Proceedings of the 33th Symposium On Ring Theory and Representation Theory

September 18th – 20th, 2000
Shimane, Japan

Edited By
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January, 2001
Naruto, Japan

第３３回環論及び表現論シンポジウム報告集

２０００年9月１８日－２０日
島根大学総合理工学部

２００１年1月
鳴門教育大学

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PREFACE

The 33th Symposium on Ring Theory and Representation Theory was held at Shimane, on September 18th - 20th, 2000.

The Volume presents thirteen articles given in the symposium. These articles contain advanced results toward the new century.

We would like to thank Professor Yasuyuki Hirano, Hidetoshi Marubayashi, Kiyoichi Oshiro, Yukio Tsushima and Kunio Yamagata for helpful suggestions concerning the symposium. Finally we wish to thank Professor A. Ueda and staffs of the department of Mathematics, shimane University, for their cooporation.

Shigeru Kobayashi

Naruto, January 2001

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SYMMETRY OF ALMOST HEREDITARY RINGS *

YOSHITOMO BABA AND HIROYUKI MIKI

In [6] an almost N-projective module is defined as a generalization of a N-projective module to characterize the lifting property. This module is further studied in the succeeding papers [4], [7], [8]. And in [10] M. Harada called a module M to be *almost projective* if M is almost N-projective for any finitely generated module N. We see that semisimple rings, serial rings, QF-rings and H-rings are well-characterized by the property of an almost projective module in [10], [11]. Using this remarkable module, in [9] he defined a *right almost hereditary ring* R, i.e., R is an artinian ring with J_R almost projective, where J is the Jacobson radical of R. On the other hand, it is well known that an artinian hereditary ring R is characterized by the following equivalent conditions:

(1) J_R is projective;
(2) ${}_RJ$ is projective;
(3) $E/Socle(E)$ is injective for any injective right R-module E;
(4) $E/Socle(E)$ is injective for any injective left R-module E.

Therefore a right almost hereditary ring is a generalization of an artinian hereditary ring. In this paper, first we characterize a right almost hereditary ring using left ideals in section 2 (we note that M. Harada already gave a structure theorem of it using right ideals in [9]). Further we consider the following generalized condition of (3):

$(\#)_r$ A factor module of E by its socle is a direct sum of an injective module and finitely generated almost injective modules for any injective right R-module E (not necessarily finitely generated).

Symmetrically we consider the left version $(\#)_l$. And we show that a ring R is a right almost hereditary ring if and only if it satisfies $(\#)_l$ using a characterization of a right almost hereditary ring given by left ideals. But M. Harada already showed that a right almost hereditary ring is not always a left almost hereditary ring in [9, p801]. That is, the equivalences (1) $\Leftrightarrow$ (4) and (2) $\Leftrightarrow$ (3) are generalized. But the other equivalences are not generalized.

In [9] he further considered the following stronger conditions than one of an almost hereditary ring :

* The detail version of this note will be submitted for publication elsewhere.

$(*)_r$ Every submodule of a finitely generated projective right R-module is almost projective.

$(**)_r$ The Jacobson radical of M is almost projective for any finitely generated almost projective right R-module M;

$(***)_r$ every submodule of a finitely generated almost projective right R-module is also almost projective.

In this paper we call an artinian ring R a *right strongly almost hereditary ring* (abbreviated *right SAH ring*) if R satisfies $(*)_r$. On the other hand, an artinian hereditary ring is also characterized by the following equivalent conditions:

(a) Every submodule of a projective right R-module is also projective;
(b) every submodule of a projective left R-module is also projective;
(c) every factor module of an injective right R-module is also injective;
(d) every factor module of an injective left R-module is also injective.

In section 3 we consider the following generalized condition of (c):

$(*^{\#})_r$ Every factor module of an injective right R-module is a direct sum of an injective module and finitely generated almost injective modules.

Similarly we define $(*^{\#})_l$ for left R-modules. The aim of Section 3 is to show that an artinian ring R is right SAH if and only if R satisfies $(*^{\#})_l$. But we see that the equivalence between a right SAH ring and an artinian ring which satisfies $(*^{\#})_r$ does not hold in general.

In [9] M. Harada also showed that an artinian ring R satisfies $(**)_r$ iff it satisfies $(***)_r$. In section 4 we consider the following generalized conditions of (c):

$(**^{\#})_r$ $M/Socle(M)$ is a direct sum of an injective module and finitely generated almost injective modules for any injective or finitely generated almost injective right R-module M;

$(***^{\#})_r$ every factor module of an injective or finitely generated almost injective right R-module is a direct sum of an injective module and finitely generated almost injective modules.

We also consider $(**^{\#})_l$ and $(***^{\#})_l$ for left R-modules. The aim of Section 4 is to show that an artinian ring R satisfies $(**)_r$ if and only if R satisfies $(**^{\#})_l$ if and only if R satisfies $(***^{\#})_l$. But we see that the equivalence between the two conditions $(**)_r$ and $(**^{\#})_r$ does not hold in general.

§1 Preliminaries

In this paper, we always assume that every ring is a basic artinian ring with identity and every module is unitary. Let R be a ring and let $P(R) = \{e_i\}_{i=1}^n$ be a complete set of pairwise orthogonal primitive idempotents in R. We denote the *Jacobson radical*, an *injective hull* and the *composition length* of a module M by $J(M)$, $E(M)$ and $|M|$, respectively. Especially, we put $J := J(R_R)$. For a module M we denote the *socle* of M by $S(M)$ and the *k-th socle* of M by $S_k(M)$ (i.e., $S_k(M)$ is a submodule of M defined by $S_k(M)/S_{k-1}(M) = S(M/S_{k-1}(M))$ inductively).

Let M and N be modules. M is called *almost N-projective* (resp. *almost N-injective*) if for any homomorphism $\phi : M \to L$ (resp. $\phi' : L \to M$) and any epimorphism $\pi : N \to L$ (resp. monomorphism $\iota : L \to N$) either there exists a homomorphism $\tilde{\phi} : M \to N$ (resp. $\tilde{\phi'} : N \to M$) such that $\phi = \pi\tilde{\phi}$ (resp. $\phi' = \tilde{\phi'}\iota$) or there exist a nonzero direct summand N' of N and a homomorphism $\theta : N' \to M$ (resp. $\theta' : M \to N'$) such that $\phi\theta = \pi i$ (resp. $\theta'\phi' = p\iota$), where i is an inclusion of N' in N (resp. p is a projection on N' of N).

A ring R is called *right* (resp. *left*) *hereditary* if every submodule of a projective right (resp. left) R-module is also projective. It is well known that a perfect or neotherian ring is right hereditary iff it is left hereditary (see, for instance, [13, Chapter 9]). So we call a right hereditary ring a *hereditary* ring since rings are artinian in this paper. Further an artinian ring R is hereditary iff J_R is projective (see, for instance, [1, 18. Exercises 10 (2)]). Furthermore an artinian ring R is hereditary iff $E/S(E)$ is injective for any injective right R-module E. We also see that R is hereditary iff E/A is injective for any submodule A of an injective module E by [1, 18. Exercises 10 (1)].

Further M is called *almost projective* (resp. *almost injective*) if M is always almost N-projective (resp. almost N-injective) for any finitely generated R-module N. The following is an important characterization of an almost projective module given by M. Harada.

Lemma 1 *([10, Corollary $1^{\#}$]). Suppose that M is an indecomposable finitely generated left R-module. Then M is almost injective but not injective if and only if there exist an indecomposable injective left R-module E and a positive integer k such that $M \cong J^kE$ and J^iE is projective for any $i = 0, \cdots, k-1$.*

And we call an artinian ring R a *right almost hereditary ring* if J is almost projective as a right R-module. By [10, Theorem 1] this definition is equivalent to the condition: $J(P)$ is almost projective for any finitely generated projective right R-module P.

A module is called *uniserial* if its lattice of submodules is a finite chain, i.e., any two submodules are comparable. An artinian ring R is called a *right serial* (resp. *co-serial*) *ring* if every indecomposable projective (resp. injective) right R-module is uniserial. And we call a ring R a *serial ring* if R is a right and left serial ring. Let $f_1, f_2, \cdots, f_n$ be primitive idempotents in a serial ring R. Then a sequence $\{f_1R, f_2R, \cdots, f_nR\}$ (resp. $\{Rf_1, Rf_2, \cdots, Rf_n\}$) of indecomposable projective right (resp. left) R-modules is called a *Kupisch series* if $f_jJ/f_jJ^2 \cong f_{j+1}R/f_{j+1}J$ (resp. $Jf_j/J^2f_j \cong Rf_{j+1}/Jf_{j+1}$) holds for any $j = 1, \cdots, n-1$. Further $\{f_1R, f_2R, \cdots, f_nR\}$ (resp. $\{Rf_1, Rf_2, \cdots, Rf_n\}$) is called a *cyclic Kupisch series* if it is a Kupisch series and $f_nJ/f_nJ^2 \cong f_1R/f_1J$ (resp. $Jf_n/J^2f_n \cong Rf_1/Jf_1$) holds. Let R be a serial ring with a Kupisch series $\{f_1R, f_2R, \cdots, f_nR\}$. If $f_nJ = 0$ and $P(R) = \{f_1, \cdots, f_n\}$, then R is called a serial ring *in the first category*. And if $\{f_1R, f_2R, \cdots, f_nR\}$ is a cyclic Kupisch series and $P(R) = \{f_1, \cdots, f_n\}$, then R is called a serial ring *in the second category*.

§2 A structure theorem for an almost hereditary ring

The following is a structure theorem for a right almost hereditary ring given by M. Harada.

Theorem 2 *([9, Theorem 1]). A ring is right almost hereditary if and only if it is a direct sum of the following rings:*

(i) Hereditary rings;

(ii) serial rings;

(iii) rings R with $P(R) = \{h_1, \cdots, h_m, f_1^{(1)}, f_2^{(1)}, \cdots, f_{n_1}^{(1)}, f_1^{(2)}, \cdots, f_{n_2}^{(2)}, f_1^{(3)}, \cdots, f_{n_k}^{(k)}\}$ *such that, for each* $l = 1, \cdots, k$ *we put* $S_l := \sum_{j=1}^{n_l} f_j^{(l)}$ *and* $p_l := |f_1^{(l)}R_R|$*, the following four conditions hold for any* $l = 1, \cdots, k$ *and* $s = 1, \cdots, m$*,*

(a) S_lRS_l *is a serial ring in the first category with* $\{f_1^{(l)}RS_l, f_2^{(l)}RS_l, \cdots, f_{n_l}^{(l)}RS_l\}$ *a Kupisch series of right* S_lRS_l*-modules,*

(b) $S_l R(1-S_l) = 0$, $(h_1 + \cdots + h_m)R(f_1^{(l)} + \cdots + f_{p_l-1}^{(l)}) \neq 0$ *and* $(h_1 + \cdots + h_m)R(f_{p_l}^{(l)} + \cdots + f_{n_l}^{(l)}) = 0$,

(c) $(h_s J/h_s J^2) f_j^{(l)} = \bar{0}$ *for any* $j \geq 2$,

we let α_l *be a positive integer such that* $f_1^{(l)}R/f_1^{(l)}J^j$ *is injective for any* j $(\geq \alpha_l + 1)$ *but* $f_1^{(l)}R/f_1^{(l)}J^{\alpha_l}$ *is not injective (see Lemma ??(3) below as for the existence of* α_l*) and put* $H := \sum_{s=1}^{m} h_s + \sum_{l=1,j=1}^{k\ \ \alpha_l} f_j^{(l)}$, *then*

(d) HRH *is a hereditary ring.*

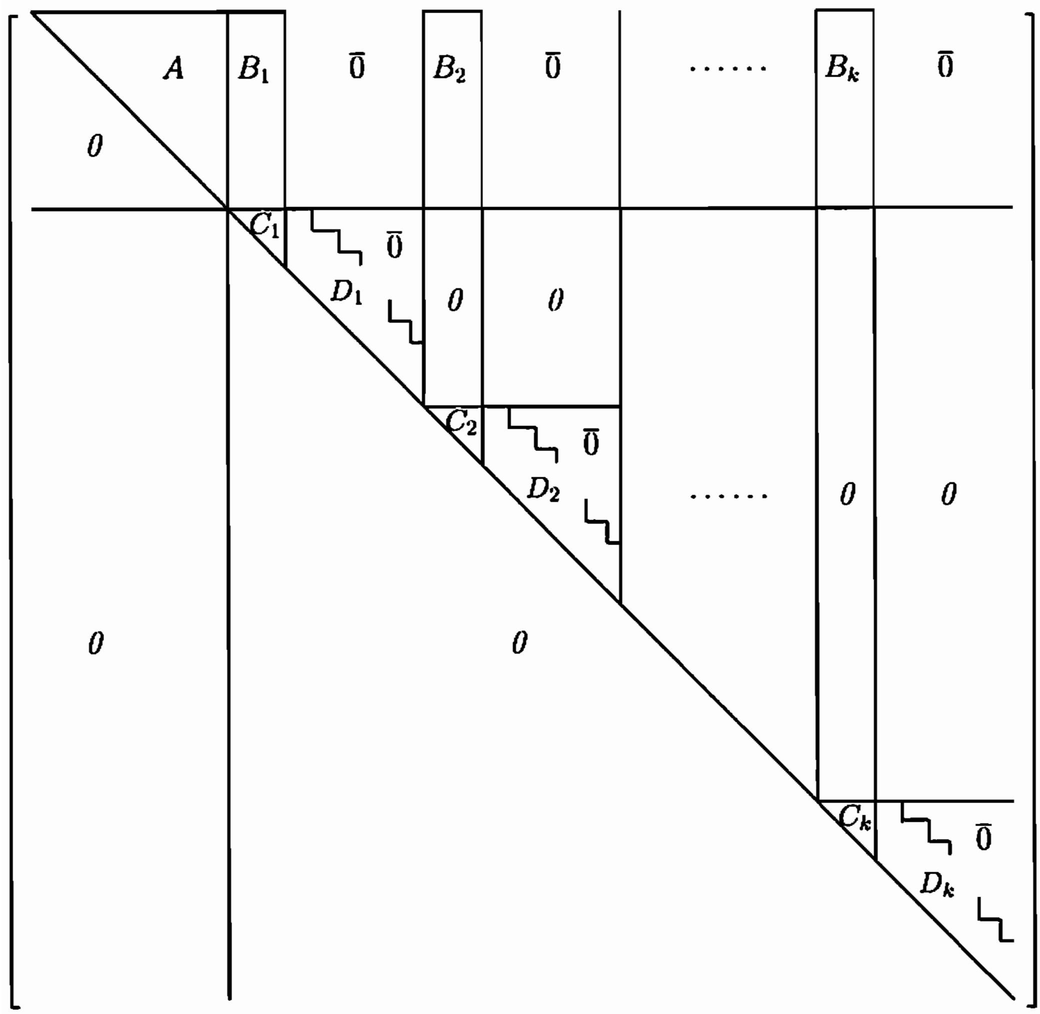
Remark 3 . *By [5] we know that a hereditary ring is represented as*

$$\begin{bmatrix} D_1 & M_{1,2} & M_{1,3} & \cdots & \cdots & M_{1,n} \\ 0 & D_2 & M_{2,3} & \cdots & \cdots & M_{2,n} \\ \vdots & & 0 & \cdots & \cdots & \\ & & & \ddots & \cdots & \\ & & & 0 & D_{n-1} & M_{n-1,n} \\ 0 & & \cdots & & 0 & D_n \end{bmatrix},$$

where $D_1, D_2, \cdots, D_n$ *are division rings and* M_{ij} *is a left* D_i*-right* D_j*-bimodule for any* i, j. *Further by [12] a serial ring in the first category is represented as the following factor ring:*

$$\left[\begin{array}{cccccccccccccccc} D & D & \cdots & \cdots & \cdots & D & \bar{0} & & & & \cdots & & & & & \bar{0} \\ \ddots & \ddots & \ddots & \cdots & \cdots & \vdots & \vdots & & & & \cdots & & & & & \bar{0} \\ & 0 & D & D & \cdots & D & \bar{0} & & & & \cdots & & & & & \bar{0} \\ & & 0 & D & D & \cdots & & D & \bar{0} & & & \cdots & & & & \bar{0} \\ & & & 0 & D & \cdots & & \vdots & \vdots & & & \cdots & & & & \bar{0} \\ \vdots & & & & 0 & \ddots & & D & \bar{0} & \cdots & & & & & & \bar{0} \\ \vdots & & & & & \ddots & & \ddots & & \cdots & \cdots & & D & \bar{0} & \cdots & \bar{0} \\ \vdots & & & & & & & \ddots & & \ddots & \cdots & & \vdots & \vdots & \cdots & \bar{0} \\ \vdots & & & & & & & & & 0 & D & \cdots & D & \bar{0} & \cdots & \bar{0} \\ & & & & & & & & & & 0 & D & \cdots & & \cdots & D \\ & & & & & & & & & & & 0 & D & & & \vdots \\ & & & & & & & & & & & & 0 & & \ddots & \vdots \\ 0 & & & & \cdots & \cdots & & \cdots & & \cdots & & & & & 0 & D \end{array}\right],$$

where D is a division ring. So a ring R in Theorem 2(iii) is represented as the following factor ring:



,

where $1_A = \sum_{l=1}^{m} h_l$, $1_{C_l} = \sum_{j=1}^{\alpha_l} f_j^{(l)}$ *and* $1_{C_l+D_l} = \sum_{j=1}^{n_l} f_j^{(l)}$ *for each* l. *Further* $HRH = A \cup (\cup_{l=1}^{k}(B_l \cup C_l))$ *and* $S_l R S_l = C_l \cup D_l$.

In Theorem 2 a right almost hereditary ring is characterized by right ideals. Here we shall characterize a ring in Theorem 2(iii) by left ideals.

First we characterize α_l in Theorem 2(iii) not using the right module structure.

Lemma 4. *Let R be a ring satisfying (a), (b) in Theorem 2(iii) and α_l as in Theorem 2(iii). Define an integer α'_l to satisfy $(h_1 + \cdots + h_m)Rf_j^{(l)} = 0$ for any $j = \alpha'_l + 1, \cdots, n_l$ but $(h_1 + \cdots + h_m)Rf_{\alpha'_l}^{(l)} \neq 0$. Then $\alpha_l = \alpha'_l$.*

Using Lemma 4 we have a lemma.

Lemma 5.

(1) Let R be a ring in Theorem 2(iii). We may assume that $h_sRh_t = 0$ for any $s > t$ by the representation form of a hereditary ring (see Remark 3). Then the following condition (e) holds:

(e) $h_sJ \cong (\oplus_{i=s+1}^{m}(h_iR)^{u_i}) \oplus (\oplus_{l=1}^{k}(f_1^{(l)}R/\ f_1^{(l)}J^{\alpha_l})^{v_l})$ as right R-modules for some non-negative integers $u_{s+1}, \cdots, u_m, v_1, \cdots, v_k$.

(2) Suppose that a ring R satisfies (a), (b), (e), then (c) and (d) hold.

Hence (a), (b), (c), (d) in Theorem 2(iii) can be replaced by (a), (b), (e).

The following gives a characterization of a ring in Theorem 2(iii) using left ideals.

Theorem 6. *Let R be a ring with $P(R) = \{h_1, \cdots, h_m, f_1^{(1)}, f_2^{(1)}, \cdots, f_{n_1}^{(1)}, f_1^{(2)}, \cdots, f_{n_2}^{(2)}, f_1^{(3)}, \cdots, f_{n_k}^{(k)}\}$. $P(R)$ satisfies (a),(b),(c),(d) in Theorem 2(iii) if and only if the following five condition hold for any $l = 1, \cdots, k$, we put $S_l := \sum_{j=1}^{n_l} f_j^{(l)}$,*

(a′) S_lRS_l is a serial ring in the first category with $\{S_lRf_{n_l}^{(l)}, S_lRf_{n_l-1}^{(l)}, \cdots, S_lRf_1^{(l)}\}$ a Kupisch series of left S_lRS_l-modules,

(b′) $S_lR(1 - S_l) = 0$ and $(h_1 + \cdots + h_m)RS_l \neq 0$,

(c′) $Jf_j^{(l)}/J^2f_j^{(l)}$ is simple as a left R-module for any $j = 2, \cdots, n_l$,

we let α'_l be the same integer as in Lemma 4 and put $H' := \sum_{s=1}^{m} h_s + \sum_{l=1,j=1}^{k,\ \alpha'_l} f_j^{(l)}$, then

(d′) $H'RH'$ is a hereditary ring , and

(f) $E({}_RRf_1^{(l)}/Jf_1^{(l)})$ is projective as a left R-module for any $l = 1, \cdots, k$.

Then we note that $\alpha'_l = \alpha_l$, and so $H' = H$ and (d′) coincides with (d), where H and (d) are as in Theorem 2(iii).

By using Theorem 6, we can show the following theorem, which is the main theorem in this paper.

Theorem 7. *R satisfies* $(\#)_l$ *if and only if R is a right almost hereditary ring.*

Remark 8 . *In [9, p801] M. Harada already showed that a right almost hereditary ring is not always a left almost hereditary ring. We shall give an example for this suggestion below. (see Example 11)*

§3 Strongly almost hereditary rings

Before considering right SAH rings, we define a special (serial) ring. A serial ring is called a *strongly serial ring* if it is a direct sum of indecomposable serial rings R with a Kupisch series $\{f_{1,1}R,\ f_{1,2}R, \cdots, f_{1,\beta_1}R,\ f_{2,1}R, \cdots, f_{m,\beta_m}R\}$ such that $|f_{i,\beta_i}R| = 2$ for any $i = 1, \cdots, m-1$ and $|f_{m,\beta_m}R| = 1$ or 2, where $P(R) = \{f_{i,j}\}_{i=1,j=1}^{m\ \ \beta_i}$ and $f_{i,j}R$ is injective iff $j = 1$. Then, if $|f_{m,\beta_m}R| = 1$ (resp. $= 2$), then R is a serial ring in the first (resp. second) category. Further we can easily check the following characterization of a strongly serial ring.

The following is a structure theorem of a right SAH ring given by M. Harada.

Theorem 9 *([9, Theorem 3]). A ring is right SAH if and only if it is a direct sum of the following rings:*

(i) Hereditary rings;

(ii) strongly serial rings;

(iii) rings R with $P(R) = \{h_1, \cdots, h_m, f_1^{(1)}, f_2^{(1)}, \cdots, f_{n_1}^{(1)}, f_1^{(2)}, \cdots, f_{n_2}^{(2)}, f_1^{(3)}, \cdots, f_{n_k}^{(k)}\}$ *such that, for each* $l = 1, \cdots, k$ *we put* $S_l := \sum_{j=1}^{n_l} f_j^{(l)}$ *and* $H := \sum_{s=1}^{m} h_s + \sum_{l=1}^{k} f_1^{(l)}$*, the following three conditions hold for any* $l = 1, \cdots, k$*:*

(x) S_lRS_l *is a strongly serial ring in the first category with a Kupisch series* $\{f_1^{(l)}RS_l,\ f_2^{(l)}RS_l, \cdots, f_{n_l}^{(l)}RS_l\}$ *of right* S_lRS_l*-modules,*

(y) $S_lR(1-S_l) = 0$, $(h_1+\cdots+h_m)Rf_1^{(l)} \neq 0$ *and* $(h_1+\cdots+h_m)R(f_2^{(l)} + \cdots + f_{n_l}^{(l)}) = 0$, *and*

(z) *HRH is a hereditary ring.*

We note that by Lemma 4 a ring in Theorem 9 (iii) coincides with a ring in Theorem 2 (iii) if it satisfies that $\alpha_l = 1$ and S_lRS_l is a strongly serial ring for any $l = 1, \cdots, k$, where α_l and S_l are as in it.

Moreover, the condition (ii) in the above Theorem is not the same as [9, Theorem 3], i.e., when R is a serial ring in the second category, he wrote that "R is a serial ring in the second category with $J^2 = 0$". But this original condition is not suitable. We give an example. Let R be a serial ring in the second category with $P(R) = \{f_1, f_2, f_3, f_4\}$ such that $\{f_1R, f_2R, f_3R, f_4R\}$ is a Kupisch series and $|f_1R| = 4$, $|f_2R| = 3$, $|f_3R| = 2$, $|f_4R| = 2$. Then R is a strongly serial ring. So it is right SAH by the following proof. But $J^2 \neq 0$. In an unpublished lecture note written by M. Harada the condition is already corrected.

The purpose of this section is to show the following theorem.

Theorem 10. *A ring R is right SAH if and only if R satisfies $(*^{\#})_l$.*

A right SAH ring does not always satisfy $(*^{\#})_r$ and a ring satisfying $(*^{\#})_r$ is not always a right SAH ring, Now we give an example.

Example 11. Consider a factor ring

$$R := \begin{bmatrix} D & D & 0 & D & \bar{0} & \bar{0} \\ 0 & D & 0 & D & \bar{0} & \bar{0} \\ 0 & 0 & D & D & \bar{0} & \bar{0} \\ 0 & 0 & 0 & D & D & \bar{0} \\ 0 & 0 & 0 & 0 & D & D \\ 0 & 0 & 0 & 0 & 0 & D \end{bmatrix},$$

where D is a division ring. And we consider that R is a ring by the ordinary addition and the multiplication of matrices. Put $H := e_1 + e_2 + e_3 + e_4$ and $S_1 := e_4 + e_5 + e_6$, where e_i is the (i,i)-matrix unit for any i.

Then HRH is a hereditary ring and S_1RS_1 is a strongly serial ring in the first category. And R is a ring in Theorem 9(iii), i.e., R is a right SAH ring.

But we claim that R does not satisfies $(*^{\#})_r$. e_4R is an injective left R-module with $e_4R/S(e_4R) \cong e_4R/e_4J$. And $e_4R/S(e_4R)$ is not injective.

Further $e_4R/S(e_4R)$ is not almost injective by [10, Corollary $1^{\#}$] since $e_1R \oplus e_3R$ is a projective cover of $E(e_4R/e_4J)$.

By Theorem 10 R satisfies $(*^{\#})_l$ but is not a left SAH ring.

§4 Stronger conditions than that of a SAH ring

The following is a structure theorem of an artinian ring which satisfies $(**)_r$ and $(***)_r$ which are stronger conditions than that of a right SAH ring:

Theorem 12 *([9, Theorem 4]). For a ring the following are equivalent:*

(a) It satisfies $(**)_r$;

(b) it satisfies $(***)_r$;

(c) it is a direct sum of the following rings:

(i) Hereditary rings which are not serial;

(ii) serial rings with the radical square zero;

(iii) rings R in Theorem 9 (iii) such that HRH is not a serial ring and $J(S_lRS_l)^2 = 0$ for any $l = 1, \cdots, k$, where H and S_l are as in Theorem 9 (iii).

The purpose of this section is to show the following theorem.

Theorem 13. *For a ring R the following are equivalent:*

(a) R satisfies $(**)_r$ ($\Leftrightarrow (***)_r$);

(b) R satisfies $(**^{\#})_l$;

(c) R satisfies $(***^{\#})_l$.

Acknowledgement

The condition $(\#)_r$ in this paper is first considered by M. Harada. He called a ring satisfying this condition a right co-almost hereditary ring and gave the structure theorem which is a dual form of Theorem 2 in an unpublished paper. The authors thank M. Harada for using results in this unpublished paper.

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D-加群の V-極小自由分解
— 動機づけを中心に

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2000 年 11 月 28 日

D-加群の自由分解はグレブナ基底の方法を用いて具体的に構成できる. 実装もいくつかあり, `Macaulay2` [2] や `kan/k0` [7] による実装が存在する. 自由分解のなかで, 同次化ワイル代数 $D^{(h)}$ 上の加群 M の自由分解で, V-filtration による $\mathrm{gr}_V(M)$ の極小自由分解を自然に導く自由分解を V-極小自由分解とよぶ. この自由分解の研究にわれわれがいたった動機を説明するのが, この文章の目的である. 講演では前半でこの動機を説明し, 後半では V-極小自由分解の技術的な話題にもふれた. 後半の話題については, [5], [6] でくわしく議論しているので, 本稿ではふれない.

なお, 拙文は de Rham コホモロジの計算と D-加群の $(-w, w)$-極小自由分解に関する, 松江での 9 月 18 日の講演および東京での 11 月 23 日の講演をもとにしている. 両方の研究集会の報告集に同一の原稿を送っているが御容赦ねがいたい.

1 de Rham コホモロジ

D 加群は図形の幾何を理解するのに活用できる. 準備のために, 20 世紀の前半からなかばにかけて展開した, de Rham コホモロジの理論を復習しておこう.

X を n 次元複素多様体とする. X を $2n$ 次元のなめらかな実多様体とみなす. Ω^p を X 上の C^∞ 級の p 形式の集合をあらわす. これは実ベクトル空間 (無限次元) とみなせる.

外微分を用いて次のような複体をつくれる ($d \circ d = 0$).

$$\cdots \longrightarrow \Omega^p \xrightarrow{d} \Omega^{p+1} \xrightarrow{d} \Omega^{p+2} \xrightarrow{d} \cdots$$

次のコホモロジ群を i-次の de Rham コホモロジ群とよぶ.

$$H^i(X, \mathbf{R}) = \frac{\mathrm{Ker}\,(\Omega^i \xrightarrow{d} \Omega^{i+1})}{\mathrm{Ker}\,(\Omega^{i-1} \xrightarrow{d} \Omega^i)}.$$

これは幾何的な特異コホモロジ群に一致する.
例: $X = \mathbf{C} \setminus \{0\}$ の場合を考えよう.

$$\omega = \frac{xdy - ydx}{x^2 + y^2} \in \Omega^1$$

は $d\omega = 0$ であり、また $\int_C \omega = 2\pi$ である. ここで C は原点の周りをまわる円周である. $d\varphi = \omega$ となる X 上の 0-form はないことを示そう. そのようなものがあるとすると, $\int_C \omega = \int_C d\varphi = \int_{\partial C} \varphi = 0$ となり矛盾が生じる. よって, ω は H^1 において 0 でない. このような議論をやっていくと, 結局 $\dim_{\mathbf{R}} H^i(X, \mathbf{R}) = 1$, $i = 0, 1$ がわかる.

2 Buchberger アルゴリズムと計算環論, 計算環論用システム

話はかわるが, 筆者 (高山) は研究は “こだわり” がないとなかなかできないもんだとおもってる. 筆者たちの場合 こだわりは, アルゴリズミックに数学的対象が計算できるかどうかということである. この de Rham コホモロジ群をアルゴリズミックに計算できないものであろうか?

こんなふうに思うのも時代的背景がある. 1980 年代, 90 年代は, グレブナ基底を求める Buchberger アルゴリズムをもとに, この実装の効率化, 環論での不変量計算の成功, システムの設計などがおおきく進んだ時代である. なんとかこういったことを de Rham コホモロジでできないかと思っても不自然ではない.

I を多項式環のイデアル, $w \in \mathbf{R}^n$ を weight としよう. イデアル I の生成元 $\{g_1, \ldots, g_m\}$ が weight w に関する I のグレブナ基底であるとは,

$$\mathrm{in}_w(I) = \langle \mathrm{in}_w(g_1), \ldots, \mathrm{in}_w(g_m) \rangle$$

であることである. ここで $\mathrm{in}_w(f)$ は多項式 f に現れる項のなかで, w に関する次数が一番大きいものを足したものである.

Buchberger アルゴリズムは, グレブナ基底を計算するアルゴリズムである. Buchberger アルゴリズムは, ユークリッドの互除法 ($w = 1, n = 1$) および行列に対するガウスの消去法 ($w_1 << w_2 << \ldots << w_n$, イデアル I は, 1 次式で生成されている) の拡張である. グレブナ基底と Buchberger アルゴリズムに対しては, いろいろといい教科書があり, たとえば [1] の本は入門として評価がたかい.

グレブナ基底でなにができるのかを, 上のアナログで説明しよう.

1. $\mathbf{k}[x_1, x_2, \ldots, x_l] \cap I$ を求めることができる (消去法). これは ガウス消去のアナログ.

2. 多項式 f_i, d が与えられたとき, $\sum a_i f_i = d$ を満たす a_i を全て求めることができる. これは, ユークリッドの互除法の応用で, 1 変数多項式環で不定方程式が解けることのアナログ.

不定方程式がとければ, 自由分解が構成できる.

$$\cdots \longrightarrow R^{b_2} \xrightarrow{\psi_2} R^{b_1} \xrightarrow{\psi_1} R \longrightarrow R/I \longrightarrow 0.$$

ここで $R = \mathbf{k}[x_1, \ldots, x_n]$, ψ_i は, 多項式を成分とする行列, $\mathrm{Ker}(\psi_i) = \mathrm{Im}(\psi_{i+1})$.

これは面白いということで, Macaulay が 1970 年代の終りから, 1980 年代の始めにかけてまず開発された. その後, CoCoA, Singular, Macaulay2 などが開発された. (asir, gb, F4 等は代数方程式を解くという別の動機から開発された.) このようなシステムの開発で, 環論における不変量たとえば, ヒルベルト多項式, 極小自由分解の betti 数などが実際に計算できるようになった.

ソフトの開発の過程で計算機科学的な問題, データ型, 言語, 分散アルゴリズムの記述など数学システム開発の問題が生じてきた. これらの本格的な基礎がない状態では, なかなか先にすすめなくなっている. そこで現在, たとえば Macaulay2 は, 数学向けの言語設計の問題に挑戦してるし, `openxm.org` は, データ型, 分散アルゴリズムなどの問題に挑戦している.

3 ワイル代数 D と Annihilating operators

以上のように, 計算可換環論は Buchberger アルゴリズムを用いていろいろな不変量計算のアルゴリズムを与えた. しかし, 多項式の世界だけでは計算できないこともあった. たとえば, $\mathbf{Q}[x, 1/f]$ は $\mathbf{Q}[x]$ 加群としては有限生成でない. これが D 加群を導入する一つの理由である.

定理 [4] $f \in \mathbf{Q}[x_1, \ldots, x_n]$ に対して, $\dim_{\mathbf{C}} H^i(\mathbf{C}^n \setminus V(f), \mathbf{C})$ は (D の計算を用いて) 計算可能.

実装: kan/sm1 または Macaulay2.

このアルゴリズムは, おおざっぱにいって, グレブナ基底の二つの応用, 消去と自由分解の構成 (1 次不定方程式を解く), を D でおこなうことにより実現された.

例: $f = x^3 - y^2z^2 + y^2 + z^2$ のとき, $H^i(\mathbf{C}^3 \setminus V(f), \mathbf{C})$ の次元 h_i は, $h_0 = h_1 = 1$, $h_2 = 0$, $h_3 = 8$. (asir を用いて `ifplot` で $y = z$ のときの 図を書いてみるとおもしろい: `ox_launch(0,"ox_plot"); ifplot(x^3-y^4+2*y^2);`).

さてコホモロジ群の次元をどのようにして計算するのか説明しよう.

Step 1: D をワイル代数 (たとえば堀田の本 [3] を参照) とするとき, 左 D 加群として,

$$\mathbf{Q}[x, 1/f] \simeq D/I$$

となる D の左イデアルを求める.

これをもとめるには, Malgrange の仕事をもとにした Oaku のアルゴリズムがある. これの概略を説明しよう.

$\mathrm{Ann}(f^s) = \{\ell \in D[s] \mid \ell \bullet f^s = 0\}$ とおく. このとき

$$\langle t - f(x), \frac{\partial f}{\partial x_i}\partial_t + \partial_{x_i} : i = 1, \ldots, n\rangle \cap \mathbf{Q}\langle t\partial_t, x, \partial_{x_1}, \ldots, \partial_{x_n}\rangle$$

で, $-t\partial_t$ を s でおきかえたものが $\mathrm{Ann}(f^s)$ と一致する. この事実は, $\delta(t - f(x))$ の Mellin 変換が f^{s-1} であるということから直観的に説明できる.

さて,

$$^\exists L(x, \partial_x, s) f^{s+1} = b(s) f^s$$

をみたす最小次数の多項式 $b(s)$ を f の Bernstein-Sato 多項式という. $b(s)$ は単項生成イデアル $\langle f, \mathrm{Ann}(f^s)\rangle \cap \mathbf{Q}[s]$ の生成元である. 例えば, 先の例の f では $b(s) = (3s+4)(s+1)(3s+5)$ となる. $-r_0$ を $b(s) = 0$ の最小の整数根とするとき, $\mathbf{Q}[x, 1/f] \simeq D\frac{1}{f^{r_0}}$ となる. $\mathrm{Ann}(f^s)|_{s \to -r_0}$ が求める I である.

Step 2: 次に $M = D/I$ の自由分解を構成する. 任意の自由分解ではだめで, weight vector $(-w, w)$ に適合している (adapted) 分解をとらないといけない. この自由分解については [5] および [6] を見よ. この自由分解を

$$\cdots \xrightarrow{\psi_2} D^{b_1} \xrightarrow{\psi_1} D^{b_0} \longrightarrow M \longrightarrow 0$$

とする. この自由分解は Buchberger アルゴリズムで計算できる. (自由分解の計算は本質的に 1 次不定方程式の解の基底を求める計算である. 前の節の Buchberger アルゴリズムでできることを参照.) b_i を betti 数とよぶ.

さて Grothendieck-Deligne の比較定理によると

$$H^i(X \setminus V(f), \mathbf{C}) \simeq H^{i-n}(\cdots \xrightarrow{1\otimes\psi_2} D/\partial D \otimes_D D^{b_1} \xrightarrow{1\otimes\psi_1} D/\partial D \otimes_D D^{b_0} \longrightarrow 0) \otimes \mathbf{C}$$

がなりたつ. 右辺は $\mathbf{Q}$ ベクトル空間の複体とみなす. D の元を成分とする行列 ψ_i の具体的表示より計算してやれば, コホモロジ群の次元がわかる. 技術的には, 複体にでてくるベクトル空間の次元が無限次元であるので, そこをうまく処理して計算しないといけない.

技術的には結局, 消去法, 一次不定方程式の解と比較定理でコホモロジの計算アルゴリズムを得たことになる.

このアルゴリズムを実際実装してみるとわかるが, なるべく betti 数の小さい weight $(-w, w)$ に適合した自由分解をつくるのが効率的計算に有利であるように思える. それが $(-w, w)$-極小自由分解の研究をはじめた最初の動機であった. $(-w, w)$-極小自由分解の betti 数自体が何か幾何的意味のある量であるかどうかはまだ答えのわからない問題である.

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Computing de Rham cohomology groups

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Goal

Theorem (1998,2000): For any $f \in \mathbf{Q}[x_1,\dots,x_n]$, $\dim_{\mathbf{C}} H^k(\mathbf{C}^n \setminus V(f), \mathbf{C})$ is computable (by Gröbner basis computation in D and $(-w,w)$-minimal free resolution).

Gröbner basis

I : an ideal of tbe ring of polynomials.

$w \in \mathbf{R}^n$: a weight.

A set of generators $\{g_1,\dots,g_m\}$ of the ideal I is called Gröbner basis when

$$\mathrm{in}_w(I) = \langle \mathrm{in}_w(g_1),\dots,\mathrm{in}_w(g_m)\rangle$$

Here, $\mathrm{in}_w(f)$ is the subsum of the highest w-order terms of f.

$$\mathrm{ord}_w(f) = \max_{\alpha\in E} w\cdot\alpha,\ f=\sum_{\alpha\in E} c_\alpha x^\alpha$$

Buchberger algorithm computes Gröbner basis.

1980's, 1990's: efficiency, invariants, systems.

Buchberger algorithm is a generalization of the Euclidean algorithm and the Gaussian elimination.

What we can do by Buchberger algorithm?

1. Elimination

$$\mathrm{k}[x_1,\dots,x_n]\cap I$$

2. Solving linear indefinite equations, Syzygy. For given f_i and d, find all a_i satisfying

$$\sum a_i f_i = d.$$

Computing free resolutions and Hilbert functions.

$\mathrm{k}\left[x,\frac{1}{f}\right]$ was difficult in computional algebra.

What is D?

$$D = \mathbf{Q}\langle x_1,\dots,x_n,\partial_1,\dots,\partial_n\rangle$$

$$x_ix_j = x_jx_i,\ \partial_i\partial_j = \partial_j\partial_i,\ \partial_ix_j = x_j\partial_i + \delta_{ij}.$$

Weyl algebra, the ring of differential operators.

For $(u,v)\in\mathbf{R}^n\times\mathbf{R}^n$, $(u_i+v_i\geq 0)$

$$\mathrm{ord}_{(u,v)}(f) = \max_{(\alpha,\beta)\in E} u\cdot\alpha+v\cdot\beta,\ f=\sum_{(\alpha,\beta)\in E} c_{\alpha\beta}x^\alpha\partial^\beta$$

Computer Demo

de Rham cohomology group

X : n-dimensional complex manifold.

