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PREFACE

The 32nd Symposium on Ring Theory and Representation Theory was held at Yamaguchi, on October 5th - 7th, 1999. The symposium and these proceedings are financially supported by Grant-in-Aid for Scientific Research (A)(1) from the Ministry of Education through the arrangements by Professor Masanori Ishida of Tohoku University.

The volume presents thirteen articles given in the symposium. These articles contain advanced results in ring theory and representation theory. We expect their developments toward the third millennium.

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Jun-ichi Miyachi

Tokyo, January 2000

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An R-module M is said to be a CS-module or extending module if, for any submodule X of M, there exists a direct summand X^* of M which is an essential extension of X. The concept of this module is a notable property of injective modules. In a glance, to control this property seems to be simple. However, even if to study the problem when finite direct sums of CS-module is CS is difficult. A major reason seems to comes from that there are several kind of CS-modules. In fact, the following CS-modules M are considered:

(A) for any submodule X of M, there is a decomposition $M = X^* \oplus M'$ such that $X \subseteq_e X^*$, where $X \subseteq_e X^*$ means that X^* is an essential extension of X.

(B) for a given decomposition $M = \oplus_I M_i$ and any submodule X of M, there exists a decomposition $M = X^* \oplus (\oplus_I M_i')$ with $X \subseteq_e X^*$ and $M_i' \subseteq M_i$.

(C) for any decomposition $M = \oplus_I M_i$ and any submodule X of M, there exists a decomposition $M = X^* \oplus (\oplus_I M_i')$ such that $X \subseteq_e X^*$ and $M_i' \subseteq M_i$.

M with the condition (A) is, of course, a usual CS-module. We say that M is a CS-module for $M = \oplus_I M_i$ if M satisfies the condition (B). And we say that M is a normal CS-module if M satisfies the condition (C), and say that M is a finite normal CS-module if M satisfies the condition (C) for any finite index set I.

Any finitely generated torsion free abelian group $G = \mathbb{Z}_1 \oplus \cdots \oplus \mathbb{Z}_n$ ($n \geq 2$) is a CS-module as a $\mathbb{Z}$-module, but not a CS-module for $G = \mathbb{Z}_1 \oplus \cdots \oplus \mathbb{Z}_n$. We emphasize that almost known CS-modules are normal CS-modules.

As we stated above, for CS-modules, the following is an open problem:

Let M be an R-module with a decomposition $M = \oplus_I M_i$ with each M_i CS. Then give a characterization for M to be a CS-module.

The detailed version of this paper will be submitted for publication elsewhere.

Our purpose of this paper is to give some results on this problem by introducing generalizing relative injectivity. Main results are following: Let $M_1, \cdots, M_n$ be CS-modules and put $P = M_1 \oplus \cdots \oplus M_n$ .Then

(1) When $n = 2$, P is CS for $P = M_1 \oplus M_2$ if and only if M_i is a generalized M_j-injective for $i \neq j$.

(2) When $n \geq 3$, P is CS for $P = M_1 \oplus \cdots \oplus M_n$ if and only if M_i is a generalized $M_j \oplus M_k$-injective for any distinct i, j, k, and, if and only if $M_i \oplus M_j$ is a generalized M_k-injective for any distinct i, j, k.

(3) When $n \geq 3$, P is normal CS if and only if each M_i is normal CS and M_i is a generalized $M_j \oplus M_k$-injective for any distinct i, j, k, and, if and only if $M_i \oplus M_j$ is a generalized M_k-injective for any distinct i, j, k.

(4) When each M_i is a uniform module, P is CS for $P = M_1 \oplus \cdots \oplus M_n$ if and only if M_i is a generalized M_j-injective for $i \neq j$.

1. Preliminaries

Throughtout this paper R is a ring with identity and modules are unitary right R-modules.

Let M be a module and N a submodule of M. $N \subseteq_e M$ (resp. $N <_{\oplus} M$) means that N is an essential submodule (resp. direct summand) of M. For $T <_{\oplus} P$, π_T denotes the projection : $P \to T$. For an element $m \in M$, by $(0 : m)$ we denotes the annihilator right ideal of m .

Let $\{M_i \mid i \in I\}$ be a family of modules and let $M = \oplus_I M_i$. A module M is said to be a CS-module for $M = \oplus_I M_i$, if for any submodule X of M, there exists $X^* \subseteq M$ and $\overline{M_i} <_{\oplus} M_i (i \in I)$ such that $X \subseteq_e X^*$ and $M = X^* \oplus (\oplus_I \overline{M_i})$.

Let $M = M_1 \oplus M_2$ and let $\varphi : M_1 \to M_2$ be a homomorphism. Put $\langle M_1 \xrightarrow{\varphi} M_2 \rangle = \{m_1 + \varphi(m_1) \mid m_1 \in M_1\}$. Then $M = M_1 \oplus M_2 = \langle M_1 \xrightarrow{\varphi} M_2 \rangle \oplus M_2$.

M and N are R-modules. M is said to be essentially N-injective, if for any submodule X of N and any homomorphism $f : X \to M$ with $Kerf \subseteq_e X$, there exists a homomorphism $f^* : N \to M$ with $f^*|_X = f$.

A module M is said to be a *quasi-continuous* if it is a CS-module with the following condition (C_3) :

(C_3) If M_1 and M_2 are direct summands of M such that $M_1 \cap M_2 = 0$, then $M_1 \oplus M_2$ is a direct summand of M.

The following is known (cf.[1, pp.16-17])

Proposition 1.1. (1) *For R-modules M and N, If M is essentially N-injective, then M is essentially K-injective for any submodule K of N.*

(2) *Let M be a module and $\{N_\lambda \mid \lambda \in \Lambda\}$ a family of modules. Then M is essentially $\oplus_\Lambda N_\lambda$-injective if and only if M is essentially N_λ-injective for all $\lambda \in \Lambda$.*

(3) *Let F be a finite set and $\{M_i \mid i \in F\}$ a family of modules. Then $\oplus_F M_i$ is essentially N-injective if and only if M_i is essentially N-injective for all $i \in F$.*

By a slight modification of the proof of [4, Theorem 1.7], we can show the following:

Proposition 1.2. *The following conditions are equivalent:*

(1) *$\oplus_\Lambda M_\lambda$ is essentially N-injective*

(2) *$\oplus_I M_i$ is essentially N-injective for every countable subset $I \subseteq \Lambda$*

(3) *M_λ is essentially N-injective for every $\lambda \in \Lambda$ and following condition (A_2') holds.*

(A_2') For every choice of $n \in N$ and $m_i \in M_i$ for distinct $\alpha_i \in \Lambda (i \in \mathbb{N})$ such that $(0 : m_i) \supseteq (0 : n)$ and $\cap_{i=1}^{\infty} Ker\varphi_i \subseteq_e nR$ for canonical homomorphism $\varphi_i : nR \to m_iR$, ascending sequence $\cap_{i \geq n}(0 : m_i)(n \in \mathbb{N})$ become stationary.

Definition 1.3. *Let M and N be modules. M is said to be generalized N-injective, if for any submodule X of N and any homomorphism $\varphi : X \to M$, there exist decompositions $N = \overline{N} \oplus \overline{\overline{N}}$, $M = \overline{M} \oplus \overline{\overline{M}}$, a homomorphism $\overline{\varphi} : \overline{N} \to \overline{M}$ and a monomorphism $\psi : \overline{\overline{M}} \to \overline{\overline{N}}$ satisfying following properties $(*)$, $(**)$.*

$(*)$ *$X \subseteq \overline{N} \oplus \psi(\overline{\overline{M}})$*

$(**)$ *For $x \in X$, we express x in $N = \overline{N} \oplus \overline{\overline{N}}$ as $x = \overline{x} + \overline{\overline{x}}$, where $\overline{x} \in \overline{N}$ and $\overline{\overline{x}} \in \overline{\overline{N}}$. Then $\varphi(x) = \overline{\varphi}(\overline{x}) + \overline{\overline{\varphi}}(\overline{\overline{x}})$, where $\overline{\overline{\varphi}} = \psi^{-1}$.*

Proposition 1.4. (1) *If M is N-injective, then M is generalized N-injective.*

(2) *If M is generalized N-injective, then M is essentially N-injective.*

Proof. (1) is clear.

(2). Let X be a submodule of N and $f : X \to M$ be a homomorphism with $Kerf \subseteq_e X$. Let Y be a submodule of N with $X \oplus Y \subseteq_e N$. Define $g : A = X \oplus Y \to M$ by $g(x + y) = f(x)$. Since $X \oplus Y \subseteq_e N$ and $Kerf \subseteq_e X$, we see $Kerg \subseteq_e N$. By assumption, there exist decompositions $M = \overline{M} \oplus \overline{\overline{M}}$ and $N = \overline{N} \oplus \overline{\overline{N}}$, a homomorphism $\overline{g} : \overline{N} \to \overline{M}$ and a monomorphism $h : \overline{\overline{M}} \to \overline{\overline{N}}$ satisfying, for $a = \overline{a} + \overline{\overline{a}}$ with $\overline{a} \in \overline{N}$ and $\overline{\overline{a}} \in \overline{\overline{N}}$, $g(a) = \overline{g}(\overline{a}) + \overline{\overline{g}}(\overline{\overline{a}})$, where $\overline{\overline{g}} = h^{-1}$. Since $Kerg \subseteq_e N$, we see $Imh = 0$ and hence $\overline{\overline{M}} = 0$.

Now define $f^* : N = \overline{N} \oplus \overline{\overline{N}} \to M$ by $f^*(\overline{n} + \overline{\overline{n}}) = \overline{g}(\overline{n})$. Then we see $f^*|_X = f$. Thus M is essentially N-injective. □

Remark *N-injective $\Rightarrow$ generalized N-injective $\Rightarrow$ essentially N-injective.*

Proposition 1.5. *If M is generalized N-injective, then M is generalized K-injective for any direct summand K of N.*

Proof. Let X be a submodule of K and $f : X \to M$ be a homomorphism. Put $N = K \oplus L$. Define $g : X \oplus L \to M$ by $g(x + l) = f(x)$. By assumption, there exists decompositions $M = \overline{M} \oplus \overline{\overline{M}}$ and $N = \overline{N} \oplus \overline{\overline{N}}$, a homomorphism $\overline{g} : \overline{N} \to \overline{M}$ and a monomorphism $h : \overline{\overline{M}} \to \overline{\overline{N}}$. Then L is a submodule of $\overline{N}$ since $g(L) = 0$ and h is a monomorphism. Thus we get

$$N = (\overline{N} \cap K) \oplus \overline{\overline{N}} \oplus L \qquad \cdots (1)$$

Since $\overline{N} \cap K <_{\oplus} K$, there exists $K' \subseteq K$ such that $K = (\overline{N} \cap K) \oplus K'$. Let $p_{K'}$ be the projection : $N \to K'$. Then we have

$$\overline{\overline{N}} \simeq K' \qquad (\text{by } p_{K'}|_{\overline{\overline{N}}})$$

Therefore the natural map $\alpha : p_{K'}(\overline{\overline{N}}) \to p_{\overline{N} \cap K}(\overline{\overline{N}})$ is a well-defined homomorphism (where $p_{\overline{N} \cap K}$ is the projection : $N \to \overline{N} \cap K$). Since $K' = p_{K'}(\overline{\overline{N}})$, we get

$$\begin{aligned} N &= (\overline{N} \cap K) \oplus K' \oplus L \\ &= (\overline{N} \cap K) \oplus \langle K' \xrightarrow{\alpha} \overline{N} \cap K \rangle \oplus L \qquad \cdots (2) \end{aligned}$$

Put $K^* = \langle K' \xrightarrow{\alpha} \overline{N} \cap K \rangle$ and let p_{K^*} be the projection : $N \to K^*$. Then by (1) and (2),

$$\overline{\overline{N}} \simeq K^* \qquad (\text{by } p_{K^*}|_{\overline{\overline{N}}})$$

Hence $h^* = p_{K^*} \circ h : \overline{\overline{M}} \to K^*$ is a monomorphism. Now for $x \in X$, we express x in $K = (\overline{N} \cap K) \oplus K^*$ as $x = \overline{n_k} + k^*$, where $\overline{n_k} \in \overline{N} \cap K$ and $k^* \in K^*$. Put $\overline{f} = \overline{g}|_{\overline{N} \cap K}$ and $\overline{\overline{f}} = h^{*-1}$. Then we see

$$f(x) = \overline{f}(\overline{n_k}) + \overline{\overline{f}}(k^*)$$

Therfore M is generalized K-injective. □

2. Direct sums of CS-modules

The one of main purpose of this paper is to show the following.

Theorem 2.1. *Let M_1 and M_2 be CS-modules and let $P = M_1 \oplus M_2$. Then P is CS for $P = M_1 \oplus M_2$ if and only if M_i is generalized M_j-injective ($i \neq j$).*

For a proof of this theorem, we need several results.

Lemma 2.2. *Let M and N be modules with $M \cap N = 0$, and let $f : M \to N$ be a homomorphism. If A is an essential submodule of M, then $\langle A \xrightarrow{f|_A} N\rangle \subseteq_e \langle M \xrightarrow{f} N\rangle$.*

In particular, $f(A) = 0$ implies $A \subseteq_e \langle M \xrightarrow{f} N\rangle$.

Proof. Evident. □

Lemma 2.3. *Let $\{M_\alpha \mid \alpha \in \Lambda\}$ be a family of modules and put $P = \oplus_\Lambda M_\alpha$. If $P = X \oplus (\oplus_\Lambda M''_\alpha)$, where $M_\alpha = M'_\alpha \oplus M''_\alpha$ ($\alpha \in \Lambda$), then $X = \oplus_{\beta\in\Lambda}\langle M'_\beta \to \oplus_\Lambda M''_\alpha\rangle$.*

Proof. Since $X \cap (\oplus_\Lambda M''_\alpha) = 0$, the natural map $f : \pi_{\oplus_\Lambda M'_\alpha}(X) \to \pi_{\oplus_\Lambda M''_\alpha}(X)$ is a homomorphism. Put $f_\alpha = f|_{M'_\alpha}$ ($\alpha \in \Lambda$). Then

$$\begin{aligned} X &= \langle \pi_{\oplus_\Lambda M'_\alpha}(X) \xrightarrow{f} \pi_{\oplus_\Lambda M''_\alpha}(X)\rangle \\ &= \langle \oplus_\Lambda M'_\alpha \xrightarrow{f} \oplus_\Lambda M''_\alpha\rangle \\ &= \oplus_{\beta\in\Lambda}\langle M'_\beta \xrightarrow{f_\beta} \oplus_\Lambda M''_\alpha\rangle \end{aligned}$$ □

Lemma 2.4. *Let M and N be modules with $M \cap N = 0$, and let $\langle M \xrightarrow{\alpha} N\rangle = X_1 \oplus X_2$. Then there exists a decomposition $M = M_1 \oplus M_2$ such that $X_i = \langle M_i \xrightarrow{\alpha|_{M_i}} N\rangle$ ($i = 1, 2$).*

Proof. Define $f : \langle M \xrightarrow{\alpha} N\rangle \to M$ by $f(m + \alpha(m)) = m$. Then f is an isomorphism, since $M \cap N = 0$. Put $M_i = f(X_i)$. Then

$$X_i = \langle M_i \xrightarrow{\alpha|_{M_i}} N\rangle.$$ □

Lemma 2.5. *Let $M = M_1 \oplus M_2$ and let X be a submodule of M. If $X_1 \subseteq_e M_1$ for $X_1 \subseteq X$, then $X \supseteq_e X_1 \oplus (M_2 \cap X)$.*

Proof. For $0 \neq x \in X$, we express x in $M = M_1 \oplus M_2$ as $x = m_1 + m_2$, where $m_1 \in M_1$ and $m_2 \in M_2$. If $m_1 \in X_1$, then $m_2 \in M_2 \cap X$. So we get $0 \neq x \in X_1 \oplus (M_2 \cap X)$. If $m_1 \notin X_1$, then there exists $r \in R$ such that $0 \neq m_1 r \in X_1$. Hence $0 \neq xr = m_1 r + m_2 r \in X_1 \oplus (M_2 \cap X)$.

Hence $X \supseteq_e X_1 \oplus (M_2 \cap X)$. □

Lemma 2.6. *Let P be a module with a decomposition $P = M_1 \oplus M_2$, where each M_i is a CS-module, and let M_i is essentially M_j-injective ($i \neq j$). Let X be a submodule of P with $X \supseteq_e X_1 \oplus X_2$ and let $M_i = T_i \oplus N_i$ with $X_i \subseteq_e T_i$ ($i = 1, 2$). Then there exist $X^* \subseteq M$ and $M'_i <_\oplus M_i$ ($i = 1, 2$) such that $X \subseteq_e X^*$ and $P = X^* \oplus (M'_1 \oplus M'_2)$.*

Proof. Put $Y_1 = (T_1 \oplus N_1 \oplus N_2) \cap X$. Since $X \supseteq_e X_1 \oplus X_2$, the natural map $f : \pi_{T_1}(Y_1) \to \pi_{N_1}(Y_1)$ is a homomorphism. Since $X_1 \subseteq_e T_1$ and $f(X_1) = 0$, we have $X_1 \subseteq_e \langle \pi_{T_1}(Y_1) \xrightarrow{f} \pi_{N_1}(Y_1)\rangle = \pi_{M_1}(Y_1)$ by Lemma 2.2. Since M_1 is a CS-module, there exists a decomposition $M_1 = Y_1^* \oplus M'_1$ with $\pi_{M_1}(Y_1) \subseteq_e Y_1^*$. Thus we see

$$\begin{array}{lll} P & = & Y_1^* \oplus T_2 \oplus M_1' \oplus N_2 \\ \cup\!\mid & & \cup\!\mid^e \quad \cup\!\mid^e \\ X & \supseteq_e & X_1 \oplus X_2 \end{array}$$

As $\pi_{M_1}(Y_1) \subseteq_e Y_1^*$, the natural map $\alpha : \pi_{T_2}(X) \to \pi_{M_1'}(X)$ is a homomorphism. Further we see $Ker\alpha \subseteq_e \pi_{T_2}(X)$ since $X_2 \subseteq Ker\alpha \subseteq \pi_{T_2}(X) \subseteq T_2$ and $X_2 \subseteq_e T_2$. Thus there exists a homomorphism $\overline{\alpha} : T_2 \to M_1'$ such that $\overline{\alpha}|_{\pi_{T_2}(X)} = \alpha$. Since $X_2 \subseteq_e T_2$ and $\overline{\alpha}(X_2) = 0$, we get $X_2 \subseteq_e \langle T_2 \xrightarrow{\overline{\alpha}} M_1' \rangle$ by Lemma 2.2. Thus we see

$$\begin{array}{lll} P & = & Y_1^* \oplus \langle T_2 \xrightarrow{\overline{\alpha}} M_1' \rangle \oplus M_1' \oplus N_2 \\ \cup\!\mid & & \cup\!\mid^e \qquad\quad \cup\!\mid^e \\ X & \supseteq_e & X_1 \quad \oplus \quad X_2 \end{array}$$

Define $\beta : M_2 = T_2 \oplus N_2 \to M_1'$ by $\beta(t_2 + n_2) = \overline{\alpha}(t_2)$. Then

$$\langle T_2 \xrightarrow{\overline{\alpha}} M_1' \rangle \oplus N_2 = \langle M_2 \xrightarrow{\beta} M_1' \rangle$$

Put $Y_2 = \langle M_2 \xrightarrow{\beta} M_1' \rangle \cap X$. Since $\langle M_2 \xrightarrow{\beta} M_1' \rangle$ is a CS-module, there exists a decomposition $\langle M_2 \xrightarrow{\beta} M_1' \rangle = Z_2' \oplus Z_2''$ with $Y_2 \subseteq_e Z_2''$. By Lemma 2.4, there exists a decomposition $M_2 = M_2' \oplus M_2''$ such that $Z_2' = \langle M_2' \to M_1' \rangle$ and $Z_2'' = \langle M_2'' \to M_1' \rangle$. Since $X_2 \subseteq_e Y_2$, we see

$$\begin{array}{lll} P & = & Y_1^* \oplus Z_2'' \oplus M_1' \oplus Z_2' \\ \cup\!\mid & & \cup\!\mid^e \quad \cup\!\mid^e \\ X & \supseteq_e & X_1 \oplus X_2 \end{array}$$

Since $X \subseteq Y_1^* \oplus \langle M_2 \to M_1' \rangle = Y_1^* \oplus Z_2' \oplus Z_2''$, $Y_2 \subseteq_e Z_2''$ and $X_1 \subseteq_e Y_1^*$, the natural map $\gamma : \pi_{Y_1^*}(X) \to \pi_{Z_2'}(X)$ is a homomorphism with $Ker\gamma \subseteq_e \pi_{Y_1^*}(X)$. Inasmuch as Z_2' is essentially Y_1^*-injective, there exists a homomorphism $\overline{\gamma} : Y_1^* \to Z_2'$ with $\overline{\gamma}|_{\pi_{Y_1^*}(X)} = \gamma$. By Lemma 2.2, we get $X_1 \subseteq_e \langle Y_1^* \xrightarrow{\overline{\gamma}} Z_2' \rangle$. So we see

$$\begin{array}{lll} P = & \langle Y_1^* \xrightarrow{\overline{\gamma}} Z_2' \rangle \oplus Z_2'' \oplus M_1' \oplus Z_2' \\ \cup\!\mid & \quad\cup\!\mid^e \qquad\quad \cup\!\mid^e \\ X & \supseteq_e \quad X_1 \quad \oplus \quad X_2 \end{array}$$

Put $X^* = \langle Y_1^* \xrightarrow{\overline{\gamma}} Z_2' \rangle \oplus Z_2''$, then $X \subseteq_e \langle Y_1^* \xrightarrow{\overline{\gamma}} Z_2' \rangle \oplus Z_2'' = X^*$. Thus, we see

$$\begin{aligned} P &= X^* \oplus M_1' \oplus Z_2' \\ &= X^* \oplus M_1' \oplus \langle M_2' \to M_1' \rangle \\ &= X^* \oplus M_1' \oplus M_2' \end{aligned}$$

□

Lemma 2.7. *Let P be a generalized T-injective CS-module and let $N <_{\oplus} P$. If $A \subseteq_e T$ and $B \subseteq_e N$ with $A \overset{\varphi}{\simeq} B$, then there exist $A' \subseteq_e A$,* *decompositions* $T = \overline{T} \oplus \overline{\overline{T}}$, $N = \overline{N} \oplus \overline{\overline{N}}$, *a homomorphism* $f : \overline{T} \to \overline{N}$ *and a monomorphism* $g : \overline{\overline{N}} \to \overline{\overline{T}}$ *satisfying following* (*), (**):

(*) $A' \subseteq \overline{T} \oplus g(\overline{\overline{N}})$

(*) *For* $a' \in A'$, *we express* a' *in* $T = \overline{T} \oplus \overline{\overline{T}}$ *as* $a' = \overline{a} + \overline{\overline{a}}$, *where* $\overline{a} \in \overline{T}$, $\overline{\overline{a}} \in \overline{\overline{T}}$. *Then* $\varphi(a') = f(\overline{a}) + g^{-1}(\overline{\overline{a}})$.

Proof. Since P is generalized T-injective and φ is a isomorphism, there exist decompositions $T = \overline{T} \oplus \overline{\overline{T}}$, $P = \overline{P} \oplus \overline{\overline{P}}$ and monomorphisms $\overline{\varphi} : \overline{T} \to \overline{P}$, $\psi : \overline{\overline{P}} \to \overline{\overline{T}}$. As $\overline{P}$ is a CS-module, there exists a decomposition $\overline{P} = \overline{P^*} \oplus S$ with $\overline{\varphi}(\overline{T}) \subseteq_e \overline{P^*}$. Put $\overline{A} = A \cap \overline{T}$, $\overline{\overline{A}} = A \cap \overline{\overline{T}}$, $\varphi(\overline{A}) = \overline{B}$ and $\varphi(\overline{\overline{A}}) = \overline{\overline{B}}$, then we see $\overline{B} = B \cap \overline{P^*} \subseteq_e \overline{P^*}$, $\overline{\overline{B}} = B \cap \overline{\overline{P}} \subseteq_e \overline{\overline{P}}$ and $\overline{B} \oplus \overline{\overline{B}} \subseteq_e \pi_N(\overline{P^*}) \oplus \pi_N(\overline{\overline{P}}) \subseteq_e N$.

As N is a CS-module, there exist a decomposition $N = K \oplus \pi_N(\overline{\overline{P}})^*$ with $\pi_N(\overline{\overline{P}}) \subseteq_e \pi_N(\overline{\overline{P}})^*$. Since $\pi_N(\overline{\overline{P}})^* \cap \pi_N(\overline{P^*}) = 0$ and $\overline{B} \oplus \overline{\overline{B}} \subseteq_e N$, the natural map $\alpha : \pi_N(\overline{\overline{P}})^* \to \pi_{\overline{\overline{P}}}(\pi_N(\overline{\overline{P}})^*)$ is an isomorphism.

Now define $h : \overline{T} \to \overline{\overline{T}}$ by $h(\overline{t}) = -\psi \circ \alpha \circ p \circ \overline{\varphi}(\overline{t})$ (where p is the projection : $P \to \pi_N(\overline{\overline{P}})^*$). Put $f = \pi_K \circ \overline{\varphi} \circ \beta : \langle \overline{T} \xrightarrow{h} \overline{\overline{T}} \rangle \to K$ and $g = \psi \circ \alpha : \pi_N(\overline{\overline{P}})^* \to \overline{\overline{T}}$, where $\beta : \langle \overline{T} \xrightarrow{h} \overline{\overline{T}} \rangle \to \overline{T}$ ($\overline{t} + h(\overline{t}) \mapsto \overline{t}$). Then we see that g is a monomorphism. For $\overline{a} + \overline{\overline{a}} \in \overline{A} \oplus \overline{\overline{A}}$, we express $\overline{a} + \overline{\overline{a}}$ in $\langle \overline{T} \xrightarrow{h} \overline{\overline{T}} \rangle \oplus \overline{\overline{T}}$ as $\overline{a} + \overline{\overline{a}} = \overline{t} + h(\overline{t}) + \overline{\overline{t}}$, where $\overline{t} + h(\overline{t}) \in \langle \overline{T} \xrightarrow{h} \overline{\overline{T}} \rangle$ and $\overline{\overline{t}} \in \overline{\overline{T}}$. Since $\psi^{-1}(h(\overline{t}))$, $\psi^{-1}(\overline{\overline{t}}) \in N \cap \overline{\overline{P}}$, we get

$$\begin{aligned}
\varphi(\overline{a} + \overline{\overline{a}}) &= \overline{\varphi}(\overline{t}) + \psi^{-1}(h(\overline{t})) + \psi^{-1}(\overline{\overline{t}}) \\
&= \pi_K(\overline{\varphi}(\overline{t})) + p(\overline{\varphi}(\overline{t})) + \psi^{-1}(h(\overline{t})) + \psi^{-1}(\overline{\overline{t}}) \\
&= \pi_K(\overline{\varphi}(\overline{t})) + p(\overline{\varphi}(\overline{t})) + (-p(\overline{\varphi}(\overline{t}))) + \alpha^{-1}(\psi^{-1}(\overline{\overline{t}})) \\
&= \pi_K \circ \overline{\varphi} \circ \beta(\overline{t} + h(\overline{t})) + (\psi \circ \alpha)^{-1}(\overline{\overline{t}}) \\
&= f(\overline{t} + h(\overline{t})) + g^{-1}(\overline{\overline{t}})
\end{aligned}$$

□

Proposition 2.8. *Let* $P = M_1 \oplus \cdots \oplus M_n$ *and let* $M_i = M_i' \oplus M_i''$ ($i = 1, \cdots, n$). *If* P *is CS for* $P = M_1 \oplus \cdots \oplus M_n$, *then* $P' = M_1' \oplus \cdots \oplus M_n'$ *is CS for* $P' = M_1' \oplus \cdots \oplus M_n'$.

Proof. Let $X \subseteq P'$. Since $P' <_{\oplus} P$, P' is a CS-module. Hence there exists a direct summand $X^* <_{\oplus} P'$ with $X \subseteq_e X^*$. By assumption, for $X^* \oplus (M_1'' \oplus \cdots \oplus M_n'') \subseteq P$, there exist $Z \subseteq P$ and $\overline{M_i} <_{\oplus} M_i$ ($i = 1, \cdots, n$) such that $X^* \oplus (M_1'' \oplus \cdots \oplus M_n'') \subseteq_e Z$ and $P = Z \oplus (\overline{M_1} \oplus \cdots \oplus \overline{M_n})$. Since $X^* \oplus (M_1'' \oplus \cdots \oplus M_n'') <_{\oplus} P$, we get

$$P = X^* \oplus (M_1'' \oplus \cdots \oplus M_n'') \oplus (\overline{M_1} \oplus \cdots \oplus \overline{M_n})$$

Since $M_i'' \cap \overline{M_i} = 0$, the natural map $\alpha_i : \pi_{M_i'}(\overline{M_i}) \to \pi_{M_i''}(\overline{M_i})$ is a homomorphism. Put $\overline{M_i'} = \pi_{M_i'}(\overline{M_i})$. Since $\overline{M_i} = \langle \overline{M_i'} \to M_i'' \rangle$, we get

$$P = X^* \oplus ((\overline{M_1'} \to M_1'') \oplus \cdots \oplus (\overline{M_n'} \to M_n'')) \oplus (M_1'' \oplus \cdots \oplus M_n'')$$
$$= X^* \oplus (\overline{M_1'} \oplus \cdots \oplus \overline{M_n'}) \oplus (M_1'' \oplus \cdots \oplus M_n'')$$

Since $X^* \oplus (\overline{M_1'} \oplus \cdots \oplus \overline{M_n'}) \subseteq P'$ and $P' \cap (M_1'' \oplus \cdots \oplus M_n'') = 0$, we see

$$P' = X^* \oplus (\overline{M_1'} \oplus \cdots \oplus \overline{M_n'})$$ □

Proof of Theorem 2.1.

" Only if " : By Proposition 2.8, it is enough to show that if $M = M_1 \oplus M_2$ is CS for $M = M_1 \oplus M_2$, then M_2 is generalized M_1-injective.

Assume that $M = M_1 \oplus M_2$ is CS for $M = M_1 \oplus M_2$. Let $A \subseteq M_1$ and $\varphi : A \to M_2$ be a homomorphism. By assumption, for $\langle A \xrightarrow{\varphi} M_2 \rangle \subseteq M$, there exist $Z <_\oplus M$ and $M_i'' <_\oplus M_i$ $(i = 1, 2)$ such that $\langle A \xrightarrow{\varphi} M_2 \rangle \subseteq_e Z$ and $M = Z \oplus M_1'' \oplus M_2''$. Put $M_i' = M_i \cap (Z \oplus M_j'')$ $(i \neq j)$. Since $M_i = M_i'' \oplus (M_i \cap (Z \oplus M_j''))$, we get

$$M = M_1' \oplus M_2' \oplus M_1'' \oplus M_2''$$
$$= \quad Z \quad \oplus M_1'' \oplus M_2''$$

Let p be the projection : $M \to Z$ and put $X = p(M_1')$, $Y = p(M_2')$. For any $x \in X$, we can express x as $x = m_1' + m_2''$, where $m_1' \in M_1'$ and $m_2'' \in M_2''$. Then $\varphi' : M_1' \to M_2''$ $(m_1' \mapsto m_2'')$ is a homomorphism since $Z \cap M_2'' = 0$. By the same argument, for any $y \in Y$, we can express y as $y = m_2' + m_1''$, where $m_2' \in M_2'$ and $m_1'' \in M_1''$. Then $\psi : M_2' \to M_1''$ $(m_2' \mapsto m_1'')$ is a homomorphism. Since $\langle A \xrightarrow{\varphi} M_2 \rangle \subseteq_e Z = X \oplus Y$, ψ is a monomorphism. Thus, for any $a \in A$, we see

$$a + \varphi(a) = x + y = m_1' + m_2' + m_1'' + m_2''$$
$$= m_1' + \varphi'(m_1') + m_1'' + \psi^{-1}(m_1'')$$

Since $A \subseteq M_1$, we get

$$a = m_1' + m_1'' \ ; \qquad \varphi(a) = \varphi'(m_1') + \psi^{-1}(m_1'')$$

Therefore M_2 is generalized M_1-injective.

" If " : Let $X \subseteq P$ and put $X_i = M_i \cap X$ $(i = 1, 2)$. Then there exist decompositions $M_i = T_i \oplus N_i$ with $X_i \subseteq_e T_i$ $(i = 1, 2)$. Put $Y_2 = (N_1 \oplus N_2) \cap X$. For $\pi_{N_i}(Y_2) \subseteq N_i$, there exist decompositions $N_i = N_i' \oplus N_i''$ with $\pi_{N_i}(Y_2) \subseteq_e N_i'$ $(i = 1, 2)$. Then the natural map $\alpha : \pi_{N_1}(Y_2) \to \pi_{N_2}(Y_2)$ is an isomorphism. By Proposition 1.5, N_2 is generalized N_1'-injective. Thus, by Lemma 2.7, for $\alpha : \pi_{N_1}(Y_2) \to \pi_{N_2}(Y_2)$, there exist decompositions $N_i' = \overline{N_i'} \oplus \overline{\overline{N_i'}}$ $(i = 1, 2)$, a homomorphism $\overline{\alpha} : \overline{N_1'} \to \overline{\overline{N_2'}}$ and a monomorphism $\beta : \overline{N_2'} \to \overline{\overline{N_1'}}$ satisfying, for any $x \in \pi_{N_1}(Y_2)$, x can be expressed as $x = \overline{x} + \overline{\overline{x}}$ with $\overline{x} \in \overline{N_1}$ and $\overline{\overline{x}} \in Im\beta$, and $\alpha(x) = \overline{\alpha}(\overline{x}) + \beta^{-1}(\overline{\overline{x}})$. Since $\pi_{N_1}(Y_2) \subseteq_e N_1'$ and α is an isomorphim, $\overline{\alpha}$ is a monomorphim. So we see

$$Y_2 = \langle \pi_{N_1}(Y_2) \xrightarrow{\alpha} \pi_{N_2}(Y_2)\rangle \subseteq_e \langle \overline{N_1'} \xhookrightarrow{\overline{\alpha}} \overline{\overline{N_2'}}\rangle \oplus \langle \overline{N_2'} \xhookrightarrow{\beta} \overline{\overline{N_1'}}\rangle$$

Put $Y_2^* = \langle \overline{N_1'} \xhookrightarrow{\overline{\alpha}} \overline{\overline{N_2'}}\rangle \oplus \langle \overline{N_2'} \xhookrightarrow{\beta} \overline{\overline{N_1'}}\rangle$. Then P is essentially Y_2^*-injective since $\overline{\alpha}$ and β are monomorphisms. In the next step, we newly put $N_i = \overline{\overline{N_i'}} \oplus N_i''$. By Lemma 2.5, we see

$$\begin{array}{lcl} P & = & T_1 \oplus T_2 \oplus Y_2^* \oplus N_1 \oplus N_2 \\ \cup\!\mid & & \cup\!\mid^e \quad \cup\!\mid^e \quad \cup\!\mid^e \\ X & \supseteq_e & X_1 \oplus X_2 \oplus Y_2 \end{array}$$

Put $P' = T_1 \oplus T_2 \oplus N_1 \oplus N_2$, $M_i' = T_i \oplus N_i$ and $X' = P' \cap X$. Then

$$\begin{array}{lcl} P' & = & T_1 \oplus T_2 \oplus N_1 \oplus N_2 \\ \cup\!\mid & & \cup\!\mid^e \quad \cup\!\mid^e \\ X' & \supseteq_e & X_1 \oplus X_2 \end{array}$$

By Lemma 2.6, there exist $Z \subsetneq P'$ and $\overline{M_i'} <_\oplus M_i'$ such that $X' \subseteq_e Z'$ and $P' = Z \oplus \overline{M_1'} \oplus \overline{M_2'}$. So we see

$$\begin{array}{lcl} P & = & Z \oplus Y_2^* \oplus \overline{M_1'} \oplus \overline{M_2'} \\ \cup\!\mid & & \cup\!\mid^e \quad \cup\!\mid^e \\ X & \supseteq_e & X' \oplus Y_2 \end{array}$$

Since $X' \cap (\overline{M_1'} \oplus \overline{M_2'}) = 0$, the natural map $\gamma : \pi_{Y_2^*}(X) \to \pi_{\overline{M_1'} \oplus \overline{M_2'}}(X)$ is a homomorphism. Now we get $Ker\gamma \subseteq_e \pi_{Y_2^*}(X)$ since $Y_2 \subseteq Ker\gamma \subseteq \pi_{Y_2^*}(X) \subseteq Y_2^*$ and $Y_2 \subseteq_e Y_2^*$. Inasmuch as $\overline{M_1'} \oplus \overline{M_2'}$ is essentially Y_2^*-injective, there exists a homomorphism $\gamma^* : Y_2^* \to \overline{M_1'} \oplus \overline{M_2'}$ with $\gamma^*|_{\pi_{Y_2^*}(X)} = \gamma$. Since $Y_2 \subseteq_e Y_2^*$ and $\gamma^*(Y_2) = 0$, $Y_2 \subseteq_e \langle Y_2^* \xrightarrow{\gamma^*} \overline{M_1'} \oplus \overline{M_2'}\rangle$ by Lemma 2.2. Hence

$$X \subseteq_e < Y_2^* \xrightarrow{\gamma^*} \overline{M_1'} \oplus \overline{M_2'} > \oplus Z$$

As a result we get a decomposition

$$P = \langle Y_2^* \xrightarrow{\gamma^*} \overline{M_1'} \oplus \overline{M_2'}\rangle \oplus Z \oplus \overline{M_1'} \oplus \overline{M_2'}$$

Therefore P is CS for $P = M_1 \oplus M_2$.

□

The following result is a consequence of Theorem 2.1 and Propositoin 2.8

Proposition 2.9. *If M and N are relative generalized injective CS-modules, then M' is generalized N-injective for all $M' <_\oplus M$.*

Theorem 2.10. *Let $M_1, \cdots, M_n$ be CS-modules and let $P = M_1 \oplus \cdots \oplus M_n$ $(n \geq 3)$. Then the following conditions are equivalent.*

(1) *P is CS for $P = M_1 \oplus \cdots \oplus M_n$*

(2) *For any $M_{i_1}, M_{i_2}, M_{i_3} \in \{M_1, \cdots, M_n\}$, $P' = M_{i_1} \oplus M_{i_2} \oplus M_{i_3}$ is CS for $P' = M_{i_1} \oplus M_{i_2} \oplus M_{i_3}$.*

Proof. (1) $\Rightarrow$ (2) holds by Proposition 2.8.

(2) $\Rightarrow$ (1). Assume that (1) holds for $n = k$, and let $P = M_1 \oplus \cdots \oplus M_k \oplus M_{k+1}$ and $M = M_k \oplus M_{k+1}$. For any $j \in \{1, \cdots, k-1\}$, $M_j \oplus M_k \oplus M_{k+1}$ is CS for $M_j \oplus M_k \oplus M_{k+1}$ by (2). So we see that M is a CS-module. Hence M_j is generalized M-injective and M is generalized M_i-injective ($j = 1, \cdots, k-1$) by Theorem 2.1. By assumption, for any $X \subseteq P$, there exist $X^* \subseteq P$, $M_j' <_{\oplus} M_j$ ($j = 1, \cdots, k-1$) and $M' <_{\oplus} M$ such that $X \subseteq_e X^*$ and $P = X^* \oplus M_1' \oplus \cdots \oplus M_{k-1}' \oplus M'$. Then

$$M = M' \oplus (X^* \oplus M_1' \oplus \cdots \oplus M_{k-1}') \cap M$$

By Theorem 2.1, M is CS for $M = M_k \oplus M_{k-1}$. So, for $(X^* \oplus M_1' \oplus \cdots \oplus M_{k-1}') \cap M <_{\oplus} M$, there exist $M_k' <_{\oplus} M_k$ and $M_{k+1}' <_{\oplus} M_{k+1}$ such that $M = M_k' \oplus M_{k+1}' \oplus (X^* \oplus M_1' \oplus \cdots \oplus M_{k-1}') \cap M$. Let p_1 and p_2 be the projections : $M \to M_k' \oplus M_{k+1}'$, $M \to (X^* \oplus M_1' \oplus \cdots \oplus M_{k-1}') \cap M$, respectively. Since $M' \cap [(X^* \oplus M_1' \oplus \cdots \oplus M_{k-1}') \cap M] = 0$, the natural map $\varphi : p_1(M') \to p_2(M')$ is a homomorphism. Then

$$\begin{aligned} M' &= \langle p_1(M') \xrightarrow{\varphi} p_2(M') \rangle \\ &= \langle M_k' \oplus M_{k+1}' \xrightarrow{\varphi} X^* \oplus M_1' \oplus \cdots \oplus M_{k-1}' \rangle \end{aligned}$$

Thus we get

$$\begin{aligned} P &= X^* \oplus M_1' \oplus \cdots \oplus M_{k-1}' \oplus M' \\ &= X^* \oplus M_1' \oplus \cdots \oplus M_{k-1}' \oplus \langle M_k' \oplus M_{k+1}' \xrightarrow{\varphi} X^* \oplus M_1' \oplus \cdots \oplus M_{k-1}' \rangle \\ &= X^* \oplus M_1' \oplus \cdots \oplus M_{k-1}' \oplus M_k' \oplus M_{k+1}' \end{aligned}$$

Therefore P is CS for $P = M_1 \oplus \cdots \oplus M_k \oplus M_{k+1}$. □

Remark *In* [2], *we announced without a proof that if $M_1, \cdots, M_n$ are CS-modules and M_i is generalized M_j-injective for $i \neq j$ then $P = M_1 \oplus \cdots \oplus M_n$ is CS for $P = M_1 \oplus \cdots \oplus M_n$. However, we must correct this statement in the present form* (2) *of theorem above.*

Lemma 2.11. *Let P be an R-module with a decomposition $P = M_1 \oplus M_2 \oplus M_3$ where M_3 is a CS-module and $P' = M_1 \oplus M_2$ is CS for $P' = M_1 \oplus M_2$, and let M_i be essentially M_j-injective ($i \neq j$). For a submodule $X \subseteq P$ with $X \supseteq_e X_1 \oplus X_2 \oplus X_3$ and decompositions $M_i = T_i \oplus N_i$ with $T_i \supseteq_e X_i$ ($i = 1, 2, 3$), there exist $X^* \subseteq P$ and $M_i' <_{\oplus} M_i$ ($i = 1, 2, 3$) such that $X \subseteq_e X^*$ and $P = X^* \oplus (M_1' \oplus M_2' \oplus M_3')$.*

Proof. Put $Y_2 = (T_1 \oplus T_2 \oplus N_1 \oplus N_2 \oplus N_3) \cap X$. Since $X \supseteq_e X_1 \oplus X_2 \oplus X_3$, the natural map $f : \pi_{T_1 \oplus T_2}(Y_2) \to \pi_{N_1 \oplus N_2}(Y_2)$ is a homomorphism. Since $T_1 \oplus T_2 \supseteq_e X_1 \oplus X_2$ and $f(X_1 \oplus X_2) = 0$, we get

$$X_1 \oplus X_2 \subseteq_e \langle \pi_{T_1 \oplus T_2}(Y_2) \xrightarrow{f} \pi_{N_1 \oplus N_2}(Y_2) \rangle = \pi_{M_1 \oplus M_2}(Y_2)$$

by Lemma 2.2. Inasmuch as P' is a CS for $P' = M_1 \oplus M_2$, there exist $Y_2^* \subseteq P'$ and $M_i' <_{\oplus} M_i$ ($i = 1, 2$) such that $\pi_{M_1 \oplus M_2}(Y_2) \subseteq_e Y_2^*$ and $P' = Y_2^* \oplus M_1' \oplus M_2'$. So we see

$$\begin{array}{lll} P = & Y_2^* & \oplus T_3 \oplus M_1' \oplus M_2' \oplus N_3 \\ \cup\!\shortmid & \cup\!\shortmid^e & \cup\!\shortmid^e \\ X \supseteq_e X_1 \oplus X_2 \oplus X_3 & & \end{array}$$

Since $\pi_{M_1 \oplus M_2}(Y_2) \subseteq_e Y_2^*$, the natural map $\alpha : \pi_{T_3}(X) \to \pi_{M_1' \oplus M_2'}(X)$ is a homomorphism. We see $Ker\alpha \subseteq_e \pi_{T_3}(X)$ since $X_3 \subseteq Ker\alpha \subseteq \pi_{T_3}(X) \subseteq T_3$ and $X_3 \subseteq_e T_3$. By Proposition 1.1, $M_1' \oplus M_2'$ is essentially T_3-injective. So there exists a homomorphism $\overline{\alpha} : T_3 \to M_1' \oplus M_2'$ such that $\overline{\alpha}|_{\pi_{T_3}(X)} = \alpha$. Since $X_3 \subseteq_e T_3$ and $\overline{\alpha}(X_3) = 0$, we see $X_3 \subseteq_e \langle T_3 \xrightarrow{\overline{\alpha}} M_1' \oplus M_2' \rangle$ by Lemma 2.2. Thus we see

$$\begin{array}{lll} P = & Y_2^* \oplus \langle T_3 \xrightarrow{\overline{\alpha}} M_1' \oplus M_2' \rangle \oplus M_1' \oplus M_2' \oplus N_3 & \\ \cup\!\shortmid & \cup\!\shortmid^e \qquad\qquad \cup\!\shortmid^e & \\ X \supseteq_e X_1 \oplus X_2 \quad \oplus \quad X_3 & & \end{array}$$

Define $\beta : M_3 = T_3 \oplus N_3 \to M_1' \oplus M_2'$ by $\beta(t_3 + n_3) = \overline{\alpha}(t_3)$. Then

$$\langle T_3 \xrightarrow{\overline{\alpha}} M_1' \oplus M_2' \rangle \oplus N_3 \;=\; \langle M_3 \xrightarrow{\beta} M_1' \oplus M_2' \rangle$$

Put $Y_3 = \langle M_3 \xrightarrow{\beta} M_1' \oplus M_2' \rangle \cap X$. Since $\langle M_3 \xrightarrow{\beta} M_1' \oplus M_2' \rangle$ is a CS-module, there exists a decomposition $\langle M_3 \xrightarrow{\beta} M_1' \oplus M_2' \rangle = Z_3' \oplus Z_3''$ with $Y_3 \subseteq_e Z_3''$. By Lemma 2.4, there exists a decomposition $M_3 = M_3' \oplus M_3''$ such that $Z_3' = \langle M_3' \to M_1' \oplus M_2' \rangle$ and $Z_3'' = \langle M_3'' \to M_1' \oplus M_2' \rangle$. Since $X_3 \subseteq_e Y_3$, we see

$$\begin{array}{lll} P = & Y_2^* & \oplus Z_3'' \oplus Z_3' \oplus M_1' \oplus M_2' \\ \cup\!\shortmid & \cup\!\shortmid^e & \cup\!\shortmid^e \\ X \supseteq_e X_1 \oplus X_2 \oplus X_3 & & \end{array}$$

Since $X \subseteq Y_2^* \oplus \langle M_3 \to M_1' \oplus M_2' \rangle = Y_2^* \oplus Z_3' \oplus Z_3''$, $Y_3 \subseteq_e Z_3''$ and $X_1 \oplus X_2 \subseteq_e Y_2^*$, the natural map $\gamma : \pi_{Y_2^*}(X) \to \pi_{Z_3'}(X)$ is a homomorphism with $Ker\gamma \subseteq_e \pi_{Y_2^*}(X)$. Since Z_3' is essentially Y_2^*-injective, there exists a homomorphism $\overline{\gamma} : Y_2^* \to Z_3'$ with $\overline{\gamma}|_{\pi_{Y_2^*}(X)} = \gamma$. By Lemma 2.2, we have $X_1 \oplus X_2 \subseteq_e \langle Y_2^* \xrightarrow{\overline{\gamma}} Z_3' \rangle$. So we see

$$\begin{array}{lll} P = \langle Y_2^* \xrightarrow{\overline{\gamma}} Z_3' \rangle \oplus Z_3'' \oplus M_1' \oplus M_2' \oplus Z_3' & & \\ \cup\!\shortmid \qquad\quad \cup\!\shortmid^e \qquad\quad \cup\!\shortmid^e & & \\ X \supseteq_e \; X_1 \oplus X_2 \,\oplus\, X_3 & & \end{array}$$

Put $X^* = \langle Y_2^* \xrightarrow{\overline{\gamma}} Z_3' \rangle \oplus Z_3''$, then $X \subseteq_e \langle Y_2^* \xrightarrow{\overline{\gamma}} Z_3' \rangle \oplus Z_3'' \;=\; X^*$. Therefore we see

$$\begin{aligned} P &= X^* \oplus M_1' \oplus M_2' \oplus Z_3' \\ &= X^* \oplus M_1' \oplus M_2' \oplus \langle M_3' \to M_1' \oplus M_2' \rangle \\ &= X^* \oplus M_1' \oplus M_2' \oplus M_3' \end{aligned}$$

□

Lemma 2.12. *Let T be a quasi-continuous module and let N_1 and N_2 be generalized T-injective modules. Let $A_1 \oplus A_2$ be an essential submodule of T*

and let B_i be an essential submodule of N_i $(i=1,2)$ such that $A_1 \oplus A_2 \overset{\alpha}{\simeq} B_1 \oplus B_2$ and $\alpha(A_i) = B_i$ $(i=1,2)$. Then there exist decompositions $T = \overline{T} \oplus \overline{\overline{T}}$ and $N_i = \overline{N_i} \oplus \overline{\overline{N_i}}$ $(i=1,2)$ such that $\langle A_1 \oplus A_2 \overset{\alpha}{\simeq} B_1 \oplus B_2 \rangle \subseteq_e \langle \overline{N_1} \hookrightarrow \overline{\overline{N_2}} \oplus \overline{\overline{T}} \rangle \oplus \langle \overline{N_2} \hookrightarrow \overline{\overline{T}} \oplus \overline{\overline{N_1}} \rangle \oplus \langle \overline{T} \hookrightarrow \overline{\overline{N_1}} \oplus \overline{\overline{N_2}} \rangle$.

