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筑波大学

Organizing Committee of The Symposium on Ring Theory and Representation Theory

The Symposium on Ring Theory and Representation Theory has been held annually in Japan and the Proceedings have been published by the organizing committee. The first Symposium was organized in 1968 by H. Tominaga, H. Tachikawa, M. Harada and S. Endo. After their retirement, in 1997, a new committee was organized for managing the Symposium. The present members of the committee are Y. Hirano (Okayama Univ.), Y. Iwanaga (Shinshu Univ.), S. Koshitani (Chiba Univ.) and K. Nishida (Shinshu Univ.).

The Proceedings of each Symposium is edited by program organizer. Anyone who wants these Proceedings should ask the program organizer of each Symposium or one of the committee members.

The next Symposium in 2002 will be held in Okayama and its program will be arranged by Y. Hirano.

Concerning several informations on ring theory group in Japan containing schedules of meetings and symposiums, you can see on the following homepage:

`http://fuji.cec.yamanashi.ac.jp/~ring/`

which is arranged by M. Sato of Yamanashi University.

Yasuo Iwanaga
Nagano, Japan
December, 2001

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PREFACE

The 34th Symposium on Ring Theory and Representation Theory was held in Maebashi on October 15th - 17th, 2001. This volume consists of fourteen articles presented at the symposium. It includes two results settling two conjectures: Solomon's second conjecture concerning zeta functions of orders, and the conjecture concerning representation dimension of artin algebras introduced by M. Auslander.

We would like to thank all speakers and their coauthors for their contributions.

A part of the financial support of the symposium was arranged by Professor Hiroaki Komatsu, Professor Kenji Nishida and Professor Atumi Watanabe. We wish to express our thanks for their arrangements.

I would like to thank Professor Yasuo Iwanaga for his helpful suggestions concerning the symposium. Finally we should like to express our gratitude to Professor Koichiro Ohtake and his students of Gunma University who contributed in the organization of the symposium.

Hisaaki Fujita
Tsukuba
January, 2002

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DERIVATIONS ON QUANTUM SPACES

Nobuyuki Fukuda

In this paper we introduce quantum analogues of separable algebras and separable extensions, and investigate thier properties. It is well-known that separable algebras are characterized as the algebras whose all derivations are inner. Based on this fact, we define quantum separable algebras as the quantum spaces whose all quantum derivations are inner. Similarly, quantum separable extensions are defined in terms of quantum derivations. We obtain quantum analogues of basic results in the theory of separable algebras.

Let K be a commutative ring, H a Hopf algebra over K with the antipode S. Throughout this paper, we abbreviate $\otimes_K$ to $\otimes$, and use the Sweedler notation.

In [5], a "quantum space" means a K-algebra A with a left H-module algebra structure and a right H^{op}-comodule algebra structure ρ that make A into a *crossed H-bimodule* (or *Yetter-Drinfeld module*), that is, it holds that

$$\sum h_{(1)}a_{(0)} \otimes h_{(2)}a_{(1)} = \sum (h_{(2)}a)_{(0)} \otimes (h_{(2)}a)_{(1)}h_{(1)}$$

for all $h \in H, a \in A$. In this paper, such a K-algebra is called a *crossed H-bimodule algebra.*

One of the most basic example of quantum spaces is (the coordinate ring of) the quantum affine space $K_q[X]$. Let K be a field. Fix $0 \neq q \in K$. The n-dimensional quantum affine space $K_q[X]$ is the K-algebra generated by the n-elements $x^1, \cdots, x^n$ subject to the relations $x^i x^j = q x^j x^i (i < j)$. Put $A = K_q[X]$. Let H be the well-known quantum deformation $\mathrm{GL}_n(q)$ of GL_n (see [1, p.91]). Thus, H is generated by the $n^2 + 1$ elements $\{t^i_j\}_{1\leq i,j\leq n}, (\det_q)^{-1}$ with the relations

$$t^k_i t^l_i = q t^l_i t^k_i, \qquad t^k_j t^k_i = q t^k_i t^k_j,$$
$$t^l_j t^k_i = t^k_i t^l_j, \qquad t^k_j t^l_i - t^l_i t^k_j = (q - q^{-1}) t^k_i t^l_j,$$

where $i < j$, $l < k$, and $\det_q$ is the quantum determinant ([1, p.91]). Further, $\mathrm{GL}_n(q)$ has a Hopf algebra structure with the comultiplication Δ and the counit ϵ such that

The final version of this paper will be submitted for publication elsewhere.

$\Delta(t^i_j) = \sum_\alpha t^i_\alpha \otimes t^\alpha_j, \varepsilon(t^i_j) = \delta^i_j$. Then A is a crossed H-bimodule algebra with the action and the coaction defined by

$$\rho(x^i) = \sum_\alpha x^\alpha \otimes S(t^i_\alpha),$$
$$h \cdot f = \sum \langle h, f_{(1)} \rangle f_{(0)}$$

for $h \in \mathrm{GL}_n(q), f \in K_q[X]$, where $\rho(f) = \sum f_{(0)} \otimes f_{(1)}$, and the bilinear form $\langle\ ,\ \rangle : \mathrm{GL}_n(q) \times \mathrm{GL}_n(q) \to K$ is a cobraided structure on $\mathrm{GL}_n(q)$ such that $\langle t^i_j, t^k_l \rangle = q(R^{-1})^{ki}_{jl}$. Here R is the R-matrix defined by

$$R^{ij}_{kl} = \delta^i_l \delta^j_k (1 + (q-1)\delta^{ij}) + (q - q^{-1})\delta^i_k \delta^j_l \theta(j,i),$$

where

$$\theta(i,j) = \begin{cases} 1 & \text{if } i > j, \\ 0 & \text{if } i \leq j. \end{cases}$$

When an A-bimodule M has a left H-module structure satisfying $h(aub) = \sum (h_{(1)}a)(h_{(2)}u)(h_{(3)}b)$ for all $h \in H, a, b \in A, u \in M$, we write $M \in {}_A\mathcal{M}(H)_A$. Clearly, $A \in {}_A\mathcal{M}(H)_A$. For an A-bimodule $M \in {}_A\mathcal{M}(H)_A$, a K-linear map $D : A \to M$ is a *quantum K-derivation* if $D(ab) = D(a)b + \sum a_{(0)}(a_{(1)} \cdot D)(b)$ for all $a, b \in A$, where $\rho(a) = \sum a_{(0)} \otimes a_{(1)}$, and $\mathrm{Hom}_K(B, M)$ is a left H-module with the action defined by $h \cdot D(a) = \sum h_{(1)}(D(S(h_{(2)})a))$ for all $h \in H, D \in \mathrm{Hom}_K(A, M), a \in A$ ([5]).

The q-difference operators are the most basic examples of quantum derivations. Let K be a field, $H = GL_q(n)$ and $A = K_q[X]$. Suppose that q^2 is not a root of unity. The quantum n-th Weyl algebra $A_n(q)$ is the K-algebra generated by $K_q[X]$ and the n elements $\partial_1, \ldots, \partial_n$ with the relations

$$\partial_i \partial_j = q^{-1} \partial_j \partial_i \qquad (i < j),$$
$$\partial_i x^j = q x^j \partial_i \qquad (i \neq j),$$
$$\partial_i x^i = 1 + q^2 x^i \partial_i + (q^2 - 1) \sum_{j>i} x^j \partial_j.$$

Then, the K-vector space isomorphism $A \cong A_n(q)/\oplus_{i=1}^n A_n(q)\partial_i$ makes $A (= K_q[X])$ into a left $A_n(q)$-module, and ∂_i acts on A as the q-difference operator:

$$\partial_i \cdot f(x^i) = \frac{f(q^2 x^i) - f(x^i)}{q^2 x^i - x^i} \qquad (f(x^i) \in K[x^i]),$$
$$\partial_i \cdot f(x^j) = 0 \qquad (f(x^j) \in K[x^j], \text{ where } j \neq i).$$

Define a K-algebra automorphism $\phi : A \to A$ by $\phi(x^i) = q^2 x^i$. Note that ϕ is H-linear and H-colinear. Let M be the A-bimodule such that $M = A$ as a left H-module, and the A-action is defined by $a \cdot b \cdot c = \phi(a)bc$ for all $a, c \in A, b \in M(= A)$. It is clear that $M \in {}_A\mathcal{M}(H)_A$. Then one can verify that each $\partial_i : A \to M$ is a quantum K-derivation. Moreover, the K-space of all quantum K-derivations equals $\oplus_{i=1}^n A\partial_i$.

This paper is organized as follows. In Section 1, we define quantum separable (crossed bimodule) algebras, in terms of quantum derivations, and obtain some results. In Section 2, we generalize the notion of quantum separability to extensions of crossed bimodule algebras. In Section 3, we compare quantum separability with usual separability under some conditions. Moreover, we give an exmaple which is separable and is not quantum separable.

Refer to [4] for theory of separable algebras.

1 Quantum separable algebras

Let $M \in {}_A\mathcal{M}(H)_A$. A quantum K-derivation $D : A \to M$ is *inner* if there exists an element $u \in M$ such that $D(a) = \sum ua - a_{(0)}(a_{(1)}u)$ for all $a \in A$.

Recall that separable algebras are characterized as the algebras whose all derivations are inner. This fact gives justification to the following definition. separable

DEFINITION 1.1. A crossed H-bimodule algebra A is *quantum separable* if, for any $M \in {}_A\mathcal{M}(H)_A$, every quantum K-derivation $D : A \to M$ is inner.

Let $\mu : A \otimes A \to A$ be the multiplication structure map of A. In other words, $\mu(a \otimes b) = ab$ for all $a, b \in A$.

The following characterization of quantum separable algebras is a quantum analogue of the well-known result ([4, Prop.10.2]).

THEOREM 1.2. *The following are equivalent.*

(1) A crossed H-bimodule algebra is quantum separable.

(2) There exists a linear map $\psi : A \to A \otimes A$ *such that* $\mu \circ \psi = \mathrm{id}_A$ *and* $\psi(auc) = \sum a_{(0)}(a_{(1)} \cdot \psi)(u)c$ *for all* $a, b \in A$, $u \in A \otimes A$.

