-manifold which has some imcompressible sub 2-manifold are called Haken-manifold.

The following result of Haken manifolds is very basic and important.

Theorem 1 (Walthausen, F. (see [4])) *Let M be a Haken manifold. Then there exists the sequence of manifolds $S^3 = M_0, M_1, \cdots M_n = M$ and imcompressible 2-manifolds $F_i^1, F_i^2 \subset \partial M_i$ such that M_{i+1} is homeomorphic to the manifold given by attaching F_i^1 and F_i^2 from M_i.*

The sequence appeared in the above theory is called hierarchy. The theory of combinatorial group theory called "HNN-extension" (see [3]) shows that for given Haken manifold M, one can solve the word problem of fundamental group of M if the hierarchy of M is found. So, we can say that the word problem of fundamental groups of Haken manifolds can be reduced to the problem of finding the "hierarchy".

3 Universal Cover of Haken Manifolds

For finding a hierarchy, we'll study universal cover of Haken manifolds and its fundamental domain.

Theorem 2 *Let $(f_i^1, f_i^2)_{i \in I}$ be the sequence of sub 2-manifolds of ∂B^3, and let M_J be the manifold given by attaching all of f_i^1 and f_i^2 in $i \in J$ from B^3 Then the following are all equivalent.*

1. *The fundamental domain of universal cover of M_I given the canonical lift of $M_\emptyset$ is simply connectded.*
2. *There exists sequence $\emptyset = I_0 \subset I_1 \subset \cdots \subset I_t = I$ such that $M_{I_0}, M_{I_1}, \cdots, M_{I_t}$ gives a hierarchy of M_I.*

Above two results shows that the problem for finding hierarchy is replaced the problem for finding simply connected fundamental domain of universal covers.

Example 1 The figure 1 shows the relationship of hierarchy and simply connected fundamental domain.

After these reduction of problems the problem of finding the hierarchy sequence is affirmatively solved with following theorem.

Theorem 3 *There exists a recursive algorism to find the simply connected fundamental domain of universal cover of given Haken manifolds.*

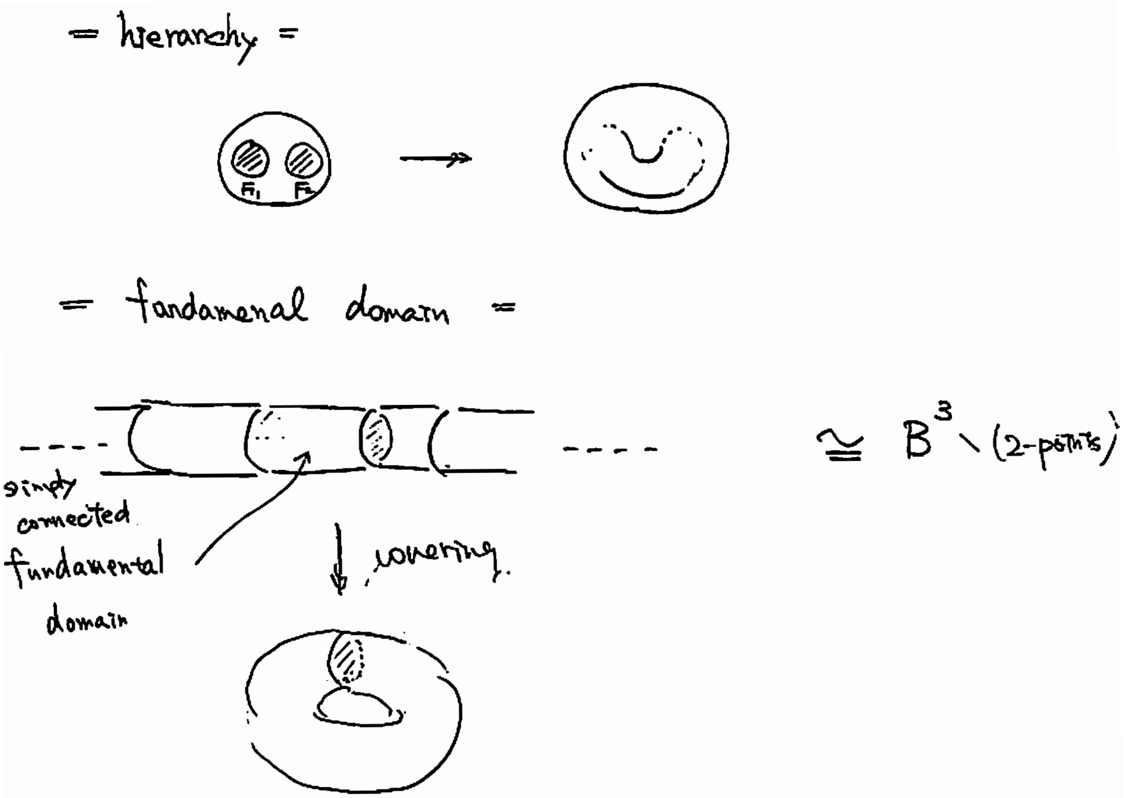

Figure 1: the hierarchy and simply connected fundamental domain of universal cover of $S^1 \times B^2$

4 Applications

We'll study of applications of above section's results. First, we solve the next important problem.

Problem 1 *Let M be a Haken manifold and λ be a some loop of ∂M. Check the existence of the sub 2-manifold F of M such that F is homeomorphic to B^2 and it's boundary coincide l.*

To combinate the classical topological theory called "loop lemma" (see [2]) and above section's results, we can get the next.

Theorem 4 *There is a recursive algorism for checking the above problem.*

Next, we try to use above theory to "Knot theory". The injection map $K : S^1 \to S^3$ is simply called knot, and given two knots K and K' are called equivalent if there exists map $F : S^1 \times I \to S^3$ such that;

1. $F(?, 0) = K$ and $F(?, 1) = K'$.
2. $F(?, t)$ is injective for any $t \in I$.

For given knot K, the sub 3-manifold E of S^3 is called knot complement of K if $S^3 \setminus E$ contains im$K$ and $S^3 \setminus E$ has B^2 bundle structure of im$K$. The knot K is called trivial if it is equivalent to the knot K' which is factors through the cannonical injection $B^2 \hookrightarrow B^3 \hookrightarrow S^3$. The following is classical knot theory (see [1]),

Theorem 5 *For given knot K and E be the kont complement of K. Then the following are all equivalent.*

1. *K is trivial.*
2. *Let l be the meridian loop of ∂E, that is the loop of ∂E of homology 0. Then there exists 2-manifold F of E such that F is homeomorphic to B^2 and it's boundary coincide l.*

The second condition of above theorem is just the first problem of this section, and we have already get the recursive algorism for checking it. So we can get the following.

Theorem 6 *There exists recursive algorism for checking the given knot is trivial or not.*

References

[1] Burde, G.; Zieschang, H. (1985): Knots. Studies in Math. 5, Walter de Gruyter

[2] Hempel, J. (1976): 3-manifolds. Ann. of Math. Studies, 86, Prinston Univ. Press.

[3] Lyndon, R. C.; Schupp, P. E. (1977): Combinatorial group theory. Ergen. Math. Grenzgeb. 89, Springer-Verlag

[4] Waldhousen, F. (1968): On irreducible 3-manifolds which are sufficiently large. Ann. of Math. 87: 56-88

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Introduction to Kuhn's theory
— Algebraic topology and representation theory —

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In a series of papers [K1-3] Nicholas J.Kuhn has developed a very interesting theory which combines algebraic topology and representation theory. In Part I, he deduces several important properties on the Steenrod algebra which are used to solve the Segal and the Sullivan conjectures from his original generic representation theory. I introduced this part at the 31st symposium. My private motivation is that my debut paper [T] is effectively used in his generic representation theory.

In the following, I reproduce my talk briefly. Refer to the original papers for details.

1. One-sided Morita theory

Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories with exact direct limits. It is well-known that if $\ell:\mathcal{A}\to\mathcal{B}$ is an exact functor having a fully faithful right adjoint then it induces an equivalence of $\mathcal{A}/\mathrm{Ker}(\ell)$ to $\mathcal{B}$.

Let k be a fixed base field. Let $\mathcal{S}$ be a small subcategory of a k-additive abelian category $\mathcal{C}$ with exact direct limits. Denote by $\mathrm{Rep}(\mathcal{S}^{op})$ the category of k-additive functors from $\mathcal{S}^{op}$ to the category of k-vector spaces. The functor $r:\mathcal{C}\to\mathrm{Rep}(\mathcal{S}^{op})$ given by

$$r(C) = \mathrm{Hom}_{\mathcal{C}}(-,C)$$

has a left adjoint ℓ. Kuhn has proved the following theorem and calls

This is in final form.

it one-sided Morita theory. The essential part is (1) $\Rightarrow$ (2) and known as Gabriel and Popescu's theorem in case $\mathcal{S}$ = one object. In my debut paper [T] I gave a very short proof of it and Kuhn uses and generalizes my argument to multi-object version.

Theorem. TFAE

(1) $\mathcal{S}$ generates $\mathcal{C}$.

(2) ℓ is exact and r is fully faithful.