Proof. As T is a quasi-continuous module, there exist decomposition $T = T_1 \oplus T_2$ such that $A_i \subseteq_e T_i$ $(i=1,2)$. By Proposition 1.5, N_i is generalized T_i-injective $(i=1,2)$. So, for $\alpha|_{A_i} : A_i \to B_i$, there exist decompositions $T_i = \overline{T_i} \oplus \overline{\overline{T_i}}$, $N_i = \overline{N_i} \oplus \overline{\overline{N_i}}$, a homomorphism $\overline{\alpha_i} : \overline{T_i} \to \overline{\overline{N_i}}$ and a monomorphism $\beta_i : \overline{N_i} \hookrightarrow \overline{\overline{T_i}}$ $(i=1,2)$ satisfying, for any $x \in A_i$, x can be expressed as $x = \overline{x} + \overline{\overline{x}}$ with $\overline{x} \in \overline{T_i}$ and $\overline{\overline{x}} \in Im\beta_i$, and $\alpha(x) = \overline{\alpha_i}(\overline{x}) + \beta_i^{-1}(\overline{\overline{x}})$. Since $A_i \subseteq_e T_i$, we get

$$\langle A_i \overset{\alpha}{\to} B_i \rangle \subseteq_e \langle \overline{T_i} \overset{\overline{\alpha_i}}{\to} \overline{\overline{N_i}} \rangle \oplus \langle \overline{N_i} \overset{\beta_i}{\hookrightarrow} \overline{\overline{T_i}} \rangle$$

Since $A_i \subseteq_e T_i$ and α is an isomorphism, $\overline{\alpha_i}$ is a monomorphism. Define $\overline{\alpha} : \overline{T_1} \oplus \overline{T_2} \to \overline{\overline{N_1}} \oplus \overline{\overline{N_2}}$ by $\overline{\alpha}(\overline{t_1} + \overline{t_2}) = \overline{\alpha_1}(\overline{t_1}) + \overline{\alpha_2}(\overline{t_2})$. Then $\overline{\alpha}$ is a monomorphism. Thus we get

$$\begin{aligned} \langle A_1 \oplus A_2 \overset{\alpha}{\to} B_1 \oplus B_2 \rangle &= \langle A_1 \overset{\alpha|_{A_1}}{\to} B_1 \rangle \oplus \langle A_2 \overset{\alpha|_{A_2}}{\to} B_2 \rangle \\ &\subseteq_e \langle \overline{T_1} \overset{\overline{\alpha_1}}{\hookrightarrow} \overline{\overline{N_1}} \rangle \oplus \langle \overline{N_1} \overset{\beta_1}{\hookrightarrow} \overline{\overline{T_1}} \rangle \oplus \langle \overline{T_2} \overset{\overline{\alpha_2}}{\hookrightarrow} \overline{\overline{N_2}} \rangle \oplus \langle \overline{N_2} \overset{\beta_2}{\hookrightarrow} \overline{\overline{T_2}} \rangle \\ &= \langle \overline{T_1} \oplus \overline{T_2} \overset{\overline{\alpha}}{\hookrightarrow} \overline{\overline{N_1}} \oplus \overline{\overline{N_2}} \rangle \oplus \langle \overline{N_1} \overset{\beta_1}{\hookrightarrow} \overline{\overline{T_1}} \oplus \overline{\overline{T_2}} \oplus \overline{\overline{N_2}} \rangle \oplus \langle \overline{N_2} \overset{\beta_2}{\hookrightarrow} \overline{\overline{T_1}} \oplus \overline{\overline{T_2}} \oplus \overline{\overline{N_1}} \rangle \end{aligned}$$

□

Remark *For Lemma 2.12, we do not know whether the statement holds if we change the assumption "Let T be a quasi-continuous module and let N_1 and N_2 be generalized T-injective modules" by "Let T, N_1, N_2 be mutually generalized injective modules". If we can change the statement in this form, (2) of Theorem 2.15 can be changed by the following*

(2′) M_i is generalized M_j-injective for $i \neq j$.

Theorem 2.13. *Let M_1 and M_2 be CS-modules and M_3 a quasi-continuous module. Put $P = M_1 \oplus M_2 \oplus M_3$. Then P is CS for $P = M_1 \oplus M_2 \oplus M_3$ if and only if M_i is mutually generalized injective for $i \neq j$.*

Proof. " Only if " holds by Proposition 2.8 and Theorem 2.1.

" If part " : Let $X \subseteq P$ and put $X_i = M_i \cap X$. Then there exist decompositions $M_i = T_i \oplus N_i$ with $X_i \subseteq_e T_i$. Put $Y_2 = (N_1 \oplus N_2) \cap X$. By the same argument as in the proof of Theorem 2.1, we see

$$\begin{array}{cccccccc} P = & T_1 \oplus & T_2 \oplus & T_3 \oplus & Y_2^* \oplus & N_1 \oplus N_2 \oplus N_3 \\ \cup\!\!| & \cup\!\!|^e & \cup\!\!|^e & \cup\!\!|^e & \cup\!\!|^e & \\ X \supseteq & X_1 \oplus & X_2 \oplus & X_3 \oplus & Y_2 & \end{array}$$

Then P is essentially Y_2^*-injective. Now we put $Y_3 = (N_1 \oplus N_2 \oplus N_3) \cap X$. Since N_3 is a CS-module, for $\pi_{N_3}(Y_3) \subseteq N_3$, there exists a decomposition

$N_3 = \overline{N_3} \oplus \overline{\overline{N_3}}$ with $\pi_{N_3}(Y_3) \subseteq_e \overline{N_3}$. By Theorem 2.1 and Proposition 2.8, $N_1 \oplus N_2$ is CS for $N_1 \oplus N_2$. Thus, for $\pi_{N_1 \oplus N_2}(Y_3) \subseteq N_1 \oplus N_2$, there exist $X^* \subseteq N_1 \oplus N_2$ and decompositions $N_i = \overline{N_i} \oplus \overline{\overline{N_i}}$ $(i = 1, 2)$ such that $\pi_{N_1 \oplus N_2}(Y_3) \subseteq_e X^*$ and $N_1 \oplus N_2 = X^* \oplus \overline{\overline{N_1}} \oplus \overline{\overline{N_2}}$. By Lemma 2.3, we get

$$X^* = \langle \overline{N_1} \to \overline{\overline{N_1}} \oplus \overline{\overline{N_2}} \rangle \oplus \langle \overline{N_2} \to \overline{\overline{N_1}} \oplus \overline{\overline{N_2}} \rangle$$

Since $Y_2 \cap (N_1 \oplus N_2) = 0$ and $X_3 \cap N_3 = 0$, the natural map α : $\pi_{N_3}(Y_3) \to \pi_{N_1 \oplus N_2}(Y_3)$ is an isomorphim. Put $B_i = \langle \overline{N_i} \to \overline{\overline{N_1}} \oplus \overline{\overline{N_2}} \rangle \cap \pi_{N_1 \oplus N_2}(Y_3)$ and $A_i = \alpha^{-1}(B_i)$ $(i = 1, 2)$. Then we get

$$\begin{array}{ccccc} \overline{N_3} & & \langle \overline{N_1} \to \overline{\overline{N_1}} \oplus \overline{\overline{N_2}} \rangle & \oplus & \langle \overline{N_2} \to \overline{\overline{N_1}} \oplus \overline{\overline{N_2}} \rangle \\ \cup\!|^e & & \cup\!|^e & & \cup\!|^e \\ A_1 \oplus A_2 & \overset{\alpha}{\simeq} & B_1 & \oplus & B_2 \end{array}$$

By Lemma 2.12 and Lemma 2.4, there exist decompositions $\overline{N_i} = \overline{N_i^*} \oplus \overline{N_i^{**}}$ $(i = 1, 2, 3)$ such that $\langle A_1 \oplus A_2 \xrightarrow{\alpha} B_1 \oplus B_2 \rangle \subseteq_e Y_3^*$,
where $Y_3^* = \langle \langle \overline{N_1^*} \to \overline{\overline{N_1}} \oplus \overline{\overline{N_2}} \rangle \hookrightarrow \langle \overline{N_2^{**}} \to \overline{\overline{N_1}} \oplus \overline{\overline{N_2}} \rangle \oplus \overline{N_3^{**}} \rangle$
$\oplus \langle \langle \overline{N_2^*} \to \overline{\overline{N_1}} \oplus \overline{\overline{N_2}} \rangle \hookrightarrow \langle \overline{N_1^{**}} \to \overline{\overline{N_1}} \oplus \overline{\overline{N_2}} \rangle \oplus \overline{N_3^{**}} \rangle$
$\oplus \langle \overline{N_3^*} \hookrightarrow . \langle \overline{N_1^{**}} \to \overline{\overline{N_1}} \oplus \overline{\overline{N_2}} \rangle \oplus \langle \overline{N_2^*} \to \overline{\overline{N_1}} \oplus \overline{\overline{N_2}} \rangle \rangle$.

Then P is essentially Y_3^*-injective. Put $Y_3' = \langle A_1 \oplus A_2 \xrightarrow{\alpha} B_1 \oplus B_2 \rangle$. Since $Y_3 = \langle \pi_{N_3}(Y_3) \xrightarrow{\alpha} \pi_{N_1 \oplus N_2}(Y_3) \rangle$ and $A_1 \oplus A_2 \subseteq_e \pi_{N_3}(Y_3)$, we have $Y_3' = \langle A_1 \oplus A_2 \xrightarrow{\alpha} B_1 \oplus B_2 \rangle \subseteq_e Y_3$ by Lemma 2.2. Then

$$\begin{aligned} N_1 \oplus N_2 \oplus N_3 &= X^* \oplus \overline{N_3} \oplus \overline{\overline{N_1}} \oplus \overline{\overline{N_2}} \oplus \overline{\overline{N_3}} \\ &= Y_3^* \oplus \langle \overline{N_1^{**}} \to \overline{\overline{N_1}} \oplus \overline{\overline{N_2}} \rangle \oplus \langle \overline{N_2^{**}} \to \overline{\overline{N_1}} \oplus \overline{\overline{N_2}} \rangle \oplus \overline{N_3^{**}} \\ &\quad \oplus \overline{\overline{N_1}} \oplus \overline{\overline{N_2}} \oplus \overline{\overline{N_3}} \\ &= Y_3^* \oplus \overline{N_1^{**}} \oplus \overline{N_2^{**}} \oplus \overline{N_3^{**}} \oplus \overline{\overline{N_1}} \oplus \overline{\overline{N_2}} \oplus \overline{\overline{N_3}} \end{aligned}$$

In the next step, we newly put $N_i = \overline{N_i^{**}} \oplus \overline{\overline{N_i}}$. By Lemma 2.5, we see

$$\begin{array}{l} P = T_1 \oplus T_2 \oplus T_3 \oplus Y_2^* \oplus Y_3^* \oplus N_1 \oplus N_2 \oplus N_3 \\ \cup\!| \quad \cup\!|^e \quad \cup\!|^e \quad \cup\!|^e \quad \cup\!|^e \quad \cup\!|^e \\ X \supseteq_e X_1 \oplus X_2 \oplus X_3 \oplus Y_2 \oplus Y_3' \end{array}$$

Put $P' = T_1 \oplus T_2 \oplus T_3 \oplus N_1 \oplus N_2 \oplus N_3$, $M_i' = T_i \oplus N_i$ and $X' = P' \cap X$. Inasmuch as $M_1 \oplus M_2$ is CS for $M_1 \oplus M_2$, $M_1' \oplus M_2'$ is CS for $M_1' \oplus M_2'$ by Proposition 2.8. Hence, by Lemma 2.11, there exist $Z \subseteq P'$ and $\overline{M_i'} <_\oplus M_i'$ $(i = 1, 2, 3)$ such that $P' = Z \oplus \overline{M_1'} \oplus \overline{M_2'} \oplus \overline{M_3'}$ and $X' \subseteq_e Z$. So we see

$$\begin{array}{l} P = Z \oplus Y_2^* \oplus Y_3^* \oplus \overline{M_1'} \oplus \overline{M_2'} \oplus \overline{M_3'} \\ \cup\!| \quad \cup\!|^e \quad \cup\!|^e \quad \cup\!|^e \\ X \supseteq_e X' \oplus Y_2 \oplus Y_3' \end{array}$$

Since $X' \cap (\overline{M_1'} \oplus \overline{M_2'} \oplus \overline{M_3'}) = 0$, the natural map $\beta : \pi_{Y_2^* \oplus Y_3^*}(X) \to \pi_{\overline{M_1'} \oplus \overline{M_2'} \oplus \overline{M_3'}}(X)$ is a homomorphism. As $Y_2 \oplus Y_3' \subseteq Ker\beta \subseteq \pi_{Y_2^* \oplus Y_3^*}(X) \subseteq Y_2^* \oplus Y_3^*$ and $Y_2 \oplus Y_3' \subseteq_e Y_2^* \oplus Y_3^*$, $Ker\beta \subseteq_e \pi_{Y_2^* \oplus Y_3^*}(X)$. Since P is essentially Y_2^*-injective and essentially Y_3^*-injective, P is essentially $Y_2^* \oplus Y_3^*$-injective. Hence $\overline{M_1'} \oplus \overline{M_2'} \oplus \overline{M_3'}$ is essentially $Y_2^* \oplus Y_3^*$-injective. Thus there exists a homomorphism $\overline{\beta} : Y_2^* \oplus Y_3^* \to \overline{M_1'} \oplus \overline{M_2'} \oplus \overline{M_3'}$ with $\overline{\beta}|_{\pi_{Y_2^* \oplus Y_3^*}(X)} = \beta$. Since $Y_2 \oplus Y_3' \subseteq_e Y_2^* \oplus Y_3^*$ and $\overline{\beta}(Y_2 \oplus Y_3') = 0$, by Lemma 2.2,

$$Y_2 \oplus Y_3' \subseteq_e \langle Y_2^* \oplus Y_3^* \xrightarrow{\overline{\beta}} \overline{M_1'} \oplus \overline{M_2'} \oplus \overline{M_3'} \rangle$$

Thus we get

$$X \subseteq_e \langle Y_2^* \oplus Y_3^* \xrightarrow{\overline{\beta}} \overline{M_1'} \oplus \overline{M_2'} \oplus \overline{M_3'} \rangle \oplus Z$$

Now we obtain a decomposition

$$P = \langle Y_2^* \oplus Y_3^* \xrightarrow{\overline{\beta}} \overline{M_1'} \oplus \overline{M_2'} \oplus \overline{M_3'} \rangle \oplus Z \oplus \overline{M_1'} \oplus \overline{M_2'} \oplus \overline{M_3'}$$

Therefore P is CS for $P = M_1 \oplus M_2 \oplus M_3$. □

By the proof of theorem above, we can obtain the following:

Proposition 2.14. *Let M_1, M_2 and M_3 be CS-modules and put $P = M_1 \oplus M_2 \oplus M_3$. Then the following conditions are equivelent:*

(1) *P is CS for $P = M_1 \oplus M_2 \oplus M_3$*

(2) *M_i is generalized $M_j \oplus M_k$-injective for $\{i, j, k\} = \{1, 2, 3\}$*

(3) *$M_i \oplus M_j$ is generalized M_k-injective for $\{i, j, k\} = \{1, 2, 3\}$*

Theorem 2.15. *Let $M_1, \cdots, M_n$ be CS-modules ($n \geq 3$) and let $P = M_1 \oplus \cdots \oplus M_n$. Then the following conditions are equivalent:*

(1) *P is CS for $P = M_1 \oplus \cdots \oplus M_n$*

(2) *M_i is generalized $M_j \oplus M_k$-injective for distinct $i, j, k \in \{1, 2, \cdots, n\}$*

(3) *$M_i \oplus M_j$ is generalized M_k-injective for distinct $i, j, k \in \{1, 2, \cdots, n\}$*

Lemma 2.16. *Let $M_1, \cdots, M_n$ be finite normal CS-modules and let $P = M_1 \oplus \cdots \oplus M_n$. If P is CS for $P = M_1 \oplus \cdots \oplus M_n$, then for any decomposition $M_i = M_i' \oplus M_i''$ ($i = 1, 2, \cdots, n$), P is CS for $P = M_1' \oplus \cdots \oplus M_n' \oplus M_1'' \oplus \cdots \oplus M_n''$.*

Proof. When $n = 1$, this statement holds, since M is a finite normal CS-module. In the case $n \geq 2$, since each M_i is a finite normal CS-module, M_i' and M_i'' are mutually generalized injective. Hence, by Theorem 2.10, it is enough to show that, for distinct $A, B, C \in \{M_1', \cdots, M_n', M_1'', \cdots, M_n''\}$, $M = A \oplus B \oplus C$ is CS for $M = A \oplus B \oplus C$.

By Proposition 2.8, if $A \subsetneq M_i$, $B \subsetneq M_j$, $C \subsetneq M_k$ for distinct i, j, k, then M is CS for $M = A \oplus B \oplus C$.

Let $A, B \subseteq M_i$ (that is $M_i = A \oplus B$), $C \subseteq M_j$ ($i \neq j$). By Theorem 2.1 and Proposition 2.8, M is CS for $M = (A \oplus B) \oplus C$. So, for a submodule X of M, there exist $X^* \subseteq M$, $\overline{A \oplus B} \subseteq A \oplus B$ and $C' \subseteq C$ such that $X \subseteq_e X^*$ and $M = X^* \oplus \overline{A \oplus B} \oplus C'$. Then

$$A \oplus B = \overline{A \oplus B} \oplus [(X^* \oplus C') \cap (A \oplus B)]$$

Since $M_i = A \oplus B$ is a finite normal CS-module, for $(X^* \oplus C') \cap (A \oplus B)$, there exist $A' <_\oplus A$ and $B' <_\oplus B$ such that $A \oplus B = [(X^* \oplus C') \cap (A \oplus B)] \oplus A' \oplus B'$. Let p_1 and p_2 be the projections : $A \oplus B \to A' \oplus B'$, $A \oplus B \to [(X^* \oplus C') \cap (A \oplus B)]$ respectively. Then the natural map $\varphi : p_1(\overline{A \oplus B}) \to p_2(\overline{A \oplus B})$ is a homomorphism since $(\overline{A \oplus B}) \cap [(X^* \oplus C') \cap (A \oplus B)] = 0$. Then

$$\overline{A \oplus B} = \langle p_1(\overline{A \oplus B}) \xrightarrow{\varphi} p_2(\overline{A \oplus B})\rangle = \langle A' \oplus B' \xrightarrow{\varphi} X^* \oplus C' \rangle$$

Thus we get

$$\begin{aligned} M &= X^* \oplus \overline{A \oplus B} \oplus C' \\ &= X^* \oplus \langle A' \oplus B' \xrightarrow{\varphi} X^* \oplus C' \rangle \oplus C' \\ &= X^* \oplus A' \oplus B' \oplus C' \end{aligned}$$

Therefore M is CS for $M = A \oplus B \oplus C$. □

Lemma 2.17. *Let M_1 and M_2 be CS-modules, $P = M_1 \oplus M_2$ and let $\varphi : M_1 \to M_2$ be a homomorphism. If P is CS for $P = M_1 \oplus M_2$, then P is CS for $P = \langle M_1 \xrightarrow{\varphi} M_2 \rangle \oplus M_2$.*

Proof. This is clear since $M_1 \simeq \langle M_1 \to M_2 \rangle$. □

Theorem 2.18. *Let M_1 and M_2 be finite normal CS-modules and put $P = M_1 \oplus M_2$. Then the following conditions are equivalent:*

(1) *P is a finite normal CS module*

(2) *P is CS for $P = M_1 \oplus M_2$*

(3) *M_i is generalized M_j-injective ($i \neq j$).*

Proof. It is enough to prove (2) ⇒ (1). Let $P = T_1 \oplus \cdots \oplus T_m$. First we consider the case $P = T_1 \oplus T_2$. By (2), we have a decomposition $P = M_1' \oplus M_2' \oplus T_2$ with some $M_i' <_\oplus M_i$, put $M_i = M_i' \oplus M_i''$. For $P = M_1' \oplus M_2' \oplus M_1'' \oplus M_2''$, let p_1 and p_2 be the projection : $P \to M_1' \oplus M_2'$, $P \to M_1'' \oplus M_2''$, respectively. Then the natural map $\alpha : p_2(T_2) \to p_1(T_2)$ is a homomorphism since $(M_1' \oplus M_2') \cap T_2 = 0$. So we get

$$T_2 = \langle p_2(T_2) \xrightarrow{\alpha} p_1(T_2) \rangle = \langle M_1'' \oplus M_2'' \xrightarrow{\alpha} M_1' \oplus M_2' \rangle$$

By Lemma 2.16 and Lemma 2.17, $P = M_1' \oplus M_2' \oplus \langle M_1'' \oplus M_2'' \xrightarrow{\alpha} M_1' \oplus M_2' \rangle = M_1' \oplus M_2' \oplus T_2$ is CS for $P = M_1' \oplus M_2' \oplus T_2$. As $T_1 \cap T_2 = 0$, the natural map $\beta : p_1(T_1) \to \pi_{T_2}(T_1)$ is a homomorphism. So we get

$$T_1 = \langle p_1(T_1) \xrightarrow{\beta} \pi_{T_2}(T_1) \rangle = \langle M_1' \oplus M_2' \xrightarrow{\beta} T_2 \rangle$$

Thus, by Lemma 2.17, P is CS for $P = T_1 \oplus T_2$.

Next we consider the case $m \geq 3$. Let $T_i, T_j, T_k \in \{T_1, \cdots, T_m\}$. Since $P = T_i \oplus (\oplus_{l \neq i} T_l)$ is CS for $P = T_i \oplus (\oplus_{l \neq i} T_l)$, T_i is generalized $T_j \oplus T_k$-injective by Theorem 2.1 and Proposition 1.5. Thus, by Theorem 2.15, P is CS for $P = T_1 \oplus \cdots \oplus T_m$.

Therefore P is a finite normal CS-module. □

By a quite similar proof, we can show the following :

Theorem 2.19. *Let $M_1, \cdots, M_n$ be finite normal CS-modules ($n \geq 3$) and let $P = M_1 \oplus \cdots \oplus M_n$. Then the following conditions are equivalent:*

(1) *P is a finite normal CS-module*

(2) *P is CS for $P = M_1 \oplus \cdots \oplus M_n$*

(3) *M_i is generalized $M_j \oplus M_k$-injective for distinct $i, j, k \in \{1, 2, \cdots, n\}$*

(4) *$M_i \oplus M_j$ is generalized M_k-injective for distinct $i, j, k \in \{1, 2, \cdots, n\}$*

Theorem 2.20. *Let $M_1, \cdots, M_n$ be quasi-continuous modules and put $P = M_1 \oplus \cdots \oplus M_n$. Then P is CS for $P = M_1 \oplus \cdots \oplus M_n$ if and only if M_i is generalized M_j-injective for $i \neq j$.*

Proof. This follows from Theorem 2.10 and Theorem 2.13. □

Corollary 2.21. *Let $M_1, \cdots, M_n$ be uniform modules and let $P = M_1 \oplus \cdots \oplus M_n$. Then P is CS for $P = M_1 \oplus \cdots \oplus M_n$ if and only if M_i is generalized M_j-injective ($i \neq j$).*

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Certain Seminormal Rings and Certain Seminormal Semigroups

MITSUO KANEMITSU [1]

Let S be an additive semigroup such that torsion-free, cancellative with identity 0 and $S \neq \{0\}$. A prime ideal of S is called divided if P is comparable to every principal ideal of S. If every prime ideal of S is divided, then S is called a divided semigroup.

We study some properties of valuation semigroups, pseudo-valuation semigroups, seminormal semigroups and divided semigroups. And we give some examples about these semigroups.

Moreover, we show that if S an atomic semigroup, then S is divided if and only if dim $S = 1$. Also, we show that a finitely generated prime ideal of S such that dim $S = 1$, then it is maximal.

ここでは，付値半群はPVS半群であること，またPVS半群は半正規半群でもありまたdivided 半群であることを述べ，これらの半群の例やある条件の下でdivided 半群となる条件や，divided素イデアルが極大イデアルとなる十分条件について述べる．

特に断らない限り半群はすべて加法で記されていて単位元0を持ち，消去的でtorsion-freeである0以外の元を持つ可換半群とする．Sを半群，$G = q(S) = \{s - s' \mid s,\ s' \in S\}$を$S$の商群とする．$S$と$G$の中間半群を$S$の**oversemigroup**という．

IをSの空でない部分集合とする．IがSの**イデアル**とは，$I + S \subset I$，即ち，Iの任意の元aとSの任意の元sに対して$a + s \in I$のときをいう．イデアルIが$I \neq S$のとき**真のイデアル**という．

Sの元uが単元であるとは，Sのある元vが存在して$u+v = 0$のときをいう．Sの単元全体をU(S)と記す．$M = S - \mathrm{U}(S)$が空集合でないとき，MはSのイデアルとなり，$M \subset I$となる任意のイデアルIに対して，$M = I$または$I = S$が成立する．このMが空集合でないときMはSの**極大イデアル**と呼ばれる．

Sの真のイデアルPが**素イデアル**とは，Sの元a, bで$a + b \in P$なら$a \in P$または$b \in P$のときをいう．極大イデアルは素イデアルである．素イデアル全体の集合をSpec(S)と記す．真のイデアルQが**準素イデアル**とは，Sの元a, bに対して，$a + b \in Q$なら$a \in Q$または$nb \in Q$（nは正の整数）のときをいう．環のときと同様に素イデアルは準素イデアルである．SのイデアルIに対して，$\mathrm{rad}(I) = \{x \in S \mid$ ある自然数nに対して，$nx \in I\}$とおく．QがSの準素イデアルのとき，$\mathrm{rad}(Q) = P$は素イデアルとなるが，このときQは**P-準素イデアル**と呼ばれる．

唯一つしか素イデアルを持たない半群Sは次元が1であるといい，dim $S = 1$と記す．

$x \in S$に対して，$(x) = x+S = \{x+a \mid a \in S\}$は$S$のイデアルをなし，$S$の**単項イデアル**という．また$a_1,\ a_2,\ \cdots,\ a_n \in S$に対して，$I = < a_1,\ a_2,\ \cdots,\ a_n >= (a_1,\ a_2,\ \cdots,\ a_n) = \cup_{i=1}^{n} (a_i) = \cup_{i=1}^{n} (a_i + S)$とおき，このイデアルを$a_1,\ a_2,\ \cdots,\ a_n$**で生成されたイデアル**という．有限個の元で生成されるイデアルを**有限生成イデアル**という．

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[1] This is an abstract and the details will be published elsewhere.

TをSのoversemigroupとする．Tの元tがS上整とは，ある正の整数nに対して$nt \in S$のときをいう．Tのすべての元がS上整であるとき，半群TはS上整であるという．Sの商群Gの元でS上整のもの全体をSの整閉包といい，$\bar{S}$と記す．

ここで記号$\mathbf{Z_n}$をn以上の整数全体の集合とする．$\mathbf{Z}$は整数全体の集合で加法群と考える．記号などの全般的な参考文献は[1]および[5]を参照せよ．

定義 1 ([1])．半群Sの素イデアルPがdividedであるとは，$P \supset (a)$または$P \subset (a)$が任意の単項イデアル(a)に対していえるときをいう．Sがdivided半群とは各素イデアルがdividedであるときをいう．

定義 2 ([2, 3, 4, 6])．半群Sの素イデアルPが強素イデアルとは，$\forall a,\ b \in S$に対して$a+P \subset b+S$または$b+S \subset a+P$が成立するときをいう．半群Sの各素イデアルが強素イデアルのとき，Sは擬付値半群（簡単にPVS）という．特にSの商群Gの任意の元αに対して，$\alpha \in S$または$-\alpha \in S$が成立するときSは付値半群という．例としては，多項式半群$\mathbf{Z}[X] = \{a+nX \mid a \in \mathbf{Z},\ n \in \mathbf{Z_0}\}$は付値半群だが，$\mathbf{Z}+\mathbf{Z_2}X = \{a+nX \mid a \in \mathbf{Z},\ n \leq 2\}$は付値半群ではない．

定義 3 ([4, 6])．半群Sが商群$G = q(S)$を持つとき，$\alpha \in G$で$2\alpha \in S$，$3\alpha \in S$なら$\alpha \in S$が成り立つとき，Sは半正規半群（簡単に，SN半群）という．

単位元1を持つ可換環Rが半正規環（簡単にSN環）とは，Rの商体Kの元αが$2\alpha \in R$かつ$3\alpha \in R$を満たすときは$\alpha \in R$がいえるときをいう．これは正規環より少し弱い環である．

(代数的閉体上の) 代数曲線の座標環において通常特異点の局所環はSN環であることが知られている．

補題1と命題2で付値半群，PVSおよびdivided半群の間の関係を述べてみよう．

補題 1. 付値半群はPVSである．

この逆は必ずしもいえない．

例 1 (菅谷ー松田)．Fをtorsion-freeアーベル群とし，Hをその部分群とする．$\alpha_0 = 0,\ \alpha_1,\ \alpha_2,\ \cdots \in F$を，負でない任意の整数$n$に対して，

$$H + \sum_{i=0}^{n} \mathbf{Z}\alpha_i \subsetneq H + \sum_{i=0}^{n+1} \mathbf{Z}\alpha_i$$

のようにとる．$F = \cup_{i=0}^{\infty}(H + \sum_{i=0}^{n} \mathbf{Z}\alpha_i)$とおき，$V = F \cup M$を考えるとこれは$M$を極大イデアルに持つ付値半群である．$S = H \cup M$はPVSだが付値半群ではない．

命題 2. PVSならSN半群でありかつdivided半群である．

divided 半群の性質について述べよう．

命題 3. Sがdivided半群とする．このとき次のことがいえる．
(1) Spec(S)は包含関係で全順序集合である．
(2) 素イデアルPに対して，ある自然数nが存在して，nPが$P-$準素イデアルになる．

次にdivided だがSN半群でない例とdivided だがPVS でない半群の例をあげる.

例 2. $S=\{0,\ 2,\ 3,\ 4,\ \cdots\}=<2,\ 3>$なる$Z_0$の加法部分半群を考える. このとき$q(S)=Z$となる. 代数曲線$y^2=x^3$の座標環

$$R=k[X,\ Y]/(Y^2-X^3)=k[U^2,\ U^3]=k[S]$$

なる半群環はSN環ではない. 従ってSもSN半群ではない. しかし, Sはdivided半群である.

例 3 (Andersonの例). $S=<(2,\ 0),\ (1,\ 1),\ (0,\ 1)>\subset Z_0\oplus Z_0$は加法半群として, $S=\{(2a+b,\ b+c)\mid a,\ b,\ c\ \in Z_0\}$, $q(S)=Z\oplus Z$となる. $\alpha=(m,\ n)\in q(S)$, $2\alpha\in S$かつ$3\alpha\in S$とする. このとき, もし$m=2t+1$なら$m\neq 0$かつ$\alpha=(m,\ n)=t(2,\ 0)+(1,\ 1)+(n-1)(0,\ 1)\in S$となる. また$m=2t$なら$\alpha=(m,\ n)=t(2,\ 0)+n(0,\ 1)\in S$である.

よって$\alpha=(m,\ n)\in S$で, Sは半正規半群である. 代数曲面$y^2=xz^2$の座標環をRとする. kを体とすると,

$$R=k[S]=k[X,\ Y,\ Z]/(XZ^2-Y^2)=k[U^2,\ UV,\ V]$$

となる. ただし, $X,\ Y,\ Z,\ U,\ V$は不定元とする. このSはdivided半群ではない.

次にdivided半群の特徴付けを述べよう.

命題 5. 次の(1) - (4)は同値である.
(1) Sはdivided半群である.
(2) Sの真である任意のイデアル$I,\ J$に対して, $I\subset \mathrm{rad}(J)$または$I\supset \mathrm{rad}(J)$.
(3) $\forall a,\ b\in S$に対して, $(a)\subset \mathrm{rad}((b))$または$(a)\supset \mathrm{rad}((b))$.
(4) $\forall a,\ b\in S$に対して, $a\mid b$または$b\mid na$となる自然数nが存在する.
(5) $\forall a,\ b\in S$に対して, $b+S\subset na+M$または$b+S\supset na+M$なる自然数nが存在する.

これを使用すると次の例がいえる.

例 4 (divided半群だがPVSでない例).
多項式半群$S=Z[X,\ Y]=\{n+mX+tY\mid n\in Z,\ m,\ t\in Z_0\}$はdivided半群であるがPVSではない.

定理7を証明するために次の補題を述べよう.

補題 6. イデアルIをSの単項イデアルでないとする. このとき, $\forall x\in I^{-1}$に対して, $x+I\subset M$が成立する.

定理 7. Pが半群Sの単項イデアルではないようなdivided素イデアルとする. このとき, $P^{-1}=P:P$は半群である.

定義 4. Sの元xがatomとは, xは単元ではなく, またもし$x=a+b$とSの元$a,\ b$で書けるならaかbのどちらか一方は単元であるときをいう. 半群Sのすべての単元でないものが有限個のSのatomの和であるときSはatom半群という.

次元1の半群に関する次の二つの定理を述べよう.

定理 8. Sが atom 半群とする. このときSが divided であることと $\dim S = 1$ であることは同値である.

定理 9は半正規半群とその整閉包の様子を与えている.

定理 9 ([6]). Sが次元1の半群とする. このときSの oversemigroup がすべて SN である必要十分条件は, SがSN かつSの整閉包$\overline{S}$が付値半群となることである.

定理10は次の二つの補題を使うと証明できる.

補題 10. sを半群Sの元とすると, sはS上整には決してならない.

補題 11. Iを半群Sの有限生成であるイデアルとする. もし$x + I \subset I$なら, xはS上整である.

定理 12. Pが有限生成である素イデアルとする. Pが divided 素イデアルならPは極大イデアルである.

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ON MAX MODULES

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ABSTRACT. This paper is a review of max modules, i.e., modules for which every nonzero submodule has a maximal submodule, and max rings, i.e., rings over which every module is max.

1. INTRODUCTION

In what follows R denotes an associative ring with identity and R-modules will be unital (and, unless stated to the contrary, left R-modules.) Given R-modules N and M, we will write $N \leq M$ (or $M \geq N$) to denote that N is a submodule of M and $N < M$ (or $M > N$) if N is a proper submodule of M.

A submodule N of a nonzero module M is called *maximal* if it is a proper submodule and is not properly contained in any other proper submodule of M.

A Zorn's Lemma argument shows that every finitely generated nonzero module has a (not necessarily unique) maximal submodule. More generally, as noted by Rant [Ra], any nonzero module M which has a minimal generating set contains a maximal submodule. (A generating subset B of M is called *minimal* if no proper subset of B generates M.) course, if $M = R$ the maximal

Let M be a nonzero R-module. A finite chain of $n+1$ submodules of M of the form

$$M = M_0 > M_1 > \cdots > M_n = 0$$

is called a *composition series of length n* for M if each factor M_{i-1}/M_i is simple for $i = 1, 2, \ldots, n$. Note that M_{i-1}/M_i is simple if and only if M_i is a maximal submodule of M_{i-1}.

The following result is classical (see, for example, [AF, §11]).

Proposition 1.1. *The following statements are equivalent for a nonzero R-module M:*

(a) *M has a composition series;*
(b) *M is both a noetherian and an artinian module;*
(c) *M is a noetherian module and every nonzero factor module of M contains a simple submodule;*

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(d) *M is an artinian module and every nonzero submodule of M contains a maximal submodule.*

Definition 1.1. A nonzero R-module M is called *semi-artinian* or a *Loewy module* if every nonzero factor module of M contains a simple submodule.

Of course, any artinian module is semi-artinian, but not conversely. (For example, one can show that the semi-artinian $\mathbb{Z}$-modules are precisely the torsion abelian groups and so the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$ is semi-artinian but not artinian.)

Definition 1.2. A nonzero module M is called *semi-noetherian* or a *max module* if each of its nonzero submodules contains a maximal submodule. (Faith [Fa1, Fa2] also uses the term *Hamsher module.*)

Of course, any noetherian module is max, but not conversely. (For example, if R is the ring of all sequences over the field $\mathbb{Z}_2$ which are eventually constant then the R-module R is max but not noetherian. More easily, over any ring R any semisimple R-module with infinitely many summands is both max and semi-artinian but neither artinian nor noetherian.)

A rephrasing of the equivalence of (b), (c) and (d) of Proposition 1.1 says that M is both artinian and noetherian iff M is both artinian and max iff M is both noetherian and semi-artinian.

In view of Hopkins-Levitzki's famous theorem that every left artinian ring is left noetherian, a natural question arising from Proposition 1.1 is: if M is a projective artinian module is M max or, equivalently, noetherian? The following example due to Fisher [Fi] answers this in the negative.

Example. Let K be a field and let V be a countably infinite dimensional vector space over K with basis $\{v_1, v_2, \dots\}$. For each $n \in \mathbb{N}$ let V_n denote the subspace of V generated by $\{v_1, v_2, \dots, v_n\}$. Define the linear transformation T on V by $T(v_1) = 0$ and $T(v_n) = v_{n-1}$ for $n > 1$. Let S be the polynomial ring $F[T]$; S may be thought of as the subring of $\mathrm{End}_K(V)$ generated by the scalar linear transformations and all powers of T. Then, as a left S-module, the submodules of V are precisely the V_n for $n \in \mathbb{N}$. Thus V is an artinian non-max S-module.

Now if we define the ring R and the left R-module M by

$$R = \begin{bmatrix} S & V \\ 0 & K \end{bmatrix} \text{ and } M = \begin{bmatrix} 0 & V \\ 0 & K \end{bmatrix}$$

then M is a cyclic projective artinian left R-module which is non-max since its proper nonzero submodules are all of the form

$$N = \begin{bmatrix} 0 & V_n \\ 0 & 0 \end{bmatrix}$$

for some $n \in \mathbb{N}$. (Note M's similarity to the quasi-cyclic abelian group $\mathbb{Z}_{p^\infty}$. In particular, M is uniserial.)

Further to this example, we note the following result from [Fi].

Proposition 1.2. *Let M be a projective artinian R-module. Then M is noetherian if either* (a) *R is commutative or* (b) *R is left hereditary or* (c) *M is a generator for the category of left R-modules.*

More examples of cyclic artinian uniserial modules which are not noetherian are given by Hartley in [Har] and Cohn in [Coh].

Facchini in [Fac] has characterized the commutative rings R for which the classes of noetherian and artinian modules coincide as those for which every artinian submodule of the injective hull of a simple R-module is finitely generated.

2. The socle series and the radical series.

Definition 2.1. We denote the socle of the module M by $\mathrm{Soc}(M)$. (This is defined to be 0 if M has no simple submodules.)

The *second socle* of M is then defined to be the submodule $\mathrm{Soc}_2(M)$ of M containing $\mathrm{Soc}(M)$ such that $\mathrm{Soc}_2(M)/\mathrm{Soc}(M) = \mathrm{Soc}(M/\mathrm{Soc}(M))$.

Letting $\mathrm{Soc}_1(M) = \mathrm{Soc}(M)$ and proceeding in this fashion, we manufacture the *socle series* or *(lower) Loewy series* of M as the ascending chain of submodules

$$0 \leq \mathrm{Soc}_1(M) \leq \mathrm{Soc}_2(M) \leq \cdots \leq \mathrm{Soc}_\alpha(M) \leq \mathrm{Soc}_{\alpha+1}(M) \leq \cdots,$$

where, for each ordinal $\alpha > 0$,

$$\mathrm{Soc}_{\alpha+1}(M)/\mathrm{Soc}_\alpha(M) = \mathrm{Soc}(M/\mathrm{Soc}_\alpha(M)),$$

and if α is a limit ordinal then

$$\mathrm{Soc}_\alpha(M) = \bigcup_{0<\beta<\alpha} \mathrm{Soc}_\beta(M).$$

Since M is a set, at some stage the socle series of M must become stationary, i.e., there is an ordinal ρ such that $\mathrm{Soc}_\alpha(M) = \mathrm{Soc}_\rho(M)$ for all ordinals $\alpha \geq \rho$.

The first result in this section is an important characterisation of semi-artinian modules. (Recall that a submodule A of a module B is called *essential* if $A \cap C \neq 0$ for each nonzero submodule C of B.)