(3) There exists $e \in A \otimes A$ *such that* $\mu(e) = 1$ *and* $ea = \sum a_{(0)}(a_{(1)} \cdot e)$ *for all* $a \in A$, *where* $A \otimes A$ *is regarded as an A-bimodule with the natural action.*

An element $e \in A \otimes A$ that satisfies condition (3) of the theorem is a quantum analogue of a separability idempotent for a separable algebra.

It is known that separable algebras over a field are finite-dimensional.

PROPOSITION 1.3. *Suppose that* K *is a field. If* H *is finite-dimensional over* K, *and a crossed H-bimodule algebra* A *is quantum separable, then* A *is finite-dimensional over* K.

2 Quantum separable algebras extension

Let B be another crossed H-bimodule algebra. If A is a crossed H-bimodule subalgebra of B, we say that B is an extension of A (as a crossed H-bimodule algebra). For $M \in {}_B\mathcal{M}(H)_B$, a quantum K-derivation $D : B \to M$ is a *quantum A-derivation* if $D(A) = 0$. A quantum A-derivation $D : B \to M$ is *inner* if there exists an element $u \in M$ such that $D(a) = ua - \sum a_{(0)}(a_{(1)}u)$ for all $a \in B$.

We generalize the notion of quantum separability to extensions.

DEFINITION 2.1. A crossed H-bimodule algebra extension B of A is *quantum separable* if, for any $M \in {}_B\mathcal{M}(H)_B$, every quantum A-derivation $D : B \to M$ is inner.

Define $\mu : B \otimes_A B \to B$ by $\mu(a \otimes b) = ab$ for all $a, b \in A$.

We obtain a generalization of Theorem 1.2 (see [4, Lemma 10.8]).

THEOREM 2.2. *The following are equivalent.*

(1) A crossed H-bimodule algebra extension B of A is quantum separable.

(2) There exists a linear map $\psi : B \to B \otimes_A B$ such that $\mu \circ \psi = \mathrm{id}_B$ and $\psi(auc) = \sum a_{(0)}(a_{(1)} \cdot \psi)(u)c$ for all $a, b \in B$, $u \in B \otimes_A B$.

(3) There exists $e \in B \otimes_A B$ such that $\mu(e) = 1$ and $ea = \sum a_{(0)}(a_{(1)} \cdot e)$ for all $a \in B$.

PROPOSITION 2.3. *Let C be an extension of B, and B an extension of A.*

(1) If C is a quantum separable extension of A, then C is a quantum separable extension of B.

(2) If C is a quantum separable extension of B, and B is a quantum separable extension of A, then C is a quantum separable extension of A.

3 Examples

Throughout this section, suppose that the antipode S of H is bijective. Let A be the opposite Hopf algebra H^{op} of H. Then, A is a crossed H-bimodule algebra with the H-action $\rightharpoonup$ and the H^{op}-coaction ρ such that

$$\rightharpoonup : H \otimes A \to A, \qquad h \rightharpoonup a = \sum S^{-1}(h_{(1)})ah_{(2)},$$

$$\rho : A \to H^{\mathrm{op}} \otimes B, \qquad \rho(a) = \sum a_{(1)} \otimes a_{(2)}.$$

See [3, Example 10.6.13]. From now on, set $A = H^{\mathrm{op}}$, and its crossed H-bimodule algebra structure is as above.

EXAMPLE 3.1. If H is commutative, the concept of quantum separable algebra coincides with that of separable algebra. In fact, in this case, the H-action is trivial.

EXAMPLE 3.2. Let G be a finite group. Suppose that H is the group algebra KG of G. In this case, the crossed H-bimodule algebra A is quantum separable if and only if A is separable as K-algebra.

Finally, we give an example of crossed H-bimodule algebras which is separable as a K-algebra but not quantum separable.

EXAMPLE 3.3. Suppose that K is an algebraically closed field of characteristic $\mathrm{ch}K \neq 2$. Let H be the only (up to isomorphism) noncommutative noncocommutative semisimple Hopf algebra of dimension 8 defined in [2]. Precisely, H is generated by the element x, y, z with the relations

$$x^2 = y^2 = 1, \qquad z^2 = \frac{1}{2}(1 + x + y - xy)$$
$$yx = xy, \qquad zx = yz, \qquad , zy = xz,$$

and its comultiplication is defined by

$$\Delta(x) = x \otimes x, \qquad \Delta(y) = y \otimes y,$$
$$\Delta(z) = \frac{1}{2}(1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z)$$
$$\varepsilon(x) = \varepsilon(y) = 1, \qquad \varepsilon(z) = 1.$$

Since K is algebraically closed, and H is semisimple as a K-algebra, it follows that $A(= H^{\mathrm{op}})$ is separable as a K-algebra. However, one can show that there exists no element $e \in A \otimes A$ with $\mu(e) = 1$ such that $ex = \sum x_{(0)}(x_{(1)} \cdot e)$ and $ey = \sum y_{(0)}(y_{(1)} \cdot e)$. Therefore A is not quantum separable.

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ON SPLITTING SUPERHEREDITARY PRERADICALS

Yasuyuki Hirano and Hisaya Tsutsui

Abstract: We determine the structure of rings R with the property that, for every right R-module M and every ideal I of R, the annihilator of I in M is a direct summand of M.

Our recent work summarized hereinabove was briefly introduced at the symposium. In this proceeding, we shall provide some further details[1].

For a unitary right R-module M and a right ideal I of R, let $P_I(M) = \sum_{f \in Hom_R(R/I, M)} \mathrm{Im}(f)$. M is then said to be *split* in P_I if $P_I(M)$ is a direct summand of M, and we shall say that P_I is *splitting* if every R-module M splits in P_I. We then consider the structure of rings R with the property that P_I is *splitting* for a certain subset of the set of all right ideals of R. Notice when P_I is splitting for all (two sided) ideals of R, our consideration is reduced to as such described in the abstract. In this case, our main result yields that R is a finite direct sum of prime fully right idempotent rings all of whose proper factor rings are semisimple Artinian.

The title of this paper suggests a torsion theoretic origin of our study. A hereditary preradical r is called superhereditary if the class of r-torsion modules is closed under direct products. By [3, I.2.E4 and I.2.E5], for an ideal I of R, P_I is superhereditary and every superhereditary preradical is of this form.

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[1] The complete version of this paper has been submitted for publication elsewhere.

Throughout, every ring will be assumed to have an identity element, and R-modules considered are unitary. For any terminology that shall not be defined in this paper, we refer Anderson-Fuller [1], Rowen [13], [14], or Stenström [15].

Our first example shows that even when P_I is splitting for every right ideal of *R*, *R* is not necessarily semisimple Artinian.

Example 1. Let *k* be a universal differential field with derivation *D* and let $R = k[y, D]$ denote the ring of differential polynomials in the indeterminates y with coefficients in k. Cozzens [5, Theorem 1.4] proved that *R* is a simple right *V*-ring and that *R* has, up to isomorphism, a unique simple right *R*-module. Hence every right *R*-module of finite length is completely reducible and injective. If I is a non-zero right ideal of *R*, then R/I is of finite length, and so R/I is completely reducible and injective. Now let *M* be a right *R*-module and let *I* be a nonzero proper right ideal of *R*. Then $P_I(M) = Soc(M)$ is injective and hence is a direct summand of *M*.

Proposition 1. *Let R and S be a pair of Morita equivalent rings. Then P_I is splitting for every ideal I of R if and only if P_J is splitting for every ideal J of S.*

For a right *R*-module *M*, let $E(M_R)$ denote the injective hull of *M*.

Lemma 1. *Let I be an ideal of R. If P_I is splitting, then R/I is a flat left R-module.*

Proof. Let *M* be an injective right R/I -module. Then we have $M = E(M_{R/I}) = P_I(E(M_R))$. But then by hypothesis, *M* is a direct summand of $E(M_R)$, and hence it is injective as a right R-module. Therefore by [15, Proposition 11.3.13], R/I is flat as a left R-module.

Proposition 2. *If P_I is splitting for every principal ideal I of R, then R is a fully right idempotent ring.*

The ring sited in Example 1 is fully right idempotent but it is not von Neumann regular. Thus, the hypothesis of Proposition 2 does not imply that the ring is von Neumann regular. Note also that the converse of Proposition 2 is false. (See Example 4).

Lemma 2. *Let I be an ideal of R. If R/I is a flat left R-module and R/I is semisimple Artinian, then P_I* is splitting.

In particular, if R is fully right idempotent, we have the following partial converse of Proposition 2.

Lemma 3. *Let R be a fully right idempotent ring. Assume that, for any nonzero ideal I of R, R/I is a semisimple Artinian ring. Then P_I is splitting for every ideal I of R.*

The proof of the following lemma is essentially the same as in the proof of [8, Lemma 5.2].

Lemma 4. *Let R be a ring, let I be an ideal of R such that $P_I(R)=0$, and let H be an ideal of R containing I. Assume that P_I is splitting. If A is a torsionless right R-module, then A/AH is a projective right R/H-module.*

Theorem 1. *Let R be a ring and let I be an ideal of R such that* $P_I(R) = 0$ *.If* P_I *is splitting, then* R/H *is a right hereditary right perfect ring for every ideal H of R containing I.*

Proof. Let H be an ideal of R containing I and let K be any right ideal of R containing H. Since K is a torsionless right R-module, K/H is a projective right ideal of R/H by Lemma 4. This implies that R/H *is* a right hereditary ring.

Let J be an infinite *set* with cardinality Card(R), and set $R_\alpha = R_R$, $A = \prod_{\alpha\in J} R_\alpha$.
By Lemma 4, A/AH is a projective right R/H -module. Thus A/AH is a direct summand of some direct sum of copies of R/H . But then by [8, Theorem 5.1], R/H is a right perfect ring.

A module M is called *CS* provided that every submodule of M is essential in a direct summand of M. M is called *completely CS* provided that every quotient of M is CS.