(3) $\mathcal{C}$ has enough injectives and if I, J are injectives in $\mathcal{C}$, then

i) r(I) is injective,

ii) $\mathrm{Hom}_{\mathcal{C}}(I,J) \xrightarrow[\sim]{r} \mathrm{Hom}_{\mathrm{Rep}}(r(I),r(J))$.

2. Generic representations

In the rest of this article, let $k = F_q$ the field of q elements. Let $\mathcal{V}(q)$ (resp. $\mathcal{V}_{fin}(q)$) be the category of vector spaces (resp. finite dimensional vector spaces) over F_q. A (not necessarily additive) functor $F: \mathcal{V}_{fin}(q) \to \mathcal{V}(q)$ is called a generic representation.

Examples.

$$T^n: V \mapsto V^{\otimes n},\ S^n: V \mapsto V^{\otimes n}/\Sigma_n,\ S_n: V \mapsto (V^{\otimes n})^{\Sigma_n}$$

where Σ_n is the symmetric group of degree n.

Let $\mathcal{F}(q)$ be the category of generic representations. It has the following structures. Let F, G be generic representations. The tensor product $F \otimes G$ is defined object-wise $(F \otimes G)(V) = F(V) \otimes G(V)$. The dual functor DF is defined by $DF(V) = F(V^*)^*$. For example, S^n and S_n are dual to each other. We have obviously

$$\mathrm{Hom}_{\mathcal{F}(q)}(F,DG) \simeq \mathrm{Hom}_{\mathcal{F}(q)}(G,DF).$$

Let $W \in \mathcal{V}_{fin}(q)$. Let $P_W(V)$ be the F_q vector space with basis Hom(W,V). P_W's form projective generators of $\mathcal{F}(q)$ since we have

$$\mathrm{Hom}_{\mathcal{F}(q)}(P_W,F) \simeq F(W).$$

We have further $P_V \otimes P_W \simeq P_{V\oplus W}$ and this yields that $P_{F_q}{}^{\otimes n}$, n = 0,1,2,...

form projective generators of $\mathcal{F}(q)$.

Dually we put $I_W = DP_W$. Since we have

$$\mathrm{Hom}_{\mathcal{F}(q)}(F, I_W) \simeq \mathrm{Hom}_{\mathcal{F}(q)}(P_W, DF) \simeq DF(W)$$

one sees that $I_{F_q}^{\otimes n}$, $n = 0,1,2,\ldots$ form injective cogenerators of $\mathcal{F}(q)$.

3. (Locally) finite functors

Let $\mathcal{A}$ be an abelian category with exact direct limits. An object A of $\mathcal{A}$ is simple (resp. finite, resp. locally finite) if $A \neq 0$ and A has no nontrivial subobjects (resp. if A has a Jordan-Hölder series of finite length, resp. if A is the union of its finite subobjects). The subcategory of all locally finite objects of $\mathcal{F}(q)$ is denoted by $\mathcal{F}_\omega(q)$. We call $F \in \mathcal{F}(q)$ weakly finite if $F: \mathcal{V}_{fin}(q) \to \mathcal{V}_{fin}(q)$. Our previous basic examples T^n, S^n, S_n, P_W, I_W are all weakly finite.

For a weakly finite functor F, let $d_F(n)$ be the dimension of $F(V_n)$, where V_n is (any) n-dimensional vector space over F_q.

Examples. 1) $F = T^r$. $d_F(n) = n^r$.

2) $F = S^r$. $d_F(n) = \binom{n+r-1}{r}$.

3) $F = I_{F_q}$ or P_{F_q}. $d_F(n) = q^n$.

Note that d_F is a polynomial function in case 1), 2) but not in 3).

Theorem. For $F \in \mathcal{F}(q)$, TFAE.

(1) F is finite.

(2) F is a subquotient of a finite direct sum of T^n's.

(3) F is weakly finite and d_F is a polynomial function.

Therefore, T^n, S^n, S_n are all finite but not P_W or I_W.

Corollary. If F is finite, then so is DF and we have $F \simeq DDF$. The tensor product of two (locally) finite functors is (locally) finite.

As a less trivial result, Kuhn has shown that I_W are locally finite. One concludes that $I_{F_q}^{\otimes n}$, $n = 0,1,2,\ldots$ form injective cogenerators of $\mathcal{F}_\omega(q)$.

4. Generic embedding theorem

Alperin (1986) proved that every finitely generated $F_q[GL(V)]$ module embeds into a finite direct sum of $S^n(V)$'s. Kuhn has proved the following generic version of this and further has shown it is equivalent to the End(V) version of Alperin's embedding theorem.

Theorem. If $F \in \mathcal{F}(q)$ is finite, F embeds into a finite direct sum of S^n's.

Since $S_n = DS^n$, we have

Corollary. S_n, $n = 0,1,2,\ldots$ generate $\mathcal{F}_\omega(q)$.

We have reached the following one-sided Morita context. We put

$$\mathcal{S}^{\bullet} = \{S^0, S^1, S^2, \ldots\}$$

$$\mathcal{S}_{\bullet} = \{S_0, S_1, S_2, \ldots\}$$

These subcategories are dual to each other by D. $M \in \mathrm{Rep}(\mathcal{S}^{\bullet})$ means a F_q-additive functor $M: S^n \longmapsto M_n \in \mathcal{V}(q)$. We think M is a graded vector space (M_n) with the following additional structure.

$$a: S^n \to S^m \text{ or } a: S_m \to S_n \text{ induces } a: M_n \to M_m.$$

If $F \in \mathcal{F}_\omega(q)$, $r(F) \in \mathrm{Rep}(\mathcal{S}^{\bullet})$ is defined by

$$r(F)_n = \mathrm{Hom}_{\mathcal{F}(q)}(S_n, F).$$

In particular we have

$$r(I_V)_n = \mathrm{Hom}_{\mathcal{F}(q)}(S_n, I_V) = DS_n(V) = S^n(V)$$

i.e., $r(I_V): S^n \mapsto S^n(V)$. The functor $r: \mathcal{F}_\omega(q) \longrightarrow \mathrm{Rep}(\mathcal{S}^{\bullet})$ has a left adjoint ℓ and we have a one-sided Morita context. The results are interpreted in terms of the Steenrod algebra yielding some important results in algebraic topology.

5. Steenrod algebra

Let C(q) be the polynomial F_q algebra in $\xi_1, \xi_2, \ldots, \xi_n, \ldots$

We give the following $\mathbb{Z}_{\leq 0}$ graded Hopf algebra structure:

$$\deg(\xi_i) = 1 - q^i,$$
$$\Delta(\xi_n) = \sum_{i+j=n} \xi_i^{q^j} \otimes \xi_j, \text{ where } \xi_0 = 1,$$
$$\varepsilon(\xi_n) = \delta_{0n}.$$

The set of monomials $\xi_1^{r_1} \ldots \xi_\ell^{r_\ell}$ with $d(\underline{r}) = r_1(q-1)+\ldots+r_\ell(q^\ell-1) = d$ forms a basis of $C(q)_{-d}$, hence it is finite dimensional.

The dual graded Hopf algebra $A(q)$ is called the Steenrod algebra. It is positively graded with $A(q)_d = C(q)_{-d}{}^*$. Let

$$\mathcal{P}(q;\underline{r}),\ d(\underline{r}) = d$$

be the dual basis of $\xi_1^{r_1} \ldots \xi_\ell^{r_\ell}$, $d(\underline{r}) = d$. We put $e(\underline{r}) = r_1+\ldots+r_\ell$.

A graded $A(q)$ module $M = \oplus_{n=0}^{\infty} M_n$ is called unstable if

$$\mathcal{P}(q;\underline{r})M_n = 0 \text{ if } e(\underline{r}) > n.$$

Let $\mathcal{U}(q)$ denote the category of unstable $A(q)$-modules.

Example.

Let $S^\bullet(V)$ be the graded vector space $(S^n(V))_n$. It is identified with the symmetric algebra of V. Define an algebra map

$$\rho: S^\bullet(V) \to S^\bullet(V)[[\xi_1,\xi_2,\ldots]] = S^\circ(V) \hat{\otimes} C(q)$$

by setting

$$\rho(x) = \sum_{n=0}^{\infty} x^{q^n} \otimes \xi_n \text{ for } x \text{ in } V.$$

Then $S^\bullet(V)$ becomes a graded $C(q)$ comodule algebra with comodule structure

$$S^n(V) \to S^{n+d}(V) \otimes C(q)_{-d},\ d,n = 0,1,2,\ldots$$

Hence it is a graded $A(q)$ module algebra. It is easy to check

$$\mathcal{P}(q;\underline{r})x_1\ldots x_n = \begin{cases} \sum_{\underline{i}} x_1^{q^{i_1}} \ldots x_n^{q^{i_n}} & \text{if } e(\underline{r}) \le n \\ 0 & \text{if } e(\underline{r}) > n \end{cases}$$

where $\underline{i}$ runs over all permutations of

$$(\underbrace{0\ldots0}_{n-e(\underline{r})}\ \underbrace{1\ldots1}_{r_1}\ \ldots\ldots\ \underbrace{\ell\ldots\ell}_{r_\ell})$$

Hence $S^\bullet(V)$ is an unstable $A(q)$ module.