Proposition 2.1. *The following statements are equivalent for a module M:*

(a) *every nonzero homomorphic image of M has an essential socle;*
(b) *M is semi-artinian;*
(c) $\mathrm{Soc}_\rho(M) = M$ *for some ordinal $\rho \geq 1$;*

(d) *there exists an ascending chain of submodules*

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_\alpha \subseteq M_{\alpha+1} \subseteq \cdots \subseteq M_\tau = M$$

where each proper containment $M_\alpha \subset M_{\alpha+1}$ *implies that the factor module* $M_{\alpha+1}/M_\alpha$ *is simple.*

We now consider the dual of the above.

Definition 2.2. We denote the radical of M by $\mathrm{Rad}(M)$. Thus $\mathrm{Rad}(M)$ is defined to be the intersection of the maximal submodules of M and this is taken as M if M has no maximal submodules.

The *second radical* of M is then defined to be the submodule $\mathrm{Rad}_2(M)$ of M given by $\mathrm{Rad}_2(M) = \mathrm{Rad}(\mathrm{Rad}(M))$.

Letting $\mathrm{Rad}_1(M) = \mathrm{Rad}(M)$ and proceeding in this fashion, we manufacture the *radical series* or *(upper) Loewy series* of M as the descending chain of submodules

$$M \geq \mathrm{Rad}_1(M) \geq \mathrm{Rad}_2(M) \geq \cdots \geq \mathrm{Rad}_\alpha(M) \geq \mathrm{Rad}_{\alpha+1}(M) \geq \cdots,$$

where, for each ordinal $\alpha > 0$,

$$\mathrm{Rad}_{\alpha+1}(M) = \mathrm{Rad}(\mathrm{Rad}_\alpha(M)),$$

and if α is a limit ordinal then

$$\mathrm{Rad}_\alpha(M) = \bigcap_{0<\beta<\alpha} \mathrm{Rad}_\beta(M).$$

Since M is a set, at some stage the radical series of M must become stationary, i.e., there is an ordinal ρ such that $\mathrm{Rad}_\alpha(M) = \mathrm{Rad}_\rho(M)$ for all ordinals $\alpha \geq \rho$.

The next result is the analogue of Proposition 2.1. (Recall that a submodule A of a module B is called *small* if $A + C < B$ for every $C < B$.)

Proposition 2.2. *The following statements are equivalent for a module M:*

(a) *every nonzero submodule of M has a small radical;*

(b) *M is max;*

(c) $\mathrm{Rad}_\rho(M) = 0$ *for some ordinal* $\rho \geq 1$*;*

(d) *there exists a descending chain of submodules*

$$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_\alpha \supseteq M_{\alpha+1} \supseteq \cdots \supseteq M_\tau = 0$$

where each proper containment $M_\alpha \supset M_{\alpha+1}$ *implies that* $M_{\alpha+1}$ *is a maximal submodule of* M_α*.*

Hirano [Hi] makes the following definition, in which $J(R)$ denotes the Jacobson radical of the ring R.

Definition 2.3. $J(R)$ is said to be *T-nilpotent on the left R-module M* if for every $x \in M$ and every sequence $a_1, a_2, \ldots$ in $J(R)$ we have $a_n a_{n-1} \cdots a_1 x = 0$ for some n.

In particular, $J(R)$ is T-nilpotent on R if and only if for every sequence $a_1, a_2, \ldots$ in $J(R)$ we have $a_n a_{n-1} \cdots a_1 = 0$ for some n, i.e., (see [AF, p. 314]) $J(R)$ is *left T-nilpotent* (also known as *left vanishing*). Using condition (c) of Proposition 2.2, Hirano has shown

Proposition 2.3. *$J(R)$ is T-nilpotent on every max left R-module M.*

While investigating injectivity conditions, Clark and Smith [CS] proved that if the ring R satisfies the property

$$\mathrm{Soc}_2(E(U)) \text{ is finitely generated for each simple left } R\text{-module } U$$

then any left semi-artinian max R-module with finitely generated socle is both artinian and noetherian. (Here $E(U)$ is the injective hull of U.)

Shock [Sh] makes a detailed study of max and semi-artinian modules vis-à-vis the noetherian and artinian conditions. In particular, he proves the following two results.

Theorem 2.4. *If M is an infinitely generated module such that M/N is max for all $N < M$ then some factor module of M has an infinitely generated socle and a zero radical.*

Theorem 2.5. *The following conditions are equivalent for a module M:*

(a) *M is noetherian;*
(b) *every factor module of M is max and has a finitely generated socle;*
(c) *in every factor module of M, every submodule has small radical and finitely generated socle;*
(d) *in every factor module of M, every submodule has finitely generated radical and finitely generated socle.*

3. Max rings.

Much of the interest in max modules comes from the following classical result due to Bass [Ba].

Theorem 3.1. *The following conditions are equivalent for a ring R:*

(a) *R is left perfect, i.e., $R/J(R)$ is semisimple artinian and $J(R)$ is left T-nilpotent;*
(b) *$R/J(R)$ is semisimple artinian and every nonzero left R-module is max;*
(c) *every flat left R-module is projective;*
(d) *R satisfies the minimum condition for principal* right *ideals;*
(e) *R contains no infinite orthogonal set of idempotents and every nonzero* right *R-module is semi-artinian;*
(f) *every left R-module has a projective cover.*

Bass conjectured in [Ba] that if every nonzero left R-module is max and R contains no infinite orthogonal set of idempotents then R is left perfect. However, both Cozzens [Coz] and Koifman [Ko] produced striking counterexamples.

On the positive side, Hamsher [Ham], Koifman [Ko] and Renault [Re] each showed that the Bass conjecture is true if R is commutative. Consequently, Armendariz and Fisher [AFi] proved the conjecture true for rings satisfying a polynomial identity while Chandran [Ch] proved it true for any ring in which every left ideal is two-sided. Yu and Xue have generalised Chandran's result by establishing the conjecture in [Yu] for rings with the property that every maximal left ideal is two-sided and in [X] for rings R in which, for every $r \in R$, there is an $n(r) \in \mathbb{N}$ such that $Rr^{n(r)}$ is a two-sided ideal. Hirano [Hi] also shows the conjecture to be true for rings of (nilpotent) bounded index modulo the Jacobson radical for which primitive factor rings are artinian. Tuganbaev, in [Tu1] and [Tu2], has also verified the conjecture for rings R of bounded index for which all left primitive factor rings are left π-regular and for rings R such that any prime factor ring of R is algebraic over its centre. (A ring R is called *left π-regular* if, for each $r \in R$, there is an $n \in \mathbb{N}$ such that $Rr^n = Rr^{n+1}$. Dischinger [Di] showed that this condition is right-left symmetric and the term *strongly π-regular* is also used.)

Definition 3.1. R is called a *left max ring* if every left R-module is max.

Of course, every left perfect ring is left max. If R is left perfect but not right perfect then we see from Theorem 3.1 that R is not right max. (The ring of row-finite $\mathbb{N} \times \mathbb{N}$ matrices over a field K is such a ring, see [Ba].)

Further examples of left max rings are given by the following theorem which is attributed toVillamayor (see also [MV]).

Theorem 3.2. *The following conditions are equivalent for a ring R:*

(a) *R is a left V-ring, i.e., every simple left R-module is injective;*

(b) *every proper left ideal of R is an intersection of maximal left ideals;*

(c) $\mathrm{Rad}(M) = 0$ *for every left R-module M.*

The counterexamples of Cozzens and Koifman mentioned above are in fact right V-rings which are also simple principal right ideal domains. Camillo in [Ca] shows that every principal right ideal domain which is also a right max ring must be simple and conjectures that they are also right V-rings.

A well-known theorem due to Kaplansky states that a commutative ring is a V-ring if and only if it is (von Neumann) regular. Of course, the Cozzens and Koifman examples show that a non-commutative V-ring need not be regular. Conversely, there is an example due to Faith of a regular right V-ring which is not a left V-ring. (See [CoF, Example 5.14] and, for further aspects of this example, see [Ca]. Faith [Fa1] also notes that this example *is* left max.)

Our next theorem is a combination of results from Faith [Fa1], Hamsher [Ham], Hirano [Hi], Koifman [Ko] and Renault [Re].

Theorem 3.3. *The following conditions are equivalent for a ring R:*

(a) *R is a left max ring;*
(b) *$R/J(R)$ is a left max ring and $J(R)$ is left T-nilpotent;*
(c) *for every simple left R-module U, $E(U)$ is a max module;*
(d) *there is a cogenerator module C for the category of left R-modules which is max;*
(e) *every nonzero quasi-injective left R-module has a maximal submodule.*

For commutative rings we have the following nice characterization, due to Faith [Fa2], Hamsher [Ham], Koifman [Ko] and Renault [Re].

Theorem 3.4. *The following conditions are equivalent for a commutative ring R:*
(a) *R is a max ring;*
(b) *$R/J(R)$ is a regular ring and $J(R)$ is T-nilpotent;*
(c) *the localization $R_{\mathcal{M}}$ at any maximal ideal $\mathcal{M}$ of R is a max ring;*
(d) *the localization $R_{\mathcal{M}}$ at any maximal ideal $\mathcal{M}$ of R is a perfect ring.*

The equivalence of conditions (a) and (b) of Theorem 3.4 has been investigated further by Markov [Ma] and Tuganbaev [Tu1]. Markov shows that if R is a ring satisfying a polynomial identity then R is a left (equivalently, right) max ring if and only if $J(R)$ is left (equivalently, right) T-nilpotent and $R/J(R)$ is left π-regular while Tuganbaev shows that R is a left max ring in which all the maximal left ideals are ideals if and only if $J(R)$ is *right* T-nilpotent and $R/J(R)$ is regular.

While left perfect rings and left V-rings are left max, there are left max rings which are neither left perfect nor a left V-ring as the following example shows.

Example. Let A denote the ring of sequences over the field $\mathbb{Z}_2$. Then A is a commutative regular ring and so a V-ring, but A is not perfect. Let G be the group of order 2. Then the group ring $R = A[G]$ is neither perfect (since its factor A is not) nor a V-ring (since the order of G is not a unit in A). However, for any maximal ideal $\mathcal{M}$ of R, if $\mathcal{N} = \mathcal{M} \cap A$ then $R_{\mathcal{M}}$ is isomorphic to $A_{\mathcal{N}}[G]$. Then, since $A_{\mathcal{N}}$ is a field and G is finite, it follows that R is locally artinian and so max by Theorem 3.4.

We now note that no ring of polynomials can be a max ring. To see this let A be any ring and let R be the ring of polynomials $A[x]$. Following Proposition 2.8 of Sarath [Sa], we construct a module over R which is not max. The construction resembles that used in Fisher's example detailed earlier.

Let S be any simple A-module with generator u, let $T = S^{(\mathbb{N})}$, the direct sum of countably many copies of S, and, for each $n \in \mathbb{N}$, let u_n be the element of T whose nth coordinate is u and all other coordinates are 0. We define an R-module multiplication on T by setting $xu_1 = 0$ and $xu_n = u_{n-1}$ for $n > 1$ and extending this in the obvious way. Then the submodule structure of the R-module T is determined as for Fisher's example and T is not a max module.

Since the commutative V-rings are precisely the commutative regular rings, a natural question, asked by Faith, is if every artinian module over *every* regular ring is max.

Goodearl [Go] answered this in the negative by showing that for any ordinal α there is a prime unit-regular ring R with a faithful module whose lattice of submodules is isomorphic to $[1, \alpha]$, the well-ordered set of all ordinals β such that $1 \le \beta \le \alpha$. Moreover he showed that there is also such a ring R with a faithful cyclic module whose lattice of submodules is anti-isomorphic to $[1, \alpha]$.

In [CaF] Camillo and Fuller establish the following theorem providing more left max rings. Here a ring R is called *right semi-artinian* if the right R-module R is semi-artinian and in this case every nonzero right R-module is semi-artinian. However, they also give an example of a left and right semi-artinian ring which is not left max.

Theorem 3.5. *Let R be a ring with the ascending chain condition on (left or right) primitive ideals. If R is right semi-artinan then R is left max.*

Consequently, any right semi-artinian ring which is either semilocal or commutative is left max.

We finish this section with a look at how max modules are used to characterise the rings given in the following definition due to Camillo and Xue [CaX].

Definition 3.2. A ring R is called *left quasi-perfect* if every left artinian R-module has a projective cover.

Camillo and Xue show that the class of left quasi-perfect rings lies strictly between the classes of semiperfect rings and left perfect rings. They give the following characterization.

Theorem 3.6. *The following conditions are equivalent for a semiperfect ring R:*

(a) *R is a left quasi-perfect ring;*
(b) *every nonzero left artinian R-module is max;*
(c) *every artinian left R-module has a composition series;*
(d) *every artinian left R-module is finitely generated.*

Cai and Xue [CX] give a refinement of the equivalences in Theorem 3.6 by replacing "artinian" in (b), (c) and (d) by "strongly artinian", where a module is said to be *strongly artinian* if each of its proper submodules has a composition series.

4. Tall modules and tall rings.

The next definition comes from Sarath [Sa]:

Definition 4.1. A module M is called *tall* if there is an $N < M$ such that both N and M/N are non-noetherian.

The ring R is called a *tall ring* if every non-noetherian R-module is tall.

Sarath proves the following:

Theorem 4.1. *The following statements are equivalent for a ring R:*

(a) *every R-module with Krull dimension is noetherian;*
(b) *R is tall;*
(c) *every non-noetherian R-module has a proper non-noetherian submodule.*

The proof of Theorem 4.1 uses the following concepts.

If M is a non-noetherian module, define submodules $G(M)$ and $H(M)$ of M by

$$G(M) = \bigcap \{N \mid N \text{ a submodule of } M \text{ with } M/N \text{ noetherian}\}$$

$$H(M) = \bigcap \{N \mid N \text{ a non-noetherian submodule of } M\}.$$

If M is noetherian then $G(M)$ and $H(M)$ are both defined to be 0.

It's not too difficult to show that $H(M) \subseteq G(M) \subseteq \mathrm{Rad}(M)$ and from this and Theorem 4.1 one can see that any max ring is tall.

Question: Is every tall ring max?

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A DUALITY FOR FINITE GROUP ACTIONS ON TENSOR CATEGORIES

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1. DUALITY

A tensor category over a field k is a k-linear monoidal category. A module over a tensor category $\mathcal{C}$ is a k-linear category $\mathcal{M}$ equipped with an associative action $\mathcal{C} \times \mathcal{M} \to \mathcal{M}$. If a group G acts on a tensor category $\mathcal{C}$, we have the tensor category $\mathcal{C}^G$ of G-invariant objects in $\mathcal{C}$ and the semi-direct product tensor category $\mathcal{C}[G]$, defined analogously to an invariant ring and a skew group ring, respectively.

Assume that G is finite and $k[G]$ is semi-simple.

Theorem 1. *There exists a one-to-one correspondence between $\mathcal{C}^G$-modules having direct summands and $\mathcal{C}[G]$-modules having direct summands.*

Here a k-linear category $\mathcal{M}$ is said to have direct summands if every idempotent endomorphism in $\mathcal{M}$ splits.

The correspondence is given by assigning to a $\mathcal{C}[G]$-module $\mathcal{M}$ the $\mathcal{C}^G$-module $\mathcal{M}^G$ consisting of G-invariant objects in $\mathcal{M}$. We could say the assignment

$$\{\mathcal{C}[G]\text{-modules with direct summands}\} \to \{\mathcal{C}^G\text{-modules with direct summands}\}$$
$$\mathcal{M} \mapsto \mathcal{M}^G$$

is an equivalence of 2-categories. Details are given in [4].

Let $\mathcal{V}$ be the category of finite dimensional k-vector spaces. Let G act on $\mathcal{V}$ trivially. Then $\mathcal{V}^G$ is the category of $k[G]$-modules and $\mathcal{V}[G]$ is the category of G-graded vector spaces. The theorem actually follows from the special case for $\mathcal{C} = \mathcal{V}$ with trivial G-action. Note that a G-graded vector space is just a $k[G]^*$-module, where $k[G]^*$ is the dual of $k[G]$. This special case is generalized to the following.

Theorem 2. *Let A be a finite dimensional semisimple co-semisimple Hopf algebra. Let $B = A^*$ be the dual Hopf algebra. Let $\mathcal{A}$ be the tensor category of finite dimensional A-modules and $\mathcal{B}$ the tensor category of finite dimensional B-modules. Then there exists a one-to-one correspondence between $\mathcal{A}$-modules having direct summands and $\mathcal{B}$-modules having direct summands.*

This may be regarded as a version of the duality of cross product constructions for Hopf algebra actions on rings ([1], [2]). Details are given in [3].

In Section 2 we give the definitions of the tensor categories $\mathcal{C}^G$ and $\mathcal{C}[G]$. In Sections 3 and 4 we give two applications of Theorem 1.

The detailed version of this paper has been submitted for publication elsewhere.

2. Definition of $\mathcal{C}^G$ and $\mathcal{C}[G]$

Notations for monoidal structures of a tensor category are as follows: $(A, B) \mapsto A.B$ denotes the tensor product operation, I the unit object, $\alpha_{A,B,C}\colon (A.B).C \to A.(B.C)$ the associativity isomorphism, $\lambda_A\colon I.A \to A$ the left unit isomorphism, $\rho_A\colon A.I \to A$ the right unit isomorphism.

For a tensor category $\mathcal{C}$, a left $\mathcal{C}$-module is a k-category $\mathcal{X}$ equipped with a bilinear functor $\mathcal{C} \times \mathcal{X} \to \mathcal{X}\colon (A, X) \mapsto A.X$ and isomorphisms of associativity $\alpha_{A,B,X}\colon (A.B).X \to A.(B.X)$ and unitality $\lambda_X\colon I.X \to X$ for $A, B \in \mathcal{C}$, $X \in \mathcal{X}$ satisfying the conditions of naturality and coherence similar to the ones for a monoidal category.

An action of a group G on a k-category $\mathcal{X}$ consists of data

- functors $\sigma_*\colon \mathcal{X} \to \mathcal{X}$ for all $\sigma \in G$
- isomorphisms $\phi(\sigma, \tau)\colon (\sigma\tau)_* \to \sigma_* \circ \tau_*$ for all $\sigma, \tau \in G$
- an isomorphism $\nu\colon 1_* \to \mathrm{Id}_{\mathcal{X}}$

which make the the following diagrams commutative for all $\sigma, \tau, \rho \in G$ and $X \in \mathcal{X}$.

$$\begin{array}{ccc} (\sigma\tau\rho)_*X & \xrightarrow{\phi(\sigma\tau,\rho)_X} & (\sigma\tau)_*\rho_*X \\ {\scriptstyle \phi(\sigma,\tau\rho)_X}\downarrow & & \downarrow{\scriptstyle \phi(\sigma,\tau)_{\rho_*X}} \\ \sigma_*(\tau\rho)_*X & \xrightarrow[\sigma_*(\phi(\tau,\rho)_X)]{} & \sigma_*\tau_*\rho_*X \end{array} \tag{1}$$

$$1_*X \underset{1_*(\nu_X)}{\overset{\phi(1,1)_X}{\rightleftarrows}} 1_*1_*X \tag{2}$$

$$1_*X \underset{\nu_{1_*X}}{\overset{\phi(1,1)_X}{\rightleftarrows}} 1_*1_*X \tag{3}$$

Here commutativity of the last two diagrams means that the opposite arrows are inverse to each other.

Let $\mathcal{X}$ be a category with G-action. The category of G-invariants in $\mathcal{X}$, denoted by $\mathcal{X}^G$, is defined as follows. An object of $\mathcal{X}^G$ is a pair (X, f), where X is an object of $\mathcal{X}$ and f is a family of isomorphisms $f(\sigma)\colon \sigma_*X \to X$ for all $\sigma \in G$ making the following diagram commutative for all $\sigma, \tau \in G$.

$$\begin{array}{ccc} (\sigma\tau)_*X & \xrightarrow{f_{\sigma\tau}} & X \\ {\scriptstyle \phi(\sigma,\tau)_X}\downarrow & & \uparrow{\scriptstyle f_\sigma} \\ \sigma_*\tau_*X & \xrightarrow[\sigma_*(f_\tau)]{} & \sigma_*X \end{array}$$

An action of G on a tensor category $\mathcal{C}$ means an action of G on $\mathcal{C}$ preserving the tensor structure. Namely it consists of data

- tensor functors $\sigma_*\colon \mathcal{C} \to \mathcal{C}$ for all $\sigma \in G$
- isomorphisms $\phi(\sigma, \tau)\colon (\sigma\tau)_* \to \sigma_* \circ \tau_*$ of tensor functors for all $\sigma, \tau \in G$
- an isomorphism $\nu\colon 1_* \to \mathrm{Id}_{\mathcal{C}}$ of tensor functors

making the diagrams (1), (2), (3) commutative (with obvious change of letters).

By the definition of a tensor functor, the above σ_* consists of

- a functor $\sigma_*: \mathcal{C} \to \mathcal{C}$
- natural isomorphisms $\psi(\sigma)_{A,B}: \sigma_*A.\sigma_*B \to \sigma_*(A.B)$ for all $A, B \in \mathcal{C}$
- an isomorphism $\iota(\sigma): I \to \sigma_*I$

making the following diagrams commutative for all $A, B, C \in \mathcal{C}$.

$$\begin{array}{ccc} (\sigma_*A.\sigma_*B).\sigma_*C & \xrightarrow{\alpha_{\sigma_*A,\sigma_*B,\sigma_*C}} & \sigma_*A.(\sigma_*B.\sigma_*C) \\ {\scriptstyle \psi(\sigma)_{A,B}.\sigma_*C}\downarrow & & \downarrow{\scriptstyle \sigma_*A.\psi(\sigma)_{B,C}} \\ \sigma_*(A.B).\sigma_*C & & \sigma_*A.\sigma_*(B.C) \\ {\scriptstyle \psi(\sigma)_{A.B,C}}\downarrow & & \downarrow{\scriptstyle \psi(\sigma)_{A,B.C}} \\ \sigma_*((A.B).C) & \xrightarrow[\sigma_*(\alpha_{A,B,C})]{} & \sigma_*(A.(B.C)) \end{array} \qquad (4)$$

$$\begin{array}{ccc} I.I & \xrightarrow{\lambda_I} & I \\ {\scriptstyle \iota(\sigma).\iota(\sigma)}\downarrow & & \downarrow{\scriptstyle \iota(\sigma)} \\ \sigma_*I.\sigma_*I \xrightarrow[\psi(\sigma)_{I,I}]{} & \sigma_*(I.I) & \xrightarrow[\sigma_*(\lambda_I)]{} \sigma_*I \end{array} \qquad (5)$$

The requirement that $\phi(\sigma, \tau)$ is a morphism of tensor functors means that the following diagram is commutative for all $A, B \in \mathcal{C}$.

$$\begin{array}{ccc} (\sigma\tau)_*A.(\sigma\tau)_*B & \xrightarrow{\phi(\sigma,\tau)_A.\phi(\sigma,\tau)_B} & \sigma_*\tau_*A.\sigma_*\tau_*B \\ & & \downarrow{\scriptstyle \psi(\sigma)_{\tau_*A,\tau_*B}} \\ {\scriptstyle \psi(\sigma\tau)_{A,B}}\downarrow & & \sigma_*(\tau_*A.\tau_*B) \\ & & \downarrow{\scriptstyle \sigma_*(\psi(\tau)_{A,B})} \\ (\sigma\tau)_*(A.B) & \xrightarrow[\phi(\sigma,\tau)_{A.B}]{} & \sigma_*\tau_*(A.B) \end{array} \qquad (6)$$

We could say that a G-action on the tensor category $\mathcal{C}$ consists of the data σ_*, $\phi(\sigma, \tau)$, ν, $\psi(\sigma)$, $\iota(\sigma)$ making the diagrams of (1)–(6) commutative.

Suppose G acts on a tensor category $\mathcal{C}$. The category $\mathcal{C}^G$ becomes a tensor category as follows. The tensor product is defined by

$$(A, f).(B, g) = (A.B, h),$$

where

$$h(\sigma) = f(\sigma).g(\sigma) \circ \psi(\sigma)^{-1}_{A,B}.$$

The associativity isomorphisms are inherited from $\mathcal{C}$.

The tensor category $\mathcal{C}[G]$ is defined as follows. We set $\mathcal{C}[G] = \bigoplus_{\sigma\in G} \mathcal{C}$ as a category. So an object of $\mathcal{C}[G]$ is expressed as $\bigoplus_{\sigma\in G}(A_\sigma, \sigma)$ with $A_\sigma \in \mathcal{C}$, and a morphism from $\bigoplus_{\sigma\in G}(A_\sigma, \sigma)$ to $\bigoplus_{\sigma\in G}(B_\sigma, \sigma)$ is expressed as $\bigoplus_{\sigma\in G}(f_\sigma, \sigma)$ with $f_\sigma \colon A_\sigma \to B_\sigma$ a morphism in $\mathcal{C}$. The tensor product operation in $\mathcal{C}[G]$ is defined by

$$(A, \sigma).(B, \tau) = (A.\sigma_* B, \sigma\tau).$$

The associativity is given by

$$\begin{array}{ccc}
((A,\sigma).(B,\tau)).(C,\rho) = (A.\sigma_* B, \sigma\tau).(C,\rho) & = & ((A.\sigma_* B).(\sigma\tau)_* C, \sigma\tau\rho) \\
\Big\downarrow \alpha_{(A,\sigma),(B,\tau),(C,\rho)} & & \Big\downarrow (\alpha(A,\sigma,B,\tau,C),\sigma\tau\rho) \\
(A,\sigma).((B,\tau).(C,\rho)) = (A,\sigma).(B.\tau_* C, \tau\rho) & = & (A.\sigma_*(B.\tau_* C)), \sigma\tau\rho)
\end{array}$$

where $\alpha(A, \sigma, B, \tau, C)$ is the composite

$$\begin{array}{c}
(A.\sigma_* B).(\sigma\tau)_* C \\
\Big\downarrow (A.\sigma_* B).\phi(\sigma,\tau)_C \\
(A.\sigma_* B).\sigma_*\tau_* C \\
\Big\downarrow \alpha_{A,\sigma_* B,\sigma_*\tau_* C} \\
A.(\sigma_* B.\sigma_*\tau_* C) \\
\Big\downarrow A.\psi(\sigma)_{B,\tau_* C} \\
A.\sigma_*(B.\tau_* C).
\end{array}$$

3. Group actions on group tensor categories

Let A be a group. The tensor category $\mathcal{V}[A]$ is just the category of A-graded vector spaces. For $a \in A$ we write the simple object (k, a) of $\mathcal{V}[A]$ as $\underline{a}$. The tensor product in $\mathcal{V}[A]$ is then given by $\underline{a}.\underline{b} = \underline{ab}$, and the isomorphisms of associativity and unitality are identities.

Let $t \colon A^3 \to k^\times$ be a 3-cocycle. The tensor category $\mathcal{V}[A, t]$ is defined as follows: It has the same underlying k-category, tensor product and unit object as $\mathcal{V}[A]$, while the isomorphisms of associativity and unitality are given by

$$\begin{aligned}
\alpha_{\underline{a},\underline{b},\underline{c}} &= t(a, b, c) 1_{\underline{abc}} \\
\lambda_{\underline{a}} &= t(1, 1, a)^{-1} 1_{\underline{a}} \\
\rho_{\underline{a}} &= t(a, 1, 1) 1_{\underline{a}}
\end{aligned}$$

for $a, b, c \in A$.

Suppose that G acts on the tensor category $\mathcal{V}[A, t]$. This amounts to specifying an action of G on A is given, denoted by $(\sigma, a) \mapsto {}^\sigma a$, and maps

$$u \colon G \times A \times A \to k^\times$$

$$v \colon G \times G \times A \to k^\times$$

such that

$$1 = \frac{t(b,c,d)t(a,bc,d)t(a,b,c)}{t(ab,c,d)t(a,b,cd)}$$
$$\frac{t(a,b,c)}{t({}^\sigma a,{}^\sigma b,{}^\sigma c)} = \frac{u(\sigma;b,c)u(\sigma;a,bc)}{u(\sigma;ab,c)u(\sigma;a,b)}$$
$$\frac{u(\sigma;{}^\tau a,{}^\tau b)u(\tau;a,b)}{u(\sigma\tau;a,b)} = \frac{v(\sigma,\tau;ab)}{v(\sigma,\tau;a)v(\sigma,\tau;b)}$$
$$\frac{v(\sigma\tau,\rho;a)v(\sigma,\tau;{}^\rho a)}{v(\tau,\rho;a)v(\sigma,\tau\rho;a)} = 1$$

for all $\sigma,\tau,\rho \in G$, $a,b,c,d \in A$.

We have $\mathcal{V}[A,t][G] = \mathcal{V}[A \rtimes G, s]$, where s is a 3-cocycle on the semi-direct product $A \rtimes G$ given by

$$s((a,\sigma),(b,\tau),(c,\rho)) = t(a,{}^\sigma b,{}^{\sigma\tau}c)u(\sigma;b,{}^\tau c)v(\sigma,\tau;c).$$

Theorem 1 for $\mathcal{C} = \mathcal{V}[A,t]$ asserts that the assignment $\mathcal{M} \mapsto \mathcal{M}^G$ yields a one-to-one correspondence between $\mathcal{V}[A \rtimes G, s]$-modules and $\mathcal{V}[A,t]^G$-modules. As an application of this, we can show

Proposition 3. *If $|A|$ and $|G|$ are coprime and t is not a coboundary, then there exists no tensor functor $\mathcal{V}[A,t]^G \to \mathcal{V}$.*

4. Group actions on matrix categories

Let $\mathcal{C}$ be the tensor category of (k^n,k^n)-bimodules for a positive integer n. An object of $\mathcal{C}$ may be expressed as an n by n matrix (V_{ij}) of vector spaces V_{ij}, and the tensor product operation is given by

$$(V_{ij}).(W_{jk}) = (\textstyle\bigoplus_j V_{ij} \otimes W_{jk})_{ik}.$$

Thus $\mathcal{C}$ is regarded as the category $\mathrm{Mat}_n(\mathcal{V})$ of matrices of vector spaces. It is also identified with the category $\mathrm{End}\,\mathcal{V}^n$ of k-linear functors $\mathcal{V}^n \to \mathcal{V}^n$.

Let $w\colon G^3 \to k^\times$ be a 3-cocycle. Let $\mathcal{X}$ be a $\mathcal{V}[G,w]$-module with underlying category equivalent to $\mathcal{V}^n$. We can show that G acts on the tensor category $\mathrm{End}\,\mathcal{X} \simeq \mathrm{Mat}_n(\mathcal{V})$ and every G-action on $\mathrm{Mat}_n(\mathcal{V})$ arises in this way.

Theorem 1 for $\mathcal{C} = \mathrm{End}\,\mathcal{X}$ gives an equivalence of 2-categories

$$\{(\mathrm{End}\,\mathcal{X})[G]\text{-modules with d.s.}\} \to \{(\mathrm{End}\,\mathcal{X})^G\text{-modules with d.s.}\}$$
$$\mathcal{M} \mapsto \mathcal{M}^G.$$

On the other hand, we have an equivalence of 2-categories

$$\{\mathcal{V}[G,w]\text{-modules}\} \to \{(\mathrm{End}\,\mathcal{X})[G]\text{-modules}\}$$
$$\mathcal{Y} \mapsto \mathrm{Hom}(\mathcal{X},\mathcal{Y}).$$

Also we have $(\mathrm{End}\,\mathcal{X})^G = \mathrm{End}_{\mathcal{V}[G,w]}\,\mathcal{X}$, the category of $\mathcal{V}[G,w]$-linear functors $\mathcal{X} \to \mathcal{X}$. Combining these together, we obtain

Proposition 4. *The assignment*

$$\{\mathcal{V}[G,w]\text{-modules with d.s.}\} \to \{\mathrm{End}_{\mathcal{V}[G,w]}\,\mathcal{X}\text{-modules with d.s.}\}$$
$$\mathcal{Y} \mapsto \mathrm{Hom}_{\mathcal{V}[G,w]}(\mathcal{X},\mathcal{Y})$$

is an equivalence of 2-categories.

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テンソル森田同値とガロア拡大
Tensor Morita Equivalences and Galois Extensions

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Abstract. The (co)modules over a Hopf algebra form naturally a tensor category, and it is known that tensor equivalences between two such categories are given by Hopf biGalois extensions. This is a survey of this relationship including some related results from [M1; M2] due to the author.

はじめに

日本の環論の２つの伝統－森田理論とガロア理論－のホップ代数における結びつきについて述べたい．

ホップ代数の著しい特徴は，その表現全体がテンソル圏をなす点にある．２つのホップ代数は，その表現圏が互いにテンソル同値のときテンソル森田（竹内）同値であると言われ，そのテンソル同値はホップ・ガロア拡大と呼ばれる，古典的ガロア拡大の一般化により与えられることが知られている．ホップ・ガロア理論はホップ代数の分野の中でも近年特に充実をみたが，この結びつきを意識して研究されることは少なかったように思われる．

このレポートの目的は，上の結びつきをできるだけ初等的に（簡略化の為ときには小さなウソもつきつつ）解説することと，それに関連した筆者による具体的な結果を紹介することにある．以下，その結果（２つある）の概要を述べさせて頂く．

歴史的に，ガロア理論はホップ化される以前に非可換化され，つまり，ある群をガロア群に持つ非可換環のガロア拡大なるものが定義さ

れた．しかし，そういったガロア拡大をすべて求めるといった問題は難しく，その種の（非自明な）結果は皆無であったように思われる．第1の結果は，上に述べた結びつきを手がかりとすれば，2面体群 D_{2n} や一般4元数群 Q_{4m} をガロア群に持つ（ある条件を満たす）体k上の非可換ガロア拡大がすべて求まるというものである．より詳しく，群環 kD_{2n}, kQ_{4m} を含むいくつかの有限次元ホップ代数のそれぞれにつき，その（正確には双対の）ホップ・ガロア拡大及びそれにテンソル森田同値なホップ代数をすべて求める．

最近4つの論文が独立にKaplansky 予想を否定し，一定次元の有限次元ホップ代数のファミリーで無限個の同形類からなるものを構成して見せた．第2の結果は，こうして得られた計4つのファミリーのいずれもがテンソル森田竹内同値を除けばただひとつのホップ代数からなる，言い換えれば1つのファミリーに属する2つのホップ代数は必ずテンソル森田竹内同値であることをいう．

以下kを基礎体にとり，簡単のためもあって断らない限りkは標数ゼロの代数的閉体であるとする．Hは（k上の）ホップ代数を表す．

1. テンソル森田同値

<u>群の表現</u>．群Gの線形表現全体または群環 kG 上の加群全体 kG-Mod はテンソル圏をなす．大ざっぱに言ってこれは，2つの表現（V_1, V_2 上）からテンソル積表現と呼ばれる表現（$V_1 \otimes V_2$ 上）が構成でき，また単位表現と呼ばれる特別な表現（k 上）が存在して，

（結合則）　$(V_1 \otimes V_2) \otimes V_3 = V_1 \otimes (V_2 \otimes V_3)$

（単位則）　$k \otimes V = V = V \otimes k$

を満たすことを意味する．ここに，テンソル積表現，単位表現は

$$g(v_1 \otimes v_2) = gv_1 \otimes gv_2, \quad g1 = 1 \quad (g \in G)$$

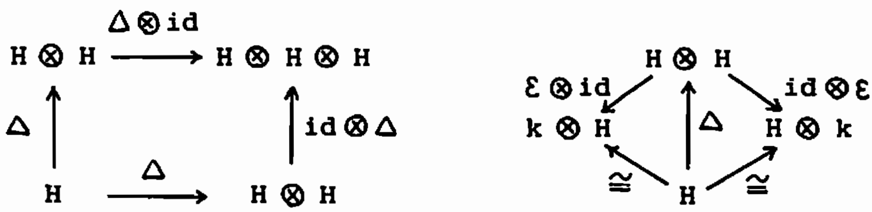
により定義されるが，これらはそれぞれ kG ⊗ kG, k による $V_1 \otimes V_2$, k への自然な加群作用を，代数射

$$\Delta : kG \longrightarrow kG \otimes kG, \quad \Delta(g) = g \otimes g$$
$$\varepsilon : kG \longrightarrow k, \quad \varepsilon(g) = 1 \qquad (g \in G)$$

により kG まで引き戻して得られる作用にほかならない．

<u>テンソル森田同値</u>．これとまったく同じアイデアで，ホップ代数H上の加群圏 H-Mod がテンソル圏をなすことがわかる．

<u>定義</u>．H = (H, Δ, ε) が<u>ホップ代数</u>であるとは，H が代数，$\Delta: H \longrightarrow H \otimes H$, $\varepsilon : H \longrightarrow k$ が代数射であって，



を可換にし，さらに群の逆元に相当するアンティポードと呼ばれる射 $S : H \longrightarrow H$ を伴うことをいう．

さて，(H-Mod, ⊗, k)はテンソル圏をなす．但し，Δ を通し H ⊗ H-加群 $V_1 \otimes V_2$ ($V_i \in$ H-Mod) を，ε を通し k-加群 k をそれぞれ H-加群と見る．実際，上の定義における左の可換図形はベクトル空間のいつもの同一視 $(V_1 \otimes V_2) \otimes V_3 = V_1 \otimes (V_2 \otimes V_3)$ が H-線形であることにほかならず，右の可換図形は $k \otimes V = V = V \otimes k$ が H-線形であることにほかならない．こうしてみると，H-Mod がテンソル圏をなすには，H は双代数でありさえすればよく，つまりアンティポードを持つ必要はない．しかし，テンソル同値とガロア拡大との好ましい結びつきを得るために，双代数でなく専らホップ代数を考える．

2つの代数が森田同値であるとはその加群圏が互いに同値（正確には k-線形同値）であることをいうのに倣い，次のように定義する．

定義．2つのホップ代数 H, L がテンソル森田同値であるとは，左（または同値に右）加群圏 H-Mod, L-Mod が互いにテンソル同値であることをいう．

注意．このとき当然，H と L は森田同値である．しかし更に H, L のいずれか（結果的に両方）が有限次元であれば，H と L は代数として同形となってしまう．

テンソル圏．ここでテンソル圏についてざっと復習しておこう．圏 $\mathcal{C}$, 函手 $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$，特別な対象 I からなる（$\mathcal{C}$, $\otimes$, I）がテンソル圏であるとは，さらに自然同値

$$a : (X \otimes Y) \otimes Z \xrightarrow{\cong} X \otimes (Y \otimes Z)$$
$$l : I \otimes X \xrightarrow{\cong} X, \quad r : X \otimes I \xrightarrow{\cong} X \qquad (X, Y, Z \in \mathcal{C})$$

が存在して，ある5角形と3角形の図形を可換にすることをいう．

MacLane の coherence 定理．この2つの図形さえ可換であれば，$\mathcal{C}$ において a, l, r, id を合成またはテンソル積して得られるどんな図形も可換である（言い換えれば，テンソル積 $X \otimes Y \otimes \ldots\ldots \otimes Z$ を結合させカッコをつける方法はさまざまあっても，そのうちのどの2つも一意に決まる標準的［つまり a, l, r, id を合成またはテンソル積して得られる］同形を以て互いに同形である）．

テンソル圏の間の函手（または同値）$\Phi : (\mathcal{C}, \otimes, I) \longrightarrow (\mathcal{C}', \otimes', I')$がテンソル函手（または同値）であるとは，

自然同値 $\Phi(X) \otimes' \Phi(Y) \xrightarrow{\cong} \Phi(X \otimes Y) \quad (X, Y \in \mathcal{C})$

と同形 $I' \xrightarrow{\cong} \Phi(I)$

が存在して，双方の構造 a, l, r と両立することをいう．（双方のテンソル積と単位対象のいわば翻訳の仕方が，上の自然同値と同形によって指定されている状態をいう．）

ところで，テンソル圏，テンソル函手はもともとモノイダル圏，モノイダル函手と呼ばれていた．講演タイトルも当初「モノイダル森田同値」としていたが，前講演者の丹原氏と最近の傾向に従い「テンソル」に改めた．また，テンソル圏という場合，$\mathcal{C}$はさらに基礎体k上の加法圏であって$\otimes$はk-双線形である場合を指すことが多い．実は，我々の意味するのもこれであり，テンソル函手または同値といえばΦはk-線形であるとする．

<u>第1の結果</u>．ホップ代数に戻ろう．ホップ代数 H(の有限次元加群圏)の Grothendieck 群はテンソル積からくる積で環（Grothendieck環）をなす．明らかに次が成り立つ．

<u>事実</u>．2つのホップ代数 H, L がテンソル森田同値であれば，それらの Grothendieck 環は一致する．

この逆は一般に成り立たない．それを示す丹原・山上の結果を述べるため，8次元非可換半単純ホップ代数をリストアップしよう．

$$kD_8, \quad kQ_8, \quad H_8$$

これらは順に，位数8の2面体群 D_8 の群環，4元数群 Q_8 の群環，Kac-Paljutkin による群環と異なるホップ代数である．ホップ代数が必ず自明な1次表現εを持つことに注意すれば，次元を勘定して，これらのいずれもが代数として $k\times k\times k\times k\times M_2(k)$ に等しいことがわかる．さらに，これらの Grothendieck 環が一致するのを見るのもやさしい．いま考えているホップ代数は半単純だから，Grothendieck環といっても，指標環，つまり既約指標全体を$\mathbb{Z}$上の自由基底に持ちテンソル積表現から決まる積を持つ環，にほかならない．

<u>定理</u>([TY])．これらのうちのどの2つもテンソル森田同値でない．

これを導くのに基とした結果も大変興味深く，固定した指標環（この論文の呼ぶフュージョン環）を持つ半単純テンソル圏を，ベクトル

空間と線形同形からなるデータで完全に記述するというものである．しかし，我々はホップ・ガロア理論の立場から上の定理を

$$kD_{2n},\quad kQ_{4m},\quad H_{4m}\quad (n > 2,\ m > 1)$$

に拡張する．これらは順に，位数 2n の 2 面体群 D_{2n}の群環，位数 4m の一般 4 元数群 Q_{4m}の群環，群環と異なる 4m 次元非可換半単純ホップ代数 H_{4m}で m = 2 のとき H_8 に一致するもの（定義は述べない）である．（一般 4 元数群はもっと狭い意味の群を指すことも多い．Q_{4m} は本により dicyclic group とも binary dihedral group とも呼ばれている．） n = 2m のとき，kD_{4m}, kQ_{4m}, H_{4m} の指標環はほぼ一致する．実際，m が偶数ならば完全に一致し，m が奇数ならば 1 次指標群の差

$$X(D_{4m}) = \mathbb{Z}/2 \oplus \mathbb{Z}/2,\qquad X(Q_{4m}) = X(H_{4m}) = \mathbb{Z}/4$$

があるのみ．結果は次のとおり．

<u>定理</u>（[M1]）．i) H = kD_8, kD_{2n}（n 奇数），kQ_{4m} or H_{4m} の場合，H にテンソル森田同値なホップ代数は H 自身に限る．

ii) kD_{4m} (m > 2) にテンソル森田同値なホップ代数が自分自身よりほかにただ 1 つ存在する．

<u>加群から余加群へ</u>．有限次元ホップ代数 H の双対 $H^* = \mathrm{Hom}_k(H, k)$ はまたホップ代数をなす．実際，H の積 $H \otimes H \longrightarrow H$, 単位元 $k \longrightarrow H$ の双対

$$H^* \otimes H^* = (H \otimes H)^* \xleftarrow{\Delta} H^*,\qquad k = k^* \xleftarrow{\varepsilon} H^*$$

が H^* の（余代数）構造を与える．

例．有限群 G の群環 kG の双対 $(kG)^*$ は G の函数環 k^G に等しい．元 g の特性函数 e_g ($e_g(h) = \delta_{g,h}$)全体が k^G の基底をなし，これらは互いに直交するベキ等元でその和は単位元に等しい．また，k^G の構

造は次で与えられる：

$$\Delta(e_g) = \sum_{h \in G} e_h \otimes e_{h^{-1}g}, \quad \varepsilon(e_g) = \delta_{1,g}.$$

Hは依然有限次元とし，Vをベクトル空間とする．随伴関係

$$\mathrm{Hom}_k(H \otimes V,\ V) = \mathrm{Hom}_k(V,\ \mathrm{Hom}_k(H,\ V)) = \mathrm{Hom}_k(V,\ V \otimes H^*)$$

において，左H-加群構造 $H \otimes V \longrightarrow V$ と右 H^*-余加群構造 $V \longrightarrow V \otimes H^*$とが丁度対応し，こうしてH-Modは右 H^*-余加群圏 Comod-H^*とテンソル圏として同一視できる．我々は後に述べる理由から，加群圏より余加群圏を好む．

一般に余代数(C, Δ, ε)上の右余加群とは，ベクトル空間Vと構造$\rho : V \longrightarrow V \otimes C$であって，

$$\begin{array}{ccc} V \otimes C & \xrightarrow{\rho \otimes id} & V \otimes C \otimes C \\ {\scriptstyle \rho}\uparrow & & \uparrow {\scriptstyle id \otimes \Delta} \\ V & \xrightarrow{\rho} & V \otimes C \end{array} \qquad \begin{array}{ccc} V \otimes C & & \\ & \searrow^{id \otimes \varepsilon} & \\ {\scriptstyle \rho}\uparrow & & V \otimes k \\ & \nearrow_{\cong} & \\ V & & \end{array}$$

を可換にするものをいう．これらは，代数A上の右加群構造を線形射$V \otimes A \longrightarrow V$で表すとき，これが満たすべき可換図形の矢印を逆転させて得られる．