X can be regarded as $2n$-dimensional smooth real manifold.

Ω^p : the space of smooth p forms on X.

$(d\circ d = 0)$.

$$\cdots\longrightarrow\Omega^p\xrightarrow{d}\Omega^{p+1}\xrightarrow{d}\Omega^{p+2}\xrightarrow{d}\cdots$$

$$H^i(X,\mathbf{C}) = \frac{\mathrm{Ker}\,(\Omega^i\xrightarrow{d}\Omega^{i+1})}{\mathrm{Im}\,(\Omega^{i-1}\xrightarrow{d}\Omega^i)}.$$

Example: $X=\mathbf{C}\setminus\{0\}$. $\omega = \frac{xdy-ydx}{x^2+y^2}\in\Omega^1$ spans the $H^1(X,\mathbf{C})$.

How to compute H^i by an algorithmic method?

Step 1. Find I such that $\mathbf{Q}[x,1/f]\simeq D/I$. (Malgrange, 70's)

$$\mathrm{Ann}\,f^s = \{\ell\in D[s]\mid \ell f^s = 0\}$$

$$\mathrm{Ann}\,f^s = \langle t-f(x), \frac{\partial f}{\partial x_i}\partial_t + \partial_{x_i}\rangle\cap\mathbf{Q}\langle t\partial_t, x, \partial_x\rangle,$$

$s\leftrightarrow -t\partial_t$ (Elimination).

Example: $f = x(1-x)$,

$$\mathrm{Ann}\,f^s = D\cdot\{x(1-x)\partial_x - s(1-2x)\}.$$

The minimal degree polynomial $b(s)$ satisfying

$$^{\exists}L(x, \partial_x, s) \in D[s],\ Lf^{s+1} = b(s)f^s$$

is called the b-function of f. (Sato, Gel'fand, Berstein, Kashiwara, ... 60's, 70's.)
Example: $f = x(1-x)$, $b(s) = s+1$, $L = (1-2x)\partial_x + 4(1+s)$.

$$\langle b(s)\rangle = \langle \mathrm{Ann}\, f^s, f\rangle \cap \mathbf{Q}[s].$$

$(Lf - b)f^s = 0$.
Let $-r_0$ be the minimal integral root of $b(s) = 0$. Then,

$$\mathbf{Q}\left[x, \frac{1}{f}\right] \simeq D\frac{1}{f^{r_0}} \simeq D/I,$$

$I = \mathrm{Ann}\, f^s|_{s=-r_0}$
Example: $D/(x(1-x)\partial_x + (1-2x))$.
Put

$$J = I|_{\partial_i \to -x_i, x_i \to \partial_i}.$$

Computer Demo

Step 2: computing $(-w, w)$-minimal free resolution of D/J

$$D^{(h)} = \mathbf{Q}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n, h\rangle,$$

$\partial_i x_i = x_i \partial_i + h^2$.
$M = D^{(h)}/J^{(h)}$, $w \in \mathbf{Z}^n_{>0}$. $D^{(h)}$ will be denoted by D.
For $m \in \mathbf{Z}^r$, define

$$\begin{aligned} & F^k_{(-w,w)}(D^r[m]) \\ = & \oplus_{i=1}^r \left\{p_i \in D \,\middle|\, \mathrm{ord}_{(-w,w)}(p_i) + m_i \le k\right\}. \end{aligned}$$

Let

$$\cdots \xrightarrow{\psi_2} D^{r_1}[m^1] \xrightarrow{\psi_1} D^{r_0}[m^0] \longrightarrow M \longrightarrow 0 \quad (A)$$

be a $(-w, w)$-graded free resolution of M. Here, $m^i \in \mathbf{Z}^{r_i}$ is a degree shift.
(A) is called $(-w, w)$-adapted (strict) when

$$F^k_{(-w,w)}((\mathrm{A}))$$

is exact for all $k \in \mathbf{Z}$ as a complex of abelian groups.
(A) is said to be $(-w, w)$-minimal when

$$\{D^{r_i}[m^i], \mathrm{gr}_{(-w,w)}(\psi_i)\}$$

is the minimal resolution.
Example:

$$\ell_1 = h\partial_1 - (x_1\partial_1 + x_2\partial_2),$$

$$\ell_2 = h\partial_2 - (x_1\partial_1 + x_2\partial_2).$$

$(-1, -1, 1, 1)$-minimal resolution

$$0 \to D^2[1,2] \xrightarrow{\psi_2} D^3[1,1,1] \xrightarrow{\psi_1} D[0] \to M[0] \to 0,$$

$$\psi_1 = \begin{pmatrix} h\partial_x - x\partial_x - y\partial_y \\ h\partial_y - x\partial_x - y\partial_y \\ x\partial_x^2 - x\partial_x\partial_y + y\partial_x\partial_y - y\partial_y^2 \end{pmatrix}$$

$$\psi_2 = \begin{pmatrix} x\partial_x - x\partial_y + y\partial_y + xh & -y\partial_y - xh & -h + x \\ -\partial_y + h & \partial_x - h & 1 \end{pmatrix}$$

Example: $f = x(1-x)$

$$0 \to D[1] \xrightarrow{\ell} D[0] \to D/D\ell \to 0.$$

$\ell = x\partial^2 - x\partial$.
$(-w, w)$-minimal resolution is computable by modifying LaScala-Stillman's algorithm to construct minimal free resolutions (Journal of Symbolic Computation, 26, 1998).

Computer Demo

Step 3: Apply the comparison theorem

Theorem (Comparison theorem of Grothendieck-Deligne)

$$\begin{aligned} & H^{n-i}(X, \mathbf{C}) \\ \simeq\ & H^{-i}\left(\cdots \xrightarrow{\psi_{j+1}} D/(x_1 D + \cdots x_n D) \otimes_D D^{r_j} \xrightarrow{\psi_j} \cdots\right) \otimes \mathbf{C} \end{aligned}$$

$(i = 0, 1, \ldots, n)$ where (D^{r_j}, ψ_j) is a resolution of D/J, which is the formal fourier transform of the differential equations for $1/f^{r_0}$.
Define a polynomial $B(s)$ by

$$\langle B(\textstyle\sum w_i x_i \partial_i)\rangle = \mathrm{in}_{(-w,w)}(I) \cap \mathbf{Q}\langle \textstyle\sum w_i x_i \partial_i\rangle.$$

Theorem: Let (D^{r_j}, ψ_j) be a $(-w, w)$-minimal free resolution of D/J. Especially, it is $(-w, w)$-adapted.

Then,

$$
\begin{aligned}
& H^{-i}\left(\cdots \xrightarrow{1\otimes\psi_{j+1}} D/xD \otimes_D D^{r_j} \xrightarrow{1\otimes\psi_j} \cdots\right) \\
\simeq \ & H^{-i}\left(\cdots \frac{F^{k_1}_{(-w,w)}(D^{r_j}[m^j])}{\langle xD, F^{k_0-1}_{(-w,w)}(D^{r_j}[m^j])\rangle} \xrightarrow{\bar{\psi}_j} \cdots\right)
\end{aligned}
$$

($xD = x_1D + \cdots + x_nD$), where k_1 is the maximal integral root of $B(s) = 0$ and k_0 is the minimal integral root of $B(s) = 0$.

The right hand side is the cohomology groups of a complex of finite dimensional vector spaces. So, it is computable.

Example: $X = \mathbf{C}^1 \setminus V(x(1-x))$.

$(-w, w) = (-1, 1)$, $B(s) = s(s-1)$.

$$
\begin{array}{ccccccc}
0 & \to & \mathbf{Q} & \xrightarrow{x\partial^2 - x\partial} & (\mathbf{Q} + \mathbf{Q}\partial) & \to & 0 \\
 & & 1 & \mapsto & 0 & & \\
 & & H^0 & & H^1 & &
\end{array}
$$

$x(x\partial^2 - x\partial_x) = \theta(\theta - 1) - x\theta$, $\theta = x\partial$.

Example: $f = x(1-x)$. $H^0 = 1, H^1 = 2$.

Example: $f = x^3 - y^2$. $H^0 = 1, H^1 = 1, H^2 = 0$.

Example: $f = x^2 - yz^2$. $H^0 = 1, H^1 = 1, H^2 = 0, H^3 = 0$.

Example: $f = x^p + y^q + xy^{q-1}$, $(p = 4, q = 5)$. $H^0 = 1, H^1 = 1, H^2 = 1$.

Example: $f = x^3 - y^2z^2 + y^2 + z^2$. $H^0 = 1, H^1 = 1, H^2 = 0, H^3 = 8$.

Computer Demo

Implementations

kan/k0: Oaku and Takayama, http://www.openxm.org

Macaulay2: D.Grayson, M.Stillman, H.Tsai, http://www.math.uiuc.edu/Macaulay2

Further developement

1. $H^i(\mathbf{C}^n \setminus V(I), \mathbf{C})$ (Uli Walter).
2. Algorithmic computation of cup product (Uli Walther).
3. $\mathcal{E}xt^i_D(M, N)$ (Uli Walther, Harrison Tsai).

Introductory reference

Saito, M., Sturmfels, B., Takayama, N., *Gröbner deformations of hypergeometric differential equations.* Algorithms and Computation in Mathematics, 6. Springer-Verlag, Berlin, 2000.

Computation by kan/k0 of the cohomology groups of $X = C^2 \setminus V(x^p + y^q + xy^{q-1})$ where $p = 4$ and $q = 5$.

```
bash$ k0
This is kan/k0 Version 1998,12/15
WARNING: This is an EXPERIMENTAL version

In(1)=Loading startup files (startup.k)   1997, 3/11.
sm1 version = 3.000726
Default ring is Z[x,h].

In(2)=load["demo.k"];;

In(3)=nonquasi(4,5);

f=x*y^4+y^5+x^4
_u,_v,_t,x,y,s
[   _u*_t-x*y^4-y^5-x^4 , _u*_v-1 ,
 _v*y^4*D_t+4*_v*x^3*D_t*Dx ,
 4*_v*x*y^3*D_t+5*_v*y^4*D_t*Dy ]
 6. 7....o 8....oooo 9o.oooo. 10.o 11oooo
 12o... 13o.oooooo 14.o.oo 15ooo.ooo
 16. 17.oo.o 18oo.o.oooo 19ooooooo
Completed (GB with sugar).
Step 1: Annhilating ideal (II)
[[-16*x^2*Dx-20*x*y*Dx-12*x*y*Dy-16*y^2*Dy-64*x-80*y ,
 -64*x*y^2*Dx-16*y^3*Dx-48*y^3*Dy+500*x*y*Dx+16*x^2*Dy
   -20*x*y*Dy+400*y^2*Dy-256*y^2+2000*y,
 64*x*y^3*Dx+80*y^4*Dx-16*y^4*Dy-2000*x^2*y*Dx
   -2500*x*y^2*Dx-64*x^3*Dy-1500*x*y^2*Dy
   -2000*y^3*Dy-8000*x*y-10000*y^2 ,
 1048576*x*y^2*Dx^2-786432*x*y^2*Dx*Dy+851968*y^3*Dx*Dy
   -589824*y^3*Dy^2-10240000*x*y*Dx^2+7372800*x*y*Dx*Dy
   -8192000*y^2*Dx*Dy-16384*x*y*Dy^2+5570560*y^2*Dy^2
   -3407872*x*y*Dx+5242880*y^2*Dx-6291456*y^2*Dy
   +21708800*x*Dx-51200000*y*Dx+49152*x*Dy
   +51200000*y*Dy-13631488*y+86835200 ]  ]

------------ Resolution Summary  --------------
Betti numbers : [    1 , 3 , 2 ]
Betti numbers of the Schreyer frame: [    1 , 5 , 5 , 1 ]
-----------------------------------------------
Step2: (-1,1)-minimal resolution (Res0)
 [
  [
    [ -16*x*Dx^2-20*x*Dx*Dy-12*y*Dx*Dy
       -16*y*Dy^2+20*Dx*h^2+28*Dy*h^2 ]
    [ 64*x*Dx*Dy^2+16*x*Dy^3+48*y*Dy^3
       -48*Dy^2*h^2+16*y*Dx^2*h+500*x*Dx*Dy*h
       -20*y*Dx*Dy*h+400*y*Dy^2*h-20*Dx*h^3-700*Dy*h^3 ]
    [ 262144*x^2*Dy^3-262144*x*y*Dy^3-1835008*x*Dx*Dy*h^2
      -196608*x*Dy^2*h^2-1376256*y*Dy^2*h^2+2752512*Dy*h^4
      -196608*y^2*Dx^2*h-2048000*x^2*Dx*Dy*h
      +573440*x*y*Dx*Dy*h+32768*y^2*Dx*Dy*h
      -1638400*x*y*Dy^2*h+393216*y^2*Dy^2*h
      -14663680*x*Dx*h^3+491520*y*Dx*h^3+2867200*x*Dy*h^3
      -12484608*y*Dy*h^3+32440320*h^5 ]
  ]
  [
    [12582912*y*Dy^2-29360128*Dy*h^2-32768000*x*Dy*h
       +106430464*y*Dy*h-235667456*h^3 ,
     3145728*y*Dx-1048576*x*Dy+4194304*y*Dy-6553600*h^2 ,
     256*Dx+64*Dy ]
    [-65536*x*Dy^2-16384*y*Dx*h-512000*x*Dy*h
       +24576*y*Dy*h+4096*h^3 ,
     -16384*x*Dx-16384*x*Dy+28672*h^2 ,
     Dy ]
  ]
 ]
sm1>sm1>Starting ox_asir server.

Step3: computing the cohomology of the truncated complex.
Roots and b-function are [[0 , 3 ]  ,
[ -6291456000*s^5+66060288000*s^4-259129344000*s^3
 +450035712000*s^2-291962880000*s ]  ]
[[[[    5 ]  , [    5 ]  , [    5 ]  ]  ,
 [[ 5 , 5 , 5 ], [ 5 , 5 , 5 ]  ]  ]  ,
 [  5 , 5 ] , [ 5 , 5 ]  ]
i = 0
dim of the i-th truncated complex = 10
i = 1
dim of the i-th truncated complex = 15
i = 2
dim of the i-th truncated complex = 6
Answer is
[[ 4, [12716605440 , 1966080*e_-32440320 ,
        -19267584*e_^2+54046573120 ]],
[3 , [128 , 256*e_ ]],
[1 , [   ]  ]  ]
In(4)=quit;
bash$ exit
```

The output means that

$$\dim H^2 = 1, \dim H^1 = 1, \dim H^0 = 1.$$

ANOTHER TRIANGULAR MATRIX RING HAVING AUSLANDER-GORENSTEIN PROPERTY

YOSHIAKI HIRANO

1. Introduction

Let R be a left and right noetherian ring, and let k be a positive integer. For a minimal injective resolution

$$0 \to R \to I^0(R) \to I^1(R) \to \cdots \to I^i(R) \to \cdots$$

of R as a right R-module, if the flat dimension $\mathrm{fd}(I^i(R)_R)$ of $I^i(R)$ is at most i for all i $(0 \le i \le k-1)$, then R is called *k-Gorenstein*. Then the definition is left-right symmetric by [2, Theorem 3.7]. Moreover, Fossum, Griffith and Reiten [2, Theorem 3.10] showed that the lower triangular matrix ring $T_2(R) = \begin{pmatrix} R & 0 \\ R & R \end{pmatrix}$ of degree 2 over R is k-Gorenstein if and only if so is R. Recently, Iwanaga and Wakamatsu [4, Theorem 8] showed that for any integer $n(\ge 2)$, the lower triangular matrix ring $T_n(R)$ of degree n over R is k-Gorenstein if and only if so is R. In this paper, we study the following triangular matrix ring.

Definition 1.1. Let n be a positive integer, and put $N = 2n$. The subset of the full matrix ring $\mathrm{M}_N(R)$ of degree N over a ring R

$$\{(a_{ij}) \in \mathrm{M}_N(R) \mid a_{ij} = 0 \ (1 \le i < j \le N),\ a_{ii-1} = 0 \text{ for odd } i \ (1 < i < N)\}$$

is a subring of $\mathrm{M}_N(R)$. This subring is denoted by $U_N(R)$.

Then we prove the following theorem.

Theorem 1.2. *Let R be a left and right noetherian ring, and let k be a positive integer. Then R is k-Gorenstein if and only if so is $U_N(R)$.*

In the proof of Theorem 1.2, we give a description of a minimal injective resolution of $U_N(R)$, from which we can prove the following corollary.

Corollary 1.3. *Let R be a ring. Assume that the injective dimension $\mathrm{id}(R_R)$ of R_R is finite. Then the following equation holds.*

$$\mathrm{id}(U_N(R)_{U_N(R)}) = \mathrm{id}(R_R) + n$$

Throughout this paper, all rings are not necessarily commutative rings with identity element. Let S be a ring, and let X be a right S-module. A minimal injective resolution of X is denoted by $0 \to X \to I^0(X) \to I^1(X) \to \cdots \to I^i(X) \to \cdots$. The flat dimension of X is denoted by $\mathrm{fd}(X_S)$.

The detailed version of this paper has been submitted for publication elsewhere.

2. Modules over a formal triangular matrix ring

In this section, we recall some basic facts concerning a minimal injective resolution of a module over a formal triangular matrix ring, which we need in the proof of the theorem. (See e.g. [1, III §2] for the beginning part.)

Let A and B be rings and M a (B, A)-bimodule, and let Λ be the formal triangular matrix ring $\Lambda = \begin{pmatrix} A & 0 \\ M & B \end{pmatrix}$. Then let $F(-)$ denote the functor $\mathrm{Hom}_A(M, -)$ from the category of right A-modules to the category of right B-modules, and let $G(-)$ denote the functor $- \otimes_B M$ from the category of right B-modules to the category of right A-modules.

Let X be a right A-module and Y a right B-module, and let $\varphi : Y \to F(X)$ be a B-homomorphism. Then the triple (X, Y, φ) is a right Λ-module, i.e., it is $X \oplus Y$ as an additive group and its Λ-operation on $X \oplus Y$ is defined by

$$(x, y) \begin{pmatrix} a & 0 \\ m & b \end{pmatrix} = (xa + \varphi(y)(m), yb)$$

for $(x, y) \in X \oplus Y$, $\begin{pmatrix} a & 0 \\ m & b \end{pmatrix} \in \Lambda$. Similarly, for a right A-module X, a right B-module Y and an A-homomorphism $\psi : G(Y) \to X$, the triple (X, Y, ψ) is a right Λ-module, whose Λ-operation on $X \oplus Y$ is defined by

$$(x, y) \begin{pmatrix} a & 0 \\ m & b \end{pmatrix} = (xa + \psi(y \otimes m), yb)$$

for $(x, y) \in X \oplus Y$, $\begin{pmatrix} a & 0 \\ m & b \end{pmatrix} \in \Lambda$. Note that for the usual isomorphism $\mu : \mathrm{Hom}_B(Y, F(X)) \cong \mathrm{Hom}_A(G(Y), X)$, $(X, Y, \varphi) = (X, Y, \psi)$ if and only if $\mu(\varphi) = \psi$.

If X is an injective right A-module, then $(X, F(X), 1)$ is an injective right Λ-module. If Y is an injective right B-module, then $(0, Y, 0)$ is an injective right Λ-module.

If M is projective as a right A-module, then a minimal injective resolution of (X, Y, φ) is given by the following lemma. (See [4, Corollary 2(1)].)

Lemma 2.1. *Assume that M is projective as a right A-module. Let $(X, Y, \varphi : Y \to F(X))$ be a right Λ-module. Then*

(1) $I^0(X, Y, \varphi) \cong (I^0(X), F(I^0(X)), 1) \oplus (0, I^0(Ker\varphi), 0)$

(2) $I^i(X, Y, \varphi) \cong (I^i(X), F(I^i(X)), 1) \oplus (0, I^{i-1}(Ker\varphi_1), 0) \quad (i \geq 1)$.

In particular, if φ is a monomorphism, then $Ker\varphi_1 \cong Coker\varphi$.

Next, if M is flat as a left B-module, we can estimate the flat dimension of a certain right Λ-module (X, Y, ψ).

Lemma 2.2. *Assume that M is flat as a left B-module. Let $(X, Y, \psi : G(Y) \to X)$ be a right Λ-module such that ψ is an epimorphism. Then for $i \geq 1$, $\mathrm{fd}((X, Y, \psi)_\Lambda) \leq i$ if and only if $\mathrm{fd}(Y_B) \leq i$ and $\mathrm{fd}(Ker\psi_A) \leq i - 1$.*

In particular, if ψ is an isomorphism, then $\mathrm{fd}((X, Y, \psi)_\Lambda) \leq i$ if and only if $\mathrm{fd}(Y_B) \leq i$.

3. Modules over a 3×3 formal triangular matrix ring

We prove Theorem 1.2 by induction on N, decomposing $U_N(R)$ into blocks. A sticky point here is that we have no decomposition of $U_N(R) = \begin{pmatrix} A & 0 \\ M & B \end{pmatrix}$ such that M is both left B-flat and right A-projective. In our proof, we decompose $U_N(R)$ in two ways. In order to clarify the relationship between two presentations of a right $U_N(R)$-module depending on the two decompositions of $U_N(R)$, in this section, we study modules over a 3×3 formal triangular matrix ring.

Let A_1, A_2 and A_3 be rings, let M_{21}, M_{31} and M_{32} be (A_2, A_1)-, (A_3, A_1)- and (A_3, A_2)-bimodules, respectively, let $\eta : M_{32} \otimes_{A_2} M_{21} \to M_{31}$ be an (A_3, A_1)-homomorphism, and let Γ be a 3×3 formal triangular matrix ring

$$\Gamma = \begin{pmatrix} A_1 & 0 & 0 \\ M_{21} & A_2 & 0 \\ M_{31} & M_{32} & A_3 \end{pmatrix}$$

with usual matrix operations.

Definition 3.1. Let X_i be a right A_i-module for $i = 1, 2, 3$, and let $\psi_{21} : X_2 \otimes_{A_2} M_{21} \to X_1$ and $\psi_{31} : X_3 \otimes_{A_3} M_{31} \to X_1$ be A_1-homomorphisms and $\psi_{32} : X_3 \otimes_{A_3} M_{32} \to X_2$ an A_2-homomorphism. Assume that $\psi_{21}, \psi_{31}, \psi_{32}$ and η satisfy equation: $\psi_{31} \circ (1_{X_3} \otimes \eta) = \psi_{21} \circ (\psi_{32} \otimes 1_{M_{21}})$. Then the additive group $X_1 \oplus X_2 \oplus X_3$ has a right Γ-module structure by defining

$$x\gamma = (x_1 a_1 + \psi_{21}(x_2 \otimes m_{21}) + \psi_{31}(x_3 \otimes m_{31}), x_2 a_2 + \psi_{32}(x_3 \otimes m_{32}), x_3 a_3)$$

for $x = (x_1, x_2, x_3) \in X_1 \oplus X_2 \oplus X_3$, $\gamma = \begin{pmatrix} a_1 & 0 & 0 \\ m_{21} & a_2 & 0 \\ m_{31} & m_{32} & a_3 \end{pmatrix} \in \Gamma$. We denote this right Γ-module by $(X_1, X_2, X_3, \psi_{21}, \psi_{31}, \psi_{32})$.

Note that any right Γ-module is isomorphic to some $(X_1, X_2, X_3, \psi_{21}, \psi_{31}, \psi_{32})$.

There are two ways to decompose Γ as a 2×2 formal triangular matrix ring. In the following Lemmas 3.2 and 3.3, we clarify the relationship between two presentations of a right Γ-module depending on the two decompositions of Γ.

Lemma 3.2. *Let* $\Gamma = \begin{pmatrix} A_1 & 0 \\ M & B \end{pmatrix}$, *where* $M = \begin{pmatrix} M_{21} \\ M_{31} \end{pmatrix}$, $B = \begin{pmatrix} A_2 & 0 \\ M_{32} & A_3 \end{pmatrix}$. *Let* $(X_1, (X_2, X_3, \psi_{32}), \psi)$ *be a right* Γ*-module, where* X_1 *is a right* A_1*-module,* $(X_2, X_3, \psi_{32} : X_3 \otimes_{A_3} M_{32} \to X_2)$ *is a right* B*-module and* $\psi : (X_2, X_3, \psi_{32}) \otimes_B M \to X_1$ *is an* A_1*-homomorphism. Then* ψ *defines* A_1*-homomorphisms* $\psi_{21} : X_2 \otimes_{A_2} M_{21} \to X_1$ *and* $\psi_{31} : X_3 \otimes_{A_3} M_{31} \to X_1$, *and*

$$(X_1, (X_2, X_3, \psi_{32}), \psi) = (X_1, X_2, X_3, \psi_{21}, \psi_{31}, \psi_{32}).$$

Lemma 3.3. *Let $(X_1, X_2, X_3, \psi_{21}, \psi_{31}, \psi_{32})$ be a right Γ-module. Let $\Gamma = \begin{pmatrix} A & 0 \\ M' & A_3 \end{pmatrix}$, where $A = \begin{pmatrix} A_1 & 0 \\ M_{21} & A_2 \end{pmatrix}$, $M' = (M_{31}, M_{32})$. Then (X_1, X_2, ψ_{21}) is a right A-module, the mapping $(\psi_{31}, \psi_{32}) : X_3 \otimes_{A_3} M' \to (X_1, X_2, \psi_{21})$ defined by*

$$x_3 \otimes (m_{31}, m_{32}) \mapsto (\psi_{31}(x_3 \otimes m_{31}), \psi_{32}(x_3 \otimes m_{32}))$$

for $x_3 \in X_3$, $(m_{31}, m_{32}) \in M'$ is an A-homomorphism, and

$$(X_1, X_2, X_3, \psi_{21}, \psi_{31}, \psi_{32}) = ((X_1, X_2, \psi_{21}), X_3, (\psi_{31}, \psi_{32})).$$

4. Proof of Theorem 1.2

In what follows, put $A = \begin{pmatrix} R & 0 \\ R & R \end{pmatrix}$, and decompose $U_N = U_N(R)$ as follows:

$$U_N = \begin{pmatrix} A & 0 \\ M & U_{N-2} \end{pmatrix} = \begin{pmatrix} A & 0 & 0 \\ M_{21} & U_{N-4} & 0 \\ M_{31} & M_{32} & A \end{pmatrix} = \begin{pmatrix} U_{N-2} & 0 \\ M' & A \end{pmatrix}.$$

(Note that M is projective as a right A-module and that M' is flat as a left A-module.) Put $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in A$. Let $e'_i \in U_{N-2}$ $(1 \le i \le N-2)$, $f_i \in U_N$ $(1 \le i \le N)$ be the matrices such that $e'_i(f_i)$ has 1 for the (i,i)-entry and 0 for the other entries.

In order to prove the "only if" part of Theorem 1.2, we need the following Lemmas 4.1 and 4.2.

A minimal injective resolution of U_N is given by the following lemma.

Lemma 4.1. (1) $I^j(f_N U_N) \cong (I^j(e_2A), \mathrm{Hom}_A(M, I^j(e_2A)), 1)$ $\quad (j \ge 0)$

(2) $I^0(f_2U_N) \cong I^0(f_NU_N)$, $I^i(f_2U_N) \cong I^i(f_NU_N) \oplus I^{i-1}(f_NU_N/f_2U_N)$ $\quad (i \ge 1)$

$I^j(f_NU_N/f_2U_N) \cong (0, I^j(e'_{N-2}U_{N-2}), 0)$ $\quad (j \ge 0)$

(3) $I^0(f_3U_N) \cong I^0(f_NU_N)$, $I^i(f_3U_N) \cong I^i(f_NU_N) \oplus I^{i-1}(f_NU_N/f_3U_N)$ $\quad (i \ge 1)$

$I^j(f_NU_N/f_3U_N) \cong (I^j(e_2A/e_1A), \mathrm{Hom}_A(M, I^j(e_2A/e_1A)), 1)$ $\quad (j \ge 0)$

(4) Let $4 \le k \le N-1$. Then

$I^0(f_kU_N) \cong I^0(f_NU_N)$, $I^i(f_kU_N) \cong I^i(f_NU_N) \oplus I^{i-1}(f_NU_N/f_kU_N)$ $\quad (i \ge 1)$

$I^j(f_NU_N/f_kU_N) \cong (0, I^j(e'_{N-2}U_{N-2}/e'_{k-2}U_{N-2}), 0)$ $\quad (j \ge 0)$

(5) $I^0(f_1U_N) \cong I^0(f_NU_N)$

$$I^i(f_1U_N) \cong I^i(f_NU_N) \oplus I^{i-1}(f_NU_N/f_3U_N) \oplus I^{i-1}(f_NU_N/f_2U_N)$$
$$\oplus \cdots \oplus I^0(f_NU_N/f_{2i+1}U_N) \oplus I^0(f_NU_N/f_{2i}U_N) \quad (1 \le i \le n-1)$$

$$I^i(f_1U_N) \cong I^i(f_NU_N) \oplus I^{i-1}(f_NU_N/f_3U_N) \oplus I^{i-1}(f_NU_N/f_2U_N)$$
$$\oplus \cdots \oplus I^{i-(n-1)}(f_NU_N/f_{N-1}U_N) \oplus I^{i-(n-1)}(f_NU_N/f_{N-2}U_N)$$
$$\oplus (0, I^{i-n}(e_2A/e_1A), 0) \quad (i \geq n)$$

Next, we give some isomorphisms, which we need in the proof of the "only if" part of Theorem 1.2.

Lemma 4.2. *Let i be a nonnegative integer.*
(1) $I^i(e_2A) \cong (I^i(R), I^i(R), \varphi_i : I^i(R) \to \mathrm{Hom}_R(R, I^i(R)))$
(2) $I^i(e_2A) \cong \mathrm{Hom}_A(M_{31}, I^i(e_2A))$
(3) $I^i(e_2A) \otimes_A M_{31} \cong I^i(e_2A)$
(4) $I^i(e_2A) \otimes_A M_{32} \cong \mathrm{Hom}_A(M_{21}, I^i(e_2A))$
(5) $\mathrm{Hom}_A(M, I^i(e_2A)) \cong I^i(e'_{N-2}U_{N-2})$

Proof of the "only if" part of Theorem 1.2. We proceed by induction on $N = 2n$. In the case that $N = 2$, this follows from [2, Theorem 3.10]. Assume that U_{N-2} is k-Gorenstein. Then we show that $\mathrm{fd}(I^i(f_jU_N)_{U_N}) \leq i$ for all j $(1 \leq j \leq N)$ and i $(0 \leq i \leq k-1)$. Here we prove the case that $j = N$. (See [3, §4] for the other cases.)

It follows from Lemma 4.1(1) that

$$I^i(f_NU_N) \cong (I^i(e_2A), \mathrm{Hom}_A(M, I^i(e_2A)), 1).$$

Let $\varepsilon : \mathrm{Hom}_A(M, I^i(e_2A)) \otimes_{U_{N-2}} M \to I^i(e_2A)$ be an evaluation map. Then

$$(I^i(e_2A), \mathrm{Hom}_A(M, I^i(e_2A)), 1) = (I^i(e_2A), \mathrm{Hom}_A(M, I^i(e_2A)), \varepsilon).$$

Since $M = M_{21} \oplus M_{31}$, $\mathrm{Hom}_A(M, I^i(e_2A))$ is expressed in the form

$$(\mathrm{Hom}_A(M_{21}, I^i(e_2A)), \mathrm{Hom}_A(M_{31}, I^i(e_2A)), \psi_{32})$$

using the decomposition of $U_{N-2} = \begin{pmatrix} U_{N-4} & 0 \\ M_{32} & A \end{pmatrix}$. Here, $\psi_{32} : \mathrm{Hom}_A(M_{31}, I^i(e_2A)) \otimes_A M_{32} \to \mathrm{Hom}_A(M_{21}, I^i(e_2A))$ is defined by $\psi_{32}(f \otimes x)(y) = f(xy)$ for $f \in \mathrm{Hom}_A(M_{31}, I^i(e_2A))$ $x \in M_{32}$, $y \in M_{21}$. Then ψ_{32} is a U_{N-4}-isomorphism by Lemma 4.2(2), (4). Using Lemma 3.2, we have

$$I^i(f_NU_N) \cong (I^i(e_2A), \mathrm{Hom}_A(M_{21}, I^i(e_2A)), \mathrm{Hom}_A(M_{31}, I^i(e_2A)), \psi_{21}, \psi_{31}, \psi_{32})$$

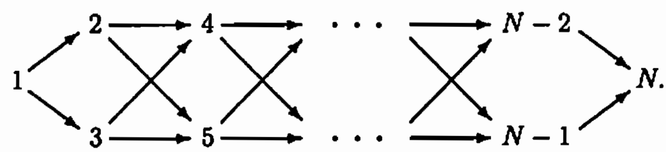
where $\psi_{t1} : \mathrm{Hom}_A(M_{t1}, I^i(e_2A)) \otimes M_{t1} \to I^i(e_2A)$ is an evaluation map for $t = 2, 3$. Then ψ_{31} is an A-isomorphism by Lemma 4.2(2), (3). Hence by Lemma 3.3, we have

$$I^i(f_NU_N) \cong ((I^i(e_2A), \mathrm{Hom}_A(M_{21}, I^i(e_2A)), \psi_{21}), \mathrm{Hom}_A(M_{31}, I^i(e_2A)), (\psi_{31}, \psi_{32})).$$

Since both ψ_{31} and ψ_{32} are isomorphisms, (ψ_{31}, ψ_{32}) is also an isomorphism. Since A is k-Gorenstein, we have $\mathrm{fd}(I^i(e_2A)_A) \leq i$, so that by Lemma 4.2(2), $\mathrm{fd}(\mathrm{Hom}_A(M_{31}, I^i(e_2A))_A) \leq i$. Then since ${}_AM'$ is flat, it follows from Lemma 2.2 that $\mathrm{fd}(I^i(f_NU_N)_{U_N}) \leq i$. □

The "if" part of Theorem 1.2 can be shown using another decomposition of $U_N(R)$. (See [3, Proposition 4.1].)

Remark. Let R be a ring, and let n be a positive integer. Put $N = 2n$. Then the ring $U_N(R)$ is defined by the following quiver (i.e., if there is an arrow from i to j, then the (j, i)-entry is R, or else the (j, i)-entry is 0):



When R is a field, we can compute a projective resolution and an injective resolution of a certain right $U_N(R)$-module. Hence, it is easily checked that $U_N(R)$ is ∞-Gorenstein. When R is a discrete valuation ring, it follows from [5, Theorem 4.6] that $U_N(R)$ is ∞-Gorenstein. There are some other quivers such that the rings given by them have Auslander-Gorenstein property.

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Some examples of S_R(H)-blocks [1]

HIEDA YOSHIMASA

1 Introduction

Let G be a finite group, p a prime divisor of the order of G and (K, R, k) a p-modular system, i.e., R is a complete discrete valuation ring with maximal ideal (π), K is the quotient field of R of characteristic 0 and $k(:= R/(\pi))$ is the residue field of R of characteristic p. Moreover, we assume that K contains the $|G|$th roots of unity.

For a subgroup H of G, $\widehat{H}$ denotes the sum of all elements of H in the group algebra $\mathfrak{o}G$, where $\mathfrak{o}$ is R, K or k and e_χ (resp. $\tilde{e}_\psi$) is the central primitive idempotent of KG (resp. KH) corresponding to $\chi \in \mathrm{Irr}(G)$ (resp. $\psi \in \mathrm{Irr}(H)$), where $\mathrm{Irr}(G)$ (resp. $\mathrm{Irr}(H)$) is the set of all irreducible K-characters of G (resp. H).

Using the above notation the *Hecke algebra* $\mathcal{H}_K(G, H, \psi)$ is $\mathrm{End}_{KG}(\tilde{e}_\psi KG)$ $(= \tilde{e}_\psi KG\tilde{e}_\psi)$ and we know that $\{e_\chi \tilde{e}_\psi; \chi \in \Phi(\psi)_H^G\}$ is the set of all central primitive idempotents of $\mathcal{H}_K(G, H, \psi)$ (in KG), where $\Phi(\psi)_H^G := \{\chi \in \mathrm{Irr}(G); (\chi|_H, \psi)_H \neq 0\}$ ([1, (11.26) Corollary]).

In this note we consider the case ψ is the trivial character 1_H of H. Then the Hecke algebra $\mathcal{H}_\mathfrak{o}(G, H, 1_H)$ equals $\mathrm{End}_{\mathfrak{o}G}(\widehat{H}\mathfrak{o}G)$ since $\tilde{e}_H := \tilde{e}_{1_H} = \widehat{H}/|H|$. So we denote it $S_\mathfrak{o}(H)$ for brevity. (We use S from the *Schur algebra*.) Here we mention that $S_R(H)/\pi S_R(H) \simeq S_k(H)$ as $\widehat{H}RG$ is a permutation module.

As $S_K(H) = K \otimes_R S_R(H)$, for a central idempotent ε of $S_R(H)$, there exists a non-empty subset β of $\Phi_H^G := \Phi(1_H)_H^G$ such that $\varepsilon = \sum_{\chi\in\beta} e_\chi \tilde{e}_H$. Here the element of this form is denoted by ε_β and if ε_β is a centrally primitive, β (or $\varepsilon_\beta S_R(H)$) is called an *$S_R(H)$-block*. Hence the set of $S_R(H)$-blocks corresponds bijectively to the set of $S_k(H)$-blocks from the above.

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[1] The final and detailed version of this note will be submitted for publication elsewhere.

On the other hand, the multiplication induces the R-algebra homomorphism $\phi : Z(RG) \longrightarrow Z(S_R(H))$. Using the map ϕ, G.R.Robinson [6] has proved that $Z(S_R(H)) \simeq \mathrm{End}_{R[G\times G]}(RG\widehat{H}RG)$ as R-algebras. Then an $S_R(H)$-block corresponds to a primitive idempotent of $\mathrm{End}_{R[G\times G]}(RG\widehat{H}RG)$. Therefore we can define a *defect group*, we denote it $\delta_H(\beta)$, for an $S_R(H)$-block β in $G \times G$.

Now we recall that for any $S_R(H)$-block β there exists the unique p-block B such that $\beta \subset \mathrm{Irr}(B)$ ([6, Lemma2.1(ii)]). Also if e_B is a block idempotent i.e., a central primitive idempotent, of RG with the condition $\phi(e_B) \neq 0$, then $\phi(e_B) = \sum_{\beta\in\mathfrak{B}} \varepsilon_\beta$, where $\mathfrak{B}$ is the suitable non-empty subset of $S_R(H)$-blocks. So $\mathrm{Irr}(B) \cap \Phi_H^G$ is a (disjoint) union of $S_R(H)$-blocks.

The purpose of this note we show some examples of $S_R(H)$-blocks and their defect groups.

The notation is almost standard. Concerning some basic facts and terminologies used here, we refer to [1] and [5] for example.

2 Preliminaries

For later use, we shall exhibit some results on $S_R(H)$-blocks, which are proved in [2], [3] or [6]. At first we notice the following remark.

Remark 1 ([6, Remark of Proposition 2.2]) *If* $H = \{1\}$, *then* $\mathrm{Irr}(B)$ *is an* $S_R(\{1\})$*-block for any p-block* B *of* G. *Moreover, a defect group of an* $S_R(\{1\})$*-block* $\mathrm{Irr}(B)$ *is the diagonal subgroup* $\delta(B)^\Delta := \{(x,x) \in G\times G; x \in \delta(B)\}$, *where* $\delta(B)$ *is a (usual) defect group of* B.

Proposition 2.1 ([6, Lemma 2.1]) *(i) For any* $S_R(H)$*-block* β *and* $x, y \in G$, $\dfrac{|\delta_H(\beta)|}{|C_G(x)||C_G(y)|}\sum_{\chi\in\beta}\chi(x)\chi(y) \in R$. *In particular,* $\dfrac{|\delta_H(\beta)|}{|G\times G|}\sum_{\chi\in\beta}\chi(1)^2 \in R$.

(ii) β *is contained in a single p-block* B *of* G *in the usual sence, and if* B *has a defect group* D, *then* $\delta_H(\beta)$ *is contained (up to conjugacy) in* $D \times D$.

Corollary 2.2 ([2, Corollary 3]) *If* $\sum_{\chi\in\beta}\chi(1)^2$ *is prime to p for an* $S_R(H)$*-block* β, *then a defect group of* β *is a Sylow p-subgroup of* $G\times G$.

As the trivial character 1_G is always in Φ_H^G for any subgroup H of G, there exists the $S_R(H)$-block, which has 1_G. So we call it the *principal* $S_R(H)$*-block* and denote it β_0.