This example means that the Steenrod algebra $A(q)$ acts on the functor $S^{\bullet}$ unstably. Thus we have a natural linear map

$$A(q)_d \longrightarrow \mathrm{Hom}_{\mathcal{F}(q)}(S^n, S^{n+d})$$

If $M \in \mathrm{Rep}(\mathcal{S}^{\circ})$, composing it with the canonical map

$$\mathrm{Hom}_{\mathcal{F}(q)}(S^n, S^{n+d}) \longrightarrow \mathrm{Hom}_{F_q}(M_n, M_{n+d})$$

one gets an unstable $A(q)$ module structure on $M = (M_n)_{n \geq 0}$. In other words we have a natural functor

$$M \mapsto (M_n)_{n \geq 0}, \ \mathrm{Rep}(\mathcal{S}^{\circ}) \to \mathcal{U}(q).$$

Theorem. This is an equivalence of categories.

This follows from the fact that the set $\mathcal{P}(q;\underline{r})$, $e(\underline{r}) \leq n$, $d(\underline{r}) = d$ forms a basis of $\mathrm{Hom}_{\mathcal{F}(q)}(S^n, S^{n+d})$ and that $\mathrm{Hom}_{\mathcal{F}(q)}(S^n, S^m) = 0$ unless $n \leq m$.

Thus we can interpret the one-sided Morita context in terms of the Steenrod algebra as follows. We have a one-sided Morita context

$$\mathcal{U}(q) \underset{r}{\overset{\ell}{\rightleftarrows}} \mathcal{F}_{\omega}(q) \qquad \ell \dashv r$$

where

$$r(F) = (\mathrm{Hom}_{\mathcal{F}(q)}(S_n, F))_{n \geq 0}$$
$$\ell(M)(V) = S_{\bullet}(V) \otimes_{A(q)} M.$$

Here we view $S^{\bullet}(V)$ as a left unstable $A(q)$-module, hence dually $S_{\circ}(V)$ as a right $A(q)$-module with negative gradation; $S_d(V)$ having degree $-d$. Note that we can identify

$$r(I_V) = S^{\bullet}(V).$$

6. Main results

Applying the one-sided Morita theorem to the previous context, one gets the following results.

Theorem. (1) $S^{\cdot}(V)$ is injective in $\mathcal{U}(q)$.

(2) If $V, W \in \mathcal{V}_{fin}(q)$, then

$$F_q[\mathrm{Hom}(V,W)] \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{U}(q)}(S^{\cdot}(V), S^{\cdot}(W)).$$

(3) ℓ is exact and r is fully faithful. ℓ induces an equivalence

$$\mathcal{U}(q)/\mathcal{N}(q) \simeq \mathcal{F}_{\omega}(q)$$

where

$$\mathcal{N}(q) = \{M \in \mathcal{U}(q) \mid \mathrm{Hom}_{\mathcal{U}(q)}(M, S^{\cdot}(V)) = 0 \ \forall V\}.$$

(2) follows from the fact

$$\mathrm{Hom}_{\mathcal{F}(q)}(I_V, I_W) \simeq \mathrm{Hom}_{\mathcal{F}(q)}(P_W, P_V) \simeq P_V(W) = F_q[\mathrm{Hom}(V,W)].$$

7. Tensor products

If M and N are unstable A(q) modules then the tensor product $M \otimes N$ is unstable with obvious gradation and module action. Thus $\mathcal{U}(q)$ and $\mathcal{F}_{\omega}(q)$ are both tensor categories. Kuhn has proved that both functors ℓ and r preserve the tensor product. In particular we have

$$(\ell M(V) \otimes \ell N(V))^* \simeq \ell(M \otimes N)(V)^*$$

and this is interpreted as

$$\mathrm{Hom}_{\mathcal{U}(q)}(M \otimes N, S^{\cdot}(V)) \simeq \mathrm{Hom}_{\mathcal{U}(q)}(M, S^{\cdot}(V)) \hat{\otimes} \mathrm{Hom}_{\mathcal{U}(q)}(N, S^{\cdot}(V))$$

for M,N in $\mathcal{U}(q)$ and V in $\mathcal{V}_{fin}(q)$.

Kuhn says that these results yield analogous results with the following substitutions:

$q \longrightarrow p$

$A(q) \longrightarrow A_p$ Steenrod algebra with Bockstein

$\mathcal{U}(q) \longrightarrow \mathcal{U}_p$ unstable A_p modules

$S^{\cdot}(V) \longrightarrow H^{\cdot}(V)$ cohomology of BV with coefficients in F_p

and the results are used in solutions to the Segal and the Sullivan conjectures.

References

[K1] N.J.Kuhn, Generic representations of the finite general linear groups and the Steenrod algebra: I, Amer.J.Math. 116(1993),327-360.

[K2] ———, ——— : II, K-theory 8(1994),395-428.

[K3] ———, ——— : III, K-theory 9(1995),273-303.

[T] M.Takeuchi, A simple proof of Gabriel and Popesco's theorem, J.Algebra 18(1971),112-113.

Principal blocks with extra-special defect groups of order 27

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Abstract. The principal 3-blocks of the groups $PGU(3,q^2)$ defined over the finite field $GF(q^2)$ satisfying $q \equiv 2$ or 5 (mod 9) are Morita equivalent to one another. The principal 3-blocks of the groups $PGL(3,q)$ satisfying $q \equiv 4$ or 7 (mod 9) are Morita equivalent to one another. The principal 3-blocks of the groups $SU(3,q^2)$ defined over the finite field $GF(q^2)$ satisfying $q \equiv 2$ or 5 (mod 9) are Morita equivalent to one another. The principal 3-blocks of the groups $SL(3,q)$ satisfying $q \equiv 4$ or 7 (mod 9) are Morita equivalent to one another. Each of these groups has a Sylow 3-subgroup isomorphic to the extra-special group of order 27 of exponent 3. This paper also contains some other results.

§ 1 Introduction

1.1. In modular representation theory there is an important conjecture due to M.Broué (Question 6.2 in [Br2]). For principal blocks this conjecture can be stated like the following:

Broué's conjecture. Let G and G' be finite groups having the same p-local structure and in particular having a common Sylow p-subgroup P. If P is abelian, is it true that their principal p-blocks $B_0(G)$ and $B_0(G')$ are derived

(This is not a final version.)

equivalent?

When P is cyclic, this conjecture is proved by M.Linckelmann [L] and J.Rickard [Ri1]. On the other hand, it is known that if P is not abelian, this is not true. (see section 6 in [Br2] for counter examples). Nevertheless, it seems that there are not so many derived category equivalence classes among the principal p-blocks of finite groups having a fixed p-local structure. Keeping this in mind, in this paper we consider the case $p = 3$ and offer some examples of Morita equivalent principal 3-blocks of an infinite series of finite groups having the same 3-local structure with the same non-abelian Sylow 3-subgroup. (Note that if two blocks are Morita equivalent, then they are derived equivalent.)

1.2. To be accurate, here we give the exact definitions of technical terms in 1.1. The definition in §4 in [Ri2] states as follows: Finite groups G and H have the same p-local structure if they have a common Sylow p-subgroup P such that whenever Q_1 and Q_2 are subgroups of P and $f: Q_1 \to Q_2$ is an isomorphism, then there is an element $g \in G$ such that $f(x) = x^g$ for all $x \in Q_1$ if and only if there is an element $h \in H$ such that $f(x) = x^h$ for all $x \in Q_1$. (In this case group theorists often say that G and H have the same fusion on p-subgroups of P .) Let ($\mathcal{K}$, $\mathcal{O}$, k) be a splitting p-modular system for all subgroups of the considering groups, that is, $\mathcal{O}$ is a complete discrete valuation ring with unique maximal ideal $\mathcal{P}$, $\mathcal{K}$ is its quotient fiels of characteristic zero and k is its residue field $\mathcal{O}/\mathcal{P}$ of prime characteristic p and we assume that $\mathcal{K}$ and k are big enough such that they are splitting fields for all subgroups of the considering groups (see § 6 in Chapter 3 in [NT]). The principal p-block $B_0(G)$ of a finite group G is the indecomposable two-sided ideal of the group ring of G over $\mathcal{O}$ to which the trivial module belongs. In this paper " modules" always mean finitely generated modules. They are left modules, unless stated otherwise.