Theorem 2. *Let R be a ring. If* P_I *is splitting for every ideal I of R, then every factor ring of R is a finite direct sum of prime rings.*

Proof. By Proposition 2, R is in particular, a semiprime ring. Let Z denote the center of R and consider the ring $R^e = R^{op} \otimes_Z R$. Then R is a right R^e-module defined by $r(a\otimes b) = arb$ for all $a\otimes b \in R^e$ and $r \in R$. In this case, I is an ideal of R if and only if I is an R^e-submodule of R. Let I and J be ideals of R with $I \subset J$. Then there is an idempotent $e \in R$ such that $l_R(I) = eR$. Since R is semiprime, e is a central idempotent of R and I is an essential submodule of the right R^e-submodule $(1-e)R$. This implies that $J = eJ \oplus (1-e)J$ and I is an essential submodule of the right R^e-submodule $(1-e)J$. This means that J is a CS-module over a ring R^e. Clearly, for every ideal A of R, P_K is splitting for every ideal K of R/A. Hence every ideal of R is completely CS as a right R^e-module. Therefore the cyclic right R^e-module R satisfies the hypotheses of [10, Theorem 1]. Hence R is a finite direct sum of uniform R^e-modules. Since R is semiprime, it now follows that R is a finite direct sum of prime rings. Obviously this is true for all factor rings of R.

There are examples of rings that are not fully right idempotent but all of whose factor rings are prime (See Example 3 below). Therefore, the converse of Theorem 2 is in general false by Proposition 2. However, we have the following necessary and sufficient condition for a ring to have the property that P_I is splitting for every ideal I.

Theorem 3. *Let R be a ring. Then the following statements are equivalent:*

1) P_I is splitting for every ideal I of R.

2) R is a finite direct sum of fully right idempotent prime rings T such that T/I is a semisimple Artinian ring for all proper nonzero ideal I of T.

Proof of 1) $\Rightarrow$ 2). By Theorem 2, R is a finite direct sum of prime rings. Hence we may assume that R is a prime ring that is not simple. By Proposition 2, R is fully right idempotent. Let I be a proper nonzero ideal of R. Since R is a prime ring, $P_I(R) = 0$. But then, by Theorem 1, R/I is a right perfect ring. As a fully right idempotent right perfect ring is semisimple Artinian, R/I is a semisimple Artinian ring.

We are now in a position to give several examples.

Example 1. Let K be a field. Consider the ring $R=\begin{pmatrix} K & 0 \\ K & K \end{pmatrix}$. In this ring, $I=\begin{pmatrix} K & 0 \\ K & 0 \end{pmatrix}$ is an ideal and R/I is a flat R-module. Since $R/I \cong K$, P_I is splitting by Lemma 2.

Example 2. Let K be a field. Let $n_1, n_2, \cdots, n_k$ be positive integers and let $n = n_1 + n_2 + \cdots + n_k$. Consider the matrix ring $M_n(K)$ and its semisimple Artinian subring $M_{n_1}(K) \oplus M_{n_2}(K) \oplus \cdots \oplus M_{n_k}(K)$. Let R be the set of countable matrices over $M_n(K)$ of the form

$$\begin{pmatrix} A_m & & 0 & \\ & a & & \\ 0 & & a & \\ & & & \ddots \end{pmatrix}$$

where $a \in M_{n_1}(K) \oplus M_{n_2}(K) \oplus \cdots \oplus M_{n_k}(K)$. and A_m is an arbitrary $m \times m$ matrix over $M_n(K)$ and m is allowed to be any integer. Then R is a von Neumann regular ring and that the set I of countable matrices of the form

$$\begin{pmatrix} A_m & & 0 & \\ & 0 & & \\ 0 & & 0 & \\ & & & \ddots \end{pmatrix}$$

is a unique minimal (nonzero) ideal of R.

Since $R/I \cong M_{n_1}(K) \oplus M_{n_2}(K) \oplus \cdots \oplus M_{n_t}(K)$, every proper homomorphic image of R is a semisimple Artinian. Hence, by Lemma 3, all right R-modules split in P_I for every ideal I of R.

For an arbitrary ring R, the next example shows that even when R/I is semisimple Artinian for every nonzero ideal I of R, P_I need not be splitting for every ideal I of R. The example also shows that the hypothesis "fully right idempotent" in Lemma 3 cannot be replaced by "fully idempotent."

Example 3. Let K be a field of characteristic zero and let $A_1(K)$ denote the first Weyl algebra over K, that is, the K-algebra generated by x, y with the relation $xy - yx = 1$. Consider the ring $R = xA_1(K) + K$. It is well-known that R is a right Noetherian domain with one nonzero proper ideal $I = xA_1(K)$. R/I is a field as it is isomorphic to K. Since $I^2 = I$, R is fully idempotent. But since R is a non-simple right Noetherian ring, it is not fully right idempotent. Hence, by Proposition 2, P_I is not splitting.

Our next example is an example of a von Neumann regular ring over which P_I is not splitting for a principal ideal.

Example 4. Denote the cardinality of a denumerable set by $\aleph_0$, and $\aleph_1$ be the smallest cardinal number greater than $\aleph_0$. Let $R = Hom_D(V, V)$ where V is a right vector space over a division ring D with $\dim_D(V) = \aleph_1$. Since R is a von Neumann regular ring, it is in particular, fully right idempotent. R has exactly two nonzero proper ideals $L = \{f \in R \mid \dim f(V) < \aleph_0\}$ and $M = \{f \in R \mid \dim f(V) < \aleph_1\}$ both of which are principal and $L \subset M$. Therefore, R is a prime fully right idempotent ring with R/L being not semisimple Artinian. Hence by Theorem 3, P_I is not splitting for every ideal of R.

We next investigate a necessary and sufficient condition for a ring to have the property that P_I is splitting for a single ideal I. A ring R is said to be *normal* if every idempotent of R is central. A ring is *completely normal* if every factor ring of R is normal. Note that a normal right hereditary ring R has no nonzero nilpotent elements.

Theorem 4. *Let R be a completely normal ring and let I be an ideal of R. Then the following statements are equivalent:*

1) P_I *is splitting.*

2) R/I *is a flat left R-module and there exists a central idempotent e of R such that* $eR \supseteq I$ *and* eR/I *is semisimple Artinian.*

Proof. 1) $\Rightarrow$ 2) By Lemma 1, R/I is a flat left R-module. Since P_I is splitting, there exists a central idempotent $f \in R$ such that $l_R(I) = fR$. Put $e = 1 - f$. Then it is clear that $eR \supseteq I$ and thus $P_I(eR) = 0$. Hence, by Theorem 1, eR/I is a right hereditary right perfect ring. Since eR/I is a normal right hereditary ring, it has no nonzero nilpotent elements. Thus, by [1, Theorem 28.4], it is semisimple Artinian.

2) $\Rightarrow$ 1) Let M be a right R-module. Then since $eR \supseteq I$, $P_I(M) = M(1-e) \oplus P_I(Me)$. But since R/I is a flat left R-module, so is eR/I and hence, by Lemma 2, $P_I(Me)$ is a direct summand of Me. Therefore, $P_I(M)$ is a direct summand of M.

A commutative ring is in particular completely normal and the assumption $P_I(R) = 0$ yields that R/I is right hereditary and right perfect. Thus the corollary below is now evident by Theorem 4.

Corollary 4. *Let R be a commutative ring and let I be an ideal of R such that* $P_I(R) = 0$. *Then* P_I *is splitting if and only if* R/I *is a flat left R-module and* R/I *is a finite direct sum of fields.*

Theorem 5. *Let R be a fully right idempotent ring and let I be an ideal of R. Then the following statements are equivalent:*

1) P_I *is splitting.*

2) There exists a central idempotent e of R such that $eR \supseteq I$ *and* eR/I *is semisimple Artinian.*

Proof. 1) $\Rightarrow$ 2) Since P_I is splitting, there exists an idempotent $f \in R$ such that $l_R(I) = fR$. Since a fully idempotent ring is semiprime, f must be central. Put $e = 1-f$. Now, as was the case in the proof of Theorem 4, $eR \supseteq I$ *and* eR/I is a right perfect ring. But since R is fully right idempotent and every ideal of eR is an ideal of R, eR is fully idempotent and hence, so is eR/I. Since any fully right idempotent ring is semiprimitive, we now conclude that eR/I is semisimple Artinian by [1, Theorem 28.4].

2) $\Rightarrow$ 1) Since R is fully right idempotent, for any a in an ideal I of R, there exists $c \in aR \subseteq I$ such that $a = ac$. Hence R/I is flat as a left R-module by [15, Proposition 11.3.13]. The result now follows by the same proof: 2) $\Rightarrow$ 1) of Theorem 4.

We now turn to consider the conditions under which a ring with the property that P_I is splitting for every ideal I of R is semisimple Artinian.

Theorem 6. *Suppose that R is a right fully bounded Noetherian ring and that P_I is splitting for every maximal ideal I of R. Then R is semisimple Artinian.*

Proof. Since a right bounded Noetherian simple ring is Artinian, by Proposition 4, it follows that every prime factor ring of R is simple Artinian. Since R is a Noetherian ring, it has only finitely many minimal prime ideals P_i ($i = 1,2,\cdots,n$) and their intersection is equal to the prime radical $B(R)$ of R. But R/P_i is simple Artinian for each i. Hence each P_i is a maximal ideal. By hypothesis P_{P_i} is splitting, and so R/P_i is flat as a left R-module. Let x be an arbitrary element of $B(R)$. By [15, Proposition 11.3.13], there exists c_i in P_i such that $xc_i = x$. Set $c = c_1 \cdot c_2 \cdot \cdots \cdot c_n$. Then c is in $B(R)$ and $xc = x$, that is $x(1-c) = 0$. Since c is nilpotent, $1-c$ is invertible. This implies $x = 0$, and therefore $B(R) = 0$. Hence R is embedded in the direct sum of R/P_i's. Since this direct sum is Artinian as a right R-module, R is also

Artinian as a right R-module.

Theorem 7. *Let R be a ring all of whose right primitive factor rings are Artinian. If P_I is splitting for every ideal I of R, then R is a semisimple Artinian ring.*

Proof. By Theorem 2, R is a finite direct sum of prime rings. Hence, without loss of generality, we may assume that R is a prime ring. By Proposition 2, R is fully right idempotent. Hence by [2, Theorem], R is also von Neumann regular. Thus by [9, Theorem 6.6], every nonzero ideals of R contains a nonzero central idempotent. Since R is prime, this implies that R is simple. As a simple ring is primitive, we now conclude that R is a simple Artinian ring.