余加群を考えること自体悪いことではなく，例えば無限次元代数 A上の局所有限次元加群はAの双対余代数 A° 上の余加群として自然に捉えることができるし，アフィン群や量子群G上の有理加群は座標環O(G)上の余加群にほかならない．

最近，2つの余代数はその余加群圏が互いに同値のとき，森田竹内同値と呼ばれるようになった．これは，余加群圏の間の同値の決定と余加群圏の特徴付けを与えた竹内光弘氏の20年以上も前の仕事 (J. Fac. Sci. Univ. Tokyo, 24(1977), 629-644) が，漸く浸透してきたことによる．この用語に倣い次のように定義しよう．

定義．２つのホップ代数 H, L がテンソル森田竹内同値，もっと簡単にテンソル竹内同値であるとは，右（または同値に左）余加群圏 Comod-H, Comod-L が互いにテンソル同値であることをいう．*)

申し遅れたが，Comod-H のテンソル構造は，$H\otimes H$-余加群 $V_1\otimes V_2$ ($V_i\in$ Comod-H)，k-余加群 k を，積 $H\otimes H\longrightarrow H$，単位元 $k\longrightarrow H$ を通し H-余加群と見ることで与えられる．

以後，テンソル森田同値でなくテンソル竹内同値を専ら扱う．

2．ホップ・ガロア理論

ホップ・ガロア拡大とは．これは次の２つを起源に持つ．

①体の（さらには可換環，非可換環の）ガロア拡大

②アフィン群の主等質空間（ＰＨＳ = principal homogeneous space）

ここでは②からのアプローチを採る．大ざっぱに言って，空間Ｘと可換代数Ｒとが対応し，さらに演算を付帯させて，群Ｇと可換ホップ代数Ｈとが対応する．この対応は反変，つまり矢印を逆転させるものであり，従って空間の直積×には可換代数のテンソル積⊗が対応する．さらに，空間Ｘを右G-空間とする作用 $X\times G\longrightarrow X$ には，可換代数Ｒを右H-余加群とする代数射 $\rho: R\longrightarrow R\otimes H$ が対応する．このような (R,ρ) は右H-余加群代数と呼ばれる．

右G-空間ＸがＰＨＳであるとは

(1) 右G-空間として $X\cong G$ を満たす

こと，と大ざっぱに定義される．これはＧの作用が推移的かつ効果的というのと同値だから，

(2) $X\times X\longleftarrow X\times G$, $(x, xg)\longleftarrow\!\shortmid (x, g)$ が全射かつ単射

*) H, L が有限次元であれば，これは H^* と L^* がテンソル森田同値というのと同値．

と言い換えられる（が，こちらの方がホントの定義により近い）．これを対応する R = (R, ρ) に翻訳すれば，

(3)　　$R\otimes R \longrightarrow R\otimes H,\quad a\otimes b \longmapsto a\rho(b)$ が全単射

となる．この条件を満たす $R\neq 0$ を（k を不変環に持つ）右H-ガロア拡大と，今度は厳密に定義する．しかし，条件(3)を見る限り，HとRは可換でなくてもよい．そこで，非可換な H, R に関してもこの定義をそのまま用いる．

例．G自身GのPHSであるのに応じ，H自身 $\Delta: H \longrightarrow H\otimes H$ を構造射として右H-ガロア拡大．これをトリビアルな右H-ガロア拡大と呼ぶ．

注意．i)上の定義は，基礎体kが任意の場合に意味を持つ．kが代数的閉体で，Hが有限生成可換代数ならば，可換な右H-ガロア拡大はHに限ることが知られている．
ii) 上の大ざっぱな定義から，右H-余加群代数Rに対し

$$R\text{が右H-ガロア拡大} \Leftrightarrow \text{右H-余加群として } R\cong H$$

が成り立つと思われるかもしれない．これは一般にはウソで，大ざっぱさのほころびが露呈してしまった．しかし，kを任意の体として Hが有限次元であれば⇒は正しい．また，H^*, Rが共に有限次元半単純であれば⇐は正しい（これはシンポジウムの折，丹原氏に質問して頂いて気付いた）．

ここで，①との関わりを述べたい．①にいう古典的ガロア拡大は，ホップ・ガロアと対比して群ガロア拡大と呼ばれることも多い．体の群ガロア拡大の場合そのガロア群は自己同形群として一意に定まるが，一般の環拡大の場合にはこれは成り立たず，有限群Gとその作用を指定した上，Gをガロア群に持つガロア拡大という言い方をする必要がある．さて，代数 $R\neq 0$ に対して次が成り立つ．

事実．R ⊃ k が有限群 G をガロア群に持つガロア拡大

⇔　R が右 k^G-ガロア拡大

ここに前述のとおり，$k^G = (kG)^*$であり，R への G の作用（を線形化した kG の作用）と k^G の余作用は随伴の関係で対応する．

これまで多くの著者がさまざまなホップ代数 H に対し，

問題「右 H-ガロア拡大をすべて求めよ」

に答えてきた．しかし，半単純な H に関する結果は皆無に等しい（特に $H = k^G$ の場合，即ち群ガロア拡大の場合にさえそうであるように思う）．H が半単純の場合には，この問題をテンソル竹内同値と結びつけて考えることが有効であると主張したい．この場合，むしろ H^* が半単純であることが直接効いて Comod-H が半単純（つまり各 H-余加群が半単純）となることがポイント．ホップ・ガロア拡大とテンソル竹内同値の問題のどちらか一方が他方を導くのではなく，互いに相補ってうまくいくというのが，この研究を通しての経験である．

両側ガロア拡大．L を H とは別のホップ代数とする．左 L-ガロア拡大というのも「右」と同様に定義される（その構造射は R ⟶ L ⊗ R となる）．

定義．(L, H)-両側ガロア拡大とは，左 L-かつ右 H-ガロア拡大 R で，L と H の余作用が可換（従って，R は特に (L, H)-両側余加群）なるものをいう．

定理([S])．右 H-ガロア拡大 R に対し，ホップ代数 L とその R への余作用 R ⟶ L ⊗ R が存在して，R は (L, H)-両側ガロア拡大となる．この L は同形を除き一意的．余作用は L の自己同形を除き（余作用を L の自己同形で捻ってもまだ条件を満たすが，その分の自由度を除けば）一意的である．

定義．この L を R の左ガロア・ホップ代数と呼ぼう．

例．トリビアルな右H-ガロア拡大Hの左ガロア・ホップ代数は明らかにH自身である．

Hは有限次元であると仮定し，Rを右H-ガロア拡大とする．前に注意したとおり，右H-余加群として $R \cong H$ となる．この同形を通し Rの積をHに移すことで，Rは接合積 ${}_\sigma H$ に等しくなる：$R = {}_\sigma H$．但し，$\sigma : H \otimes H \longrightarrow k$ は2-コサイクル条件を満たし畳込み積に関し可逆な線形形式であり，${}_\sigma H$ は右H-余加群Hに新しい積

$$a \cdot b \text{ (in } {}_\sigma H) = \sum \sigma(a_1, b_1) a_2 b_2 \text{ (in } H)$$

を導入して得られる右H-余加群代数である．${}_\sigma H$ の左ガロア・ホップ代数は何だろうか？答は H^σ と書かれ，Hのコサイクル変形と呼ばれる，土井幸雄氏に依るホップ代数である．これは，余代数Hに新しい積

$$a \circ b \text{ (in } H^\sigma) = \sum \sigma(a_1, b_1) a_2 b_2 \, \sigma^{-1}(a_3, b_3) \text{ (in } H)$$

を導入して得られるホップ代数である．ここに，σ^{-1} はσの畳込み積に関する逆である．これは H^σ の2-コサイクルと見なせて，$(H^\sigma)^{\sigma^{-1}} = H$ が成り立つから，「Hと H^σ とは互いにコサイクル変形である」という言い方が許される．${}_\sigma H$ が左 H^σ-ガロア拡大を確かめるのに，HのΔが与える ${}_\sigma H \longrightarrow H^\sigma \otimes {}_\sigma H$ が代数射であるのを見るのが本質的だが，これはやさしい（${}_\sigma H$ のσと H^σ の σ^{-1}が打ち消し合う）．

上で，昨今かなり浸透してきた Sweedler の記法

$$\Delta(a) = \sum a_1 \otimes a_2,$$

$$(\Delta \otimes \mathrm{id}) \circ \Delta(a) = (\mathrm{id} \otimes \Delta) \circ \Delta(a) = \sum a_1 \otimes a_2 \otimes a_3$$

を用いた．余代数の余積Δから見ると，代数の積の記法 ab は簡潔で羨ましい．負けずにΔも elementwise に扱えるようにと，上の記法が考案された．

テンソル竹内同値との関わり．これは次で与えられる．

定理（[S]）．2つのホップ代数 H, L に対し，次の i), ii) は同値である．

i) H と L はテンソル竹内同値である．

ii) (L, H)- 両側ガロア拡大が存在する．

もし次の iii) が成り立てば，i), ii) が成り立つ．H と L のいずれか一方（結果的に両方）が有限次元であれば，逆も成り立ち i)-iii)は互いに同値となる．

iii) H と L は互いにコサイクル変形である．

i), ii) が成り立つとき，さらに両側ガロア拡大とテンソル同値の間に 1 対 1 対応が存在する．実際，R を (L, H)-両側ガロア拡大とすると，余テンソル積 $\square_H$ の与える函手

$$\Phi_R : \text{H-Comod} \longrightarrow \text{L-Comod}, \quad \Phi_R(V) = R \square_H V$$

が，R の積，単位元からくるテンソル構造

$$(R \square_H V_1) \otimes (R \square_H V_2) \xrightarrow{\cong} R \square_H (V_1 \otimes V_2), \qquad k \xrightarrow{\cong} R \square_H k$$

と共にテンソル同値与える．そして，(L, H)-両側ガロア拡大全体とテンソル同値 H-Comod $\xrightarrow{\approx}$ L-Comod 全体とが $R \longmapsto \Phi_R$ により同形を除き 1 対 1 に対応する．一般に，右 H-余加群 (W, ρ) と左 H-余加群 (V, λ)の余テンソル積 $W \square_H V$ は線形射の差核

$$W \square_H V = \mathrm{Ker}(W \otimes V \underset{id \otimes \lambda}{\overset{\rho \otimes id}{\rightrightarrows}} W \otimes H \otimes V)$$

で定義される．

いま見たように，余加群圏の間のテンソル同値を与えるのは両側ガロア拡大である．これは両側余加群ではあるものの，もとより代数であるのと，構造射が代数射である点が勝り代数的である．一方，加群圏の間のテンソル同値は然るべき条件を満たす両側加群かつ余代数で

与えられ，余代数的である（森田同値が両側加群のテンソル積で与えられ，テンソル構造が余代数構造で与えられる）．ホップ代数びととて人の子，余代数よりも代数の方が自然で扱いやすい．代数であれば線形代数が効くし，生成元と関係式による素朴で強力な構成法も使える．テンソル同値が代数的という点にこそ，余加群圏をより好む理由がある．

<u>第1の結果の精密化</u>． kD_{2n} らにテンソル森田同値なホップ代数をすべて求めた第1の結果は，いまや $k^{D_{2n}}$ らのガロア拡大に関する結果を示す次の表に吸収される．

ホップ代数 H		非トリビアルな右 H-ガロア拡大の個数	左ガロア・ホップ代数
$k^{D_{2n}}$（n 奇数） $k^{Q_{4m}}$, H^*_{4m}		なし	—
k^{D_8}		2	共に k^{D_8}
$k^{D_{4m}}$	2 < m 奇数	1	J_{4m}
	2 < m 偶数	2	共に J_{4m}
J_{4m}	2 < m 奇数	1	$k^{D_{4m}}$
	2 < m 偶数	2	$k^{D_{4m}}$, J_{4m}

例えば，$k^{D_{4m}}$（2 < m 偶数）の行を見て欲しい．この場合，右 $k^{D_{4m}}$-ガロア拡大，つまり D_{4m} をガロア群に持つ k のガロア拡大，が $k^{D_{4m}}$ よりほかに丁度2つ存在し，それらの左ガロア・ホップ代数が共に J_{4m} であることが示されている．ここに，J_{4m} は新しい 4m 次元半単

純ホップ代数で，非可換であり群環と異なる（定義は述べない）．従って，これが $k^{D_{4m}}$ とテンソル竹内同値で $k^{D_{4m}}$ と異なる唯一のホップ代数である（2 < m 奇数の場合もそう）．第1の結果と比べられたい．

3. 否定されたKaplansky予想を弁護する

<u>予想とその現状</u>．1974年，Kaplansky はシカゴでホップ代数の講義を行い，翌年その講義録を出版した．その付録に掲げられたいくつかの予想は，その後の研究の指針となったが，ここでいう予想は 10 番目に挙げられた次のものである．

<u>Kaplansky 予想</u>．与えられた自然数を次元にもつ有限次元ホップ代数は，同形を除き有限個しか存在しないだろう．

最初の貢献は，1996年ルーマニアの若い研究者によってなされた．

<u>定理</u>(Stefan)．「有限次元ホップ代数」を「有限次元半単純ホップ代数」に限定すれば予想は正しい．

しかし，最近になって4つの論文が独立に予想を否定した．

<u>定理</u>（独立4論文）．予想は一般には正しくない．

4つの論文とは次をさす．

S. Gelaki, J. Algebra 209(1998)

N. Andruskiewitsch and H.-J. Schneider, 上に同じ

M. Beattie et al., Invent math. 136(1999)

E. Müller, Proc. London Math. Soc., to appear

各論文は予想の反例として，無限個の同形類からなる同一次元のホップ代数のファミリーを1つずつ構成した．ところで，上の論文のう

ち最初の3つのプレプリントは偶然にも 1997 年秋，わずか数週間の差で出された．3つの反例もまたよく似ており，わずかの差ながら一般性の点で上から下の順になっている．（さらに最初の2組に至っては，同じ雑誌の同じ巻号に前後並んで掲載されている．何の因果か[TY]まで同じ号に載っている．） Müller の反例は趣を全く異にするもので，後で紹介するとおり量子群の範疇に属すものである．

<u>弁護</u>．さて，こうして否定されてしまった予想を次の形で弁護する．

<u>定理</u>([M2])．上で得られた4つのファミリーのどれもが，テンソル竹内同値を除けばただ1つのホップ代数からなる．

これを状況証拠に，予想を次のように修正したくなる．

<u>Kaplansky 予想の修正</u>．与えられた自然数を次元にもつ有限次元ホップ代数は，テンソル竹内（または同値に，森田）同値を除き有限個しか存在しないだろう．

上の定理が基礎とする結果は，量子群の言葉で表現するとわかりやすい．量子群といっても一定の定義があるわけでないが，その本体がホップ代数である点は皆一致している．群より一般の，さまざまな程度の対称性を備えたホップ代数を，それぞれにそう呼んでいる．

ここではごく形式的に，（k 上の）量子群の圏を（k 上の）ホップ代数の圏の双対圏として定義し，アフィン群の記号を借りて，量子群 G に対応するホップ代数を O(G) と書く．対応は反変だから，G の部分群または商群 G' には，それぞれ O(G) の商または部分ホップ代数 O(G') が対応する．（位数 m の）有限量子群は（m 次元の）有限次元ホップ代数に対応する量子群として定義する．また，有理 G-加群の圏 G-Mod は O(G)-余加群の圏 Comod-O(G) そのものとし，G-量子ＰＨＳは O(G)-ガロア拡大のことと理解する．

定理([M2]). $i = 1, 2$ に対し量子群の pullback 図形

$$\begin{array}{ccc} G & \twoheadrightarrow & \bar{G} \\ \uparrow & & \uparrow \\ G_i & \longrightarrow & \bar{G}_i \end{array}$$

が与えられたとする．もし，$\bar{G}_1$ と $\bar{G}_2$ が $\bar{G}$ において共役，つまり有理点 $g \in \bar{G}(k)$ $(= \mathrm{Alg}(O(\bar{G}), k))$が存在して $\bar{G}_1 = g\bar{G}_2g^{-1}$を満たす，ならば，$G_1$-Mod と G_2-Mod はテンソル同値，即ち $O(G_1)$ と $O(G_2)$ はテンソル竹内同値となる．

E. Müller の反例．これは，量子一般線形群 $GL_q(n)$ の有限量子部分群をすべて求めるという彼の仕事の中で提示された．以下の記述の仕方は竹内氏のアイデアに依る．

奇数 $N > 2$ を固定し，$q \in k$ を 1 の原始 N 乗根とする．2×2 特殊線形群 $SL(2)$ の N 次巡回部分群 Γ に対し，量子群 G_Γ を pullback 図形

$$\begin{array}{ccc} SL_q(2) & \twoheadrightarrow & SL(2) \\ \uparrow & & \uparrow \\ G_\Gamma & \twoheadrightarrow & \Gamma \end{array}$$

を以て定義する．一般に，代数群とくに有限群は，その座標環を対応するホップ代数に持つ量子群と見做せることに注意しよう．上の図形における上段の射は，量子特殊線形群 $SL_q(2)$ からの量子フロベニウス射を表す．Müllerは，こうして得られた $\{G_\Gamma\}_\Gamma$ が無限個の同形類からなる位数 N^4 の有限量子群のファミリーであることを示した．しかし前定理によると，このうちのどの 2 つも互いにテンソル竹内同値である．実際，線形代数によりどの Γ も対角行列 $\mathrm{diag}(q, q^{-1})$ の生成する部分群に共役となるからである．

残りの 3 つの反例はどれも pointed ホップ代数（単純余加群が必ず

1次元であるようなホップ代数）からなる．これらに関する結果も前定理を用いて導かれるが，もう少し一般化して「有限次元pointed ホップ代数Hはある条件を満たせば，Hの余根基フィルターから得られる次数つきホップ代数 gr H とテンソル竹内同値となる」という形で述べることができる([M2]).

<u>前定理の証明のアイデア</u>．これは次に示すとおり，極めて単純．記号を定理にあるとおりとする．

G_1と G_2 がテンソル竹内同値をいうのに，(G_1, G_2)-両側量子ＰＨＳの存在を示す．$\overline{X} := \overline{G}_1 g = g\overline{G}_2$ が $(\overline{G}_1, \overline{G}_2)$-両側量子ＰＨＳであることは，先のＰＨＳの定義から明らか．量子空間の pullback 図形

$$\begin{array}{ccc} G & \twoheadrightarrow & \overline{G} \\ \uparrow & & \uparrow \\ X & \longrightarrow & \overline{X} \end{array}$$

を作ると，X が求めていたものであることが示される．

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SOME EXTENSIONS OF QUASI-BAER RINGS

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Throughout R denotes an associative ring with identity. Let n be a positive integer. Then $M_n(R)$ and $T_n(R)$ denote the ring of $n \times n$ matrices over R and the ring of $n \times n$ upper triangular matrices over R, respectively.

Kaplansky [9] introduced Baer rings to abstract various properties of rings of operators on a Hilbert space. Clark [7] introduced the quasi-Baer rings and characterized a finite dimensional quasi-Baer ring over an algebraically closed field as a twisted matrix units semigroup algebra. Further works on quasi-Baer rings appear in [2], [3], [4], [8] and [12]. In this note we state some results on extensions of Baer rings and quasi-Baer rings.

A *Baer ring* is a ring in which the left annihilator of every subset is generated by an idempotent (see [9]). A ring R is called *quasi-Baer* if the left annihilator of every right ideal of R is generated by an idempotent. Note that the definitions of Baer and quasi-Baer rings are left-right symmetric.

First we state some examples of Baer rings and quasi-Baer rings.

Examples of Baer rings. 1) Rings with no zero-divisors are Baer rings.

2) A ring R is called a right p.p.ring if every principal right ideal of R is projective. A right p.p.ring with no infinite set of orthogonal idempotents is a Baer ring (see Small [13]). In particular, hereditary Noetherian rings are Baer rings.

3) Let V be a vector space over a field K. Then the endomorphism ring $\mathrm{End}_K(V)$ is a Baer ring. More generally, right (or left) self-injective regular rings are Baer rings (see Mewborn [10, Proposition 4.1]).

4) The algebra of all bounded operators on a Hilbert space is a Baer ring.

Of course Baer rings are quasi-Baer. Recall that an R-module M is called a CS-module if every submodule is essential in a direct summand of M.

The detail version of this paper will be submitted for publication elsewhere.

Examples of quasi-Baer rings. 1) Every prime ring is a quasi-Baer ring.

2) A semiprime ring R is quasi-Baer if and only if R is a CS-module over the ring $R \otimes_Z R^{op}$.

3) For any prime ring R, the ring $T_n(R)$ is quasi-Baer for any positive integer n.

Next, we give some examples which show that the class of Baer rings is not closed under some extensions. To state those examples, we need the following facts.

Let R be a commutative domain and let n be an integer greater than 1. Then the ring $M_n(R)$ is a Baer ring if and only if R is semihereditary (see Kaplansky [9, p.17]).

For a commutative ring R, the polynomial ring $R[x]$ is semihereditary if and only if R is von Neumann regular (see Camillo [6, Theorem]).

For an integer n greater than 1, $T_n(R)$ is Baer if and only if R is a division ring (see [9, p.16]).

Examples of extensions of Baer rings. 1) $\mathbf{Z}[x]$ is a Baer ring, but $A = M_2(\mathbf{Z}[x])$ is not a Baer ring. In fact, the principal left ideal of A generated by $A = \begin{pmatrix} 2 & 0 \\ x & 0 \end{pmatrix}$ is not projective.

2) $R = M_2(\mathbf{Z})$ is a Baer ring, but $R[x](= M_2(\mathbf{Z}[x])$ is not a Baer ring.

3)The ring $\mathbf{Z}$ of integers is a Baer ring, but $T_2(\mathbf{Z})$ is not a Baer ring.

P. Pollingher and A. Zaks [12] gave a necessary and sufficient condition for a matirx ring $M_n(R)$ to be Baer. They also proved the following: Let R be a quasi-Baer ring and let n be a positive integer. Then $M_n(R)$ and $T_n(R)$ are quasi-Baer rings.

Clark [7] proved that if R is a quasi-Baer ring and if e is an idempotent then eRe is also a quasi-Baer rring. Hence we have the following.

Proposition 1. *The class of quasi-Baer rings is Morita stable.*

The polynomial rings over a Baer ring was first considered by Armendariz. We call a ring R *reduced* if it has no nonzero nilpotent elements. Armendariz [1] proved that if R is a reduced Baer ring, then the polynomial ring $R[x]$ is also a reduced Baer ring.

As pointed out in the examples above, $R[x]$ is not necessarily Baer even if R is Baer. Recently G.F. Birkenmeier, J. Y. Kim and J. K. Park [5] proved that if R is a quasi-Baer ring, then the polynomial ring $R[x]$ is also a quasi-Baer ring.

To state a generalization of this result, we introduce the following notion: A monoid M is said to be *ordered* if the elements of M are linearly oredered with respect to the relation $<$ and that, for for all $x, y, z \in G$, $x < y$ implies $zx < zy$ and $xz < yz$.

We consider a monoid ring RG of an ordered group G over a ring R. Obviously any submonoid of an ordered group is an ordered monoid.

Examples of ordered groups. 1) Torsion-free nilpotent groups are ordered groups. (see [11, Lemma 13.1.6]).

2) Free groups are ordered groups (see [11, Corollary 13.2.8]).

Polynomial rings, Laurent polynomial rings and free rings are monoid rings of ordered groups.

Theorem 1. *Let G be an ordered monoid. Then the monoid ring RG is a quasi-Baer ring if and only if R is a quasi-Baer ring.*

In a reduced ring R, left and right annihilators of any subset S of R coincide. Hence a reduced quasi-Baer ring is a Baer ring Thus we have the following corollary.

Corollary 1. *Let R be a ring and let G be an ordered monoid. Then the monoid ring RG is a reduced Baer ring if and only if R is a reduced Baer ring.*

Let R be a ring, let G be a group and assume G act on R by means of a homomorphism into the automorphism group of R. We denote by r^g the image of $r \in R$ under $g \in G$. The skew group ring $R * G$ is a ring which as a left R-module is free with basis G and multiplication defined by the rule $gr = r^g g$. Then R may be considered as a left $R * G$-module as follows: for any $a \in R$ and any $\sum_g r_g g \in R * G$, define $(\sum r_g g) \cdot a = \sum r_g a^g \in R$. A ring R is called a *G-quasi-Baer ring* if, for any $R * G$-submodule I of R, the left annihilator of I is generated by an idempotent. When G is a cyclic group generated by σ, a G-quasi-Baer ring is simply called a *σ-quasi-Baer ring.*

Theorem 2. *Let R be a ring and let G be an ordered group acting on R. Then $R*G$ is a quasi-Baer ring if and only if R is a G-quasi-Baer ring.*

As a special case of this theorem, we obtain

Corollary 2. *Let R be a ring and let σ be an automorphim of R. Then $R[x, x^{-1}; \sigma]$ is a quasi-Baer ring if and only if R is a σ-quasi-Baer ring.*

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INVERSE AND DIRECT IMAGES FOR QUANTUM WEYL ALGEBRAS

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0. Introduction

In [16] Wess and Zumino gave a method for constructing noncommutative differential calculus (or de Rham complex) on the quantum affine space associated to a Hecke symmetry R. Also, they constructed the corresponding algebra of linear differential operators. Since the algebra of linear differential operators on the n-dimensional affine space is the n-th Weyl algebra, this algebra is regarded as a quantum analogue of the Weyl algebra, and called the *quantum Weyl algebra* (associated to R).

Let $R_{q,P}$ be the multiparameter R-matrix of the quantum deformation of GL_n parameterized by a scalar q and an $n \times n$ matrix $P = (p_{ij})$ in [3]. For the quantum Weyl algebra $A_n(q, P)$ associated to $R_{q,P}$, Demidov [6] and Rigal [13] consider quantum versions of classical theory of the Weyl algebras including Bernstein's inequality. And, some ring-theoretic properties of $A_n(q, P)$ have been studied in [1, 2, 7, 8, 9] etc. In [9] Jordan constructed a simple localization $B_n(q, P)$ of $A_n(q, P)$, which is a better analogue of the Weyl algebra A_n from the point of view of noncommutative ring theory.

The purpose of this note is to define an analogue of the inverse and direct images for the quantum Weyl algebra $A_n(q, P)$, and to investigate their properties. In particular, we prove a quantum analogue of *Kashiwara's theorem*, and consider preservation of holonomicity under inverse and direct images.

Throughout this note we fix a ground field K, which is assumed to be of characteristic $\mathrm{ch}\, k \neq 2$, and let q be a nonzero element of K such that q^2 is not a root of unity.

In this note, we use the terminology and the results of [12] for noncommutative ring theory, and refer to [4, 5] for the theory of the Weyl algebras, and [10] for the facts concerning Hopf algebras and quantum groups.

The detailed version of this paper will be submitted for publication elsewhere.

1. Preliminaries

Let V be an n-dimensional vector space. Assume that a non-degenerate linear transformation $R : V \otimes V \to V \otimes V$ is a Hecke symmetry, that is, satisfies the Yang-Baxter equation

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23},$$

and the Hecke condition

$$(R - q)(R + q^{-1}) = 0$$

for some $q \in K\backslash\{0\}$, where $R_{12} = R\otimes \mathrm{id}_V$, $R_{23} = \mathrm{id}_V \otimes R$. For fixed basis $\{e_1, \cdots, e_n\}$ of V, we write

$$R(e_i \otimes e_j) = R_{ij}^{kl} e_k \otimes e_l \quad (R_{ij}^{kl} \in K).$$

The *quantum affine space* $K_R[X]$ associated to a Hecke symmetry R is the K-algebra generated by $x^1, \cdots, x^n$ with relations

$$R_{kl}^{ij} x^k x^l = q x^i x^j.$$

In [16] Wess and Zumino construct examples of noncommutative differential calculus on the quantum affine space.

DEFINITION 1.1 ([16]). The *Wess-Zumino differential calculus* $\Omega(R)$ on $K_R[X]$ is the $K_R[X]$-ring with generators $\xi^1, \cdots, \xi^n$ subject to the relations

$$\xi^i \xi^j = -qR_{kl}^{ij}\xi^k\xi^l, \quad x^i\xi^j = qR_{kl}^{ij}\xi^k x^l.$$

Put $\Omega^0 = K_R[X]$, $\Omega^1 = \oplus_{i=1}^n K_R[X]\xi^i$ and $\Omega^l = (\Omega^1)^l$. Then $\Omega(R) = \oplus_{l\geq 0}\Omega^l$ is a *differential graded algebra* (for short, *DG-algebra*) with a differentail map $d : \Omega(R) \to \Omega(R)$ such that $d(x^i) = \xi^i$.

DEFINITION 1.2 ([16, 8]). The *quantum Weyl algebra* $A_n(R)$ associated to R is defined as the $K_R[X]$-ring generated by $\partial_1, \cdots, \partial_n$ with the relations

$$R_{ji}^{lk}\partial_k\partial_l = q\partial_i\partial_j, \quad \partial_i x^j = \delta_i^j + qR_{ik}^{jl}x^k\partial_l.$$

In addition, commutation relations between ∂_i and ξ^j are given by

$$\partial_i \xi^j = q^{-1}(R^{-1})_{il}^{jk}\xi^l\partial_k.$$

EXAMPLE 1.3. Let $P = (p_{ij})$ be an $n \times n$ matrix over $K \setminus \{0\}$ such that $p_{ii} = 1$, $p_{ji} = p_{ij}^{-1}$ for each i, j. Define the multiparameter R-matrix $R_{q,P}$ by

$$(R_{q,P})_{kl}^{ij} = \delta_l^i\delta_k^j(p_{ij} + (q - p_{ij})\delta^{ij}) + (q - q^{-1})\delta_k^i\delta_l^j\theta(j, i),$$

where

$$\theta(i,j) = \begin{cases} 1 & \text{if } i > j, \\ 0 & \text{if } i \le j. \end{cases}$$

We write $\Omega(q,P)$ and $A_n(q,P)$ for $\Omega(R_{q,P})$ and $A_n(R_{q,P})$, respectively. The relations of $\Omega(q,P)$ are

$$(\xi^i)^2 = 0, \quad \xi^i\xi^j = -q^{-1}p_{ij}\xi^j\xi^i \qquad (i<j),$$
$$x^i\xi^j = qp_{ij}\xi^j x^i + (q^2-1)\xi^i x^j \qquad (i<j),$$
$$x^i\xi^j = qp_{ij}\xi^j x^i \qquad (i>j),$$
$$x^i\xi^i = q^2\xi^i x^i.$$

The relations of $A_n(q,P)$ are given by

$$\partial_i\partial_j = q^{-1}p_{ij}\partial_j\partial_i \quad (i<j), \quad \partial_i x^j = qp_{ji}x^j\partial_i \quad (i \neq j),$$
$$\partial_i x^i = 1 + q^2 x^i\partial_i + (q^2-1)\sum_{j>i} x^j\partial_j.$$

In [9] Jordan constructed a simple localization of $A_n(q,P)$. For $1 \le i \le n$, let $z_i = \partial_i x^i - x^i\partial_i (= 1 + (q^2-1)\sum_{j\ge i} x^j\partial_j)$. The subset $\mathcal{Z} = \{z_1^{\alpha_1}\cdots z_n^{\alpha_n}\}_{\alpha_1,\cdots,\alpha_n\ge 0}$ is an Ore sets in $A_n(q,P)$ [9, 3.1]. We denote by $B_n(q,P)$ the localization of $A_n(q,P)$ at $\mathcal{Z}$. In [9, Thm. 3.2] it is proved that the localization $B_n(q,P)$ is simple of Krull and global dimension n like the Weyl algebra A_n in characteristic zero.

We say that an element u of a left $A_n(q,P)$-module M is *$\mathcal{Z}$-torsion* if there exists $w \in \mathcal{Z}$ such that $wu = 0$. For a left $A_n(q,P)$-module M, let $T(M)$ be the submodule consisting of the $\mathcal{Z}$-torsion elements.

Bernstein's inequality [13, Thm.3(c)]. (1) For a finitely generated nonzero left $B_n(q,P)$-module N, its Gelfand-Kirillov dimension $\mathrm{GKdim}_{B_n(q,P)}(N) \ge n$.

(2) For a finitely generated nonzero left $A_n(q,P)$-module M, the Gelfand-Kirillov dimension $\mathrm{GKdim}_{A_n(q,P)}(M/T(M)) \ge n$.

Following [13] we say that a finitely generated left $B_n(q,P)$-module N is *holonomic* if $N = 0$ or $\mathrm{GKdim}_{B_n(q,P)}(N) = n$. We say that a finitely generated left $A_n(q,P)$-module M is *holonomoc* if $M/T(M) = 0$ or $\mathrm{GKdim}_{A_n(q,P)}(M/T(M)) = n$.

From the relations of $A_n(q,P)$ described in Example 1.3, one sees that

$$A_n(q,P)/\sum_{i=1}^{n} A_n(q,P)\partial_i \cong K_{q,P}[X] \qquad \text{(as } K\text{-vector spaces)}.$$

Via this linear isomorphism, $K_{q,P}[X]$ has a left $A_n(q,P)$-module structure. Then, ∂_i acts on $K_{q,P}[X]$ as the q-difference operator:

$$\partial_i \cdot f(x^i) = \frac{f(q^2x^i) - f(x^i)}{q^2x^i - x^i} \qquad (f(x^i) \in K[x^i]),$$
$$\partial_i \cdot f(x^j) = 0 \qquad (f(x^j) \in K[x^j], \text{ where } j \neq i).$$

Note that $K_{q,P}[X]$ naturally becomes a left $B_n(q,P)$-module. Similary, the K-subalgebra $K_{q,P}[\partial]$ of $A_n(q,P)$ generated by $\partial_1, \cdots, \partial_n$ has a left $B_n(q,P)$-module structure via the linear isomorphism

$$A_n(q,P)/\sum_{i=1}^{n} A_n(q,P)x^i \cong K_{q,P}[\partial].$$

Both the left $A_n(q,P)$-modules $K_{q,P}[X]$ and $K_{q,P}[\partial]$ are holonomic.

2. Quantum Matrix Group Action and Coaction on Quantum Weyl Algebras

DEFINITION 2.1. Let R be a Hecke symmetry. $M(R)$ is the K-algebra with n^2 generators t^i_j $(1 \leq i,j \leq n)$ subject to the relations

$$R^{ij}_{\alpha\beta} t^\alpha_k t^\beta_l = R^{\alpha\beta}_{kl} t^i_\alpha t^j_\beta.$$

$M(R)$ has a bialgebra structure with the comultiplication Δ and the counit ε such that

$$\Delta(t^i_j) = t^i_\alpha \otimes t^\alpha_j, \qquad \varepsilon(t^i_j) = \delta^i_j.$$

Denote by $H(R)$ the Hopf envelope of M. Thus there exists a bialgebra morphism $\Psi : M(R) \to H(R)$ such that, for any bialgebra morphism $\psi : M(R) \to H$ with H being a Hopf algebra, there exists a Hopf algebra morphism $\bar{\psi} : H(R) \to H$ with $\psi = \bar{\psi} \circ \Psi$. Such a Hopf algebra $H(R)$ always exists. See [11, Ch.7] for details.

The bialgebra $M(R)$ has a *cobraided structure* $\langle\ ,\ \rangle : M(R) \times M(R) \to K$ such that

$$\langle t^i_j, t^k_l \rangle = q(R^{-1})^{ki}_{jl}.$$

Thus $\langle\ ,\ \rangle$ is a bilinear, and satisfies that

$$\langle a, bc\rangle = \langle a_{(1)}, c\rangle\langle a_{(2)}, b\rangle, \quad \langle ab, c\rangle = \langle a, c_{(1)}\rangle\langle b, c_{(2)}\rangle,$$
$$b_{(1)}a_{(1)}\langle a_{(2)}, b_{(2)}\rangle = \langle a_{(1)}, b_{(1)}\rangle a_{(2)}b_{(2)}$$

for all $a,b,c \in M(R)$, where we use the Sweedler notation $\Delta(a) = a_{(1)} \otimes a_{(2)}$ etc. See [10, Thm. VIII.6.4].

Throughout this note we assume that the cobraided structure is extended to $H(R)$. This assumption holds for $R_{q,P}$.

LEMMA 2.2. *(1) There exists a right $H(R)^{\mathrm{op}}$-comodule algebra structure ρ on $A_n(R)$ such that*

$$\rho(x^i) = x^a \otimes S(t^i_a), \quad \rho(\partial_i) = \partial_a \otimes t^a_i,$$

for $1 \le i \le n$, where S denotes the antipode of the Hopf algebra $H(R)$.

(2) There exists a left $H(R)$-module algebra structure on $A_n(R)$ such that

$$h \cdot D = \langle h, D_{(1)} \rangle D_{(0)}$$

for $h \in H(R)$, $D \in A_n(R)$, where $\rho(D) = D_{(0)} \otimes D_{(1)}$.

Define the K-algebra automorphism $\varphi : A_n(R) \to A_n(R)$ by

$$\varphi(x^i) = q^2 x^i, \quad \varphi(\partial_i) = q^{-2}\partial_i \qquad (1 \le i \le n).$$

For $k \ge 0$, following [14, 15], we define the twisted bracket $[\ ,\ ] : A_n(R) \times A_n(R) \to A_n(R)$ by

$$[D, D'] = DD' - \varphi(D'_{(0)})(D'_{(1)} \cdot D) \qquad (D, D' \in A_n(R)).$$

One can verify that, $[\partial_i, f] \in K_R[X]$ for all $f \in K_R[X]$. Using this twisted bracket, the left $A_n(R)$-action on $K_R[X]$ is described as follows:

$$x^i \cdot f = x^i f, \quad \partial_i \cdot f = [\partial_i, f] \qquad (f \in K_R[X]).$$

Futher, the following twisted Leibniz rule holds:

$$\partial_i(fg) = \partial_i(f)g + \varphi(f_{(0)})(f_{(1)} \cdot \partial_i)(g) \qquad (f, g \in K_R[X]).$$

3. Inverse and Direct Images of Modules over Quantum Weyl Algebras

The main purpose of this section is to define a quantum analogue of the inverse and direct images for quantum Weyl algebras. We refer to [4, 5] for the inverse and direct images for the classical Weyl algebra.

Fix another nonnegative integer m. Let V' be an m-dimensional vector space, and $R' : V' \otimes V' \to V' \otimes V'$ a Hecke symmetry for q. For the algebras $A_m(R')$ the variables and derivatives are denoted by $y^1, \cdots, y^m$ and $\partial'_1, \cdots \partial'_m$, respectively.

Let $F : \Omega(R') \to \Omega(R)$ be a DG-algebra morphism. Thus, in particular, the restriction of F to $K_{R'}[Y]$ is a K-algebra morphism from $K_{R'}[Y]$ to $K_R[X]$. Then $K_R[X]$ has a right $K_{R'}[Y]$-module structure via F.

Let M be a left $A_m(R')$-module (so M is also a left $K_{R'}[Y]$-module).

DEFINITION 3.1. The *inverse image* of M under F is the left $K_R[X]$-module

$$F^*M = K_R[X] \otimes_{K_{R'}[Y]} M.$$

THEOREM 3.2. *For any left $A_m(R')$-module M, the inverse image F^*M of M under F is a left $A_n(R)$-module with the action defined by*

$$x^i \cdot (f \otimes u) = x^i f \otimes u,$$
$$\partial_i \cdot (f \otimes u) = \partial_i(f) \otimes u + (\partial_i f - [\partial_i, f])(F^l) \otimes \partial_l' u$$

for $f \in K_R[X]$, $u \in M$, where $F^l = F(y^l)$.

The inverse image $F^*A_m(q, P')$ naturally becomes a $A_n(q, P)$-$A_m(q, P')$ bimodule. Following classical notation, we denote this bimodule by $D_{X\to Y}$. Then it follows that

$$F^*M \cong D_{X\to Y} \otimes_{A_m(q,P')} M$$

for any left $A_m(q, P')$-module M.

Given a DG-algebra morphism $F : \Omega(q, P') \to \Omega(q, P)$, we obtain a DG-algebra morphism $\Omega(q^{-1}, (P')^t) \to \Omega(q^{-1}, P^t)$ such that $y^j \mapsto F^j$ for $1 \leq j \leq m$, where P^t and $(P')^t$ are the transposed matrix of P and P', respectively. We also denote this morphism by F. Then by the above way we obtain the $A_n(q^{-1}, P^t)$-$A_m(q^{-1}, (P')^t)$-bimodule $D_{X\to Y}(= F^*A_m(q^{-1}, (P')^t))$. Define $D_{Y\leftarrow X}$ to be the $A_m(q, P')$-$A_n(q, P)$ bimodule such that $D_{Y\leftarrow X} = D_{X\to Y}$ as a K-vector space, and that $A_m(q, P')$-$A_n(q, P)$ bimodule action is defined by

$$D' * v * D = \tau(D) \cdot v \cdot \tau(D')$$

for $D \in A_n(q, P), D' \in A_m(q, P')$ and $v \in D_{Y\leftarrow X}$, where τ is the K-algebra anti-isomorphism $\tau : A_n(q, P) \to A_n(q^{-1}, P^t)$ such that

$$\tau(x^i) = x^i, \quad \tau(\partial_i) = -q^{-2(n-i+1)}\partial_i \qquad (1 \leq i \leq n),$$

and $\cdot$ denotes the $A_n(q^{-1}, P^t)$-$A_m(q^{-1}, (P')^t)$ bimodule action on $D_{X\to Y}$.

DEFINITION 3.3. Let M be a left $A_n(q, P)$-module. The *direct image* F_*M of M under F is the left $A_m(q, P')$-module $D_{Y\leftarrow X} \otimes_{A_n(q,P)} M$.

DEFINITION 3.4. Fix nonnegative integers n and m. Let $P = (p_{ij})_{1\leq i,j\leq n+m}$ be an $(n + m) \times (n + m)$ matrix as in Example 1.3. For $\Omega(q, P)$, the variables and the differentials are denoted by

$$x^1, \cdots, x^n, y^1, \cdots, y^m, \quad \text{and} \quad \xi^1, \cdots, \xi^n, \eta^1, \cdots, \eta^m$$

instead of $x^1, \cdots, x^{n+m}$ and $\xi^1, \cdots, \xi^{n+m}$, where y^j (resp. η^j) plays the role of x^{n+j} (resp. ξ^{n+j}) for $1 \leq j \leq m$.

(1) Denote by P' the $m \times m$ matrix with (i,j)-entry $p_{n+i,n+j}$. The variables and the differentials of $\Omega(q,P')$ are denoted by $y^1,\cdots,y^m$ and $\eta^1,\cdots,\eta^m$, respectively.

Define the DG-algebra morphism $\pi : \Omega(q,P') \to \Omega(q,P)$ by

$$\pi(y^j) = y^j, \quad \pi(\eta^j) = \eta^j \qquad (1 \leq j \leq m).$$

(2) Let P'' be the $n \times n$ matrix with (i,j)-entry $p_{i,j}$. For $\Omega(q,P'')$, we denote the variables and differentials by $x^1,\cdots,x^n$ and $\xi^1,\cdots,\xi^m$, respectively.

Define the DG-algebra morphism $\iota : \Omega(q,P) \to \Omega(q,P'')$ by

$$\iota(x^i) = x^i, \quad \pi(\xi^i) = \xi^i \qquad (1 \leq i \leq n),$$
$$\iota(y^j) = 0, \quad \pi(\eta^j) = 0 \qquad (1 \leq j \leq m).$$

LEMMA 3.5. *Let notation be as in Definition 3.4.*

(1) If M is a finitely generated left $A_m(q,P')$-module, then π^*M is a finitely generated $A_{n+m}(q,P)$-module, and

$$\mathrm{GKdim}_{A_{n+m}(q,P)}(\pi^*M) = \mathrm{GKdim}_{A_m(q,P')}(M) + n.$$

(2) If M is a finitely generated left $B_m(q,P')$-module, then π^*M is a finitely generated $B_{n+m}(q,P)$-module, and

$$\mathrm{GKdim}_{B_{n+m}(q,P)}(\pi^*M) = \mathrm{GKdim}_{B_m(q,P')}(M) + n.$$

In particular, M is holonomic if and only if π^*M is holonomic.

(3) If M is a finitely generated left $A_n(q,P'')$-module, then ι_*M is a finitely generated $A_{n+m}(q,P)$-module, and

$$\mathrm{GKdim}_{A_{n+m}(q,P)}(\iota_*M) = \mathrm{GKdim}_{A_n(q,P'')}(M) + m.$$

(4) If M is a finitely generated left $B_n(q,P'')$-module, then ι_*M is a finitely generated $B_{n+m}(q,P)$-module, and

$$\mathrm{GKdim}_{B_{n+m}(q,P)}(\iota_*M) = \mathrm{GKdim}_{B_n(q,P'')}(M) + m.$$

In particular, M is holonomic if and only if ι_*M is holonomic.