1.3. Our starting point is the following two theorems stating on the principal 3-blocks of finite groups having non-cyclic abelian Sylow 3-subgroups. More precisely, the groups in Theorem 1.4 and Theorem 1.5 have a common Sylow 3-subgroup isomorphic to an elementary abelian group $Z_3 \times Z_3$ of order 9 and have a common normalizer H of this common Sylow 3-subgroup, which is isomorphic to the semi-direct product $(Z_3 \times Z_3) \rtimes Q_8$ of $Z_3 \times Z_3$ by a quaternion group Q_8 of order 8 with the faithful action. Then all of them have the same 3-local structure. Actually $PSU(3,2^2)$ is isomorphic to H and $B_0(H) = \mathcal{O}H$. Let G be a group in Theorem 1.4 or Theorem 1.5 and set

$$G' = \begin{cases} H \cong PSU(3,2^2) & \text{if } G \text{ is a group in Theorem 1.4 ,} \\ PSL(3,4) & \text{if } G \text{ is a group in Theorem 1.5.} \end{cases}$$

Then a Morita equivalence between $B_0(G')$ and $B_0(G)$ is given by functors

$$M \underset{B_0(G')}{\otimes} - \quad \text{and} \quad M^* \underset{B_0(G)}{\otimes} -$$

with a suitable indecomposable (G,G')-bimodule M and its $\mathcal{O}$ -dual M^*. Furthermore, M is $\Delta(Q)$-projective trivial source module as $G \times G'$-module, where $\Delta(Q)$ is the diagonal group $\{ (x,x) \mid x \in Q \}$ and Q is the common Sylow 3-subgroup above. Hence M induces also a Morita equivalence between $B_0(C_{G'}(R))$ and $B_0(C_G(R))$ for each 3-subgroup R of Q via a Brauer morphism , and then this equivalence is a Rickard's splendid equivalence (see Theorem 4.1 in [Ri2]) and also a Puig equivalence (see Puig and Scott's Theorem 1.6 in [M]). We remark that M is the unique indecomposable non-projective direct summand of $B_0(G) \underset{\mathcal{O}H}{\otimes} B_0(G')$. I heard that T.Okuyama has proved that $B_0(PSU(3,2^2))$ and $B_0(PSL(3,4))$ are derived equivalent to each other and then all principal

3-blocks in Theorem 1.4 and Theorem 1.5 are derived equivalent to one another.

Theorem 1.4. (S.Koshitani and N.Kunugi 1997 [KK]) Let G be a projective special unitary group $PSU(3,q^2)$ defined over the finite field $GF(q^2)$ satisfying $q \equiv 2$ or 5 (mod 9) . Then $B_0(G)$ is Morita equivalent to $B_0(PSU(3,2^2))$.

Theorem 1.5. (N.Kunugi 1997 [K]) Let G be a projective special linear group $PSL(3,q)$ defined over the finite field $GF(q)$ satisfying $q \equiv 4$ or 7 (mod 9). Then $B_0(G)$ is Morita equivalent to $B_0(PSL(3,4))$.

1.6. Note that each projective general unitary group $PGU(3,q^2)$ contains $PSU(3,q^2)$ as a normal subgroup with index 3, when $3 \mid q+1$, and each projective general linear group $PGL(3,q)$ contains $PSL(3,q)$ as a normal subgroup with index 3 , when $3 \mid q-1$. Let M(3) be the extra-special group of order 27 of exponent 3. The projective general unitary groups $PGU(3,q^2)$ satisfying $q \equiv 2$ or 5 (mod 9) and the projective general linear groups $PGL(3,q)$ satisfying $q \equiv 4$ or 7 (mod 9) have a common Sylow 3-subgroup P which is isomorphic to M(3) and have the same 3-local structure: More precisely, we have $P = \langle Q,w \rangle$ and $Q \cong Z_3 \times Z_3$ with a 3-element w and these groups above have a common normalizer H of Q and H contains a common normalizer of P and H controls the fusion on 3-subgroups of P in these groups. Furthermore, H is isomorphic to the semi-direct product $(Z_3 \times Z_3) \rtimes SL(2,3)$ with the faithful action. Actually, $PGU(3,2^2)$ is isomorphic to H and $B_0(H) = \mathcal{O}H$.

Theorem 1.7. (N.Kunugi and Y.Usami 1998 [KU])

(i) Let G be a projective general unitary group $PGU(3,q^2)$ defined over the finite field $GF(q^2)$ satisfying $q \equiv 2$ or 5 (mod 9). Then $B_0(G)$ is Morita equivalent to $B_0(PGU(3,2^2))$.

(ii) Let G be a projective general linear group PGL(3,q) defined over the finite field GF(q) satisfying $q \equiv 4$ or 7 (mod 9). Then $B_0(G)$ is Morita equivalent to $B_0(PGL(3,4))$.

1.8. Let G be a group in Theorem 1.7 and set

$$G' = \begin{cases} H \cong PGU(3,2^2) & \text{if } G \text{ is a group in (i)} \\ PGL(3,4) & \text{if } G \text{ is a group in (ii).} \end{cases}$$

Then each Morita equivalence between $B_0(G')$ and $B_0(G)$ in Theorem 1.7 is given by functors

$$M \underset{B_0(G')}{\otimes} - \quad \text{and} \quad M^* \underset{B_0(G)}{\otimes} -$$

with a suitable indecomposable (G,G')-bimodule M and its $\mathcal{O}$-dual M^*. Furthermore, M is a $\Delta(P)$-projective trivial source module as a $G \times G'$ module. Hence M also induces Morita equivalence between $B_0(C_{G'}(R))$ and $B_0(C_G(R))$ for each 3-subgroup R of P via a Brauer morphism and then this equivalence is a Rickard's splendid equivalence and also a Puig equivalence. We remark that M is an indecomposable direct summand of $B_0(G) \underset{\mathcal{O}H}{\otimes} B_0(G')$ and any other indecomposable direct summand is $\Delta(\langle w \rangle)$-projective. When we investigate the principal p-blocks we may assume that the maximal normal p'-subgroup $O_{p'}(G)$ of G is trivial, since $O_{p'}(G)$ is contained in the kernel of any G-module lying in the principal p-block of G. Using the classification of the finite simple groups, we can list up all finite groups G with $O_{3'}(G) = 1$ and having the same 3-local structure as that of $PGU(3,2^2)$: namely, G is either a group in Theorem 1.7 or G is an extension of G_0 with some G_0 among the groups in Theorem 1.7 by a field automorphism and in such case $B_0(G)$ is isomorphic to $B_0(G_0)$. (More precisely, they are Morita equivalent by a suitable bimodule having the same property stated above.) Then by Theorem 1.7 we can conclude that there are at most two derived category equivalence classes of the principal 3-blocks of the finite groups having the same 3-local structure as that of $PGU(3,2^2)$ (using the classification of finite simple groups). Outline of the proof of Theorem 1.7 is given in § 1 in [U].

1.9. Note that when $3 \mid q+1$, the special unitary group $SU(3,q^2)$ defined over the finite field $GF(q^2)$ contains a normal subgroup of order 3 as its center and $PSU(3,q^2)$ is the factor group of it by this normal subgroup. Similarly, when $3 \mid q-1$, the special linear group $SL(3,q)$ defined over the finite field $GF(q)$ contains a normal subgroup of order 3 as its center and $PSL(3,q)$ is the factor group of it by this normal subgroup. Furthermore, the special unitary groups $SU(3,q^2)$ satisfying $q \equiv 2$ or 5 (mod 9) and the special linear groups $SL(3,q)$ satisfying $q \equiv 4$ or 7 (mod 9) have a common Sylow 3-subgroup P isomorphic to M(3). These groups also have a common normalizer H of P which is isomorphic to the semi-direct product $M(3) \rtimes Q_8$ with the faithful action. Actually, $SU(3,2^2)$ is isomorphic to H and $B_0(H) = \mathcal{O}H$. Any two of these groups have the same 3-local structure.

Theorem 1.10 (Y.Usami)

(i) Let G be a special unitary group $SU(3,q^2)$ defined over the finite field $GF(q^2)$ satisfying $q \equiv 2$ or 5 (mod 9). Then $B_0(G)$ is Morita equivalent to $B_0(SU(3,2^2))$.

(ii) Let G be a special linear group $SL(3,q)$ defined over the finite field $GF(q)$ satisfying $q \equiv 4$ or 7 (mod 9). Then $B_0(G)$ is Morita equivalent to $B_0(SL(3,4))$.

1.11. Note that the projective general unitary group $PGU(3,q^2)$ is the group obtained from the general unitary group $GU(3,q^2)$ on factoring this group by its center of order q+1. Also note that the projective linear group PGL(3,q) is the group obtained from general linear group GL(3,q) on factoring this group by its center of order q-1. Furthermore, the general unitary groups $GU(3,q^2)$ defined over the finite field $GF(q^2)$ satisfying $q \equiv 2$ or 5 (mod 9) and the general linear groups GL(3,q) defined over the finite field GF(q) satisfying $q \equiv 4$ or 7 (mod 9) have a common Sylow 3-subgroup of order 81 and also have the same 3-local structure.

Theorem 1.12. (Y.Usami)
(i) Let G be a general unitary group $GU(3,q^2)$ defined over the finite field $GF(q^2)$ satisfying $q \equiv 2$ or 5 (mod 9). Then $B_0(G)$ is Morita equivalent to $B_0(GU(3,2^2))$.
(ii) Let G be a general linear group GL(3,q) defined over the finite field GF(q) satisfying $q \equiv 4$ or 7 (mod 9). Then $B_0(G)$ is Morita equivalent to $B_0(GL(3,4))$.