Proposition 2.3 ([6, Lemma 2.3(iii)] and [2, Proposition 4]) *For the principal* $S_R(H)$*-block* β_0, $\beta_0 = \{1_G\}$ *if and only if* H *contains a Sylow* p*-subgroup of* G. *Moreover, in the above case a defect group of* β_0 *is a Sylow* p*-subgroup of* $G \times G$.

Proposition 2.4 ([6, Corollary 2.4]) *If* H *is normal in* G, *then the* $S_R(H)$*-blocks of* G *are precisely the* p*-blocks of* $R[G/H]$.

In the rest of this section we assume that H is a p'-subgroup of G and consider only those p-blocks such that $\phi(e_B) \neq 0$.

In this case $\tilde{e}_H \in RG$, i.e., $\widehat{H}RG = \tilde{e}_H RG$ is a projective RG-module and kH is a semisimple k-algebra.

Now for any $\varphi \in \mathrm{IBr}(G)$, let S_φ (resp. P_φ) be an irreducible kG-module (resp. an indecomposable projective RG-module) corresponding to φ. Also, we let $\Psi_H^G := \{\varphi \in \mathrm{IBr}(G); k_H | S_{\varphi\downarrow H}\}$. Note that $\Psi_H^G = \{\varphi \in \mathrm{IBr}(G); P_\varphi | \tilde{e}_H RG\}$. So we can define $\beta^* := \{\varphi \in \mathrm{IBr}(B); P_\varphi | \varepsilon_\beta(\tilde{e}_H RG)\}$ correponding to an $S_R(H)$-block β.

Therefore the decomposition matrix D_B of B has the following form :([3])

$$D_B = \left(\begin{array}{cccc|c} D_{\beta_0} & 0 & \cdots & 0 & * \\ 0 & D_{\beta_1} & \cdots & 0 & * \\ \vdots & \vdots & \ddots & \vdots & * \\ 0 & 0 & \cdots & D_{\beta_t} & * \\ \hline 0 & 0 & \cdots & 0 & * \\ \vdots & \vdots & \cdots & \vdots & * \\ 0 & 0 & \cdots & 0 & * \end{array}\right) \tag{2.1}$$

From the form of the decomposition matrix (2.1), we get the following orthogonality relation for the $S_R(H)$-block.

Theorem 2.5 ([3, Theorem 5]) *Let* H *be a* p'*-subgroup of* G *and* β *an* $S_R(H)$*-block. Then we have*

$$\sum_{\chi \in \beta} \chi(x\tilde{e}_H)\chi(y) = 0$$

for any $y \in G - G_{p'}$ *and* $x \in G_{p'}$ *such that* $\langle x, H\rangle$ *is a* p'*-subgroup.*

3 Some examples of $S_R(H)$-blocks

In this section we show some $S_R(H)$-blocks and their defect groups.

Now $\mathfrak{S}_n$ (resp. $\mathfrak{A}_n$) denotes the symmetric group (resp. the alternating group) of degree n and we denote the irreducible characters of $\mathfrak{S}_n$ the same notations corresponding to the Young diagrams. ($[n]$ means the trivial character $1_{\mathfrak{S}_n}$ for example.) Also, B_0 is the principal p-block of G.

We have already got the trivial subgroup case $H = \{1\}$([Remark 1]). So at first we check the other trivial subgroup case $H = G$.

Example 1 *Let $H := G$ and* $\mathrm{Char}k = p$.

(1) $\Phi_H^G = \{1_G\}$.

(2) $\Phi_H^G = \beta_0$ *and* $\delta_G(\beta_0) \in Syl_p(G \times G)$.

(Proof) (2) holds by Proposition 2.3. □

From the trivial subgroup cases we get the next example.

Example 2 $G := C_p$ *the cyclic group of order* p *and* $\mathrm{Char}k = p$.

(1) $\Phi_H^G = \begin{cases} \{1_G\} = \beta_0 & (H = G) \\ \mathrm{Irr}(G) = \cup_{B \in Bl_p(G)} \mathrm{Irr}(B) & (H = \{1_G\}) \end{cases}$.

(2) $\delta_H(\beta_0) = \begin{cases} G \times G & (H = G) \\ G^\Delta & (H = \{1_G\}) \end{cases}$.

(Proof) As G has only trivial subgroups, the statements follow from Remark 1 and Example 1. □

We know the following two examples.

Example 3 ([2, Example 7] *Let $G := \mathfrak{S}_3$ and $H := \langle (1, 2) \rangle$*.

(1) $\Phi_H^G = \{[3], [2, 1]\}$.

(2) *(a) If* $p = 2$*, then* $\Phi_H^G = \beta_0 \cup \beta_1$*, where* $\beta_0 = \{[3]\}, \beta_1 = \{[2,1]\}$ *and* $\delta_H(\beta_0) \in Syl_2(G \times G), \delta_H(\beta_1) = \{1\} \times \{1\}$

(b) If $p = 3$*, then* $\Phi_H^G (= \mathrm{Irr}(B_0) \cap \Phi_H^G) = \beta_0$ *and* $\delta_H(\beta_0) \in Syl_3(G \times G)$.

Example 4 ([2, Example 8] *Let* $G := \mathfrak{S}_3$ *and* $H := \langle (1,2,3) \rangle$.

(1) $\Phi_H^G = \{[3], [1^3]\}$.

(2) *(a) If* $p = 2$*, then* $\Phi_H^G (= \mathrm{Irr}(B_0)) = \beta_0$ *and* $\delta_H(\beta_0) =_{G \times G} \delta(B_0)^{\Delta}$.

(b) If $p = 3$*, then* $\Phi_H^G = \beta_0 \cup \beta_0'$*, where* $\beta_0 = \{[3]\}, \beta_0' = \{[1^3]\}$ *and* $\delta_H(\beta) \in Syl_3(G \times G)$*, where* β *is* β_0 *or* β_0'.

The next example correponds to [2, Example 12].

Example 5 *Let* $G := \mathfrak{A}_n$ *and* $H := \mathfrak{A}_{n-1}$, $(n \geq 4)$.

(1) $\Phi_H^G = \{1_G, \chi\}$*, where* $\chi(1) = n - 1$.

(2) *(a) If* p *does not divide* n*, then* $\beta_0 = \{1_G\}$ *and* $\delta_H(\beta_0) \in Syl_p(G \times G)$.

(b) If p *divides* n*, then* $\beta_0 = \{1_G, \chi\} (= \Phi_H^G)$.
In particular, if p *is odd prime, then* $\delta_H(\beta_0) \in Syl_p(G \times G)$.

(Proof) Put $\tilde{G} := \mathfrak{S}_n$. As $1_G = [n]|_G$ and $\chi = [n-1,1]|_G$ for the irreducible characters $[n], [n-1,1]$ of $\tilde{G}$ and Corollary 2.2, the assertions hold. □

For the principal $S_R(H)$-block of $\mathfrak{S}_p$ satisfies the following.

Example 6 ([2, Example 11] *Let* $G := \mathfrak{S}_p, H := \mathfrak{S}_t \, (1 \leq t \leq p)$ *and* $\mathrm{Char} k = p$.

(1) $\beta_0 = \mathrm{Irr}(B_0) \cap \Phi_H^G = \{[p-i, 1^i]; 0 \leq i \leq p-t\}$.

(2) $\delta_H(\beta_0) =_{G \times G} \begin{cases} P^{\Delta} & t = 1 \\ P \times P & 2 \leq t \leq p \end{cases}$,

where P *is a Sylow* p*-subgroup of* G.

Moreover, we show the next example if p is an odd prime.

Example 7 *Let p be an odd prime, $G := \mathfrak{S}_p$ and $H := \mathfrak{A}_t$ $(\frac{p+1}{2} < t \leq p)$.*

(1) $\Phi_H^G \cap \mathrm{Irr}(B_0) = \{[p-i, 1^i]; 0 \leq i \leq p-t\} \cup \{[j+1, 1^{p-j-1}]; 0 \leq j \leq p-t\}$.

(2) $\beta_0 = \{[p-i, 1^i]; 0 \leq i \leq p-t\}$ *and* $\delta_H(\beta_0) \in Syl_p(G \times G)$.

(Proof) Put $\widetilde{H} := \mathfrak{S}_t$.

(1) As $\Phi_H^{\widetilde{H}} = \{1_{\widetilde{H}}, \mathrm{sgn}_{\widetilde{H}}\}$, $\mathrm{Irr}(B_0) \cap \Phi_H^G = \mathrm{Irr}(B_0) \cap (\Phi_{\widetilde{H}}^G \cup \Phi(\mathrm{sgn}_{\widetilde{H}})_{\widetilde{H}}^G)$. Here $\mathrm{Irr}(B_0) \cap \Phi_{\widetilde{H}}^G = \{[p-i, 1^i]; 0 \leq i \leq p-t\}, \mathrm{Irr}(B_0) \cap \Phi(\mathrm{sgn}_{\widetilde{H}})_{\widetilde{H}}^G = \{[j+1, 1^{p-j-1}]; 0 \leq j \leq p-t\}$ since $\mathrm{Irr}(B_0) = \{[p-i, 1^i]; 0 \leq i \leq p-1\}$ and Branching theorem ([4, Theorem 9.2]). Then the assertion holds.

(2) As $p-t+1 < t$ from the assumption, there exists some $\chi \in \mathrm{Irr}(B_0)$ which is not in Φ_H^G. Therefore $\mathrm{Irr}(B_0) \cap \Phi_H^G = \beta_0 \cup \beta_1$ by Branching theorem and the form of the decomposition matrix (2.1), where $\beta_0 = \{[p-i, 1^i]; 0 \leq i \leq p-t\}, \beta_1 = \{[j+1, 1^{p-j-1}]; 0 \leq j \leq p-t\}$.

Moreover, the later half holds from (2) and Corollary 2.2 as $\chi(1) \equiv \pm 1 (\mathrm{mod}\ p)$ for any $\chi \in \mathrm{Irr}(B_0)$. □

Remark 2 *If $3 \leq t \leq \frac{p+1}{2}$ in the above example, then $\beta_0 = \mathrm{Irr}(B_0)$.*

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On t-structures and torsion theories induced by compact objects

Yoshiaki Kato

ABSTRACT. First, we show that a compact object C in a triangulated category, which satisfies suitable conditions, induces a t-structure. Second, in an abelian category we show that a complex $P^{\cdot}$ of small projective objects of term length two, which satisfies suitable conditions, induces a torsion theory. In the case of module categories, using a torsion theory, we give equivalent conditions for $P^{\cdot}$ to be a tilting complex. Finally, in the case of artin algebras, we give one to one correspondence between tilting complexes of term length two and torsion theories with certain conditions.

0. Introduction

This note is a summary of my joint work with Hoshino and Miyachi ([HKM]).

In the representation theory of finite dimensional algebras, torsion theories were studied by several authors in connection with classical tilting modules. For these torsion theories, there are equivalences between torsion (resp., torsionfree) classes and torsionfree (resp., torsion) classes, which is known as Theorem of Brenner and Butler ([HR]). One of the authors gave one to one correspondence between classical tilting modules and torsion theories with certain conditions ([Ho1], [Ho2]). But in the case of a self-injective algebra A, tilting modules are essentially isomorphic to A. In [Ri], Rickard introduced the notion of tilting complexes as a generalization of tilting modules, and showed that these complexes induce equivalences between derived categories of module categories. Tilting complexes of term length two are often studied in the case of self-injective algebras (e.g. [Hl], [HK]). On the other hand, for triangulated categories, Beilinson, Bernstein and Deligne introduced the notions of t-structures and admissible abelian subcategories, and studied relationships between them ([BBD]). In this paper, first, we deal with a compact object C in a triangulated category, and study a t-structure induced by C. Second, in an abelian category $\mathcal{A}$ we deal with a complex $P^{\cdot}$ of small projective objects of term length two and study a torsion theory induced by $P^{\cdot}$.

In Section 1, we show that a compact object C in a triangulated category $\mathcal{T}$, which satisfies suitable conditions, induces a t-structure $(\mathcal{T}^{\leq 0}(C), \mathcal{T}^{\geq 0}(C))$, and its core $\mathcal{T}^{0}(C)$ is equivalent to the category $\operatorname{Mod} B$ of left B-modules, where $B = \operatorname{End}_{\mathcal{T}}(C)^{\mathrm{op}}$ (Theorem 1.3). In Section 2, we define subcategories $\mathcal{X}(P^{\cdot})$, $\mathcal{Y}(P^{\cdot})$ of an abelian category $\mathcal{A}$ satisfying the condition Ab4, and show when $(\mathcal{X}(P^{\cdot}), \mathcal{Y}(P^{\cdot}))$ is a torsion theory (Theorem 2.10). Furthermore, we show that if $P^{\cdot}$ induces a torsion theory $(\mathcal{X}(P^{\cdot}), \mathcal{Y}(P^{\cdot}))$ for $\mathcal{A}$, then the core $\mathsf{D}(\mathcal{A})^{0}(P^{\cdot})$ is admissible abelian, and then there is a torsion theory $(\mathcal{Y}(P^{\cdot})[1], \mathcal{X}(P^{\cdot}))$ for $\mathsf{D}(\mathcal{A})^{0}(P^{\cdot})$ (Theorem 2.15). In Section 3, we apply results of Section 2 to module categories. We characterize a torsion theory for the category $\operatorname{Mod} A$ of left A-modules,

The detailed version of this paper will be submitted for publication elsewhere.

and for its core $\mathsf{D}(\mathrm{Mod}\,A)^0(P^\cdot)$ (Theorems 3.5 and 3.8). Furthermore, using a torsion theory, we give equivalent conditions for $P^\cdot$ to be a tilting complex (Corollary 3.6). In Section 4, We show that, if $P^\cdot$ is a tilting complex, then it induces equivalences between torsion theories for $\mathrm{Mod}\,A$ and for $\mathrm{Mod}\,B$, where $B = \mathrm{End}_{\mathsf{D}(\mathrm{Mod}\,A)}(P^\cdot)^{\mathrm{op}}$ (Theorem 4.4). In Section 5, in the case of artin algebras, if a torsion theory $(\mathcal{X}, \mathcal{Y})$ satisfies certain conditions, then there exists a tilting complex $P^\cdot$ of term length two such that a torsion theory $(\mathcal{X}, \mathcal{Y})$ coincides with $(\mathcal{X}(P^\cdot), \mathcal{Y}(P^\cdot))$ (Theorem 5.8). As a consequence, we have one to one correspondence between tilting complexes of term length two and torsion theories with certain conditions (Corollary 3.7, Propositions 5.5, 5.7 and Theorem 5.8).

1. t-structures induced by compact objects

In this section, we deal with a triangulated category $\mathcal{T}$ and its full subcategory $\mathcal{C}$. We will call $\mathcal{C}$ admissible abelian provided that $\mathrm{Hom}_{\mathcal{T}}(\mathcal{C}, \mathcal{C}[n]) = 0$ for $n < 0$, and that all morphisms in $\mathcal{C}$ are $\mathcal{C}$-admissible in the sense of [BBD], 1.2.3. In this case, according to [BBD], Proposition 1.2.4, $\mathcal{C}$ is an abelian category. A triangulated category $\mathcal{T}$ is said to contain direct sums if direct sums of objects indexed by any set exist in $\mathcal{T}$. An object C of $\mathcal{T}$ is called compact if $\mathrm{Hom}_{\mathcal{T}}(C, -)$ commutes with direct sums. Furthermore, a collection $\mathcal{S}$ of compact objects of $\mathcal{T}$ is called a generating set provided that $X = 0$ whenever $\mathrm{Hom}_{\mathcal{T}}(\mathcal{S}, X) = 0$, and that $\mathcal{S}$ is stable under suspension (see [Ne] for details). For an object $C \in \mathcal{T}$ and an integer n, we denote by $\mathcal{T}^{\geq n}(C)$ (resp., $\mathcal{T}^{\leq n}(C)$) the full subcategory of $\mathcal{T}$ consisting of $X \in \mathcal{T}$ with $\mathrm{Hom}_{\mathcal{T}}(C, X[i]) = 0$ for $i < n$ (resp., $i > n$), and set $\mathcal{T}^0(C) = \mathcal{T}^{\leq 0}(C) \cap \mathcal{T}^{\geq 0}(C)$.

For an abelian category $\mathcal{A}$, we denote by $\mathsf{C}(\mathcal{A})$ the category of complexes of $\mathcal{A}$, and denote by $\mathsf{D}(\mathcal{A})$ (resp., $\mathsf{D}^+(\mathcal{A})$, $\mathsf{D}^-(\mathcal{A})$, $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$) the derived category of complexes of $\mathcal{A}$ (resp., complexes of $\mathcal{A}$ with bounded below homologies, complexes of $\mathcal{A}$ with bounded above homologies, complexes of $\mathcal{A}$ with bounded homologies). For an additive category $\mathcal{B}$, we denote by $\mathsf{K}(\mathcal{B})$ (resp., $\mathsf{K}^-(\mathcal{B})$, $\mathsf{K}^{\mathrm{b}}(\mathcal{B})$) the homotopy category of complexes of $\mathcal{B}$ (resp., bounded above complexes of $\mathcal{B}$, bounded complexes of $\mathcal{B}$) (see [RD] for details).

Proposition 1.1. *Let $\mathcal{T}$ be a triangulated category which contains direct sums, C a compact object satisfying $\mathrm{Hom}_{\mathcal{T}}(C, C[n]) = 0$ for $n > 0$. Then for any $r \in \mathbb{Z}$ and any object $X \in \mathcal{T}$, there exist an object $X^{\geq r} \in \mathcal{T}^{\geq r}(C)$ and a morphism $\alpha^{\geq r} : X \to X^{\geq r}$ in $\mathcal{T}$ such that*

(i) *for any $i \geq r$, $\mathrm{Hom}_{\mathcal{T}}(C, \alpha^{\geq r}[i])$ is an isomorphism,*

(ii) *for every object $Y \in \mathcal{T}^{\geq r}(C)$, $\mathrm{Hom}_{\mathcal{T}}(\alpha^{\geq r}, Y)$ is an isomorphism.*

Definition 1.2 ([BBD]). Let $\mathcal{T}$ be a triangulated category. For full subcategories $\mathcal{T}^{\leq 0}$ and $\mathcal{T}^{\geq 0}$, $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ is called a t-structure on $\mathcal{T}$ provided that

(i) $\mathrm{Hom}_{\mathcal{T}}(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 1}) = 0$;

(ii) $\mathcal{T}^{\leq 0} \subset \mathcal{T}^{\leq 1}$ and $\mathcal{T}^{\geq 0} \supset \mathcal{T}^{\geq 1}$;

(iii) for any $X \in \mathcal{T}$, there exists a distinguished triangle $X' \to X \to X'' \to$ with $X' \in \mathcal{T}^{\leq 0}$ and $X'' \in \mathcal{T}^{\geq 1}$,

where $\mathcal{T}^{\leq n} = \mathcal{T}^{\leq 0}[-n]$ and $\mathcal{T}^{\geq n} = \mathcal{T}^{\geq 0}[-n]$.

A t-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ on $\mathcal{T}$ is called non-degenerate if $\bigcap_{n \in \mathbb{Z}} \mathcal{T}^{\leq n} = \bigcap_{n \in \mathbb{Z}} \mathcal{T}^{\geq n} = \{0\}$.

Theorem 1.3. *Let $\mathcal{T}$ be a triangulated category which contains direct sums, C a compact object satisfying* $\mathrm{Hom}_{\mathcal{T}}(C, C[n]) = 0$ *for* $n > 0$, *and* $B = \mathrm{End}_{\mathcal{T}}(C)^{\mathrm{op}}$. *If* $\{C[i] : i \in \mathbb{Z}\}$ *is a generating set, then the following hold.*

(1) $(\mathcal{T}^{\leq 0}(C), \mathcal{T}^{\geq 0}(C))$ *is a non-degenerate t-structure on* $\mathcal{T}$.

(2) $\mathcal{T}^0(C)$ *is admissible abelian.*

(3) *The functor*

$$\mathrm{Hom}_{\mathcal{T}}(C, -) : \mathcal{T}^0(C) \to \mathsf{Mod}\, B$$

is an equivalence.

Remark 1.4. Under the condition of Theorem 1.3, according to [BBD], Proposition 1.3.3, there exists a functor $(-)^{\geq n} : \mathcal{T} \to \mathcal{T}^{\geq n}(C)$ (resp., $(-)^{\leq n} : \mathcal{T} \to \mathcal{T}^{\leq n}(C)$) which is the right (resp., left) adjoint of the natural embedding functor $\mathcal{T}^{\geq n}(C) \to \mathcal{T}$ (resp., $\mathcal{T}^{\leq n}(C) \to \mathcal{T}$).

For an object C in a triangulated category $\mathcal{T}$ and integers $s \leq t$, let $\mathcal{T}^{[s]}(C) = \mathcal{T}^0(C)[-s]$, $\mathcal{T}^{[s,t]}(C) = \mathcal{T}^{\leq t}(C) \cap \mathcal{T}^{\geq s}(C)$, and $\mathcal{T}^{\mathrm{b}}(C) = (\bigcup_{n \in \mathbb{Z}} \mathcal{T}^{\leq n}(C)) \cap (\bigcup_{n \in \mathbb{Z}} \mathcal{T}^{\geq n}(C))$. An object M of an abelian category $\mathcal{A}$ is called small provided that $\mathrm{Hom}_{\mathcal{A}}(M, -)$ commutes with direct sums in $\mathcal{A}$.

Corollary 1.5. *Let $\mathcal{A}$ be an abelian category satisfying the condition Ab4 (i.e. direct sums of exact sequences are exact) and $T^{\cdot}$ a bounded complex of small projective objects of $\mathcal{A}$ satisfying*

(i) $\{T^{\cdot}[i] : i \in \mathbb{Z}\}$ *is a generating set for* $\mathsf{D}(\mathcal{A})$,

(ii) $\mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(T^{\cdot}, T^{\cdot}[i]) = 0$ *for* $i \neq 0$.

If either of the following conditions (1) or (2) is satisfied, then we have an equivalence of triangulated categories

$$\mathsf{D}(\mathcal{A})^{\mathrm{b}}(T^{\cdot}) \cong \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B),$$

where $B = \mathrm{End}_{\mathsf{D}(\mathcal{A})}(T^{\cdot})^{\mathrm{op}}$.

(1) *$\mathcal{A}$ has enough projectives.*

(2) *$\mathcal{A}$ has enough injectives and* $\mathsf{D}(\mathcal{A})^{\geq 0}(T^{\cdot}) \subset \mathsf{D}^{+}(\mathcal{A})$.

Moreover, if $\mathsf{D}(\mathcal{A})^0(T^{\cdot}) \subset \mathsf{D}^{\mathrm{b}}(\mathcal{A})$, *then we have an equivalence*

$$\mathsf{D}^{\mathrm{b}}(\mathcal{A}) \cong \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B).$$

2. Torsion theories for abelian categories

Throughout this section, we fix the following notation. Let $\mathcal{A}$ be an abelian category satisfying the condition Ab4, and let $d_P^{-1} : P^{-1} \to P^0$ be a morphism in $\mathcal{A}$ with the P^i being small projective objects of $\mathcal{A}$, and denote by $P^{\cdot}$ the mapping cone of d_P^{-1}. We set $\mathcal{C}(P^{\cdot}) = \mathsf{D}(\mathcal{A})^0(P^{\cdot})$, $B = \mathrm{End}_{\mathsf{D}(\mathcal{A})}(P^{\cdot})^{\mathrm{op}}$, and define a pair of full subcategories of $\mathcal{A}$

$$\mathcal{X}(P^{\cdot}) = \{X \in \mathcal{A} : \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(P^{\cdot}, X[1]) = 0\},$$
$$\mathcal{Y}(P^{\cdot}) = \{X \in \mathcal{A} : \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(P^{\cdot}, X) = 0\}.$$

For any $X \in \mathcal{A}$, we define a subobject of X

$$\tau(X) = \sum\nolimits_{f \in \mathrm{Hom}_{\mathcal{A}}(\mathrm{H}^0(P^\cdot), X)} \mathrm{Im}\, f$$

and an exact sequence in $\mathcal{A}$

$$(e_X) : 0 \to \tau(X) \xrightarrow{j_X} X \to \pi(X) \to 0.$$

Remark 2.1. It is easy to see that $P^\cdot$ is a compact object of $\mathsf{D}(\mathcal{A})$, and we have $\mathcal{X}(P^\cdot) = \mathsf{D}(\mathcal{A})^{\leq 0}(P^\cdot) \cap \mathcal{A}$ and $\mathcal{Y}(P^\cdot) = \mathsf{D}(\mathcal{A})^{\geq 1}(P^\cdot) \cap \mathcal{A}$.

Lemma 2.2. *For any $X \in \mathcal{A}$, the following hold.*
(1) $\mathrm{Ker}(\mathrm{Hom}_{\mathcal{A}}(d_P^{-1}, X)) \cong \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(P^\cdot, X)$.
(2) $\mathrm{Cok}(\mathrm{Hom}_{\mathcal{A}}(d_P^{-1}, X)) \cong \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(P^\cdot, X[1])$.

Lemma 2.3. *For any $X \in \mathcal{A}$, the following hold.*
(1) $\mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(P^\cdot, X[n]) = 0$ *for $n > 1$ and $n < 0$.*
(2) $\mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(P^\cdot, X) \cong \mathrm{Hom}_{\mathcal{A}}(\mathrm{H}^0(P^\cdot), X)$.

Lemma 2.4. *The following hold.*
(1) *$\mathcal{X}(P^\cdot)$ is closed under factor objects and direct sums.*
(2) *$\mathcal{Y}(P^\cdot)$ is closed under subobjects.*
(3) *For any $X \in \mathcal{A}$, $\mathrm{Hom}_{\mathcal{A}}(\mathrm{H}^0(P^\cdot), j_X)$ is an isomorphism.*

Lemma 2.5. *For any $X^\cdot \in \mathsf{D}(\mathcal{A})$ and $n \in \mathbb{Z}$, we have a functorial exact sequence*

$$0 \to \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(P^\cdot, \mathrm{H}^{n-1}(X^\cdot)[1]) \to \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(P^\cdot, X^\cdot[n]) \to \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(P^\cdot, \mathrm{H}^n(X^\cdot)) \to 0.$$

Moreover, the above short exact sequence commutes with direct sums.

Lemma 2.6. *The following are equivalent.*
(1) *$\{P^\cdot[i] : i \in \mathbb{Z}\}$ is a generating set for $\mathsf{D}(\mathcal{A})$.*
(2) $\mathcal{X}(P^\cdot) \cap \mathcal{Y}(P^\cdot) = \{0\}$.

Lemma 2.7. *The following hold.*
(1) $\mathrm{H}^0(P^\cdot) \in \mathcal{X}(P^\cdot)$ *if and only if* $\mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(P^\cdot, P^\cdot[i]) = 0$ *for all $i > 0$.*
(2) $\mathrm{H}^{-1}(P^\cdot) \in \mathcal{Y}(P^\cdot)$ *if and only if* $\mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(P^\cdot, P^\cdot[i]) = 0$ *for all $i < 0$.*

Definition 2.8. A pair $(\mathcal{X}, \mathcal{Y})$ of full subcategories $\mathcal{X}, \mathcal{Y}$ in an abelian category $\mathcal{A}$ is called a torsion theory for $\mathcal{A}$ provided that the following conditions are satisfied (see e.g. [Di] for details):
(i) $\mathcal{X} \cap \mathcal{Y} = \{0\}$;
(ii) $\mathcal{X}$ is closed under factor objects;
(iii) $\mathcal{Y}$ is closed under subobjects;
(iv) for any object X of $\mathcal{A}$, there exists an exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{A}$ with $X' \in \mathcal{X}$ and $X'' \in \mathcal{Y}$.

Remark 2.9. Let $\mathcal{A}$ be an abelian category and $(\mathcal{X}, \mathcal{Y})$ a torsion theory for $\mathcal{A}$. Then for any $Z \in \mathcal{A}$, the following hold.

(1) $Z \in \mathcal{X}$ if and only if $\mathrm{Hom}_{\mathcal{A}}(Z, \mathcal{Y}) = 0$.
(2) $Z \in \mathcal{Y}$ if and only if $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X}, Z) = 0$.

Theorem 2.10. *The following are equivalent.*
(1) $\{P^{\cdot}[i] : i \in \mathbb{Z}\}$ *is a generating set for* $\mathsf{D}(\mathcal{A})$ *and* $\mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(P^{\cdot}, P^{\cdot}[i]) = 0$ *for all* $i > 0$.
(2) $\mathcal{X}(P^{\cdot}) \cap \mathcal{Y}(P^{\cdot}) = \{0\}$ *and* $\mathrm{H}^0(P^{\cdot}) \in \mathcal{X}(P^{\cdot})$.
(3) $\mathcal{X}(P^{\cdot}) \cap \mathcal{Y}(P^{\cdot}) = \{0\}$ *and* $\tau(X) \in \mathcal{X}(P^{\cdot})$, $\pi(X) \in \mathcal{Y}(P^{\cdot})$ *for all* $X \in \mathcal{A}$.
(4) $(\mathcal{X}(P^{\cdot}), \mathcal{Y}(P^{\cdot}))$ *is a torsion theory for* $\mathcal{A}$.

Definition 2.11. For a complex $X^{\cdot} = (X^i, d^i)$, we define the following truncations:

$$\begin{aligned}
\sigma_{>n}(X^{\cdot}) &: \ldots \to 0 \to \mathrm{Im}\, d^n \to X^{n+1} \to X^{n+2} \to \ldots, \\
\sigma_{\leq n}(X^{\cdot}) &: \ldots \to X^{n-2} \to X^{n-1} \to \mathrm{Ker}\, d^n \to 0 \to \ldots, \\
\sigma'_{\geq n}(X^{\cdot}) &: \ldots \to 0 \to \mathrm{Cok}\, d^{n-1} \to X^{n+1} \to X^{n+2} \to \ldots, \\
\sigma'_{<n}(X^{\cdot}) &: \ldots \to X^{n-2} \to X^{n-1} \to \mathrm{Im}\, d^{n-1} \to 0 \to \ldots.
\end{aligned}$$

Lemma 2.12. *For any* $X^{\cdot} \in \mathsf{D}(\mathcal{A})$ *with* $\mathrm{H}^n(X^{\cdot}) = 0$ *for* $n > 0$ *and* $n < -1$, *there exists a distinguished triangle in* $\mathsf{D}(\mathcal{A})$ *of the form*

$$\mathrm{H}^{-1}(X^{\cdot})[1] \to X^{\cdot} \to \mathrm{H}^0(X^{\cdot}) \to .$$

Lemma 2.13. *Assume* $\mathcal{X}(P^{\cdot}) \cap \mathcal{Y}(P^{\cdot}) = \{0\}$. *Then for any* $X^{\cdot} \in \mathsf{D}(\mathcal{A})$, *the following are equivalent.*
(1) $X^{\cdot} \in \mathcal{C}(P^{\cdot})$.
(2) $\mathrm{H}^n(X^{\cdot}) = 0$ *for* $n > 0$ *and* $n < -1$, $\mathrm{H}^0(X^{\cdot}) \in \mathcal{X}(P^{\cdot})$ *and* $\mathrm{H}^{-1}(X^{\cdot}) \in \mathcal{Y}(P^{\cdot})$.

Remark 2.14. Let $\mathcal{A}$ be an abelian category and $\mathcal{X}, \mathcal{Y}$ full subcategories of $\mathcal{A}$. Then the pair $(\mathcal{X}, \mathcal{Y})$ is a torsion theory for $\mathcal{A}$ if and only if the following two conditions are satisfied:
(i) $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X}, \mathcal{Y}) = 0$;
(ii) for any object X in $\mathcal{A}$, there exists an exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{A}$ with $X' \in \mathcal{X}$ and $X'' \in \mathcal{Y}$.

Theorem 2.15. *Assume* $\mathcal{X}(P^{\cdot}) \cap \mathcal{Y}(P^{\cdot}) = \{0\}$ *and* $\mathrm{H}^0(P^{\cdot}) \in \mathcal{X}(P^{\cdot})$. *Then the following hold.*
(1) $\mathcal{C}(P^{\cdot})$ *is admissible abelian.*
(2) *The functor*

$$\mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(P^{\cdot}, -) : \mathcal{C}(P^{\cdot}) \to \mathsf{Mod}\, B$$

is an equivalence.
(3) $(\mathcal{Y}(P^{\cdot})[1], \mathcal{X}(P^{\cdot}))$ *is a torsion theory for* $\mathcal{C}(P^{\cdot})$.

Proposition 2.16. *Assume* $P^{\cdot}$ *satisfies the conditions*
(i) $\{P^{\cdot}[i] : i \in \mathbb{Z}\}$ *is a generating set for* $\mathsf{D}(\mathcal{A})$,
(ii) $\mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(P^{\cdot}, P^{\cdot}[i]) = 0$ *for* $i \neq 0$.

If $\mathcal{A}$ has either enough projectives or enough injectives, then we have an equivalence of triangulated categories

$$\mathsf{D}^{\mathsf{b}}(\mathcal{A}) \cong \mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\, B).$$

3. Torsion theories for module categories

In this section, we apply results of Section 2 to the case of module categories. In and after this section, R is a commutative ring and I is an injective cogenerator in the category of R-modules. We set $D = \mathrm{Hom}_R(-, I)$. Let A be an R-algebra and denote by $\mathsf{Proj}\, A$ (resp., $\mathsf{proj}\, A$) the full additive subcategory of $\mathsf{Mod}\, A$ consisting of projective (resp., finitely generated projective) modules. We denote by A^{op} the opposite ring of A and consider right A-modules as left A^{op}-modules. Also, we denote by $(-)^*$ both the A-dual functors $\mathrm{Hom}_A(-, A)$ and set $\nu = D \circ (-)^*$.

It is well known that, in a module category, the small projective objects are just the finitely generated projective modules. In the following, we deal with the case where $\mathcal{A} = \mathsf{Mod}\, A$ and use the same notation as in Section 2.

Lemma 3.1. *For any $X \in \mathsf{Mod}\, A$, we have*

$$\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, A)}(P^{\cdot}, X[1]) \cong \mathrm{H}^1((P^{\cdot})^*)\otimes_A X.$$

Lemma 3.2. *The following hold.*
(1) $\mathcal{X}(P^{\cdot}) = \mathrm{Ker}(\mathrm{H}^1((P^{\cdot})^*)\otimes_A -)$.
(2) $\mathcal{Y}(P^{\cdot}) = \mathrm{Ker}(\mathrm{Hom}_A(\mathrm{H}^0(P^{\cdot}), -))$.

Lemma 3.3. *The following hold.*
(1) $D(\mathrm{H}^1((P^{\cdot})^*)) \cong \mathrm{H}^{-1}(\nu(P^{\cdot}))$.
(2) $\mathcal{X}(P^{\cdot}) = \mathrm{Ker}(\mathrm{Hom}_A(-, \mathrm{H}^{-1}(\nu(P^{\cdot}))))$ *and hence* $\mathrm{H}^0(P^{\cdot}) \in \mathcal{X}(P^{\cdot})$ *if and only if* $\mathrm{H}^{-1}(\nu(P^{\cdot})) \in \mathcal{Y}(P^{\cdot})$.
(3) $\mathrm{Ker}(\mathrm{Tor}_1^A(\mathrm{H}^1((P^{\cdot})^*), -)) = \mathrm{Ker}(\mathrm{Ext}_A^1(-, \mathrm{H}^{-1}(\nu(P^{\cdot}))))$.

Lemma 3.4. *The following hold.*
(1) $\mathcal{X}(P^{\cdot}) \subset \mathrm{Ker}(\mathrm{Ext}_A^1(\mathrm{H}^0(P^{\cdot}), -))$.
(2) $\mathcal{Y}(P^{\cdot}) \subset \mathrm{Ker}(\mathrm{Tor}_1^A(\mathrm{H}^1((P^{\cdot})^*), -))$.

Theorem 3.5. *The following are equivalent.*
(1) $\mathcal{X}(P^{\cdot}) \cap \mathcal{Y}(P^{\cdot}) = \{0\}$ *and* $\mathrm{H}^0(P^{\cdot}) \in \mathcal{X}(P^{\cdot})$.
(2) $\mathcal{X}(P^{\cdot}) \cap \mathcal{Y}(P^{\cdot}) = \{0\}$ *and* $\tau(X) \in \mathcal{X}(P^{\cdot})$, $\pi(X) \in \mathcal{Y}(P^{\cdot})$ *for all* $X \in \mathsf{Mod}\, A$.
(3) $(\mathcal{X}(P^{\cdot}), \mathcal{Y}(P^{\cdot}))$ *is a torsion theory for* $\mathsf{Mod}\, A$.
(4) $\mathcal{X}(P^{\cdot})$ *consists of the modules generated by* $\mathrm{H}^0(P^{\cdot})$ *and* $\mathcal{Y}(P^{\cdot})$ *consists of the modules cogenerated by* $\mathrm{H}^{-1}(\nu(P^{\cdot}))$.

Corollary 3.6. *The following are equivalent.*
(1) $P^{\cdot}$ *is a tilting complex.*
(2) $\mathcal{X}(P^{\cdot}) \cap \mathcal{Y}(P^{\cdot}) = \{0\}, \mathrm{H}^0(P^{\cdot}) \in \mathcal{X}(P^{\cdot})$ *and* $\mathrm{H}^{-1}(P^{\cdot}) \in \mathcal{Y}(P^{\cdot})$.
(3) $(\mathcal{X}(P^{\cdot}), \mathcal{Y}(P^{\cdot}))$ *is a torsion theory for* $\mathsf{Mod}\, A$ *and* $\mathrm{H}^{-1}(P^{\cdot}) \in \mathcal{Y}(P^{\cdot})$.

For an object X in an additive category $\mathcal{B}$, we denote by $\mathsf{add}(X)$ the full subcategory of $\mathcal{B}$ consisting of objects which are direct summands of finite direct sums of copies of X.

Corollary 3.7. *For any tilting complexes $P_1^{\cdot} : P_1^{-1} \to P_1^0$, $P_2^{\cdot} : P_2^{-1} \to P_2^0$ for A of term length two, the following are equivalent.*
(1) $(\mathcal{X}(P_1^{\cdot}), \mathcal{Y}(P_1^{\cdot})) = (\mathcal{X}(P_2^{\cdot}), \mathcal{Y}(P_2^{\cdot}))$.
(2) $\mathsf{add}(P_1^{\cdot}) = \mathsf{add}(P_2^{\cdot})$ *in* $\mathsf{K}^{\mathrm{b}}(\mathsf{Proj}\, A)$.

Theorem 3.8. *Assume $\mathcal{X}(P^{\cdot}) \cap \mathcal{Y}(P^{\cdot}) = \{0\}$ and $\mathrm{H}^0(P^{\cdot}) \in \mathcal{X}(P^{\cdot})$. Then the following hold.*
(1) $\{P^{\cdot}[i] : i \in \mathbb{Z}\}$ *is a generating set for* $\mathsf{D}(\mathsf{Mod}\, A)$.
(2) $\mathcal{C}(P^{\cdot})$ *is admissible abelian.*
(3) $(\mathcal{Y}(P^{\cdot})[1], \mathcal{X}(P^{\cdot}))$ *is a torsion theory for* $\mathcal{C}(P^{\cdot})$.
(4) *The functor*

$$\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, A)}(P^{\cdot}, -) : \mathcal{C}(P^{\cdot}) \to \mathsf{Mod}\, B$$

is an equivalence.

Remark 3.9. The following are equivalent.
(1) $P^{\cdot}$ is a tilting complex.
(2) $\mathcal{X}(P^{\cdot}) \cap \mathcal{Y}(P^{\cdot}) = \{0\}$ and $P^{\cdot} \in \mathcal{C}(P^{\cdot})$.