Corollary 6. *Let R be a ring satisfying a polynomial identity. If P_I is splitting for every ideal I of R, then R is a semisimple Artinian ring.*

A ring R is called a *biregular ring* if every principal ideal is generated by a central idempotent. For a biregular ring, it is evident that P_I is splitting for every principal ideal I of R. Noting that the hypothesis "*P_I is splitting for every ideal I of R*" in Theorem 7 cannot be replaced by "*P_I is splitting for every principal ideal I of R*," we hereby insert the following conjecture.

Conjecture 1. *Let R be a ring all of whose right primitive factor rings are Artinian. If P_I is splitting for every principal ideal I of R, then R is a biregular ring.*

Theorem 8. *Let R be a fully right bounded ring. If P_I is splitting for every right ideal I of R, then R is a semisimple Artinian ring.*

Proof. We shall assume that R is right primitive and prove that it is simple Artinian. The result then follows by Theorem 7. Since R is right bounded, it follow that R has a minimal right ideal I. Consider now that the direct sum $I^{(\mathbb{N})}$ of countably infinite copies of I and let $E = E(I^{(\mathbb{N})})$ denote its injective hull. Then one can show that $P_M^I(E) = I^{(\mathbb{N})}$. By hypothesis, $I^{(\mathbb{N})}$ is then a direct summand of E and thereby, it is injective. If RI is not finitely generated as a right ideal, then there is an epimorphism $\phi : RI \to I^{(\mathbb{N})}$. Since $I^{(\mathbb{N})}$ is injective, ϕ extends to a homomorphism $\overline{\phi} : R \to I^{(\mathbb{N})}$. But then, $I^{(\mathbb{N})} = \mathrm{Im}\,\phi \subseteq \mathrm{Im}\,\overline{\phi} = \overline{\phi}(1)R$. Since $\overline{\phi}(1)R$ is contained in a direct sum of finitely many copies of I, this is a contradiction. Therefore RI is a finitely generated right ideal. Now, since I is an idempotent, we may write $RI = a_1I + a_2I + \cdots + a_mI$ for some $a_1, a_2, \ldots, a_m$ in R. Thus RI is a homomorphic image of the direct sum of m copies of I and therefore, it is completely reducible. This will yield that $RI = eR$ for some central idempotent e in R. As R is prime, it is now evident that $RI = R$ and therefore, R is in fact a simple Artinian ring.

Corollary 7. *Let R be a right semi-Artinian ring. If P_I is splitting for every right ideal I of R, then R is a semisimple Artinian ring.*

Proposition 4. *Assume that R has, up to isomorphism, finitely many simple right R-modules. If P_I is splitting for every right ideal I of R, then R is a finite direct sum of* *simple right Noetherian right V-rings..*

We now conclude our paper by a conjecture for the general structure of rings over which P_I is splitting for every right ideal I of R.

Conjecture 2. If P_I is splitting for every right ideal I of R, then R is a finite direct sum of simple right V-rings.

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FINITENESS OF REPRESENTATION DIMENSION [1]

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M. Auslander introduced a concept of representation dimension of artin algebras in [A], which was a trial to give a reasonable way of measuring homologically how far an artin algebra is from being of finite representation type. His methods given there have been effectively applied not only for the representation theory of artin algebras, but also for the theory of quasi-hereditary algebras of Cline-Parshall-Scott [CPS] by Dlab and Ringel in [DR2]. Unfortunately, much seems to be unknown about representation dimension itself, especially whether any artin algebra has a finite representation dimension or not. In §1.3, we will give a positive answer to this question by showing that any module is a direct summand of some module whose endomorphism ring is quasi-hereditary. These were conjectured by Ringel and Yamagata [X2].

Our method is to construct certain filtration of subcategories of $\operatorname{mod}\Lambda$ (§2.2). We will formulate it in terms of rejective subcategories (§2.1), which was effectively applied in [I1] to study the representation theory of orders (see §2.1.2) and give a characterization of their finite Auslander-Reiten quivers in [I2]. Our filtration is an analogy of preprojective partition given by Auslander and Smalo [AS], which was related to quasi-hereditary algebras by Dlab and Ringel [DR3]. In [I3], our method will be applied to solve Solomon's second conjecture on zeta functions of orders.

1 In this paper, any module is assumed to be a left module. For an artin algebra Λ over R, let $\operatorname{mod}\Lambda$ (resp. $\operatorname{pr}\Lambda$) be the category of finitely generated left Λ-modules (resp. projective Λ-modules), J_Λ the Jacobson radical of Λ, $\operatorname{dom.dim}\Lambda$ the dominant dimension of Λ [T], $\mathrm{I}_\Lambda(X)$ the injective hull of the Λ-module X and $(\ )^* := \operatorname{Hom}_R(\ ,\mathrm{I}_R(R/J_R)) : \operatorname{mod}\Lambda \leftrightarrow \operatorname{mod}\Lambda^{op}$ the duality. For $X \in \operatorname{mod}\Lambda$, we denote by $\operatorname{add}X$ the full subcategory of $\operatorname{mod}\Lambda$ consisting of direct summands of a finite direct sum of X.

1.1 The *representation dimension* of an artin algebra Λ is defined by $\operatorname{rep.dim}\Lambda := \inf\{\operatorname{gl.dim}\Gamma \mid \Gamma \in A(\Lambda)\}$, where $A(\Lambda)$ is the collection of all artin algebras Γ such that $\operatorname{dom.dim}\Gamma \geq 2$ and $\operatorname{End}_\Gamma(\mathrm{I}_\Gamma(\Gamma))$ is Morita-equivalent to Λ. We collect some known results which will not be used in this paper.

[1] The detailed version of this paper will be submitted elsewhere.

(1) $\mathrm{rep.dim}\,\Lambda = \inf\{\mathrm{gl.dim}\,\mathrm{End}_\Lambda(M) \mid M \in \mathrm{mod}\,\Lambda$ such that $\Lambda \oplus \Lambda^* \in \mathrm{add}\,M\}$ holds (see [A] for (1)–(4)).

(2) Λ is of finite representation type if and only if $\mathrm{rep.dim}\,\Lambda \leq 2$.

(3) If $\mathrm{gl.dim}\,\Lambda \leq 1$, then $\mathrm{rep.dim}\,\Lambda \leq 3$.

(4) If Λ is selfinjective, then $\mathrm{rep.dim}\,\Lambda \leq$ (the Loewy length of Λ).

(5) If R is a perfect field, then $\mathrm{rep.dim}(\Lambda \otimes_R \Gamma) \leq \mathrm{rep.dim}\,\Lambda + \mathrm{rep.dim}\,\Gamma$ [X1].

(6) $\mathrm{rep.dim}\,\mathrm{T}_2(\Lambda) \leq \mathrm{rep.dim}\,\Lambda + 2$ [FGR].

1.2 Let Λ be an artin algebra and I an 2-sided ideal of Λ. We call I a *heredity ideal* of Λ if $I^2 = I \in \mathrm{pr}\,\Lambda$ and $IJ_\Lambda I = 0$ hold.

We call Λ a *quasi-hereditary algebra* if there exists a chain $0 = I_m \subseteq I_{m-1} \subseteq \cdots \subseteq I_0 = \Lambda$ such that I_{n-1}/I_n is a heredity ideal of Λ/I_n for any n $(0 < n \leq m)$. In this case, $\mathrm{gl.dim}\,\Lambda \leq 2m - 2$ holds by [DR1], and $\mathrm{mod}\,\Lambda$ forms a *highest weight category* [CPS].

1.3 Main Theorem (Proof in 2.3) *Let Λ be an artin algebra.*

(1) Any $M \in \mathrm{mod}\,\Lambda$ is a direct summand of some $N \in \mathrm{mod}\,\Lambda$ such that $\mathrm{End}_\Lambda(N)$ is a quasi-hereditary algebra.

(2) $\mathrm{rep.dim}\,\Lambda$ has a finite value which is not greater than $2l - 2$, where l is the length of a $(\Lambda, \mathrm{End}_\Lambda(\Lambda \oplus \Lambda^*))$-module $\Lambda \oplus \Lambda^*$.

2 Rejective subcategories In the rest, any subcategory $\mathcal{C}'$ of an additive category $\mathcal{C}$ is assumed to be full and closed under direct sums. Let $\mathcal{J}_\mathcal{C}$ be the Jacobson radical of $\mathcal{C}$ and $[\mathcal{C}']$ the ideal of $\mathcal{C}$ consisting of morphisms which factor through some object in $\mathcal{C}'$. Thus $\mathcal{J}_\mathcal{C}(X, X)$ forms the Jacobson radical of the ring $\mathcal{C}(X, X)$ for any $X \in \mathcal{C}$.

2.1 Let $\mathcal{C}$ be an additive category and $\mathcal{C}'$ a subcategory of $\mathcal{C}$. Then $\mathcal{C}'$ is called a *right rejective subcategory* of $\mathcal{C}$ if the inclusion functor $\mathcal{C}' \to \mathcal{C}$ has a right adjoint $\mathbb{F} : \mathcal{C} \to \mathcal{C}'$ with a counit ϵ [HS] such that ϵ_X is a monomorphism for any $X \in \mathcal{C}$ (cf. [I1]5.1). In other word, $\mathcal{C}(\ ,\mathbb{F}(X)) \xrightarrow{\epsilon_X} [\mathcal{C}'](\ ,X)$ is an isomorphism on $\mathcal{C}$ for any $X \in \mathcal{C}$.

If $\Gamma := \mathcal{C}(M, M)$ is an artin algebra for an additive generator M of $\mathcal{C}$, then a bijection $\{\mathcal{C}' :$ right rejective subcategory of $\mathcal{C}$ such that $\mathcal{J}_{\mathcal{C}'} = 0\} \to \{I :$ heredity ideal of $\Gamma\}$ is given by $\mathcal{C}' \mapsto I := [\mathcal{C}'](M, M)$, and its inverse is given by $I \mapsto \mathcal{C}'$ for a subcategory $\mathcal{C}'$ of $\mathcal{C}$ such that $I = [\mathcal{C}'](M, M)$.