§ 2 Outline of the proof of Theorem 1.10 and Theorem 1.12

2.1. Let G be a group in Theorem 1.10 and Z be its center of order 3 and set

$$G' = \begin{cases} H \cong SU(3,2^2) & \text{if G is a group in (i)} \\ SL(3,4) & \text{if G is a group in (ii)} \end{cases}$$

and also set $\bar{G} = G/Z$, $\bar{G}' = G'/Z$ and $\bar{H} = H/Z$. Let $\bar{M}$ be a $(\bar{G},\bar{G}')$-bimodule which induces a Morita equivalence between $B_0(\bar{G}')$ and $B_0(\bar{G})$. Then by the remark in 1.3 $\bar{M}$ is obtained from a $(\bar{G},\bar{G}')$-bimodule $B_0(\bar{G}) \otimes_{\mathcal{O}\bar{H}} B_0(\bar{G}')$ by subtracting all indecomposable projective direct summands. On the other hand, any indecomposable direct summand of $B_0(G) \otimes_{\mathcal{O}H} B_0(G')$ is a trivial source module and contains $\Delta(Z)$ in its kernel and then contains $\Delta(Z)$ in its vertex. Let M be the (G,G')-bimodule obtained from $B_0(G) \otimes_{\mathcal{O}H} B_0(G')$ by subtracting all indecomposable direct summands with vertices $\Delta(Z)$. Now the character of any direct summand of $B_0(G) \otimes_{\mathcal{O}H} B_0(G')$ is the sum of the first part and the second part; namely, the first part is a $\mathbb{Z}$-linear combination of products of an irreducible character of $B_0(G)$ containing Z in its kernel and an irreducible character of $B_0(G')$ containing Z in its kernel, and the second part is a $\mathbb{Z}$-linear combination of products of an irreducible character of $B_0(G)$ not containing Z in its kernel and an irreducible character of $B_0(G')$ not containing Z in its kernel. Let χ be an irreducible character of $B_0(G)$ containing Z in its kernel and χ' be an irreducible character of $B_0(G')$ containing Z in its kernel. The multiplicity of $\chi\chi'$ in the character of $B_0(G) \otimes_{\mathcal{O}H} B_0(G')$ can be calculated by $(\chi_{\downarrow H} , \chi'_{\downarrow H})_H$ and then the first part of the character of $B_0(G) \otimes_{\mathcal{O}H} B_0(G')$ coincides with the character of $B_0(\bar{G}) \otimes_{\mathcal{O}\bar{H}} B_0(\bar{G}')$ (Here $\chi_{\downarrow H}$ is the restriction of χ to H.) Let S be a simple $B_0(G)$-module (namely, a simple $B_0(\bar{G})$-module) and S' be a simple $B_0(G')$-module (namely, a simple $B_0(\bar{G}')$-module). Let $P(S \otimes_{\mathcal{O}} S')$ be the projective cover of $S \otimes_{\mathcal{O}} S'$ as a $(G \times G')/\Delta(Z)$-module and $p(S \otimes_{\mathcal{O}} S')$ be the projective cover of $S \otimes_{\mathcal{O}} S'$ as a $\bar{G} \times \bar{G}'$-module . Note that $P(S \otimes_{\mathcal{O}} S')$ is an indecomposable $G \times G'$-module with vertex $\Delta(Z)$ and the first part of its character coincides with the character of $p(S \otimes_{\mathcal{O}} S')$. We can show that the multiplicity of $P(S\otimes_{\mathcal{O}} S')$ in $B_o(G)\otimes_{\mathcal{O}H} B_o(G')$ is equial to the multiplicity of $p(S\otimes_{\mathcal{O}} S')$ in $B_o(\bar{G})\otimes_{\mathcal{O}\bar{H}} B_o(\bar{G}')$ using G.R.Robinson's results on projective summands of induced modules (Theorem 3 in [Ro]). Consequently we can show the first part of the character of M coincides with the character of $\bar{M}$ and

$$\bar{M} = \mathcal{O}\bar{G} \otimes_{\mathcal{O}G} M \otimes_{\mathcal{O}G'} \mathcal{O}\bar{G} \tag{2.1}$$

and then by Lemma 10.2.11 in [Ru] we can conclude that M and its $\mathcal{O}$-dual M^* induce a Morita equivalence between $B_o(G')$ and $B_o(G)$.

2.2. The center of any group in Theorem 1.12 is a direct product of the maximal normal 3'-subgroup and a normal subgroup of order 3. Let G be the factor group of a group in Theorem 1.12 by its maximal normal 3'-subgroup and let Z be a normal subgroup of G of order 3 which is the center of G. Now $< Z,Q,w >$ is a common Sylow 3-subgroup of these groups G with elementary abelian group Q of order 9 and a 3-element w. Note that $< Z,Q,w >/Z$ is a common Sylow 3-subgroup of the groups in Theorem 1.7. These groups G have a common normalizer H of ZQ. set

$$G' = \begin{cases} H \cong GU(3,2^2)/O_{3'}(GU(3,2^2)) & \text{if } G \text{ is a group in (i)} \\ GL(3,4)/O_{3'}(GL(3,4)) & \text{if } G \text{ is a group in (ii),} \end{cases}$$

and set $\bar{G} = G/Z$, $\bar{G}' = G'/Z$ and $\bar{H} = H/Z$. Let $\bar{M}$ be a $(\bar{G},\bar{G}')$-bimodule which induces a Morita equivalence between $B_o(\bar{G}')$ and $B_o(\bar{G})$. Then by the remark in 1.8 $\bar{M}$ is obtained from $B_o(\bar{G})\otimes_{\mathcal{O}\bar{H}} B_o(\bar{G}')$ by subtracting all indecomposeble $\Delta(<w>)$-projective direct summands. On the other hand, any indecomposable direct summand of $B_o(G)\otimes_{\mathcal{O}H} B_o(G')$ is a trivial source module and contains $\Delta(Z)$ in its vertex as in 2.1. We define M as the (G,G')-bimodule obtained from

$B_0(G) \otimes_{\mathcal{O}H} B_0(G')$ by subtracting all indecomposable $\Delta(Z\langle w\rangle)$-projective direct summands. Here we have to investigate the indecomposable $G\times G'$-modules with vertices $\Delta(Z\langle w\rangle)$ applying Theorem 3.2 in [Br1] and the Green correspondence. We also have to study the relation between them and the indecomposable $\bar{G}\times\bar{G}'$-modules with vertices $\Delta(\langle w\rangle)$. As in 2.1 we can show a close relation between the characters of M and $\bar{M}$ and obtain (2.1) and then by Lemma 10.2.11 in [Ru] we get the conclusion.

2.3. I did not mention the results in Theorem 1.10 (ii) and Theorem 1.12 at the symposium in November. Actually, they are obtained after the symposium. Since they are new and I do not have enough time to check them carefully again and again now, so I have to confess that I feel uneasy about the correctness of these results.

References

[Br1] M.Broué, On Scott modules and p-permutation modules, Proc.A.M.S.93 (1985) 401-408

[Br2] M.Broué, Isométries parfaites, types de blocs, catégories dérivées, Astérisque 181-182 (1990) 61-92.

[KK] S.Koshitani and N.Kunugi, The principal 3-blocks of the 3-dimensional projective special unitary groups in non-defining characteristic , submitted.

[K] N.Kunugi, Morita equivalent 3-blocks of the 3-dimensional projective special linear groups, submitted.

[KU] N.Kunugi and Y.Usami, The principal 3-blocks of the 3-dimensional projective general linear and projective general unitary groups and Morita equivalences , submitted.

[L] M.Linckelmann, Derived equivalence for cyclic blocks over a p-adic ring, Math.Z.207 (1991), 293-304

[M] A.Marcus, On equivalences between blocks of group algebras: reduction to the simple components, J.Algebra 184 (1996) 372-396.

[NT] .H.Nagao and Y.Tsushima,Representations of finite groups (Academic Press, New york, 1989).

[Ri1] J.Rickard,Derived categories and stable equivalence, J.of Pure and Appl. Algebra 61 (1989),303-317.

[Ri2] J.Rickard, Splendid equivalences: Derived categories and permutation modules, Proc. London Math. Soc. 72 (1996) 331-358.

[Ro] G.R.Robinson,. On projective summands of induced modules, J.Algebra 122 (1989) 106-111.

[RU] R.Rouquier, The derived category of blocks with cyclic defect groups in : S.König and A.Zimmermann, Derived Equivalences for Group Rings, Lecture Notes in Math 1685, Springer (1998) 199-220

[U] Y.Usami , Principal blocks with extra-special defect groups of order 27, 数理研 講究録 シンポジウム 京都 1998.9月

Lattice の cancellation と Eichler condition

山崎愛一

R: Dedekind domain, $\mathfrak{p} \in \operatorname{Max} R$($R$の maximal ideal 全体) に対し、$R_{\mathfrak{p}} := R$の $\mathfrak{p}$-adic completion $\widehat{R} := \prod_{\mathfrak{p} \in \operatorname{Max} R} R_{\mathfrak{p}}$ とする。また R-加群 X に対し、$X_{\mathfrak{p}} := X \otimes_R R_{\mathfrak{p}}$, $\widehat{X} := X \otimes_R \widehat{R}$とおく。

L が R-lattice とは有限生成 projective R-module であること。Λが R-order とは R-lattice で 1 を持つ環であること。L がΛ-lattice とは R-lattice であってΛ-module であること。

L, L'をΛ-lattice として、$\widehat{L'} \simeq \widehat{L}$ のとき L と L'は局所同型と言い、$L \simeq_{loc} L'$と記す。