Example 3.10 (cf. [HK]). Let A be a finite dimensional algebra over a field k given by a quiver

$$\begin{array}{ccc} 1 & \xrightarrow{\alpha} & 2 \\ {\scriptstyle\delta}\uparrow & & \downarrow{\scriptstyle\beta} \\ 4 & \xleftarrow[\gamma]{} & 3 \end{array}$$

with relations $\beta\alpha = \gamma\beta = \delta\gamma = \alpha\delta = 0$. For each vertex i, we denote by $S(i), P(i)$ the corresponding simple and indecomposable projective left A-modules, respectively. Define a complex $P^{\cdot}$ as the mapping cone of the homomorphism

$$d_P^{-1} = \left[\begin{smallmatrix} f & 0 & 0 & 0 \\ 0 & 0 & g & 0 \end{smallmatrix}\right] : P(2)^2 \oplus P(4)^2 \to P(1) \oplus P(3),$$

where f and g denote the right multiplications of α and γ, respectively. Then $P^{\cdot}$ is not a tilting complex. However, $P^{\cdot}$ satisfies the assumption of Theorem 3.8 and hence we have an equivalence of abelian categories

$$\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, A)}(P^{\cdot}, -) : \mathcal{C}(P^{\cdot}) \to \mathsf{Mod}\, B,$$

where $B = \mathrm{End}_{\mathsf{D}(\mathsf{Mod}\, A)}(P^{\cdot})^{\mathrm{op}}$ is a finite dimensional k-algebra given by a quiver

$$1 \leftarrow 2 \qquad 3 \leftarrow 4.$$

There exist exact sequences in $\mathcal{C}(P^{\cdot})$ of the form

$$0 \to S(1) \to S(2)[1] \to P(1)[1] \to 0, \quad 0 \to S(3) \to S(4)[1] \to P(3)[1] \to 0,$$

and these objects and morphisms generate $\mathcal{C}(P^{\cdot})$.

4. Equivalences between torsion theories

Throughout this section, $P^\cdot$ is assumed to be a tilting complex. Then there exists an equivalence of triangulated categories

$$F : \mathsf{D}^-(\mathsf{Mod}\,B) \to \mathsf{D}^-(\mathsf{Mod}\,A)$$

such that $F(B) = P^\cdot$. Let $G : \mathsf{D}^-(\mathsf{Mod}\,A) \to \mathsf{D}^-(\mathsf{Mod}\,B)$ be a quasi-inverse of F. For any $n \in \mathbb{Z}$, we have ring homomorphisms

$$B \to \mathrm{End}_A(\mathrm{H}^n(P^\cdot))^{\mathrm{op}} \quad \text{and} \quad B \to \mathrm{End}_A(\mathrm{H}^n((P^\cdot)^*)).$$

In particular, $\mathrm{H}^0(P^\cdot)$ is an A-B-bimodule and $\mathrm{H}^1((P^\cdot)^*)$ is a B-A-bimodule.

Lemma 4.1. *The following hold.*
(1) *For any $X^\cdot \in \mathcal{C}(P^\cdot)$, we have $G(X^\cdot) \cong \mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\,A)}(P^\cdot, X^\cdot)$.*
(2) *We have an equivalence*

$$\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\,A)}(P^\cdot, -) : \mathcal{C}(P^\cdot) \to \mathsf{Mod}\,B$$

whose quasi-inverse is given by the restriction of F to $\mathsf{Mod}\,B$.

Lemma 4.2. *There exists a tilting complex $Q^\cdot \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,B)$ such that*
(i) $Q^\cdot \cong G(A)$,
(ii) $Q^i = 0$ *for* $i > 1$ *and* $i < 0$,
(iii) $\mathrm{H}^i(Q^\cdot) \cong \mathrm{H}^i((P^\cdot)^*)$ *for* $0 \le i \le 1$,
(iv) $\mathrm{H}^i(\mathrm{Hom}^\cdot_B(Q^\cdot, B)) \cong \mathrm{H}^i(P^\cdot)$ *for* $-1 \le i \le 0$.

Lemma 4.3. *For any $M \in \mathsf{Mod}\,B$, the following hold.*
(1) $\mathrm{H}^i(F(M)) = 0$ *for* $i > 0$ *and* $i < -1$.
(2) $\mathrm{H}^0(F(M)) \cong \mathrm{H}^0(P^\cdot) \otimes_B M$.
(3) $\mathrm{H}^{-1}(F(M)) \cong \mathrm{Hom}_B(\mathrm{H}^1((P^\cdot)^*), M)$.

Theorem 4.4. *Define a pair of full subcategories of* $\mathsf{Mod}\,B$

$$\mathcal{U}(P^\cdot) = \mathrm{Ker}(\mathrm{H}^0(P^\cdot)\otimes_B -), \quad \mathcal{V}(P^\cdot) = \mathrm{Ker}(\mathrm{Hom}_B(\mathrm{H}^1((P^\cdot)^*), -)).$$

Then the following hold.
(1) *$(\mathcal{U}(P^\cdot), \mathcal{V}(P^\cdot))$ is a torsion theory for* $\mathsf{Mod}\,B$.
(2) *We have a pair of functors*

$$\mathrm{Hom}_A(\mathrm{H}^0(P^\cdot), -) : \mathcal{X}(P^\cdot) \to \mathcal{V}(P^\cdot), \quad \mathrm{H}^0(P^\cdot)\otimes_B - : \mathcal{V}(P^\cdot) \to \mathcal{X}(P^\cdot)$$

which define an equivalence.
(3) *We have a pair of functors*

$$\mathrm{H}^1((P^\cdot)^*)\otimes_A - : \mathcal{Y}(P^\cdot) \to \mathcal{U}(P^\cdot), \quad \mathrm{Hom}_B(\mathrm{H}^1((P^\cdot)^*), -) : \mathcal{U}(P^\cdot) \to \mathcal{Y}(P^\cdot)$$

which define an equivalence.

Definition 4.5. Let $(\mathcal{U}, \mathcal{V})$ be a torsion theory for an abelian category $\mathcal{A}$. Then $(\mathcal{U}, \mathcal{V})$ is called splitting if $\mathrm{Ext}^1_{\mathcal{A}}(\mathcal{V}, \mathcal{U}) = 0$.

For a left A-module M, we denote by proj dim $_A M$ (resp., inj dim $_A M$) the projective (resp., the injective) dimension of M.

Proposition 4.6. *The torsion theory* $(\mathcal{U}(P^\cdot), \mathcal{V}(P^\cdot))$ *for* $\operatorname{Mod} B$ *is splitting if and only if* $\operatorname{Ext}_A^2(\mathcal{X}(P^\cdot), \mathcal{Y}(P^\cdot)) = 0$. *In particular,* $(\mathcal{U}(P^\cdot), \mathcal{V}(P^\cdot))$ *is splitting if either* proj dim $X \leq 1$ *for all* $X \in \mathcal{X}(P^\cdot)$ *or* inj dim $Y \leq 1$ *for all* $Y \in \mathcal{Y}(P^\cdot)$.

5. Torsion theories for artin algebras

In this section, we deal with the case where R is a commutative artin ring, I is an injective envelope of an R-module $R/\operatorname{rad}(R)$ and A is a finitely generated R-module. We denote by $\operatorname{mod} A$ the full abelian subcategory of $\operatorname{Mod} A$ consisting of finitely generated modules. Note that $\mathrm{H}^n(P^\cdot), \mathrm{H}^n(\nu(P^\cdot)) \in \operatorname{mod} A$ for all $n \in \mathbb{Z}$. We set

$$\mathcal{X}_c(P^\cdot) = \mathcal{X}(P^\cdot) \cap \operatorname{mod} A \quad \text{and} \quad \mathcal{Y}_c(P^\cdot) = \mathcal{Y}(P^\cdot) \cap \operatorname{mod} A.$$

Proposition 5.1. *The following are equivalent.*
(1) $\mathcal{X}_c(P^\cdot) \cap \mathcal{Y}_c(P^\cdot) = \{0\}$ *and* $\mathrm{H}^0(P^\cdot) \in \mathcal{X}_c(P^\cdot)$.
(2) $\mathcal{X}_c(P^\cdot) \cap \mathcal{Y}_c(P^\cdot) = \{0\}$ *and* $\tau(X) \in \mathcal{X}_c(P^\cdot)$, $\pi(X) \in \mathcal{Y}_c(P^\cdot)$ *for all* $X \in \operatorname{mod} A$.
(3) $(\mathcal{X}_c(P^\cdot), \mathcal{Y}_c(P^\cdot))$ *is a torsion theory for* $\operatorname{mod} A$.
(4) $\mathcal{X}_c(P^\cdot)$ *consists of the modules generated by* $\mathrm{H}^0(P^\cdot)$ *and* $\mathcal{Y}_c(P^\cdot)$ *consists of the modules cogenerated by* $\mathrm{H}^{-1}(\nu(P^\cdot))$.

Lemma 5.2. *The following are equivalent.*
(1) $\{P^\cdot[i] : i \in \mathbb{Z}\}$ *is a generating set for* $\mathrm{D}(\operatorname{mod} A)$.
(2) $\mathcal{X}_c(P^\cdot) \cap \mathcal{Y}_c(P^\cdot) = \{0\}$.

Lemma 5.3. *The following hold.*
(1) *If* $DA \in \mathcal{X}_c(P^\cdot)$, *then* $\mathrm{H}^{-1}(P^\cdot) = 0$, *i.e.* $P^\cdot \cong \mathrm{H}^0(P^\cdot)$ *in* $\mathrm{D}(\operatorname{mod} A)$.
(2) $\mathrm{H}^0(\nu(P^\cdot)) \in \mathcal{X}_c(P^\cdot)$ *if and only if* $\mathrm{H}^{-1}(P^\cdot) \in \mathcal{Y}_c(P^\cdot)$.

Lemma 5.4. *Assume* $\mathcal{X}_c(P^\cdot) \cap \mathcal{Y}_c(P^\cdot) = \{0\}$ *and* $\mathrm{H}^0(P^\cdot) \in \mathcal{X}_c(P^\cdot)$. *Then the following are equivalent.*
(1) $\mathrm{H}^0(\nu(P^\cdot)) \in \mathcal{X}_c(P^\cdot)$.
(2) $\mathcal{X}_c(P^\cdot)$ *is stable under* $DA \otimes_A -$.
(3) $\mathrm{H}^{-1}(P^\cdot) \in \mathcal{Y}_c(P^\cdot)$.
(4) $\mathcal{Y}_c(P^\cdot)$ *is stable under* $\operatorname{Hom}_A(DA, -)$.

Proposition 5.5. *The following are equivalent.*
(1) $P^\cdot$ *is a tilting complex.*
(2) $\mathcal{X}_c(P^\cdot) \cap \mathcal{Y}_c(P^\cdot) = \{0\}$, $\mathrm{H}^0(P^\cdot) \in \mathcal{X}_c(P^\cdot)$ *and* $\mathrm{H}^{-1}(P^\cdot) \in \mathcal{Y}_c(P^\cdot)$.
(3) $(\mathcal{X}_c(P^\cdot), \mathcal{Y}_c(P^\cdot))$ *is a torsion theory for* $\operatorname{mod} A$ *and* $\mathrm{H}^{-1}(P^\cdot) \in \mathcal{Y}_c(P^\cdot)$.
(4) $(\mathcal{X}_c(P^\cdot), \mathcal{Y}_c(P^\cdot))$ *is a torsion theory for* $\operatorname{mod} A$ *and* $\mathcal{X}_c(P^\cdot)$ *is stable under* $DA \otimes_A -$.
(5) $(\mathcal{X}_c(P^\cdot), \mathcal{Y}_c(P^\cdot))$ *is a torsion theory for* $\operatorname{mod} A$ *and* $\mathcal{Y}_c(P^\cdot)$ *is stable under* $\operatorname{Hom}_A(DA, -)$.

Definition 5.6. Let $\mathcal{A}$ be an abelian category and $\mathcal{C}$ a full subcategory of $\mathcal{A}$ closed under extensions. Then an object $X \in \mathcal{C}$ is called Ext-projective (resp., Ext-injective) if $\mathrm{Ext}^1_{\mathcal{A}}(X, \mathcal{C}) = 0$ (resp., $\mathrm{Ext}^1_{\mathcal{A}}(\mathcal{C}, X) = 0$).

Proposition 5.7. *Assume $P^{\cdot}$ is a tilting complex. Then the following hold.*
(1) $\mathrm{H}^0(P^{\cdot}) \in \mathcal{X}_c(P^{\cdot})$ *is* Ext-*projective and generates* $\mathcal{X}_c(P^{\cdot})$.
(2) $\mathrm{H}^{-1}(\nu(P^{\cdot})) \in \mathcal{Y}_c(P^{\cdot})$ *is* Ext-*injective and cogenerates* $\mathcal{Y}_c(P^{\cdot})$.

Theorem 5.8. *Let $(\mathcal{X}, \mathcal{Y})$ be a torsion theory for* $\mathrm{mod}\, A$ *such that $\mathcal{X}$ contains an* Ext-*projective module X which generates $\mathcal{X}$, $\mathcal{Y}$ contains an* Ext-*injective module Y which cogenerates $\mathcal{Y}$, and $\mathcal{X}$ is stable under $DA \otimes_A -$. Let $M^{\cdot}_X$ be a minimal projective presentation of X and $N^{\cdot}_Y$ a minimal injective presentation of Y. Then*

$$P^{\cdot} = M^{\cdot}_X \oplus \mathrm{Hom}^{\cdot}_A(DA, N^{\cdot}_Y)[1]$$

is a tilting complex such that $\mathcal{X} = \mathcal{X}_c(P^{\cdot})$ and $\mathcal{Y} = \mathcal{Y}_c(P^{\cdot})$.

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Almost self-duality and H-rings *

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Abstract

Recently we pointed out that left H-rings (Harada rings) do not have a self-duality in general. In this note, we prove that every left H-ring has an almost self-duality, which is a generalization of self-dualities.

「すべての左 H 環が self-duality をもつか」という問題は，「すべての QF 環が中山自己同型写像をもつか」という問題と同値であることが，加戸・大城 [2] によって証明されていたが，最近筆者は [3] において，Kraemer [4] の構成した weakly symmetric self-duality をもたない QF 環の例が，中山自己同型写像をもたない QF 環の例であることを注意し，それを用いて，実際に self-duality をもたない左 H 環の例を構成した．したがって，一般には左 H 環は self-duality をもたない．しかし，すべての左 H 環は self-duality の一般化である almost self-duality と呼ばれる duality をもつことを証明できたので (定理 B)，この結果について報告する．

以下，すべての環は単位元をもち，すべての加群は単位的であるとする．加群 M に対して，その radical, socle, top を，それぞれ $J(M)$, $S(M)$, $T(M)$ で表す．

1 Almost self-duality

まず最初に，Morita duality に関するいくつかの定義を与えよう．両側加群 ${}_AU_B$ が **Morita duality を定めるとは**，U-dual functor の対 $\mathrm{Hom}_A(U,-) : A\text{-Mod} \rightleftarrows \text{Mod-}B : \mathrm{Hom}_B(U,-)$ が，それぞれ A-Mod，Mod-B の充満部分圏 $\mathcal{A}$，$\mathcal{B}$ で，次の条件を満たすものの間の duality を定めることをいう: (1) $A \in \mathcal{A}$，$B \in \mathcal{B}$，(2) $\mathcal{A}$，$\mathcal{B}$ は全射準同型像で閉じている．ただし，A-Mod，Mod-B はそれぞれ左 A，右 B 加群全体の圏を表す．よく知られているように，${}_AU_B$ が Morita duality を定めることは，${}_AU_B$ が忠実かつ平衡的で，${}_AU$, U_B がそれぞれ入射的余生成素であることと同値である．特に $A = B$ であるとき，${}_AU_A$ は **self-duality を定める**という．Morita duality を定める両側加群 ${}_AU_B$ が存在するとき，A は**左 Morita duality をもつ**といい，A は B に**左 Morita dual** であ

*The detailed version of this note will be submitted for publication elsewhere.

るという．環の列 $A_0 = A, A_1, \ldots, A_n = B$ で各 A_i は A_{i+1} に左 Morita dual であるようなものが存在するとき，A は B に左 **almost Morita dual** であるということにする．特に A が自分自身に左 almost Morita dual であるとき，A は **almost self-duality** をもつという (Simson [9] 参照)．同様に，右 almost Morita dual の概念も定義されるが，almost self-duality の存在は左右どちらの almost Morita dual を用いても同じであることを注意しておく．定義より，almost self-duality は self-duality の一般化である．

最初に述べたここでの目的を詳述するために，次の例から始めよう．

例 1. [1] と [10] の結果より，互いに非同型な環 $A_1, A_2, \ldots, A_5$ と Morita duality を定める両側加群 ${}_{A_1}U_{1A_2}, {}_{A_2}U_{2A_3}, \ldots, {}_{A_5}U_{5A_1}$ で，

$$A_i \cong \begin{pmatrix} D & V_i \\ 0 & D \end{pmatrix}$$

となるものが存在する．ただし，D はある斜体で V_i は (D, D) 両側加群である．いま，

$$R_1 = \begin{pmatrix} A_1 & U_1 & 0 & 0 & 0 & 0 \\ 0 & A_2 & U_2 & 0 & 0 & 0 \\ 0 & 0 & A_3 & U_3 & 0 & 0 \\ 0 & 0 & 0 & A_4 & U_4 & U_4 \\ U_5 & 0 & 0 & 0 & A_5 & A_5 \\ U_5 & 0 & 0 & 0 & J(A_5) & A_5 \end{pmatrix},$$

$$R_2 = \begin{pmatrix} A_2 & U_2 & 0 & 0 & 0 & 0 \\ 0 & A_3 & U_3 & 0 & 0 & 0 \\ 0 & 0 & A_4 & U_4 & 0 & 0 \\ 0 & 0 & 0 & A_5 & U_5 & U_5 \\ U_1 & 0 & 0 & 0 & A_1 & A_1 \\ U_1 & 0 & 0 & 0 & J(A_1) & A_1 \end{pmatrix},$$

$$\cdots$$

$$R_5 = \begin{pmatrix} A_5 & U_5 & 0 & 0 & 0 & 0 \\ 0 & A_1 & U_1 & 0 & 0 & 0 \\ 0 & 0 & A_2 & U_2 & 0 & 0 \\ 0 & 0 & 0 & A_3 & U_3 & U_3 \\ U_4 & 0 & 0 & 0 & A_4 & A_4 \\ U_4 & 0 & 0 & 0 & J(A_4) & A_4 \end{pmatrix}$$

とおく．このとき，$R_1, R_2, \ldots, R_5$ は互いに非同型な左 H 環で，各 $i = 1, 2, 3, 4$ に対して R_i は R_{i+1} に，R_5 は R_1 に左 Morita dual である．したがって，各 R_i は self-duality をもたないが almost self-duality をもつ．

本稿の目的は，上の例において各 R_i が almost self-duality をもつことが，これらの環における固有の現象ではなく，すべての左 H 環において成り立つことを概説することである．

両側加群 ${}_AU_B$ は Morita duality を定めるとする．このとき，それぞれ A，B の直交原始巾等元の基本集合 $\{e_1, \ldots, e_n\}$，$\{f_1, \ldots, f_n\}$ で，$S(Uf_i) \cong T(Ae_i)$，$S(e_iU) \cong T(f_iB)$ $(i = 1, \ldots, n)$ を満たすものが存在する．

補題 2 ([5, Corollary 3.4])．上の設定の元で，I を $\{1, \ldots, n\}$ の空でない部分集合とし，$e = \sum_{i \in I} e_i$，$f = \sum_{i \in I} f_i$ とおく．このとき，両側加群 ${}_{eAe}eUf_{fBf}$ も Morita duality を定める．

これを使えば，次の命題が得られる．

命題 3. 環 A の 0 でない巾等元 e に対して，もし A が almost self-duality をもてば，eAe も almost self-duality をもつ．

証明. $\{e_1, e_2, \ldots, e_n\}$ を A の直交原始巾等元の基本集合とする．$I \subset \{1, \ldots, n\}$ に対して，$g_I = \sum_{i \in I} e_i$ とおく．almost self-duality の存在は森田同値によって不変であるから，$e = g_{I_0}$，$I_0 \subset \{1, \ldots, n\}$ として良い．$m = |I_0|$ とおく．ただし，$|*|$ は $*$ の集合としての位数を表す．A は almost self-duality をもつから，補題 2 を繰り返し用いれば，$g_{I_0}Ag_{I_0}$ が $g_{I_1}Ag_{I_1}$ に左 almost Morita dual となるような $I_1 \subset \{1, \ldots, n\}$ が存在することが分かる．ここで $|I_1| = m$ である．以下同様にして，$\{1, \ldots, n\}$ の部分集合の列 $I_0, I_1, I_2, \ldots$ で，各 $g_{I_i}Ag_{I_i}$ は $g_{I_{i+1}}Ag_{I_{i+1}}$ に左 almost Morita dual，$|I_i| = m$ であるようなものが存在する．このとき，任意の $i < j$ に対して，$g_{I_i}Ag_{I_i}$ は $g_{I_j}Ag_{I_j}$ に左 almost Morita dual である．$\{1, \ldots, n\}$ の位数 m の部分集合は有限個しかないから，$I_i = I_j$ であるような $i < j$ が存在する．したがって，$g_{I_i}Ag_{I_i} = g_{I_j}Ag_{I_j}$ は almost self-duality をもつから，$eAe = g_{I_0}Ag_{I_0}$ も almost self-duality をもつ．□

注意 4. 補題 2 と命題 3 より，環 A における Morita duality や almost self-duality の存在は，巾等元 $e \in A$ について環 eAe に遺伝する．しかし，次の例が示すように，self-duality については，これは成り立たない．

例 5 (**Kraemer** [4])．環 $A_1, A_2, \ldots, A_5$ と両側加群 ${}_{A_1}U_{1A_2}, {}_{A_2}U_{2A_3}, \ldots, {}_{A_5}U_{5A_1}$ は例 1 の通りとする．$A = A_1 \times \cdots \times A_5$ を環直積，$U = U_1 \bigoplus \cdots \bigoplus U_5$ を加群の直和とすれば，U は Morita duality を定める (A, A) 両側加群となるから，A は self-duality をもつ．しかし，$e = (1, 0, \ldots, 0) \in A$ とおけば，e は中心的巾等元であるが，$eAe = A_1$ は self-duality をもたない．

命題 3 から，次の almost self-duality の存在の特徴付けが得られる．

定理 A．（アルチン）環 A が almost self-duality をもつための必要十分条件は，両側 PF 環 (QF 環)R と R の巾等元 e で $A \cong eRe$ となるものが存在することである．

証明. ($\Rightarrow$) $A_1 = A$ とおく．仮定より Morita duality を定める両側加群 ${}_{A_1}U_{1A_2}, {}_{A_2}U_{2A_3}, \ldots, {}_{A_n}U_{nA_1}$ が存在する．例 5 と同様に，$B = A_1 \times A_2 \times \cdots \times A_n$，$V = U_1 \bigoplus U_2 \bigoplus \cdots \bigoplus U_n$ とおく．このとき B の V による trivial extension $R = B \ltimes V$ と，B の巾等元 $(1, 0, \ldots, 0)$

に対応する R の巾等元 e を考えれば，Faith の定理 (例えば [11, Theorem 10.7] 参照) より R は両側 PF 環で，$A \cong eRe$ である．

($\Leftarrow$) 命題 3 より明らか． □

注意 6. (アルチン) 環 A が両側加群 U によって定められる self-duality をもつとき (特に A が体上有限次元多元環のとき)，A の U による trivial extension $R = A \ltimes U$ は両側 PF 環 (QF 環) となり，A は R の剰余環 $R/(0 \ltimes U)$ と同型であることはよく知られている．上の定理はこの事実の類似と見ることができる．特に

$$R = \begin{pmatrix} A & U \\ U & A \end{pmatrix}, \quad e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

とおけば，($U \times U = 0$ によって積を定めて)R は両側 PF 環 (QF 環) で，$A \cong eRe$ である．

${}_AU_B$ を Morita duality を定める両側加群とする．このとき，右 annihilator を 2 回取る対応は，A と B の両側イデアル全体の間の束同型を与える:

$$\mathcal{S}({}_AA_A) \cong \mathcal{S}({}_BB_B)\ ;\quad K \mapsto r_Br_U(K).$$

この対応について，次の性質が成り立つことが分かる．

補題 7. (1) $r_Br_U(AeA) = BfB$. (ただし，e, f は補題 2 と同じ．)

(2) $r_Br_U(S(A_A)) = S(B_B)$.

(3) A の任意の両側イデアル K, L に対して，$r_Br_U(KL) = (r_Br_U(K)) \cdot (r_Br_U(L))$.

この補題より次を得る．

補題 8. 環 A は almost self-duality をもつとし，e を A の巾等元とする．K が AeA, $S(A_A)$, $AeAS(A_A)$ のいずれかの形の両側イデアルのとき，剰余環 A/K も almost self-duality をもつ．

証明. 一般に ${}_AU_B$ が Morita duality を定めるとき，A の任意の真のイデアル K に対して，$(A/K, B/r_Br_U(K))$ 両側加群 $r_U(K)$ は Morita duality を定める ([11, Corollary 2.5])．したがって，特に $K = S(A_A)$ のとき，補題 7(2) より，$A/S(A_A)$ は $B/S(B_B)$ に左 Morita dual であるから，A が almost self-duality をもてば $A/S(A_A)$ も almost self-duality をもつ．

$K = AeA$ のときは，補題 7(1) と命題 3 の証明と同様な議論を用いて，A が almost self-duality をもてば A/AeA も almost self-duality をもつことが分かる．

$K = AeAS(A_A)$ のときは，上の 2 つの場合と補題 7(3) より分かる． □

注意 9. 上の証明で述べたように，環 A の Morita duality の存在は剰余環に遺伝する．しかし，self-duality については，たとえ剰余環が環直和因子によるものであっても，これは成り立たない．実際，例 5 の A において $K = (1-e)A$ とすれば，$1-e$ は中心的巾等元であるが，$A/K \cong A_1$ は self-duality をもたない．上の補題は，特別な形の剰余環については almost self-duality の存在が遺伝することを示している．

この節の最後に，環 A と巾等元 $e \in A$ によって定まる，ある種の環拡大を定義し，duality に関する性質を述べておく．この拡大は左 H 環の構造の理論において重要な役割を果たす，

A を基本的半完全環，$e \in A$ を巾等元とする．このとき，

$$A_e = \begin{pmatrix} A & Ae \\ eJ(A) & eAe \end{pmatrix}$$

と定義すると，行列の積によって A_e も基本的半完全環となる．これは [2, p.404] の $W_i(R)$ に相当するものである．

$$A \cong \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} A_e \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

であることを注意しておく．

${}_AU_B$ は Morita duality を定めるとし，A, B は基本的であると仮定する．$S({}_AU) = S(U_B)$ が成り立つから，これらを単に $S(U)$ で表す．$\{e_1, \dots, e_n\}$，$\{f_1, \dots, f_n\}$ をそれぞれ A，B の直交原始巾等元の完全集合で，$S(Uf_i) \cong T(Ae_i)$，$S(e_iU) \cong T(f_iB)$ $(i = 1, \dots, n)$ を満たすものとする．I を $\{1, \dots, n\}$ の空でない部分集合とし，$e = \sum_{i \in I} e_i$，$f = \sum_{i \in I} f_i$ とおく．このとき

$$U_{e,f} = \begin{pmatrix} U & Uf/S(U)f \\ eU & eUf \end{pmatrix}$$

と定義する．

命題 10. 上の設定で，$U_{e,f}$ は Morita duality を定める (A_e, B_f) 両側加群である．

これを使えば次を示すことができる．

命題 11. A を基本的半完全環，$e \in A$ を巾等元とするとき，A_e が Morita duality (almost self-duality) をもつための必要十分条件は，A が Morita duality (almost self-duality) をもつことである．

注意 12. 例 1, 5 の設定で，B を trivial extension $A \ltimes U$ によって定め，f を A の巾等元 $(0, 0, \dots, 1)$ に対応する B の巾等元とする．このとき例 1 の R_1 は B_f と同型である．B は QF 環であるから self-duality をもつが，例 1 で注意したように，R_1 は self-duality をもたない．したがって，命題 11 における $(\Leftarrow)$ は self-duality については成り立たない．

2 H 環

それでは，H 環の議論に入ろう．左アルチン環 A は，次の条件を満たす直交原始巾等元の基本集合 $\{e_{ij} \mid 1 \le i \le m,\ 1 \le j \le n(i)\}$ をもつとき，左 H 環 (**左 Harada 環**) であると言われる．

(1) 任意の $i = 1, \ldots, m$ に対して $e_i A_A$ は入射的である.
(2) 任意の $i = 1, \ldots, m$, $j = 2, \ldots, n(i)$ に対して，右 A 加群として $e_{ij}A \cong J(e_{i,j-1}A)$ である.

左 H 環は QF 環や単列環の一般化であり，QF 環や単列環は self-duality をもつ．しかしながら冒頭で触れたように，左 H 環は必ずしも self-duality をもつとは限らない.

先ほど定義した環 A_e との関係を述べた次の 3 つの命題は，左 H 環の構造を記述するために基本的かつ重要である．本質的には，大城氏の一連の論文 [6, 7, 8] で述べられている.

命題 13. A を基本的左 H 環，$e \in A$ を巾等元とするとき，A_e も基本的左 H 環である.

命題 14 ([7, Theorem 2] 参照). A を基本的左 H 環，$f \in A$ を fA_A が入射的でないような原始巾等元とするとき，$(1-f)A(1-f)$ も基本的左 H 環である.

命題 15 ([7, Theorems 1,2] 参照). A を基本的左 H 環，$e, f \in A$ を $fA \cong J(eA)$ であるような原始巾等元とし，$A' = (1-f)A(1-f)$，$\tilde{A} = A'_e$ とおく．このとき，次の条件を満たす全射環準同型写像 $\phi : \tilde{A} \to A$ が存在する.

(1) $\mathrm{Ker}(\phi) \leq S(\tilde{A}_{\tilde{A}})$.
(2) ϕ が同型写像 $\Leftrightarrow$ ${}_A Ae$ は入射的でない.

標語的な言い方をすれば，命題 13 は左 H 環に原始巾等元を「付け加えて」も再び左 H 環になり，命題 14 は左 H 環から (入射的な直既約射影的右加群に対応しない) 原始巾等元を「取り除いて」も再び左 H 環になることを示している．そして命題 15 は左 H 環から (入射的な直既約射影的右加群に対応しない) 原始巾等元を「取り除いた」後，適当な原始巾等元を「付け加え」ると，元の左 H 環が (ほぼ) 復元できることを示している．これらの命題を繰り返し用いれば,「任意の左 H 環は QF 環の適当な拡大とその剰余環によって構成できる」([8, p.118] 参照) のである.

したがって，これらの操作とうまく調和する環の性質は，元の左 H 環から (入射的な直既約射影的右加群に対応しない) 原始巾等元を次々に「取り除いて」得られる QF 環に帰着して証明することができる．定理 B の証明が示すように，almost self-duality の存在はこのような性質の一つなのである.

主定理 (定理 B) を示すために，もう一つだけ補題を用意しておく.

補題 16. A を基本的左 H 環，K を $S(A_A)$ に含まれるイデアルとするとき，もし A が almost self-duality をもてば，剰余環 A/K も almost self-duality をもつ.

証明. K を A のイデアルで $S(A_A)$ に含まれるものとすると，左 H 環の性質から，ある巾等元 $e \in A$ が存在して $K = AeAS(A_A)$ と書けることが分かる．したがって，補題 8 より A/K も almost self-duality をもつ. □

定理B．すべての左H環はalmost self-dualityをもつ．

証明．Aを左H環とする．almost self-dualityの存在は森田同値によって保たれるから，Aは基本的であるとして良い．Aの完全集合に含まれる直交原始巾等元の個数nによる数学的帰納法で証明する．

$n=1$のとき，左H環Aは局所的QF環であるから，self-dualityしたがってalmost self-dualityをもつ．

$n>1$とし，完全集合に含まれる直交原始巾等元の個数がnより小さい基本的左H環はalmost self-dualityをもつと仮定する．すべての原始巾等元$f \in A$に対してfA_Aが入射的の場合，AはQF環であるから，self-dualityしたがってalmost self-dualityをもつ．fA_Aが入射的でないような原始巾等元$f \in A$が存在する場合，Aが左H環であることより，原始巾等元$e \in A$で$fA \cong J(eA)$を満たすようなものが存在する．$A' = (1-f)A(1-f)$，$\tilde{A} = A'_e$とおく．命題14よりA'も左H環であるから，帰納法の仮定よりA'はalmost self-dualityをもつ．ゆえに命題11と命題13より，$\tilde{A}$もalmost self-dualityをもつ左H環である．したがって命題15と補題16より$A \cong \tilde{A}/\mathrm{Ker}(\phi)$もalmost self-dualityをもつ． □

この定理と定理Aより，

系 17. すべての左H環Aに対して，$A \cong eRe$となるようなQF環Rと巾等元$e \in R$が存在する．

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DERIVED EQUIVALENCES FOR BLOCKS OF FINITE GROUPS

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1. Introduction

Let G be a finite group. Let k be an algebraically closed field of characteristic $p > 0$. In modular representation theory of finite groups, it is important to investigate relations between representations of G and representations of $N_G(P)$ where P is a p-subgroup of G. There is a well-known conjecture due to Broué.

Conjecture(Broué [1, 4.9 Conjecture]) Let G be a finite group with abelian Sylow p-subgroup P. Then the principal block of kG and the principal block of $kN_G(P)$ are derived equivalent.

The purpose of this paper is to give an example checked this conjecture. In §2 and §3 we list known results for derived equivalences for symmetric algebras and for blocks of finite groups. In §4 we state the main result and explain it.

2. Derived equivalences for symmetric algebras

Let k be an algebraically closed field of characteristic $p > 0$. We assume all algebras considered are finite-dimensional symmetric algebras, all modules are finite-dimensional right modules and all complexes are bounded.

For algebras A and B, an (A, B)-bimodule M is said to be *exact* if M is projective as a left A-module and a right B-module.

Theorem 2.1 (Rickard [8, Theorem 3.3]) *Let A and B be algebras. Then the following conditions are equivalent.*

(i) *A and B are derived equivalent.*

(ii) *There exists a complex $X^\bullet$ of exact (A, B)-bimodules such that*

(a) *$X^\bullet \otimes_B X^{\bullet\vee} \cong A$ in the homotopy category of (A, A)-bimodules*

(b) *$X^{\bullet\vee} \otimes_A X^\bullet \cong B$ in the homotopy category of (B, B)-bimodules.*

A complex $X^\bullet$ in the above theorem is called *a split-endomorphism two-sided tilting complex.*

Theorem 2.2 (Rickard [8]) *Let A and B be algebras. If A and B are derived equivalent then there exists an exact (A, B)-bimodule L inducing a stably equivalence of Morita type between A and B.*

The detailed version of this paper will appear in J. Algebra.

Theorem 2.3 (Linckelmann [4, Theorem 2.1]) *Let A and B be indecomposable non-simple algebras. Let M be an (A, B)-bimodule inducing a stable equivalence of Morita type between A and B.*

Then A and B are Morita equivalent if and only if for any simple A-module S the B-module $S \otimes_A M$ is again simple.

The following method to show an existence of a derived equivalence between two algebras A and B is given by T. Okuyama in [5]

(i) Give an (A, B)-bimodule N inducing a stable equivalence of Morita type between A and B.

(ii) Determine the structure of $U \otimes_A N$ for each simple A-module U.

(iii) Construct an algebra C such that C is derived equivalent to B and $U \otimes_A N \otimes_B L$ is simple for each simple A-module U, where L is a bimodule inducing a stable equivalence between B and C (see Theorem 2.2).

Then we can conclude A and B are derived equivalent since A and C are Morita equivalent by Theorem 2.3.

3. Splendid equivalences and stable equivalences

Let G be a finite group and P a Sylow p-subgroup of G. We assume P is abelian and set $H = N_G(P)$. We denote by $B_0(G)$ the principal block of kG. A kG-module V is called a *p-permutation module* if it is a direct summand of a direct sum of permutation modules.

Definition 1 A complex $X^\bullet$ of $(B_0(G), B_0(H))$-bimodules is called *a splendid tilting complex* if

(i) $X^\bullet$ is a split endomorphism two-sided tilting complex

(ii) each term of $X^\bullet$ is a $\Delta(P)$-projective p-permutation $k[G \times H]$-module.

If there exists a splendid tilting complex of $(B_0(G), B_0(H))$-modules we say $B_0(G)$ and $B_0(H)$ are *splendidly equivalent*. We also call a bounded complex satisfying the condition (ii) of Definition 1 *splendid complex*.

For kG-module V, we set

$$V(Q) = V^Q / \sum_{R<Q} \mathrm{Tr}_R^Q(V^R),$$

where V^Q is the set of Q-fixed points in V and $\mathrm{Tr}_R^Q : V^R \to V^Q$ is the trace map.

The following theorem shows an importance of the notion of splendid equivalences.

Theorem 3.1 (Rickard [9, Theorem 4.1]) *Let $X^\bullet$ be a splendid tilting complex of $(B_0(G), B_0(H))$-bimodules. Then for every subgroup Q of P, the complex $X^\bullet(\Delta(Q))$ is a splendid tilting complex of $(B_0(C_G(Q)), B_0(C_H(Q)))$-bimodules.*

The converse does not hold. However Gollan-Okuyama obtain a stable equivalence between $B_0(G)$ and $B_0(H)$ from a splendid equivalence between $B_0(C_G(Q))$ and $B_0(C_H(Q))$ in certain situation . More generally the following is shown.

Theorem 3.2 (see [11, Theorem 5.6]) *Let $X^\bullet$ be a splendid complex of $(B_0(G), B_0(H))$-bimodules. If the complex $X^\bullet(\Delta(Q))$ is a splendid tilting complex for each $1 \neq Q \leq P$, then $B_0(G)$ and $B_0(H)$ are stable equivalent.*

4. Main Theorem

In this section we state our main theorem. This is a joint work with K. Waki.

Assumption

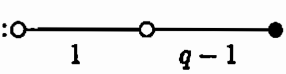
- k : an algebraically closed field of characteristic $r > 3$.
- $G = SU(3, q^2)$, $q + 1 = r^a s (a > 0)$ $r \nmid s$.
- $P \in \mathrm{Syl}_r(G)$, $P \cong C_{r^a} \times C_{r^a}$.
- $H = N_G(P)$, $N_G(P)/C_G(P) \cong S_3$.
- B : the Borel subgroup of G.
- $Q \in \mathrm{Syl}_r(B)$, $Q \cong C_{r^a}$.

Main Theorem [3, Theorem 1.1] *The principal blocks $B_0(G)$ and $B_0(H)$ are derived equivalent.*

Lemma 4.1 (i) *For $1 \neq R \leq P$ and $R \not\leq_H Q$, we have $C_G(R) = C_G(P)$.*

(ii) *For $1 \neq Q' \leq Q$, we have $N_G(Q') = C_G(Q') = C_G(Q) = N_G(Q)$. Moreover we have $C_G(Q) \cong Q \times G_1 (\cong U(2, q^2))$ for some subgroup G_1.*

Let $H_1 = G_1 \cap H$ and let Q_1 be a Sylow p-subgroup of of G_1 and H_1. Then $Q_1 \cong C_{p^a}$, so $B_0(G_1)$ and $B_0(H_1)$ are Brauer tree algebras and their Brauer tree are teh following.

$B_0(G_1)$: 

$B_0(H_1)$: 

where $\{1, q-1\}$ is the set of the simple $B_0(G_1)$-modules and $\{1, 1'\}$ is the set of the simple $B_0(H_1)$-modules. Therefore we have the following from Rouquier's result.

Lemma 4.2 (Rouquier [10]) *Let Z^0 be the Green correspondent of $B_0(G_1)$ with respect to $(G_1 \times G_1, \Delta(Q_1), G_1 \times H_1)$ There exists a splendid tilting complex of $k[G_1 \times H_1]$-modules of the form*

$$Z^\bullet : 0 \longrightarrow P_{q-1}^* \otimes P_{1'} \longrightarrow Z^0 \longrightarrow 0.$$

Using Gollan-Okuyama's result[2, §1] we obtain the following(see also Theorem 3.2).

Proposition 4.3 *There exists an exact $(B_0(G), B_0(H))$-bimodule N which induces a stably equivalence of Morita type between $B_0(G)$ and $B_0(H)$.*

The construction of the bimodule N given in [2] is as follows.

Let X be the summand of $(P_{q-1} \otimes P_{1'})_{\Delta(Q)(G_1 \times H_1)}\uparrow^{G\times H}$ with vertex $\Delta(Q)$ and let M be the summand of $B_0(G)\downarrow_{G\times H}$ with vertex $\Delta(P)$. Then there exists a complex

$$X^\bullet : 0 \longrightarrow X \xrightarrow{\gamma} M \longrightarrow 0$$

such that $X^\bullet(\Delta Q) = Z^\bullet{}_{\Delta(Q)(G_1\times H_1)}\uparrow^{C_G(Q)\times C_H(Q)}$. The bimodule N is defined by the following exact sequence

$$0 \longrightarrow X \xrightarrow{(\gamma,\tau)} M \oplus \mathrm{proj} \longrightarrow N \longrightarrow 0.$$

In fact by Lemma 4.1, for $1 \neq R \leq P$, if $R \leq_H Q$ then $X^\bullet(\Delta R) = X^\bullet(\Delta Q)$ and if $R \not\leq_H Q$ then $X^\bullet(\Delta R) = M(\Delta R)$ since $X(\Delta R) = 0$. Since $Z^\bullet{}_{\Delta(Q)(G_1\times H_1)}\uparrow^{C_G(Q)\times C_H(Q)}$ is a splendid tilting complex by a result in [2], $B_0(G)$ and $B_0(H)$ are stably equivalent(see Theorem 3.2).