LEMMA 3.6. *Let n, m and r be nonnegative integers, $P = (p_{ij})_{1\leq i,j\leq n}$, $P' = (p'_{ij})_{1\leq i,j\leq m}$ and $P'' = (p''_{ij})_{1\leq i,j\leq r}$ matrices as in Example 1.3. Given two DG-algebra morphisms $F : \Omega(q,P') \to \Omega(q,P)$ and $G : \Omega(q,P'') \to \Omega(q,P')$.*

(1) For a left $A_r(q,P'')$-module M,

$$(F \circ G)^*(M) \cong (F^* \circ G^*)(M) \qquad \text{(as left } A_n(q,P)\text{-modules)},$$

(2) For a left $A_n(q,P)$-module M,

$$(F \circ G)_*(M) \cong (G_* \circ F_*)(M) \qquad \text{(as left } A_r(q,P'')\text{-modules)}.$$

4. Kashiwara's Theorem for Quantum Weyl Algebras

In this section we give an analogue of Kashiwara's theorem for quantum Weyl algebras. See [5, Cor.17.3.2; 4, Thm.V.3.1.6] for Kashiwara's theorem for the Weyl algebras in characteristic zero. We deal with the category of $B_n(q,P)$-modules instead of the category of $A_n(q,P)$-modules.

Throughout this section, we use the notations in Definition 3.4.

Let M be a left $B_{n+m}(q,P)$-module. Following classical notation, we put

$$\Gamma_{[H]}(M) = \{m \in M \mid (y^j)^s m = 0 \ (j = 1, \cdots, m) \text{ for some } s \in N\},$$

which becomes a $B_{n+m}(q,P)$-submodule of M. In classical case, H denotes the hyperplane $\{y^1 = \cdots = y^m = 0\}$. We say that M is *supported* by H if $M = \Gamma_{[H]}(M)$. Put

$$M_0 = \{u \in M \mid y^j u = 0 \quad (j = 1, \cdots, m)\},$$

which is a $B_n(q,P'')$-submodule of $\Gamma_{[H]}(M)$.

Denote by $\mathcal{M}^{n+m}$ (resp. $\mathcal{M}^n$) the category of left modules over $B_{n+m}(q,P)$ (resp. $B_n(q,P'')$), and the full subcategory of $\mathcal{M}^{n+m}$ (resp. $\mathcal{M}^n$) consisting of all finitely generated modules is denoted by $\mathcal{M}^{n+m}_{fg}$ (resp. $\mathcal{M}^n_{fg}$). And, $\mathcal{H}^{n+m}$ (resp. $\mathcal{H}^n$) denotes the full subcategory of $\mathcal{M}^{n+m}_{fg}$ (resp. $\mathcal{M}^n_{fg}$) whose objects are holonomic modules. We denote the full subcategory of $\mathcal{M}^{n+m}$ (resp. $\mathcal{M}^{n+m}_{fg}$, $\mathcal{H}^{n+m}$) consisting of $B_{n+m}(q,P)$-modules supported by H by $\mathcal{M}^{n+m}_H$ (resp. $\mathcal{M}^{n+m}_{fg,H}$, $\mathcal{H}^{n+m}_H$).

Note that, if M is a left B_n-module, then $\iota_* M$ is naturally a left B_{n+m}-module.

THEOREM 4.3. *Let $\iota : \Omega(q,P) \to \Omega(q,P'')$ be the DG-algebra morphism defined by*

$$\iota(x^i) = x^i, \quad \iota(y^j) = 0, \quad \iota(\xi^i) = \xi^i, \quad \iota(\eta^j) = 0 \qquad (1 \le i \le n, 1 \le j \le m).$$

The functor ι_* defines an equivalence of the category $\mathcal{M}^n$ (resp. $\mathcal{M}^n_{fg}$, $\mathcal{H}^n$) with the category $\mathcal{M}^{n+m}_H$ (resp. $\mathcal{M}^{n+m}_{fg,H}$, $\mathcal{H}^{n+m}_H$). Furthermore, its inverse is the functor $M \mapsto M_0$.

5. Preservation of Holonomicity

In this section we consider whether, for any DG-algebra morphism F, the inverse and direct image functors F^* and F_* preserve holonomicity. We continue to use the notations in Definition 3.4.

We define basic DG-algebra morphisms including generalizations of the morphisms in Definition 3.4.

DEFINITION 5.1. Assume that $1 \leq r \leq n$. Let $P = (p_{ij})_{1 \leq i,j \leq n}$ be a $n \times n$ matrix as in Example 1.3. Given an r-tuple $\mathbf{i} = (i_1, \cdots, i_r)$ with $i_1 < \cdots < i_r$, we denote by $P_{\mathbf{i}}$ the $r \times r$ matrix whose (k,l)-entry is p_{i_k, i_l}. The generators of $\Omega(q, P_{\mathbf{i}})$ is denoted by $y^1, \cdots, y^r, \eta^1, \cdots, \eta^r$.

(1) The DG-algebra morphism $\pi_{\mathbf{i}} : \Omega(q, P_{\mathbf{i}}) \to \Omega(q, P)$ is defined by

$$\pi_{\mathbf{i}}(y^s) = x^{i_s}, \quad \pi_{\mathbf{i}}(\eta^s) = \xi^{i_s} \qquad (1 \leq s \leq r).$$

(2) The DG-algebra morphism $\iota_{\mathbf{i}} : \Omega(q, P) \to \Omega(q, P_{\mathbf{i}})$ is defined by

$$\iota_{\mathbf{i}}(x^{i_s}) = y^s, \quad \iota_{\mathbf{i}}(\xi^{i_s}) = \eta^s \qquad (1 \leq s \leq r),$$
$$\iota_{\mathbf{i}}(x^i) = 0, \quad \iota_{\mathbf{i}}(\xi^i) = 0 \qquad (i \notin \{i_1, \cdots, i_r\}).$$

(3) For an n-tuple $\mathbf{c} = (c_1, \cdots, c_n)$ such that each $c_i \in K \setminus \{0\}$, we define the DG-algebra morphism $m_{\mathbf{c}} : \Omega(q, P) \to \Omega(q, P)$ by

$$m_{\mathbf{c}}(x^i) = c_i x^i, \quad m_{\mathbf{c}}(\xi^i) = c_i \xi^i \qquad (1 \leq i \leq n).$$

(4) Assume that $n = 1$. For $0 \neq c \in K$, we define the DG-algebra morphism $E_c : \Omega(q, P) \to K$ by

$$E_c(x) = c, \quad E_c(\xi) = 0.$$

PROPOSITION 5.2. *The functors $(\pi_{\mathbf{i}})^*$, $(\pi_{\mathbf{i}})_*$, $(\iota_{\mathbf{i}})^*$, $(\iota_{\mathbf{i}})_*$, $(m_{\mathbf{c}})^*$, $(m_{\mathbf{c}})_*$ $(E_c)^*$ and $(E_c)_*$ preserve the holonomicity.*

Finally we consider the preservation of holonomicity under the inverse and direct images in the simplest case.

From now on, if P is the $n \times n$ matrix whose entries are all 1, we write Ω_n for $\Omega(q, P)$ and denote $A_n(q, P)$ by A_n^q.

PROPOSITION 5.3. *Let $F : \Omega_m \to \Omega_n$ be a DG-algebra morphism. Then F is a composition of the DG-morphisms in Definition 5.1.*

Combining Proposition 5.3 with Proposition 5.2 we obtain the following result:

THEOREM 5.4. *Let $F : \Omega_m \to \Omega_n$ be a DG-algebra morphism. We regard F^* (resp. F_*) as a functor from the category of left A_m^q-modules (resp. A_n^q-modules) to that of left A_n^q-modules (resp. A_m^q-modules). Then, both the functor F^* and F_* preserve the holonomicity.*

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WEAKLY SYMMETRIC ALGEBRAS INDUCED FROM REPETITIVE ALGEBRAS

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Abstract. We shall sketch some relateions between the following two classes of selfinjective algebras: one is of Hochschild extension algebras of some algebras by their injective cogenerators, and the other is of selfinjective algebras obtained as orbit algebras of some repetitive algebras.

1. Preliminaries

Let K be a fixed field and A a finite dimensional basic and connected associative K-algebra with an identity, unless otherwise stated. By modules we mean finitely generated left modules, and denote by $\operatorname{mod} A$ the category of finitely generated left A-modules and $\underline{\operatorname{mod}}\, A$ the *stable module category* of $\operatorname{mod} A$. Recall that the objects of $\underline{\operatorname{mod}}\, A$ are the objects of $\operatorname{mod} A$, and for any two objects M and N in $\underline{\operatorname{mod}}\, A$ the space of morphisms from M to N in $\underline{\operatorname{mod}}\, A$ is the quotient $\underline{\operatorname{Hom}}_A(M, N) = \operatorname{Hom}_A(M, N)/\mathcal{P}(M, N)$, where $\mathcal{P}(M, N)$ is the subspace of $\operatorname{Hom}_A(M, N)$ consisting of all A-homomorphisms which factorize through projective A-modules. If A is selfinjective then the left socle and the right socle of A coincide, and we denote them by $\operatorname{soc} A$. Two selfinjective algebras A and Λ are said to be *socle equivalent* if the factor algebras $A/\operatorname{soc} A \simeq \Lambda/\operatorname{soc}\Lambda$ are isomorphic.

We denote by $D : \operatorname{mod} A \to \operatorname{mod} A^{op}$ the standard duality $\operatorname{Hom}_K(-, K)$, where A^{op} is the opposite algebra to A. Let α be an automorphism of A.

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For a left A-module M, ${}_\alpha M$ denotes the left A-module obtained from M by changing the operation of A as follows : $a \cdot m = \alpha(a)m$ for each $a \in A$ and $m \in M$. Similarly, for a right A-module N, N_α induced for the right A-module N. A-bimodule Q is said to be a *duality module* if Q is isomorphic to DA_α as A-bimodules for some automorphism α of A, that is, Q defines the duality functors $\mathrm{Hom}_A(-, Q) : \mathrm{mod}\, A \to \mathrm{mod}\, A^{op}$ and $\mathrm{Hom}_{A^{op}}(-, Q) : \mathrm{mod}\, A^{op} \to \mathrm{mod}\, A$. If A is a selfinjective, then A is isomorphic to DA_α as A-bimodules for some automorphism α of A. Such an automorphism α is called a *Nakayama automorphism* of an algebra A. Note that DA_α and DA_β are isomorphic as A-bimodules if $\beta = \theta_u \alpha$ for automorphisms α and β of A and for some invertible element u, where $\theta_u(a) = u^{-1}au$ $(a \in A)$. Let $\{e_i\}_{i=1}^n$ be a complete set of orthogonal primitive idempotents of A. Then there is a permutation π on $\{1, \ldots, n\}$ such that $\mathrm{soc}(Qe_i) \simeq \mathrm{top}(Ae_{\pi(i)})$ for a duality A-module Q. Such a permutation π is called the *Nakayama permutation* of the duality module Q (or the automorphism α). An algebra A is called *symmetric* if A is selfinjective and the Nakayama automorphism of A is an inner automorphism, that is, $A \simeq DA$ as A-bimodules. An algebra A is called *weakly symmetric* if A is selfinjective and the Nakayama permutation of A is the identity permutation. From the definition, a symmetric algebra is weakly symmetric (see [25] for details).

The *repetitive algebra* $\widehat{A}_\alpha$ of A by A-bimodule $Q = \mathrm{Hom}(A, K)_\alpha$ is defined by the following infinite dimensional matrix algebra (locally bounded, without identity) :

$$\widehat{A}_\alpha = \begin{pmatrix} \ddots & \ddots & & & 0 \\ & A_{m-1} & Q_{m-1} & & \\ & & A_m & Q_m & \\ & & & A_{m+1} & \ddots \\ 0 & & & & \ddots \end{pmatrix}.$$

Here, $A_m = A$ and $Q_m = Q$ for all $m \in \mathbb{Z}$ and, by definition, the matrices in $\widehat{A}_\alpha$ have only finitely many nonzero elements, the addition is the usual addition of matrices, and the multiplication is induced by the canonical maps $A \otimes_A Q \to Q$, $Q \otimes_A A \to Q$ and $Q \otimes Q \to 0$ [9]. If Q is the standard duality module, namely α is the identity map, then $\widehat{A}_\alpha$ is simply denoted by $\widehat{A}$. An automorphism φ of $\widehat{A}_\alpha$ is said to be *positive* if $\varphi(A_m) \subseteq \sum_{i \geq m}(A_i \oplus Q_i)$ for all $m \in \mathbb{Z}$ and an isomorphism $\varphi : \widehat{A}_\alpha \to \widehat{A}_\beta$ is said to be *of degree zero* if $\varphi(A_m) = A_m$ for all $m \in \mathbb{Z}$, where α and β are automorphisms of A.

2. Automorphisms of repetitive algebras of degree zero

In the representation theory of finite dimensional associateive selfinjective algebras over a field K, an important role is played by selfinjective algebras of the form $\widehat{A}/(\varphi\nu)$, where $\widehat{A}$ is the repetitive algebra of an algebra A, (φ) is an automorphism of degree zero, and ν is the Nakayama automorphism of $\widehat{A}$ ([17], [20], [23], [6], and [10]). In this section we are describing the automorphisms of $\widehat{A}$ of degree zero by making use of automorphisms of A and invertible elements of A.

For a given automorphism α of A, we take automorphisms σ_m of A_m and invertible elements u_m of A_m which satisfy the following equations :

$$\sigma_m \alpha = \alpha \sigma_m$$
$$\sigma_{m+1} = \alpha \theta_{u_m} \sigma_m$$

for all $m \in \mathbb{Z}$, where $\theta_{u_m}(a_m) = u_m^{-1} a_m u_m$ for all $a_m \in A_m$ and $m \in \mathbb{Z}$. We set $\boldsymbol{\sigma} = (\sigma_m)$ and $\boldsymbol{u} = (u_m)$.

Definition 2.1. *A K-linear map $\varphi_{\boldsymbol{\sigma},\boldsymbol{u}} : \widehat{A}_\alpha \to \widehat{A}$ is defined by the following two properties.*

(1) $\varphi_{\boldsymbol{\sigma},\boldsymbol{u}}(a_m) = \sigma_m(a_m) \in A_m$ *for* $a_m \in A_m$.

(2) $\varphi_{\boldsymbol{\sigma},\boldsymbol{u}}(f_m) = f_m \sigma_m^{-1} L_{\alpha(u_m)} \in DA_m$ *for* $f_m \in (DA_\alpha)_m$.

Here, $L_{\alpha(u_m)} : A_m \to A_m$ *is the left multiplication map of* $\alpha(u_m)$*, that is,* $L_{\alpha(u_m)}(a_m) = \alpha(u_m) a_m$ *for* $a_m \in A_m$.

Obviously, $\varphi_{\boldsymbol{\sigma},\boldsymbol{u}}$ is an algebra isomorphism of degree zero. Conversely, it follows from the following proposition that any automorphism φ of degree zero is of the form given in Definition 2.1 [13].

Proposition 2.2. *Let $\varphi : \widehat{A}_\alpha \to \widehat{A}$ be an isomorphism of degree zero such that $\varphi|_{A_m}\alpha = \alpha\varphi|_{A_m}$ for any $m \in \mathbb{Z}$. Then $\varphi = \varphi_{\boldsymbol{\sigma},\boldsymbol{u}}$ for some sets $\boldsymbol{\sigma} = (\sigma_m)$ of automorphisms of A and $\boldsymbol{u} = (u_m)$ of invertible elements of A.*

Consider the case when $\sigma_m = \alpha^m$ and $u_m = 1$. The following fact then follows [24, Proposition 2.3].

Theorem 2.3. *For any automorphism α of A, $\widehat{A}_\alpha$ is isomorphic to the repetitive algebra $\widehat{A}$ of A by the standard duality module.*

3. Weakly symmetric algebras

We consider a basic connected artin K-algebra R with a complete set $\{e_i\}_{i\in I}$, not necessarily finite set, of orthogonal primitive idempotents of R. The algebra R is considered as a (locally bounded) K-*category* whose set of objects is the fixed set $\{e_i\}_{i\in I}$, and the K-module of morphisms $\mathrm{Hom}_R(e_i, e_j)$ from e_i to e_j is equal e_jRe_i for each $i, j \in I$. We first recall some definitions from [3], [7].

A K-category R is called *locally bounded* [3] if R satisfies the following conditions :

(1) distinct objects of R are not isomorphic,

(2) the algebras $R(x, x)$ are local,

(3) for each object x of R, $\sum_{y\in R} |R(x, y)|$ and $\sum_{y\in R} |R(y, x)|$ are finite.

Here, for a K-module V, we denote by $|V|$ its length over K. A functor $F : R \to \Lambda$ between two locally bounded K-categories R and Λ is called a *covering functor* if the induced maps

$$\bigoplus_{F(y)=a} R(x, y) \to \Lambda(F(x), a) \text{ and } \bigoplus_{F(y)=a} R(y, x) \to \Lambda(a, F(x))$$

are isomorphisms for all objects $x \in R$ and $a \in \Lambda$. Let R be a locally bounded K-category and G a group of K-linear automorphisms of R. Assume that G acts *freely* on the objects of R, that is, $gx \neq x$ for each object x of R and for $g \neq 1$ in G. Then, we have the *orbit category* R/G [7] whose object is the orbit of G in the set of objects of R. We denote by x^G the G-orbit of $x \in R$. A morphism $f : a \to b$ between two objects in R/G is a family $f = ({}_yf_x) \in \prod_{x,y} R(x, y)$, where x, y range over a, b, i.e. $x^G = a, y^G = b$, respectively, such that f satisfies the relation $g({}_yf_x) = {}_{gy}f_{gx}$ for all $g \in G$ and all x, y. The composition ef of $f : a \to b$ and $e : b \to c$ in R/G is defined by ${}_zef_x = \sum_{y\in b} {}_ze_{yy}f_x$. Note that $\sum_{y\in b} {}_ze_{yy}f_x$ is a finite sum, because R is locally bounded. Also, we have the canonical covering functor $F: R \to R/G$ which assigns to each object x of R its G-orbit x^G, and to a morphism $\xi \in R(x,y)$ the family $F\xi = ({}_{hy}F\xi_{gx})_{g,h\in G}$ such that ${}_{hy}F\xi_{gx} = g\xi$ if $g = h$ and ${}_{hy}F\xi_{gx} = 0$ if $g \neq h$. Moreover, F is *universal* with respect to the property $Fg = F$ for each $g \in G$, that is, each functor $E: R \to \Lambda$ which satisfies $Eg = E$ for each $g \in G$ admits a unique factorization $E = HF$ for some functor $H: R/G \to \Lambda$. A K-linear functor $E: R \to \Lambda$, such that $Eg = E$ for all $g \in G$, induces an isomorphism $R/G \simeq \Lambda$ if and only if E is surjective on the objects and G acts transitively on the fiber $E^{-1}(a)$ of each object a of Λ. If this is the case, the functor $E: R \to \Lambda$ is called a *Galois covering*. Clearly, the above functor $F: R \to R/G$ is a Galois covering (see [3], [7] for details).

Let A be a finite dimensional K-algebra. For an automorphism φ of repetitive algebra $\widehat{A}$, we denote by (φ) the automorphism group of $\widehat{A}$ generated by φ. An automorphism φ of $\widehat{A}$ is said to be a *category automorphism* if φ fixes some complete set $\mathcal{E}$ of orthogonal primitive idempotents of $\widehat{A}$, namely, φ is a functor of $\widehat{A}$ considered as a category with some object set $\mathcal{E}$ consisting of a complete set of orthogonal primitive idempotents. An automorphism φ of $\widehat{A}$ is said to be *admissible* if φ is a category automorphism of the category $\widehat{A}$ such that (φ) acts freely on the object set and has finitely many (φ)-orbits. Then we can define the orbit category $\widehat{A}/(\varphi)$ which is a finite dimensional selfinjective algebra. Let $Q = DA_\alpha$ and $\nu: \widehat{A}_\alpha \to \widehat{A}_\alpha$ be the Nakayama automorphism of $\widehat{A}_\alpha$, that is, the restrictions of ν to A_m and Q_m induce the identity maps $\nu|_{A_m}: A_m \to A_{m+1}$ and $\nu|_{Q_m}: Q_m \to Q_{m+1}$. In fact, there is an A-bimodule isomorphism

$$\Phi: \widehat{A}_\alpha \to (D\widehat{A}_\alpha)_\nu,\ x \mapsto (y \mapsto \psi(yx))$$

for all matrices $x, y \in \widehat{A}$, where $\psi: \widehat{A} \to K$ is the K-linear map given by $\psi(x) = \sum_{m\in\mathbb{Z}} f_m(1_A)$ for any $x = \bigoplus_{m\in\mathbb{Z}}(a_m \oplus f_m) \in \widehat{A}$ $(a_m \in A_m, f_m \in Q_m)$. Obviously the automorphism $\varphi\nu$ of $\widehat{A}$ is admissible for any automorphism φ of $\widehat{A}$ of degree zero. Then we have trivial extension algebras $A \ltimes DA_{\varphi_m}$ (see Section 2). The following facts relates the orbit algebras to those trivial extension algebras [13].

Theorem 3.1. *For an automorphism φ of $\widehat{A}$ of degree zero, there is an isomorphism $\widehat{A}/(\varphi\nu) \simeq A \ltimes (DA)_{\varphi_m}$ for any $m \in \mathbb{Z}$, where φ_m is the restriction of φ to A_m.*

For a positive automorphism φ of $\widehat{A}$ such that $\varphi\nu$ is admissible, we have the following equivalent conditions which characterize the weakly symmetric algebras of the form $\widehat{A}/(\varphi\nu)$ [13].

Theorem 3.2. *Let φ be a positive automorphism of $\widehat{A}$ such that $\varphi\nu$ is admissible. Then the following statements are equivalent.*

(1) *$\widehat{A}/(\varphi\nu)$ is a weakly symmetric algebra.*

(2) *$\widehat{A}/(\varphi\nu) \simeq A \ltimes (DA)_\alpha$, where α is an automorphism of A with the identity Nakayama permutation.*

(3) *The automorphism φ is of degree zero, and the restriction map φ_m of φ to A_m is an automorphism of A with the identity Nakayama permutation, for all $m \in \mathbb{Z}$.*

Proof. The implication (3) $\Rightarrow$ (2) is an immediate consequence of Theorem 3.1. The Nakayama permutation of DA_α coincides with the Nakayama permutation of $A \ltimes (DA)_\alpha$ ([25, Proposition 2.5.1]), hence (2) implies (1). Finally, (1) $\Rightarrow$ (3) follows from [13, Theorem 2.2]. □

An important class of algebras in the above theorem is given by idnetity automorphisms of A. The following fact is also stated in [6] for wild algebras.

Corollary 3.3. *Let φ be a positive automorphism of $\widehat{A}$ such that $\varphi\nu$ is admissible. Then the following statements are equivalent.*

(1) *$\widehat{A}/(\varphi\nu)$ is a symmetric algebra.*

(2) *$\widehat{A}/(\varphi\nu) \simeq A \ltimes DA$.*

(3) *The automorphism φ is of degree zero, and the restriction map φ_m is an inner automorphism of A for all $m \in \mathbb{Z}$.*

4. Module categories over weakly symmetric algebras

Let Λ be a finite dimensional K-algebra with a complete set $\{e_i | 1 \leq i \leq s\}$ of orthogonal primitive idempotents of Λ. Let I be an ideal of Λ, $A = \Lambda/I$ and e is an idempotent of Λ such that $e + I$ is an identity of A. We may assume that $e = e_1 + \cdots + e_t$ for some $t \leq s$, and $\{e_i | 1 \leq i \leq t\}$ is the subset of

$\{e_i | 1 \leq i \leq s\}$ consisting of all idempotents e_i which are not in I. Then such an idempotent e is uniquely determined by I up to an inner automorphism of Λ, and we call it a *residual identity* of A. Note that $A \simeq e\Lambda e/eIe$, $1-e \in I$ and eI is an A-module.

Theorem 4.1 **([23]).** *Let Λ be a basic and connected finite dimensional selfinjective K-algebra. Let I be an ideal of Λ, $A = \Lambda/I$, and e a residual identity of A. Assume that the ordinary quiver of A has no oriented cycles, $IeI = 0$ and eI is an injective cogenerator in* $\operatorname{mod} A$. *Then Λ is socle equivalent to an algebra $\widehat{A}/G$ where G is an infinite cyclic group of automorphisms of $\widehat{A}$ generated by $\varphi\nu_{\widehat{A}}$, for some positive automorphism φ of $\widehat{A}$. Moreover, if K is an algebraically closed field, then Λ is isomorphic to $\widehat{A}/G$.*

Idea of the proof. If K is an algebraically closed field, the canonical algebra epimorphism $e\Lambda e \to e\Lambda e/eIe$ splits because the Hochschild cohomology $\mathrm{H}^2(e\Lambda e/eIe, e\Lambda e) = 0$ [21, Theorem 3.2]. Therefore $e\Lambda e \simeq e\Lambda e/eIe \ltimes eIe \simeq \widehat{e\Lambda e_\alpha}/(\nu_{\widehat{e\Lambda e}})$. If K is an arbitrary field, however,the canonical algebra epimorphism $e\Lambda e \to e\Lambda e/eIe$ does not split in general (cf. [23, Example 4.2]). Hence we have to consider the selfinjective algebra $\Lambda[I]$ which is $A \oplus I$ as a K-space, and whose multiplication is given by $(b,x)(b',x') = (bb', bx' + xb' + xx')$ for all $b, b' \in A$ and $x, x' \in I$. Note that $I = \{(0,x) | x \in I\}$ is an ideal of $\Lambda[I]$, $IeI = 0$, $eI = r_{\Lambda[I]}(I)$ and the canonical algebra epimorphism $e\Lambda[I]e \to e\Lambda[I]e/eIe$ splits. Then it is shown in [21, Theorem 4.1] that Λ and $\Lambda[I]$ are socle equivalent (see Proposition 4.3 below), and by [23, Theorem3.8] we can construct the infinite cyclic group G such that $\widehat{A}/G \simeq \Lambda[I]$.

For an algebra R, we denote by Γ_R the Auslander-Reiten quiver of R, and by τ_R and τ_R^- the Auslander-Reiten translations DTr and TrD, respectively. We identify the vertex of Γ_R with the isoclass of the corresponding indecomposable R-modules. Ay a component of Γ_R, we mean a connected component of Γ_R. A subquiver $\mathcal{C}$ of Γ_R is called *right stable* (respectively, *left stable*) if τ_R^- (respectively, τ_R) is defined on all modules in $\mathcal{C}$. A subquiver $\mathcal{C}$ is said to be *non-periodic* if $\mathcal{C}$ does not contain τ_R-periodic modules, that is, modules X with $X = \tau_R^m X$ for some $m \geq 1$. Following [18] a subquiver $\mathcal{C}$ of Γ_R is said to be *generalized standard* if $\operatorname{rad}^\infty(X,Y) = 0$ for all modules X and Y in $\mathcal{C}$, where $\operatorname{rad}^\infty(X,Y)$ is the intersection of all finite power $\operatorname{rad}^m(X,Y)$, for $m \geq 1$, of the radical $\operatorname{rad}(X,Y)$ of $\operatorname{Hom}_R(X,Y)$. Finally, the *right annihilator* $r_R(\mathcal{C})$ of a subquiver $\mathcal{C}$ of Γ_R in R is the intersection of the right annihilators $r_R(X)$ of all modules X in $\mathcal{C}$. Clearly, $r_R(\mathcal{C})$ is an ideal of R. Similarly, $l_R(\mathcal{C})$ denotes the *left annihilator* of $\mathcal{C}$ in R.

Now, let H be a basic and connected hereditary K-algebra, Δ the (valued) ordinary quiver of H, and n the number of vertices in Δ. We take a *multiplicity-free tilting H-module* T, that is, $\mathrm{Ext}^1_H(T,T)=0$ and T is a direct sum of n pairwise nonisomorphic indecomposable H-modules (see [2], [8]). Then $A=\mathrm{End}_H(T)$ is called a *tilted algebra of type* Δ.

We consider the repetitive algebra $\widehat{A}$ of A and an infinite cyclic group G acting freely on the objects and with finitely many orbits. Then $R=\widehat{A}/G$ is a selfinjective K-algebra and we have a Galois covering $F:\widehat{A}\to R$ with group G. $F_\lambda:\mathrm{mod}\,\widehat{A}\to\mathrm{mod}\,R$ denotes the push-down functor induced by F [3]. Now, assume that Δ is not a Dynkin quiver. Then

$$\Gamma_{\widehat{A}}=\bigvee_{m\in\mathbb{Z}}(\mathcal{X}_p\vee\mathcal{R}_p)$$

where, for each $p\in\mathbb{Z}$, $\mathcal{R}_p$ is a family of components whose stable parts are tubes if Δ is Euclidean or of type $\mathbb{Z}\mathbb{A}_\infty$ if Δ is not Euclidean (namely, A is wild), and $\mathcal{X}_p$ is a component with the stable part of the form $\mathbb{Z}\Delta$. See [6], and also [1], [17], [14]. Further, it holds that $\mathrm{Hom}_{\widehat{A}}(\mathcal{R}_p,\mathcal{X}_p)=0$, $\mathrm{Hom}_{\widehat{A}}(\mathcal{X}_p\vee\mathcal{R}_p,\mathcal{X}_q\vee\mathcal{R}_q)=0$, $\nu(\mathcal{R}_p)=\mathcal{R}_{p+2}$ and $\nu(\mathcal{X}_p)=\mathcal{X}_{p+2}$ for $p,q\in\mathbb{Z},p>q$. Since a group G of automorphisms of $\widehat{A}$ acts freely on the indecomposable projective $\widehat{A}$-modules, it also acts freely on the components of $\Gamma_{\widehat{A}}$. Moreover, $\widehat{A}$ is *locally-support finite* [5], that is, for each object x of $\widehat{A}$ the full subcategory of $\widehat{A}$ consisting of the supports of indecomposable finitely generated $\widehat{A}$-modules having x in its support has finitely many objects. Consequently, applying [7] and [4], we conclude that the push-down functor $F_\lambda:\mathrm{mod}\,\widehat{A}\to\mathrm{mod}\,R$ is dense and preserves the Auslander-Reiten sequences. Therefore, Γ_R is obtained from $\Gamma_{\widehat{A}}$ by identifying (via F_λ) $\mathcal{R}_p$ with $\mathcal{R}_{p+m}$ and $\mathcal{X}_p$ with $\mathcal{X}_{p+m}$, for some $m\geq 1$ and all $p\in\mathbb{Z}$. Thus Γ_R is of the form

$$F_\lambda(\mathcal{X}_0\vee\mathcal{R}_0)\vee F_\lambda(\mathcal{X}_1\vee\mathcal{R}_1)\vee\cdots\vee F_\lambda(\mathcal{X}_{m-1}\vee\mathcal{R}_{m-1}).$$

The following proposition was proved in [23].

Proposition 4.2. *The following are equivalent for a basic and connected finite dimensional selfinjective K-algebra Λ.*

(1) *The Auslander-Reiten quiver Γ_Λ admits a non-periodic generalized standard right stable full translation subquiver which is closed under successors in Γ_Λ.*

(2) *The selfinjecitve algebra* Λ *is socle equivalent to* $\widehat{A}/(\varphi\nu_{\widehat{A}})$, *where* A *is a tilted* K*-algebra not of Dynkin type,* φ *is a positive automorphism of* $\widehat{A}$ *and* $\varphi\nu$ *is admissible.*

Moreover, if K *is an algebraically closed field, we may replace in the above equivalences "socle equivalence" to "isomorphic".*

Proof. Note that, if Λ_1 and Λ_2 are two selfinjective algebras and Λ_1 is socle equivalent to Λ_2, then we have the induced equivalent functor

$$\Phi : \mathrm{mod}(\Lambda_1/\operatorname{soc}\Lambda_1) \to \mathrm{mod}(\Lambda_2/\operatorname{soc}\Lambda_2),$$

thus for a component $\mathcal{C}$ of Γ_{Λ_1}, $\mathcal{C}'$ and $\Phi(\mathcal{C}')$ are the same forms, where $\mathcal{C}'$ is the stable part of a component $\mathcal{C}$. Moreover, there is a component $\mathcal{D}$ of Γ_{Λ_2} such that $\mathcal{D}' = \Phi(\mathcal{C}')$ because Λ_2 is selfinjective.

Now, assume that (2) holds. For a $F_\lambda(\mathcal{X}_p)$ for $p \in \mathbb{Z}$, it contains a subquiver which satisfies the required condition of (1) (see [23, Proposition5.1]).

Next, we shall prove that (1) implies (2). Assume that Γ_Λ admits a non-periodic generalized standard right stable full translation subquiver $\mathcal{C}$ which is closed under successors in Γ_Λ. Since Λ is selfinjective, $\mathcal{C}$ has no projective modules and oriented cycles. Applying [11] and [19, Lemma 2], we get that $\mathcal{C}$ contains a full translation subquiver $\mathcal{D}$ of the form $(-\mathbb{N})\Delta$, for some finite valued quiver Δ without oriented cycles, which is closed under successors in Γ_Λ. Let $I = r_\Lambda(\mathcal{D})$ be the annihilator of $\mathcal{D}$ in Λ, $A = \Lambda/I$ and e a residual identity of A. It follows from [21, Theorem 5.1 and Proposition 5.3] that $IeI = 0$, Ie is an injective cogenerator in $\mathrm{mod}\, A$, and A is a tilted algebra having a complete slice of type Δ (in the sense of [16, (4.2)]) formed by modules from $\mathcal{D}$. Consequently, it follows from the proposition 4.1 that (1) implies (2). □

Recall that the ideal I of Λ is *deforming* if the ordinary quiver of $A = \Lambda/I$ has no oriented cycles and $eIe = r_{e\Lambda e}(I) = l_{e\Lambda e}(I)$. The following is proved in [22, Theorem 3].

Proposition 4.3. *Let* Λ *be a finite dimensional selfinjective algebra with a deforming ideal* I. *Then the algebra* Λ *and* $\Lambda[I]$ *are stably equivalent.*

In the situation of Proposition 4.2, we set $I = l_\Lambda(\mathcal{D})$. This ideal I is deforming. Therefore, applying Theorem 3.2 and Proposition 4.3, we have the following theorem. See [22].

Theorem 4.4. *The following conditions are equivalent for a weakly symmetric finite dimensional K-algebra Λ.*

(1) *The Auslander-Reiten quiver Γ_Λ of Λ admits a non-periodic generalized standard left (respectively right) stable full translation subquiver which is closed under predecessors (respectively successors) in Γ_Λ.*

(2) *There is a stably equivalence $\underline{\mathrm{mod}}\,\Lambda \simeq \underline{\mathrm{mod}}\,\widehat{A}/(\varphi\nu)$ for a tilted K-algebra A not of Dynkin type, φ a positive automorphism of $\widehat{A}$ and $\varphi\nu$ is admissible.*

(3) *There is a stably equivalence $\underline{\mathrm{mod}}\,\Lambda \simeq \underline{\mathrm{mod}}\,A \ltimes (DA)_\alpha$ for a tilted K-algebra A not of Dynkin type and α is an automorphism of A.*

(4) *The algebra Λ is socle equivalent to $\widehat{A}/(\varphi\nu)$ for a tilted K-algebra A not of Dynkin type, φ a positive automorphism of $\widehat{A}$ and $\varphi\nu$ is admissible.*

(5) *The algebras Λ is socle equivalent to $A \ltimes DA_\alpha$ for a tilted K-algebra A not of Dynkin type and α is an automorphism of A.*

Moreover, if K is an algebraically closed field, we may replace in the above equivalences "socle equivalence" to "isomorphic".

We can not replace "socle equivalence" to "isomorphic" if the base field is arbitrary field. Indeed, we consider the following field K that is not algebraically closed.

Example 4.5 ([12]). Let $K = \mathbb{Z}_2(a, b, c)$ be the rational function field with three variables a, b, c over the prime field $\mathbb{Z}_2$. Let $L = K[X, Y, Z]/(X^2 - a, Y^2 - b, Z^2 - c)$ be the factor ring of the polynomial ring $K[X, Y, Z]$, and x, y and z denote the residue class of X, Y and Z in L, respectively. We can define a 2-cocycle $\kappa : L \times L \to L$ by the following equation:

$$\kappa(x^l y^m z^n, x^{l'} y^{m'} z^{n'}) = x^{l+l'-1} y^{m+m'-1} z^{n+n'-1}(lm'z + mn'xy),$$

where any of l, m, n, l', m', n' is 0 or 1.

We consider the Hochschild extension algebra T of L by $L \simeq DL$ corresponding to the 2-cocycle $\kappa : L \times L \to L$. Observe that the algebra T is weakly symmetric and socle equivalent to the split extension algebra $L \ltimes DL$. It follows from [24] [25, p. 864] that T is stably equivalnet to $L \ltimes DL$. Moreover, T has the following properties: it is a non-symmetric

Hochshild extension algebra by the standard duality module, and it is not isomporphic to selfinjective algebras of the form $\widehat{A}/(\varphi\nu)$ for an algebra A and a positive automorphism φ of $\widehat{A}$ ([13, Proposition 2.4]). □

We considered, in this notes, two kinds of selfinjective algebras, namely, Hochschild extension algebras by the standard duality modules, and selfinjective algebras of the form $\widehat{A}/(\varphi\nu)$ for an algebra A and a positive automorphism φ of $\widehat{A}$. How different are those classes of selfinjecitve algebras? Corollary 3.3 shows that symmetric algebras in the classes coincide and, in fact, we can know the difference of the classes by concrete algebras. See [13]. Those algebras are over an algebraically closed field, excepting for the case in the above example. Thus we conclude this article by posing the following problem.

Problem. Find an algebra over an algebraically closed field which satisfies the above two properties in Example 4.5.

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The Auslander-Reiten quiver, modules over artinian rings, pure-semisimplicity and Artin's problems on division ring extensions[1]

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Abstract. Classification problems in representation theory of artinian rings R and finite dimensional K-algebras R over an algebraically closed field K are studied by means of the Auslander-Reiten quiver of R, the Jacobson radical of the category mod(R) of finitely generated right R-modules and the Ziegler spectrum of R. In particular the tameness of K-algebras R and the pure semisimplicity of right artinian rings R are studied by means of vanishing properties of the transfinite Jacobson radical chain (2.2) of the category mod(R). The pure semisimplicity conjecture for right artinian rings R is discussed in relation with a class of Artin's problems on division ring extensions, and the vanishing of the square of the infinite Jacobson radical (2.3) of mod(R). It is shown how potential counter-examples R to the pure semisimplicity conjecture of length $\ell(R_R)$ two or three can look like, and the shape of their Auslander-Reiten quivers is described.

1. Introduction

Let R be a ring with an identity element. We denote by $J(R)$ the Jacobson radical of R, by Mod(R) the category of right R-modules and by mod(R) the full subcategory of Mod(R) formed by finitely generated right R-modules. If R is a right artinian ring then by Fitting's lemma (see [1]) the endomorphism ring $E_X = \text{End}(X)$ of any indecomposable module X in mod(R) is local, that is, it has a unique two-sided maximal ideal. It follows that the category mod(R) has the Krull-Schmidt property in the sense that every module X of mod(R) has a direct sum decomposition $X \cong X_1 \oplus \ldots \oplus X_m$, where $X_1, \ldots, X_m$ are indecomposable modules of mod(R), and every such a decomposition is unique up to isomorphism.

We recall that R is said to be of **finite representation type** (or representation-finite) if R is both left and right artinian and the number of isomorphism classes of finitely generated indecomposable right (and left) R-modules is finite. A ring R is called **right pure semisimple** [31] if every right R-module is a direct sum of finitely presented modules, or equivalently, if R is right artinian and every right R-module is algebraically compact (i.e. pure-injective) [30], [32]. If R is of finite representation type then R is right pure semisimple. The converse implication called the **pure semisimplicity conjecture** [34], [37] is still an open problem (see [33], [37]–[40]), but it is proved for finite dimensional

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algebras over a field [3] and for arbitrary PI-rings, that is, rings satisfying a polynomial identity (see [15] and [25]).

Let R be a finite dimensional algebra over an algebraically closed field K. We recall that R is said to be of **tame representation type** (or representation-tame) if, for any dimension $d < \infty$, there exists a finite number of $K[t]$-R-bimodules M_j, $1 \leq j \leq n_d$, which are finitely generated and free as left modules over $K[t]$, the polynomial algebra in one variable, and all but finitely many isomorphism classes of indecomposable right R-modules of dimension d are represented by modules of the form $K[t]/(t-\lambda)\otimes_{K[t]} M_j$ for some scalars $\lambda \in K$ and some j. Moreover, if there is a common bound for the numbers n_d of such $K[t]$-R-bimodules M_j for all dimensions d, the representation-tame algebra R is called **domestic** (see [35, Section 14.4] and [46, (2.1)]). Obviously any algebra of finite representation type is representation-tame and domestic.

Given an integer $d \geq 1$ we define $\mu_R(d)$ to be the minimal number n_d of bimodules M_j satisfying the conditions above. The representation-tame algebra R is defined to be of **polynomial growth** if there exists an integer $g \geq 1$ such that $\mu_R(d) \leq d^g$ for all integers $d \geq 2$ The algebra R is said to be of **wild representation type** if for every finitely generated K-algebra Λ there exists an exact representation embedding functor $F : \mathrm{mod}(\Lambda) \to \mathrm{mod}(R)$, i.e. F carries indecomposable modules to indecomposable ones and respects the isomorphism classes, that is, $X \cong Y$ iff $F(X) \cong F(Y)$ (see [35, Chapter 14], [36]). By a result of Drozd [12], the tame-wild dichotomy holds, that is, algebras of tame representation type are not of wild representation type and every finite dimensional K-algebra R over an algebraically closed field K is either of tame representation type or of wild representation type.

Let R be a right artinian ring. We recall that an R-homomorphism $f : X \to Y$ between indecomposable modules X and Y in $\mathrm{mod}(R)$ is said to be **irreducible homomorphism** if f is not an isomorphism and f is not of the form $\sum_{j=1}^{m} f_j g_j$, where $g_j : X \to X_j$, $f_j : X_j \to Y$ are non-isomorphisms and $X_1, \ldots, X_m$ are indecomposable modules in $\mathrm{mod}(R)$ (see [4], [35, Section 11.1]).

With any right artinian ring R we associate the **Auslander-Reiten quiver** Γ_R of R (more precisely, of the category $\mathrm{mod}(R)$) defined as follows. The vertices of Γ_R are the isomorphism classes $[X]$ of the indecomposable modules X in $\mathrm{mod}(R)$. There exists an arrow $[X] \to [Y]$ in Γ_R if and only if there exists an irreducible R-homomorphism $X \to Y$. The quiver Γ_R is obviously a disjoint union of its connected components.

In the representation theory of artinian rings R we are mainly interested in the following problems.

(PR1) *Classify the indecomposable modules in* $\mathrm{mod}(R)$, *list them and parameterize them geometrically by a suitable set.*

(PR2) *Give an explicit description of indecomposable modules* X *in* $\mathrm{mod}(R)$ *and their endomorphism rings* $E_X = \mathrm{End}(X)$ *by means of generators and relations.*

(PR3) *Determine the* E_X-E_Y*-bimodule structure of the hom-group* $\mathrm{Hom}_R(Y, X)$ *and of the extension group* $\mathrm{Ext}^1_R(Y, X)$ *for any pair of indecomposable modules* X *and* Y *in* $\mathrm{mod}(R)$.

(PR4) *Given a module in* $\mathrm{mod}(R)$, *find its decomposition into a direct sum of indecomposable modules.*

(PR5) *Determine the structure of the category* mod(R), *the structure of the Auslander-Reiten quiver* Γ_R *of* mod(R), *describe the shapes of connected components of the quiver* Γ_R *and describe the transfinite Jacobson radical chain* (2.2) *of the category* mod(R) *defined in Section 2.*

(PR6) *Construct a class of right pure semisimple rings* R *of infinite representation type. Investigate their Auslander-Reiten quivers* Γ_R *and the nilpotency of the infinite Jacobson radical* rad_R^∞ (2.3) *of the category* mod(R).

(PR7) *Determine the representation type of* mod(R) (*finite, wild, tame, domestic, polynomial growth*) *in the sense defined above, in case* R *is a finite dimensional* K*-algebra over an algebraically closed field* K.

(PR8) *Does the structure of the category* mod(R) *depend on the topological properties of the Ziegler spectrum* Zsp(R) *of* R (*see* [16] *and Section* 2).

In this article we present criteria for artinian rings R to be of finite representation type in terms of the Auslander-Reiten quiver Γ_R of R (see [4], [35]) and of the Jacobson radical $\mathrm{rad}_R = \mathrm{rad}(\mathrm{mod}(R))$ (2.1) of the category mod(R) (see [4], [35]). We also present a characterization of hereditary artinian rings R of finite representation type in terms of the Coxeter valued diagram associated with R. The existence of such rings R of the non-crystalographical Coxeter-Dynkin type $\mathbb{H}_3$, $\mathbb{H}_4$, $\mathbb{I}_2(5)$ and $\mathbb{I}_2(m)$ for $7 \leq m < \infty$ (see Table 3.7 of Section 3) is discussed in relation with a class of generalized Artin's problems for division ring extensions (see [26], [37]-[39]).

We show in Theorem 2.8 that tame domestic strongly simply connected finite dimensional algebras R over an algebraically closed field can be characterized by the vanishing properties of the transfinite chain (2.2) of the Jacobson radical rad_R of the module category mod(R) defined in Section 2. We also look at the generically tameness [10] and pure semiesimplicity of artinian rings in a connection with their Ziegler spectrum of Zsp(R) of R.

In Sections 4 and 5 we discuss the open problem of existence of representation-infinite right pure semisimple hereditary rings and local rings of length three and two, respectively. We describe in Section 5 a class of generalised Artin's problems for division ring extensions in connection with existence of right artinian local rings R for which the Auslander-Reiten quiver Γ_R is infinite and connected. Several open problems and conjectures are also presented.