もちろん $L \simeq L' \Rightarrow L \simeq_{loc} L'$

$$^{\exists}n \geq 1, L \oplus L^{\oplus n} \simeq L' \oplus L^{\oplus n}$$

が成り立つとき、L と L'は安定同型（stably isomorph）と言う。直既約分解の一意性が成り立つ場合は安定同型と同型が同じになることは自明であるが、今考えているのは直既約分解の一意性が成り立たない場合である。定義からは明らかではないが、後で説明するように安定同型性は同値関係になる。

一般に Bを 1 をもつ環とし、B係数の一般線形群 $GL(n, B)$ を考える。elementary matrix subgroup を $E_n(B)$ とする。すなわち

$$e_{ij} = ((i,j) \text{ 要素のみ } 1, \text{ 他は } 0 \text{ の行列}) \text{ とおいて}$$

$$E_n(B) = \langle 1 + x e_{ij} | x \in B, \quad 1 \leq i,j \leq n, \quad i \neq j \rangle$$

($\langle \ \rangle$ は、この集合の生成する部分群を意味する。)

また$\begin{pmatrix} B^\times & \\ & 1_{n-1} \end{pmatrix}$と$B^\times$を同一視して、$B^\times \subset GL(n,B)$とみなす。
Bにある条件を仮定する（KがDedekind domainの商体でBが有限次元K-algebraのとき、BもB_pも$\widehat{B}$もこの条件をみたす)。
この仮定のもとに次の定理が成り立つ。
定理(Vaserstein)　$E_n(B) \cap B^\times$は$n(\geq 2)$にdependせず、下の$\widetilde{E}(B)$に等しい。

$$\widetilde{E}(B) = \left\langle (1+xy)(1+yx)^{-1} \mid x \in B, y \in B, 1+xy \in B^\times \right\rangle$$

$$\text{また}\quad \widetilde{E}(B) \supset [B^\times, B^\times]\ (=B^\times\text{の交換子群})$$

KはRの商体として$A = \Lambda \otimes_R K, L$は$\Lambda$-latticeとし、$V = L \otimes_R K$（$= L$で張られる$K$-ベクトル空間）とする。$V$は左$A$-moduleになる。$B = \mathrm{End}_A V$,　$\Gamma = \mathrm{End}_\Lambda L$はそれぞれ$K$-algebra, R-orderになる。$L \simeq_{loc} L'$のとき、それぞれの張るK-ベクトル空間は左A-moduleとして同型である。(従って、これを同一視してVとすると$B = \mathrm{End}_A V$も共通とみなせる)。

$$L \simeq_{loc} L' \Longleftrightarrow {}^{\exists}\sigma \in \widehat{B}^\times,\quad \widehat{L}' = \sigma(\widehat{L})$$

$$L = L' \Longleftrightarrow \text{上の}\sigma\text{について}\sigma \in \widehat{\Gamma}^\times \qquad (\Gamma = \mathrm{End}_\Lambda L)$$

$$L \simeq L' \Longleftrightarrow {}^{\exists}\sigma \in B^\times,\quad L' = \sigma(L)$$

従ってLに局所同型な同型類の集合$\mathcal{G}(L)$（これをLのgenusと言う）は$B^\times \backslash \widehat{B}^\times / \widehat{\Gamma}^\times$と一対一に対応する。

LとL'が安定同型$\Longleftrightarrow {}^{\exists}\sigma \in B^\times \widetilde{E}(\widehat{B}) \subset \widehat{B}^\times, \widehat{L}' = \sigma(\widehat{L})$

$\widetilde{E}(\widehat{B})$が$\widehat{B}^\times$の正規部分群であることから$B^\times \widetilde{E}(\widehat{B})$は群をなす。よって安定同型性は同値関係になる。

Lと安定同型な同型類の集合を$\mathcal{F}(L)$, その元数を$f(L)$とする。またLに局所同型な安定同型類の集合を$C(L)$, その元数を$c(L)$とする。

$$g(L) = \sum_{L' \in C(L)} f(L')$$

$^\forall L' \in C(L)$, $f(L') = 1$ のとき、すべての安定同型類について cancellation 成立を意味し、このとき完全 cancellation 成立と言う。また$^\forall L' \in C(L)$, $f(L') > 1$ のとき、すべての安定同型類について cancellation 不成立を意味し、このとき完全非 cancellation と言う。

Λを R-order とするとき、与えられたΛ-lattice L, Mに対し次の cancellation law (jc) が成り立つ条件を見出すことは基本的問題である。

(jc)　L と Mが安定同型$\Rightarrow L \simeq M$.

(jc) 成立 $\Longleftrightarrow f(L) = 1$

Λ-lattice$L$ に対し$\Gamma := \mathrm{End}_\Lambda L$, $B := K \otimes_R \Gamma$, Bは K上 central simple のとき $B^1 := \{b \in B^\times |$ reduced norm $b = 1\} \subset \widehat{B}^1 \subset \widehat{B}^\times$ とおき、条件 (a1) を次のように定める。

(a1)　B^1は$\widehat{B}^1$で dense.

すると Rの商体 Kが代数的数体, Bは K上 central simple と仮定して次の Jacobinski-Swan の定理が成り立つ。

(JCT)　(a1) $\Rightarrow$ (jc) が成り立つ。

他方 Eichler-Kneser の強近似定理 (SAT) とは、K, Bについて上と同じ仮定のもとに

(SAT)　(a1) $\Leftrightarrow$ (ec):=($^\exists v \notin \mathrm{Max}\, R$ s.t. $B \otimes_K K_v$ is not division)

が成り立つことである。

上の両定理は、Kが有限体上の一変数関数体の場合も Swan により同じ結果が得られた。

数年前土方教授は、B^1の代わりに Vaserstein の群$\widetilde{E}(B)$ (一般の環 Bに対し$\widetilde{E}(B) := \langle (1+xy)(1+yx)^{-1} | x, y \in B,\ 1+xy \in B^\times \rangle$ として定義される) を用いて条件 (a1) を次の (a) でおきかえると (JCT) は、一般の Dedekind 環 Rについて B, Kについての仮定なしで成り立つことを示した。

(a)　$\widetilde{E}(B)$ は$\widetilde{E}(\widehat{B})$ の中で dense.

そこで一般 Dedekind 環上Λ-lattice の理論を Kが A-field の場合と同様に深めること、特に (a) に対する (SAT) の一般化を求めることが重要な問題として浮上した。

筆者はこの問題を考察して、次の結果を得た。

(a) の他に次の二つの approximation property を考える。$\overline{(\ )}$は$\widehat{B}^\times$での closure として

(a′) $\widetilde{E}(\widehat{B}) \subset \overline{B^{\times}}$.

(a″) $\widetilde{E}(\widehat{B}) \subset \overline{\widehat{R}^{\times} B^{\times}}$.

上記三つの approximation property の間には (a) ⇒ (a′) ⇒ (a″) の関係がある。

__Proposition 1__ (jc) for $L \Leftrightarrow \widetilde{E}(\widehat{B}) \subset \widehat{\Gamma}^{\times} B^{\times}$

__Proposition 2__ (jc) for $^{\forall}L$ s.t. $K \otimes_R \mathrm{End}_{\Lambda} L \simeq B \Leftrightarrow$ (a″) for B.

これは Endomorphism ring の言葉による (jc) の完全な特徴付けである。

__Theorem 1__ $B/J(B) = \bigoplus_{i=1}^{m} M_{n_i}(D_i)$, $J(B)$ は Jacobson 根基, $n_i = 1 (1 \leq i \leq r)$, $n_i \geq 2 (r < i \leq m)$ と書いたとき

(a′) for $B \Leftrightarrow$ (a′) for $D_i (1 \leq i \leq r)$

(a) (resp. (a″)) for $B \Rightarrow$ (a) (resp. (a″)) for $D_i (1 \leq i \leq r)$.

Bが simple algebra のとき、Bの center を L,Rの L での整閉包を R_Lとする。R_Lは R上有限生成とする。

(a) (resp. (a′)) for $B/R \Leftrightarrow$ (a) (resp. (a′)) for B/R_L.

(a″) for $B/R \Rightarrow$ (a″) for B/R_L.

(従って (a′) は central division algebra の場合に帰着)

これは一般の Bに対する approximation property を、central division な Bに対する それに帰着させる手段を与える。以下の定理 2 と 3 では Bは central division とする。

__Theorem 2__ Kが PF 体 (すなわち product formula をもつ体) のとき (a″) ⇒ (ec).

証明法は付値の不等式による。Kneser の証明と違ってコンパクト性を用いないために、PF 体の場合にまで一般化できた。

__Theorem 3__ Kが実数体 $\mathbb{R}$ 上の一変数代数関数体のとき (ec) ⇒ (a) (従って (a) ⇔ (a′) ⇔ (a″) ⇔ (ec))

証明法は、群論的考察を用いて、$K = \mathbb{R}(X)$ の場合に帰着させる。$K = \mathbb{R}(X)$ の場合は筆者の論文 [1] により解決済み。

[1] Strong Approximation Theorem for Division Algebras over $\mathbb{R}(X)$.
Journal of Mathematical Society of Japan. Vol. 49. No.3 (1997) 455-467

[2] Cancellation of Lattices and Approximation Properties of Division Algebras.

Journal of Mathematics, Kyoto University. Vol. 36. No.4 (1996) 857-867 （博士論文）

この研究は、土方教授の下記の研究 [3] に端を発し、それと密接に関連している。その他の関連論文は [1],[2] の references を参照。

[3] H.Hijikata: On the decomposition of lattices over orders.