As stated in §2, we must determine $U \otimes_{B_0(G)} N$ for each simple $B_0(G)$-module U. We use the exact sequence

$$0 \longrightarrow U \otimes_{B_0(G)} X \xrightarrow{(\gamma_U,\tau_U)} U \otimes_{B_0(G)} M \oplus \mathrm{proj} \longrightarrow U \otimes_{B_0(G)} N \longrightarrow 0.$$

Therefore we first determine $U \otimes_{B_0(G)} X$ and $U \otimes_{B_0(G)} M$ and next determine the image of the map (γ_U, τ_U) and finally determine $U \otimes_{B_0(G)} N = \mathrm{coker}(\gamma_U, \tau_U)$.

Let $S_0 = k_G$, S_1 and S_2 be the simple $B_0(G)$-modules and let $T_0 = k_H$, T_1 and T_2 be the simple $B_0(H)$-modules, where $\dim S_1 = q^2 - q$, $\dim S_2 = (q-1)(q^2-q+1)$, $\dim T_1 = 1$ and $\dim T_2 = 2$. The following is a key lemma.

Lemma 4.4 (i) $S_0 \otimes_{B_0(G)} N = T_0$.

(ii) *There is an exact sequence*

$$0 \longrightarrow \Omega^{-1}\begin{pmatrix} T_0 \\ T_2 \end{pmatrix} \longrightarrow \Omega(S_1 \otimes_{B_0(G)} N) \longrightarrow T_0 \longrightarrow 0$$

Remark 1 (1) The structure of $S_1 \otimes_{B_0(G)} M$ is discussed in [6].

(2) We have no infomation about $S_2 \otimes_{B_0(G)} M$. However we may consider the complex

$$X_{S_2}{}^\bullet : 0 \longrightarrow \begin{pmatrix} S_1 \\ S_0 \\ S_1 \end{pmatrix} \longrightarrow \Omega^{-2}(S_0) \longrightarrow S_1 \longrightarrow 0$$

instead of the simple module S_2 (Note that $X_{S_2}{}^\bullet \cong S_2$ in the derived category)

Now we can construct the following correspondences.

$$\begin{array}{ccccccccc}
\mathrm{mod}\ A & \xrightarrow{\otimes_A N} & \mathrm{mod}\ B_0 & \xrightarrow{\otimes_{B_0} N_1} & \mathrm{mod}\ B_1 & \xrightarrow{\otimes_{B_1} N_2} & \mathrm{mod}\ B_2 & \xrightarrow{\otimes_{B_2} N_3} & \mathrm{mod}\ B_3 \\
S_0 & \longmapsto & T_0 & \longmapsto & T_0^{(1)} & \longmapsto & T_0^{(2)} & \longmapsto & T_0^{(3)} \\
S_1 & \longmapsto & \cdot & \longmapsto & \Omega^{-1}\begin{pmatrix} T_0^{(1)} \\ T_2^{(1)} \end{pmatrix} & \longmapsto & \Omega^{-1}\begin{pmatrix} T_0^{(2)} \\ T_2^{(2)} \end{pmatrix} & \longmapsto & T_2^{(3)} \\
S_2 & \longmapsto & \cdot & \longmapsto & \cdot & \longmapsto & \cdot & \longmapsto & T_1^{(3)}
\end{array}$$

where $A = B_0(G)$, $B_0 = B_0(H)$, B_1, B_2, B_3 are all derived equivalent, N_i is a (B_{i-1}, B_i)-bimodule inducing a stable equivalence between B_{i-1} and B_i for $i = 1, 2, 3$ and $\{T_i^{(j)} | i = 0, 1, 2\}$ is the set of the simple B_j-modules. Therefore we can conclude that $B_0(G)$ and $B_0(H)$ are derived equivalent. Moreover we can know that $B_0(G)$ and $B_0(H)$ are splendid equivalent.

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HIGH ORDER KÄHLER MODULES OF NONCOMMUTATIVE RING EXTENSIONS

HIROAKI KOMATSU

ABSTRACT. We construct the high order Kähler modules of noncommutative ring extensions B/A and show their fundamental properties. Our Kähler modules represent not only high order left derivations for one-sided modules but also high order central derivations for bimodules, which are usual derivations. This new viewpoint enables us to prove new results which were not known even though B is an algebra over a commutative ring A. These are the decomposition of Kähler modules by an idempotent element (§5), exact sequences of Kähler modules and the Kähler modules of factor rings (§6), and the relation to separable extensions (§8). In particular, our exact sequences of high order Kähler modules were not known even though B is commutative.

非可換環の拡大に対して, 高次 Kähler 加群を構成し, その基本的な性質を紹介する.[1] 我々の Kähler 加群は, 片側加群の高次 left derivation を表現するばかりでなく, 両側加群の通常の derivation のうちの特別なものである高次 central derivation をも表現する. この新しい視点によって, 多元環の場合においても知られていなかった事実を発見することが可能となった. それらは, Kähler 加群の冪等元による分解 (§5), Kähler 加群の完全系列と剰余環の Kähler 加群 (§6), 分離拡大との関連 (§8) 等に現れる. 特に, 我々の得た高次 Kähler 加群の完全系列は, 可換環の場合にも知られていなかったものである.

本稿で扱う環はすべて単位元 1 を持ち, 環準同型写像は 1 を 1 に移すものとする. 加群はすべて 1 が恒等的に作用しているものとする.

記号 0.1. 本稿を通じて次の記号を用いる.

B/A 環拡大 (即ち, 環準同型写像 $A \to B$ が与えられたということ)

$J = \operatorname{Ker}\big(B \otimes_A B \ni x \otimes y \longmapsto xy \in B\big)$

$\delta : B \longrightarrow J, \quad \delta(x) = 1 \otimes x - x \otimes 1$

1. 高次 KÄHLER 加群いろいろ

Kähler 加群は, いろいろな体系で考察されている. 可換環の derivation を表現する 1 次 Kähler 加群は, 古くから研究されてきたものである. 1967 年, Osborn [17] は, 高次微分作用素を代数的に研究するために, 可換環の拡大 B/A に対し, 高次 derivation を表現する加群である高次 Kähler 加群を導入した.

$\Omega^n_{B/A} = J/J^{n+1}$: n 次 Kähler 加群

$d^n_{B/A} : B \longrightarrow \Omega^n_{B/A}, \quad d^n_{B/A}(x) = \overline{\delta(x)}$: n 次 Kähler derivation

[1]The detailed version of this paper has been submitted for publication elsewhere.

実際の Osborn の定義はこれよりも一般的であるが, 多くの研究が上の定義に基づいている. Heyneman と Sweedler も [4] において上記の定義を導入している. 日本では中井らの一連の研究があり, [14] にまとめられている.

1970 年, 服部 [3] は Osborn の定義を, B が可換環 A 上の非可換多元環の場合にまで拡張した. 服部は, 可換環の理論では見落とされていた Kähler 加群の両側 B 加群構造に着目し, 高次 derivation が両側加群の通常の derivation と密接に関係することを示した. Heyneman と Sweedler の定義も, [21] において Sweedler 自身によって, 非可換多元環にまで拡張された. そして, 多元環の分離性, 純非分離性と Kähler 加群との関連につて研究が行われた. しかし, Sweedler は Kähler 加群の両側加群構造の重要性を見落としており, 十分な成果は得られていなかった.

最近, 筆者 [10] は, 高次 Kähler 加群の定義を, B/A が非可換環の拡大の場合にまで拡張したが, 服部と同じく両側 B 加群構造に着目することにより, 多元環の場合にも知られていなかった事実を証明することができた. 本稿ではそれらについて紹介する.

本稿で扱う非可換化とは異なる方向として, Kähler 加群の量子化も行われている. Bell [1] が導入した歪微分作用素 (環同型写像による捻りが入ったもの) に対応する Kähler 加群が平野・那須・津田 [5] において研究されている. また, Verbovetsky [22] では, ホップ代数が作用する多元環の量子化された微分作用素が考察され, Kähler 加群に近いジェット加群が研究されている.

最後に, 中島 [16] は, derivation の一般化である generalized derivation を表現する加群を構成している.

2. 高次微分作用素とジェット加群

記号 2.1. ${}_BM_B$ とする.

1. $m \in M$, $x \in B$ に対し, $[m, x] = mx - xm$ とおく.
2. $S \subseteq M$ に対し, $\{ [s, x] \mid s \in S,\ x \in B \}$ で生成される M の $\mathbb{Z}$ 部分加群を $[S, B]$ であらわす.
3. $[S, B]_0$ は S で生成された M の部分 $\mathbb{Z}$ 加群とし, $[S, B]_n = \big[[S, B]_{n-1}, B \big]$ $(n = 1, 2, \ldots)$ と帰納的に定める.

些細なことだが, $B[S, B] = [S, B]B$ が成り立つので, $B[S, B]$ は M の両側部分加群を成している. 因みに, $m \in M$, $x_1, \ldots, x_n \in B$ に対し,

$$\big[\cdots \big[[m, x_1], x_2 \big], \cdots, x_n \big] = \sum_{i_1 < \cdots < i_r} (-1)^r x_{i_r} \cdots x_{i_1} m x_1 \cdots \check{x}_{i_1} \cdots \check{x}_{i_r} \cdots x_n$$

と展開される. ただし, $\check{x}_{i_j}$ は x_{i_j} を取り除くことを意味する.

定義 2.2. ${}_BM$, ${}_BN$, 自然数 n に対し,

$$\mathcal{D}^n_{B/A}(M, N) = \big\{ \varphi \in \mathrm{Hom}_{\mathbb{Z}}(M, N) \mid [\varphi, B]_{n+1} = [\varphi, A] = 0 \big\}$$

とおく．この集合の要素を n 次微分作用素とよぶ．ただし，$[\ ,\ ]$ は $\mathrm{Hom}_{\mathbb{Z}}(M,\ N)$ の自然な両側 B 加群構造による．

本稿では，個々の微分作用素よりも全体の集合を問題とする．明らかに，

$$\mathrm{Hom}_B(M,\ N) = \mathcal{D}^0_{B/A}(M,\ N) \subseteq \mathcal{D}^1_{B/A}(M,\ N) \subseteq \cdots \subseteq \mathrm{Hom}_A(M,\ N)$$

となっている．

定理 2.3. 任意の ${}_BM,\ {}_BN$ に対し，次が成り立つ．

$$\begin{array}{rcl} \mathrm{Hom}_B\left(\dfrac{B\otimes_A B}{B\,[1\otimes 1,\ B]_{n+1}}\otimes_B M,\ N\right) & \simeq & \mathcal{D}^n_{B/A}(M,\ N) \\ f & \longmapsto & f(\overline{1\otimes 1}\otimes -) \end{array}$$

ここに現れる $\dfrac{B\otimes_A B}{B\,[1\otimes 1,\ B]_{n+1}}$ を n 次ジェット加群とよぶ．

また，写像の合成積 $\circ$ について，$\mathcal{D}^n_{B/A}(M,\ N)\circ\mathcal{D}^m_{B/A}(L,\ M)\subseteq\mathcal{D}^{m+n}_{B/A}(L,\ N)$ が成立するので，

$$\mathcal{D}_{B/A}(M) = \bigcup_{n=0}^{\infty}\mathcal{D}^n_{B/A}(M,\ M)$$

は環を成すことがわかる．これは微分作用素の環とよばれている．例えば，B が標数 0 の体 A 上の多項式環の場合，$\mathcal{D}_{B/A}(B)$ は Weyl 代数である．

3. 高次 Derivation と Kähler 加群

定義 3.1. ${}_BM$ に対し，$\mathrm{LDer}^n_A(B,\ M) = \left\{\varphi\in\mathcal{D}^n_{B/A}(B,\ M)\ \middle|\ \varphi(1)=0\right\}$ とおく．この要素を **n 次 left A-derivation** とよぶ．

特に，$\varphi\in\mathrm{Hom}_A(B,\ M)$ が 1 次 left A-derivation であるためには，$\varphi(xy)=x\varphi(y)+y\varphi(x)$ $(\forall x,\ y\in B)$ を満たすことが必要十分である．可換環の場合，これを単に A-derivation とよんでいる．

定義 3.2. $n\geq 0$ を整数とする．

$$\Omega^n_{B/A} = \frac{J}{B\,[1\otimes 1,\ B]_{n+1}} \quad : \quad n \text{ 次 Kähler 加群}$$

$$d^n_{B/A} : B \longrightarrow \Omega^n_{B/A},\quad x\longmapsto\overline{\delta(x)} \quad : \quad n \text{ 次 Kähler derivation}$$

$J=B\,[1\otimes 1,\ B]$ であるから，もしも B が可換環ならば，$J^{n+1}=B\,[1\otimes 1,\ B]_{n+1}$ となり，上の定義は可換環の場合の拡張になっていることがわかる．可換環の理論では，$\Omega^n_{B/A}$ を片側加群として取り扱っているのだが，これから本稿で見てゆくように，$\Omega^n_{B/A}$ が自然に備えている両側 B 加群構造が非常に重要なのである．両側加群構造から導かれるものの一つに $\mathrm{LDer}^n_A(B,\ M)$ の左 B 加群構造がある．これは，$n=1$ の場合ではあるが，本元・小松 [7] において半素環の biderivation の特徴付けを行う際に，必要であった．

定義 3.3. ${}_BM$ とする.

1. $\mathrm{Hom}_A(B, M)$ に新しい B 加群構造を導入する.

$$(b * \varphi)(x) = \varphi(xb) - x\varphi(b) + xb\varphi(1) \qquad \big(\varphi \in \mathrm{Hom}_A(B, M);\ b,\ x \in B\big)$$

こうして得られる左 B 加群を $\widehat{\mathrm{Hom}}_A(B, M)$ であらわす.

2. $\mathrm{Hom}_A^0(B, M) = \big\{ \varphi \in \mathrm{Hom}_A(B, M) \,\big|\, \varphi(1) = 0 \big\}$ とおく.

補題 3.4. 任意の ${}_BM$ に対し, 次が成り立つ.

1. ${}_B\widehat{\mathrm{Hom}}_A(B, M) = \mathrm{Hom}_A^0(B, M) \oplus \mathrm{Hom}_B(B, M)$
2. $\mathcal{D}^n_{B/A}(B, M)$ は $\widehat{\mathrm{Hom}}_A(B, M)$ の部分加群であり, B 加群としての直和分解 $\mathcal{D}^n_{B/A}(B, M) = \mathrm{LDer}^n_A(B, M) \oplus \mathrm{Hom}_B(B, M)$ をもつ.

この構造のもとで, 次の基本的な結果を得る.

定理 3.5. 任意の ${}_BM$ に対し, 次が成り立つ.

$$\begin{array}{rcl} {}_B\mathrm{Hom}_B(\Omega^n_{B/A}, M) & \simeq & \mathrm{LDer}^n_A(B, M) \\ f & \longmapsto & f d^n_{B/A} \end{array}$$

これから, $\mathrm{LDer}^n_A(B, M)$ の B 加群構造は, $\Omega^n_{B/A}$ の右 B 加群構造から導かれたものであることがわかる. これに関連した構造として, 次のものがある.

定義 3.6. $B \otimes_A B$ に新しい両側 B 加群構造を導入する.

$$b * (x \otimes y) = bx \otimes y$$
$$(x \otimes y) * b = x \otimes yb - xy \otimes b + xyb \otimes 1$$

こうして得られる両側 B 加群を $\widehat{B \otimes_A B}$ であらわす.

このとき, ${}_B\widehat{B \otimes_A B}_B \simeq J \oplus B$ であり, 更に次を得る.

補題 3.7. 任意の ${}_BM$ に対して, 次が成立する.

1. ${}_B\mathrm{Hom}_B(\widehat{B \otimes_A B}, M) \simeq \widehat{\mathrm{Hom}}_A(B, M),\quad f \longmapsto f(- \otimes 1)$
2. ${}_B\mathrm{Hom}_B(J, M) \simeq \mathrm{Hom}_A^0(B, M),\quad f \longmapsto f\delta$

4. 両側加群の Derivation とのかかわり

本節では, Kähler 加群の両側加群構造が重要であることを示す, 最も基本的な結果を述べる.

定義 4.1. ${}_BM_B$ とする.

1. $\mathrm{Der}_A(B, M) = \big\{ d \in \mathrm{Hom}_{A\text{-}A}(B, M) \,\big|\, d(xy) = xd(y) + d(x)y \ \ (\forall x,\ y \in B) \big\}$ とおく. この集合の要素を **A-derivation** とよぶ.

2. $\mathrm{CDer}_A^n(B, M) = \{ d \in \mathrm{Der}_A(B, M) \mid [d(B), B]_n = 0 \}$ とおく. この集合の要素を n 次 **central A-derivation** とよぶ.

次の結果が, 本稿の中で最も基本的なものである.

定理 4.2. 任意の ${}_BM_B$ に対し, 次が成り立つ.

$$\begin{array}{rcl} \mathrm{Hom}_{B\text{-}B}(\Omega^n_{B/A}, M) & \simeq & \mathrm{CDer}_A^n(B, M) \\ f & \longmapsto & f d^n_{B/A} \end{array}$$

この定理を圏論的に言えば, 関手 $\mathrm{CDer}_A^n(B, -) : B \otimes_{\mathbb{Z}} B^{op}\text{-Mod} \to \mathbb{Z}\text{-Mod}$ が, 対 $(\Omega^n_{B/A},\ d^n_{B/A})$ で表現されるということであり, 証明はこの視点からなされる. 任意の $d \in \mathrm{Der}_A(B, M)$ と任意の $f \in \mathrm{Hom}_{B\text{-}B}(M, N)$ との合成写像 fd は A-derivation であるから, $\mathrm{Der}_A(B, -)$ は $\mathrm{Hom}_{A\text{-}A}(B, -) : B \otimes_{\mathbb{Z}} B^{op}\text{-Mod} \longrightarrow \mathbb{Z}\text{-Mod}$ の部分関手とみなせる. 同様に, $\mathrm{CDer}_A^n(B, -)$ は $\mathrm{Der}_A(B, -)$ の部分関手とみなせる.

一般に, (共変) 関手 $F : A\text{-Mod} \to A'\text{-Mod}$ が, ある A-A' 両側加群 U から得られる関手 $\mathrm{Hom}_A(U, -) : A\text{-Mod} \to A'\text{-Mod}$ と自然同値であるとき, F は表現可能であるという. 自然同値 $\eta : \mathrm{Hom}_A(U, -) \to F$ があるとき, $v = \eta_U(1_U) \in F(U)$ とおけば, $\eta_M(f) = F(f)(v)$ $(\forall\, {}_AM,\ \forall f \in \mathrm{Hom}_A(U, M))$ が成り立つので, F は (U, v) で表現されるという.

定理 4.2 は, 次の二つの補題を用いて証明される. 補題 4.3 は古くから知られている事実である.

補題 4.3. 関手 $\mathrm{Der}_A(B, -)$ は (J, δ) で表現される.

補題 4.4. $F : A\text{-Mod} \to A'\text{-Mod}$ は (U, v) で表現される関手とする. G は F の部分関手で, 左完全, かつ直積を保存するものとする. このとき, 集合 $\{ X \subseteq {}_AU \mid F(t_X)(v) \in G(U/X) \}$ は最小元 V を持つ. ここで, $t_X : U \to U/X$ は自然な写像をあらわす. そして, $V \subseteq {}_AU_{A'}$ であり, G は $(U/V,\ F(t_V)(v))$ で表現される.

定理 3.5 と定理 4.2 から, 次がわかる.

系 4.5. 任意の ${}_BM_B$ に対して, $\mathrm{CDer}_A^n(B, M) \subseteq \mathrm{LDer}_A^n(B, M)$.

これを見ると, 高次 central A-derivation は高次 left A-derivation のうちの特殊なものであるとも言えるのだが, 次の定理は逆の見方を示しており, 興味深い. この定理のように, 高次 left A-derivation に通常の A-derivation を対応させる方法は, 既に服部 [3] において考察されていたのであるが, 対応の値域を明示することができていなかった. それが central A-derivation を用いてあらわすことができるようになったのである.

定理 4.6. 任意の ${}_BM$ に対し, 次が成り立つ.

$$\begin{array}{rcl} \mathrm{LDer}_A^n(B, M) & \simeq & \mathrm{CDer}_A^n(B, \mathrm{Hom}_{\mathbb{Z}}(B, M)) \\ \varphi & \longmapsto & [\varphi, -] \end{array}$$

5. 冪等元による分解

定理 5.1. B の冪等元 e に対し,

$A_1 = eB(1-e)Be \cup eAe$ で生成された eBe の部分環

$A_2 = (1-e)BeB(1-e) \cup (1-e)A(1-e)$ で生成された $(1-e)B(1-e)$ の部分環

とおくと, $\Omega^n_{B/A} \simeq \Omega^n_{eBe/A_1} \times \Omega^n_{(1-e)B(1-e)/A_2}$ である.

要するに, $eB(1-e)$, $(1-e)Be$ という非可換環らしいところが消えてしまう. この定理は次からわかる.

補題 5.2. $d \in \mathrm{CDer}^n_A(B, M)$ と B の冪等元 e に対し, 次が成立する.

1. $d(e) = 0$
2. $em = me \quad \big(\forall m \in Bd(B)\big)$
3. $d\big(B[e, B]\big) = 0$

例 5.3. $B = \mathrm{M}_r(R)$ 行列環 $(r > 1) \implies \Omega^n_{B/A} = 0 \quad (\forall A, \forall n)$

6. Kähler 加群の完全系列

次の図式の左側の四角形は環準同型写像の可換図形であるとする. このとき, $d^n_{B'/A'}\sigma \in \mathrm{CDer}^n_A(B, \Omega^n_{B'/A'})$ だから, 定理 4.2 より, 右側の四角形が可換になるような両側 B 準同型写像 $\Omega^n_{\sigma/\rho} : \Omega^n_{B/A} \to \Omega^n_{B'/A'}$ が存在する.

$$\begin{array}{ccccc} A & \longrightarrow & B & \xrightarrow{d^n_{B/A}} & \Omega^n_{B/A} \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\Omega^n_{\sigma/\rho}} \\ A' & \longrightarrow & B' & \xrightarrow[d^n_{B'/A'}]{} & \Omega^n_{B'/A'} \end{array}$$

以下では, 環準同型写像 $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ と自然数 n を固定して考える.

任意の ${}_CM_C$ に対し, 次の完全系列を得る.

(1) $$0 \to \mathrm{CDer}^n_B(C, M) \xrightarrow{\subseteq} \mathrm{CDer}^n_A(C, M) \xrightarrow{\text{制限}} \mathrm{CDer}^n_A(B, M) \qquad \text{完全}$$

しかし, ${}_CM$ に対し, 次の系列は完全とは限らない.

$$0 \to \mathrm{LDer}^n_B(C, M) \xrightarrow{\subseteq} \mathrm{LDer}^n_A(C, M) \xrightarrow{\text{制限}} \mathrm{LDer}^n_A(B, M)$$

ただし, $n = 1$ の場合は, 上系列も完全であり, 次の左 C 加群の完全系列が導かれる.

$$\begin{array}{ccccccc} C \otimes_B \Omega^1_{B/A} & \longrightarrow & \Omega^1_{C/A} & \xrightarrow{\Omega^1_{1/\alpha}} & \Omega^1_{C/B} & \longrightarrow & 0 \qquad \text{完全} \\ c \otimes \omega & \longmapsto & c\,\Omega^n_{\beta/1}(\omega) & & & & \end{array}$$

これは, 可換環の場合に詳しく研究された. 同様にして, 完全系列 (1) から次の結果が得られる. これは $\Omega^n_{B/A}$ の両側加群構造なしには, 考えようもないことである.

定理 6.1. 次は両側 C 加群の完全系列である.

$$\begin{array}{ccccccc} C\otimes_B \Omega^n_{B/A}\otimes_B C & \xrightarrow{F} & \Omega^n_{C/A} & \xrightarrow{\Omega^n_{1/\alpha}} & \Omega^n_{C/B} & \longrightarrow & 0 \qquad \text{完全} \\ c\otimes\omega\otimes c' & \longmapsto & c\Omega^n_{\beta/1}(\omega)c' & & & & \end{array}$$

系 6.2. $\Omega^n_{C/B} \simeq \dfrac{\Omega^n_{C/A}}{Cd^n_{C/A}\beta(B)C}$

定理 6.3. B のイデアル I による剰余環を $\overline{B}$ とし, $\beta: B\to\overline{B}$ を自然な写像とする. このとき, 次は両側 $\overline{B}$ 加群の完全系列である.

$$\begin{array}{ccccccc} I/I^2 & \longrightarrow & \overline{B}\otimes_B \Omega^n_{B/A}\otimes_B \overline{B} & \xrightarrow{F} & \Omega^n_{\overline{B}/A} & \longrightarrow & 0 \qquad \text{完全} \\ x+I^2 & \longmapsto & 1\otimes d^n_{B/A}(x)\otimes 1 & & & & \end{array}$$

系 6.4. 定理6.3の記号のもとで,

$$\Omega^n_{\overline{B}/A} \simeq \Omega^n_{B/(\alpha(A)+I)} \simeq \frac{\Omega^n_{B/A}}{Bd^n_{B/A}(I)B}, \quad Bd^n_{B/A}(I)B = d^n_{B/A}(I) + I\Omega^n_{B/A} + \Omega^n_{B/A}I$$

7. 高次 Derivation の延長

本節でも, 環準同型写像 $A\xrightarrow{\alpha}B\xrightarrow{\beta}C$ と自然数 n を固定して考える. B の central または left A-derivation を C まで延長できるかという問題を考える. 次は容易である.

補題 7.1. 定理6.1の $F: C\otimes_B\Omega^n_{B/A}\otimes_B C\to\Omega^n_{C/A}$, $c\otimes\omega\otimes c'\mapsto c\Omega^n_{\beta/1}(\omega)c'$ について, 次が成り立つ.

1. B から両側 C 加群への n 次 central A-derivation はすべて C まで延長できることと, F が split する単射であることとは同値である.
2. B から両側 C 加群への n 次 central A-derivation の C への延長は高々一つしかないことと, F が全射であることとは同値である.

補題 7.2. $f: C\otimes_B\Omega^n_{B/A}\to\Omega^n_{C/A}$, $c\otimes\omega\mapsto c\Omega^n_{\beta/1}(\omega)$ について, 次が成り立つ.

1. B から左 C 加群への n 次 left A-derivation はすべて C まで延長できることと, f が split する単射であることとは同値である.
2. B から左 C 加群への n 次 left A-derivation の C への延長は高々一つしかないことと, f が全射であることとは同値である.

上の F と f のかかわりとして, 次の結果がある.

定理 7.3. 次は同値である.

1. F は全射.
2. f は全射.
3. $\Omega^n_{C/B}=0$.

可換環の場合は, 次の結果が知られている.

定理 7.4. (中井・石橋 [15]) C/B が可換環の分離拡大であり, ${}_BC$ が平坦であるならば, f は同型写像である.

これに相当する結果を目標にしているのだが, まだ成果は得られていない.

8. 分離拡大と純非分離拡大

定義 8.1.

1. (平田・菅野 [6], 宮下 [11]) B/A : **分離拡大** $\Longleftrightarrow$ J が ${}_BB \otimes_A B_B$ の直和因子
2. (Sweedler [20]) B/A : **純非分離拡大** $\Longleftrightarrow$ J が ${}_BB \otimes_A B_B$ の small 部分加群

B/A が体の有限次拡大の場合, 上の定義は通常の定義と一致する.
分離拡大については, 次の問題に取り組んでいる.

問題 8.2. B/A が分離拡大ならば, 全ての自然数 n に対して $\Omega^n_{B/A} = 0$ であるか.

小松 [8] において, $n = 1$ に限定すれば正しいことが示された. B が可換環 A 上の多元環のときは, Sweedler [21] において, 「$\Omega^1_{B/A} = 0 \Longrightarrow \Omega^n_{B/A} = 0\ (\forall n)$」が示されている. 従って, 多元環の場合は, 問題 8.2 は正しい. また, B が可換環で, A の上に有限生成ならば, 問題 8.2 の逆が成立することも知られている (中井 [14]).
純非分離拡大については, 次の問題に取り組んでいる.

問題 8.3. $B \otimes_A B$ において $[1 \otimes 1, B]_n = 0$ となる自然数 n があれば, B/A は純非分離拡大か.

これについては, 次の結果がある.

補題 8.4. B/A が純非分離拡大ならば, 次が成り立つ.

1. $[B, B^A]$ は B の素根基に含まれる.
2. 任意の ${}_BM_B$ に対し, $[BM^AB, B^A]$ は ${}_BM_B$ の Jacobson 根基に含まれる.

ここでは, 次の記号が用いられている.

記号 8.5. ${}_AM_A$ に対し, $M^A = \left\{ m \in M \mid [m, A] = 0 \right\}$ とおく.

補題 8.4 には B^A が現れているので, 問題 8.3 は一般には正しくないように思われる. 多元環の場合は, うまくいって次を得る.

定理 8.6. B が可換環 A 上の多元環の場合, 問題 8.3 は正しい.

体上の有限次元多元環の場合は, 問題 8.3 の逆が成立することが知られている (Sweedler [21]).
上記の問題に関連した結果をあげる.

命題 8.7. 自然数 n に対し, 次は同値である.

1. $\Omega^n_{B/A} = 0$
2. $\mathrm{CDer}^n_A(B, M) = 0 \qquad (\forall {}_BM_B)$
3. $\mathrm{LDer}^n_A(B, M) = 0 \qquad (\forall {}_BM)$
4. $\mathcal{D}^n_{B/A}(M, N) = \mathrm{Hom}_B(M, N) \qquad (\forall {}_BM, {}_BN)$

命題 8.8. 整数 $n \geq 0$ に対し, 次は同値である.

1. $B \otimes_A B$ において $[1 \otimes 1, B]_{n+1} = 0$
2. $[M^A, B]_{n+1} = 0 \qquad (\forall {}_BM_B)$
3. $\mathrm{CDer}^n_A(B, M) = \mathrm{Der}_A(B, M) \qquad (\forall {}_BM_B)$
4. $\mathrm{LDer}^n_A(B, M) = \mathrm{Hom}^0_A(B, M) \qquad (\forall {}_BM)$
5. $\mathcal{D}^n_{B/A}(M, N) = \mathrm{Hom}_A(M, N) \qquad (\forall {}_BM, {}_BN)$

系 8.9. 環準同型写像 $\alpha : A \to B$ に対し, 次は同値である.

1. α は環の圏における epimorphism
2. $J = 0$
3. $M^A = M^B \qquad (\forall {}_BM_B)$
4. $\mathrm{Hom}_A(M, N) = \mathrm{Hom}_B(M, N) \qquad (\forall {}_BM, {}_BN)$
5. $\mathrm{Der}_A(B, M) = 0 \qquad (\forall {}_BM_B)$

この系は既知であるが, 2 ～ 5 の同値性は, 命題8.7 と命題8.8 を合わせることによっても導かれる.

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Crossed Product Orders over Valuation Rings*

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Abstract

Let V be a commutative valuation domain of arbitrary Krull-dimension (rank), with quotient field F, and let K be a finite Galois extension of F with group G, and S the integral closure of V in K. If in the crossed product algebra $K * G$ the 2-cocycle takes values in the group of units of S, then one can form in a natural way a "crossed product order" $S * G \subseteq K * G$. In light of recent results by H. Marubayashi and Z. Yi on homological dimension of crossed products, we would like to discuss necessary and/or sufficient valuation-theoretic conditions, on the extension K/F, for the V-order $S * G$ to be semihereditary, or maximal, or Azumaya over V.

In this paper all rings are associative with a unit element. If A is a ring, $J(A)$ will denote its Jacobson radical and the residue ring $A/J(A)$ will be denoted by $\overline{A}$. A ring A is called *left hereditary* (resp. *left semihereditary*) if every left ideal (resp. every finitely generated left ideal) of A is projective as a left A-module. An analogous definition holds for *right hereditary* (resp. *right semihereditary*) rings. A ring is called *hereditary* (resp. *semihereditary*) if it is both left and right hereditary (resp. semihereditary). Let V be a commutative domain with quotient field F and let Q be a finite-dimensional F-algebra. A subring R of Q is said to be an *order* in Q if $RF = Q$. If $V \subseteq Z(R)$ then R is said to be a *V-order* if in addition R is integral over V. If R is maximal with respect to inclusion among V-orders of Q then R is called a *maximal* V-order (or just maximal order if the context is clear).

In this paper, V will denote a valuation ring of *arbitrary* Krull dimension, unless stated otherwise, with quotient field F, and K/F will be a finite Galois extension with group G. Let S be the integral closure of V in K, and $U(S)$ its group of units. Now consider a normalised two-cocycle $f : G \times G \mapsto U(S)$, that is, a function satisfying $\sigma(f(\tau,\gamma))f(\sigma,\tau\gamma) = f(\sigma,\tau)f(\sigma\tau,\gamma)$ for all $\sigma,\tau,\gamma \in G$ and $f(1,\sigma) = f(\sigma,1) = 1$ for all $\sigma \in G$. From such a cocycle we can form a crossed product order, given by $S * G = \sum_{\sigma\in G} Sx_\sigma$ with the usual rules of multiplication ($x_\sigma s = \sigma(s)x_\sigma$ for all $s \in S, \sigma \in G$ and $x_\sigma x_\tau = f(\sigma,\tau)x_{\sigma\tau}$). Since for all $s \in S$ and $\sigma \in G$ we have that $(sx_\sigma)^{|G|} = \prod_{i=0}^{|G|-1} \sigma^i(s)f(\sigma^i,\sigma) \in S$, and S is integral over V, it follows from [1, Thm. 2.3] that $S * G$ is a V-order in $K * G$. If $f = 1$, then $S * G$ becomes a *skew group ring*, denoted by $S \circ G$.

All notions regarding valuation theory are as defined in [4].

A lot of the theory of crossed product orders is known when V is a DVR. For example, in [2, Cor. A.5 & Prop. A.6] it was proved that $S \circ G$ is a maximal order if and only if (K, W) is unramified over (F, V) and in [9] it was shown that if K/F is tamely ramified

*The complete version of this manuscript will appear in the *Bull. London Math. Soc.*

then $S * G$ is hereditary. The converse of the last statement, namely that if $S * G$ is hereditary then K/F is tamely ramified, does not hold, as will be shown in this paper, unless the residue ring $\overline{V}$ is perfect [5, Thm. 2] or the factor set f is trivial [3, Cor. 3.6]. In this paper, we aim to generalise these results to the case when V is not necessarily a DVR and the factor set f is not necessarily trivial. We employ recent results by Marubayashi and Yi [6, 10] on homological dimension of crossed products, and this greatly simplifies our proofs.

The author wishes to thank Hidetoshi Marubayashi for bringing to his attention recent literature on this subject, and for his careful reading of the complete manuscript.

Since S is a commutative semihereditary semilocal ring, the following theorem is essentially a restatement of [6, Thm. 2.9].

Theorem 1 *Let V be an arbitrary valuation ring. Then $S \circ G$ is semihereditary if and only if K/F is tamely ramified and defectless.*

Given $x \in K$, we define its *trace* w.r.t. the extension K/F by the usual formula: $t_{K/F}(x) = \sum_{\sigma \in G} \sigma(x)$. Note that since V is integrally closed, we have that $t_{K/F}(S) \subseteq V$. Now the proof by Rosen [8, Thm. 40.13] carries over to the case when V is not necessarily a DVR to establish that $S \circ G$ is semihereditary if and only if there exists in S an element of trace 1. Note the terminology difference: what [8] refers to as *twisted group rings* are skew group rings in our case, since the twisting is trivial, i.e., $f = 1$. Our terminology usage is in line with that of [7].

Thus we obtain the following result of independent interest, which generalises [3, Thm. 3.2].

Corollary 1 *If (K, W) is a finite Galois extension of (F, V), and S is the integral closure of V in K, then K/F is tamely ramified and defectless if and only if $t_{K/F}(S) = V$.*

The following theorem is a generalisation of two classical results by Williamson and Harada. The theorem is proved by modifying, whenever necessary, the proofs of [9, Prop. 1.3] and [5, Thm. 2]. Here and elsewhere the results in [6, 10] enable us to extend the classical theory to the non-Noetherian setting.

Theorem 2 *We have:*

(a) *If K/F is tamely ramified and defectless, then $J(S * G) = J(S) * G$ and $S * G$ is semihereditary.*

(b) *If $S * G$ is semihereditary and $\overline{V}$ is a perfect field, then K/F is tamely ramified and defectless provided that V is a DVR or $J(V)$ is a non-principal ideal of V.*

REMARK. By combining Theorems 1 and 2(a), we have generalised the main theorem in Section 1 of [9], which states that when V is a DVR, then $S * G$ is hereditary *for every factor set f* if and only if K/F is tamely ramified. We recall that in the classical setting, K/F is always defectless.

We will see examples at the end of the paper which suggest that Theorem 2 may not be improved beyond the form it currently is in.

Corollary 2 *Let K/F be tamely ramified and defectless, and suppose $J(V)$ is a non-principal ideal of V. Then*

(a) *$S * G$ is a semihereditary maximal order, and*

(b) *if V is rank-1, then $S \circ G \cong End_V(S)$.*

REMARK. We suspect that for any V, $End_V(S)$ is a V-order in $M_n(F)$, where $n = |G|$. If this is indeed the case, then $S \circ G \cong End_V(S)$ whenever $S \circ G$ is semihereditary and $J(V)$ is not a principal ideal of V. When V is a principal ideal of V, one cannot always determine if $S * G$ is a maximal order, unless the cocycle f is explicitly known (see the examples below).

Recall that a V-order R of a central simple F-algebra, where V is a valuation ring of F, is *Azumaya over V* if it is a finitely generated V-module with $R/J(V)R$ a central simple $\overline{V}$-algebra.

Theorem 3 *The order $S * G$ is Azumaya over V if and only if K/F is unramified and defectless.*

The converse of this statement, namely that when K/F is unramified and defectless then $S * G$ is Azumaya over V, is more or less well known.

We end by giving examples, in every characteristic, that exhibit some limitations to this theory of crossed product orders.

EXAMPLE 1. Let $F = \mathbb{Q}(t)$, a function field in one variable over the field of the rationals. Let U_1 be the t-adic valuation ring of F, and set $V = \{x \in U_1 \mid x + J(U_1) \in \mathbb{Z}_2\}$, where $\mathbb{Z}_2$ is the 2-adic valuation ring of $\mathbb{Q}$. Let $K = F(\sqrt{t})$, a cyclic extension of F with group $G = \langle \sigma \rangle$. Let W be the unique extension of V to K. Let $f \in Z^2(G, U(W))$ be defined by $f(\sigma, \sigma) = -1, f(1, \sigma) = f(\sigma, 1) = f(1, 1) = 1$.

It turns out that $S * G$ is an invariant valuation ring of $K * G$, hence is semihereditary. However, although $\overline{V}$ is a perfect field, K/F is not tamely ramified.

Now let W_1 be the unique extension of U_1 to K. Then $W_1 * G$ is an invariant valuation ring, being an overring of $S * G$ in $K * G$. We see that while $W_1 * G$ is a maximal U_1-order, in the classical sense of the term, (K, W_1) is not unramified over (F, U_1), and $W_1 \circ G$ is not a maximal U_1-order.

The following example, communicated to the author by P. Morandi, illustrates a similar phenomenon in positive characteristic.

EXAMPLE 2. Let L be a field of characteristic $p > 0$, let $F = L((x))((y))$ be the iterated Laurent series field in two variables over L, and let $K = F(t)$ be the cyclic extension of F satisfying $t^p - t = 1/y$, with group $G = \langle \sigma \rangle$. Let V be the standard rank-2 valuation ring of F, and W the extension of V to K. Let $f \in Z^2(G, U(W))$ be defined by $f(\sigma^i, \sigma^j) = 1$ for $i + j < p$, and $f(\sigma^i, \sigma^j) = 1 - x$ otherwise, where $0 \leq i, j < p$.

Again, it turns out that $S * G$ is an invariant valuation ring of $K * G$, hence is semihereditary. But K/F is not tamely ramified. Note that we may choose L $(= \overline{V})$ to be a perfect field.