PROOF We only show the former assertion. First, $I := [\mathcal{C}'](M,M)$ is isomorphic to a projective Γ-module $\mathcal{C}(M,\mathbb{F}(M))$, and $I^2 = I$ holds by $[\mathcal{C}']^2 = [\mathcal{C}']$. Moreover, $IJ_\Gamma I = 0$ holds by $\mathcal{J}_{\mathcal{C}'} = 0$.■

2.1.1 Let $\mathcal{C}'$ be a right rejective subcategory of $\mathcal{C}$ and $\mathcal{C}''$ a subcategory of $\mathcal{C}'$. Then $\mathcal{C}'/[\mathcal{C}'']$ is a right rejective subcategory of $\mathcal{C}/[\mathcal{C}'']$ since the isomorphism $\mathcal{C}(\ ,\mathbb{F}(X)) \xrightarrow{\epsilon_X} [\mathcal{C}'](\ ,X)$ induces an isomorphism $[\mathcal{C}''](\ ,\mathbb{F}(X)) \xrightarrow{\epsilon_X} [\mathcal{C}''](\ ,X)$. Moreover, if $\mathcal{C}''$ is a right rejective subcategory of $\mathcal{C}'$, then it is a right rejective subcategory of $\mathcal{C}$.

2.1.2 Remark In this subsection, we will explain the relationship with overrings [I1], which will not be used in this paper.

(1) Define a *left rejective subcategory* by the dual of 2.1. We call $\mathcal{C}'$ a *rejective subcategory* of $\mathcal{C}$ if it is right and left rejective subcategory of $\mathcal{C}$.

(2) Let R be a complete regular local ring of dimension $d \geq 0$ with the quotient field K. An R-algebra Λ is called an *R-order* if it is finitely generated free as an R-module. Assume that Λ is an R-order. A left Λ-module L is called a *Λ-lattice* if it is finitely generated free as an R-module. We denote by $\operatorname{lat}\Lambda$ the category of Λ-lattices. Another R-order Γ is called an *overring* of Λ if $(K \otimes_R \Lambda)/I \supset \Gamma \supseteq (\Lambda + I)/I$ holds for an ideal I of $K \otimes_R \Lambda$. Then the natural morphism $\Lambda \to \Gamma$ induces a full faithful functor $\operatorname{lat}\Gamma \to \operatorname{lat}\Lambda$.

Then the following fundamental fact in [I1] asserts that right rejective subcategories can be regarded as a generalization of overrings.

Proposition *Let R be a complete regular local ring of dimension $d \leq 2$, Λ an R-order and $\mathcal{C}'$ a subcategory of $\mathcal{C} := \operatorname{lat}\Lambda$. Then $\mathcal{C}'$ is a rejective subcategory of $\mathcal{C}$ if and only if $\mathcal{C}' = \operatorname{lat}\Gamma$ for an overring Γ of Λ. In this case, the inclusion functor has the right adjoint $\operatorname{Hom}_\Lambda(\Gamma,\ )$ and the left adjoint $(\Gamma\otimes_\Lambda\ )^{**}$ for $(\ )^* := \operatorname{Hom}_R(\ ,R)$.*

2.2 Let $\mathcal{C}$ be an additive category. Then $0 = \mathcal{C}_m \subseteq \mathcal{C}_{m-1} \subseteq \cdots \subseteq \mathcal{C}_0 = \mathcal{C}$ is called a *right rejective chain* if $\mathcal{J}_{\mathcal{C}_n/[\mathcal{C}_{n+1}]} = 0$ holds and $\mathcal{C}_{n+1}$ is a right rejective subcategory of $\mathcal{C}_n$ for any n $(0 \leq n < m)$. In this case, $\mathcal{C}_{n'}/[\mathcal{C}_{n''}]$ is a right rejective subcategory of $\mathcal{C}_n/[\mathcal{C}_{n''}]$ for any $n'' \leq n' \leq n$ by 2.1.1.

If $\Gamma := \mathcal{C}(M,M)$ is an artin algebra for an additive generator M of $\mathcal{C}$, then Γ is a quasi-hereditary algebra with a heredity chain $0 = [\mathcal{C}_m](M,M) \subseteq [\mathcal{C}_{m-1}](M,M) \subseteq \cdots \subseteq [\mathcal{C}_0](M,M) = \Gamma$.

2.2.1 Proof of 2.2 $\mathcal{C}_{m-1}$ is also a right rejective subcategory of $\mathcal{C}$. By 2.1, $I := [\mathcal{C}_{m-1}](M, M)$ is a heredity ideal of Γ. Since $0 = \mathcal{C}_{m-1}/[\mathcal{C}_{m-1}] \subseteq \mathcal{C}_{m-2}/[\mathcal{C}_{m-1}] \subseteq \cdots \subseteq \mathcal{C}_0/[\mathcal{C}_{m-1}] = \mathcal{C}/[\mathcal{C}_{m-1}]$ is again a right rejective chain, we obtain the assertion inductively.∎

2.3 Our result 1.3 immediately follows from the following lemma (Put $M := \Lambda \oplus \Lambda^*$ for (2)).

Lemma *Let* Λ *be an artin algebra and* $M \in \operatorname{mod}\Lambda$. *Put* $M_0 := M$, $M_{n+1} := M_n J_{\operatorname{End}_\Lambda(M_n)} \subsetneq M_n$ *and take large* m *such that* $M_m = 0$. *Then* $0 = \mathcal{C}_m \subseteq \mathcal{C}_{m-1} \subseteq \cdots \subseteq \mathcal{C}_0 = \mathcal{C}$ *gives a right rejective chain for* $\mathcal{C}_n := \operatorname{add}\bigoplus_{l=n}^{m-1} M_l$. *Thus* $\Gamma := \operatorname{End}_\Lambda(N)$ *is a quasi-hereditary algebra for* $N := \bigoplus_{l=0}^{m-1} M_l$ *such that* $\operatorname{gl.dim}\Gamma \leq 2m-2$.

Proof (i) For any $n < l$, there exists a surjection $f_{n,l} \in \mathcal{J}_{\operatorname{mod}\Lambda}^{l-n}(\bigoplus M_n, M_l)$.

(ii) Define a functor $\mathbb{F}_n : \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda$ by $\mathbb{F}_n(X) := \sum_{Y \in \mathcal{C}_n,\ f \in \mathcal{J}_{\operatorname{mod}\Lambda}(Y,X)} f(Y)$. Then a natural transformation $\epsilon : \mathbb{F}_n \to 1$ is defined by the inclusion $\epsilon_X : \mathbb{F}_n(X) \to X$. For $X \in \mathcal{C}_{n+1}$, we obtain $\mathbb{F}_n(X) = X$ and $\epsilon_X = 1_X$ by (i). Moreover, $\mathbb{F}_n(M_n) = M_n J_{\operatorname{End}_\Lambda(M_n)} = M_{n+1}$ holds and $\operatorname{Hom}_\Lambda(M_l, M_{n+1}) \xrightarrow{\epsilon_{M_n}} \operatorname{Hom}_\Lambda(M_l, M_n)$ is an isomorphism for any $n < l$ by (i). Thus $\mathbb{F}_n$ gives a right adjoint of the inclusion $\mathcal{C}_{n+1} \to \mathcal{C}_n$ with a counit ϵ. Since $\operatorname{Hom}_\Lambda(M_l, M_{n+1}) \xrightarrow{\epsilon_{M_n}} \mathcal{J}_{\operatorname{mod}\Lambda}(M_l, M_n)$ is an isomorphism for any $n \leq l$ by (i), $\mathcal{J}_{\mathcal{C}_n/[\mathcal{C}_{n+1}]} = 0$ holds. Thus our chain is right rejective. Now $\operatorname{gl.dim}\Gamma \leq 2m-2$ follows by 1.2.∎

3 Question There still remains a problem to determine the subset $\{\operatorname{rep.dim}\Lambda \mid \Lambda \text{ is an artin algebra}\}$ of $\mathbb{N}$.

(1) I don't know an example of Λ such that $\operatorname{rep.dim}\Lambda > 3$.

(2) In [IT]0.8, Igusa and Todorov obtained the following interesting result.

Proposition *Let* Γ *be an artin algebra with* $\operatorname{gl.dim}\Gamma \leq 3$ *and* $P \in \operatorname{pr}\Gamma$. *Then* $\Lambda := \operatorname{End}_\Gamma(P)$ *satisfies* $\operatorname{fin.dim}\Lambda < \infty$.

As an easy conclusion, we obtain that $\operatorname{rep.dim}\Lambda \leq 3$ implies $\operatorname{fin.dim}\Lambda < \infty$. (We may put $\Gamma := \operatorname{End}_\Lambda(M)$ and $P := \operatorname{Hom}_\Lambda(M, \Lambda)$ for $M \in \operatorname{mod}\Lambda$ such that $\Lambda \oplus \Lambda^* \in \operatorname{add} M$ and $\operatorname{End}_\Lambda(M) = \operatorname{rep.dim}\Lambda \leq 3$.) Thus, from the viewpoint of the finitistic global dimension conjecture, it is an interesting question whether any artin algebra Λ satisfies $\operatorname{rep.dim}\Lambda \leq 3$ or not [A]. If rep.dim $\Lambda \leq 3$ always holds, then the finitistic global dimension conjecture, (general) Nakayama conjecture [AR] e.t.c. follow.

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A PROOF OF SOLOMON'S SECOND CONJECTURE ON LOCAL ZETA FUNCTIONS OF ORDERS [1]

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Let R be the ring $\mathbb{Z}$ of integers or its p-adic completion $\mathbb{Z}_p$, and K its quotient field. For an R-order Λ in a semisimple K-algebra A, its *Solomon zeta function* is defined by $\zeta_\Lambda(s) := \sum_L (\Lambda : L)^{-s}$ where L is a left ideal of Λ such that $(\Lambda : L) < \infty$ and s is a complex variable [S1]. Then ζ_Λ converges in the half-plane $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > \dim_K A\}$, and it can be shown that ζ_Λ admits analytic continuation to a meromorphic function of s [S1]. Later, Bushnell and Reiner developed the adelic approach for Solomon zeta functions ([BR1][BR2] e.t.c.). Moreover, they applied Solomon zeta functions to generalize the prime ideal theorem [BR3] and the asymptotic distribution formula of ideals [BR4] and so on.