Throughout this paper we shall use freely the ring and module terminology introduced in [1], and the representation theory terminology introduced in the monographs [4], [14] and [35]. In particular, an artinian ring R is called an **artin algebra** if the center $Z(R)$ of R is a commutative artinian ring and R viewed as a module over $Z(R)$ is finitely generated.

We recall that a ring R is said to be **connected** if R is not decomposable into a product of rings; and R is said to be **basic**, if $R/J(R) \cong F_1 \times \cdots \times F_m$, where $F_1, \ldots, F_m$ are division rings. Without loss of generality we shall assume in this paper that a ring R is connected and basic, if R is right artinian. This will not restrict the generality of our considerations, because any right artinian ring R is Morita equivalent with a connected and basic ring.

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2. A tranfinite Jacobson radical chain, tameness, pure semisimlicity and Ziegler spectrum

Throughout this section we suppose that R is a connected basic right artinian ring. Following Mitchell [21] we view the category $\mathrm{mod}(R)$ as a "ring with several objects". In particular, by the **Jacobson radical** of the category $\mathrm{mod}(R)$ (in the sense of Kelly [17]) we shall mean the two-sided ideal

$$\text{(2.1)} \qquad \mathrm{rad}_R = \mathrm{rad}(\mathrm{mod}\, R)$$

of $\mathrm{mod}(R)$ consisting of all R-homomorphisms $f : X \to Y$ in $\mathrm{mod}(R)$ such that the R-endomorphism $\mathrm{id}_X - gf$ is invertible for every $g \in \mathrm{Hom}_R(Y, X)$, or equivalently, rad_R is the intersection of all two-sided maximal ideals of the category $\mathrm{mod}(R)$. Given two modules X and Y in $\mathrm{mod}(R)$ we set

$$\mathrm{rad}_R(X,Y) = \mathrm{rad}_R \bigcap \mathrm{Hom}_R(X,Y)$$

In particular $\mathrm{rad}_R(X,X)$ is the Jacobson radical of the endomorphism ring $\mathrm{End}(X)$ of X.

It follows from Mitchell [21] (see also [4] and [35]) that if R is right artinian then rad_R is generated by all non-invertible homomorphisms between indecomposable modules in $\mathrm{mod}\, R$. This follows from the well-known fact that if $\mathrm{End}(X)$ is a local ring and Y is an indecomposable module then the group $\mathrm{rad}_R(X,Y)$ consists of all non-invertible homomorphisms (see [2], [4], [21], [31, Lemma 1.1]).

It was shown in [46] and [43] that it is useful to study the representation type properties of $\mathrm{mod}\, R$ by the vanishing properties of the transfinite Jacobson radical chain

(2.2)

$$\begin{array}{lllllllll} \mathrm{mod}\, R & \supseteq & \mathrm{rad}(\mathrm{mod}\, R) & \supseteq & \mathrm{rad}^2(\mathrm{mod}\, R) & \supseteq & \ldots \supseteq & \mathrm{rad}^\infty(\mathrm{mod}\, R) & \supseteq \\ & & & \supseteq & (\mathrm{rad}^\infty(\mathrm{mod}\, R))^2 & \supseteq & \ldots \supseteq & (\mathrm{rad}^\infty(\mathrm{mod}\, R))^\infty & \supseteq \ldots \end{array}$$

where $\mathrm{rad}^j(\mathrm{mod}\, R)$ is the j-th power of the Jacobson radical $\mathrm{rad}(\mathrm{mod}\, R)$ of the category $\mathrm{mod}\, R$, for $j \geq 1$,

$$\text{(2.3)} \qquad \mathrm{rad}_R^\infty = \mathrm{rad}^\infty(\mathrm{mod}\, R) = \bigcap_{j=1}^{\infty} \mathrm{rad}^j(\mathrm{mod}\, R)$$

$(\mathrm{rad}^\infty(\mathrm{mod}\, R))^m$ is the m-th power of the ideal $\mathrm{rad}^\infty(\mathrm{mod}\, R)$, for $m \geq 1$, and

$$\text{(2.4)} \qquad (\mathrm{rad}_R^\infty)^\infty = (\mathrm{rad}^\infty(\mathrm{mod}\, R))^\infty = \bigcap_{m=1}^{\infty} (\mathrm{rad}^\infty(\mathrm{mod}\, R))^m$$

The higher powers of $(\mathrm{rad}^\infty(\mathrm{mod}\, R))^\infty$ are defined in a natural way.

In this section we show a role of the transfinite Jacobson radical chain (2.2) in the distinction between the finite representation type, right pure semisimplicity, domestic tame representation type and non-domestic tame representation type (see [18], [40], [43] and [46]).

Similarly to the theory of commutative local rings an important role in the study of the category mod(R) is played by a minimal set of generators of the Jacobson radical rad_R. They are represented by irreducible homomorphisms between indecomposable modules (see Introduction), because an R-homomorphism $f : X \to Y$ between indecomposable modules X and Y in mod(R) is irreducible if and only if f belongs to $\mathrm{rad}_R(X,Y) \setminus \mathrm{rad}_R^2(X,Y)$, or equivalently, the residue class of f in the factor group $\mathrm{rad}_R(X,Y)/\mathrm{rad}_R^2(X,Y)$ is not zero (see [4], [35, Section 11.1]). It follows that the factor ideal $\mathrm{rad}_R/\mathrm{rad}_R^2$ defines the Auslander-Reiten quiver Γ_R of R. Conversely, Γ_R contains a lot of important information abaut the factor $\mathrm{rad}_R/\mathrm{rad}_R^2$, and sometimes uniquely determines the ideal rad_R and the category mod(R).

The reader is referred to [4], [14], [23] and [35] for examples of Auslander-Reiten quivers Γ_R of finite dimensional algebras R. A description of the Auslander-Reiten quiver of the representation-tame hereditary K-algebras

$$\Lambda_1 = \begin{pmatrix} K & K^2 \\ 0 & K \end{pmatrix} \quad \text{and} \quad \Lambda_2 = \begin{pmatrix} K & 0 & 0 & 0 & K \\ 0 & K & 0 & 0 & K \\ 0 & 0 & K & 0 & K \\ 0 & 0 & 0 & K & K \\ 0 & 0 & 0 & 0 & K \end{pmatrix} \subseteq \mathbb{M}_5(K)$$

can be found in [14], [23] and [35, Example 11.109, Theorem 15.51].

The following Auslander's criterion for the finite representation type is of great importance (see [4, Section VIII.2] and [35, Section 11.8] for the proof).

THEOREM 2.5 (Auslander 1978). *Assume that R is a connected artin algebra. If the Auslander-Reiten quiver Γ_R of R has a finite component C then $\Gamma_R = C$ and R is of finite representation type.*

It was proved in [5] that if R is a finite dimensional algebra over an algebraically closed field K of characteristic different from 2 and R is of finite representation type then the Auslander-Reiten quiver Γ_R of R determines R and the category mod(R) uniquely up to Morita equivalence (see also [14]).

For self-injective algebras R this fact was proved earlier by C. Riedtmann (see [5] and [14]). Moreover, in [22] Riedtmann has shown that this fact does not hold for representation-finite K-algebras R, if the characteristic of K is equal to 2.

Later A. Skowroński has constructed in [44, Example 4.7] two finite dimensional self-injective algebras A and B over an algebraically closed field K of characteristic 3 such that A and B are representation-tame of polynomial growth, A and B are not Morita equivalent and their Auslander-Reiten quivers Γ_A and Γ_B are isomorphic. If K is as above and $K\langle X,Y\rangle$ is the K-algebra of polynomials in two noncommuting indeterminates X and Y, then the family of K-algebras

$$R_\lambda = K\langle X,Y\rangle/(X^2,Y^2,XY-\lambda YX), \quad \text{where} \quad \lambda \in K \setminus \{0\} \tag{2.6}$$

has the following properties (see [44]):

(a) R_λ is a local self-injective K-algebra of tame representation type, for any $\lambda \in K \setminus \{0\}$.

(b) There exists an algebra isomorphism $R_\lambda/\mathrm{soc}(R_\lambda) \cong R_\mu/\mathrm{soc}(R_\mu)$ for all $\lambda, \mu \in K \setminus \{0\}$.

(c) All algebras R_λ have isomorphic Auslander-Reiten quiver.

(d) The K-agebras R_λ and R_μ are isomorphic (or equivalently, Morita equivalent) if and only if $\lambda = \mu$ or $\lambda = \frac{1}{\mu}$.

Representation theory of artin algebras is well developed by applying the Auslander-Reiten theory (see [4]). A basic role in this case is played by the fact that for any artin algebra R every indecomposable non-projective module X in $\mathrm{mod}(R)$ admits a left almost split sequence $0 \to X'' \to X' \to X \to 0$, and every indecomposable non-injective module Y in $\mathrm{mod}(R)$ admits a right almost split sequence $0 \to Y \to Y' \to Y'' \to 0$. Since arbitrary artinian rings do not have the above properties, their representation theory is much more complicated.

In relation with the remarks stated above, the following problems seem to be of importance (see [37]–[40]).

PROBLEM 2.7. *Characterize basic connected right artinian rings R for which the Auslander-Reiten quiver Γ_R of R has a finite projective component C, i.e. every vertex $[X]$ of C is represented by an indecomposable projective R-module X.* □

PROBLEM 2.8. *Characterize basic connected right artinian rings R for which the Auslander-Reiten quiver Γ_R of R is a disjoint union of two components, where one is finite and one is infinite.* □

PROBLEM 2.9. *Characterize basic connected right artinian rings R for which the infinite radical* rad_R^∞ *of the category* $\mathrm{mod}(R)$ *is non-zero, whereas its square* $(\mathrm{rad}_R^\infty)^2$ *is zero.* □

If we restrict the consideration to artin algebras R the solution of the Problems 2.7–2.9 follows from Theorem 2.6 and Theorem 2.10 stated below. Namely, it follows from Theorem 2.6 that a basic artin algebra R satisfies the condition required in 2.7 if and only if R is a product of division rings. Furthermore, it follows from Theorems 2.6 and 2.10 that there is no artin algebra R satisfying the conditions required in 2.8 or in 2.9.

In Section 3 we shall discuss the Problems 2.7–2.9 for hereditary right artinian rings R which are not artin algebras.

There is no connected artin algebra R satisfying the conditions required in 2.8 or in 2.9, because of the following result proved in [2], [31] and [6].

THEOREM 2.10. *Assume that R is a connected artin algebra and let* rad_R *be the Jacobson radical of the category* $\mathrm{mod}(R)$. *The following conditions are equivalent:*

(a) *R is of finite representation type.*

(b) *There exists $m \geq 1$ such that* $\mathrm{rad}_R^m = 0$.

(c) $\mathrm{rad}_R^\infty = 0$.

(d) $(\mathrm{rad}_R^\infty)^2 = 0$.

Outline of proof. The implication (a)⇒(b) follows from the well-known fact that there is an equivalence of categories $\mathrm{mod}(R) \cong \mathrm{mod}(\mathcal{A}_R)$, where $\mathcal{A}_R$ is the Auslander ring of $\mathrm{mod}\, R$ (see [4] and [35, Section 11.2]), that is

$$\mathcal{A}_R = \mathrm{End}(Y_1 \oplus \cdots \oplus Y_r)$$

where $Y_1, \dots, Y_r$ is a complete set of representatives of the isomorphism classes of indecomposable modules in $\mathrm{mod}\, R$. Since the Jacobson radical of the ring $\mathcal{A}_R$ is nilpotent then the two-sided ideal $\mathrm{rad}_R = \mathrm{rad}(\mathrm{mod}\, R)$ of $\mathrm{mod}(R)$ is also nilpotent and (b) follows.

The implications (b)⇒(c)⇒(d) are obvious. In order to prove (c)⇒(a) note that given an indecomposable module X in $\mathrm{mod}\, R$ the covariant functor $h^X = \mathrm{Hom}_R(-, X) : \mathrm{mod}\, R \to \mathcal{Ab}$ from $\mathrm{mod}\, R$ to the category $\mathcal{Ab}$ of abelian groups is of finite length, because of the sequence

$$h^X \supseteq \mathrm{rad}_R(-,X) \supseteq \mathrm{rad}_R^2(-,X) \supseteq \dots \supseteq \mathrm{rad}_R^{m-1}(-,X) \supseteq \mathrm{rad}_R^m(-,X) = 0$$

for which the factors $\mathrm{rad}_R^{j-1}(-,X)/\mathrm{rad}_R^j(-,X)$ are semisimple of finite length (apply [2]). Then an application of the main result of [2] yields (a). The implication (d)⇒(a) is the main result of [6] and will complete the proof. □

PROBLEM 2.11. *We do not know if Theorem* 2.10 *remains valid for arbitrary artinian PI rings R (see* [40, Problem 3.3]) and [43, Problem 4.6]). □

Now we shall present a relation established in [43] between domesticy and some vanishing properties of the transfinite radical chain (2.2) for a class of finite dimensional K-algebras of tame representation type.

Following [45] the algebra R is said to be **strongly simply connected** if the Gabriel quiver of R (see [4]) has no oriented cycle and for any convex subcategory C of R, the Hochschild cohomology group $H^1(C) = H^1(C, C)$ vanishes.

The following analogue of Theorem 2.10 for finite dimensional K-algebras was proved in [43, Theorem 3.3].

THEOREM 2.12. *Assume that K is an algebraically closed field and R is a connected finite dimensional K-algebra and let* rad_R *be the Jacobson radical* (2.1) *of the category* $\mathrm{mod}(R)$.

(a) *If the square of the infinite radical* $(\mathrm{rad}_R^\infty)^\infty$ (2.3) *is zero then R is of tame representation type.*

(b) *If R is strongly simply connected then the following conditions are equivalent.*

(i) *The algebra R is representation-tame and domestic.*

(ii) $(\mathrm{rad}_R^\infty)^m = 0$ *for some* $m \geq 1$.

(iii) $(\mathrm{rad}_R^\infty)^\infty = 0$.

(iv) *The ideal* rad_R^∞ *is right T-nilpotent, that is, for every sequence*

$$X_1 \xrightarrow{f_1} X_2 \to \cdots \to X_m \xrightarrow{f_m} X_{m+1} \to \cdots$$

of modules $X_1, X_2, \dots$ in $\mathrm{mod}\, R$ *connected by homomorphisms $f_1, f_2, \dots$ from* rad_R^∞ *there exists $m \geq 2$ such that* $f_m f_{m-1} \cdots f_2 f_1 = 0$ *(see* [30]).

(v) *The square of the ideal* $(\mathrm{rad}_R^\infty)^\infty$ (2.4) *is zero.* □

PROBLEM 2.13. *We do not know if Theorem* 2.12 *remains valid for arbitrary finite dimensional* K*-algebra* R (*see* [43, Problems 4.3 and 4.4]).

A positive solution of the Problem 2.13 for the class of special biserial algebras R was recently given by J. Schröer [29].

We finish this section by some remarks on generically tame rings R studied in [10] in connection with Ziegler spectrum $\mathrm{Zsp}(R)$ of R. Assume that R is an arbitrary right artinian ring. Following [10] a right R-module M is said to be **generic** if M is indecomposable of infinite length and M is of finite endolength, that is, M viewed as a left module over the endomorphism ring $\mathrm{End}(M)$ is of finite length. The ring R is said to be right **generically tame** if for each $d \in \mathbb{N}$ there are only finitely many isomorphism classes of generic right R-modules of enolength d.

It was shown in [10, Theorem 4.4] that any finite dimensional K-algebra R over an algebraically closed field K is representation-tame if and only if R is generically tame, and according to [10, Theorem 4.5] the K-algebra R is representation-finite if and only if there is no generic R-module.

In connection with this result Herzog [16, p. 556] has rephrased the second Brauer-Thrall conjecture (see [5], [35]) for artin algebras as follows.

PROBLEM 2.14 (The Second Brauer-Thrall Conjecture). *If R is an artin algebra of infinite representation type, then there exists a generic R-module.* □

It is clear that there is no generic right R-module over any right pure semisimple ring R. Therefore for arbitrary right artinian rings the following problem arises naturally.

PROBLEM 2.15. *Let R be a right artinian ring such that there is no generic right R-module. Is R right pure semisimple?* □

We recall that with any ring R a quasi-compact topological space $\mathrm{Zsp}(R)$, called a (right) **Ziegler spectrum** of R, is associated as follows (see [16] and [19]). The points of $\mathrm{Zsp}(R)$ are the isomorphism classes of indecomposable algebraically compact (pure-injective) right R-modules. A topology basis of $\mathrm{Zsp}(R)$ consists of open sets

$$\mathcal{O}_F = \{M \in \mathrm{Zsp}(R);\ F(M) \neq 0\}$$

where $F : \mathrm{Mod}(R) \to \mathcal{A}b$ runs through all covariant additive functors from the category $\mathrm{Mod}(R)$ to the category $\mathcal{A}b$ of abelian groups such that F commute with arbitrary products and directed limits.

One can easily prove by applying [16] that a right artinian ring R is representation-finite if and only if the topology on $\mathrm{Zsp}(R)$ is discrete. The following problem seems to be interesting.

PROBLEM 2.16. *Give a characterisation of right pure semisimple rings by means of topological properties of the Ziegler spectrum* $\mathrm{Zsp}(R)$ *of* R. □

In connection with generically tameness the following interesting characterisation of tame algebras was given in [20].

THEOREM 2.17. *Let R be a finite dimensional K-algebra over an algebraically closed field K. Given $n \geq 1$ denote by $\mathrm{ind}_n(R)$ the subset of the Ziegler spectrum $\mathbf{Zsp}(R)$ of R defined by the indecomposable right R-modules of dimension n. Then R is representation-tame if and only if for every $n \geq 1$ the Ziegler closure of $\mathrm{ind}_n(R)$ in $\mathbf{Zsp}(R)$ consists only finitely many elements which do not belong to $\mathrm{ind}_n(R)$.* □

3. Hereditary right pure semisimple rings and Artin's problems on division ring extensions

Let us recall that the following **pure semisimplicity conjecture**

($\mathbf{pss}_R$) *A right pure semisimple ring R is of finite representation type*

remains an open problem (see [3], [31], [33], [34], [37]–[40]). The reader is referred to the author's expository paper [39] for a basic background and historical comments on the pure semisimplicity conjecture.

Let us start this section by recalling that the pure semisimplicity conjecture reduces to hereditary right artinian rings, or to the descending chain condition of left ideals for any right pure semisimple ring R.

THEOREM 3.1. *The following statements are equivalent:*

(a) *The pure semisimplicity conjecture holds for every ring R.*

(a′) *The pure semisimplicity conjecture holds for every hereditary ring R.*

(a″) *The pure semisimplicity conjecture holds for every hereditary ring of the form $R_M = \begin{pmatrix} F & {}_FM_G \\ 0 & G \end{pmatrix}$, where F, G are division rings and ${}_FM_G$ is a simple F-G-bimodule ${}_FM_G$.*

(b) *Every right pure semisimple ring R is left artinian.*

(b′) *Every right pure semisimple hereditary ring R is left artinian.*

(b″) *If F, G are division rings and ${}_FM_G$ is a simple F-G-bimodule ${}_FM_G$ such that the ring $R_M = \begin{pmatrix} F & {}_FM_G \\ 0 & G \end{pmatrix}$ is right pure semisimple, then $\dim_F M$ is finite.*

(c) *For any pair of division rings F and G, and for any simple F-G-bimodule ${}_FM_G$ such that* $\dim M_G$ *is finite and* $\dim_F M = \infty$ *one can construct an indecomposable right module of infinite length over the hereditary ring $R_M = \begin{pmatrix} F & {}_FM_G \\ 0 & G \end{pmatrix}$.*

(d) *For any pair of division rings F and G, and for any simple F-G-bimodule ${}_FM_G$ such that* $\dim M_G$ *is finite and* $\dim_F M = \infty$ *one can construct a sequence*

$$X_1 \xrightarrow{f_1} X_2 \to \cdots \to X_m \xrightarrow{f_m} X_{m+1} \to \cdots$$

of indecomposable right R_M-modules $X_1, X_2, \ldots$ of finite length connected by non-isomorphisms $f_1, f_2, \ldots$ such that $f_m f_{m-1} \cdots f_2 f_1 \neq 0$ for any $m > 1$.

Proof. The equivalence of (a′), (a″), (b′), and (b″) was established in [34, Theorem 3.3]. By [37, Theorem 3.6 and Corollary 5.1], the statements (a), (a″), (c) and (d) are equivalent. The implications (a)⇒(a′) and (b)⇒(b″) are obvious. □

Actually we hope that there exists a big class of counter-examples to the pure semisimplicity conjecture. If this is the case then according to Theorem 3.1 there are counter-examples the form $R_M = \begin{pmatrix} F & {}_FM_G \\ 0 & G \end{pmatrix}$, where F, G are division rings and ${}_FM_G$ is a simple F-G-bimodule ${}_FM_G$.

It was shown by the author in [37], [38], [41] and [41] how a construction of such potential counter-examples R_M depends on a generalized Artin problem for division ring extensions, which is much more difficult than the Artin problem for division ring extensions solved by Cohn in [8] and by Schofield in [27]. In order to explain this idea we recall from [39] definitions and some facts on Artin's problems and bimodule Artin's problems.

Following P.M. Cohn [7], [8], [9] (see also [26]) by an Artin's problem on division ring extensions we mean the following one:

ARTIN PROBLEM. *For any pair of numbers $n, m \in \mathbb{N} \cup \{\infty\}$ construct a pair of division rings $F \subseteq G$ such that*

$$\dim G_F = n \quad \textit{and} \quad \dim {}_FG = m$$

where $\dim G_F$ *and* $\dim {}_FG$ *mean the dimension of the F-vector space G viewed as a right and as a left F-module, respectively.* □

We know from a letter of P.M. Cohn that the problem was stated by E. Artin at a Conference in Princeton in 1946 (or even earlier). However, S. Lang and J. Tate who in the edition of Artin's Collected Works list all problems raised by E. Artin (to their knowledge) do not list the above one (see [39, p. 349]).

In [7] and [8] P. M. Cohn has constructed a pair of division rings $F \subseteq G$ such that $\dim G_F = 2$ and $\dim_F G \geq 3$, but he was not able to decide if the dimension $\dim_F G$ is finite or infinite. In fact he has indicated (without detailed proof) another construction of division rings $F \subseteq G$ such that $\dim G_F = 2$ and $\dim_F G = \infty$ (see the second footnote in [8, p. 418]). Five years later, for any integer $n \geq 2$ he constructed in [9] such a pair $F \subseteq G$ with $\dim G_F = n$ and $m = \dim_F G = \infty$ solving the Artin Problem for any $n \geq 2$ and $m = \infty$.

In 1983 A. Schofield (then a student of P.M. Cohn) solved the Artin Problem for any pair $m \geq 2$ and $n \geq 2$ of integers. The result was published in [26] and [27]. In [28], by applying the construction method introduced in [27], Schofield solved a more complicated problem stated in [11] by proving the following result.

THEOREM 3.2. *There exists a pair of division rings $F \subseteq G$ such that*

$$\dim G_F = 2, \quad \dim {}_FG = 3, \quad \dim_G \operatorname{Hom}_F({}_FG_G, F) = 1$$

and the F-dimension of the G-dual space of $\operatorname{Hom}_F({}_FG_G, F)$ *is equal* 2. *In this case there exists a ring isomorphism $G \cong F$.* □

This is a solution of the Artin Problem for $n = 2$, and $m = 3$ completed by some additional conditions (see Problem 3.5 below), and it solves the bimodule Artin problem for the dimension-sequence $(2, 1, 3, 1, 2)$ of length 5 (see [39, Definition 2.3]). The number 1 following 2 in sequence $(2, 1, 3, 1, 2)$ means just the obvious equality $\dim G_G = 1$.

We recall from [11] that the set

$$\mathcal{D} = \mathcal{D}_2 \cup \mathcal{D}_3 \cup \cdots \cup \mathcal{D}_s \cup \cdots$$

of **dimension-sequences** $(d_1, \ldots, d_s)$, $s \geq 2$, is defined inductively to be the minimal set satisfying the following two conditions:

(i) $\mathcal{D}_2 = \{(0,0)\}$ and $\mathcal{D}_3 = \{(1,1,1)\}$,

(ii) if the set $\mathcal{D}_s$ is defined we define $\mathcal{D}_{s+1}$ to be the set of all sequences of the form

$$(d_1, \ldots, d_{i-1}, d_i + 1,\ 1\ , d_{i+1} + 1, d_{i+2}, \ldots, d_s)$$

where $(d_1, \ldots, d_s) \in \mathcal{D}_s$ and $i = 1, \ldots, s-1$.

We note that for each m the set $\mathcal{D}_s$ of dimension-sequences of length s is closed under the action of cyclic permutations. It is easy to see that the sequence $(2,1,3,1,2)$ shown above belongs to $\mathcal{D}_5$. A number theoretic description of the set $\mathcal{D}$ is given in [39, 2.2].

Given an F-G-bimodule ${}_FN_G$ we set

$$\text{l.dim}(N) = \dim_F N \quad \text{and} \quad \text{r.dim}(N) = \dim N_G$$

We define the right dualisation and the left dualisation of ${}_FN_G$ to be the G-F-bimodule

$$N^{*r} = \text{Hom}_G({}_FN_G,\ G) \quad \text{and} \quad N^{*l} = \text{Hom}_F({}_FN_G,\ F)$$

respectively (see [11]). To any bimodule ${}_FM_G$ we associate a sequence of iterated right dualisations of ${}_FM_G$ by setting

$$M^{(0)} = M \quad \text{and} \quad M^{(j)} = (M^{(j+1)})^{*r}$$

for $j \leq -1$. The sequence of iterated left dualisations of ${}_FM_G$ is defined by the formula

$$M^{(j)} = (M^{(j-1)})^{*l} \quad \text{for} \quad j \geq 1$$

Given $s \geq 2$ we set

$$\mathbf{d}_s({}_FM_G) = (d_0^M, \ldots, d_{s-1}^M)$$

where $d_0^M = \dim M_F$ and $d_j^M = \text{r.dim}\, M^{(j)}$ for $j \geq 1$.

A relation between the set $\mathcal{D}$ of dimension-sequences and hereditary rings of finite representation type is given by the following result proved in [11].

THEOREM 3.3. *Let F and G be division rings and assume that ${}_FM_G$ is a non-zero F-G-bimodule. The hereditary ring $R_M = \begin{pmatrix} F & {}_FM_G \\ 0 & G \end{pmatrix}$ is of finite representation type and has precisely $s \geq 3$ pairwise non-isomorphic indecomposable right modules of finite length if and only if the sequence $\mathrm{d}_s({}_FM_G) = (d_0^M, \ldots, d_{s-1}^M)$ belongs to $\mathcal{D}_s$.* □

It was shown in [37] and [39, Section 2.3] that the existence of a pair of division rings F, G and an F-G-bimodule ${}_FM_G$ such that the sequence $\mathbf{d}_s({}_FM_G) = (d_0^M, \ldots, d_{s-1}^M)$ belongs to $\mathcal{D}_s$ is equivalent to an Artin Problem subject to some additional conditions given by the relation $\mathbf{d}_s({}_FM_G) \in \mathcal{D}_s$. It follows from Theorem 3.2 that there exists ${}_FM_G$ such that the sequence $\mathbf{d}_5({}_FM_G)$ equals $(2,1,3,1,2)$.

As a consequence of Theorem 3.3 we get (see [11]).

COROLLARY 3.4. *Assume that $F \subseteq G$ is an embedding of division rings such that $\dim G_F \leq \dim {}_FG$, and let $R_G = \begin{pmatrix} F & G \\ 0 & G \end{pmatrix}$.*

(a) *The category* $\mathrm{mod}(R_G)$ *has exactly* 3 *indecomposable modules up to isomorphism if and only if* $F = G$.

(b) *The category* $\mathrm{mod}(R_G)$ *has exactly* 4 *indecomposable modules up to isomorphism if and only if* $\dim G_F = \dim {}_FG = 2$.

(c) *The category* $\mathrm{mod}(R_G)$ *has exactly* 5 *indecomposable modules up to isomorphism if and only if* $\dim G_F = 2$, $\dim {}_FG = 3$ *and* $\dim_G \mathrm{Hom}_F({}_FG_G, F) = 1$, *or equivalently, if and only if the sequence* $\mathrm{d}_5({}_FG_G)$ *is equal to* $(1,3,1,2,2)$. *In this case there exists a ring isomorphism* $G \cong F$. □

It follows from Theorem 3.2 that there exists a pair of division rings $F \subsetneq G$ such that $\mathrm{d}_5({}_FG_G) = (1,3,1,2,2)$ and therefore the category $\mathrm{mod}(\begin{smallmatrix} F & G \\ 0 & G \end{smallmatrix})$ has exactly 5 indecomposable modules up to isomorphism. However the following generalized Artin's problem is still unsolved.

PROBLEM 3.5. *For any* $s \geq 7$ *construct a pair of division rings* $F \subsetneq G$ *such that the category* $\mathrm{mod}(\begin{smallmatrix} F & G \\ 0 & G \end{smallmatrix})$ *has exactly* s *indecomposable modules up to isomorphism, or equivalently, the sequence* $\mathrm{d}_s({}_FG_G) = (d_0^G, \ldots, d_{s-1}^G)$ *belongs to* $\mathcal{D}_s$. □

Following [11, Section 4] we associate with any basic right artinian hereditary ring R the Coxeter valued diagram $(\mathcal{C}_R, \mathrm{m})$ as follows. Let $F_1, F_2 \ldots, F_n$ be division rings such that

$$R/J(R) \cong F_1 \times \cdots \times F_n$$

The diagram $(\mathcal{C}_R, \mathrm{m})$ is the valued quiver with vertices $1, 2, \ldots, n$ corresponding to the division rings $F_1, F_2 \ldots, F_n$. There exists a valued arrow

$$i \bullet \xrightarrow{\ \mathrm{m}_{ij}\ } \bullet\ j$$

in $(\mathcal{C}_R, \mathrm{m})$ if and only if the F_i-F_j-bimodule

$${}_iM_j = F_i(J(R)/J(R)^2)F_j$$

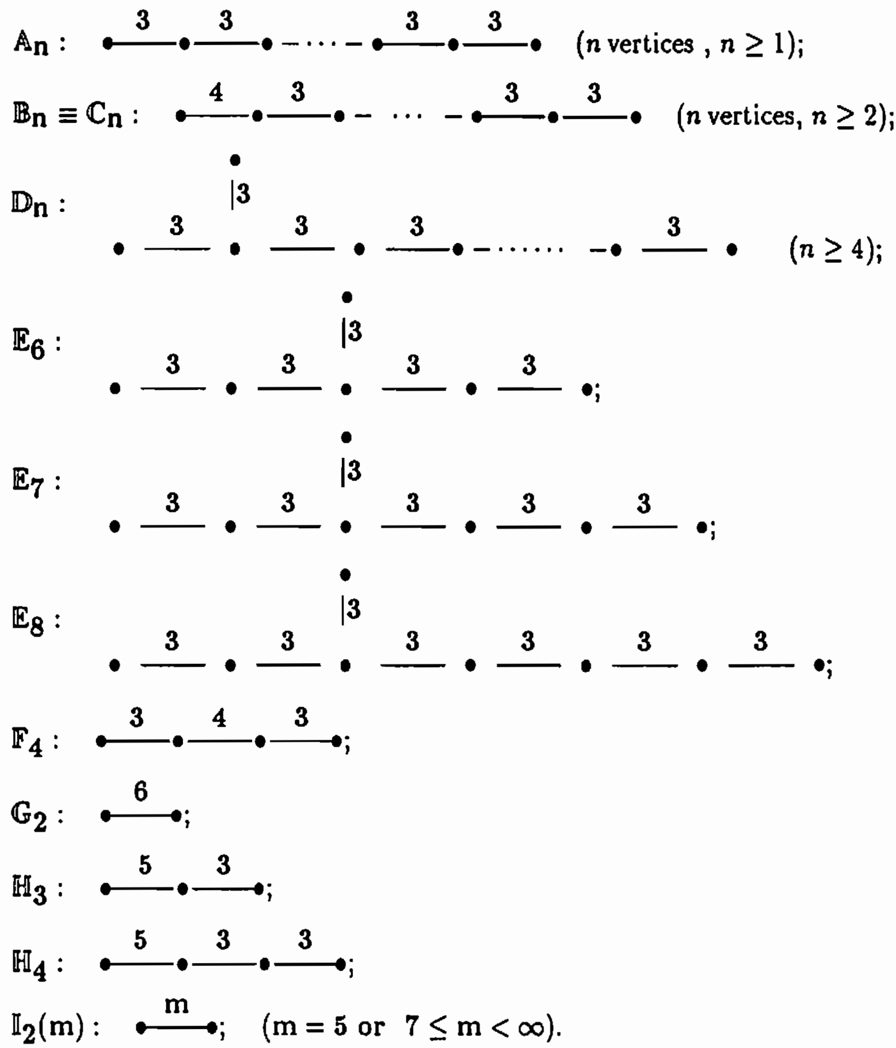
is not zero and the ring $\begin{pmatrix} F_i & {}_iM_j \\ 0 & F_j \end{pmatrix}$ has exactly $\mathrm{m}_{ij} \geq 3$ indecomposable modules up to isomorphism.

The following classification theorem was proved in [11] for representation-finite hereditary rings, and was completed in [41, Theorem] for right pure semisimple hereditary rings.

THEOREM 3.6. (a) *A connected basic hereditary right artinian ring* R *is of finite representation type if and only if the Coxeter valued diagram* $(\mathcal{C}_R, \mathrm{m})$ *associated with* R *above is any of the Coxeter-Dynkin diagrams* $\mathbb{A}_n$, $\mathbb{B}_n (= \mathbb{C}_n)$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$, $\mathbb{F}_4$, $\mathbb{G}_2$, $\mathbb{H}_3$, $\mathbb{H}_4$, $\mathbb{I}_2(p)$ *with* $p \geq 5$, $p \neq 6$, *(with any orientation) which classify the irreducible Coxeter groups (see Table* 3.7 *below).*

(b) *Let* R *be a connected basic hereditary right pure semisimple ring. Then either the ring* R *is of finite representation type and the Coxeter valued diagram* $(\mathcal{C}_R, \mathrm{m})$ *of* R *is any of the Coxeter-Dynkin diagrams* $\mathbb{A}_n$, $\mathbb{B}_n (= \mathbb{C}_n)$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$, $\mathbb{F}_4$, $\mathbb{G}_2$, $\mathbb{H}_3$, $\mathbb{H}_4$, $\mathbb{I}_2(p)$ *with* $p \geq 5$, $p \neq 6$, *of Table* 3.7, *or else* R *is of infinite representation type and the Coxeter valued diagram* $(\mathcal{C}_R, \mathrm{m})$ *contains the arrow* $\bullet \xrightarrow{\ \infty\ } \bullet$. □

TABLE 3.7. **Coxeter-Dynkin diagrams**



We finish this section by recalling from [42, Corollary 3.9] the following result.

PROPOSITION 3.8. (a) *For any Coxeter-Dynkin diagram* Δ *of Table* 3.7 *which is different from* $\mathbb{I}_2(\mathrm{m})$, *where* $\mathrm{m} \geq 7$, *there exists a connected basic hereditary artinian ring* R *such that* $(C_R, \mathrm{m}) = \Delta$.

(b) *If* Δ *is any of the crystallographic Coxeter-Dynkin diagrams of Tables* 3.7 (*that is,* Δ *is different from* $\mathbb{H}_3$, $\mathbb{H}_4$, $\mathbb{I}_2(5)$ *and* $\mathbb{I}_2(\mathrm{m})$, *where* $\mathrm{m} \geq 7$), *then there exists a hereditary artin algebra* R *such that* $(C_R, \mathrm{m}) = \Delta$.

(c) *There is no hereditary artinian PI-ring* R *such that the associated Coxeter valued diagram* (C_R, m) *is any of the non-crystallographic Coxeter-Dynkin diagrams* $\mathbb{H}_3$, $\mathbb{H}_4$, $\mathbb{I}_2(5)$ *and* $\mathbb{I}_2(\mathrm{m})$, *where* $\mathrm{m} \geq 7$. □

It follows from Proposition 3.8 that the existence of a hereditary artinian ring R such that the Coxeter diagram (C_R, m) of R is of the form $\mathbb{I}_2(\mathrm{m})$, where $\mathrm{m} \geq 7$, is an open problem being equivalent with the following one (see [42]). □

RESTRICTED ARTIN PROBLEM 3.9. *For any integer $s \geq 7$ and any dimension-sequence $v = (v_1, \dots, v_s) \in \mathcal{D}_s$, with $v_1 = 1$, construct a pair of division rings $F \subseteq G$ such that the sequence $\mathrm{d}_s({}_FG_G) = (d_0^G, \dots, d_{s-1}^G)$ is equal to v.* □

4. Potential counter-examples with two components in their Auslander-Reiten quiver

In order to formulate a solution of the Problems 2.7, 2.8 and 2.9 for a class of hereditary right artinian rings R we recall some notation introduced in [41].

Following [38], to any F-G-bimodule ${}_FM_G$ for which there exists an integer $m \geq 0$ such that $d_j^M = \mathrm{r.dim}\, M^{(j)}$ is finite for all $j \leq m$ and $d_{m+1}^M = \mathrm{r.dim}\, M^{(m+1)} = \infty$ we associate the **infinite dimension-sequence**

$$(4.1) \qquad \mathrm{d}_{-\infty}({}_FM_G) = (\dots, d_{-j}(M), \dots, d_{-2}(M), d_{-1}(M), d_0(M), \infty)$$

where $d_{-j}(M) = d_{-j+m}^M$ for $j \geq 0$. The following definition was introduced in [41].

DEFINITION 4.2. *The set of* **pure semisimple infinite dimension-sequences** *is the set*

$$\mathcal{DS}_{pss} = \mathcal{DS}_{pss}^{(1)} \cup \mathcal{DS}_{pss}^{(2)}$$

where the sets $\mathcal{DS}_{pss}^{(1)}$ and $\mathcal{DS}_{pss}^{(2)}$ are defined as follows.

The set $\mathcal{DS}_{pss}^{(1)}$ is a minimal set of sequences

$$v = (\dots, v_{-m}, v_{-m+1}, \dots, v_{-2}, v_{-1}, v_0, \infty)$$

with $v_{-j} \in \mathbb{N}$ non-zero for any $j \in \mathbb{N}$, satisfying the following two conditions:

(i) $\omega = (\dots, 2, 2, \dots, 2, 2, 1, \infty) \in \mathcal{DS}_{pss}^{(1)}$;

(ii) *if $v = (\dots, v_{-m}, \dots, v_{-1}, v_0, \infty)$ is a sequence in $\mathcal{DS}_{pss}^{(1)}$ then all sequences of the form*

$$(4.3) \qquad \xi_{-m}(v) = (\dots, v_{-m-1}, 1 + v_{-m}, 1, 1 + v_{-m+1}, v_{-m+2}, \dots, v_{-2}, v_{-1}, v_0, \infty)$$

belong to $\mathcal{DS}_{pss}^{(1)}$, for all $m \geq 1$.

Given a dimension-sequence $u = (\dots, u_{-j}, u_{-j+1}, \dots, u_{-2}, u_{-1}, u_0, \infty)$ in $\mathcal{DS}_{pss}^{(1)}$ we define the **depth** *of u to be the minimal integer $\ell(u) \geq 0$ such that $u_{-j} = 2$ for all $j \geq 1 + \ell(u)$.*

A sequence $v = (\dots, v_{-m}, v_{-m+1}, \dots, v_{-2}, v_{-1}, v_0, \infty)$ belongs to $\mathcal{DS}_{pss}^{(2)}$ if there exists a sequence of positive integers $j_1, j_2, \dots, j_s, \dots$ such that

(a) *for every m the set $\{s \in \mathbb{N};\ j_s = m\}$ is finite,*

(b) $\lim_{s\to\infty} \xi_{-j_s}\xi_{-j_{s-1}} \cdots \xi_{-j_1}(\omega) = v$, *where*

$$\lim_{s\to\infty} w^{(s)} = w$$

means that there exists a sequence $0 < r_1 < r_2 < \cdots < r_s < \cdots$ of positive integers such that $w_0^{(s)} = w_0$, $w_{-1}^{(s)} = w_{-1}, \dots, w_{-r_s}^{(s)} = w_{-r_s}$,

(c) *for every $s \geq 0$ there is $r_s > s$ such that $j_{r_s} \geq 1 + \ell(\xi_{-j_{r_s-1}}\xi_{-j_{r_s-2}} \cdots \xi_{-j_1}(\omega))$.* □

Note that each of the countably many sequences

$$(\dots,2,2,\dots,2,2,3,1,2,\infty),$$
$$(\dots,2,2,\dots,2,2,3,1,4,1,2,\infty),$$
$$(\dots,2,2,\dots,2,2,3,1,4,1,4,1,2,\infty),$$
$$(\dots,2,2,\dots,2,2,3,1,4,1,4,1,4,1,2,\infty),$$
$$\vdots \quad \vdots \quad\quad \vdots \quad\quad\quad\quad\quad\quad \ddots$$

belongs to $\mathcal{DS}^{(1)}_{pss}$.

One shows that the following two sequences $(\dots,1,4,1,4,\dots,1,4,1,4,2,2,2,1,5,\infty)$ and

$$(\dots,y_{-m-1},y_{-m},\dots,y_{-5},y_{-4},4,1,2,4,2,1,4,2,2,2,1,\infty)$$

belong to $\mathcal{DS}^{(2)}_{pss}$, where y_{-4} is the sequence $3,4,1,2,2,4$ of length 6 and y_{-m} is the sequence $3,m,1,2,\dots,2,4$ of length $m+2$, for $m\geq 5$. The sequence

$$w=(\dots,1,4,1,4,\dots,1,4,1,4,1,4,1,3,\infty)$$

does not belong to $\mathcal{DS}_{pss}$ (see [41, Example 4.8]).

By [41, Lemma 4.9], the cardinality of the set $\mathcal{DS}^{(2)}_{pss}$ is the continuum $\mathfrak{c}=2^{\aleph_0}$.

In connection with Problems 2.7–2.9 we have the following result (see [42]).

THEOREM 4.4. *Let $R_M=\begin{pmatrix} F & {}_FM_G \\ 0 & G\end{pmatrix}$, where F, G are division rings and ${}_FM_G$ is a non-zero F-G-bimodule. Assume that every indecomposable non-projective module X in* mod(R) *admits an almost split sequence $0\to X''\to X'\to X\to 0$. Then the following conditions are equivalent:*

(a) *The Auslander-Reiten quiver Γ_{R_M} of R_M is a disjoint union of two components, where one is finite and the other one is infinite.*

(b) *There exist an integer $m\geq 0$ such that* $\mathrm{r.dim}\, M^{(m+1)}=\infty$, $\mathrm{r.dim}\, M^{(j)}$ *is finite for all $j\leq m$ and the Auslander-Reiten quiver Γ_{R_M} of R_M is a disjoint union of two components.*

(c) *There exist an integer $m\geq 0$ such that* $\mathrm{r.dim}\, M^{(m+1)}=\infty$, $\mathrm{r.dim}\, M^{(j)}$ *is finite for all $j\leq m$ and the infinite dimension-sequence* $\mathrm{d}_{-\infty}({}_FM_G)$ (4.1) *belongs to the set* $\mathcal{DS}_{pss}=\mathcal{DS}^{(1)}_{pss}\cup\mathcal{DS}^{(2)}_{pss}$.

(d) *The infinite radical* $\mathrm{rad}^{\infty}_{R_M}$ *of the category* mod(R_M) *is non-zero, whereas its square* $(\mathrm{rad}^{\infty}_{R_M})^2$ *is zero.*

COROLLARY 4.5. *Assume that F, G are division rings and ${}_FM_G$ is an F-G-bimodule such that the infinite dimension-sequence* $\mathrm{d}_{-\infty}({}_FM_G)$ (4.1) *belongs to the set* $\mathcal{DS}_{pss}=\mathcal{DS}^{(1)}_{pss}\cup\mathcal{DS}^{(2)}_{pss}$. *Then the ring $R_M=\begin{pmatrix} F & {}_FM_G \\ 0 & G\end{pmatrix}$ is right pure semisimple and representation-infinite, that is, $R_M=\begin{pmatrix} F & {}_FM_G \\ 0 & G\end{pmatrix}$ is a counter-example to the pure semisimplicity conjecture (see* [37], [38]). □

REMARK 4.6. Since each of the sequences v in $\mathcal{DS}_{pss}$ contains at least one coordinate equal to 1 then the existence of ${}_FM_G$ such that the infinite dimension-sequence $\mathrm{d}_{-\infty}({}_FM_G)=v$ belongs to $\mathcal{DS}_{pss}$ is equivalent to a generalized Artin's problem on division ring extensions (apply the arguments on page 356 of [39]).