Let U_1 be the DVR of F containing V, and W_1 the extension of U_1 to K. Then $W_1 * G$ is an invariant valuation ring, being an overring of $S*G$ in $K*G$. Observe that (K, W_1) is not tamely ramified over (F, U_1). Thus the converse of the result by Williamson [**9**, Prop. 1.4] does not hold, and that we may not drop the perfectness assumption in Harada's result [**5**, Thm. 2]. Also, note that while $W_1 * G$ is a classical maximal order, $W_1 \circ G$ is not even hereditary.

We conclude that properties of the order $S * G$ cannot always be solely determined by the nature of the extension K/F, but that one has to consider the 2-cocycle f as well, and conversely.

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PRESENTATIONS OF TORUS INVARIANTS IN PARALLELED LINEAR HULLS AND THEIR APPLICATIONS

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ABSTRACT. Let G be an algebraic group such that G^0 is an albegebraic torus defined over an algebraically closed field K of characteristic zero and suppose that $G = Z_G(T)$. Let R be an affine factorial domain over K with the trivial unit group on which G acts rationally as K-automorphisms. For a linear character χ of G, we study on the R^G-module R_χ of all χ-invariants and generalize the criterion in [N1] for R_χ to be a free R^G-module of rank one, without the assumption on finiteness of G. Furthermore, we extend H.P. Kraft's presentation of torus invariants (cf. [Kr, W]) to this R with the G-action. This result can be applied to the study on coregular representations of reductive algebraic groups of certain types. In fact, we determine coregular representations of reductive groups whose semisimple parts are simple algebraic groups of type F_4.

1. Introduction

In this paper, all algebraic varieties are defined over an algebraically closed field K of characteristic zero. Without specifying, G (resp. T) will always stand for a reductive algebraic group (resp. connected algebraic torus). For an affine variety X, $\mathcal{O}(X)$ denotes the K-algebra of all regular functions on X. When a regular action of G on an affine variety X (abbr. (X, G)) is given, we say X is a G-variety and define $\mathcal{O}(X)^G$ to be the K-subalgebra consisting of all invariants of G in $\mathcal{O}(X)$. The action (X, G) is said to be stable, if X contains a non-empty open subset consisting of closed G-orbits. Recall that X is said to be conical, if $\mathcal{O}(X)$ is equipped with a positive graduation $\mathcal{O}(X) = \bigoplus_{i\geq 0} \mathcal{O}(X)_i$ such that $\mathcal{O}(X)_0 = K$. In this case, we say that an action (X, G) is conical, if the induced action G preserves the graduation of $\mathcal{O}(X)$. We denote by $X//G$ the affine variety associated with $\mathcal{O}(X)^G$, i.e., the algebraic quotient of X under the action of G and by $\pi_{X,G}$ the quotient map $X \to X/G$. In the case where $\mathcal{O}(X)^G$ is affine, the action (X, G) is said to be cofree (resp. equidimensional), if $\mathcal{O}(X)$ is $\mathcal{O}(X)^G$-free (resp. if $X \to X/G$ is equidimensional). When $\mathcal{O}(X)^G$ is a polynomial ring over K, (X, G) is called coregular.

Let $\mathfrak{X}(G)$ stand for the rational linear character group of (not necessarily connected) G over K which is regarded as an additive group. For any $\chi \in \mathfrak{X}(G)$, we set

$$\mathcal{O}(X)_\chi = \{x \in \mathcal{O}(X) \mid \sigma(x) = \chi(\sigma) \cdot x \text{ for any } \sigma \in G\},$$

whose elements are called χ-invariants or semi-invariants of G relative to χ in $\mathcal{O}(X)$. Clearly $\mathcal{O}(X)_\chi$ is an $\mathcal{O}(X)^G$-module. We have already shown

Theorem 1.1 (cf. [N4, N6]). *Suppose that G is connected and let X be an affine conical factorial variety with a conical action of G. If the action of G on X is equidimensional, then $\mathcal{O}(X)^G$ is factorial and $\mathcal{O}(X)_\chi$ is $\mathcal{O}(X)^G$-free, for any $\chi \in \mathfrak{X}(G)$ such that $\mathcal{O}(X)_\chi \cdot \mathcal{O}(X)_{-\chi} \neq \{0\}$.*

In [N1], we have obtained a criterion $\mathcal{O}(X)_\chi$ to be a free $\mathcal{O}(X)^G$-module of rank one in terms of the special semi-invariant g_χ under the assumption that G is finite. Then we come up with

Problem 1.2. What is a generalization of the criterion mentioned above without finiteness of G ?

This problem seems to be closely related to the following example which is generalized in [W] for representations of tori:

1991 *Mathematics Subject Classification.* 14L30, 20G05, 13A50, 13B15.

Key words and phrases. semigroup rings, algebraic tori, invariant theory, factorial domains, class groups.

This is an expository paper on the recent results of the author and the detailed proofs of some results shall be published elsewhere.

Example 1.3 (H.P. Kraft [Kr]). Let $V = K^3$ and suppose that $T_1 = K^\times$ acts on V via the linear representation

$$T_1 \ni t \longmapsto \begin{pmatrix} t^{-\alpha} & 0 & 0 \\ 0 & t^\beta & 0 \\ 0 & 0 & t^\gamma \end{pmatrix} \in GL_3(K)$$

for $\alpha, \beta, \gamma \in \mathbb{N}$ such that $\gcd(\alpha, \beta, \gamma) = 1$. Let W be a T_1-submodule K^2 of V deleting the first coordinate with the T_1-action

$$T_1 \ni t \longmapsto \begin{pmatrix} t^\beta & 0 \\ 0 & t^\gamma \end{pmatrix} \in GL_2(K)$$

and put $T_{1,\alpha} = \{s \in T_1 \mid s^\alpha = 1\}$. Then we have a natural isomorphism $\mathcal{O}(V^\bullet)^{T_1} \cong \mathcal{O}(W^\bullet)^{T_{1,\alpha}}$.

Concerning this example, it is natural to ask

Problem 1.4. What is the reduced expression of semigroup rings ? Moreover, can we generalize Example 1.3 to in the case where affine factorial varieties with torus actions?

In Sect. 2, we summarize our partial answers to these problems. Our results can be applied to invariant theory of certain representations of reductive groups over the complex number field $\mathbb{C}$.

We denote by G' the commutator subgroup of G and use any of the notations ρ, (ρ, G) or (V, G) to denote a finite dimensional linear representation $\rho \colon G \to GL(V)$ over $\mathbb{C}$. For a closed subgroup H of G, let $((V, G), H)$ denote the representation $\rho|_H$. A representation (V, G) is defined to be relatively stable, if the natural action of G on $V//G'$ is stable (for properties on relative stability, see [N6]).

As a partial affirmative answer to the conjecture brought up by V. L. Popov [P1] and by V. G. Kac [K] on equidimensional representations, we announce in [N5] the following

Theorem 1.5. *Suppose that G is a connected reductive algebraic group whose commutator subgroup is a simple algebraic group. If a finite-dimensional linear representation (V, G) is equidimensional and relatively stable, then it is cofree.*

In the case that G' is non-orthogonal symplectic, it has already been obtained as a somewhat general result of [N3]. When G itself is simple, the result similar to this is shown by Popov [P2] and O. M. Adamovich [A], and, moreover, cofree representations of G are determined by [P1] and G. W. Schwarz [S1, S2].

For a representation V of a reductive algebraic group G, the affine G/G'-variety $V//G'$ is an affine factorial variety with the torus G/G'-action. Under this circumstance, we study on the divisor class group $\mathrm{Cl}((V//G')//(G/G'))$ in Sect. 2. Consequently the classification of coregular representations of certain reductive groups can be reduced to Theorem 1.5 and [Sm]. In fact, we can apply the answer to Problem 1.4 to classifying coregular representations of G such that G' is a simple algebraic group of exceptional types. The proof for F_4 is given in Sect. 3 and the author comes up with Problem 3.1 which generalizes this result.

2. Paralleled Linear Hulls of Torus Invariants

In this section, let X be an affine G-variety with the trivial unit group, i.e., $U(\mathcal{O}(X)) = K^\times$ and let R denote $\mathcal{O}(X)$.

For $\mathfrak{P} \in \mathrm{Spec}(R)$, we put $D_G\mathfrak{P}) = \{\sigma \in G \mid \sigma(\mathfrak{P}) = \mathfrak{P}\}$ and

$$I_G(\mathfrak{P}) = \{\sigma \in D_G(\mathfrak{P}) \mid \sigma|_{R/\mathfrak{P}} = \mathrm{Id}_{R/\mathfrak{P}}\},$$

which are called the decomposition (resp. inertia) group at $\mathfrak{P}$ as in the case of finite group actions. Let $\mathrm{Ht}_1(X, G)$ be

$$\{\mathfrak{P} \in \mathrm{Spec}(\mathcal{O}(X)) \mid \mathrm{ht}(\mathfrak{P} \cap \mathcal{O}(X)^G) = \mathrm{ht}(\mathfrak{P}) = 1\}.$$

Theorem 2.1 (cf. [N7]). *Suppose that G^0 is a torus. The following conditions are equivalent:*

(1) $G = Z_G(G^0)$.

(2) *The equalities*

$$\mathrm{e}(\xi, \pi_{X,H}(\xi)) = \sharp(I_H(\xi)|_X) \quad (\forall \xi \in \mathrm{Ht}_1(X, H))$$

for reduced ramification indices hold for any closed subgroup H of G containing $Z_G(G^0)$ and for any affine normal irreducible H-variety X whose action is effective and stable.

(3) *For any H and X as in (2), there exists a normal finite subgroup N of H such that $(X//N, H/N)$ is divisorially unramified, i.e., $e(\eta, \pi_{X//N,H/N}(\eta)) = 1$ for any $\eta \in \mathrm{Ht}_1(X//N, H/N)$.*

The proof of our main result in this section is based on this theorem, and so, from now on to the end of Theorem 2.5, we suppose that $G^0 = T$ and $G = Z_G(T)$.

Let $\mathrm{Ht}_1(R) = \{\mathfrak{P} \in \mathrm{Spec}(R) \mid \mathrm{ht}(\mathfrak{P}) = 1\}$ and

$$\mathrm{Ht}_1(R)_\mathfrak{p} = \{\mathfrak{Q} \in \mathrm{Ht}_1(R) \mid \mathfrak{Q} \cap R^G = \mathfrak{p}\}$$

for $\mathfrak{p} \in \mathrm{Ht}_1(R^T)$. We say that a subset Γ of $\mathrm{Ht}_1(R)$ is G-removable, if $\Gamma \subseteq \mathrm{Ht}_1(R)^T$ and

$$\Gamma \cap \mathrm{Ht}_1(R)_\mathfrak{p} \subsetneq \mathrm{Ht}_1(R)_\mathfrak{p}$$

for all $\mathfrak{p} \in \mathrm{Ht}_1(R^T)$, where $\mathrm{Ht}_1(R)^T$ denotes the subset consisting of all T-invariant prime ideals in $\mathrm{Ht}_1(R)$. It should be noted that $\mathrm{Ht}_1(R)_\mathfrak{p} \neq \emptyset$ (cf. [N2]). For a subset Γ of $\mathrm{Ht}_1(R)$, let G_Γ denote the intersection $\cap_{Rf \in \Gamma} G_f$ of stabilizers G_f's of G. Let $\mathcal{R}_X(G)$ denote the subgroup of G generated by all $I_G(\mathfrak{Q})$, $\mathfrak{Q} \in \mathrm{Ht}_1(R)$. Then $\mathcal{R}_X(G)$ is a normal subgroup of G and it is finite on X (cf. [N7]).

Theorem 2.2. *Suppose that X is factorial and (X, G) is a stable action. Then*

$$\mathrm{Cl}(R^G) \cong \mathrm{Cl}(R^L) \cong \mathrm{Cl}(R^{G_\Gamma})$$

for any nonempty G-removable subset Γ of $\mathrm{Ht}_1(R)$. Especially if Γ is a maximal G-removable subset, then these class groups are canonically isomorphic to $\mathfrak{X}(G_\Gamma/\mathcal{R}_X(G_\Gamma))$.

For $\chi \in \mathfrak{X}(G)$, let λ be $\chi|_{\mathcal{R}_X(G)} \in \mathfrak{X}(\mathcal{R}(G))$. In [N1], we associate the element g_λ in R with λ satisfying $R_\lambda = R^{\mathcal{R}_X(G)} \cdot g_\lambda$. Since g_λ is a semi-invariant of G, let $\tilde{\chi}$ be a character in $\mathfrak{X}(G)$ such that $g_\lambda \in R_{\tilde{\chi}}$. Then we generalize the main result of [N2, Theorem 2.9] as follows:

Theorem 2.3. *Suppose X is factorial and (X, G) is a stable action. For any $\chi \in \mathfrak{X}(G)$, the following conditions are equivalent:*

(1) *R_χ is a free R^G-module of rank one.*

(2) *There exists a nonzero element f in $R_{\chi - \tilde{\chi}}$ such that the set*

$$\{\mathfrak{Q} \in \mathrm{Ht}_1(R) \mid v_\mathfrak{Q}(f) > 0\}$$

is G-removable.

We choose a finite-dimensional G-submodule V with a K-basis consisting of prime semi-invariants of T which generates R as a K-algebra. Let $\phi\colon \mathcal{O}(V^*)(= \mathrm{Sym}(V)) \to R$ be the canonical epimorphism associated with the embedding $\phi^*\colon X \hookrightarrow V^*$. A pair $\Delta = (W, \{w_i\})$ is said to be a G-admissible couple in V, if W is a G-submodule of V, $\{w_i \mid i \in I\}$ are semi-invariants of T in W and the equalities

$$W + \sum_{i \in I} K w_i = W \oplus \bigoplus_{i \in I} K w_i = V$$

hold. For this Δ, let $q_\Delta\colon \mathcal{O}(V^*) \to \mathcal{O}(W^*)$ be the K-epimorphism defined by $q_\Delta|_W = \mathrm{Id}_W$ and $q_\Delta(w_i) = 1$ ($i \in I$). We say that $\Delta = (W, \{w_i\})$ is a paralleled linear hull (abbr. PLH) for (X, G), if there exists a K-morphism $\Phi_\Delta\colon \mathcal{O}(W^*)^{T(w_i)} \to R^T$ and q_Δ induces an epimorphism $\mathcal{O}(V^*)^T \to \mathcal{O}(W^*)^{T(w_i)}$ such that $\Phi_\Delta \circ q_\Delta = \phi$ on $\mathcal{O}(V^*)^T$.

Using Theorem 2.3, we obtain

Propisition 2.4. *Suppose that X is factorial and (X, G) is stable. Let $\Delta = (W, \{w_i\})$ be a G-admissible couple of V. Then Δ is a PLH for (R, G) if and only if so is for (V, G).*

From this, we can prove the following theorem which partially generalize Theorem 2.2.

Theorem 2.5. *Suppose that X is factorial and (X,G) is stable. Let Δ be a PLH for (X,G). Then:*

1. *q_Δ induces the isomorphism $\mathcal{O}(V^*)^T \cong \mathcal{O}(W^*)^{\cap_i T_{w_i}}$ and $\mathcal{O}(V^*)^G \cong \mathcal{O}(W^*)^{\cap_i G_{w_i}}$.*
2. *Suppose that Δ is minimal such that Δ is a PLH for (X,G). If G is diagonal or if (X,G) is a conical action of a conical variety and V is a homogeneous minimal submodule of R, then*

$$\mathrm{Cl}(R^G) \cong \mathrm{Cl}(\mathcal{O}(V^*)^G) \cong \mathfrak{X}(G/\mathcal{R}_X(G)).$$

For an algebraic action (X,G) on an affine normal variety X, a prime element f in $\mathcal{O}(X)$ which is a semi-invariant of G is said to be (X,G)-blowing up (abbr. BU), if $\mathrm{ht}((f)\cap \mathcal{O}(X)^G) \geq 2$ and, otherwise, f is said to be (X,G)-no blowing up (abbr. NBU).

Hereafter we suppose that G is a connected reductive algebraic group over $\mathbb{C}$ and (V,G) is a finite-dimensional complex representation of G. Let V_i, $1 \leq i \leq n$, be irreducible components of (V,G) satisfying $V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$. We identify $\mathcal{O}(V)$ with $\mathcal{O}(V_1)\otimes\mathcal{O}(V_2)\otimes\cdots\otimes\mathcal{O}(V_n)$ and regard this as a $\mathbb{Z}_0^n$-graded algebra over $\mathbb{C}$ in a natural way, where $\mathbb{Z}_0$ denotes the additive monoid of non-negative integers. For any subset J of $\{1,\ldots,n\}$, $\mathcal{O}(\oplus_{j\in J}V_j)$ is usually regarded as a $\mathbb{C}$-subalgebra of $\mathcal{O}(V)$ through the canonical projection $V \to \oplus_{j\in J}V_j$. The vector space $\mathcal{O}(V)_+^{G'}/(\mathcal{O}(V)_+^{G'})^2$ is $\mathbb{Z}_0^n$-graded and a G/G'-module. So, there is a minimal system of $\mathbb{Z}_0^n$-homogeneous generators of $\mathcal{O}(V)^{G'}$ consisting of semi-invariants of G.

In this section, let $\{f_1,\ldots f_m, g_1,\ldots,g_l\}$ denote a minimal system of multihomogeneous generators of $\mathcal{O}(V)^{G'}$ consisting of semi-invariants of G and suppose that f_i, $1 \leq i \leq m$, are (V,G)-BU and g_j, $1 \leq j \leq l$, are (V,G)-NBU.

Recall that the representation (V,G) is defined to be relatively equidimensional, if the action $(V//G', G)$ is equidimensional. Applying Theorem 2.2 to $X = V//G'$ with a torus G/G'-action, we have

Theorem 2.6. *Suppose that (V,G) is relatively stable.*

1. *If, for a subset J of $\{1,\ldots,m\}$, f_j $(j \in J)$ are algebraically independent over*

$$\mathbb{C}[\{f_j \mid j \in \{1,\ldots,m\}\backslash J\} \cup \{g_1,\ldots,g_l\}]$$

denoted to B, then

$$B^{(\cap_{j\in J}G_{f_j})} \cong \mathcal{O}(V)^G$$

and f_j, $j \in \{1,\ldots,m\}\backslash J$, are $(\mathrm{Spec}(B), \cap_{j\in J}G_{f_j})$-BU and g_i, $1 \leq i \leq l$, are $(\mathrm{Spec}(B), \cap_{j\in J}G_{f_j})$-NBU.

2. *If $V//G$ is factorial, then $(V, \cap_{i=1}^m G_{f_i})$ is relatively equidimensional.*

The next result follows from Theorem 2.6:

Theorem 2.7. *Suppose that (V,G) is relatively stable and coregular. Then there exist a subset I_1 of $\{1,\ldots,l\}$ and homogeneous semi-invariants $h_i \in \mathcal{O}(V)^{(\cap_{j=1}^m G_{f_j})}$ of G indexed by the set I_1 satisfying the following conditions:*

1. *$\mathbb{C}[\{f_1,\ldots,f_m\} \cup \{h_j \mid j \in I_1\}] \subseteq \mathcal{O}(V)$, which is denoted to A, is a polynomial ring over $\mathbb{C}$ of dimension $\sharp(I_1) + m$.*
2. *$A_{\prod_{i=1}^m f_i} = (\mathcal{O}(V)^{(\cap_{j=1}^m G_{f_j})})_{\prod_{i=1}^m f_i}$.*
3. *$A^G = \mathcal{O}(V)^G$.*

We say that a homogeneous element $f \in \mathcal{O}(V)^{G'}$ is a mixed invariant of G', if $f \notin \mathcal{O}(V_i) \hookrightarrow \mathcal{O}(V)$ for any i and say that f is a fundamental invariant of G', unless $f \in (\mathcal{O}(V)_+^{G'})^2$

Proposition 2.8. *Suppose that (V,G) is relatively stable and $\mathcal{O}(V_i)^{G'} \neq \mathbb{C}$ for any i. Let $f \in \mathcal{O}(V)^{G'}$ be a fundamental homogeneous invariant of G'. If f is mixed and a semi-invariant of G, then* $\mathrm{ht}((\mathcal{O}(V)\cdot f)^G) = 1$.

Proof. Suppose that the assertion is false. Then $f \in \mathcal{O}(V)_\chi$ for some $\chi \in \mathfrak{X}(G)$. Since f is mixed, we have a non-zero vector $(a_1,\ldots,a_n) \in \mathbb{Z}_0^n$ such that

$$\chi = \sum_{i=1}^n a_i \cdot \chi_i,$$

where χ_i satisfies $V_{\chi_i} \supseteq V_i$. Then there is a number $l \in \mathbb{N}$ such that $\mathcal{O}(V)_{l\cdot\chi_i} \ni f_i^{s_i}$ for some $s_i \in \mathbb{N}$ and all i. However, by Theorem 2.3 , we have

$$\mathcal{O}(V)_{l\cdot\chi} = \mathcal{O}(V)^G \cdot f^l.$$

Thus $\prod_{i=1}^n (f_i^{s_i})^{a_i}$ is divisible by f in $\mathcal{O}(V)$, which is a contradiction. □

Corollary 2.9. *Suppose that (V,G) is relatively stable and $\mathcal{O}(V_i)^{G'} \neq \mathbb{C}$ for all i. Then there exists a subset J of $\{1,\ldots,n\}$ satisfying the following conditions;*

(1) $\mathcal{O}(V) \hookleftarrow \mathcal{O}(\oplus_{j\in J}V_j)^{G'} = \mathbb{C}[f_1,\ldots,f_m]$
(2) $\mathcal{O}(V_i)^{G'} = \mathbb{C}[f_i]$ *for* $\supset\in J$
(3) $\mathcal{O}(\oplus_{j\in J}V_j)^{G'} \not\ni g_i$ *for* $1 \leq i \leq l$.

Cosequently $(\oplus_{j\in J}V_j, G')$ *is coregular.*

Proof. Let J be denote the set consisting of all i's such that $\mathcal{O}(V_i) \cap \{f_1,\ldots,f_m\} \neq \emptyset$. Suppose the assertion (1) is false. Then, for a subset J' of J, we may assume that $\mathcal{O}(\oplus_{j\in J'}V_j)^{G'} \not\cong \otimes_{j\in J'}\mathcal{O}(V_j)^{G'}$ and $\mathcal{O}(\oplus_{j\in J_0}V_j)^{G'} \cong \otimes_{j\in J_0}\mathcal{O}(V_j)^{G'}$ for any proper subset J_0 of J'. Exchanging indices of f_j, we assume that $\mathcal{O}(V_j)^{G'} \ni f_j$. There is a fundamental homogeneous invariant h of G' in $\mathcal{O}(\oplus_{j\in J'}V_j)^{G'}$ which is not contained in $\otimes_{j\in J_0}\mathcal{O}(V_j)^{G'}$. Let u be the maximal index in J' and put

$$Q = \bigcap_{j\in J'\setminus\{u\}} G_{f_j}.$$

Then $h^t \in \mathcal{O}(V)^Q$ for some $t \in \mathbb{N}$. By Theorem 2.6, we see that $\mathrm{ht}((f_u)^Q) \geq 2$ and

$$\mathcal{O}(V)_{k\cdot\chi_u|_Q} = \mathcal{O}(V)^Q \cdot f_u^k$$

for any $k \in \mathbb{N}$. Because h^t is contained in $\sum_{k\in\mathbb{N}}\mathcal{O}(V)_{k\cdot\chi_u|_Q}$, we see that h^t is divisible by f_u in $\mathcal{O}(V)$, which is a contradiction.

From (1) and (2), we only note that $\mathcal{O}(\oplus_{j\in J}V_j)^{G'} \cong \otimes_{j\in J}\mathcal{O}(V_j)^{G'}$, which is a polynomial ring generated by an algebraically independent system $\{f_1,\ldots f_m\}$ over $\mathbb{C}$. □

3. Classification of Coregular Representations

A representation (V,G) is said to be quasi-coregular, if there is a closed subgroup L of $GL(V)$ such that $G|_V$ is a subgroup of L of finite index and (V,L) is coregular.

In this sction, we will consider the following problem and give an affirmative answer to this in a special case.

Problem 3.1. Suppose that G' is a simple algebraic group and that each irreducible component of (V,G) has a nontrivial closed G'-orbit. Furthermore suppose that (V,G) is relatively stable and coregular. Then, is there a closed subgroup L of G containing G' such that $[L : G'] < \infty$ and $((V,G),L)$ is coregular? Especially, is $((V,G),G')$ quasi-coregular?

A representation (V,G) is defined to be relatively irredundant along trivial parts, if $(V//G',G)$ is non-trivial and ($V^{G'} = \{0\}$ or) $G|_{V//G'}$ is never equal to the inner direct product

$$(\cap_{x\in(V/U)//G'}(G|_{V//G'})_x) \times (\cap_{x\in U}(G|_{V//G'})_x)),$$

for any nonzero subspace U of $V^{G'}$.

Proposition 3.2. *Suppose that*

$$G|_{V//G'} = (\bigcap_{x\in(V/U)//G'}(G|_{V//G'})_x) \times (\bigcap_{x\in U}(G|_{V//G'})_x)$$

for a subspace U of $V^{G'}$. If (V,G) is relatively equidimensional and relatively stable, then so are both actions $(V,\cap_{x\in(V/U)//G'}(G|_{V//G'})_x)$ and $(V,\cap_{x\in U}(G|_{V//G'})_x)$.

Proof. Put $X = V//G'$ and $T = G/G'$. Since (X,T) is a conical equidimensional stable action on a factorial conical variety, we can apply the structure theorem (e.g., Theorem 1.1) of cofree actions to this case and obtain the assertion. □

Corollary 3.3. *Suppose that G is connected and $(V_{G'}//G', G)$ is non-trivial. If (V, G) is relatively equidimensional and relatively stable, then there exists a closed connected subgroup N of G satisfying the conditions as follows;*

(1) $N \supseteq G'$ *and* $N|_{V_{G'}//G'} = G|_{V_{G'}//G'}$
(2) (V, N) *is relatively equidimensional and relatively stable*
(3) $(V/V^N, N)$ *is relatively irredundant along trivial parts.* □

In [N5], we briefly announce the next result which is fundamental in the study on Problem 3.1 for exceptional types. The number of basic representations is defined in [T].

Theorem 3.4. *Suppose that G' is a simple algebraic group of exceptional types. If (V, G) is relatively equidimensional, relatively stable and relatively irredundant along trivial parts, then the representation $(V_{G'}, G')$ is can be identified with one of the following representataions; $(m \cdot \Phi_1, \mathrm{G}_2)$ $(m \le 2)$,(Φ_1, F_4), $(m \cdot \Phi_1 \oplus \Phi_5, \mathrm{E}_6)$ $(m \le 1)$, (Φ_1, E_6), and (Φ_1, E_7).*

In this paper, we give the following affirmative answer to Problem 3.1 in the special case.

Theorem 3.5. *Suppose that G' is a simple algebraic group of type F_4. If (V, G) is relatively stable and coregular, then $((V, G), G')$ is coregular.*

Let f_i $(1 \le i \le m)$, g_j $(1 \le j \le n)$ and y_k $(1 \le k \le l)$ be $\mathbb{Z}_0^2$-graded elements consisting of semi-invariants of G in $\mathcal{O}(V)^{G'} \cong \mathcal{O}(V_{G'})^{G'} \otimes \mathcal{O}(V^{G'})$ such that $\{f_1, \ldots, f_m\} \cup \{g_1, \ldots, g_n\}$ is a minimal system of generators of $\mathcal{O}(V_{G'})^{G'}$ and $\{y_1, \ldots, y_l\}$ is a $\mathbb{C}$-basis of $(V^{G'})^*$. Suppose that all f_i are $(V//G', G)$-BU of codimension one and all g_j are $(V//G', G)$-NBU of codimension one. Moreover assume that $\{y_j \mid 1 \le j \le l'\}$ is the set consisting of all elements in y_j's which are $(V//G', G)$-NBU of codimension one. Set $M = \cap_{l' < j \le l} G_{y_j}$ and

$$H = \Big(\bigcap_{1 \le i \le m} G_{f_i}\Big) \cap \Big(\bigcap_{l' < j \le l} G_{y_j}\Big).$$

By (1) of Theorem 2.6, we immediately have

Lemma 3.6. *Suppose that (V, G) is relatively stable. Then (V, G) is coregular if and only if so is (V, M).* □

In order to show Theorem 3.5, by Theorem 3.4, we may suppose the following condition:

(3.7) For any closed normal connected subgroup N of G containing G' such that $((V, G), N)$ is relatively equidimensional, $((V, G), N)$ is not relatively irredundant along trivial parts.

Lemma 3.8. *Suppose that (3.7) and (V, G) is relatively stable and coregular. Then*

(1) $(V_{G'}//G', H^0)$ *is trivial.*
(2) $((V, G), M)$ *is coregular.*
(3) *Both* $((V, G), H)$ *and* $((V, G), M^0)$ *are relatively equidimensional.*
(4) $((V^{G'}, G), H^0)$ *is a stable action.*
(5) *Both* $V//H^0$ *and* $V//H$ *are factorial.*

Proof. Since $(V_{G'}//G', H^0)$ is equidimensional and is not relatively irredundant along trivial parts, by Corollary 3.3, $(V_{G'}//G', H^0)$ is trivial. The triviality of $(V_{G'}//G', H^0)$ implies (3). The assertion (2) is easy and by this and the main result in [N4], we see that $(V//G', H^0)$ is cofree. Because the quotient morphism $V//G' \to V//H^0$ is NBU of codimension one (cf. Theorem 2.4), applying §2 of [N4] to the cofree action $(V//G', H^0)$, we see that $\mathrm{Cl}(V//H^0) = \{0\}$. □

Proposition 3.9. *Suppose that the condition (3.7) holds and (V, G) is relatively stable and coregular. Then, for any closed G'-orbit $G' \cdot x$ in the principal open subset$(V_{G'})_{f_1 \cdot f_2 \cdot \cdots \cdot f_m}$ of $V_{G'}$, the slice representation $((V_{G'})_x, H_x)$ of $((V_{G'}, G), H)$ at x is coregular and the slice representation $((V_{G'})_x, G'_x)$ of $(V_{G'}, G')$ at x is quasi-coregular.*

Proof. By (1) of Lemma 3.8, we easily see that, for $x \in V_{G'}$, $G' \cdot x$ is closed if and only if so is $H \cdot x$. From Theorem 2.7 and the slice etale (cf. [L, S3]), we deduce that the slice representation (V_x, H_x) of

$((V,G),H)$, which is also denoted to $(V,H)_x$, is coregular. There is a finite subgroup H_1 of $\mathrm{Stab}_H(V^{G'})$ normalizing G' such that $H_1 \cdot G' \triangleleft H$ and

$$H_1 \cdot G'|_{V_{G'}//G'} = H|_{V_G'//G'}.$$

Since

$$(V_x, H_x) \cong ((V_{G'})_x \oplus V^{G'}, H_x),$$

the slice representation $((V_{G'})_x, (H_1 \cdot G')_x)$ of $(V_{G'}, H_1 \cdot G')$ is coregular (cf. [S1, S3]). Thus $((V,(H_1 \cdot G')_x), G'_x)$ is quasi-coregular. As $(H_1 \cdot G')/G'$ is finite, we must have

$$(V, H_1 \cdot G')_x \cong T_x(V)/T_x((H_1 \cdot G') \cdot x) \cong T_x(V)/T_x(G' \cdot x) \cong (V,G)_x.$$

Thus the slice representation $(V,G')_x$ of (V,G) is quasi-coregular. □

Corollary 3.10. *Suppose that (3.7) holds and $m = 1$. Let U be the irreducible subrepresentation of (V,G) satisfying* $\mathrm{Sym}(U^*)^{G'} = \mathbb{C}[f_1]$. *Then, for a non-trivial closed isotropy subgroup L of G' on U, the quotient representation $((V_{G'}/U,G),L)$ is quasi-coregular.*

Proof. We identify $(U \oplus V_{G'}/U, G)$ with $(V_{G'},G)$ and, through this decomposition, regard $\mathcal{O}(U)$ as a $\mathbb{C}$-subalgebra of $\mathcal{O}(V)$. Let $G' \cdot x$ be a non-trivial closed orbit in U. Then $G' \cdot x \subseteq U_{f_1}$. By Proposition 3.9, the slice representation $(V_{G'}, H)_x$ is coregular and hence, since

$$(V_{G'}, H) \cong (U,H)_x \oplus ((V_{G'}/U,G), H_x),$$

the quotient representation $((V_{G'}/U,G),H_x)$ coregular. Then, as in the proof of Proposition 3.9, we similarly see that $((V_{G'}/U, H_x), G'_x)$ is quasi-coregular. □

Suppose that $V_{G'} = V_1 \oplus V_2 \oplus \cdots \oplus V_s$ for irreducible representations (V_i,G) and regard f_i and g_j are $\mathbb{Z}_0^s$-homogeneous elements in $\mathcal{O}(V_{G'}) \cong \mathcal{O}(V_1) \otimes_{\mathbb{C}} \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathcal{O}(V_s)$ in a natural way. Let I_{BU} denotes the index set consisting of all i's such that $\mathcal{O}(V_i)^{G'}$, which are regarded as $\mathbb{C}$-subalgebras in $\mathcal{O}(V)^{G'}$, contain the $(V//G', G/G')$-BU elements of codimension one which are members of a minimal system of homogeneous generators in $\mathcal{O}(V)^{G'}$.

Proposition 3.11. *Suppose that (3.7) holds and G' is a simple algebraic group of type* F_4. *If I_{BU} is non-empty, then $(\oplus_{i\in I_{BU}} V_i, G')$ is irreducible and its simply connected covering can be identified with* (Φ_1, F_4).

Proof. For $i \in I_{BU}$, we have $\dim(V_i//G') = 1$, and by this equality and [KPV], see that the simply connected covering of (V_i, G') is equivalent to (Φ_1, F_4). We see that the isomorphism

$$\mathcal{O}(\bigoplus_{i\in I_{BU}} V_i)^{G'} \cong \bigotimes_{i\in I_{BU}} \mathcal{O}(V_i)^{G'}$$

in Corollary 2.9 implies $\sharp(I_{BU}) = 1$. □

Consequently, Theorem 3.5 is the consequence of Theorem 3.4, the next result and [S1].

Proposition 3.12. *Under the same assumption as in Theorem 3.5, we suppose that (3.7) holds and $m \geq 1$. Then $m = 1$ and* $(V_{G'}, \mathrm{F}_4) \subseteq (2 \cdot \Phi_1, \mathrm{F}_4)$.

Proof. By Proposition 3.11, we must have $((V_1,G),G') = (\Phi_1, \mathrm{F}_4)$ and $m = 1$. Let $G' \cdot x$ be a principal closed orbit of (Φ_1, F_2). Then by [P2] and [S1], we see that G'_x is connected and can identify G'_x with D_4 (cf. [EL]), which satisfies

$$((\Phi_1, \mathrm{F}_4), \mathrm{D}_4) = (\Phi_1 \oplus \Phi_3 \oplus \Phi_4 \oplus 2\theta_1, \mathrm{D}_4).$$

Suppose that $(V_{G'}/V_1, G')$ contains $(\Phi_2 = \mathrm{Ad}, \mathrm{F}_4)$. Clearly $((\mathrm{Ad}, \mathrm{F}_4), \mathrm{D}_4) \supseteq (\mathrm{Ad}, \mathrm{D}_4)$ and by Chevalley's restriction theorem, we have

$$\dim(((\mathrm{Ad}, \mathrm{F}_4), \mathrm{D}_4)/(\mathrm{Ad}, \mathrm{D}_4))^{\mathrm{D}_4} \leq \dim(\mathrm{Ad}, \mathrm{F}_4)^{T_4} - \dim(\mathrm{Ad}, \mathrm{D}_4)^{T_4} = 0,$$

where T_4 denotes the maximal torus of $\mathbf{F}_4$ in $\mathbf{D}_4$. Since $((\mathrm{Ad},\mathbf{F}_4),\mathbf{D}_4)$ must be coregular (cf. Corollary 3.10) and $(\mathrm{Ad}\oplus\Phi_1,\mathbf{D}_4)$ is maximal coregular (cf. [S1]), we deduce

$$((\mathrm{Ad},\mathbf{F}_4),\mathbf{D}_4)_{\mathbf{D}_4}\subsetneq(\mathrm{Ad}\oplus\Phi_1,\mathbf{D}_4).$$

This implies $\dim((\mathrm{Ad},\mathbf{D}_4)_{\mathbf{D}_4})\leq 36$, which conflicts with $\dim\mathbf{F}_4=52$ and $\dim((\mathrm{Ad},\mathbf{F}_4)^{\mathbf{F}_4})=0$.

Since $((\Phi_1\cdot\psi,\mathbf{F}_4),\mathbf{D}_4)$ contains the non-coregular representation $(\Phi_1\cdot\delta\oplus\Phi_3\cdot\delta\oplus\Phi_4\cdot\delta,\mathbf{D}_4)$ for a nontrivial irreducible subrepresentation $(\delta,\mathbf{D}_4)$ of $((\psi,\mathbf{F}_4),\mathbf{D}_4)$, we see that

$$(V_{G'}/V_1,\mathbf{F}_4)\not\supseteq(\Phi_1\cdot\psi,\mathbf{F}_4),$$

for any irreducible $(\psi,\mathbf{F}_4)$. Obviously $(\delta\cdot\mathrm{Ad},\mathbf{D}_4)$ is non-coregular for any irreducible $(\delta,\mathbf{D}_4)$, and hence

$$(V_{G'},\mathbf{F}_4)\not\supseteq(\Phi_2\cdot\psi,\mathbf{F}_4).$$

Consequently each irreducible component of $(V_{G'}/V_1,\mathbf{F}_4)$ is isomorphic to $(\Phi_1,\mathbf{F}_4)$, if it is non-trivial. On the other hand

$$((2\cdot\Phi_1,\mathbf{F}_4)=(2\cdot\Phi_1\oplus 2\cdot\Phi_3\oplus 2\cdot\Phi_4\oplus 6\theta_1,\mathbf{D}_4)$$

is not coregular, and we must see that $(V_{G'},G')$ is a subrepresentation of $(2\cdot\Phi_1,\mathbf{F}_4)$. □

Remark 3.13. As in the proof of Proposition 3.12, we can similarly show that Problem 3.1 is affirmative, under the assumption that G' is of exceptional type.

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環の表現論的性質とホモロジー代数的性質の関係について[1]

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Abstract

We will study relationship between representation theoretic conditions and homological conditions for rings, which is in the same direction of [**A1**][**ARo**]. We will treat some kind of Auslander conditions and τ-categories which are additive categories closely related to translation quivers. As an application, we will obtain characterizations of finite Auslander-Reiten quivers.

0 導入 加法圏 $\mathcal{C}$ が **Krull-Schmidt** であるとは、任意の $\mathcal{C}$ の object が、自己準同型環が local である様な object の有限直和に同型である事を意味する。$\mathcal{C}$ の object M が加法生成元であるとは、任意の $\mathcal{C}$ の object がある M^n の直和因子に同型である事を意味する。環 Λ に対し、有限生成射影 Λ-加群の圏を $\mathrm{pr}\,\Lambda$ と表わす時、次の一対一対応が得られる。

$$\begin{array}{ccccc} \mathcal{C} & \in & (\text{加法生成元 } M \text{ を持つ Krull-Schmidt 圏})/\text{同値} & \ni & \mathrm{pr}\,\Lambda \\ \downarrow & & & & \uparrow \\ \mathcal{C}(M,M) & \in & (\text{semiperfect 環})/\text{森田同値} & \ni & \Lambda \end{array}$$

本文の目的を一言で述べると、この対応における以下の条件の関係を調べる事にあり、[**ARo**][**A1**] 等の思想の延長にある。主定理は §6 で与えられる。

(1) $\mathcal{C}$ に関する表現論的条件。
(2) Λ に関するホモロジー代数的条件。
(3) $\mathcal{C}$ に関する圏論的条件。
(4) $\mathrm{A}(\mathcal{C})$ に関する組み合わせ的条件。

(1) では、§5 で定義する様な order 上の lattice の圏 ([**A2**]) としての意味付けを扱う。(2) では、§1 で定義する様な Λ の自己入射分解に関する性質 ([**AR**][**FGR**]) を扱う。(3) では、主に simple $\mathcal{C}$-加群の最小射影分解に関する条件を扱う。特に §3 で定義する τ-圏 ([**I3**]) と呼ばれる類を扱う。(4) における $\mathrm{A}(\mathcal{C})$ は、τ-圏 $\mathcal{C}$ に対して定まる不変量 **AR quiver** (AR=Auslander-Reiten) を表わし、§4 で定義が与えられる。それは同じく §4 で定義される、**translation quiver** と呼ばれる有向グラフの構造を持つ、純組み合わせ論的な対象である。

0.1 τ-圏を導入する動機を簡単に述べる。§5.2 で見る様に、R-order Λ が isolated singularity で $\dim R \leq 2$ の時 $\mathrm{lat}\,\Lambda$ は τ-圏であるが、真の動機は別にある。Igusa-Todorov は、"translation quiver の、斜体と両側加群による代数的実現" である **τ-species** ([**IT2**] では modulated translation quiver) を導入した。各 τ-species Q に対しては、Q の **mesh** 圏と呼ばれる τ-圏 $\hat{\mathbb{M}}(Q)$ が定義され、それは表現論において [**R**][**BG**][**IT2**] 等で良く知られたものである。§8 において、τ-species とその mesh 圏を簡単に復習するが、§8.5 は τ-圏が mesh 圏を研究するために自然な土台である事を意味しており、その事が τ-圏を導入する真の動機である。

0.2 完備離散付値環上の有限表現型 order に関しては、多くの結果が知られており、例えば幾つかの class ついては、完全な分類が成されている ([**DK2**][**HN3**][**RR**][**ZK**][**S**] 等)。しかし、全く制限を付けずに分類を与える事は非常に難しいと思われる。本文では、§6.3, §6.4 において AR quiver を用いた approach を試みるが、その観点は分類論とは少

[1] The detailed version of this paper will be submitted for publication elsewhere.