For the case $R = \mathbb{Z}$, we have the *Euler product formula* $\zeta_\Lambda = \prod_{p:\mathrm{prime}} \zeta_{\Lambda_p}$ for $\Lambda_p := \mathbb{Z}_p \otimes_{\mathbb{Z}} \Lambda$ [S1]. For a maximal overorder Γ of Λ, one can describe ζ_Γ by using Dedekind zeta function [BR2]§2. Since $\{p : \text{prime} \mid \Lambda_p \neq \Gamma_p\}$ is a finite set, the difference between ζ_Λ and ζ_Γ appear at only finitely many primes. Thus, in the rest, we will study the case $R = \mathbb{Z}_p$.

1 Local case In the rest, let R be a complete discrete valuation ring with the residue field k and the quotient field K, and Λ an R-order in a semisimple K-algebra A. We assume that k is a finite field with p elements. Since Λ is not necessarily commutative, it will be more natural to define zeta function for modules.

1.1 Definition For an A-module V of finite length, we denote by $\mathfrak{L}_\Lambda(V)$ the set of full Λ-lattices in V. Then $\overline{\mathfrak{L}}_\Lambda(V) := \mathfrak{L}_\Lambda(V)/\simeq$ is a finite set by Jordan-Zassenhaus Theorem [CR]. For $L, M \in \overline{\mathfrak{L}}_\Lambda(V)$, Solomon [S1] studied

a partial zeta function $\quad Z_\Lambda(L, M; s) := \sum_{N \subseteq L,\ N \simeq M} (L : N)^{-s}$

and the $n \times n$-matrix $\quad \mathrm{Z}_\Lambda(V; s) := (Z_\Lambda(L, M; s))_{L,M \in \overline{\mathfrak{L}}_\Lambda(V)} \quad (n := \#\overline{\mathfrak{L}}_\Lambda(V))$.

He proved that $\mathrm{Z}_\Lambda(V; s)$ has an inverse matrix in $\mathrm{M}_n(\mathbb{Z}[p^{-s}])$ by a combinatorial argument (Möbius inversion), so $Z_\Lambda(L, M; s)$ is a rational function of p^{-s}. Moreover, he gave the following conjectures in [S2].

[1] The detailed version of this paper will be submitted elsewhere.

(1) $Z_\Lambda(L, M; s)/\det \mathrm{Z}_\Gamma(V; s) \in \mathbb{Z}[p^{-s}]$ for a maximal order Γ in A.

(2) $\det \mathrm{Z}_\Lambda(V; s)$ should be the finite product $\prod_i (1 - p^{a_i - b_i s})^{-1}$ with some $a_i \in \mathbb{N}_{\geq 0}$ and $b_i \in \mathbb{N}_{>0}$.

His first conjecture (1) was proved in [**BR1**] by their analytic approach. However his second conjecture (2) seems to be still open, although a special case when Λ is hereditary was proved by Denert [D]. In this paper, by purely ring theoretic method, we will give an explicit description of $\det \mathrm{Z}_\Lambda(V; s)$ for general Λ in §2.1, which implies the second conjecture. A key idea of our proof is to consider certain filtration of the category of Λ-lattices (§2.3) and use a reduction to smaller categories (§3.2). Our filtration was already used in [I1] (see §2.3.1 below).

2 Let $\mathrm{lat}\,\Lambda$ be the category of Λ-lattices, $\mathrm{mod}\,A$ the category of finitely generated A-modules, and $\widetilde{(\ )} := (\ ) \otimes_R K : \mathrm{lat}\,\Lambda \to \mathrm{mod}\,A$. Put $A = \prod_{j=1}^r A_j$ for simple algebras A_j. Let e_j be the identity of A_j, Γ_j a maximal overorder of $e_j\Lambda$ in A_j, $\Gamma := \prod_{j=1}^r \Gamma_j$, S_j a simple A_j-module, and G_j a simple Γ_j-module. Then $S_1, \ldots, S_r$ (respectively, $G_1, \ldots, G_r$) give a complete representatives of the isomorphism classes of simple A-modules (respectively, simple Γ-modules). For $X \in \mathrm{mod}\,A$, we denote by $l_j(X)$ the multiplicity of S_j as a composition factor of X. Put $q_j := \#\,\mathrm{End}_\Gamma(G_j) = p^{\dim_k \mathrm{End}_\Gamma(G_j)}$ $(1 \leq j \leq r)$. Notice that q_j does not depend on a choice of Γ_j since any maximal order in A_j is conjugate to Γ_j.

2.1 Main Theorem *Let Λ be an R-order in a semisimple algebra A, $V \in \mathrm{mod}\,A$ and $V_j := V/S_j^{l_j(V)}$ $(1 \leq j \leq r)$. Then*

$$\det \mathrm{Z}_\Lambda(V; s) = \prod_{j=1}^{r} \prod_{n=0}^{l_j(V)-1} (1 - q_j^{n - l_j(A)s})^{-\#\overline{\mathfrak{L}}_\Lambda(S_j^n \oplus V_j)}.$$

More generally, let $\mathcal{C}$ be a right-closed subcategory of $\mathrm{lat}\,\Lambda$ *(defined in 2.2 below) such that $\mathcal{C} \supseteq \mathrm{lat}\,\Gamma$ for a maximal overorder Γ of Λ. Then*

$$\det \mathrm{Z}_{\Lambda,\mathcal{C}}(V; s) = \prod_{j=1}^{r} \prod_{n=0}^{l_j(V)-1} (1 - q_j^{n - l_j(A)s})^{-\#\overline{\mathfrak{L}}_{\mathcal{C}}(S_j^n \oplus V_j)}.$$

2.2 Since we will prove 2.1 inductively, we need a categorical generalization $\mathrm{Z}_{\Lambda,\mathcal{C}}$ of Z_Λ defined as follows: In the rest, any subcategory $\mathcal{C}$ of $\mathrm{lat}\,\Lambda$ is assumed to be *full, closed under isomorphisms, direct sums and direct summands.* We denote by $\operatorname{ind}\mathcal{C}$ the set of isomorphism classes of indecomposable objects in $\mathcal{C}$. Thus the correspondence $\mathcal{C} \mapsto \operatorname{ind}\mathcal{C}$ gives a bijection from subcategories of $\operatorname{lat}\Lambda$ to subsets of $\operatorname{ind}(\operatorname{lat}\Lambda)$, and the inverse is denoted by $\mathcal{S} \mapsto \operatorname{add}\mathcal{S}$. We denote by $\mathcal{J}_{\operatorname{lat}\Lambda}$ the Jacobson radical of the category $\operatorname{lat}\Lambda$. Thus $\mathcal{J}_{\operatorname{lat}\Lambda}(X,X)$ is the Jacobson radical of $\operatorname{End}_\Lambda(X)$ for any $X \in \operatorname{lat}\Lambda$.

We denote by $\mathfrak{L}_{\mathcal{C}}(V)$ (respectively, $\overline{\mathfrak{L}}_{\mathcal{C}}(V)$) the subset of $\mathfrak{L}_\Lambda(V)$ (respectively, $\overline{\mathfrak{L}}_\Lambda(V)$) consisting of objects in $\mathcal{C}$. Put $\mathrm{Z}_{\Lambda,\mathcal{C}}(V;s) := (Z_\Lambda(L,M;s))_{L,M\in\overline{\mathfrak{L}}_{\mathcal{C}}(V)}$. As in the proof of [I1]2.3, define a functor $\mathbb{F}_{\mathcal{C}} : \operatorname{lat}\Lambda \to \operatorname{lat}\Lambda$ by

$$\mathbb{F}_{\mathcal{C}}(X) := \sum_{Y\in\mathcal{C},\ f\in\mathcal{J}_{\operatorname{lat}\Lambda}(Y,X)} f(Y) \subseteq X.$$

We call $\mathcal{C}$ *right-closed* if $\#\operatorname{ind}\mathcal{C} < \infty$ and $\mathbb{F}_{\mathcal{C}'}(X) \in \mathcal{C}$ holds for any subcategory $\mathcal{C}'$ of $\mathcal{C}$ and $X \in \mathcal{C}'$. For example, $\operatorname{lat}_V\Lambda := \operatorname{add}\{X \in \operatorname{ind}(\operatorname{lat}\Lambda) \mid \widetilde{X} \subseteq V\}$ is right-closed for any $V \in \operatorname{mod} A$.

2.3 Proposition *Let* Γ *be a maximal overorder of* Λ *and* $\mathcal{C}$ *a right-closed subcategory of* $\operatorname{lat}\Lambda$ *such that* $\mathcal{C} \supseteq \operatorname{lat}\Gamma$*. Then there exists a chain* $\mathcal{C} = \mathcal{C}_m \supset \mathcal{C}_{m-1} \supset \cdots \supset \mathcal{C}_0 = \operatorname{lat}\Gamma$ *consisting of right-closed subcategories such that* $\operatorname{ind}\mathcal{C}_n - \operatorname{ind}\mathcal{C}_{n-1} = \{X_n\}$ *and* $\mathbb{F}_{\mathcal{C}_n}(X_n) \in \mathcal{C}_{n-1}$ $(0 < n \leq m)$*.*

2.3.1 Remark In 2.3, let M be an additive generator of $\mathcal{C}$. It is not difficult to show that the above chain satisfies the conditions of right rejective chains (in [I1]2.2) except $\mathcal{C}_0 = 0$. In particular, $\operatorname{End}_\Lambda(M)$ is a quasi-hereditary order in the sense of König and Wiedemann [**KW**] with a heredity chain $\operatorname{End}_\Lambda(M) = [\mathcal{C}_m](M,M) \supset [\mathcal{C}_{m-1}](M,M) \supset \cdots \supset [\mathcal{C}_0](M,M)$.

2.3.2 Let $\mathcal{C}$ a subcategory of $\operatorname{lat}\Lambda$ such that $\operatorname{ind}\mathcal{C} < \infty$ and $\mathcal{C} \supseteq \operatorname{lat}\Gamma$. If $\mathbb{F}_{\mathcal{C}}(X) \simeq X$ holds for any $X \in \operatorname{ind}\mathcal{C} - \operatorname{ind}(\operatorname{lat}\Gamma)$, then $\mathcal{C} = \operatorname{lat}\Gamma$.