We hope that there exists a sequence $v \in \mathcal{DS}_{pss}$, a pair F, G of division rings and an F-G-bimodule ${}_FM_G$ such that $\mathrm{d}_{-\infty}({}_FM_G) = v$. However this is still an open problem. □

5. Special trivial extensions

By [37, Remark 2.4], every basic ring R of length $\ell(R_R)$ two or three which is a potential counter-example to the pure semisimplicity conjecture is either hereditary with two non-isomorphic simple modules (and consequently of the form $R \cong R_M = \begin{pmatrix} F & {}_FM_G \\ 0 & G \end{pmatrix}$ studied in Section 4), or else R is a local ring and $J(R)^2 = 0$. In this section we investigate a class of such local rings R of length $\ell(R_R)$ two, which are trivial extensions of division rings.

Assume that F, G are division rings, ${}_FM_G$ is F-G-bimodule and $\sigma : F \to G$ is a ring isomorphism. The **trivial extension** (see [47])

$$T_M = F \ltimes {}_FM_G \tag{5.1}$$

of F by ${}_FM_G$ is defined to be the ring with an identity element whose additive group is $F \oplus M$ and the multiplication is defined by the formula $(x, m)(x', m') = (xx', xm' + m\sigma(x')$.

Our main result of this section is the following.

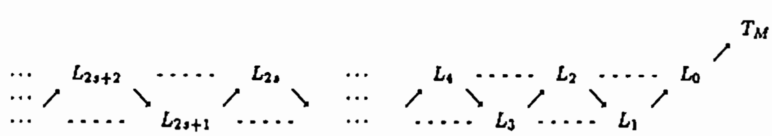
THEOREM 5.2. *Assume that F, G are division rings, ${}_FM_G$ is F-G-bimodule such that* $\dim {}_FM = \infty$, *the associated infinite dimension-sequence* $\mathrm{d}_{-\infty}({}_FM_G)$ (4.2) *belongs to $\mathcal{DS}_{pss} = \mathcal{DS}^{(1)}_{pss} \cup \mathcal{DS}^{(2)}_{pss}$ and $F \cong G$. Let $T_M = F \ltimes {}_FM_G$ be the trivial extension ring* (5.1). *Then the following statements hold.*

(a) *The ring $T_M = F \ltimes_F M_G$ is not self-injective and has infinite global dimension. T_M is a local ring such that the right T_M-module $J(T_M)$ has a direct sum decomposition $J(T_M) \cong L_0^{\dim M_G}$, where L_0 is a unique simple T_M-module. The length $\ell(T_M)$ of the right T_M-module T_M is $1 + \dim M_G$,*

(b) *The ring T_M is right pure semisimple and of infinite representation type.*

(c) *The Auslander-Reiten quiver of the category* $\mathrm{mod}(T_M)$ *is connected and has the form*

(5.3)



where L_0 is a unique simple right T_M-module and L_1 is the injective envelope of L_0.

(d) *For any $m \geq 2$ there exists an almost split sequence in* $\mathrm{mod}(T_M)$

$$0 \longrightarrow L_m \longrightarrow (L_{m-1})^{d^M_{-m}} \longrightarrow L_{m-2} \longrightarrow 0. \tag{3.8}$$

where $d^M_{-m} = \mathrm{r.dim} M^{(-m)}$ for $m \geq 0$. The module L_1 is injective. There is no almost split sequence in $\mathrm{mod}(T_M)$ *starting from an indecomposable module L if and only if L is isomorphic to any of the modules L_0, L_1 or T_M.*

(e) *The infinite Jacobson radical* $\mathrm{rad}^\infty(\mathrm{mod}\, T_M)$ *of the category* $\mathrm{mod}(T_M)$ *is generated by all homomorphisms $\varepsilon : T_M \to L_m$, with $m \geq 0$.*

(f) $(\mathrm{rad}^\infty(\mathrm{mod}\,T_M))^2 = 0$.

(g) *If* $\mathrm{d}_{-\infty}({}_FM_G) = \omega = (\ldots,2,2,\ldots,2,2,2,1,\infty)$ *then* $J(T_M) \cong L_0$, $\ell(T_H) = 2$, $\ell(L_j) = 2j-1$ *for* $j \geq 0$ *and all irreducible homomorphisms* $L_m \to L_{m-1}$ *are surjective.*

(h) *For every* $s \geq 1$ *the number of indecomposable modules in* $\mathrm{mod}(T_M)$ *of length* s *is* 0 *or* 1.

(i) *For any* $m \geq 1$ *there exists an exact sequence*

$$0 \longrightarrow L_0 \xrightarrow{u_m^+} T_M^m \longrightarrow Y_m \longrightarrow 0 \tag{5.4}$$

where u_m^+ *is an irreducible homomorphism and* Y_m *is a unique indecomposable* T_M*-module with* $\ell(Y_m/Y_mJ(T_M)) = m$ *and* $\ell(Y_mJ(T_M)) = m(\dim M_G) - 1)$. *Here* $\ell(Z)$ *means the length of a right* $T_M/J(T_M)$*-module* Z.

Proof. Since $\dim {}_FM = \infty$ then [40, Proposition 4.17], Theorem 6.1 and Corollary 6.2 apply to the hereditary ring $R_M = \begin{pmatrix} F & {}_FM_G \\ 0 & G \end{pmatrix}$.

Note that $J(T_M) = (0, {}_FM_G)$, $R/J(T_M) = F$ and $J(T_M)^2 = 0$. Then the right T_M-module $J(T_M)$ is not projective, T_M is a local ring of infinite global dimension and the ring

$$\Lambda(T_M) = \begin{pmatrix} T_M/J & {}_{(T_M/J)}J_{(T_M/J)} \\ 0 & R/J \end{pmatrix}$$

is isomorphic with R_M, where $J = J(T_M)$ is viewed as a (T_M/J)-(T_M/J)-bimodule in a natural way.

Following Gabriel [13] we associate with T_M the reduction functor

$$\mathbb{F} : \mathrm{mod}(T_M) \longrightarrow \mathrm{mod}(\Lambda(R)) \cong \mathrm{mod}(R_M) \tag{5.5}$$

defined by attaching to any module Y in $\mathrm{mod}(T_M)$ the triple $\mathbb{F}(Y) = (Y', Y'', t)$, where $Y' = Y/YJ$, $Y'' = YJ$ are viewed as right T_M/J-modules and $t : Y' \otimes_{T_M/J} J_{T_M/J} \to Y''_{T_M/J}$ is a T_M/J-homomorphism defined by formula $t(\overline{y} \otimes r) = y \cdot r$ for $\overline{y} = y + J$ and $r \in J$. The triple $\mathbb{F}(Y)$ is viewed as a right $\Lambda(T_M)$-module in a natural way. If $f : Y \to Z$ is a T_M-homomorphism we set $\mathbb{F}(f) = (f', f'')$, where $f'' : Y'' \to Z''$ is the restriction of f to $Y'' = YJ$ and $f' : Y' \to Z'$ is the R/J-homomorphism induced by f.

By standard arguments we easily show that the functor $\mathbb{F}$ has the following properties (see [13] and [4, Section X.2]).

(i) $\mathbb{F}$ is full and establishes a representation equivalence between $\mathrm{mod}(T_M)$ and the category $\mathrm{Im}\,\mathbb{F}$.

(ii) A right $\Lambda(T_M)$-module X belongs to $\mathrm{Im}\,\mathbb{F}$ if and only if X has no non-zero summand isomorphic to a simple projective right $\Lambda(T_M)$-module.

(iii) The functor $\mathbb{F}$ preserves the indecomposability, projectivity and the length.

(iv) $\mathbb{F}$ carries a homomorphism $f : Y \to Z$ to zero if and only if $\mathrm{Im}\, f \subseteq ZJ$.

(v) For any pair Y, Z of indecomposable modules in $\mathrm{mod}(T_M)$ the functor $\mathbb{F}$ induces ring isomorphisms

$$\mathrm{End}(Y)/J\mathrm{End}(Y) \cong \mathrm{End}(\mathbb{F}Y)/J\mathrm{End}(\mathbb{F}Y), \quad \mathrm{End}(Z)/J\mathrm{End}(Z) \cong \mathrm{End}(\mathbb{F}Z)/J\mathrm{End}(\mathbb{F}Z)$$

and an End(Y)/JEnd(Y)-End(Z)/JEnd(Z)-bimodule isomorphism

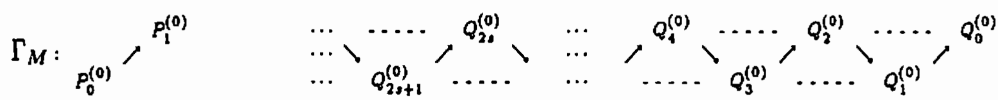
$$\mathrm{Irr}(Y, Z) \cong \mathrm{Irr}(\mathbb{F}Y, \mathbb{F}Z)$$

(vi) The ring T_M is right pure semisimple (resp. of finite representation type) if and only if $\Lambda(T_M) \cong R_M$ is right pure semisimple (resp. of finite representation type).

In particular the functor $\mathbb{F}$ carries the irreducible homomorphisms to irreducible homomorphisms. By [40, Proposition 4.17] (see also Corollary 6.2) the hereditary ring

$$\Lambda(T_M) \cong R_M = \begin{pmatrix} F & {}_FM_G \\ 0 & G \end{pmatrix}$$

is right pure semisimple and representation-infinite. Furthermore, the Auslander-Reiten quiver $\Gamma(\mathrm{mod}\, R_M)$ consists of two connected components: the preinjective one $\mathcal{Q}_M$ and the preprojective one $\mathcal{P}_M$ described in [41, (2.8) and (2.10)]. By [41, Proposition 2.6], the Auslander-Reiten quiver of mod(R_M) has the form



It follows that the ring T_M is right pure semisimple and representation-infinite. Hence T_M is not self-injective, because otherwise T_M should be representation-finite (see [15, Corollary 5.3], [40, Corollary 2.9]) and we get a contradiction.

By the properties (i)–(vi) above the preinjective component $\mathcal{Q}_M$ of $\Gamma(\mathrm{mod}(R_M))$ corresponds to the part of the Auslander-Reiten quiver of mod(T_M). Given $m \in \mathbb{N}$ we consider the T_M-module

$$L_m = \mathbb{F}(Q_m^{(0)})$$

corresponding to $Q_m^{(0)}$ via the functor $\mathbb{F}$. It follows from (iii) that the module L_0 is simple. Since the inclusion $\mathrm{soc}(T_H) = J(T_H) \hookrightarrow T_H$ is an irreducible homomorphism and T_H has a unique simple module up to isomorphism then L_0 is a direct summand of $J(T_M)$ and there is an irreducible homomorphism $L_0 \to T_H$. The preprojective component $\mathcal{P}_M$ of $\Gamma(\mathrm{mod}(R_M))$ consists of two projective modules $(0, G) = P_0 \hookrightarrow P_1 = (F, M_G)$. It follows from the properties (i)–(iii) that $\mathbb{F}(T_H) \cong P_1$ and P_0 is not in the image of $\mathbb{F}$. Consequently the Auslander-Reiten quiver of T_M is connected and has the shape shown in (b).

In view of the properties (i)–(iv) above, the remaining part of the theorem follows from [40, Proposition 4.17] and [38, Proposition 3.6]. The details are left to the reader. □

In connection with [37, Remark 2.4] the following observaion is useful.

COROLLARY 5.6. *If $F \subset G$ are division rings such that $F \cong G$, $\dim_F G = \infty$ and the associated infinite dimension-sequence $\mathrm{d}_{-\infty}({}_FG_G)$ (4.1) of the F-G-bimodule ${}_FG_G$ belongs to $\mathcal{DS}_{pss} = \mathcal{DS}_{pss}^{(1)} \cup \mathcal{DS}_{pss}^{(2)}$ then the local ring $T_G = F \ltimes {}_FG_G$ is a counter-example to the* *pure semisimplicity conjecture of length two. The global dimension of T_G is infinite and the Auslander-Reiten quiver of* $\mathrm{mod}(T_G)$ *is connected.*

Proof. Apply Theorem 5.2. □

REMARK 5.7. Since for any $v = (\ldots, v_{-m}, \ldots, v_{-1}, v_0, \infty) \in \mathcal{DS}_{pss}$ there exists $j \geq 1$ such that $v_{-j} = 1$, then according to [37, Remark 4.5] the existence of an F-G-bimodule ${}_FM_G$ such that $\mathrm{d}_{-\infty}({}_FM_G) = v$ is an infinite version of the Artin problem for division ring extensions studied in [8], [27], [37] and [38] (see [37, Section 4]). In the situation studied in Theorem 5.2 we also assume that $F \cong G$.

We hope that, by applying a modification of the bimodule amalgam rings construction of Schofield [26, Chapter 13], one can construct a division ring embedding $F \subseteq G \cong F$ such that $\mathrm{d}_{-\infty}({}_FG_G) = v$ for some of the dimension-sequences $v \in \mathcal{DS}_{pss}$. □

The following two interesting problems stated in [41, Problem 4.21] and [40, Problem 3.2] remain unsolved.

PROBLEM 5.8. *Assume that F, G are division rings, ${}_FM_G$ is F-G-bimodule such that the associated infinite dimension-sequence $\mathrm{d}_{-\infty}({}_FM_G)$ (3.2) belongs to the set $\mathcal{DS}_{pss} = \mathcal{DS}^{(1)}_{pss} \cup \mathcal{DS}^{(2)}_{pss}$.*

(a) *Find a decomposition of the right R_M-module*

$$L(\mathcal{Q}_M) = \prod_{m\geq 0}^{\infty} Q^{(0)}_m / \bigoplus_{m\geq 0}^{\infty} Q^{(0)}_m \tag{5.9}$$

in a direct sum of indecomposable modules, where $Q^{(0)}_m, Q^{(0)}_1, Q^{(0)}_2, \ldots$ are the preinjective modules shown in [41, (2.8)] *(see also the shape of Γ_M sown above).*

(b) *Give a characterization of F-G-bimodules ${}_FM_G$ for which the R_M-module $L(\mathcal{Q}_M)$ is projective.* □

PROBLEM 5.10 [40]. *Give a characterisation of semiperfect rings R for which every right R-module is pure-projective or pure-injective. Is every such a ring R right artinian, or right pure semisimple?* □

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THE DERIVED PICARD GROUP AND REPRESENTATIONS OF QUIVERS

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ABSTRACT. The derived Picard group DPic(A) of a ring A is the group of auto-equivalences of the derived category $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\,A)$ induced by tilting complexes. When A is either local or commutative, DPic(A) is a product of the usual (noncommutative) Picard group Pic(A) and a cyclic group; in this sense DPic(A) has a noncommutative geometry content.

In this talk we consider the case where A is the path algebra of a finite quiver $\vec{\Delta}$ over a field k (i.e. A is a finite dimensional hereditary k-algebra). There is a natural action of DPic(A) on a certain infinite quiver. This action is faithful when $\vec{\Delta}$ is a tree, and otherwise a connected linear algebraic group may occur as a factor of DPic(A). At any rate we get an effective description of DPic(A); examples will be shown.

In the hereditary case DPic(A) coincides with the group of triangle auto-equivalences of the derived category of A-modules. This means that we can calculate the groups of auto-equivalences of various derived categories occurring, for example, in noncommutative geometry. (Joint work with J. Miyachi.)

1. INTRODUCTION

Let k be a field, A a k-algebra, and let $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\,A)$ be the derived category of A-modules. The derived Picard group $\mathrm{DPic}_k(A)$ is the group of auto-equivalences of $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\,A)$ generated by tilting complexes.

The group $\mathrm{DPic}_k(A)$ becomes interesting when A is nonlocal and noncommutative. This takes us into the realm of noncommutative algebraic geometry.

Today I will talk about the case when A is a finite dimensional k-algebra. For a hereditary algebra A we can completely describe the structure of the group $\mathrm{DPic}_k(A)$. This is a survey of joint work with J. Miyachi. Full details are in the paper [MY].

2. SOME BACKGROUND

Suppose k is a field and A is a (unital, associative) k-algebra. Let $\mathsf{Mod}\,A$ be the category of left A-modules. Recall that Morita Theory tells us what are all k-linear auto-equivalences $F : \mathsf{Mod}\,A \to \mathsf{Mod}\,A$. They are (up to isomorphism) $FM = P \otimes_A M$, where P is a k-central invertible bimodule (i.e. $A \cong P \otimes_A Q \cong Q \otimes_A P$ for some bimodule Q). We call the group generated by the invertible bimodules P (with operation $P \otimes_A Q$) the (noncommutative) Picard group of A, denoted $\mathrm{Pic}_k(A)$.

Remark 2.1. If A is commutative then $\mathrm{Pic}_k(A)$ contains the automorphism group $\mathrm{Aut}_k(A)$ and the usual (commutative) Picard group $\mathrm{Pic}_A(A)$ as subgroups.

Complete version of the manuscript submitted to a journal.

Now let us look at the bounded derived category $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)$. We will not attempt to say too much about it now. All we need to know are a few facts. The objects of $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)$ are the bounded complexes of A-modules

$$M^{\cdot} = \left(\cdots \to 0 \to M^p \xrightarrow{\delta} M^{p+1} \xrightarrow{\delta} \cdots \to M^q \to 0 \to \cdots\right)$$

where $\delta \circ \delta = 0$. The category $\mathsf{Mod}\, A$ embeds in $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)$ as the complexes concentrated in degree 0. And $M[n]$ is the complex with M in degree $-n$. If $T^{\cdot}$ is a complex of bimodules, then one can define the derived tensor product $T^{\cdot} \otimes^{\mathrm{L}}_A M^{\cdot}$. We call $T^{\cdot}$ a tilting complex if the functor $FM^{\cdot} = T^{\cdot} \otimes^{\mathrm{L}}_A M^{\cdot}$ is an auto-equivalence of $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)$. Tilting complexes were considered by Rickard and Keller.

Example 2.2. If P is an invertible bimodule then $P[n]$ is a tilting complex.

Definition 2.3. The derived Picard group $\mathrm{DPic}_k(A)$ is the group of isomorphism classes of tilting complexes, with operation $T^{\cdot} \otimes^{\mathrm{L}}_A S^{\cdot}$.

One reason to study $\mathrm{DPic}_k(A)$ is:

Theorem 2.4 (Y.). *$\mathrm{DPic}_k(A)$ parameterizes the isomorphism classes of dualizing complexes over A.*

We note that $\mathrm{DPic}_k(A)$ contains two natural subgroups: a copy of $\mathbb{Z}$ (represented by the complexes $A[n]$), and $\mathrm{Pic}_k(A)$.

Theorem 2.5 (Y., Zimmermann-Rouquier-...). *If A is either local or commutative, then* $\mathrm{DPic}_k(A) = \mathrm{Pic}_k(A) \times \mathbb{Z}$.

This however is not the general situation:

Example 2.6 (Y). Consider the smallest nonlocal, noncommutative k-algebra $A = \begin{bmatrix} k & k \\ 0 & k \end{bmatrix}$. The bimodule $A^* := \mathrm{Hom}_k(A, k)$ turns out to be a tilting complex. Let τ be the class of $A^*[-1]$ in $\mathrm{DPic}(A)$. Then one can show that $\tau \notin \mathrm{Pic}_k(A) \times \mathbb{Z}$.

This raises:

Problem 2.7. Study the structure of the group $\mathrm{DPic}_k(A)$ and its relation to the geometry of the noncommutative ring A.

Let us write $\mathrm{Out}^{\mathrm{tr}}_k(\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A))$ for the group of k-linear triangle auto-equivalences of $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)$. The next problem is open:

Problem 2.8. Is $\mathrm{DPic}_k(A) = \mathrm{Out}^{\mathrm{tr}}_k(\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A))$?

Recall that A is hereditary if $\mathrm{gl.dim}\, A = 1$, or if every left of right ideal is a projective module.

Theorem 2.9 (Miyachi and Y.). *If A is hereditary then the answer to the previous problem is positive.*

3. Quivers and Path Algebras

We will concentrate on a finite dimensional k-algebra A. Let us assume that k is algebraically closed, and that A is basic (i.e. $A/\mathfrak{p} = k$ for every prime ideal). Since every k-algebra is Morita equivalent to a basic one the latter is really no restriction.

The geometry of A is described by a quiver $\vec{\Delta}$. Here is a nonorthodox definition of $\vec{\Delta}$. The set of vertices of $\vec{\Delta}$ is $\operatorname{Spec} A$, namely the set of prime ideals. Let $\mathfrak{r}$ be the Jacobson radical of A, and for each $\mathfrak{p}_i \in \operatorname{Spec} A$ let $e_i \in A/\mathfrak{r}$ be the corresponding central idempotent. So

$$\mathfrak{r}/\mathfrak{r}^2 = \bigoplus_{i,j} e_i(\mathfrak{r}/\mathfrak{r}^2)e_j \cong \bigoplus_{i,j} \frac{\mathfrak{p}_i \cap \mathfrak{p}_j}{\mathfrak{p}_i\mathfrak{p}_j}$$

as A-A-bimodules. Define $d_{i,j} := \dim_k \frac{\mathfrak{p}_i \cap \mathfrak{p}_j}{\mathfrak{p}_i\mathfrak{p}_j}$. Then there are $d_{i,j}$ arrows $\mathfrak{p}_i \to \mathfrak{p}_j$. The connected components of $\vec{\Delta}$ are called cliques, and they control Öre localization.

Example 3.1. Take $A = \begin{bmatrix} k & k \\ 0 & k \end{bmatrix}$. The prime ideals are $\mathfrak{p}_1 = \begin{bmatrix} 0 & k \\ 0 & k \end{bmatrix}$ and $\mathfrak{p}_2 = \begin{bmatrix} k & k \\ 0 & 0 \end{bmatrix}$. Clearly $\frac{\mathfrak{p}_1 \cap \mathfrak{p}_2}{\mathfrak{p}_1\mathfrak{p}_2} = \begin{bmatrix} 0 & k \\ 0 & 0 \end{bmatrix}$. So the quiver $\vec{\Delta}$ is $\underset{\mathfrak{p}_1}{\bullet} \longrightarrow \underset{\mathfrak{p}_2}{\bullet}$, a Dynkin quiver of type A_2.

On the other hand, given a finite quiver $\vec{\Delta}$ we can construct an algebra, called the path algebra $k\vec{\Delta}$. A path in $\vec{\Delta}$ is either a trivial path e_x for each vertex x, or a sequence of arrows $\alpha_1\alpha_2\cdots\alpha_m$, such that α_{i+1} starts where α_i ends:

$$\underset{x_1}{\bullet} \xrightarrow{\alpha_1} \underset{x_2}{\bullet} \xrightarrow{\alpha_2} \underset{x_3}{\bullet} \xrightarrow{\alpha_3} \underset{x_4}{\bullet}$$

The paths form a basis of $k\vec{\Delta}$ as vector space, and multiplication of paths is by concatenation if possible, or 0.

Example 3.2. Take the quiver $\vec{\Delta} = \underset{x_1}{\bullet} \xrightarrow{\alpha} \underset{x_2}{\bullet}$. Then e_{x_1}, e_{x_2} are orthogonal idempotents; $e_{x_1}\alpha = \alpha = \alpha e_{x_2}$; and $\alpha e_{x_1} = \alpha^2 = e_{x_2}\alpha = 0$. Thus $k\vec{\Delta} \cong \begin{bmatrix} k & k \\ 0 & k \end{bmatrix}$, with

$$e_{x_1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; e_{x_2} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \; \alpha = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Here is a classical structure theorem for finite dimensional algebras.

Theorem 3.3 (Gabriel). *Let A be a basic finite dimensional k-algebra with quiver $\vec{\Delta}$. Then*

1. *$A \cong k\vec{\Delta}/I$ for an ideal $I \subset \mathfrak{a}^2$, where $\mathfrak{a} \subset k\vec{\Delta}$ is the ideal generated by the arrows.*
2. *$I = 0$ iff A is hereditary.*

Some remarks:

Remark 3.4. According to the definitions of Cuntz-Quillen, a finite dimensional algebra A is smooth iff it is hereditary.

Remark 3.5. The surjection $\vec{\Delta} \twoheadrightarrow A$ should be viewed as a noncommutative analog of the Cohen theorem for commutative rings: $k[[t_1, \ldots, t_n]] \twoheadrightarrow A$. Indeed the arrows in $k\vec{\Delta}$ are lifts of a basis of $\mathfrak{r}/\mathfrak{r}^2$.

Remark 3.6. Beilinson showed that the derived category $\mathsf{D}^{\mathrm{b}}(\mathsf{Coh}\,\mathbf{P}^1)$ of coherent sheaves on the projective line $\mathbf{P}^1$ is equivalent to $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,k\vec{\Delta}_2)$, for a certain quiver $\vec{\Delta}_2$. According to Kontsevich-Rosenberg one has $\mathsf{D}^{\mathrm{b}}(\mathsf{Coh}\,\mathbf{NP}^n) \approx \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,k\vec{\Delta}_{n+1})$, where $\mathbf{NP}^n$ is their noncommutative projective n-space, so $\mathrm{Out}_k^{\mathrm{tr}}(\mathsf{D}^{\mathrm{b}}(\mathsf{Coh}\,\mathbf{NP}^n)) \cong \mathrm{DPic}_k(k\vec{\Delta}_{n+1})$.

4. The Auslander-Reiten Quiver

¿From now on $A = k\vec{\Delta}$ and the quiver $\vec{\Delta}$ is connected. An A-module is also called a representation of $\vec{\Delta}$. As customary we write $\mathsf{mod}\,A$ for the category of finitely generated A-modules (=finite dimensional representations).

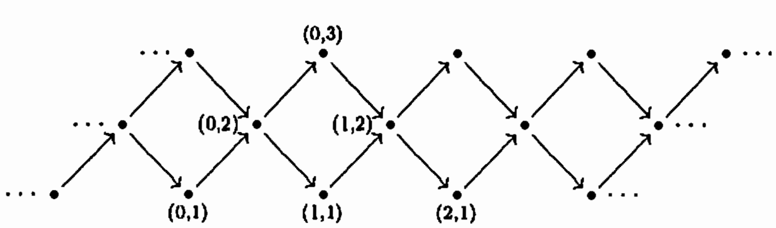
One of the important invariants of the algebra A is the set of isomorphism classes of indecomposable modules. If this set is finite A is said to have finite representation type.

Theorem 4.1 (Gabriel). *A has finite representation type iff $\vec{\Delta}$ is a Dynkin quiver of type A_n, D_n, E_6, E_7 or E_8.*

The Auslander-Reiten quiver $\vec{\Gamma}(\mathsf{mod}\,A)$ is defined as follows. Its vertices are the isomorphism classes of indecomposable A-modules, and there are d arrows $x \to y$, where d is the dimension of the space of irreducible homomorphisms $\mathrm{Irr}(M_x, M_y)$ (this is like $\mathfrak{r}/\mathfrak{r}^2$).

Happel showed how to define the Auslander-Reiten quiver $\vec{\Gamma} = \vec{\Gamma}(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A))$. This is an infinite quiver. Usually it is very complicated, but it contains a nice subquiver denoted $\vec{\mathbb{Z}}\vec{\Delta}$ that is easy to describe ($\vec{\mathbb{Z}}\vec{\Delta}$ was introduced by Riedtmann). When $\vec{\Delta}$ is a Dynkin quiver, then in fact $\vec{\Gamma}(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)) = \vec{\mathbb{Z}}\vec{\Delta}$.

Example 4.2. Take $\vec{\Delta} = \underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \underset{3}{\bullet}$. Then $\vec{\Gamma}(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)) = \vec{\mathbb{Z}}\vec{\Delta}$ is



The vertices in $\vec{\Gamma}(\mathsf{mod}\,A)$ are labeled.

The group $\mathrm{DPic}_k(A)$ acts naturally on the category $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)$, and hence on the quiver $\vec{\Gamma}(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A))$ (by quiver automorphisms). The element $\tau \in \mathrm{DPic}_k(A)$, represented by the tilting complex $A^*[-1]$, is called the translation. In the example above it is a shift by 1 to the left. Using this action we proved the following theorems. Let $\mathrm{Aut}(\vec{\mathbb{Z}}\vec{\Delta})$ denote the group of quiver automorphisms.

Theorem 4.3 (Miyachi and Y.). *Suppose A has finite representation type. Then*

$$\mathrm{DPic}_k(A) \cong \mathrm{Aut}(\mathbb{Z}\vec{\Delta})^{\langle\tau\rangle}.$$

Example 4.4. Let $\sigma \in \mathrm{DPic}_k(A)$ be represented by $A[1]$. If $\vec{\Delta}$ is of Dynkin type A_n then $\mathrm{DPic}_k(A)$ is abelian, generated by τ and σ, with one relation $\tau^{n+1} = \sigma^{-2}$.

We denote by $\mathrm{Aut}((\mathbb{Z}\vec{\Delta})_0; d)$ the group of permutations of the vertex set $(\mathbb{Z}\vec{\Delta})_0$ that preserve the arrow-multiplicity d.

Theorem 4.5 (Miyachi and Y.). *If A has infinite representation type then there is an isomorphism of groups*

$$\mathrm{DPic}_k(A) \cong \left(\mathrm{Aut}((\mathbb{Z}\vec{\Delta})_0; d)^{\langle\tau\rangle} \ltimes \mathrm{Pic}_k^0(A)\right) \times \mathbb{Z}.$$

$\mathrm{Pic}_k^0(A)$ is a connected linear algebraic group, and is trivial when $\vec{\Delta}$ is a tree.

To conclude,

Problem 4.6. Can this analysis of $\mathrm{DPic}_k(A)$ be carried out for other finite dimensional algebras A? We note that that the quiver $\vec{\Gamma}(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A))$ exists whenever $\mathrm{gl.dim}\, A < \infty$.

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On Auslander-Reiten sequences for irreducible lattices over integral group rings

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ABSTRACT. Let G be a finite group and $\mathcal{O}$ a complete discrete valuation ring of characteristic 0 with the maximal ideal (π) and the residue field $k = \mathcal{O}/(\pi)$ of characteristic $p > 0$. Let L be a non-projective absolutely irreducible $\mathcal{O}G$-lattice such that $L/\pi L$ is indecomposable. Then the middle term of the Auslander-Reiten sequence terminating in L is projective or indecomposable.

Gは有限群とする．pはGの位数$|G|$を割り切るある素数とし，$(K, \mathcal{O}, k)$はp-モジュラー系とする．すなわち，$\mathcal{O}$は標数0の完備離散付値環で，その極大イデアルを(π)で表したとき，剰余体$k = \mathcal{O}/(\pi)$は標数pの体であり，Kは$\mathcal{O}$の商体であるとする．

ここではRで$\mathcal{O}$またはkを表し，RG-加群といえばR-上自由で有限生成なものを意味するものとする．特に$\mathcal{O}G$-加群とは$\mathcal{O}G$-latticeを意味し，射影的／入射的については$\mathcal{O}G$-latticeのなすカテゴリーで考えることにする．

ところで，$\mathcal{O}G$-lattice Lが<u>既約（irreducible）</u>とは，$K \otimes_{\mathcal{O}} L$が既約な$KG$-加群のときに云う．いいかえれば，$L$がある既約な$KG$-加群の$\mathcal{O}$-形式となっていることである．次が報告したい主定理である．

定理 [JKM]　Lは射影的でない絶対既約な$\mathcal{O}G$-latticeで，$L/\pi L$は直既約なkG-加群であるものとする．このとき，Lの Auslander-Reiten 列の中間項は射影的か，もしくは直既約である．

This is a part of the joint paper [JKM] with A. Jones and G. O. Michler, which will be submitted for publication elsewhere.

ここで絶対既約とは，K の任意の拡大体 $\tilde{K}$ に対し $\tilde{K}\otimes_{\mathcal{O}} L$ が既約 $\tilde{K}G$-加群のときを云う．群の通常表現（標数0の体 K 上の表現）においては，K が1の原始 $|G|$-乗根を含めば，どんな既約 KG-加群も絶対既約になることが知られている．また，任意の既約な KG-加群 T に対して T の $\mathcal{O}$-形式 L で $L/\pi L$ が直既約なものが存在することが，Thompson[Tho] により知られている．

§1では群環の加群の Auslander-Reiten 列の基本的事項を思い出し，§2では主定理の証明のための準備として Knörr，Carlson-Jonesが導入した " virtually irreducible lattice " や " exponent " についての結果を述べる．主定理の証明は§3で与える．

群環の Auslander-Reiten 理論については，Benson[B]，Erdmann[E]，Roggenkamp [R2] に詳しいので参照して下さい．

§1　群環の Auslander-Reiten 列

群環 RG（R は $\mathcal{O}$ または k）の加群の完全列 $\mathcal{E}: 0 \to Z \to Y \xrightarrow{f} X \to 0$ は次の3つの条件を満たすとき Auslander-Reiten 列（概分裂列）とよばれる：

（1）X と Z はともに直既約；

（2）$\mathcal{E}$ は分裂していない；

（3）任意の split-epi でない準同型写像 $g: W \to X$ に対し，ある $h: W \to Y$ が存在して $g = fh$ が成り立つ．

Auslander-Reiten 列の存在は，アルティン環の場合は Auslander-Reiten によって，そして order の場合は Roggenkamp らによって示された．

定理　任意の射影的でない直既約 RG-加群 X に対し，X を最終項とする Auslander-Reiten 列 $0 \to Z \to Y \to X \to 0$ が一意的に存在する．

特に X の Auslander-Reiten 列の最初の項 Z は一意的に決まるので $Z = \tau X$ と表すことにする．（τ は Auslander-Reiten translation とよばれる．）$R = \mathcal{O}$ のときは $\tau = \Omega$ であり，$R = k$ のときは $\tau = \Omega^2$ である．ここで Ω は Hellar 作用素，すなわち ΩX は X の射影被覆 $0 \to \Omega X \to P_X \to X \to 0$ の核である．また中間項は $Y = m(X)$ と表すことにする．

これから $R=\mathcal{O}$ の場合に Roggenkamp，Thévenaz らによる Auslander-Reiten 列の構成法を紹介したい．

X，Y を $\mathcal{O}G$-lattice とする．$\mathcal{O}G$-準同型写像 $\varphi : X \to Y$ がある射影加群を経由するとき，φ を射影的と云う．また $\mathrm{ProjHom}_{\mathcal{O}G}(X,Y)$ で X から Y への射影的準同型写像のすべてのなす部分空間を表すことにする．そして

$$\underline{\mathrm{Hom}}_{\mathcal{O}G}(X,Y) := \mathrm{Hom}_{\mathcal{O}G}(X,Y)/\mathrm{ProjHom}_{\mathcal{O}G}(X,Y)$$

とおく．一般に，射影的ではない直既約 $\mathcal{O}G$-lattice X について，$\underline{\mathrm{Hom}}_{\mathcal{O}G}(X,X)$ は simple socle を持つことが確かめられる．この socle の生成元を ρ としたとき，ρ と X の projective cover による pull back として X の Auslander-Reiten 列が構成される：

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega X & \longrightarrow & m(X) & \longrightarrow & X & \longrightarrow & 0 \quad \text{: AR 列} \\ & & \| & & \downarrow & \text{pull back} & \downarrow \rho & & \\ 0 & \longrightarrow & \Omega X & \longrightarrow & P_X & \longrightarrow & X & \longrightarrow & 0 \quad \text{: projective cover} \end{array}$$

§ 2　Exponent, Virtually Irreducible Lattices と既約写像

この節では Carlson-Jones が定義した "exponent" と，Knörr が定義した "virtually irreducible lattice"，およびそれらに関する事実の説明をしたい．

一般に $\mathcal{O}G$-準同型 $\varphi : X \to Y$ に対し $|G|\varphi$ は射影的である．すなわち，

$$|G|\mathrm{Hom}_{\mathcal{O}G}(X,Y) \subseteq \mathrm{ProjHom}_{\mathcal{O}G}(X,Y)$$

であって，$\underline{\mathrm{Hom}}_{\mathcal{O}G}(X,Y)$ はトーション $\mathcal{O}$-加群である．X の恒等写像を Id_X で表すことにする．次の定義は Carlson-Jones[CJ] による．

定義　$\mathcal{O}G$-lattice X について，$\pi^a \cdot \mathrm{Id}_X$ は射影的であるが，$\pi^{a-1} \cdot \mathrm{Id}_X$ は射影的ではないとき，$\exp(X) = \pi^a$ と書き，X の <u>exponent</u> とよぶ．

$\underline{\mathrm{Hom}}_{\mathcal{O}G}(X,X) \cong \underline{\mathrm{Hom}}_{\mathcal{O}G}(\Omega X,\Omega X)$ （環として同型）なので，次がいえる．

補題 1　$\exp(X) = \exp(\Omega X)$.

さて，X が射影的でない直既約加群ならば $\underline{\mathrm{Hom}}_{\mathcal{O}G}(X,X)$ の socle は simple であった．この socle に関して，Carlson-Jones[CJ] は次のような概念をも導入した．

定義　X は射影的でない直既約な $\mathcal{O}G$-lattice とし，$\exp(X) = \pi^a$ とする．X が <u>property E を持つ</u>とは，

$$\pi^{a-1} \cdot \underline{\mathrm{Hom}}_{\mathcal{O}G}(X,X) = \mathrm{Soc}(\underline{\mathrm{Hom}}_{\mathcal{O}G}(X,X))$$

が成り立っているときを云う．いいかえれば，$\pi^{a-1} \cdot \mathrm{Id}_X$ が $\mathrm{Soc}(\underline{\mathrm{Hom}}_{\mathcal{O}G}(X,X))$ の生成元になっているとき，X は property E を持つと云う．

補題 2　絶対既約な $\mathcal{O}G$-lattice L は property E を持つ．

証明　$K \otimes_{\mathcal{O}} L$ は絶対既約な KG-加群なので $\mathrm{End}_{KG}(K \otimes_{\mathcal{O}} L) \cong K$ である．一方 $L = 1 \otimes_{\mathcal{O}} L \subset K \otimes_{\mathcal{O}} L$ なので，$\mathrm{End}_{\mathcal{O}G}(L)$ の任意の元は $\mathrm{End}_{KG}(K \otimes_{\mathcal{O}} L)$ に拡張できて，$K \otimes_{\mathcal{O}} \mathrm{End}_{\mathcal{O}G}(L) = \mathrm{End}_{KG}(K \otimes_{\mathcal{O}} L) \cong K$．よって $\mathrm{End}_{\mathcal{O}G}(L) \cong \mathcal{O}$ であり，$\mathrm{End}_{\mathcal{O}G}(L)$ の任意の元は恒等写像のスカラー倍とわかる．□

ほかにも property E を持つ $\mathcal{O}G$-lattice として次のものがある．

例 [CJ, Cor. 2.9]　（１）X が直既約で $\mathrm{rank}_{\mathcal{O}}(X)$ が p で割り切れなければ，X は property E を持つ．

（２）$(\pi^n) = |G| \cdot \mathcal{O}$ のとき，X が直既約で $\exp(X) = \pi^n$ ならば，X は property E を持つ．

（３）X が直既約で property E を持てば，ΩX も property E を持つ．

他方 Knörr は次のような virtually irreducible という概念を導入した．

定義　$\mathcal{O}G$-lattice X は次の条件※をみたすとき <u>virtually irreducible</u> と云う：

（条件※）任意の $\varphi \in \mathrm{Hom}_{\mathcal{O}G}(X,X)$ に対し，$\mathrm{Tr}(\varphi) \in \mathrm{rank}_{\mathcal{O}}(X)\cdot\mathcal{O}$ であり，かつ $\mathrm{Tr}(\varphi) \in \pi\mathrm{rank}_{\mathcal{O}}(X)\cdot\mathcal{O}$ となる必要十分条件は φ が非正則元であること．（ここで $\mathrm{Tr}(\varphi)$ は，φ を X のある $\mathcal{O}$-基底に関して行列表示したときのトレースを表す.）

実はこの virtually irreducible と，property E はほぼ同値であることが Carlson-Jones によって示された．

命題 3 [CJ, Remark 4.5]　X が virtually irreducible であるための必要十分条件は，X が絶対直既約で property E を持つことである．

さらに Carlson-Jones は exponent と Auslander-Reiten 列に関して次の結果を得た．

定理 4 [CJ, Theorem 2.4]　X は射影的でない直既約 $\mathcal{O}G$-lattice で $0 \to \Omega X \to m(X) \to X \to 0$ は Auslander-Reiten 列とする．$\exp(X) = \pi^a$，$\exp(m(X)) = \pi^b$ とおく．このとき次は同値：

（1）X は property E を持つ．

（2）$b < a$.

この定理を使うと，冒頭で述べた主定理の証明の鍵となる次の事実が示される．

補題 5　V，W はともに virtually irreducible であるとする．このとき V から W への既約写像は存在しない．

証明　ある既約写像 $f : V \to W$ が存在すると仮定してみる．すると W で終わる Auslander-Reiten 列の中間項の直和因子として V が現われる：$0 \to \Omega W \to V \oplus \cdots \to W \to 0$．いま $\exp(W) = \pi^a$，$\exp(V) = \pi^b$ とおくと定理 4 より $b < a$ である．次に V で終わる Auslander-Reiten 列 $0 \to \Omega V \to \Omega W \oplus \cdots \to V \to 0$ を考えるとその中間項に ΩW が現われる．補題 1 から $\exp(\Omega W) = \pi^a$ なので，定理 4 から $a < b$ となってしまい，矛盾．□

§ 3　主定理の証明

この節で冒頭で述べた主定理の証明をする．

$\underline{\mathrm{Hom}}_{\mathcal{O}G}(L,L)$ は simple socle を持つが，その生成元 ρ と L の射影被覆による pull back として L の AR 列が構成される：

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega L & \longrightarrow & m(L) & \longrightarrow & L & \longrightarrow & 0 \\ & & \| & & \downarrow & \text{pull back} & \downarrow \rho & & \\ 0 & \longrightarrow & \Omega L & \longrightarrow & P_L & \longrightarrow & L & \longrightarrow & 0 \end{array}$$

いま L は絶対既約なので，$\rho = \pi^n \cdot \mathrm{Id}_L$ $(\exists n)$ と表される．

Case I.　$n = 0$ のとき：　射影被覆が Auslander-Reiten 列である．特に中間項は射影的．

Case II.　$n \neq 0$ のとき：　$m(L)$ が直既約であることを示そう．直既約ではないと仮定してみる．$m(L) = V \oplus W$ とおく．上の pull back を mod (π) で reduction して得られる kG-加群の pull back は，$\overline{\pi^n \cdot \mathrm{Id}_L}$ が 0-map なので，分裂する．よって $m(L)/\pi m(L) = V/\pi V \oplus W/\pi W \cong L/\pi L \oplus \Omega L/\pi\Omega L$ を得る．$W/\pi W \cong L/\pi L$ とする．このとき $\mathrm{rank}_{\mathcal{O}} W = \mathrm{rank}_{\mathcal{O}} L$ より既約写像 $W \to L$ は単射である．(既約写像は単射かまたは全射であるが，しかし同型ではないことに注意する．) よって特に W は絶対既約で，W，L ともに virtually irreducible であることになる．しかしこれは補題 5 に矛盾する．

これで主定理の証明ができた．

例　$\mathcal{O}_G$ を自明な $\mathcal{O}G$-lattice とすると，$m(\mathcal{O}_G)$ は直既約．

§ 4　射影加群と Auslander-Reiten 列

いままでに見てきたように，Auslander-Reiten 列の中間項について，例えば直既約か否かを判定することさえも容易ではないように思われる．この節では，中間項に射影加群が現われるような Auslander-Reiten 列について考察し，特に中間項の projective-free part について調べてみたい．

まず $R = k$ のとき，すなわちモジュラー表現の場合を考える．S を単純 kG-加群とし，P_S を S の射影被覆とする．群多元環 kG は対称多元環で，$\mathrm{Top}(P_S)($:=

$P_S/\mathrm{Rad}(P_S)$) $\cong S \cong \mathrm{Soc}(P_S)$ となっている．そして $\mathrm{Rad}(P_S)$ は直既約で P_S の唯一の極大部分 kG-加群であって，

$$\mathcal{S} : 0 \to \mathrm{Rad}(P_S) \to P_S \oplus \mathrm{Rad}(P_S)/\mathrm{Soc}(P_S) \to P_S/\mathrm{Soc}(P_S) \to 0$$

が，P_S が中間項に現われるような唯一の Auslander-Reiten 列であり，standard 列とよばれている．

Standard 列 $\mathcal{S}$ は次のようにも表わされる：

$$\mathcal{S} : 0 \to \Omega S \to P_S \oplus \mathrm{Rad}(P_S)/\mathrm{Soc}(P_S) \to \Omega^{-1} S \to 0$$

よって単純加群 S で終わる AR 列は次のように書ける：

$$0 \to \Omega^2 S \to \Omega(\mathrm{Rad}(P_S)/\mathrm{Soc}(P_S)) \to S \to 0$$

注意　実は $\mathrm{Rad}(P_S)/\mathrm{Soc}(P_S)$ が decomposable なときもある．実際，$p = 5$ で $G = F_4(2)$（F_4 型の Chevalley 群）のとき分解行列が Hiss[H] によって計算されているが，その結果からある単純 kG-加群 S に対して $\mathrm{Rad}(P_S)/\mathrm{Soc}(P_S)$ が直既約にならないことがわかる．

以上，kG-加群の場合を見てきたが，次に $\mathcal{O}G$-lattice の場合を考えたい．

一般に射影的 kG-加群 P_S は liftable（持ち上げ可能）である．即ち，ある射影的 $\mathcal{O}G$-lattice Q_S が存在して，$\overline{Q_S}(:= Q_S/\pi Q_S) \cong P_S$．また，任意の射影的な直既約 $\mathcal{O}G$-lattice は，ある射影的な直既約 kG-加群 P_S の lift（持ち上げ）となっている．よって，射影的な直既約 $\mathcal{O}G$-lattice と射影的な直既約 kG-加群との間には，持ち上げを通して，1 対 1 の対応がある．

以下，Q_S は射影的な直既約 $\mathcal{O}G$-lattice で，$Q_S/\pi Q_S \cong P_S$ であるものとする．群環 $\mathcal{O}G$ の Jacobson 根基を $J(\mathcal{O}G)$ とおき，$J_S := Q_S J(\mathcal{O}G)$ とおく．また B を，Q_S が属する $\mathcal{O}G$-ブロックとする.(ここでブロックとは，群環 $\mathcal{O}G$ を両側イデアルとして直既約分解したときの成分のことである.)