し異なる。即ち、§6.3 において次の問題 (P_1) に対する解答を与えるが [I3]、それはホモロジー代数的条件と表現論的条件を比較する事によって成される。

(P_d) d 次元完備正則局所環上の order Λ の AR quiver $\mathbb{A}(\mathrm{lat}\,\Lambda)$ として実現される様な有限 translation quiver を組み合わせ的に特徴付けよ。

(P_0) 及び (P_2) の解はそれぞれ [IT2] 及び [RV] で与えられ、(P_1) に対する部分的な結果も [W] で与えられているが、§6.4 で見る様に、一般の (P_1) では他の場合には生じない本質的な困難に突き当たる。それは §3, §7, §8 で得られた τ-圏（特に mesh 圏）に関する結果を用いて克服される。

一方、代数的閉体上の有限次代数の理論においては、代数群的発想から生じた Gabriel quiver、即ち環を "quiver with relations" と捉える手法が多く用いられており、それは hereditary 代数に対して最も有効に働く。残念ながら完備離散付値環上の order は、多くの重要な場合に "quiver with relations" としては捉え難く、Gabriel quiver の手法はあまり効果的に機能しない様に思われるため、Gabriel quiver とは（quiver という点は同じではあるが）思想から異なる AR quiver により重要性が感じられる。また有限表現型有限次代数に対し、Gabriel quiver による approach は [BGRS] まででほぼ尽されている様に思われるが、[IT2] を押し進めた AR quiver の観点から approach する事は Hall 代数 ([Ri]) 等と関係した非常に興味深いテーマである様に思われる。本文では (P_0) に関係した結果を §6.1 で述べる。

0.3 §5 で見る様に、order に対しては overorder なる概念が存在する。order Λ の overorder Γ に対し、$\mathrm{lat}\,\Gamma$ は $\mathrm{lat}\,\Lambda$ の部分圏になるが、この二つを比較する事は order の表現論において基本的な手法の一つである。この様な overorder から来る $\mathrm{lat}\,\Lambda$ の部分圏は圏論的に定式化する事ができ、rejected 部分圏と呼ばれる (§3.1,§5.3(2))。それらは §7 において AR quiver における組み合わせ的意味付けを含めてより深く調べられる。特に、mesh 圏に対する rejected 部分圏は、§6.3 の証明において本質的な役割を果たす。また多くの場合、§6.3 と §7 を組み合わせる事により、$\mathbb{A}(\mathrm{lat}\,\Lambda)$ から Λ を回復する事ができ (§6.5)、その定式化は重要な問題であると思われる。

§7 では coartinian (§2) な rejected 部分圏のみに限っているが、そうでない場合は [I2] で扱われ、その結果には translation quiver 上の additive function が現れ、[B][RV][RVo] の結果とも深く関係している事を注意しておく。

0.4 本文の主な結果においては、上記 (2) において $\mathrm{gl.dim}\,\Lambda \leq 2$ という条件が付随している。これをより高い大域次元の場合で考察する事は非常に興味深い問題であると思われる。例えば、Cohen-Macaulay approximation や有限表現型高 (Krull) 次元 order は何がしかのきっかけを与えてくれるのでは無いかと考えている。

0.5 記号 集合 Q に対し、Q で生成される自由 $\mathbb{Z}$-加群 (resp. 自由 abelian monoid) を $\mathbb{Z}Q$ (resp. $\mathbb{N}Q$) 表し、Q を正規直交基底とする $\mathbb{Z}Q$ 上の内積を $(\ ,\ )$ で表す。一方、$X \in \mathbb{Z}Q$ に対し $\mathrm{supp}\,X := \{Y \in Q \mid (X, Y) \neq 0\}$ とおき、$X_+, X_- \in \mathbb{N}Q$ を $X = X_+ - X_-$ 及び $\mathrm{supp}\,X_+ \cap \mathrm{supp}\,X_- = \emptyset$ で定義する。例えば、Krull-Schmidt 圏 $\mathcal{C}$ における object の同型類は $\mathbb{N}\mathrm{J}(\mathcal{C})$ で与えられる。また、環 Λ の Jacobson radical を J_Λ と表す。

1 ホモロジー代数的条件

Λ を noether 環、$0 \to \Lambda \to I^0 \to I^1 \to \cdots$ を Λ-加群 Λ の最小入射分解とする。Λ が (l, n)-**条件** $(l, n \geq 0)$ を満たすとは、$\mathrm{fd}\, I^i < l$ が任意の i $(0 \leq i < n)$ で成立する事 [I3]。更に、Λ が k-**Gorenstein** $(k > 0)$ であるとは、(l, l)-条件が任意の l $(0 < l \leq k)$

で満たされる事を意味し、Λ が **Auslander regular** (resp. **Auslander-Gorenstein**) であるとは、$\mathrm{gl.dim}\,\Lambda < \infty$ (resp. $\mathrm{id}_{\Lambda}\Lambda < \infty$ and $\mathrm{id}\,\Lambda_{\Lambda} < \infty$) かつ任意の $k > 0$ に対して k-Gorenstein である事を意味する [**AR**][**FRG**]。

1.1 注意 (1) Λ が (l, n)-条件を満たす必要十分条件は、任意の Λ^{op}-加群 X 及び $\mathrm{Ext}^l_{\Lambda}(X, \Lambda)$ の任意の部分 Λ-加群 Y に対し、$\mathrm{Ext}^i_{\Lambda}(Y, \Lambda) = 0$ が $0 \leq i < n$ で成立する事である。

(2) (l, n)-条件自体は左右対称ではないが、次が成立する。(ii) は、k-Gorenstein 環の左右対称性 [**FGR**] 及び dominant 次元の左右対称性 [**H**] の一般化を与える。

1.1.1 定理 Λ を noether 環とする。

(i) $l, n > 0$ に対し、Λ が (l, n)-条件を満たし、Λ^{op} が $(l.l)$-条件を満たせば、Λ^{op} も (l, n)-条件を満たす。

(ii) 整数列 $0 = a_0 \leq a_1 \leq \cdots \leq a_n$ で $a_i \leq i$ $(0 \leq i \leq n)$ を満たすものに対し、次の条件 $(*)$ を Λ が満たす事と Λ^{op} が満たす事は同値。

$(*)$ $0 \to \Lambda \to I^0 \to I^1 \to \cdots$ を Λ-加群 Λ の最小入射分解とする時、$\mathrm{fd}\, I^i \leq a_i$ が $0 \leq i \leq n$ で成立。

2 圏上の加群に関する基本事項

以下 $\mathcal{C}$ は Krull-Schmidt 圏 で skeletally small、即ち object の同型類全体が集合をなすと仮定し、$\daleth(\mathcal{C})$ で直規約 object の同型類のなす集合を表す事にする。$\mathcal{C}(X, Y)$ で X から Y への morphism の成す集合を表し、$fg \in \mathcal{C}(X, Z)$ で $f \in \mathcal{C}(X, Y)$ と $g \in \mathcal{C}(Y, Z)$ の合成を表す。

$\mathcal{C}$-**加群** とは、$\mathcal{C}$ から abelian group の圏 $\mathcal{A}\mathrm{b}$ への contravariant 加法関手を意味する。$\mathcal{C}$-加群 M と M' に対し、$\mathrm{Hom}(M, M')$ で M から M' への natural transformation の成す集合を表す事により、$\mathcal{C}$-加群の圏 $\mathrm{Mod}\,\mathcal{C}$ を得る [**A1**]。

関手 $H^{\mathcal{C}} : \mathcal{C} \to \mathrm{Mod}\,\mathcal{C}$ 及び $H_{\mathcal{C}} : \mathcal{C} \to \mathrm{Mod}\,\mathcal{C}^{op}$ を $H^{\mathcal{C}}_X := \mathcal{C}(\ , X)$ 及び $H^X_{\mathcal{C}} := \mathcal{C}(X, \ )$ で定義する。$\mathcal{C}$-加群 M が有限生成であるとは、ある $X \in \mathcal{C}$ に対し完全列 $H^{\mathcal{C}}_X \to M \to 0$ が存在する事を意味する。この時 $H^{\mathcal{C}}$ (resp. $H_{\mathcal{C}}$) は、$\mathcal{C}$ から有限生成射影 $\mathcal{C}$-加群 (resp. $\mathcal{C}^{op}$-加群) の成す圏への同値を与える事が米田の補題により分かる。

以下、$\mathcal{J}_{\mathcal{C}}$ で $\mathcal{C}$ の **Jacobson radical**、即ち $\mathcal{C}$ の ideal で、任意の $X \in \mathcal{C}$ に対し $\mathcal{J}_{\mathcal{C}}(\ , X)$ (resp. $\mathcal{J}_{\mathcal{C}}(X, \ )$) が $H^{\mathcal{C}}_X$ (resp. $H^X_{\mathcal{C}}$) の **radical** (＝全ての極大部分加群の交わり) となる様なものである。この時、関手 $S^{\mathcal{C}} : \mathcal{C} \to \mathrm{Mod}\,\mathcal{C}$ 及び $S_{\mathcal{C}} : \mathcal{C} \to \mathrm{Mod}\,\mathcal{C}^{op}$ を $S^{\mathcal{C}}_X := H^{\mathcal{C}}_X / \mathcal{J}_{\mathcal{C}}(\ , X)$ 及び $S^X_{\mathcal{C}} := H^{\mathcal{C}}_X / \mathcal{J}_{\mathcal{C}}(X, \ )$ で定義する。すると $S^{\mathcal{C}}$ (resp. $S_{\mathcal{C}}$) は、$\daleth(\mathcal{C})$ から simple $\mathcal{C}$-加群 (resp. $\mathcal{C}^{op}$-加群) の同型類の成す集合への全単射を導く。以下簡単のため $H^{\mathcal{C}}_X$ (resp. $H^X_{\mathcal{C}}$, $S^{\mathcal{C}}_X$, $S^X_{\mathcal{C}}$) を H_X (resp. H^X, S_X, S^X) で表す。

$\mathrm{pd}\, M$ (resp. $\mathrm{fd}\, M$) で $\mathcal{C}$-加群 M の射影次元 (resp. 平坦次元) を表し、$n \geq 0$ に対して、$\daleth^+_n(\mathcal{C})$ (resp. $\daleth^-_n(\mathcal{C})$) で $\mathrm{pd}\, S_X \leq n$ (resp. $\mathrm{pd}\, S^X \leq n$) なる $X \in \daleth(\mathcal{C})$ 全体を表す。

以下では、"部分圏" という時には、full でかつ、同型、直和及び直和因子に関して閉じているもののみを扱う事にする。ゆえに $\mathcal{C}$ の部分圏と $\daleth(\mathcal{C})$ の部分集合が $\mathcal{C}' \mapsto \daleth(\mathcal{C}')$ により一対一に対応し、その逆を $S \mapsto \mathrm{add}\, S$ で表わす事にする。

$\mathcal{C}$ の部分圏 $\mathcal{C}'$ に対し、$I_{\mathcal{C}'}$ で $\mathcal{C}'$ の object を factor through するような morphism 全体からなる $\mathcal{C}$ の ideal を表わす事にする。この時、quotient category $\overline{\mathcal{C}} := \mathcal{C} / \mathcal{C}'$ を $Ob(\overline{\mathcal{C}}) := Ob(\mathcal{C})$ 及び $\overline{\mathcal{C}}(X, Y) := \mathcal{C}(X, Y)/I_{\mathcal{C}'}(X, Y)$ $(X, Y \in \mathcal{C})$ で定義する。

$\mathcal{C}$ が **left artinian** (resp. **right artinian**) であるとは、任意の $X \in \mathcal{C}$ に対し H_X の長

さが有限である事。$\mathcal{C}$ が artinian であるとは、$\mathcal{C}$ が左右ともに artinian である事。$\mathcal{C}/\mathcal{C}'$ が artinian の時、$\mathcal{C}'$ を $\mathcal{C}$ の **coartinian 部分圏**と呼ぶ。$\mathfrak{I}(\mathcal{C})$ の部分集合 S が **artinian** であるとは、$\mathcal{C}' := \mathrm{add}(\mathfrak{I}(\mathcal{C}) - S)$ が $\mathcal{C}$ の coartinian 部分圏である事を意味する。

3 圏論的条件

3.1 $\mathcal{C}$ の部分圏 $\mathcal{C}'$ が **rejected** であるとは、次が成立する事である (cf. 5.6)。
inclusion $\mathcal{C}' \to \mathcal{C}$ は right adjoint $(\ )^- : \mathcal{C} \to \mathcal{C}'$ 及び left adjoint $(\ )^+ : \mathcal{C} \to \mathcal{C}'$ を持ち、$(\ )^-$ に対応する counit を ϵ^- とすると、任意の $X \in \mathcal{C}$ に対して ϵ_X^- は monomorphism であり、$(\ )^+$ に対応する unit を ϵ^+ とすると、任意の $X \in \mathcal{C}$ に対して ϵ_X^+ は epimorphism。
この時、$\mathfrak{I}(\mathcal{C})$ の部分集合 $\mathfrak{I}(\mathcal{C}) - \mathfrak{I}(\mathcal{C}')$ は **rejectable** と呼ばれる。

3.2 (1) Krull-Schmidt 圏 $\mathcal{C}$ が **τ-圏** であるとは、任意の $X \in \mathcal{C}$ に対し、complex $(X] = (\tau^+X \overset{\nu_X^+}{\to} \theta^+X \overset{\mu_X^+}{\to} X)$ 及び $[X) = (X \overset{\mu_X^-}{\to} \theta^-X \overset{\nu_X^-}{\to} \tau^-X)$ が存在して、$H_{\tau^+X} \to H_{\theta^+X} \to H_X \to S_X \to 0$ 及び $H^{\tau^-X} \to H^{\theta^-X} \to H^X \to S^X \to 0$ は最小射影分解を与え、$H^{\theta^+X} \to H^{\tau^+X} \to S^{\tau^+X} \to 0$ 及び $H_{\theta^-X} \to H_{\tau^-X} \to S_{\tau^-X} \to 0$ は完全となる事である。更に、任意の $X \in \mathcal{C}$ に対して $\mathrm{pd}\, S_X \leq 2$ と $\mathrm{pd}\, S^X \leq 2$ が成立する時、$\mathcal{C}$ は **regular τ-圏**と呼ばれる。便宜上 $(X]$ 及び $[X)$ は complex の同型類全体を表すものとし、より詳しく $(X]_{\mathcal{C}}$ 及び $[X)_{\mathcal{C}}$ と表す事もある。
(2) 定義より直ちに、§0 の対応において、$\mathcal{C}$ が regular τ-圏である事と、$\mathrm{gl.dim}\,\Lambda \leq 2$ かつ $\mathrm{Ext}_\Lambda^2(\Lambda/J_\Lambda, \Lambda)$ が semisimple である事は同値である。
(3) τ-圏 $\mathcal{C}$ においては、任意の $X \in \mathfrak{I}(\mathcal{C}) - \mathfrak{I}_1^+(\mathcal{C})$ (resp. $X \in \mathfrak{I}(\mathcal{C}) - \mathfrak{I}_1^-(\mathcal{C})$) に対し、$(X] = [\tau^+X)$ (resp. $[X) = (\tau^-X]$) が成立する事が示される。特に τ^+ は $\mathfrak{I}(\mathcal{C}) - \mathfrak{I}_1^+(\mathcal{C})$ と $\mathfrak{I}(\mathcal{C}) - \mathfrak{I}_1^-(\mathcal{C})$ との間の全単射を与え、τ^- はその逆を与える。
(4) $\mathcal{C}'$ を τ-圏 $\mathcal{C}$ の部分圏とする。この時、$\overline{\mathcal{C}} := \mathcal{C}/\mathcal{C}'$ も τ-圏であることが容易に分かる。更に、$X \in \mathfrak{I}(\mathcal{C}) - \mathfrak{I}(\mathcal{C}')$ に対し、$\overline{\theta^+X} \neq 0$ ならば $(\overline{X}]_{\overline{\mathcal{C}}} = \overline{(X]_{\mathcal{C}}}$ であり、$\overline{\theta^+X} = 0$ ならば $(\overline{X}]_{\overline{\mathcal{C}}} = (0 \to 0 \to \overline{X})$ である。

3.3 定義 $\mathcal{C}$ を、加法生成元を持つ τ-圏で $\bigcap_{i \geq 0} \mathcal{J}_{\mathcal{C}}^i = 0$ なるものとする。
(1) $A, B \in \mathfrak{I}(\mathcal{C})$ 及び $n \geq 0$ に対し、(A, B) が距離 n の **Nakayama pair** であるとは、次の可換図式が存在して、$a_0 = \mu_A^-$, $a_n = \mu_B^+$ 及び $(Y_{i-1}] = [X_i) = (X_i \xrightarrow{(-a_i\ g_i)} Y_i \oplus X_{i-1} \xrightarrow{\binom{f_i}{a_{i-1}}} Y_{i-1})$ が任意の i $(1 \leq i \leq n)$ に対して成立する事である。

$$
\begin{array}{ccccccccc}
Y_0 & \xleftarrow{f_1} & Y_1 & \xleftarrow{f_2} \cdots \xleftarrow{f_{n-1}} & Y_{n-1} & \xleftarrow{f_n} & B = Y_n \\
\uparrow{\scriptstyle a_0} & & \uparrow{\scriptstyle a_1} & \cdots & \uparrow{\scriptstyle a_{n-1}} & & \uparrow{\scriptstyle a_n} \\
A = X_0 & \xleftarrow{g_1} & X_1 & \xleftarrow{g_2} \cdots \xleftarrow{g_{n-1}} & X_{n-1} & \xleftarrow{g_n} & X_n
\end{array}
$$

この時、$B = n^-(A)$ (resp. $A = n^+(B)$) と表す事にすると、$n^-(A)$ (resp. $n^+(B)$) は存在すれば一意的である事が分かる。更に、$s^-(A) := \bigcup_{i=0}^n \mathrm{supp}\, X_i$ 及び $s^+(B) := \bigcup_{i=0}^n \mathrm{supp}\, Y_i$ とおく。
(2) τ-圏 $\mathcal{C}$ が **orderlike** であるとは、n^- が全単射 $\mathfrak{I}_1^-(\mathcal{C}) \to \mathfrak{I}_1^+(\mathcal{C})$ を導き、かつ $\mathfrak{I}(\mathcal{C}) = \bigcup_{A \in \mathfrak{I}_1^-(\mathcal{C})} s^-(A)$ が成立する事である。
"Nakayama pair" 及び "orderlike" の名称の理由は §5.3(2) において明らかになる。

3.4 次の定理は τ-圏において基本的であり [I3][IT1]、例えば $\mathcal{C}$ の圏論的性質を $\mathbb{A}(\mathcal{C})$ (§4) の組み合わせ的性質に翻訳するのに用いられる。

定理 $\mathcal{C}$ を τ-圏、$a \in \mathcal{J}_{\mathcal{C}}(X_0, Y_0)$, $L := \operatorname{Cok} H_a$ とし、任意の $f \in \mathcal{J}_{\mathcal{C}}^2(X,Y)$ に対して $a+f$ が a に complex として同型であると仮定する。この時以下が成立する。

(1) a の **ladder** と呼ばれる、次の可換図式と $U_i \in \mathcal{C}$ 及び $h_i \in \mathcal{C}(U_i, Z_{i-1})$ が存在して、$a = \binom{b_0}{0} \in \mathcal{C}(Z_0 \oplus U_0, Y_0)$ 及び $(Y_{i-1}] = (Z_i \oplus U_i \xrightarrow{\begin{pmatrix} b_i & -g_i \\ 0 & h_i \end{pmatrix}} Y_i \oplus Z_{i-1} \xrightarrow{\begin{pmatrix} f_i \\ b_{i-1} \end{pmatrix}} Y_{i-1})$ が任意の i に対して成立する。

$$\begin{array}{ccccccccc}
Y_0 & \xleftarrow{f_1} & Y_1 & \xleftarrow{f_2} & Y_2 & \xleftarrow{f_3} & Y_3 & \xleftarrow{f_4} & \cdots \\
\uparrow{\scriptstyle b_0} & & \uparrow{\scriptstyle b_1} & & \uparrow{\scriptstyle b_2} & & \uparrow{\scriptstyle b_3} & & \cdots \\
Z_0 & \xleftarrow{g_1} & Z_1 & \xleftarrow{g_2} & Z_2 & \xleftarrow{g_3} & Z_3 & \xleftarrow{g_4} & \cdots
\end{array}$$

(2) (1) より導かれる次の可換図式において、各列は $\mathcal{J}_{\mathcal{C}}^n L$ の最小射影分解を与え、$\mathcal{J}_{\mathcal{C}}^{n-1} L \to \mathcal{J}_{\mathcal{C}}^n L$ は自然な inclusion である。

$$\begin{array}{ccccccccc}
L & \longleftarrow & \mathcal{J}_{\mathcal{C}} L & \longleftarrow & \mathcal{J}_{\mathcal{C}}^2 L & \longleftarrow & \mathcal{J}_{\mathcal{C}}^3 L & \longleftarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \cdots \\
H_{Y_0} & \xleftarrow{H_{f_1}} & H_{Y_1} & \xleftarrow{H_{f_2}} & H_{Y_2} & \xleftarrow{H_{f_3}} & H_{Y_3} & \xleftarrow{H_{f_4}} & \cdots \\
\uparrow{\scriptstyle H_{b_0}} & & \uparrow{\scriptstyle H_{b_1}} & & \uparrow{\scriptstyle H_{b_2}} & & \uparrow{\scriptstyle H_{b_3}} & & \cdots \\
H_{Z_0} & \xleftarrow{H_{g_1}} & H_{Z_1} & \xleftarrow{H_{g_2}} & H_{Z_2} & \xleftarrow{H_{g_3}} & H_{Z_3} & \xleftarrow{H_{g_4}} & \cdots
\end{array}$$

(3) L が semisimple ならば、各 Y_i 及び Z_i は、$Z_0 = X_0 - (\theta^+ Y_0 - X_0)_-$ 及び次の **recursion fomula** より得られる (§0.5)。

$$Y_i = \theta^+ Y_{i-1} - Z_{i-1}, \quad Z_i = \tau^+ Y_{i-1} - (\theta^+ Y_i - \tau^+ Y_{i-1})_- \quad (i > 0)$$

特に以下の系が得られる。

(4) $\mathcal{C}$ が left artinian である必要十分条件は、任意の $X \in \mathfrak{I}(\mathcal{C})$ に対して $Y_0 := 0$, $Y_1 := X$ 及び $Y_i := (\theta^+ Y_{i-1} - \tau^+ Y_{i-2})_+$ $(i \geq 2)$ とおく時、十分大きな i に対して $Y_i = 0$ が成立する事。

(5) $\bigcap_{i \geq 0} \mathcal{J}_{\mathcal{C}}^i = 0$ の仮定の元、$\mathcal{C}$ が regular τ-圏である必要十分条件は、任意の $X \in \mathfrak{I}(\mathcal{C})$ に対して $Y_0 := 0$, $Y_1 := X$ 及び $Y_i := \theta^+ Y_{i-1} - \tau^+ Y_{i-2}$ $(i \geq 2)$ とおく時、任意の $i \geq 0$ に対して $Y_i \geq 0$ が成立する事。

4 組み合わせ的対象

(1) $Q = (Q, Q^p, Q^i, \tau^+, d, d')$ が **translation quiver** であるとは、Q^p と Q^i は集合 Q の部分集合で、τ^+ は全単射 $Q - Q^p \to Q - Q^i$ であり、d と d' は写像 $Q \times Q \to \mathbb{N}_{\geq 0}$ で $d(Y,X) = d'(\tau^+ X, Y)$ が任意の $X \in Q - Q^p$ と $Y \in Q$ に対して成立する事を意味する。

通常、Q は以下のように有向グラフとして表される。Q を点の集合とし、任意の $X, Y \in Q$ で $d(X,Y) \neq 0$ を満たすものに対し、valued arrows $X \xrightarrow{(d(X,Y), d'(X,Y))} Y$ を描き、任意の $X \in Q - Q^p$ に対して X から $\tau^+ X$ への dotted arrow を描く。valuation $(1,1)$ は省略する。

更に、translation quiver Q が **admissible** であるとは、写像 $c: Q \to \mathbb{N}_{>0}$ が存在して、任意の $X, Y \in Q$ に対して $c(X)d(X,Y) = d'(X,Y)c(Y)$ が存在する事。

(2) τ-圏 $\mathcal{C}$ に対し、**AR quiver** と呼ばれる translation quiver $\mathbb{A}(\mathcal{C}) = (Q, Q^p, Q^i, \tau^+, d, d')$ を $Q := \mathfrak{I}(\mathcal{C})$, $Q^p := \mathfrak{I}_1^+(\mathcal{C})$, $Q^i := \mathfrak{I}_1^-(\mathcal{C})$, $d(X,Y) := \langle \theta^+ Y, X \rangle$ 及び $d'(X,Y) := \langle \theta^- X, Y \rangle$ で定義する (§0.5)。$\mathbb{A}(\mathcal{C})$ は $(X]$ と $[X)$ の各項を視覚的に表示する。

(3) translation quiver $Q = (Q, Q^p, Q^i, \tau^+, d, d')$ に対し、$\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}Q)$ の元 θ^+, θ^-, τ^+ 及び τ^- を以下で定義する。$X \in Q$ に対し $\theta^+ X := \sum_{Y \in Q} d(Y,X)Y$, $\theta^- X := \sum_{Y \in Q} d'(X,Y)Y$、$X \in Q^p$ に対し $\tau^+ X := 0$、$X \in Q - Q^i$ に対し $\tau^- X := (\tau^+)^{-1}(X)$、$X \in Q^i$ に対し $\tau^- X := 0$ とおく。

Q の部分集合 S 及び $f \in \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}Q)$ に対し、$f_{Q/S} \in \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}(Q-S))$ を $f_{Q/S}(X) := f(X)|_{Q-S}$ で定義する。但し $|_{Q-S} : \mathbb{Z}Q \to \mathbb{Z}(Q-S)$ は natural projection を表す。

5 表現論的対象

以下、R を完備正則局所環、$d := \dim R \geq 0$、K を商体とする。R-代数 Λ が **R-order** であるとは、R-加群として有限生成自由である事を意味する。以下、Λ を R-order とする。左 Λ-加群 L が **Λ-lattice** であるとは、R-加群として有限生成自由である事を意味する。Λ-lattice の圏を $\mathrm{lat}\,\Lambda$ で表す時、それが Krull-Schmidt 圏である事は良く知られている。一方、$(\ )^* = \mathrm{Hom}_R(\ ,R)$ は $\mathrm{lat}\,\Lambda$ と $\mathrm{lat}\,\Lambda^{op}$ の間の duality を与える。$\mathrm{rin}\,\Lambda := (\mathrm{pr}\,\Lambda^{op})^*$ を **relative injective** Λ-lattice の圏と呼ぶ。例えば $d = 0$ (R が体) の時は、R-order は有限次 R-代数と同じ意味であり、$\mathrm{lat}\,\Lambda$ は有限生成 Λ-加群の圏 $\mathrm{mod}\,\Lambda$ と一致する。

R-order Λ と R の商体 K に対して、$\tilde{\Lambda} := \Lambda \otimes_R K$ は有限次 K-代数である。R-order Λ が **isolated singularity** であるとは、$\mathrm{gl.dim}\,\Lambda \otimes_R R_{\mathfrak{p}} = \mathrm{ht}\,\mathfrak{p}$ が任意の R の non-maximal prime ideal $\mathfrak{p}$ に対して成立する事。重要な結果として、R-order Λ が isolated singularity である必要十分条件は、$\mathrm{lat}\,\Lambda$ が AR sequence を持つ事である事が知られている **[A2]**。これは次の定理の (2) を意味する。

5.1 定理 Λ を R-order とし、$d := \dim R$ とおく。

(1) 任意の $X \in \mathfrak{I}(\mathrm{pr}\,\Lambda)$ (resp. $X \in \mathfrak{I}(\mathrm{rin}\,\Lambda)$) に対して、$S_X$ (resp. S^X) は各項が有限生成である様な最小射影分解を持つ。$d > 0$ ならば $\mathrm{pd}\,S_X = d$ (resp. $\mathrm{pd}\,S^X = d$) が成立し、$d = 0$ ならば $\mathrm{pd}\,S_X \leq 1$ (resp. $\mathrm{pd}\,S^X \leq 1$) が成立する。

(2) Λ が isolated singularity である必要十分条件は、任意の $X \in \mathfrak{I}(\mathrm{lat}\,\Lambda) - \mathfrak{I}(\mathrm{pr}\,\Lambda)$ (resp. $X \in \mathfrak{I}(\mathrm{lat}\,\Lambda) - \mathfrak{I}(\mathrm{rin}\,\Lambda)$) に対して S_X (resp. S^X) は各項が有限生成である様な最小射影分解を持つ事である。この時 $\mathrm{pd}\,S_X = 2$ (resp. $\mathrm{pd}\,S^X = 2$) が成立する。

この定理から、容易に次の系が得られる。

5.2 系 R-order Λ が isolated singularity で $\dim R \leq 2$ ならば、$\mathrm{lat}\,\Lambda$ は regular τ-圏である。この時、各 $X \in \mathfrak{I}(\mathrm{lat}\,\Lambda) - \mathfrak{I}(\mathrm{pr}\,\Lambda)$ (resp. $X \in \mathfrak{I}(\mathrm{lat}\,\Lambda) - \mathfrak{I}(\mathrm{rin}\,\Lambda)$) に対して、$(X]$ (resp. $[X)$) は **AR sequence** と呼ばれる。

5.3 例 §5.2 の仮定の元、Λ の AR quiver $\mathbb{A}(\mathrm{lat}\,\Lambda)$ が §4(2) により定義される。

(1) $Q := \mathbb{A}(\mathrm{lat}\,\Lambda)$ とおく時、$d \leq 1$ ならば $Q^p = \mathfrak{I}(\mathrm{pr}\,\Lambda)$ 及び $Q^i = \mathfrak{I}(\mathrm{rin}\,\Lambda)$ であり、$d = 2$ ならば $Q^p = Q^i = \emptyset$ である。更に、$k := R/J_R$, $c(X) := \dim_k \mathrm{End}_\Lambda(X)/J_{\mathrm{End}_\Lambda(X)}$ とおく事により、Q は admissible である。

(2) $d = 1$ とし、$\mathcal{C} := \mathrm{lat}\,\Lambda$ とおく。Λ が有限表現型 R-order ならば、$\mathcal{C}$ は orderlike τ-圏であり、$n^+ : \mathfrak{I}_1^+(\mathcal{C}) \to \mathfrak{I}_1^-(\mathcal{C})$ は中山関手 $\mathrm{Hom}_\Lambda(\ ,\Lambda)^*$ により与えられる。更に、$A \in \mathfrak{I}_1^-(\mathcal{C})$

(resp. $B \in \daleth_1^+(\mathcal{C})$) 及び $X \in \daleth(\mathcal{C})$ に対し、$X \in s^-(A)$ (resp. $X \in s^+(B)$) が成立する必要十分条件は、X の injective hull $X \to I$ (resp. projective cover $P \to X$) が $A \le I$ (resp. $B \le P$) を満たす事である。

5.4 R-order Γ が Λ の **overorder** (resp. **overring**) であるとは、$\tilde{\Lambda} \supset \Gamma \supseteq \Lambda$ (resp. $\tilde{\Lambda}$ のある ideal I に対して $\tilde{\Lambda}/I \supset \Gamma \supseteq (\Lambda + I)/I$) が成立する事。これらの時、自然な写像 $\Lambda \to \Gamma$ は full faithful 関手 $\operatorname{lat}\Gamma \to \operatorname{lat}\Lambda$ を導き、$\operatorname{lat}\Gamma$ は $\operatorname{lat}\Lambda$ の部分圏とみなされる。次の命題により、(lat Λ の overring)、(lat Λ の rejected 部分圏) 及び ($\daleth(\operatorname{lat}\Lambda)$ の rejectable subset) が一対一に対応する事が分かる。

5.5 命題 Λ を R-order で $\dim R \le 2$ と仮定する。この時、$\operatorname{lat}\Lambda$ の部分圏 $\mathcal{C}'$ が rejected である必要十分条件は、ある overring Γ が存在して、$\mathcal{C}' = \operatorname{lat}\Gamma$ となる事である。この時、inclusion $\operatorname{lat}\Gamma \to \operatorname{lat}\Lambda$ の right adjoint は $(\ )^- := \operatorname{Hom}_\Lambda(\Gamma,\ )$、left adjoint は $(\ )^+ := (\Gamma \otimes_\Lambda\ )^{**}$ で与えられる。

5.6 $d = 0$、Λ を有限次 R-代数とする。$\operatorname{mod}\Lambda$ の部分圏 $\mathcal{T}$, $\mathcal{F}$ に対し、$(\mathcal{T}, \mathcal{F})$ が **torsion theory** であるとは、inclusion $\mathcal{T} \to \operatorname{mod}\Lambda$ (resp. $\mathcal{F} \to \operatorname{mod}\Lambda$) は right adjoint $(\ )^- : \operatorname{mod}\Lambda \to \mathcal{T}$ (resp. left adjoint $(\ )^+ : \operatorname{mod}\Lambda \to \mathcal{F}$) を持ち、$0 \to X^- \to X \to X^+ \to 0$ が任意の $X \in \operatorname{mod}\Lambda$ に対し完全となる事 (cf. 3.1)。この時、$\mathcal{T}$ (resp. $\mathcal{F}$) が商加群 (resp. 部分加群) に関して閉じている事は容易に分かる。$(\mathcal{T}, \mathcal{F})$ が **hereditary torsion theory** であるとは、$\mathcal{T}$ が部分加群に関しても閉じている事を意味する。

6 主定理 R を完備正則局所環とする。

6.1 $\dim R = 0$ の時の主定理は次である。

定理 $\dim R = 0$、Γ を有限次 R-代数、$0 \to \Gamma \to I^0 \to I^1 \to \cdots$ を Γ-加群 Γ の最小入射分解とし、$\mathcal{C} := \operatorname{pr}\Gamma$ とおく。この時、各 i $(1 \le i \le 4)$ に対し次の条件 (1-i), (2-i), (3-i) 及び (4-i) は同値である。

(1-i)(表現論的条件)

(1-1): 有限次 R-代数 Λ 及び $\operatorname{mod}\Lambda$ 上の torsion theory $(\mathcal{T}, \mathcal{F})$ が存在して $\mathcal{C}$ は $\mathcal{F}$ に同値となる (§5.6)。

(1-2): (1-1) かつ $(\mathcal{T}, \mathcal{F})$ は hereditary torsion theory。

(1-3): (1-1) かつ $\mathcal{F}$ は socle が射影的である様な Λ-加群の成す圏。

(1-4): (1-1) かつ $\mathcal{F} = \operatorname{mod}\Lambda$。

(2-i)(ホモロジー代数的条件) $\operatorname{gl.dim}\Gamma \le 2$ でかつ

(2-1) Γ 及び Γ^{op} は $(2,2)$-条件を満たす (§1)。

(2-2) Γ は Auslander regular。

(2-3) Γ は Auslander regular かつ I^1 は 0 でない射影加群を直和因子に持たない。

(2-4) Γ は $(1,2)$-条件を満たす。(この時 Γ は **Auslander 代数**と呼ばれる。)

(3-i)(圏論的条件)

(3-1): $\mathcal{C}$ は regular τ-圏。

(3-2): (3-1) かつ n^- は写像 $\daleth_1^-(\mathcal{C}) - \daleth_0^-(\mathcal{C}) \to \daleth(\mathcal{C})$ を与える。

(3-3): (3-1) かつ n^- は写像 $\daleth_1^-(\mathcal{C}) - \daleth_0^-(\mathcal{C}) \to \daleth_1^+(\mathcal{C})$ を与える。

(3-4): (3-1) かつ n^- は写像 $\daleth_1^-(\mathcal{C}) - \daleth_0^-(\mathcal{C}) \to \daleth(\mathcal{C}) - \daleth_1^+(\mathcal{C})$ を与える。

(4-i)(組み合わせ的条件)

(4-1) $\mathcal{C}$ は τ-圏で §3.4(5) にある条件が満たされる。

(4-2): (4-1) かつ、任意の $X \in \mathfrak{I}_1^-(\mathcal{C}) - \mathfrak{I}_0^-(\mathcal{C})$ に対し、$Y_0 := \theta^- X$, $Y_1 := \theta^+\theta^- X - X$ 及び $Y_i := \theta^+ Y_{i-1} - \tau^+ Y_{i-2}$ $(i \geq 2)$ とおく時、ある $n \geq 0$ が存在して $Y_i \in \mathbb{N}(\mathfrak{I}(\mathcal{C}) - \mathfrak{I}_1^+(\mathcal{C}))$ $(0 \leq i < n)$ 及び $Y_{n+1} = 0$ が成立する。

(4-3): (4-2) において $Y_n \in \mathfrak{I}_1^+(\mathcal{C})$ が成立する。

(4-4): (4-2) において $Y_n \in \mathfrak{I}(\mathcal{C}) - \mathfrak{I}_1^+(\mathcal{C})$ が成立する。

特に、(1-4) と (4-4) の同値性と §3.4(4) を組みせるて、§0.2 の (P_0) の解を得る **[IT2]**。

6.2 $\dim R = 1$ の時の主定理は次である。ここで、R-order Λ が **Auslander order** であるとは、$\mathrm{gl.dim}\,\Lambda \leq 2$ かつ Λ^* の最小射影分解 $0 \to P_1 \to P_0 \to \Lambda^* \to 0$ が $P_0 \in \mathrm{rin}\,\Lambda$ を満たす事。但し、**[ARo]** と異なり、Λ が isolated singularity である事を仮定しない事に注意。

定理 $\dim R = 1$、Γ を R-order とし $\mathcal{C} := \mathrm{pr}\,\Gamma$ とおく。この時、次の (2)(2′)(3) は同値であり、もし Γ が isolated singularity ならば (1) も同値である。

(1)(表現論的条件) 有限表現型 R-order Λ が存在して、$\mathcal{C}$ は $\mathrm{lat}\,\Lambda$ に同値。

(2)(ホモロジー代数的条件) Γ は Auslander order。

(2′)(ホモロジー代数的条件) $\mathrm{gl.dim}\,\Gamma \leq 2$ かつ Γ は Auslander regular。

(3)(圏論的条件) $\mathcal{C}$ は τ-圏。

これは、§0.2 の (P_1) については何も主張していない。しかし (P_1) に解答を与える次の定理の証明において重要な役割を果たす。

6.3 定理 有限個の点より成る translation quiver Q に対し、次の (1)-(3) は同値。

(1)(表現論的条件) $\mathbb{A}(\mathrm{lat}\,\Lambda) = Q$ となる完備離散付値環 R 及び R-order Λ が存在する。

(2)(圏論的条件) $\mathbb{A}(\mathcal{C}) = Q$ を満たすある（全ての）τ-圏 $\mathcal{C}$ に対し、$\mathcal{C}$ は orderlike (§3.3) でありかつ $\mathcal{C} / \mathfrak{I}_1^+(\mathcal{C})$ は artinian。