Journal of Mathematical Society of Japan. Vol. 49. No.3 (1997) 431-437

参考

Q 上四元数体 H については (ec) が成り立たない。従って cancellation law も成り立たない。この場合、安定同型類が同型類にどのように分かれるかを Swan が具体的に調べている。[4] 以下、その概略。

$C_n = Q(\zeta_n)$, $\zeta_n = e^{\frac{2\pi i}{n}}$ を n 次円分体、R_nを C_nの index 2 の実部分体とする。n が奇数のとき $C_n = C_{2n}$だから、簡単のため n は偶数とする。また $C_2 = Q$ は自明だから $n \geq 4$ とする。

$H_n = C_n \coprod C_n j$, $j^2 = -1$, ,$^\forall z \in C_n$, $jz = \bar{z}j$ （$\bar{\ }$は複素共役）により H_nは R_n上 central division algebra(Quaternion algebra) になる。H_nの Z-order で R_nの主整環 Rを含むものは R-order でもある。今、H_nの極大 R-order Λ（unique ではない）に限定して、Λ が cancellation property をもつかどうかを調べる。結果は次の通り。（この結果を導くには、円分体の具体的知識（類数、単数群）が必要である。）

(1) $n = 4, 6, 8, 10, 14, 18$ については安定同型類も同型類もただ一つで cancellation 成立。$n = 12, 30$ については安定同型類は 2 つあるが完全 cancellation 成立。

(2) $n = 20, 24, 42$ については、安定同型類は 2 つあり、一方は cancellation 成立、他方は不成立で 2 つの同型類に分かれる。

(3) その他の n については完全非 cancellation。

Swan はさらに極大でない R-order についても調べている。いずれにしろ、n が大きければつねに非 cancellation である（一般に$\Lambda \subset \Lambda'$でΛが cancellation property をもてば Λ'もそう。従って極大 R-order が完全非 cancellation ならば、すべての R-order が完全非 cancellation である)。

（実）quaternion algebra H について、$H^{\times}$に含まれる非可換有限部分群 G を考える。必然的に $G \subset H^{(1)}$であり $H^{(1)}/(\pm 1) \simeq SU(2)/(\pm 1) \simeq SO(3)$ だから、G は三次元回転群 $SO(3)$ の非可換有限部分群の逆像として得られる。後者は多面体群（polyhedral group）と呼ばれ、二面体群 D_{2n}、四面体群 T、八面体群 O、二十面体群 Iですべてである。対応する $H^{(1)}$の有限部分群を$\tilde{D}_{2n}, \tilde{T}, \tilde{O}, \tilde{I}$とする（これを binary polyhedral group と言う）。このうちの一つを G として、G で生成される H の有限生成 sub Z-module $\mathbb{Z}(G)$ は Q-algebra $\mathbb{Q}(G)$ の Z-order である（$\mathbb{Q}(\tilde{D}_{2n}) = H_{2n}$, $\mathbb{Q}(\tilde{O}) = H_8$, $\mathbb{Q}(\tilde{I}) = H_{10}$, しかし $\mathbb{Q}(\tilde{T})$ は H_6 ではなくて $H_6' = (C_6, \sigma, -2)$ である)。

Swan は $\mathbb{Z}(G)$ の cancellation について詳しく調べている。結果は$\tilde{D}_4$, $\tilde{D}_6$, $\tilde{D}_8$, $\tilde{D}_{10}$, $\tilde{T}$, $\tilde{O}$, $\tilde{I}$ について cancellation 成立。それ以外の$\tilde{D}_{2n}$については不成立。

$\tilde{D}_{12}$, $\tilde{D}_{14}$, $\tilde{D}_{18}$ について不成立の状況を各安定同型類ごとに調べている（$\tilde{D}_{12}$については一部の安定同型類については cancellation 成立。詳しくは安定同型類は 4 つあって、$f(L)$ は 1,1,3,3 である)。

その他$\widetilde{D_4} \times C_2$（$C_2$は 2 次巡回群）などについて調べている（この場合 cancellation 不成立）。

[4] R.G.Swan: *Projective modules over binary polyhedral groups.* J.reine angew.Math. **342**(1983)66–172

NON-COMMUTATIVE GALOIS THEORY AND ACTIONS OF HOPF ALGEBRAS

TADASHI YANAI*

1. Introduction and notation

This note is a summary of the author's recent results on a study concerning Galois correspondence theory for prime rings and actions of finite dimensional pointed Hopf algebras, started in [Y]. This consideration is based on Kharchenko's Galois theory for prime rings [K] established in 1970's and Milinski's result [Mi], which generalizes the theory of linear differential identities with automorphisms of prime rings.

In the first place, we list the notation which we use here. A detailed definition can be seen in [K, M, S].

Let R be a prime ring and $\mathcal{F}$ the set of all nonzero ideals of R. We denote by Q the symmetric Martindale quotient ring of R and by K the center of Q. It is well known that Q is a prime ring and K is a field.

In this note, all algebras, coalgebras, Hopf algebras, and tensor products are assumed to be defined over a fixed field k, unless mentioned otherwise.

Let H be a finite dimensional pointed Hopf algebra with comultiplication Δ, counit ε, and antipode S. As we assumed that H is finite dimensional, S is bijective. We denote the inverse map of S by $\bar{S}$. $G(H)$ represents the set of all group-like elements of H.

Throughout, we assume that Q is a k-algebra and H acts on Q continuously, i.e. for any $h \in H$ and $I \in \mathcal{F}$, there exists $J \in \mathcal{F}$ with $h \cdot J \subseteq I$.

For nonempty subsets A, B in the smash product algebra $Q\#H$, we set $A^B = \{a \in A | ab = ba \text{ for all } b \in B\}$. The following fact is implied easily.

Proposition 1 [Y, Lemma 2.4]. *R^H coincides the subring of all invariants in R:*

$$R^H = \{r \in R | h \cdot r = \varepsilon(h)r \textit{ for all } h \in H\}.$$

Furthermore, we assume that $(Q\#H)^R = K$. In this case, we say that the action of H is *X-outer* (due to [Mi]).

2. Correspondence mappings

A subring $U \subseteq R$ containing R^H is called *rationally complete* if for $a \in R$ and a nonzero ideal I of U, $aI \subseteq U$ implies $a \in U$. A rationally complete subring satisfies the following.

The detailed version of this paper will be submitted for publication elsewhere.

*The author is supported by Grant-in-Aid for Scientific Research (No. 10640052).

Proposition 2 [Y, Proposition 2.5]. *If a rationally complete subring U contains a nonzero ideal of R, then $U = R$.*

The following fact enables us to determine a correspondence which generalizes the usual Galois correspondence.

Lemma A [Mi, Y]. .

(1) *K is stable under the action of H.*
(2) *$K\#H$ is a right H-comodule algebra.*
(3) *For $R^H \subseteq U \subseteq R$, $(Q\#H)^U$ is a right H-comodule subalgebra of $K\#H$.*
(4) *For $K \subseteq \Lambda \subseteq K\#H$, R^Λ is a rationally complete subring.*

For a rationally complete subring U of R containing R^H and a right comodule subalgebra Λ of $K\#H$ containing K, we define $\Phi(U) = (K\#H)^U$ and $\Psi(\Lambda) = R^\Lambda$. Then, by Lemma A, Φ and Ψ determine the following correspondence:

$$\left\{ U \,\middle|\, \begin{array}{l} R^H \subseteq U \subseteq R, \\ \text{rationally complete subring} \end{array} \right\} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \left\{ \Lambda \,\middle|\, \begin{array}{l} K \subseteq \Lambda \subseteq K\#H, \\ \text{right comodule subalgebra} \end{array} \right\}.$$

Our principal objective is to show that the above correspondence is one to one. In concrete, we intend to verify the following two equations:

$$(*) \qquad \left\{ \begin{array}{l} \Psi(\Phi(U)) = U \\ \Phi(\Psi(\Lambda)) = \Lambda \end{array} \right.$$

If $H = \mathrm{k}G$, where G is a finite group of automorphisms, this correspondence essentially represents the usual Galois correspondence, and in this case, we know by Kharchenko's result [K, Theorem 3.10.2] that Φ and Ψ give a one to one correspondence.

As another example so that the above correspondence is one to one, we consider the action of the following Hopf algebra.

Example. Let $N \geq 2$ be an integer and ζ a root of the N-th cyclotomic polynomial over $\mathbb{Z}$. Assume that k contains ζ. We define the Hopf algebra A_N, which is called *the Taft Hopf algebra*, as follows:

(1) as an algebra, A_N is generated by generators X, Y and relations $X^N = 1, Y^N = 0, YX = \zeta XY$ over k.
(2) Comultiplication is given by $\Delta(X) = X \otimes X$, $\Delta(Y) = 1 \otimes Y + Y \otimes X$,
(3) counit by $\varepsilon(X) = 1$, $\varepsilon(Y) = 0$, and
(4) antipode by $S(X) = X^{N-1}, S(Y) = -\zeta^{N-1}X^{N-1}Y$.