一般には J_S は直既約とは限らない．しかし Wiedemann[W] の結果を利用すれば次のことがわかる．

補題 [Ka, Proposition 3]　もし B が無限表現型であれば J_S は直既約である．

ところで，$Q_S/\pi Q_S \cong P_S$ の socle は simple である．よって Q_S を真に含むような $K \otimes_{\mathcal{O}} Q_S$ の $\mathcal{O}G$-部分加群のなかで極小なものが一意的に存在するが，それを I_S とおく：

$$
\begin{array}{ll}
K \otimes_{\mathcal{O}} Q_S & \\
\quad | & \\
\quad I_S & : \text{unique minimal overmodule of } Q_S \\
\quad | & \\
\quad Q_S & \\
\quad | & \\
\quad J_S & : \text{unique maximal submodule of } Q_S
\end{array}
$$

補題 [Ka, Lemma 4]　B は無限表現型であるとする．このとき

（１）$J_S \cong \Omega I_S$．特に I_S は直既約である．

（２）I_S で終わる Auslander-Reiten 列は Q_S が現われる唯一の Auslander-Reiten 列である．

上の事実に関連して，もっと一般に $\mathcal{O}$-order Λ の " bijective " 直既約 Λ-lattice Q について，次の事実が示されていることを西田憲司氏から教えていただいた．(ここで Q が bijective とは，projective かつ injective のときを云う．)

命題 [HN, 2.3.2]　Q を bijective 直既約 Λ-lattice とする．Q の（唯一存在する）極大 Λ-submodule を $'Q$ とし，また Q の（唯一存在する）極小 Λ-overmodule を Q' とおく，$Q' \supset Q \supset {'Q}$．このとき，

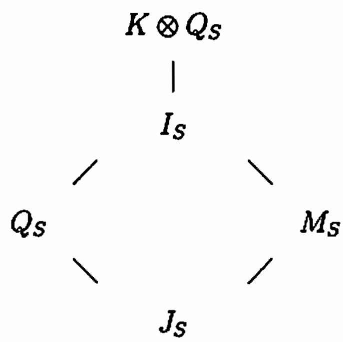
（１）$'Q$：直既約　$\Longleftrightarrow$　Q'：直既約．

（２）もし $'Q$ が直既約でなければ，$'Q$ は２つの射影的でない直既約 Λ-lattices の直和である．

さて，$\mathcal{A} : 0 \to J_S \to Q_S \oplus M_S \to I_S \to 0$ を Q_S が現われる Auslander-Reiten 列とする．もし S が liftable ならば（即ち，ある $\mathcal{O}G$-lattice L が存在して $L/\pi L \cong S$

となるとき）次のことがいえる.

注意　S は liftable であるとする．また Q_S の属する $\mathcal{O}G$-ブロックは無限表現型であるとする、このとき M_S は，I_S に真に含まれ，かつ J_S を真に含むような $\mathcal{O}G$-lattices のなかで射影的ではない唯一のものである：



さらに P_S が現われる standard Auslander-Reiten 列とは次のような関係もある．

定理 [Ka, Theorem 9]　S は liftable であるとする．また Q_S の属する $\mathcal{O}G$-ブロックは無限表現型であるとする．このとき，Q_S が現われる $\mathcal{O}G$-lattices の Auslander-Reiten 列 $\mathcal{A}$ を mod (π) で reduction して得られる kG-加群の完全列 $\overline{\mathcal{A}}$ は，P_S が中間項に現われる standard 列と分裂列 $0 \to S \to S \oplus S \to S \to 0$ との直和となる．

最後に M_S が直既約とならないときもあることを注意しておく．例えば $|G| = p^2$, $(\pi) = p \cdot \mathcal{O}$ のとき，自明な単純 kG-加群 $S = k_G$ に対し，M_S は直既約ではなく，自明な $\mathcal{O}G$-lattice $\mathcal{O}_G$ がその直和因子として現われる．

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SOME REMARKS ON (M, N)-INJECTIVE MODULES

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Let P and Q be rings, and ${}_PM$, N_Q and ${}_PU_Q$ a left P-module, a right Q-module and a P-Q-bimodule, respectively, and let $\varphi : M \times N \to U$ be a P-Q-bilinear map. Then we say that $({}_PM, N_Q)$ (or $({}_PM, N_Q; U)$) is a pair with respect to φ or simply a pair (see [10], [11] or [1, Section 24]). For elements $x \in M$, $y \in N$ and subsets $X \subset M$, $Y \subset N$, by xy we denote the element $\varphi(x, y)$, and by $r_N(X)$ (resp. $\ell_M(Y)$) we denote the right (resp. left) annihilator module $\{y \in N \mid Xy = 0\}$ $(\leq N_Q)$ (resp. $\{x \in M \mid xY = 0\}$ $(\leq {}_PM)$). We say that a pair $({}_PM, N_Q)$ satisfies ℓ-ann (resp. r-ann) if for any submodule X of ${}_PM$ (resp. any submodule Y of N_Q), $X = \ell_M r_N(X)$ (resp. $Y = r_N \ell_M(Y)$) holds and $({}_PM, N_Q)$ is a dual pair if $({}_PM, N_Q)$ satisfies ℓ-ann and r-ann (see [10]).

Let ${}_PM_R$ be a P-R-bimodule and f an idempotent of R and put $Q = fRf$. Then a pair $({}_PM, Rf_Q)$ always signifies the pair with respect to the P-Q-bilinear map $\varphi: M \times Rf \to Mf$ via $\varphi(x, af) = xaf$; $x \in M$, $af \in Rf$.

Let $({}_PM, N_Q; U)$ be a pair. Then U_Q is said to be (M, N)-injective if the following condition (*) holds for any submodule K of N_Q and any homomorphism $\theta : K \to U$.

(*) $\theta : K \to U$ is given by left multiplication by an element of M.

Moreover U_Q is said to be (M, N)-F-injective (resp. (M, N)-cyclic-injective) if (*) holds for any K $(\leq N_Q)$ which is finitely generated (resp. cyclic) and any homomorphism $\theta : K \to U$, and U_Q is said to be (M, N)-FI-injective (resp. (M, N)-CI-injective or (M, N)-simple-injective) if (*) holds for any K $(\leq N_Q)$ and any homomorphism $\theta : K \to U$ whose image is finitely generated (resp. cyclic or simple).

Let ${}_PM_R$ and L_R be a P-R-bimodule and a right R-module, respectively, and let $({}_PL^*, L_R; M)$ be a pair with respect to a natural map $\eta : L^* \times L \to M$, where ${}_PL^* = \mathrm{Hom}_R(L, M)$. Then (L^*, L)-injectivity of M_R implies L-injectivity of M_R and in particular (M,

The detailed version of this paper will be submitted for publication elsewhere.

R)-injectivity of M_R implies injectivity of M_R.

Ikeda and Nakayama [8, Thoerem 1] and Xue [16, Lemma 3] have shown that a right Q-module U_Q is Q-F-injective (i.e. (U, Q)-F-injective) if and only if $\ell_U r_Q(a) = Ua$ and $\ell_U(I \cap K) = \ell_U(I) + \ell_U(K)$ hold for any $a \in Q$ and any finitely generated right ideals I and K of Q.

On the other hand, as generalizations of [7, Proposition 5.2], in [9, Lemmas 1.13 and 2.4] Kurata and Hashimoto have pointed out that in case ${}_PU_Q$ is a right dual bimodule (resp. a dual bimodule), U_Q is Q-FI-injective (resp. (P, U)-FI-injective), where ${}_PU_Q$ is called a right dual bimodule if $({}_PU, Q_Q)$ is a dual pair, and ${}_PU_Q$ is called a dual bimodule if both $({}_PP, U_Q)$ and $({}_PU, Q_Q)$ are dual pairs.

For a pair $({}_PM, N_Q; U)$, in [2, Theorem 12] Anh, Herbera and Menini have shown that if both ${}_PU$ and U_Q are (M, N)-simple-injective and AB5* modules whose socles satisfy certain conditions, then both ${}_PU$ and U_Q are (M, N)-FI-injective and in [2, Proposition 14] they have shown that under certain situation, U_Q is (M, N)-injective if and only if ${}_PM$ is linearly compact.

In this note for a pair $({}_PM, N_Q; U)$, we shall study properties on (M, N)-simple-injective modules, (M, N)-FI-injective modules and (M, N)-injective modules in relation to [8], [16], [7], [9] and [2] above.

Throughout this note, every ring has an identity and every module is unitary. Let $({}_PM, N_Q; U)$ be a pair, $x \in M, Z \leq Y \leq N_Q$ and $\theta : Y \to U$ a homomorphism. Then, by $\hat{x} : N \to U$ we denote the left multiplication map by x and by $\theta|_Z$ we denote the restriction map of θ to Z.

The following lemmas are essentially due to Ikeda and Nakayama [8].

Lemma 1 (see [8, Theorem 1] and [2, Theorem 12]). Let $({}_PM, N_Q; U)$ be a pair. Then the following conditions (1) and (2) are equivalent.

(1) U_Q is (M, N)-cyclic-injective.

(2) $\ell_U r_Q(y) = My$ for any $y \in N$.

Moreover in case $r_N(M) = 0$, the conditions are equivalent to the following condition (3).

(3) $\ell_U r_Q(My) = My$ for any $y \in N$.

Lemma 2 (see [8, Theorem 1]). Let $({}_PM, N_Q; U)$ be a pair and $Y_i \leq N_Q$ $(i = 1, 2)$. Then the following are equivalent.

(1) If $\theta : Y_1 + Y_2 \to U$ is a homomorphism such that $\theta|_{Y_i} = \hat{x}_i$ for some elements $x_i \in M$ $(i = 1, 2)$, then $\theta = \hat{x}$ for some element $x \in M$.

(2) If $\theta : Y_1 + Y_2 \to U$ is a homomorphism such that $\theta|_{Y_1} = \hat{x}_1$ for some element $x_1 \in M$ and $Y_2 \leq \mathrm{Ker}\,\theta$, then $\theta = \hat{x}$ some element $x \in M$.

(3) $\ell_M(Y_1 \cap Y_2) = \ell_M(Y_1) + \ell_M(Y_2)$.

Remark 1. Let $\theta : Y \to U$ be a homomorphism such that $\mathrm{Im}\,\theta$ is cyclic (resp. finitely generated). Then there exist submodules Y_1 and Y_2 of Y such that Y_1 is cyclic (resp. finitely generated), $Y_2 \leq \mathrm{Ker}\,\theta$ and $Y = Y_1 + Y_2$.

By Lemmas 1 and 2 and Remark 1, we have the following theorems.

Theorem 3 (see [8, Thoerem 1] and [16, Lemma 3]). Let $({}_PM, N_Q; U)$ be a pair. Then the following are equivalent.

(1) U_Q is (M, N)-F-injective.

(2) (i) $\ell_U r_Q(y) = My$ for any element $y \in N$.

(ii) $\ell_M(Y_1 \cap Y_2) = \ell_M(Y_1) + \ell_M(Y_2)$ for any finitely generated submodules Y_1 and Y_2 of N_Q.

(3) $\ell_U(y^{-1}K) = \ell_M(K)y$ for any element $y \in N$ and any finitly generated submodule K of N_Q, where $y^{-1}K$ denotes the right ideal $\{a \in Q \mid ya \in K\}$ of Q

Theorem 4. Let $({}_PM, N_Q; U)$ be a pair. Then the following conditions are equivalent.

(1) U_Q is (M, N)-FI-injective.

(2) U_Q is (M, N)-CI-injective.

(3) (i) $\ell_U r_Q(y) = My$ for any element $y \in N$.

(ii) $\ell_M(Y_1 \cap Y_2) = \ell_M(Y_1) + \ell_M(Y_2)$ for any finitely generated (cyclic) submodule Y_1 and any submodule Y_2 of N_Q.

(4) $\ell_U(y^{-1}K) = \ell_M(K)y$ for any element $y \in N$ and any submodule K of N_Q, where $y^{-1}K$ denotes the right ideal $\{a \in Q \mid ya \in K\}$ of Q

If $({}_PM, N_Q)$ is a dual pair, then $\ell_M(Y_1 \cap Y_2) = \ell_M(Y_1) + \ell_M(Y_2)$ holds for any submodules Y_1 and Y_2 of N. Therefore by Theorem 4 and Lemma 1, we have the following.

Corollary 5. Let $({}_PM, N_Q; U)$ be a dual pair. If the pair $({}_PU, Q_Q)$ satisfies ℓ-ann, then U_Q is (M, N)-FI-injective.

Remark 2. Taking $M = U$, $N = Q$; or $M = P$, $N = U$ in Corollary 5, we obtain the statements [9, Lemmas 1.13 and 2.4] mentioned in the introduction.

As a statement similar to Theorems 3 and 4, we have the following.

Proposition 6. Let $({}_PM, N_Q; U)$ be a pair. Then the following conditions are equivalent.

(1) U_Q is (M, N)-simple-injective.

(2) (i) $\ell_U r_Q(y) \cap \mathrm{Soc}(U_Q) \leq My$ for any $y \in N$.

(ii) $\ell_M(yQ \cap K) = \ell_M(yQ) + \ell_M(K)$ for any $y \in N$ and any $K \leq N_Q$ such that $(yQ + K)/K$ is simple.

(3) $\ell_U(y^{-1}K) = \ell_M(K)y$ for any $y \in N$ and any $K \leq N_Q$ such that $(yQ + K)/K$ is simple.

The following proposition is essentially due to Nicholson and Yousif [13] and Anh, Herbera and Menini [2].

Proposition 7 (see [13, Lemma 4.2] and [2, Theorem 12]). Let $({}_PM, N_Q; U)$ be a pair and assume that U-duals of simple factor modules of submodules of N_Q are simple as left P-modules. Then the following are equivalent.

(1) U_Q is (M, N)-simple-injective.

(2) $({}_PM, N_Q)$ satisfies r-ann.

The following lemma is shown by applying the proof of [5, Lemma 2.1].

Lemma 8. Let ${}_PM_R$ be a bimodule and f an idempotent of R with $\ell_M(Rf) = 0$ and put $Q = fRf$. Then Mf_Q is (M, Rf)-simple-injective if and only if M_R is (M, R)-simple-injective (i.e. M_R is R-simple-injective).

By Proposition 7 and Lemma 8 we have the following corollary, in which the equivalence of (1) and (3) provides Theorem 2.4 in [10] with another proof.

Corollary 9 (see [10, Theorem 2.4]). Let ${}_PM_R$ be a bimodule and f an idempoptent of R with $\ell_M(Rf) = 0$ and put $Q = fRf$. Assume that Mf-duals of simple right Q-modules are simple as left P-modules. Then the following are equivalent.

(1) M_R is R-simple-injective.

(2) Mf_Q is (M, Rf)-simple-injective.

(3) $({}_PM, Rf_Q)$ satisfies r-ann.

Let ${}_PM$ be a left P-module. Then a family $\{L_i\}_{i\in I}$ of submodules of M is called an inverse system of M if for any elements i and j of I, there exists an element k of I such that $L_k \le L_i \cap L_j$. A module ${}_PM$ is said to be AB5* if for any inverse system $\{L_i\}_{i\in I}$ of M and any submodule K of M, $\bigcap_{i\in I}(K + L_i) = K + \bigcap_{i\in I} L_i$ holds. If $({}_PM, N_Q)$ is a dual pair for some module N_Q, then ${}_PM$ is clearly AB5* (see e.g. [14]). Moreover by [3, Theorem 6] (or [4, Lemma 2.2]) the converse also holds, so a module ${}_PM$ is AB5* if and only if there exist a ring Q and a right Q-module N_Q such that $({}_PM, N_Q)$ is a dual pair.

The following theorem is obtained by a slight modification of the proof of [2, Theorem 12].

Theorem 10 (see [2, Theorem 12]). Let $({}_PM, N_Q; U)$ be a pair such that U_Q has essential socle and assume that ${}_PU$ is AB5*. Then the following are equivalent.

(1) U_Q is (M, N)-simple-injective.

(2) U_Q is (M, N)-FI-injective.

Let $({}_PM, N_Q)$ be a pair. Then by $A_\ell(M, N)$ we denote the class $\{X \le {}_PM \mid X = \ell_M r_N(X)\}$ of submodules of ${}_PM$.

Let ${}_PM$ be a module and A a class of submodules of ${}_PM$. Then ${}_PM$ is said to be A-linearly compact if any finitely solvable system $(x_i, X_i)_{i\in I}$ of ${}_PM$ with $X_i \in A$ is solvable (see e.g. [15] for the definition of "finitely solvable system").

As a characterization of an (M, N)-injective module, we have the following theorem, which is essentially due to [12, Lemma 4], [6, Theorem 2], [16, Lemma 5 and Proposition 6] and [2, Proposition 14].

Theorem 11 (see [12], [6], [16] and [2]). Let $({}_PM, N_Q; U)$ be a pair. Then the following are equivalent.

(1) U_Q is (M, N)-injective.

(2) (i) U_Q is (M, N)-F-injective.

(ii) ${}_PM$ is A_ℓ(M, N)-linearly compact.

By a modification of [2, Proposition 14], we have the following theorem.

Theorem 12 (see [2, Proposition 14]). Let $({}_PM, N_Q; U)$ be a pair such that U_Q has essential socle. Then the following are equivalent.

(1) U_Q is (M, N)-injective.

(2) (i) U_Q is (M, N)-simple-injective.

(ii) ${}_PM$ is A_ℓ(M, N)-linearly compact.

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Examples of QF rings without Nakayama automorphism and H-rings without self-duality *

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Abstract

By investigating the structure of H-rings deeply, Kado and Oshiro [5] proved that the following three conditions are equivalent:

(A) Every basic left H-ring has a Nakayama isomorphism.
(B) Every basic QF ring has a Nakayama automorphism.
(C) Every left H-ring has a self-duality.

On the other hand, Kraemer [6] had constructed an example of a QF ring that does not have a weakly symmetric self-duality. In this note, we shall point out that the Kraemer's example is an example of a QF ring without Nakayama automorphism and give some other examples of QF rings without Nakayama automorphism, and by giving a necessary and sufficient condition for a special type of H-rings to have a self-duality, we shall give concrete examples of H-rings without self-duality.

「すべての H 環が self-duality をもつか」という問題は, 「すべての QF 環が中山自己同型写像をもつか」という問題と同値であることが, 加戸・大城 [5] によって証明されているが, 後者の問題の解答も我々には知られていなかった. 一方, Kraemer は [6] において, weakly symmetric self-duality をもたない QF 環の例を構成していた. この報告集では, この Kraemer の例が, 中山自己同型写像をもたない QF 環の例になっていることを注意するとともに, いくつかの中山自己同型写像をもたない QF 環の例を与える. また特別な形の H 環が self-duality をもつための必要十分条件を示し, それと中山自己同型写像をもたない QF 環を用いて, 具体的に self-duality をもたない H 環の例を構成する.

この報告集において, すべての環は単位元をもち, すべての加群は単位的であるとする. 加群 M に対して, その入射包絡, radical, socle, top を, それぞれ, $E(M)$, $J(M)$, $S(M)$, $T(M)$ で表す.

よく知られているように, 両側加群 ${}_RU_S$ が **Morita duality** を定めるとは, ${}_RU_S$ が忠実かつ平衡的で, ${}_RU$, U_S がそれぞれ入射余生成素であることをいう. 特に $R = S$ で

*The detailed version of this note will be submitted for publication elsewhere.

あるとき, ${}_RU_R$ は self-duality を定めるという. Morita duality を定める両側加群が存在するとき, R は左 Morita duality をもつといい, R は S に左 Morita dual であるという.

self-duality を定める両側加群 ${}_RU_R$ において, R の任意の原始巾等元 e に対して, $T({}_RRe) \cong {}_R\mathrm{Hom}_R(T(eR_R), U_R)$ が成り立つとき, ${}_RU_R$ は weakly symmetric self-duality を定めるという. この条件は左右対称的である. $\mathrm{Hom}_R(T(eR_R), U_R) \cong S({}_RUe)$ であるから, ${}_RU_R$ が weakly symmetric self-duality を定めることと, R の任意の原始巾等元 e に対して $T({}_RRe) \cong S({}_RUe)$ が成り立つこととは同値である. weakly symmetric self-duality を定める両側加群が存在するとき, 環 R は weakly symmetric self-duality をもつという. ([6, p.12] 参照.)

注意 1. (1) よく知られているように, すべての artin algebra R は weakly symmetric self-duality をもつ. 実際, K を R の中心とし, $E = E(T(K))$ を K-Mod の極小入射余生成素とするとき, $U = \mathrm{Hom}_K(R, E)$ は自然に R-両側加群となり, weakly symmetric self-duality を定める. したがって, 命題 2 より特に, 任意の artin algebra は中山同型写像 (定義は後述) をもつ.

(2) 正則両側加群 ${}_RR_R$ が weakly symmetric self-duality を定めるような QF 環は, weakly symmetric QF 環として知られている. これは中山置換が恒等的であることと同値である. weakly symmetric でない QF 環であっても, weakly symmetric self-duality をもつ場合がある.

[5] にしたがって, 中山同型写像 (Nakayama isomorphism) の定義を与えよう. R を左 Morita duality をもつ基本的半完全環とする. $\{e_1, e_2, \ldots, e_n\}$ をその直交原始巾等元の完全集合とし, $S = \mathrm{End}_R\left(\bigoplus_{i=1}^n E\left(T\left(Re_i\right)\right)\right)$ を R-Mod の極小入射余生成素の自己準同型環とする. f_i を射影 $\bigoplus_{i=1}^n E(T(Re_i)) \to E(T(Re_i))$ に対応する S の巾等元とする. 環同型写像 $\tau : R \to S$ は, $\tau(e_i) = f_i$ $(i = 1, 2, \ldots, n)$ を満たすとき, **中山同型写像**であるという. [7, p.42] より中山同型写像が存在するかどうかは, 直交巾等元の完全集合の取り方によらない. ([5, Remark in p.387] 参照.)

次の命題は, weakly symmetric self-duality の存在と中山同型写像の存在は同値であることを示している. ([4, Proposition 3.1] において Haack はアルチン環について証明したが, 同じ方法が通用する.)

命題 2 ([4, Proposition 3.1]). R を左 Morita duality をもつ基本的半完全環とし, ${}_RU$ を R-Mod の極小入射余生成素とする. このとき次の条件は同値である.

(1) R は weakly symmetric self-duality をもつ.

(2) R は中山同型写像をもつ.

(3) R の任意の原始巾等元 e に対して, $U\tau(e) \cong E(T(Re))$ を満たすような環同型写像 $\tau : R \to \mathrm{End}({}_RU)$ が存在する.

QF 環における中山置換と中山自己同型写像の概念を思い出しておこう. R を QF 環, $\{e_1, e_2, \ldots, e_n\}$ をその直交原始巾等元の基本集合とするとき, $S(e_iR) \cong T(\sigma(e_i)R)$,

$S(R\sigma(e_i)) \cong T(Re_i)$ $(i = 1, 2, \ldots, n)$ を満たすような $\{e_1, e_2, \ldots, e_n\}$ の置換 σ が存在する. これを R の中山置換という. R の自己同型写像 τ は, 中山置換を引き起こすとき, 中山自己同型写像と呼ばれる. 命題2より, 基本的QF環 R が中山自己同型写像をもつことと weakly symmetric self-duality をもつこととは同値である.

次の命題において, 特別な形のQF環が中山自己同型写像をもつための条件を与える. 以下記述を簡単にするために, ある正整数 m に対して, 整数 i の m による最小正剰余を $[i]$ で表すことにする. また, 環準同型写像 $\alpha : A \to A'$, $\beta : B \to B'$ と両側加群 ${}_AM_B$, ${}_{A'}M'_{B'}$ が与えられたとき, 加法的準同型写像 $\phi : M \to M'$ は, $\phi(amb) = \alpha(a)\phi(m)\beta(b)$ $(a \in A,\ b \in B,\ m \in M)$ を満たすならば, ϕ は (α, β)-**semilinear** であるということにする.

命題 3. $A_1, A_2, \ldots, A_m$ $(m \geq 2)$ を基本的アルチン環とし, ${}_{A_1}U_{1A_2}, {}_{A_2}U_{2A_3}, \ldots, {}_{A_m}U_{mA_1}$ を Morita duality を定める両側加群とする.

$$R = \begin{pmatrix} A_1 & U_1 & 0 & \ldots & 0 & 0 \\ 0 & A_2 & U_2 & \ldots & 0 & 0 \\ 0 & 0 & A_3 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & A_{m-1} & U_{m-1} \\ U_m & 0 & 0 & \ldots & 0 & A_m \end{pmatrix}$$

とおき, R の環構造を通常の行列の作用と $U_iU_j = 0$ $(1 \leq i, j \leq m)$ によって定める. このとき

(1) R はQF環である.

(2) R が中山自己同型写像をもつための必要十分条件は, 各 $i = 1, 2, \ldots, m$ について, R の任意の原始巾等元 e に対して $T(A_ie) \cong S(U_i\tau_i(e))$ が成り立つような環同型写像 $\tau_i : A_i \to A_{[i+1]}$ と, $(\tau_i, \tau_{[i+1]})$-semilinear な同型写像 $\phi_i : U_i \to U_{[i+1]}$ が存在することである.

この命題の証明において, (1) については i-pair を用いて R の入射性を確かめればよく, (2) については命題2を使えば良い.

注意 4. 命題3において, $A = A_1 \times A_2 \times \cdots \times A_m$ を環直積とし, $U = U_1 \bigoplus U_2 \bigoplus \cdots \bigoplus U_m$ とおけば, U は自然に A-両側加群となり, ${}_AU_A$ はself-dualityを定める. したがって, Faith の結果 (例えば [9, Theorem 10.7] 参照) より, A の U による trivial extension はQF環となる. 実は R はこの trivial extension と同型である.

それでは, Kraemer の例を示そう. そのため一つの記号を導入する. C, D を環とする. このとき両側加群 ${}_CM_D$ に対して両側加群の列 C_i を次のように帰納的に定義する: $M_1 = {}_CM_D$ とおき, $i = 2, 3, \ldots$ に対しては,

$$M_i = \begin{cases} {}_C\operatorname{Hom}_C({}_DM_{i-1C}, {}_CC_C)_D & (i\text{ が奇数のとき}), \\ {}_D\operatorname{Hom}_D({}_CM_{i-1D}, {}_DD_D)_C & (i\text{ が偶数のとき}) \end{cases}$$

と定める.

例 5 ([6, Remark 6.6]). [8] と [3] の結果より, 斜体の拡大 $C > D$ で次の条件を満たすものが存在する. (いわゆる Artin's problem の反例 (の一つ). [6, Theorems 6.1 and 6.2]) 参照.)

(1) $\dim({}_DC) = 2$, $\dim(C_D) = 3$.

(2) 環同型写像 $\lambda : D \to C$, $\mu : C \to D$ が存在する.

(3) (λ, μ)-semilinear な同型写像 $\phi : {}_DC_{6C} \to {}_CC_{1D}$ が存在する.

(4) $(a_1, a_2, a_3, a_4, a_5) = (3, 1, 2, 2, 1)$ かつ $(b_1, b_2, b_3, b_4, b_5) = (1, 3, 1, 2, 2)$, ただし各 a_i と b_i は, それぞれ C_i の右と左の次元を表す.

このとき, [6, Lemma 6.3] によって,

(5) $\psi(c_7)(c_1) = \mu(c_7(\phi^{-1}(c_1)))$ $(c_7 \in C_7, c_1 \in C_1)$ によって定義される写像 $\psi : {}_CC_{7D} \to {}_DC_{2C}$ は (μ, λ)-semilinear な同型写像である.

いま

$$A_i = \begin{pmatrix} C & C_i \\ 0 & D \end{pmatrix} \ (i \text{ が奇数のとき}),\quad A_i = \begin{pmatrix} D & C_i \\ 0 & C \end{pmatrix} \ (i \text{ が偶数のとき})$$

とおくと, これらは上三角行列環で, さらに

$$U_i = \begin{pmatrix} D & 0 \\ C_i & C \end{pmatrix} \ (i \text{ が奇数のとき}),\quad U_i = \begin{pmatrix} C & 0 \\ C_i & D \end{pmatrix} \ (i \text{ が偶数のとき})$$

とおけば, [9, Corollary 10.3] より, U_{i+1} は duality を定める (A_{i+2}, A_i)-両側加群となる. λ, μ, ϕ, ψ を用いれば, 環として $A_6 \cong A_1$, $A_7 \cong A_2$ であることが分かる. したがって U_5 を Morita dality を定める (A_1, A_4)-両側加群と見ることができる. 同様に U_1 を Morita duality を定める (A_2, A_5)-両側加群と見る. いま,

$$R = \begin{pmatrix} A_5 & U_4 & 0 & 0 & 0 \\ 0 & A_3 & U_2 & 0 & 0 \\ 0 & 0 & A_1 & U_5 & 0 \\ 0 & 0 & 0 & A_4 & U_3 \\ U_1 & 0 & 0 & 0 & A_2 \end{pmatrix}$$

とおく. C_i の次元 a_i, b_i を用いれば, すべての A_i $(i = 1, 2, \ldots, 5)$ は互いに非同型であることが分かる. したがって, 命題 3 より, R は中山自己同型写像をもたない QF 環である. この R は [6, Remark 6.6] の環であり, weakly symmetric self-duality をもたない QF 環の例として与えられた. R は 10 個の単純加群の同型類をもつ.

例 6. A を self-duality はもつが weakly symmetric self-duality はもたないアルチン環とする (例 5, 7 参照). このとき, 命題 3 より, (2 次以上の) 環

$$R = \begin{pmatrix} A & U & 0 & \cdots & 0 & 0 \\ 0 & A & U & \cdots & 0 & 0 \\ 0 & 0 & A & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A & U \\ U & 0 & 0 & \cdots & 0 & A \end{pmatrix}$$

は中山自己同型写像をもたない QF 環である.

例5と同じ材料を使って, 例5よりも単純加群の個数の少ない中山自己同型写像をもたない QF 環の例を与えよう.

例 7. 例5と同じ設定の元で,

$$A = \begin{pmatrix} C & C_1 & 0 & 0 & 0 \\ 0 & D & C_2 & 0 & 0 \\ 0 & 0 & C & C_3 & 0 \\ 0 & 0 & 0 & D & C_4 \\ C_5 & 0 & 0 & 0 & C \end{pmatrix}, \quad B = \begin{pmatrix} C & C_3 & 0 & 0 & 0 \\ 0 & D & C_4 & 0 & 0 \\ 0 & 0 & C & C_5 & 0 \\ 0 & 0 & 0 & D & C_6 \\ C_7 & 0 & 0 & 0 & C \end{pmatrix},$$

$$U = \begin{pmatrix} C & 0 & 0 & 0 & C_6 \\ C_2 & D & 0 & 0 & 0 \\ 0 & C_3 & C & 0 & 0 \\ 0 & 0 & C_4 & D & 0 \\ 0 & 0 & 0 & C_5 & C \end{pmatrix}$$

とおく. ただし, A における C_5, U における C_6 と B における C_7 は, 環同型写像 λ^{-1} : $C \to D$ を用いて, C-両側加群と見る. このとき, [6, Theorem 6.4] より U は Morita duality を定める (B, A)- 加群となり,

$$\begin{pmatrix} a & b & 0 & 0 & 0 \\ 0 & c & d & 0 & 0 \\ 0 & 0 & e & f & 0 \\ 0 & 0 & 0 & g & h \\ j & 0 & 0 & 0 & i \end{pmatrix} \mapsto \begin{pmatrix} \lambda(g) & \phi(h) & 0 & 0 & 0 \\ 0 & \mu(i) & \psi(j) & 0 & 0 \\ 0 & 0 & a & b & 0 \\ 0 & 0 & 0 & c & d \\ f & 0 & 0 & 0 & e \end{pmatrix}$$

によって定義される環同型写像 $B \to A$ が存在する. したがって, U を self-duality を定める A-両側加群と見ることができる. R を A の U による trivial extension とする. このとき, [9, Theorem 10.7] より R は QF 環である. e'_i を A の (i,i)-行列単位とし, e_i を e'_i に対応する R の巾等元とする. "i" $(i = 1, 2, \ldots, 5)$ によって, $T(e_iR)$ または $T(Re_i)$ に同型な composition factor を表すことにする. C_i の次元 a_i, b_i を使えば, 直既約射影加群 R_R, ${}_RR$ の Loewy series は次の通りであることが分かる.

R_R

$$\begin{array}{c} 1 \\ 2\,2\,2\,3\,3 \\ 4 \end{array} \qquad \begin{array}{c} 2 \\ 3\,4 \\ 5 \end{array} \qquad \begin{array}{c} 3 \\ 4\,4\,5\,5\,5 \\ 1 \end{array} \qquad \begin{array}{c} 4 \\ 5\,5\,1 \\ 2 \end{array} \qquad \begin{array}{c} 5 \\ 1\,2\,2 \\ 3 \end{array}$$

${}_RR$

$$\begin{array}{c} 1 \\ 4\,4\,4\,5\,5 \\ 3 \end{array} \qquad \begin{array}{c} 2 \\ 5\,1 \\ 4 \end{array} \qquad \begin{array}{c} 3 \\ 1\,1\,2\,2\,2 \\ 5 \end{array} \qquad \begin{array}{c} 4 \\ 2\,2\,3 \\ 1 \end{array} \qquad \begin{array}{c} 5 \\ 3\,4\,4 \\ 2 \end{array}$$

したがって, R の中山置換は $e_i \mapsto e_{[i+3]}$ で巡回的である. しかし, e_1R と e_4R の組成列の長さは異なっている. ゆえに, $\tau(e_1) = e_4$ を満たす R の自己同型写像は存在しない. したがって R は中山自己同型写像をもたない.

これらの中山自己同型写像をもたないQF環の例を用いて, self-dualityをもたないH環の例を構成しよう. まずH環の定義を思い出しておく.

左アルチン環 R は, 次の条件を満たす直交原始巾等元の基本集合 $\{e_{ij} | 1 \leq i \leq m, 1 \leq j \leq n(i)\}$ が存在するとき, 左H環 (左Harada環) であるという.

(1) 各 $e_{i1}R_R$ は入射的である.

(2) $e_{ij}R \cong J(e_{i,j-1}R_R)$ $(j = 2, 3, \ldots, n(i))$.

右H環も同様に定義される. 左H環かつ右H環を両側H環と呼ぶ ([5] 参照). もし, R が左H環であれば, [5, Proposition 3.2] より, 極小入射余生成素 $\bigoplus_{i,j} E(T(Re_{ij}))$ は有限生成であるから, R は $\mathrm{End}_R(\bigoplus_{i,j} E(T(Re_{ij})))$ に左Morita dualであることを注意しておく ([1, Theorem 30.4]).

冒頭で述べたように, [5] において, 3条件 (A), (B), (C) は同値であることが示されているが, 上の例から条件 (A), (B) は成り立たないから, self-dualityをもたない左H環が存在するはずである. [5, Proposition 3.3] の特別な場合と見ることができる次の定理は, このようなH環の具体例を生み出す. これを述べるために, 記号を導入しよう.

R を環, I をその直交原始巾等元の完全集合とする. I の空でない部分集合 K に対して, $e_K = \sum_{e \in K} e$ とおく. また, R の零でない巾等元 e に対して,

$$R_e = \begin{pmatrix} eRe & eRe & eR(1-e) \\ J(eRe) & eRe & eR(1-e) \\ (1-e)Re & (1-e)Re & (1-e)R(1-e) \end{pmatrix}$$

とおく. R_e は通常の行列の演算により環となる.

定理 8. R を基本的QF環とし, I をその直交原始巾等元の完全集合, σ を I 上の中山置換とする. I の任意の空でない部分集合 K に対して, $e = e_K$, $e' = e_{\sigma(K)}$ とおく. このとき,

(1) R_e は両側H環である.

(2) R_e は $R_{e'}$ に左Morita dualである.

(3) R_e がself-dualityをもつための必要十分条件は, $\tau(e) = e'$ を満たす R の環同型写像 τ が存在することである.

証明の概略 $A = eRe$, $B = (1-e)R(1-e)$, $U = eR(1-e)$, $V = (1-e)Re$ とし, $\Lambda = R_e$ とおく. このとき

$$R = \begin{pmatrix} A & U \\ V & B \end{pmatrix}, \ \Lambda = \begin{pmatrix} A & A & U \\ J(A) & A & U \\ V & V & B \end{pmatrix}$$

である．I_A, I_B をそれぞれ A, B の直交原始巾等元の完全集合とする．I を集合の直和 $I_A \dot{\cup} I_B$ と見なし，中山置換 σ を $I_A \dot{\cup} I_B$ 上の置換と見なす．$X, Y \in \{A, B\}$ に対して，$I_{XY} = \{f \in I_X | \sigma(f) \in I_Y\}$, $g_{XY} = \sum_{f \in I_{XY}} \sigma(f)$ とおく．$I_A = I_{AA} \dot{\cup} I_{AB}$, $I_B = I_{BA} \dot{\cup} I_{BB}$ である．さらに

$$f_1 = \begin{pmatrix} 1_A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ f_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1_A & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ f_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1_B \end{pmatrix},$$

$$g_A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & g_{AA} & 0 \\ 0 & 0 & g_{AB} \end{pmatrix}, \ g_B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & g_{BA} & 0 \\ 0 & 0 & g_{BB} \end{pmatrix}$$

とおく．

(1) i-pair について調べ，[5, Proposition 3.2] を用いれば，${}_\Lambda \Lambda f_2$, ${}_\Lambda \Lambda f_3$ は入射的，${}_\Lambda \Lambda f_1 \cong J({}_\Lambda \Lambda f_2)$，また，$f_1 \Lambda_\Lambda$, $f_3 \Lambda_\Lambda$ は入射的，$f_2 \Lambda_\Lambda \cong J(f_1 \Lambda_\Lambda)$ であることが分かる．したがって Λ は両側 H 環である．

(2) やはり i-pair について調べ，[5, Proposition 3.2] を用いれば，同型 $E(T(\Lambda f_1)) \cong \Lambda g_A$, $E(T(\Lambda f_2)) \cong \Lambda g_A / S(\Lambda g_A)$, $E(T(\Lambda f_3)) \cong \Lambda g_B$ が成り立つことが分かる．R の行列表現において，

$$e' = \begin{pmatrix} g_{AA} & 0 \\ 0 & g_{AB} \end{pmatrix}, \ 1 - e' = \begin{pmatrix} g_{BA} & 0 \\ 0 & g_{BB} \end{pmatrix}$$

であることに注意し，この同型を用いれば，極小入射余生成素 $\bigoplus_{i=1}^{3} E(T(\Lambda f_i))$ の自己準同型環は $R_{e'}$ と同型であることが示せる．

(3) $A' = e'Re'$, $B' = (1-e')R(1-e')$, $U' = e'R(1-e')$, $V' = (1-e')Re'$ とし，$\Lambda' = R_{e'}$ とおく．Λ' の巾等元を

$$f'_1 = \begin{pmatrix} 1_{A'} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ f'_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1_{A'} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ f'_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1_{B'} \end{pmatrix}$$

とおく．

($\Rightarrow$) Λ が self-duality をもつと仮定する．(2) と Λ (と Λ') 上の直既約射影加群の入射性より，[7, p.42] を用いれば，環同型写像 $\rho : \Lambda \to \Lambda'$ で，$\rho(f_i) = f'_i$ $(i = 1, 2, 3)$ を満たすものが存在することが分かる．したがって，Λ と Λ' の行列表現より，ρ は R の環自己同型写像 τ で $\tau(e) = e'$ を満たすものを導く．

($\Leftarrow$) R の環自己同型写像 τ で $\tau(e) = e'$ を満たすものが存在すると仮定する．$\rho : R_e \to R_{e'}$ を行列成分表示を用いて $\rho((r_{ij})) = (\tau(r_{ij}))$ によって定めれば，仮定よりこれは環同型写像であることが分かる．したがって (2) より R_e は self-duality をもつ．

注意 9. 後の例が示すように，上の定理において R_e は必ずしも self-duality をもつとは限らない．すなわち，一般に $R_e \ncong R_{e'}$ である．しかし，中山置換 σ の位数は有限である

から, R_e から始めて左 Morita dual の環をとることを有限回繰り返せば, 最初の環 R_e に戻る.

次の系は定理 8 の特別な場合である.

系 10. R を基本的 QF 環, I をその直交巾等元の完全集合, σ を I 上の中山置換とする. I の分割 $I = I_1 \dot{\cup} U_2 \dot{\cup} \cdots \dot{\cup} I_m$ で $\sigma(I_i) = I_{[i+1]}$ $(i = 1, 2, \ldots, m)$ を満たすものが存在すると仮定する. $e_i = e_{I_i}$, $R_{ij} = e_i R e_j$ とし,

$$\Lambda = \begin{pmatrix} R_{11} & R_{11} & R_{12} & \ldots & R_{1m} \\ J(R_{11}) & R_{11} & R_{12} & \ldots & R_{1m} \\ R_{21} & R_{21} & R_{22} & \ldots & R_{2m} \\ \vdots & \vdots & \vdots & & \vdots \\ R_{m1} & R_{m1} & R_{m2} & \ldots & R_{mm} \end{pmatrix}$$

とおく. このとき,

(1) Λ は両側 H 環である.

(2) Λ は

$$\Gamma = \begin{pmatrix} R_{22} & R_{22} & R_{23} & \ldots & R_{2m} & R_{21} \\ J(R_{22}) & R_{22} & R_{23} & \ldots & R_{2m} & R_{21} \\ R_{32} & R_{32} & R_{33} & \ldots & R_{3m} & R_{31} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ R_{m2} & R_{m2} & R_{m3} & \ldots & R_{mm} & R_{m1} \\ R_{12} & R_{12} & R_{13} & \ldots & R_{1m} & R_{11} \end{pmatrix}$$

に左 Morita dual である.

(3) Λ が self-duality をもつための必要十分条件は, $\tau(e_1) = e_2$ を満たす R の環同型写像 τ が存在することである.

この報告集の最後として, 前半の中山自己同型写像をもたない QF 環の例を用いて self-duality をもたない H 環の具体例を構成する.

例 11. A_i, U_i $(i = 1, 2, \ldots, 5)$ を例 5 と同じとする. したがって, ${}_{A_{[i+2]}}U_{[i+1]}{}_{A_i}$ は Morita duality を定め, すべての $A_1, A_2, \ldots, A_5$ は互いに非同型である.

$$\Lambda = \begin{pmatrix} A_5 & A_5 & U_4 & 0 & 0 & 0 \\ J(A_5) & A_5 & U_4 & 0 & 0 & 0 \\ 0 & 0 & A_3 & U_2 & 0 & 0 \\ 0 & 0 & 0 & A_1 & U_5 & 0 \\ 0 & 0 & 0 & 0 & A_4 & U_3 \\ U_1 & U_1 & 0 & 0 & 0 & A_2 \end{pmatrix}$$

とおく. A_5 と A_3 は非同型であるから, 系 10 より Λ は self-duality をもたない両側 H 環である.

各 A_i は 2×2 行列環であるから, Λ は 12×12 行列環である. e_i を (i, i)-行列単位とし, 単純加群 $T(e_i R)$ または $T(Re_i)$ を "i" で表せば, 直既約射影 Λ 加群の Loewy series は次の通りである.

Λ_Λ

1	2	3	4	5	6	7	8	9	10	11	12
3	4	2	5 5	6 6	7	8 8 8	9	10 10	11 11	12	1 1 1
2	5 5	4	6	7	8	9	10	11	12	1	3 3 3
4	6	5								3	2
5											4

${}_\Lambda\Lambda$

1	2	3	4	5	6	7	8	9	10	11	12
12	3 3	1	2	4 4	5	6 6 6	7	8 8	9 9	10	11 11 11
11	1 1	12	3 3	2 2	4	5	6	7	8	9	10
	12	11	1 1	3	2						
			12	1							

例 12. R を例 7 の中山自己同型写像をもたない QF 環とし, $e_1, e_2, \dots, e_5$ をその直交原始巾等元とする. R の中山置換は $e_i \mapsto e_{[i+3]}$ で巡回的である.

$$\Lambda = R_{e_1} = \begin{pmatrix} e_1Re_1 & e_1Re_1 & e_1R(1-e_1) \\ J(e_1Re_1) & e_1Re_1 & e_1R(1-e_1) \\ (1-e_1)Re_1 & (1-e_1)Re_1 & (1-e_1)R(1-e_1) \end{pmatrix}$$

とおく. Λ は 6 個の単純加群の同型類をもつ. 例 7 より e_1R と e_4R の組成列の長さは異なっている. したがって Theorem 8 より Λ は self-duality をもたない両側 H 環である.

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