Proof Put $\mathcal{S} := \{X \in \operatorname{ind}\mathcal{C} \mid \mathbb{F}_{\mathcal{C}}(X) \subsetneq X\}$. For any $L \in \mathcal{S} - \operatorname{ind}(\operatorname{lat}\Gamma)$, let f be the composition of an isomorphism $L \to \mathbb{F}_{\mathcal{C}}(L)$ and the natural inclusion $\mathbb{F}_{\mathcal{C}}(L) \to L$. Since f is in the radical of $\operatorname{End}_\Lambda(L)$ and $\operatorname{Hom}_\Lambda(\Gamma, L) = \operatorname{Hom}_\Lambda(\Gamma, L)f$ holds, we obtain $\operatorname{Hom}_\Lambda(\Gamma, L) = 0$ by Nakayama's Lemma, a contradiction. Thus $\mathcal{S} \subseteq \operatorname{ind}(\operatorname{lat}\Gamma)$ holds. Put $M := \bigoplus_{X\in\operatorname{ind}\mathcal{C}-\mathcal{S}} X$ and $N := \mathbb{F}_{\operatorname{add}\mathcal{S}}(M)$, which are right $\operatorname{End}_\Lambda(M)$-modules. Since $M = \mathbb{F}_{\mathcal{C}}(M) = MJ + N$ holds for the Jacobson radical J of $\operatorname{End}_\Lambda(M)$, we obtain $M = N$

by Nakayama's Lemma. Since $\mathcal{S}$ is closed under surjections, we obtain $M \in \operatorname{add}\mathcal{S}$. Thus $M = 0$ and $\operatorname{ind}\mathcal{C} = \mathcal{S} \subseteq \operatorname{ind}(\operatorname{lat}\Gamma)$ holds.∎

2.3.3 Proof of 2.3 Assume $\mathcal{C} \supsetneq \operatorname{lat}\Gamma$. There exists $X \in \operatorname{ind}\mathcal{C} - \operatorname{ind}(\operatorname{lat}\Gamma)$ such that $\mathbb{F}_{\mathcal{C}}(X) \not\simeq X$ by 2.3.2. Put $\mathcal{C}' := \operatorname{add}(\operatorname{ind}\mathcal{C} - \{X\})$, then $\mathbb{F}_{\mathcal{C}}(X) \in \mathcal{C}'$ holds. For any subcategory $\mathcal{C}''$ of $\mathcal{C}'$ and $L \in \mathcal{C}''$, X is not a direct summand of $\mathbb{F}_{\mathcal{C}''}(L)$ $(\in \mathcal{C})$. Hence $\mathbb{F}_{\mathcal{C}''}(L) \in \mathcal{C}'$ holds. Thus $\mathcal{C}'$ is right-closed again.∎

3 In the rest, we will give a proof of 2.1 *under the assumption* $r = 1$, *namely A is simple.* Thanks to this assumption, our calculation becomes much simpler that the general one in [I2]. Put $S := S_1$, $G := G_1$, $l := l_1$ and $q := q_1$. Moreover, put $(L : M) := (L : L \cap M) \cdot (M : L \cap M)^{-1}$ for $V \in \operatorname{mod} K$ and $L, M \in \mathfrak{L}_R(V)$ for simplicity. This symbol is skew-symmetric, and satisfies $(L : M) \cdot (M : N) = (L : N)$.

3.1 For any $N \in \operatorname{lat}\Lambda$, there exists a map $b^N : \mathfrak{L}_\Lambda(V) \to \mathbb{R}_{>0}$ such that

$$b^N_L \cdot (b^N_M)^{-1} = (\operatorname{Hom}_\Lambda(N, L) : \operatorname{Hom}_\Lambda(N, M)) \cdot (L : M)^{-l(\widetilde{N})/l(A)}$$

holds for any $L, M \in \mathfrak{L}_\Lambda(V)$.

Proof For simplicity, put ${}_\Lambda(L, M) := \operatorname{Hom}_\Lambda(L, M)$. Fix $X \in \mathfrak{L}_\Lambda(V)$. For $L \in \mathfrak{L}_\Lambda(V)$, put $b^N_L := ({}_\Lambda(N, L) : {}_\Lambda(N, X)) \cdot (L : X)^{-l(\widetilde{N})/l(A)}$. We only have to show that $L \simeq M$ implies $b^N_L = b^N_M$, namely $({}_\Lambda(N, L) : {}_\Lambda(N, M)) = (L : M)^{l(\widetilde{N})/l(A)}$. This is true for $N = \Lambda$. Now we assume $\widetilde{N} = \widetilde{N'}$. Since $({}_\Lambda(N, L) : {}_\Lambda(N', L)) = ({}_\Lambda(N, M) : {}_\Lambda(N', M))$ holds, we obtain $({}_\Lambda(N, L) : {}_\Lambda(N, M)) = ({}_\Lambda(N', L) : {}_\Lambda(N', M))$. Thus the assertion follows easily.∎

3.2 Lemma *Let $\mathcal{C}$ be a subcategory of* $\operatorname{lat}\Lambda$ *and* $X \in \operatorname{ind}\mathcal{C}$. *Assume that* $Y := \mathbb{F}_{\mathcal{C}}(X)$ *satisfies* $X \supsetneq Y \in \mathcal{C}$. *Put* $\mathcal{C}' := \operatorname{add}(\operatorname{ind}\mathcal{C} - \{X\})$.

(1) For any $W \in \mathcal{C}$ and $L \in \operatorname{lat}\Lambda$ such that $\widetilde{X \oplus L} \simeq \widetilde{W}$,

$$Z_\Lambda(X \oplus L, W; s) - (X : Y)^{-s} \cdot Z_\Lambda(Y \oplus L, W; s)$$
$$= \begin{cases} b^X_L \cdot (b^X_M)^{-1} \cdot Z_\Lambda(L, M; s - l(\widetilde{X})/l(A)) & \text{if } W \simeq X \oplus M \text{ for some } M. \\ 0 & \text{otherwise.} \end{cases}$$

(2) For any $V \in \operatorname{mod} A$ such that $\widetilde{X} \subseteq V$,

$$(1 - \mathbf{C}) \cdot \mathbf{Z}_{\Lambda,\mathcal{C}}(V; s) = \begin{pmatrix} \mathbf{Z}_{\Lambda,\mathcal{C}'}(V; s) & * \\ O & \mathbf{B} \cdot \mathbf{Z}_{\Lambda,\mathcal{C}}(V/\widetilde{X}; s - l(\widetilde{X})/l(A)) \cdot \mathbf{B}^{-1} \end{pmatrix}$$

holds, where B *and* C *are the matrices such that* $\mathrm{B}_{L,L} = b_L^X$ *and* $\mathrm{C}_{X\oplus L, Y\oplus L} = (X:Y)^{-s}$ *for any* $L \in \overline{\mathfrak{L}}_{\mathcal{C}}(V/\widetilde{X})$ *and other entries are* 0.

PROOF (1) Let $\mathfrak{L} := \{Z \subseteq X\oplus L \mid Z \simeq W\}$ and $p_Z := (Z \subseteq X\oplus L \xrightarrow{\binom{1}{0}} X)$ the composition for $Z \in \mathfrak{L}$. Put $\mathfrak{L}_1 := \{Z \in \mathfrak{L} \mid p_Z$ is not a split epimorphism$\}$. Since $Z \subseteq Y \oplus L$ holds for any $Z \in \mathfrak{L}_1$, we obtain

$$\sum_{Z\in\mathfrak{L}_1} (X\oplus L : Z)^{-s} = (X:Y)^{-s} \cdot Z_\Lambda(Y\oplus L, W; s).$$

Assume $\mathfrak{L} \neq \mathfrak{L}_1$. Then we can put $W \simeq X \oplus M$. For any $N \subseteq L$ such that $N \simeq M$, let $i_N \in \mathrm{Hom}_\Lambda(N, L)$ be the natural inclusion and $\mathfrak{L}_N := \{Z \in \mathfrak{L} - \mathfrak{L}_1 \mid Z \cap L = N\}$. Define a map $\phi_N : \mathrm{Hom}_\Lambda(X, L) \to \mathfrak{L}_N$ by $\phi_N(f) := (\left(\begin{smallmatrix} 1_X & f \\ 0 & i_N \end{smallmatrix}\right) : X \oplus N \subseteq X \oplus L)$ for $f \in \mathrm{Hom}_\Lambda(X, L)$. It is easily checked that $\phi_N(f) = \phi_N(g)$ holds if and only if $f - g$ factors through i_N. Thus we obtain an injection $\phi_N : \mathrm{Hom}_\Lambda(X, L)/\mathrm{Hom}_\Lambda(X, N) \to \mathfrak{L}_N$. Moreover, consider the following commutative diagram for any $Z \in \mathfrak{L}_N$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L & \xrightarrow{(0\ 1)} & X\oplus L & \xrightarrow{\binom{1}{0}} & X & \longrightarrow & 0 \\
 & & \cup^{i_N} & & \cup^{(i_1\ i_2)} & & \| & & \\
0 & \longrightarrow & N & \longrightarrow & Z & \xrightarrow{p_Z} & X & \longrightarrow & 0
\end{array}$$

Taking q such that $qp_Z = 1_X$, we can easily show that $\phi_N(qi_2) = Z$ holds. Hence ϕ_N is a bijection. Now (1) follows from

$$\begin{aligned}
\sum_{Z\in\mathfrak{L}-\mathfrak{L}_1} (X\oplus L : Z)^{-s} &= \sum_{N\subseteq L,\ N\simeq M}\ \sum_{Z\in\mathfrak{L}_N} (X\oplus L : Z)^{-s} \\
&= \sum_{N\subseteq L,\ N\simeq M} (\mathrm{Hom}_\Lambda(X, L) : \mathrm{Hom}_\Lambda(X, N)) \cdot (L:N)^{-s} \\
&\overset{3.1}{=} b_L^X \cdot (b_M^X)^{-1} \cdot \sum_{N\subseteq L,\ N\simeq M} (L:N)^{l(\widetilde{X})/l(A)-s} \\
&= b_L^X \cdot (b_M^X)^{-1} \cdot Z_\Lambda(L, M; s - l(\widetilde{X})/l(A)).
\end{aligned}$$