If $H = A_N$ acts on Q in a continuous and X-outer way, then it can be computed that both of $(*)$ hold. Moreover, we know that there exist the following types of rationally complete subrings and corresponding right comodule subalgebras:

(i) $R \longleftrightarrow K$,
(ii) $\{r \in R | X^n \cdot r = r\} \longleftrightarrow K < X^n >$, the subalgebra of $K\#H$ generated by X^n over K, where n is a divisor of N with $n \neq N$,
(iii) $\{r \in R | X^n \cdot r = r, Y \cdot r = \alpha(r - X \cdot r)\} \longleftrightarrow K < X^n, Y + \alpha X >$, where n is a divisor of N with $n \neq 1$ and $\alpha \in K$, and
(iv) $R^H \longleftrightarrow K\#H$.

Above example generalizes [Y, Theorem 6.10]. More generally, we have the following result, which is the main theorem in this note.

Main Theorem.
If $K = K^H$, then both of the equations in $(*)$ hold.

3. Related results and proof of the Main Theorem

Here, we state several lemmas related to our correspondence theory and needed to describe the outline of the proof of the Main Theorem.

For $a, x \in Q$ and $h \in H$, it is easy to see that Q is a left $Q\#H$-module via $(a\#h) \cdot x = a(h \cdot r)$ and a right $Q\#H$-module via $x \triangleleft (a\#h) = \bar{S}h \cdot (xa)$.

The following Lemma is a modification of [Mi, Theorem 4.1], which generalizes the theory of linear differential identities with automorphisms [K, Theorem 2.2.2] to the actions of pointed Hopf algebras.

Lemma B [Mi]. *For $I \in \mathcal{F}$, define a mapping $\mu : Q \otimes_K Q\#H \to \mathrm{Hom}(I, Q)$ by $q \otimes_K \xi \mapsto (r \mapsto \xi \cdot (rq))$ for $q \in Q$, $\xi \in Q\#H$, and $r \in I$. Then, μ is injective.*

The following Lemma is a development of [Mi, Theorem 2.3].

Lemma C. *Let U be a subring of R containing R^H and M a nonzero (R, U)-subbimodule of $Q\#H$. Then, there exist $a \in R$, $\sigma \in G(H)$, and $\xi \in \Phi(U)$ such that $0 \neq (a\#\sigma)\xi \in M$.*

For a left integral t in H, we apply Lemma C to the (R, U)-subbimodule RtU of $Q\#H$ and have a nonzero element $(a\#1)\xi \in RStU$, where $a \in R$ and $\xi \in \Phi(U)$. Write $\sum_{i=1}^{n} r_i S t s_i = (a\#1)\xi$ for $r_i \in R$ and $s_i \in U$. Then, applying this element via $\triangleleft$ for $x \in Q$, we have a Hopf version of [MP, Proposition 4].

Corollary. *Let t be a nonzero left integral of H. Then, there exist $a, r_i \in R$, $s_i \in U$ $(i = 1, \ldots, n)$, and $\xi \in \Phi(U)$ with $\sum_{i=1}^{n} t \cdot (xr_i)s_i = (xa) \triangleleft \xi$ for all $x \in Q$.*

Let B be a right coideal subalgebra of H. We define $\int_B^\ell = \{b \in B | b'b = \varepsilon(b')b \text{ for all } b' \in B\}$ and $\int_B^r = \{b \in B | bb' = \varepsilon(b')b \text{ for all } b' \in B\}$. We call an element in $\int_B^\ell$ (resp. $\int_B^r$) *a left* (resp. *right*) *integral* of B.

Since we assumed that H is finite dimensional pointed, we have the following.

Lemma D [Ma1, Ma2, Ko]. *Let $B \subseteq B'$ be right coideal subalgebras of H. Then, we have the followings.*

(1) *B is a Frobenius algebra.*
(2) *B' is free as a left B-module and as a right B-module.*
(3) *$\int_B^\ell$ and $\int_B^r$ are 1-dimensional $\mathbf{k}$-spaces.*
(4) *B is generated by a nonzero element in $\int_B^\ell$ as a right H-comodule.*

As a consequence of the above result, we have

Lemma E. *For* $0 \neq b^{(\ell)} \in \int_B^\ell$ *and* $0 \neq b^{(r)} \in \int_B^r$*, there exists* $h \in H$ *so that* $Sb^{(\ell)} = hb^{(r)}$.

Proof. For $b^{(\ell)} \in \int_B^\ell$ and $b' \in B$, we have

$$Sb^{(\ell)}\varepsilon(b') = \sum Sb^{(\ell)}Sb'_1b'_2 = \sum S(b'_1b^{(\ell)})b'_2 = \sum S(\varepsilon(b'_1)b^{(\ell)})b'_2 = Sb^{(\ell)}b'.$$

Let $\{h_i\}$ be a right B-basis for H. Then, $Sb^{(\ell)} = \sum h_ib_i$ for some $b_i \in B$. By the above formula, we have $\sum h_ib_ib' = \sum h_ib_i\varepsilon(b')$ for $b' \in B$ and so, $b_i \in \int_B^r$ for all i. As $\int_B^r$ is a 1-dimensional k-space, $b_i = \alpha_i b^{(r)}$ for some $\alpha_i \in$ k. Hence, setting $h = \sum \alpha_i h_i$, we have the conclusion. □

Now, we are ready to describe the outline of the proof of the Main Theorem. Hereafter, we assume that $K = K^H$. Then, $K\#H = K \otimes H$ is a Hopf algebra over K. We can assume $K =$ k without the loss of generality.

Proof of the Main Theorem (sketch).

$(\Phi(\Psi(\Lambda)) = \Lambda)$

Note that $\Lambda \subseteq \Phi(\Psi(\Lambda))$ are right coideal subalgebras of H. Let $\xi \in \int_\Lambda^\ell$, $I \in \mathcal{F}$ with $\xi \cdot I \subseteq R$, and $\eta \in \Phi(\Psi(\Lambda))$. For $r \in I$, we have $\xi \cdot r \in \Psi(\Lambda)$ and

$$(\eta\xi) \cdot r = \eta \cdot (\xi \cdot r) = \varepsilon(\eta)(\xi \cdot r).$$

Hence, $(\eta\xi - \varepsilon(\eta)\xi) \cdot I = 0$. By Lemma B, it is implied that $\eta\xi = \varepsilon(\eta)\xi$ and ξ is also a left integral of $\Phi(\Psi(\Lambda))$. Then, by Lemma D(4), it follows that $\Phi(\Psi(\Lambda)) = \Lambda$.

$(\Psi(\Phi(U)) = U$:)

First, we note that h in Lemma E is a group-like element in this case. Using this fact and the Corollary to Lemma C, we have elements $a, r_i \in R$, $s_i \in U$ $(i = 1, \ldots, n)$, and $\xi \in \int_{\Phi(U)}^\ell$ so that $(0 \neq) \sum_{i=1}^n r_iSts_i = (a\#1)S\xi$. Let $I \in \mathcal{F}$ with $t \cdot I \subseteq R$. For $x \in I$, we have

$$x \triangleleft \sum_{i=1}^n r_iSts_i = \sum_{i=1}^n t \cdot (xr_i)s_i \in U$$

as $t \cdot I \subseteq R^H$. On the other hand, for $u \in \Psi(\Phi(U))$,

$$u(x \triangleleft (a\#1)S\xi) = u\xi \cdot (xa) = \xi \cdot (uxa) \in \xi \cdot (Ia).$$

Hence, $A = \xi \cdot (Ia)$ is a left ideal of $\Psi(\Phi(U))$ contained in U. $A \neq 0$ is guaranteed by Lemma B. Considering R^{op} and H^{cop}, we have a nonzero right ideal B of $\Psi(\Phi(U))$ contained in U. Then, AUB is a nonzero ideal of $\Psi(\Phi(U))$ contained in U and $\Psi(\Phi(U)) = U$ is implied by Proposition 1. □

It is expected that the method to prove the Main Theorem is also effective to consider the unsolved case $K \neq K^H$.

References

[K] V. K. Kharchenko, *Automorphisms and derivations of associative rings*, Kluwer, Dordrecht, 1991.

[Ko] M. Koppinen, *Coideal subalgebras in Hopf algebras: freeness, integrals, smash products*, Comm. Algebra 21 (1993), 427–444.

[Ma1] A. Masuoka, *Freeness of Hopf algebras over coradical subalgebras*, Comm. Algebra 20 (1992), 1353–1373.

[Ma2] ———, *Coideal subalgebras in finite Hopf algebras*, J. Algebra 163 (1994), 819–831.

[Mi] A. Milinski, *Actions of pointed Hopf algebras on prime algebras*, Comm. Algebra 23 (1995), 313–333.

[M] S. Montgomery, *Hopf algebras and their actions on rings, CBMS Regional Conf. Ser. in Math.* 82, AMS, Providence, R. I., 1993.

[MP] ——— and D. S. Passman, *Outer Galois theory of prime rings*, Rocky Mt. J. Math. 14 (1984), 305–318.

[S] M. E. Sweedler, *Hopf algebras*, Benjamin, New York, 1969.

[Y] T. Yanai, *Correspondence theory of Kharchenko and X-outer actions of pointed Hopf algebras*, Comm. Algebra 25 (1997), 1713–1740.

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