(3)(組み合わせ的条件) Q は admissible であり次の条件が満たされる (§4(3))。

(i) 任意の $X \in Q^i$ に対し、$Y_0 := \theta^- X$, $Y_1 := \theta^+\theta^- X - X$, $Y_i := \theta^+ Y_{i-1} - \tau^+ Y_{i-2}$ $(i \geq 2)$ 及び $t(X) := \bigcup_{i \geq 0} \mathrm{supp}\, Y_i$ とおく。この時ある $n \geq 0$ が存在して $Y_i \in \mathbb{N}(Q - Q^p)$ $(0 \leq i < n)$, $Y_n \in Q^p$ 及び $Y_{n+1} = 0$ が成立し、更に $Q = \bigcup_{X \in Q^i} t(X)$ となる。

(ii) 任意の $X \in Q^i$ に対し、$Y_0 := 0$, $Y_1 := X$ 及び $Y_i := (\theta^+_{Q/Q^p} Y_{i-1} - \tau^+_{Q/Q^p} Y_{i-2})_+$ $(i \geq 2)$ とおく時、十分大きな i に対して $Y_i = 0$ が成立。

6.4 定理 §6.3 の証明の概略 Krull-Schmidt 圏 $\mathcal{C}$ が **order 構造**を持つとは、完備離散付値環 R 及び R-order Γ が存在して、$\mathcal{C} = \mathrm{pr}\,\Gamma$ となる（即ち §0 の対応において $\mathcal{C}$ が Γ に対応する）事とする。(2)⇔(3) は §3.4 より従い、(1)⇒(2) は §5.3(2) より従うので、(2)⇒(1) のみ示せば良い。Q は admissible なので、τ-species $\mathcal{Q}$ で $|\mathcal{Q}| = Q$ を満たすものがとれる。この時ある order Λ が存在して、$\widehat{\mathbb{M}}(\mathcal{Q})$ が $\mathrm{lat}\,\Lambda$ に同値となる事を示さねばならない。§6.2 により、$\widehat{\mathbb{M}}(\mathcal{Q})$ が order 構造を持つ事のみいえば良いのだが、それが本質的に最も難しい部分であり、$\dim R = 1$ 特有の問題である。

6.4.1 §3.2 で見た様に、完備離散付値環上の order Λ が isolated singularity ならば $\mathrm{lat}\,\Lambda$ は τ-圏である。逆に次の補題は、τ-圏 $\mathcal{C}$ が order 構造を持ちかつ（対応する order が）isolated singularity ならば、ある完備離散付値環上の order Λ が存在して、$\mathcal{C}$ は $\mathrm{lat}\,\Lambda$

に同値になる事を示す [I3]。その証明は、§7 の結果の応用として、rejected 部分圏への reduction を用いてなされる。

補題 $\mathcal{C}$ を completely graded (§8.5) orderlike τ-圏、$k[[x]]$ を体 k 上の巾級数環とする。更に、$\mathcal{C}$ は k-圏であり、任意の $X \in \mathfrak{I}(\mathcal{C})$ に対して $\mathcal{C} / \mathcal{J}_{\mathcal{C}}(X,X)$ は k の有限次拡大であり、$\mathcal{C} / \mathfrak{I}_1^+(\mathcal{C})$ は artinian である事を仮定する。この時 $\mathcal{C}$ は $k[[x]]$-order 構造を持つ。

6.5 注意 §6.3 の応用として、§7 における一般 Rejection と組み合わせる事により、tree type が classical Dynkin diagram である様な有限表現型 Gorenstein order の完全な分類を与える事ができる。

7 一般 Rejection

この商では §3.1 の述語を用いる。$\mathfrak{I}(\mathrm{lat}\,\Lambda)$ の一点よりなる部分集合の rejectability に関しては、Drozd-Kirichencko による次の必要十分条件が知られており [DK1]、それは Bass order の理論の基礎となる重要な補題である [DKR][HN1][HN2]。

7.1 (DK Rejection Lemma) Λ を R-order で $\dim R \leq 1$ とする。$X \in \mathfrak{I}(\mathrm{lat}\,\Lambda)$ に対し、$\{X\}$ が rejectable である必要十分条件は、$X \in \mathrm{pr}\,\Lambda \cap \mathrm{rin}\,\Lambda$ となる事である。

この章では $\dim R \leq 1$ の場合にこの結果を一般化し、§7.2において、$\mathfrak{I}(\mathrm{lat}\,\Lambda)$ の artinian subset (§2) の rejectability を、$\mathbb{A}(\mathrm{lat}\,\Lambda)$ の組み合わせ的な言葉を用いて表わす [I3]。その結果は、lattice の圏だけではなく、一般の τ-圏に対して成立し、そのおかげで §6.4.1 の証明を得る事が出来る。

ここで、rejectable set $S = \mathfrak{I}(\mathrm{lat}\,\Lambda) - \mathfrak{I}(\mathrm{lat}\,\Gamma)$ が artinian である必要十分条件は、$\#S < \infty$ かつ ($d = 1$ ならば Γ は Λ の overorder) である事を注意しておく。興味深い事実として、$\mathfrak{I}(\mathcal{C})$ の artinian subset S の rejectability は、$\mathbb{A}(\mathcal{C})$ を S 上に制限して得られる translation quiver 及び S の部分集合 $S \cap \mathfrak{I}_1^+(\mathcal{C})$ と $S \cap \mathfrak{I}_1^-(\mathcal{C})$ のみに依存して決まることが分かる。

7.2 定理 $\mathcal{C}$ を τ-圏, $\mathcal{C}'$ を $\mathcal{C}$ の coartinian 部分圏とし、$\overline{\mathcal{C}} := \mathcal{C} / \mathcal{C}'$ とおく。この時以下の条件は同値。

(表現論的条件) $\mathcal{C}'$ は $\mathcal{C}$ の rejected 部分圏。

(圏論的条件) 任意の $X \in \mathfrak{I}(\mathcal{C}) - \mathfrak{I}_1^-(\mathcal{C})$ に対し μ_X^- は $\overline{\mathcal{C}}$ において monomorphism であり、任意の $X \in \mathfrak{I}(\mathcal{C}) - \mathfrak{I}_1^+(\mathcal{C})$ に対し μ_X^+ は $\overline{\mathcal{C}}$ において epimorphism。

(組み合わせ的条件) 次の (i) 及び (ii) が成立する。

(i) 任意の $X \in \mathfrak{I}(\mathcal{C}) - \mathfrak{I}_1^-(\mathcal{C})$ に対し、$Y_0 := \theta_{\overline{\mathcal{C}}}^- X$, $Y_1 := \theta_{\overline{\mathcal{C}}}^+ \theta_{\overline{\mathcal{C}}}^- X - X$ 及び $Y_i := \theta_{\overline{\mathcal{C}}}^+ Y_{i-1} - \tau_{\overline{\mathcal{C}}}^+ Y_{i-2}$ ($i \geq 2$) とおく時、任意の $i \geq 0$ に対して $Y_i \geq 0$ が成立する。

(ii) 任意の $X \in \mathfrak{I}(\mathcal{C}) - \mathfrak{I}_1^+(\mathcal{C})$ に対し、$Y_0 := \theta_{\overline{\mathcal{C}}}^+ X$, $Y_1 := \theta_{\overline{\mathcal{C}}}^- \theta_{\overline{\mathcal{C}}}^+ X - X$ 及び $Y_i := \theta_{\overline{\mathcal{C}}}^- Y_{i-1} - \tau_{\overline{\mathcal{C}}}^- Y_{i-2}$ ($i \geq 2$) とおく時、任意の $i \geq 0$ に対して $Y_i \geq 0$ が成立する。

7.3 rejected 部分圏の他に、次の概念が役に立つ事が多い。$\mathcal{C}$ の部分圏 $\mathcal{C}'$ が **trivial** であるとは、$\mathcal{C}'$ を含む $\mathcal{C}$ の rejected 部分圏は $\mathcal{C}$ に限る事を意味する。

定理 [I1][I3] $\mathcal{C}$ を τ-圏, $\mathcal{C}'$ を $\mathcal{C}$ の coartinian 部分圏とし、$\overline{\mathcal{C}} := \mathcal{C} / \mathcal{C}'$ とおく。この時以下の条件は同値。

(表現論的条件) $\mathcal{C}'$ は $\mathcal{C}$ の trivial 部分圏。

(圏論的条件) 任意の $P \in \mathfrak{I}_1^+(\mathcal{C})$ 及び $I \in \mathfrak{I}_1^-(\mathcal{C})$ に対して $\overline{\mathcal{C}}(P, I) = 0$ が成立。

(組み合わせ的条件) 任意の $X \in \mathfrak{I}_1^-(\mathcal{C})$ に対し、$Y_0 := X$, $Y_1 := \theta_{\mathcal{C}}^+ X$ 及び $Y_i := (\theta_{\mathcal{C}}^+ Y_{i-1} - \tau_{\mathcal{C}}^+ Y_{i-2})_+$ $(i \geq 2)$ とおく時、任意の $i \geq 0$ に対して $Y_i|_{\mathfrak{I}_1^+(\mathcal{C})} = 0$ が成立する。

7.4 例 (1) $\mathfrak{I}(\mathcal{C})$ の rejectable subset S が **minimal** であるとは、S に含まれる $\mathfrak{I}(\mathcal{C})$ の rejectable subset は、S と空集合のみである事とする。この時、$S \cap \mathfrak{I}(\mathrm{pr}\,\Lambda)$ 及び $S \cap \mathfrak{I}(\mathrm{rin}\,\Lambda)$ は一点よりなる事が §7.3 より容易に分かる。

(2) Λ を R-order、$\dim R = 1$ とし、S を $\mathfrak{I}(\mathrm{lat}\,\Lambda)$ の部分集合で $\#S \leq 4$ なるものとする。この時 S が minimal artinian rejectable subset である必要十分条件は、S が次のいずれかの形をしている事である。ここで、$\{P\} := S \cap \mathfrak{I}(\mathrm{pr}\,\Lambda)$ 及び $\{I\} := S \cap \mathfrak{I}(\mathrm{rin}\,\Lambda)$ とおく。

(1)	$\overset{P=I}{\bullet}$	(DK Rejection)
(2)	$\overset{P}{\bullet}\longrightarrow\overset{I}{\bullet}$	
(3)	$\overset{P}{\bullet}\xrightarrow{(a\ b)}\bullet\longrightarrow\overset{I}{\bullet}$	$ab \leq 2$
(4)	$\overset{P}{\bullet}\longrightarrow\bullet\xrightarrow{(a\ b)}\overset{I}{\bullet}$	$ab \leq 2$
(5)	$\overset{P}{\bullet}\xrightarrow{(a\ b)}\bullet\xrightarrow{(b\ a)}\overset{I}{\bullet}$	$P = \tau^+ I, ab \leq 3$
(6)	$\overset{P}{\bullet}\xrightarrow{(a\ b)}\bullet\longrightarrow\bullet\longrightarrow\overset{I}{\bullet}$	$ab \leq 2$
(7)	$\overset{P}{\bullet}\longrightarrow\bullet\xrightarrow{(a\ b)}\bullet\longrightarrow\overset{I}{\bullet}$	$ab \leq 2$
(8)	$\overset{P}{\bullet}\longrightarrow\bullet\longrightarrow\bullet\xrightarrow{(a\ b)}\overset{I}{\bullet}$	$ab \leq 2$
(9)	$\overset{P}{\bullet}\xrightarrow{(a\ b)}\bullet\xrightarrow{(b\ a)}\overset{X}{\bullet}\longrightarrow\overset{I}{\bullet}$	$P = \tau^+ X, ab \leq 3$
(10)	$\overset{P}{\bullet}\longrightarrow\overset{X}{\bullet}\xrightarrow{(a\ b)}\bullet\xrightarrow{(b\ a)}\overset{I}{\bullet}$	$X = \tau^+ I, ab \leq 3$
(11)	$\begin{array}{ccccc} \overset{P}{\bullet} & \longrightarrow & \bullet & \longrightarrow & \overset{I}{\bullet} \\ & & \downarrow\uparrow & & \\ & & X\bullet & & \end{array}$	$P = \tau^+ I, X = \tau^+ X$
(12)	$\begin{array}{ccc} \bullet & \longrightarrow & \overset{I}{\bullet} \\ \uparrow & & \uparrow \\ P\bullet & \longrightarrow & \bullet \end{array}$	$P = \tau^+ I$

8 付録 τ-species とその mesh 圏について簡単に復習する [IT2][I3]。§0.5 の記号を用いる。

8.1 $\mathcal{Q} = (Q, D_X, {}_X M_Y)$ が **species** であるとは、Q は集合、各 $X \in Q$ に対して D_X は斜体であり、各 $X, Y \in Q$ に対して ${}_X M_Y$ は (D_X, D_Y)-両側加群で $\sum_{X \in Q} \dim_{D_X} {}_X M_Y < \infty$ 及び $\sum_{Y \in Q} \dim_{D_Y} {}_X M_Y < \infty$ を満たすものである事。

この時、異なる $X, Y \in Q$ に対して $P_0(X, Y) := 0$、$P_0(X, X) := D_X$、

$$P_n(X, Y) := \bigoplus_{Z_1, \cdots, Z_{n-1} \in Q} {}_X M_{Z_1} \otimes_{D_{Z_1}} \cdots \otimes_{D_{Z_{n-1}}} {}_{Z_{n-1}} M_Y \ (n > 0) \text{ 及び}$$
$$P_n(A, B) := \prod_{X, Y \in Q} \mathrm{Mat}_{\langle A, X \rangle, \langle B, Y \rangle}(P_n(X, Y)) \ (A, B \in \mathbb{N}Q, n \geq 0) \text{ とおく。}$$

自然な写像 $P_n(X,Y)\times P_m(Y,Z)\to P_{n+m}(X,Z)$, $(f,g)\mapsto fg:=f\otimes g$ $(X,Y,Z\in Q)$ から行列積を用いて自然な写像 $P_n(A,B)\times P_m(B,C)\to P_{n+m}(A,C)$ $(A,B,C\in \mathbb{N}Q)$ を得る。

圏 $\hat{\mathbb{P}}(Q)$ 及び $\mathbb{P}(Q)$ を、$Ob(\hat{\mathbb{P}}(Q))=Ob(\mathbb{P}(Q)):=\mathbb{N}Q$, $\hat{\mathbb{P}}(Q)(A,B):=\prod_{n\geq 0}P_n(A,B)$ 及び $\mathbb{P}(Q)(A,B):=\bigoplus_{n\geq 0}P_n(A,B)$ $(A,B\in\mathbb{N}Q)$ で定義する。合成は $(f_n)_{n\geq 0}\cdot(g_n)_{n\geq 0}:=(\sum_{i=0}^{n}f_ig_{n-i})_{n\geq 0}$ により定める。

8.2 命題 species Q に対し、$\hat{\mathbb{P}}(Q)$ は Krull-Schmidt 圏であり、Q の **tensor 圏**と呼ばれる。

8.3 $Q=(Q,Q^p,Q^i,D_X,{}_XM_Y,\tau^+,a,b)$ が **τ-species** であるとは、$(Q,D_X,{}_XM_Y)$ は species、Q^p 及び Q^i は Q の部分集合、τ^+ は全単射 $Q-Q^p\to Q-Q^i$ であり、任意の $X\in Q-Q^p$, $Y\in Q$ に対して $a_X:D_X\to D_{\tau^+X}$ は環同型かつ $b_{X,Y}:\mathrm{Hom}_{D_Y}({}_{\tau^+X}M_Y,D_Y)\to {}_YM_X$ は (D_Y,D_X)-加群の同型であることを意味する。ここで ${}_{\tau^+X}M_Y$ は a_X を通して D_X-加群と見なされている。

$d(X,Y):=\dim_{D_X}{}_XM_Y$ 及び $d'(X,Y):=\dim_{D_Y}{}_XM_Y$ とおく事により、Q の **underlying quiver** と呼ばれる translation quiver $|Q|:=(Q,Q^p,Q^i,\tau^+,d,d')$ を得る。

この時、$\mathrm{Hom}_{D_Y}(\mathrm{Hom}_{D_Y}({}_{\tau^+X}M_Y,D_Y),{}_YM_X)={}_{\tau^+X}M_Y\otimes_{D_Y}{}_YM_X$ であるので、$b_{X,Y}$ は ${}_{\tau^+X}M_Y\otimes_{D_Y}{}_YM_X$ の元と見なされる。$\gamma(X):=\sum_{Y\in Q}b_{X,Y}\in P_2(\tau^+X,X)$ とおくと、$b_{X,Y}$ は D_X^{op}-加群の準同型なので、任意の $f\in D_X$ に対して $\gamma(X)f=a_X(f)\gamma(X)$ が成立する。I を $\{\gamma(X)|X\in Q\}$ で生成される $\mathbb{P}(Q)$ の ideal とすると、$I=\bigoplus_{n\geq 0}I_n$ $(I_n\subseteq P_n)$ と表わされる。$\hat{\mathbb{P}}(Q)$ の ideal $\hat{I}:=\prod_{n\geq 0}I_n$ に対し、$\hat{\mathbb{M}}(Q):=\hat{\mathbb{P}}(Q)/\hat{I}$ とおく。

8.4 命題 τ-species Q に対し、$\hat{\mathbb{M}}(Q)$ は τ-圏であり、Q の **mesh 圏**と呼ばれる。

8.5 Krull-Schmidt 圏 $\mathcal{C}$ に対し、**associated completely graded 圏** $\hat{\mathbb{G}}(\mathcal{C})$ を $\hat{\mathbb{G}}(\mathcal{C}):=\prod_{i\geq 0}\mathcal{J}_{\mathcal{C}}^i/\mathcal{J}_{\mathcal{C}}^{i+1}$ で定義する。もし $\mathcal{C}$ が $\hat{\mathbb{G}}(\mathcal{C})$ に同値ならば、$\mathcal{C}$ は **completely graded** であると呼ばれる。

定義は略すが、各 τ-圏 $\mathcal{C}$ に対し、**AR species** と呼ばれる τ-species $\hat{\mathbb{A}}(\mathcal{C})$ を定義する事ができ、$|\hat{\mathbb{A}}(\mathcal{C})|=\mathbb{A}(\mathcal{C})$ が成立する。

定理 [I3] (1) $\mathcal{C}$ が τ-圏 ならば、$\hat{\mathbb{G}}(\mathcal{C})$ も τ-圏である。

(2) 任意の τ-species Q に対し $\hat{\mathbb{A}}(\hat{\mathbb{M}}(Q))=Q$ が成立し、任意の τ-圏 $\mathcal{C}$ に対し $\hat{\mathbb{M}}(\hat{\mathbb{A}}(\mathcal{C}))=\hat{\mathbb{G}}(\mathcal{C})$ が成立する。特に、$\hat{\mathbb{A}}$ は τ-species 全体と completely graded τ-圏全体の間の一対一対応を与え、$\hat{\mathbb{M}}$ はその逆を与える。

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Morita equivalences for general linear groups in non-defining characteristic

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§1 Preliminary The result of this paper is a joint work with **Akihiko Hida** . Let G be a finite group, and $(K, \mathcal{O}, \mathbf{k})$ be a splitting ℓ-modular system for G. Here $\mathrm{char}(K) = 0, \mathrm{char}(\mathbf{k}) = \ell > 0$. For $R \in \{\mathcal{O}, \mathbf{k}\}$, let $B_0(RG)$ be the principal block of RG.
$\mathfrak{S}_n$ denotes the symmetric group on n letters. $\mathbb{F}_q$ denotes a field with q elements with $\ell \nmid q$. Let natural numbers $e(q)$ and $r(q)$ be as follows:

$$e(q) := \mathbf{Min}\{\ i \in \mathbb{N} \mid q^i \equiv 1 \pmod{\ell}\ \},$$

$$r(q) := \mathbf{Max}\{\ r \in \mathbb{N} \mid \ell^r \mid q^{e(q)} - 1\ \}:\ \text{the } \ell\text{-part of } q^{e(q)} - 1.$$

Let A and B be blocks ideals. "$A \sim_M B$" means that A is Morita (Puig) equivalent to B. "$A \sim_d B$" means that A is derived (splendid Rickard) equivalent to B (see [29],[30]).

We use results on representation theory of finite general linear groups in non-defining characteristic due to Fong-Srinivasan and Dipper-James (see [14], [15], [9],[10],[11],[12], [13],[17]).

§2 Motivations We wish to prove the following conjectures:

Conjecture 2.1 (Broué). [2], [3], [5] *Let B be an ℓ-block ideal of G with abelian defect group D. Then B and its Brauer correspondent in $\mathcal{N}_G(D)$ are derived equivalent?*

Conjecture 2.2 (James). [18] *Suppose that char*$(\mathbf{k}) = \ell > n$ *and* $e(q) = e$. *Let ζ a primitive e-th root of unity in $\mathbb{C}$. Then, the decomposition matrix of Dipper-James Schur algebra $S_\zeta(n, r)_{\mathbb{C}}$ over $\mathbb{C}$ is equal to that of Dipper-James Schur algebra $S_{\overline{q}}(n, r)_{\mathbf{k}}$ over* $\mathbf{k}$ *?*

Let $\mathbf{G}$ be a connected reductive algebraic group over $\mathbb{F}_q$ with a Frobenius map F. We assume that the centre of $\mathbf{G}$ is connected. Let ℓ be a prime number with $\ell \nmid q$.

[0]The detailed version of this paper will be submitted for "Doctor thesis at Chiba Univ., Japan".

Lusztig series

The following is so-called Lusztig series:

$$\mathcal{E}(\mathbf{G}^F, \{s\}) := \bigcup_{(\mathbf{T},\theta)} \{\chi \in \widehat{\mathbf{G}}^F \mid \langle \chi, R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle \neq 0\}.$$

Here, the above pair $(\mathbf{T}, \theta)$ runs $s_1 \in \{s\}$ and $\theta \in \widehat{\mathbf{T}}^F \leftrightarrow s_1 \in \mathbf{T}^{*F}$, and $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ is a generalized Deligne-Lusztig character.

Its modular version is given as follows:

For a semisimple ℓ'-element $s \in \mathbf{G}^{*F^*}$, let

$$\mathcal{E}_\ell(\mathbf{G}^F, \{s\}) := \bigcup_t \mathcal{E}(\mathbf{G}^F, \{st\}), \quad t \in (C_{\mathbf{G}^*}(s)^{F^*})_\ell.$$

Theorem 2.3 (Broué-Michel). [4] *Each set $\mathcal{E}_\ell(\mathbf{G}^F, \{s\})$ is a union of ℓ-block of $\mathbf{G}^F$.*

Definition 1. *An ℓ-block B as an algebra is unipotent, if there exists $\chi \in \mathcal{E}_\ell(\mathbf{G}^F, \{1\})$ such that χ belongs to B. In particular, $B_0(\mathcal{O}\mathbf{G}^F)$ is unipotent.*

Theorem 2.4 (Bonnafé-Rouquier). [1] $\mathcal{E}_\ell(\mathbf{G}^F, \{s\}) \sim_M \mathcal{E}_\ell(C_{\mathbf{G}^*}(s)^{*F}, \{1\})$ *as ℓ-block ideals. (i.e. If a block B_s belongs to $\mathcal{E}_\ell(\mathbf{G}^F, \{s\})$, then there exists a unipotent block B_1' of $C_{\mathbf{G}^*}(s)^{*F}$ such that B_s and B_1' are Morita equivalent (not Puig equivalent in general).)*

In particular, for finite general linear groups we may concentrate unipotent blocks by this theorem.

We want to classify the block ideals of $\mathbf{kG}(\mathbb{F}_q)$, up to Morita equivalence, and recover its structure as algebras from some small subgroups. So, we wish to prove the following conjecture:

Conjecture 2.5. *If $e(q) = e(q'), r(q) = r(q')$ then for any unipotent block ideal B of $\mathbf{G}(\mathbb{F}_q)$ there exists a unipotent block ideal B' of $\mathbf{G}(\mathbb{F}_{q'})$ such that $B \sim_M B'$ by an exact ℓ-permutation (B, B')-bimodule. This equivalence preserves the natural indices of modules.*

In this article we deal the special case concerning these three conjectures for finite general linear groups.

§3 Abacus and $[w{:}k]$-pairs

Definition 2. *For a k-core τ and a non-negative integer w, let $\Lambda_{k,w,\tau}$ be the set of paritions of $kw + |\tau|$ whose k-core is τ.*

Given partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$, define $\beta = (\beta_1, \beta_2, \dots)$ as follows:

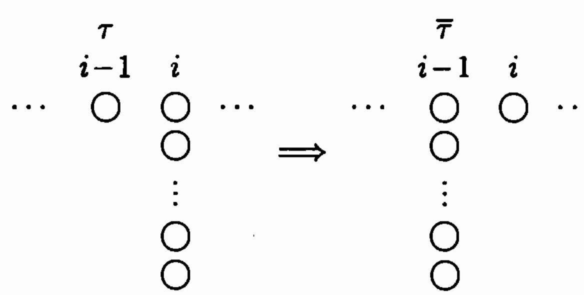
$$\beta_i := r - i + \lambda_i (1 \leq i \leq r).$$

We call this β an r-element β-set for λ.

Definition 3 (Scopes). *For non-negative integers m and m-core $\tau = (\tau_1, \dots, \tau_r)$, let Γ be the r-element β-set for τ, and suppose that when Γ is displayed on an abacus with m-runners there are k more than beads in the i-th column than in the $(i-1)$-th column. Let m-core $\overline{\tau}$ be displayed by an r-element β-set $\overline{\Gamma}$ satisfying*

$$\begin{aligned} \overline{\Gamma}_j &= \Gamma_j \qquad \text{for } j \neq i, i-1 \\ \overline{\Gamma}_i &= \Gamma_{i-1} \\ \overline{\Gamma}_{i-1} &= \Gamma_i, \end{aligned}$$

where Γ_j is the number of beads on the j-th runner in the abacus configulation for Γ. In these situation, we shall say that $\Lambda_{m,w,\tau}$ and $\Lambda_{m,w,\overline{\tau}}$ form a Scopes $[w:k]$-pair.



Scopes has proved the following:

Theorem 3.1 (Scopes). [32] *If $\Lambda_{p,w,\tau}$ and $\Lambda_{p,w,\nu}$ form a $[w:k]$-pair with $k \geq w$, then p-blocks $B^{w,\tau}$ and $B^{w,\nu}$ of symmetric groups are Puig equivalent.*

By Jost we also know the following:

Theorem 3.2 (Jost). *[21] If $\Lambda_{e,w,\tau}$ and $\Lambda_{e,w,\nu}$ form a $[w:k]$-pair with $k \geq w$, then unipotent ℓ-blocks $B_{w,\tau}$ and $B_{w,\nu}$ are Puig equivalent.*

Example 1. *If B is a unipotent block of $GL_n(q)$ with e-weight 2, then one of the following holds:*

1. $B \cong B_0(\mathbf{k}GL_{2e}(q))$.

2. *$(B, \overline{B})$ forms $[2:1]$-pair for some unipotent block $\overline{B}$ of $\mathbf{k}GL_{n-1}(q)$. (Actually, these blocks are derived equivalent to its Brauer correpondent of the ℓ-local subgroup. (Hida-Miyachi(1999)) (The method we used is different from J. Chuang's for $\mathfrak{S}_n$)*

3. *$B \sim_M B'$ for some unipotent block B' of $\mathbf{k}GL_m(q)$ with $m < n$.*

§4 A core ρ and results of J. Chuang and R. Kessar

Definition 4 (Chuang-Kessar-Rouquier). *[8] Let ρ be the e-core which satisfies the following property : ρ has an abacus configulation in which each runner other than the leftmost one (the 0-th runner) has at least $w-1$ more beads than the runners to its immediate left.*

Chuang and Kessar consider the following setting up:

$e = p > w$.

$r := |\rho|$.

$G := \mathfrak{S}_{pw+r}$.

$B^{w,\rho}$: the p-block of $\mathbf{k}G$ with p-weight w and p-core ρ.

$D :=$ a defect group of $B^{w,\rho}$.

$N := \mathfrak{S}_p \wr \mathfrak{S}_w \supset D$.

$L := \mathfrak{S}_p \times \cdots \times \mathfrak{S}_p \times \mathfrak{S}_r$.

$H := (\mathfrak{S}_p \wr \mathfrak{S}_w) \times \mathfrak{S}_r \supset \mathcal{N}_G(D)$.

$\mathcal{O}Hf$:=the Brauer correspondent of $B^{w,\rho}$ in H.

Let X be the Green correspondent of $B^{w,\rho}$ in $G \times H$ with respect to $(G \times G, \Delta(D), G \times H)$. Chuang and Kessar have proved the following:

Theorem 4.1 (Chuang-Kessar). [8] *Suppose that $p > w$. Then, we get an isomorphism*

$$\mathcal{O}Hf \cong \mathrm{End}_G(X_{\mathcal{O}})$$

by checking $\mathrm{rank}_{\mathcal{O}}(\mathrm{End}_G(X)) \leq w! \cdot \mathrm{rank}_{\mathcal{O}}(\mathcal{O}Lf)$. *In particular, $\mathcal{O}Hf$ is Morita equivalent to $B^{w,\rho}$.*

Remark 1. *1. X is exact.*

2. $\mathcal{O}Hf \to \mathrm{End}_G(X)$ is a split $(\mathcal{O}Hf, \mathcal{O}Hf)$-monomorphism.

3. $w!\,\mathrm{rank}_{\mathcal{O}}(\mathcal{O}Lf) = \mathrm{rank}_{\mathcal{O}}(\mathcal{O}Hf)$.

4. By Marcus [23] $\mathcal{O}Hf \sim_d B_0(\mathcal{O}N)$.

5. $(D^\lambda \otimes_{B^{w,\rho}} X)\downarrow_L$ is known, but $D^\lambda \otimes_{B^{w,\rho}} X$ is not known.

§5 Chuang-Kessar type theorem We assume that $char(\mathbf{k}) = \ell > w$. Choose a prime power q with $e(q) = e$. Just mimicking Chuang and Kessar's setting up, we consider the following:

$r := |\rho|$.

$G(q) := GL_{ew+r}(q)$.

$B^{w,\rho}(q)$: the unipotent ℓ-block of $\mathbf{k}G(q)$ with e-weight w and e-core ρ.

$D(q) :=$ a defect group of $B^{w,\rho}$.

$N(q) := GL_e(q) \wr \mathfrak{S}_w \supset D(q)$.

$L(q) := GL_e(q) \times \cdots \times GL_e(q) \times GL_r(q)$.

$H_w(q) := (GL_e(q) \wr \mathfrak{S}_w) \times GL_r(q) \supset \mathcal{N}_G(D(q))$.

$\mathcal{O}H_w(q)f_q$:=the Brauer correspondent of $B_{w,\rho}(q)$ in $H_w(q)$.

Once we believe that an anology of Chuang-Kessar theorem holds for finite general linear groups, we can easily prove the following:

Proposition 5.1. **(An anology of Chuang-Kessar theorem)** *Let $X(q)$ be the Green correspondent of $B^{w,\rho}(q)$ in $G(q)\times H_w(q)$ with respect to $(G(q)\times G(q), \Delta(D(q)), G(q)\times H_w(q))$. Then, we get an isomorphism*

$$\mathcal{O}H_w(q)f_q \cong \mathrm{End}_{G(q)}(X_{\mathcal{O}}(q))$$

by checking $\mathrm{rank}_{\mathcal{O}}(\mathrm{End}_{G(q)}(X_{\mathcal{O}}(q))) \leq w!\cdot \mathrm{rank}_{\mathcal{O}}(\mathcal{O}L(q)f_q)$. *In particular, $\mathcal{O}H_w(q)f_q$ is Morita equivalent to $B_{w,\rho}(q)$.*

Remark 2. *One must consider not only unipotent characters but also characters indexed by semisimple ℓ-elements. We can know these characters by [9]. We also need some results by [15] in order to mimic Chuang and Kessar's argument.*

§6 Indices of the simple $B_0(GL_e(q)\wr \mathfrak{S}_w)$-modules In this section we reformulate indices of the simple $B_0(GL_e(q)\wr \mathfrak{S}_w)$-modules to fit that of $B_{w,\rho}(q)$ via the equivalence in Proposition 5.1. For $i=1,2,\ldots,e$ let $\nu_i=(i,1^{e-i})\vdash e$. The principal block $B_0(\mathbf{k}GL_e(q))$ has e non-isomorphic irreducible modules

$$\{\ D_{\mathbf{k},q}(\nu_i)\mid i=1,2,\ldots,w\}.$$

Fix $R\in\{K,\mathbf{k}\}$. Let $\boldsymbol{n}$ be an e-tuple non-negative integer of w. i.e. $\sum_{i=1}^{e}\boldsymbol{n}_i=w$. $S_{R,q}(\boldsymbol{n}) := \bigotimes_i (S_{R,q}(\nu_i)^{\otimes \boldsymbol{n}_i})$ is an $R[GL_e(q)^{\times w}]$-module. In particular, $S_{K,q}(\boldsymbol{n})$ is a simple $K[GL_e(q)^{\times w}]$-module. The parabolic subgroup $\mathfrak{S}_{\boldsymbol{n}}$ act on $S_{R,q}(\boldsymbol{n})$. So, $S_{R,q}(\boldsymbol{n})$ is an $R[L_{(e^w)}\rtimes \mathfrak{S}_{\boldsymbol{n}}]$-module. $\mathrm{Ind}_{L_{(e^w)}}^{L_{(e^w)}\rtimes\mathfrak{S}_{\boldsymbol{n}}} S_{R,q}(\boldsymbol{n})$ is decomposed into $\bigoplus_{\boldsymbol{\lambda}\vdash\boldsymbol{n}}(S_{R,q}(\boldsymbol{n})\otimes_R(\dim_R S_R^{\boldsymbol{\lambda}})\cdot S_R^{\boldsymbol{\lambda}})$ where S_R^{μ} means the Specht module of $R[\mathfrak{S}_{|\mu|}]$ corresponding to μ, $S_R^{\boldsymbol{\lambda}}=\bigotimes_i S_R^{\boldsymbol{\lambda}_i}$ and $S_{R,q}(\boldsymbol{n})\otimes_R S_R^{\boldsymbol{\lambda}}$ is the inner tensor product of $R[L_{(e^w)}\rtimes\mathfrak{S}_{\boldsymbol{n}}]$-modules $S_{R,q}(\boldsymbol{n})$ and $S_R^{\boldsymbol{\lambda}}$.

Let

$$T_R^{\boldsymbol{\lambda}_i}=\begin{cases} S_R^{\boldsymbol{\lambda}_i} & \text{if } i+e \text{ is even,}\\ S_R^{\boldsymbol{\lambda}_i'} & \text{if } i+e \text{ is odd.}\end{cases}$$

Here, $\boldsymbol{\lambda}_i'$ is the conjugate partition of $\boldsymbol{\lambda}_i$. Let $T_R^{\boldsymbol{\lambda}}=\bigotimes_i T_R^{\boldsymbol{\lambda}_i}$.

For $\boldsymbol{\lambda}\vdash\boldsymbol{n}$ let $U_{R,q}(\boldsymbol{\lambda})$ be $\mathrm{Ind}_{L_{(e^w)}\rtimes\mathfrak{S}_{\boldsymbol{n}}}^{GL_e(q)\wr\mathfrak{S}_w}(S_{R,q}(\boldsymbol{n})\otimes T_R^{\boldsymbol{\lambda}})$, and Let $U_{\mathbf{k},q}(\boldsymbol{\lambda})^{\rho}$ be the $R[H_w(q)]$-module $U_{R,q}(\boldsymbol{\lambda})\otimes_R S_{R,q}(\rho)$.

Moreover, one can construct a module by using

$$\{ D_{\mathbf{k},q}(\nu_i) \mid i = 1, 2, \ldots, e\}$$

instead of $\mathbf{k}[GL_e(q)]$-modules $\{ S_{\mathbf{k},q}(\nu_i) \mid i = 1, 2, \ldots, e\}$. We denote it by $V_{\mathbf{k},q}(\boldsymbol{\lambda})^\rho$.

§7 Results Now we can state our main results of this article as follows:

Theorem 7.1 (Hida-Miyachi). *For any simple $B_{w,\rho}(q)$-module $D_{\mathbf{k},q}(\lambda)$, the Green correspondent $D_{\mathbf{k},q}(\lambda) \otimes_{\mathbf{k}G} X(q)$ of $D_{\mathbf{k},q}(\lambda)$ is independent of q in the following sense:*

Assume that $e(q) = e(q')$ and $r(q) = r(q')$. Let $\mathcal{M}_{q,q'}$ be the canonical $(\mathbf{k}H_w(q)f_q, \mathbf{k}H(q')f_{q'})$-bimodule which induces $\mathbf{k}H_w(q)f_q \sim_M \mathbf{k}H(q')f_{q'}$, due to A. Marcus. Then

$$D_{\mathbf{k},q}(\lambda) \otimes_{B_{w,\rho}(q)} X(q) \otimes_{\mathbf{k}H_w(q)f_q} \mathcal{M}_{q,q'} \otimes_{\mathbf{k}H(q')f_{q'}} X(q')^\vee \cong D_{\mathbf{k},q'}(\lambda).$$

Actually, $D_{\mathbf{k},q}(\lambda) \otimes_{B_{w,\rho}(q)} X(q) \cong V_{\mathbf{k},q}(\overline{\lambda})^\rho$. Here, $\overline{\lambda}$ is the e-quotient of λ. Moreover, we know the decomposition numbers corresponding to the e-core ρ :

$$d_{\lambda,\mu} = d_{\overline{\lambda},\overline{\mu}} = [U_{\mathbf{k},q}(\overline{\lambda}) : V_{\mathbf{k},q}(\overline{\mu})].$$

(*The other parts of $B_{w,\rho}(q)$ can be calculated by Dipper-James theory.*)

Remark 3. *First we can determine the Green correspondents of simple $B_{2,\rho}(q)$-modules in $H_2(q)$ finding two trivial source modules of $B_{2,\rho}(q)$, using the decomposition numbers for Hecke algebras of type* **A** *by [28] and [20], chasing the image of Mullineux-Kleshchev map [25, p.120], using properties of Specht modules [17] and induction on $\Lambda_{e,2,\rho}$.*

Next we can determine the Green correspondents of simple $B_{w,\rho}(q)$-modules in $H_w(q)$ using induction on w and some commutative diagrams among $B_{w,\rho}(q)$,$B_0(GL_e(q)) \otimes B_{w-1,\rho}(q)$ and their Brauer correspondents.

In order to prove $B_{w,\rho}(q) \sim_M B_{w,\rho}(q')$ with the property in the above theorem we use [14],[23], and [31].

Corollary 7.2. *If there exist a sequence of e-cores*

$$\rho = \tau^0, \tau^1, \ldots, \tau^s$$

such that Λ_{e,w,τ^i} and $\Lambda_{e,w,\tau^{i+1}}$ form a $[w : k_i]$-pair with $k_i \geq w - 1$, Broué's conjecture is true for $B_{w,\tau^s}(q)$.

Theorem 7.3 (Hida-Miyachi). *Assume that $e = e(q) = e(q')$ and $r(q) = r(q')$. If there exist a sequence of e-cores*

$$\rho = \tau^0, \tau^1, \dots, \tau^s$$

such that Λ_{e,w,τ^i} and $\Lambda_{e,w,\tau^{i+1}}$ form a $[w : k_i]$-pair with $k_i \geq w - 1$, then

$$B_{w,\tau^s}(q) \sim_M B_{w,\tau^s}(q').$$

Here, each $[w : w-1]$-pair is a derived (spelendid) equivalence between two unipotent blocks. Moreover, the above Morita equivalence preserves natural indices (partitions) of modules. (i.e. The simple module $D_{\mathbf{k},q}(\mu)$ (resp. the "Specht"like module $S_{\mathbf{k},q}(\mu)$, the Young module $X_q(\mu)$, PIM $P_q(\mu)$) indexed by a partition μ corresponds to $D_{\mathbf{k},q'}(\mu)$ (resp. $S_{\mathbf{k},q'}(\mu), X_{q'}(\mu)$, $P_{q'}(\mu)$).)

Remark 4. *Just mimicking an argument in [7], constructing a generalization of [33] and using Theorem 7.1, we deduce the above results. (see also [27]).*

§8 Remarks Some conjectures on quantized decomposition numbers [22] and radical series of Specht modules for $\Lambda_{e,w,\rho}$ will be described in "Topics on Combinatrial Representation Theory" organized by T. Nakajima. (The first announcement of this was stated in the author's lecture "On the unipotent blocks of finite general linear groups" at a conference " Algébres de Hecke affines et groups réductifs (CIRM,Luminy,16-20 octobre 2000)" .)

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