(2) Immediate from (1).∎

3.3 Proof of 2.1 (i) We will show 2.1 for $\mathcal{C} := \mathrm{lat}\,\Gamma$ by induction on the length of V. We can put $\mathrm{ind}\,\mathcal{C} = \{X\}$ and $l(\widetilde{X}) = 1$. Since $\#\overline{\mathfrak{L}}_{\mathcal{C}}(V) = 1$

holds for any $V \in \bmod A$, we only have to show $\det \mathrm{Z}_{\Lambda,\mathcal{C}}(V;s) = \prod_{n=0}^{l(V)-1}(1 - q^{n-l(A)s})^{-1}$. We apply 3.2(2), where $1 - \mathrm{C} = 1 - q^{-l(A)s}$ and $\overline{\mathfrak{L}}_{\mathcal{C}'}(V) = \emptyset$ holds.

$$\begin{aligned}\det \mathrm{Z}_{\Lambda,\mathcal{C}}(V;s) &\overset{3.2(2)}{=} \det((1-q^{-l(A)s})^{-1} \cdot \mathrm{B} \cdot \mathrm{Z}_{\Lambda,\mathcal{C}}(V/\widetilde{X}; s - 1/l(A)) \cdot \mathrm{B}^{-1}) \\ &= (1-q^{-l(A)s})^{-1} \cdot \prod_{n=0}^{l(V/\widetilde{X})-1} (1 - q^{n-l(A)(s-1/l(A))})^{-1} \\ &= \prod_{n=0}^{l(V)-1} (1-q^{n-l(A)s})^{-1}\end{aligned}$$

(ii) Take a filtration $\mathcal{C} = \mathcal{C}_m \supset \mathcal{C}_{m-1} \supset \cdots \supset \mathcal{C}_0 = \operatorname{lat}\Gamma$ in 2.3. We assume that 2.1 holds for $\mathcal{C}' := \mathcal{C}_{m-1}$. We will show 2.1 for $\mathcal{C}$ by induction on the length of V. We apply 3.2(2) for $X := X_m$, where $\det(1 - \mathrm{C}) = 1$ holds by $X \not\simeq \mathbb{F}_{\mathcal{C}}(X) = Y$.

$$\begin{aligned}&\det \mathrm{Z}_{\Lambda,\mathcal{C}}(V;s) \overset{3.2(2)}{=} \det \mathrm{Z}_{\Lambda,\mathcal{C}'}(V;s) \cdot \det \mathrm{Z}_{\Lambda,\mathcal{C}}(V/\widetilde{X}; s - l(\widetilde{X})/l(A)) \\ =& \prod_{n=0}^{l(V)-1} (1-q^{n-l(A)s})^{-\#\overline{\mathfrak{L}}_{\mathcal{C}'}(S^n)} \cdot \prod_{n=0}^{l(V/\widetilde{X})-1} (1-q^{n-l(A)(s-l(\widetilde{X})/l(A))})^{-\#\overline{\mathfrak{L}}_{\mathcal{C}}(S^n)} \\ =& \prod_{n=0}^{l(V)-1} (1-q^{n-l(A)s})^{-\#\overline{\mathfrak{L}}_{\mathcal{C}'}(S^n)-\#\overline{\mathfrak{L}}_{\mathcal{C}}(S^{n-l(\widetilde{X})})} \\ =& \prod_{n=0}^{l(V)-1} (1-q^{n-l(A)s})^{-\#\overline{\mathfrak{L}}_{\mathcal{C}}(S^n)} \blacksquare\end{aligned}$$

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ON DERIVED EQUIVALENT COHERENT RINGS

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For a selfinjective artin algebra A, the projectively stable category $\underline{\mathrm{mod}}\,A$ of finitely presented left A-modules has the structure of a triangulated category ([4]) and the canonical functor $\underline{\mathrm{mod}}\,A \to \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\,A)/\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\,A)_{\mathrm{fpd}}$ is an equivalence of triangulated categories ([8, 11]). In particular, for derived equivalent artin algebras A, B there exists an equivalence of triangulated categories $\underline{\mathrm{mod}}\,A \cong \underline{\mathrm{mod}}\,B$. Furthermore, if A, B are derived equivalent finite dimensional selfinjective algebras over a field, then there exist bimodules ${}_BM_A$ and ${}_AN_B$ such that the functors $M \otimes_A -$ and $N \otimes_B -$ induce an equivalence of triangulated categories $\underline{\mathrm{mod}}\,A \cong \underline{\mathrm{mod}}\,B$ ([12]). Our aim of this note is to generalize these results.

In section 1, we show that if $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathrm{proj}\,A)$ is a tilting complex with $B = \mathrm{End}_{\mathsf{D}(\mathrm{Mod}\,A)}(P^\bullet)^{\mathrm{op}}$, then the difference between the left global (resp., selfinjective) dimensions of A and B is less than the term length of $P^\bullet$.

In section 2, for a left and right coherent ring A we define the full triangulated subcategory $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\,A)_{\mathrm{fGd}}$ of $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\,A)$ consisting of complexes of finite Gorenstein dimension and show that the functors $R\mathrm{Hom}^\bullet_A(-, A)$ define a duality between $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\,A)_{\mathrm{fGd}}$ and $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\,A^{\mathrm{op}})_{\mathrm{fGd}}$.

In section 3, we deal with the projectively stable category $\underline{\mathcal{G}}(\mathrm{mod}\,A)$ of modules of Gorenstein dimension zero over a left and right coherent ring A. Then, as announced by Avramov [2], $\underline{\mathcal{G}}(\mathrm{mod}\,A)$ has the structure of a triangulated category and the canonical functor F: $\underline{\mathcal{G}}(\mathrm{mod}\,A) \to \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\,A)/\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\,A)_{\mathrm{fpd}}$ induces an equivalence of triangulated categories $\underline{\mathcal{G}}(\mathrm{mod}\,A) \cong \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\,A)_{\mathrm{fGd}}/\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\,A)_{\mathrm{fpd}}$. Furthermore, we show that if A, B are derived equivalent left and right coherent rings, and if either $\mathrm{inj\,dim}\,{}_AA < \infty$ or $\mathrm{inj\,dim}\,A_A < \infty$, then $\underline{\mathcal{G}}(\mathrm{mod}\,A) \cong \underline{\mathcal{G}}(\mathrm{mod}\,B)$ as triangulated categories.

In section 4, we show that if A, B are derived equivalent finite dimensional algebras over a field, and if $\mathrm{inj\,dim}\,{}_AA = \mathrm{inj\,dim}\,A_A < \infty$, then there exist bimodules ${}_BM_A$ and ${}_AN_B$ such that the functors $M \otimes_A -$ and $N \otimes_B -$ induce an equivalence of triangulated categories $\underline{\mathcal{G}}(\mathrm{mod}\,A) \cong \underline{\mathcal{G}}(\mathrm{mod}\,B)$.

Throughout this note, rings are associative rings with identity and modules are unitary modules. For a ring A, we denote by $\mathrm{Mod}\,A$ the category of left A-modules and by $\mathrm{Proj}\,A$ the full subcategory of $\mathrm{Mod}\,A$ consisting of projective modules. Also, we denote by $\mathrm{mod}\,A$ the full subcategory of $\mathrm{Mod}\,A$ consisting of finitely presented modules and by $\mathrm{proj}\,A$ the full subcategory of $\mathrm{mod}\,A$ consisting of projective modules. For a ring A, we denote by A^{op} the opposite ring of A and consider right A-modules as left A^{op}-modules. For an abelian category $\mathcal{A}$, we denote by $\mathsf{D}(\mathcal{A})$ the derived category of cochain complexes over $\mathcal{A}$ and by $\mathsf{D}^-(\mathcal{A})$, $\mathsf{D}^+(\mathcal{A})$ and $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$ the full triangulated subcategories of $\mathsf{D}(\mathcal{A})$ consisting of complexes with bounded above, bounded below and bounded cohomology, respectively.

The detailed version of this paper will be submitted for publication elsewhere.

For an additive category $\mathcal{B}$, we denote by $\mathsf{K}(\mathcal{B})$ the homotopy category of cochain complexes over $\mathcal{B}$ and by $\mathsf{K}^-(\mathcal{B})$, $\mathsf{K}^+(\mathcal{B})$ and $\mathsf{K}^b(\mathcal{B})$ the full triangulated subcategories of $\mathsf{K}(\mathcal{B})$ consisting of bounded above, bounded below and bounded complexes, respectively. Also, for an additive full subcategory $\mathcal{B}$ of an abelian category, we denote by $K^{-,b}(\mathcal{B})$ the full triangulated subcategory of $\mathsf{K}^-(\mathcal{B})$ consisting of complexes with bounded cohomology. We refer to [5, 14, 3] for basic results in the theory of derived categories and to [10, 12] for definitions and basic results in the theory of tilting complexes.

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1. Selfinjective dimensions of derived equivalent rings

In the following, we denote by $\mathsf{Z}^i(X^\bullet)$, $\mathsf{Z}'^i(X^\bullet)$ and $\mathsf{H}^i(X^\bullet)$ the i-th cycle, the i-th cocycle and the i-th cohomology of a complex $X^\bullet$, respectively.

Definition 1.1 ([5]). For a complex $X^\bullet$, we define the following truncations:

$$\begin{aligned}
\sigma_{\le n}(X^\bullet)&: \cdots \to X^{n-2} \to X^{n-1} \to \mathsf{Z}^n(X^\bullet) \to 0 \to \cdots,\\
\sigma'_{\ge n}(X^\bullet)&: \cdots \to 0 \to \mathsf{Z}'^n(X^\bullet) \to X^{n+1} \to X^{n+2} \to \cdots,\\
\tau_{\le n}(X^\bullet)&: \cdots \to X^{n-2} \to X^{n-1} \to X^n \to 0 \to \cdots,\\
\tau_{\ge n}(X^\bullet)&: \cdots \to 0 \to X^n \to X^{n+1} \to X^{n+2} \to \cdots.
\end{aligned}$$

Definition 1.2. For a complex $P^\bullet \in \mathsf{K}^b(\operatorname{proj} A)$, we denote by $\mathcal{C}(P^\bullet)$ the full subcategory of $\mathsf{D}(\operatorname{Mod} A)$ consisting of complexes $X^\bullet$ with $\operatorname{Hom}_{\mathsf{D}(\operatorname{Mod} A)}(P^\bullet, X^\bullet[i]) = 0$ for $i \neq 0$.

Remark 1.3. Let $P^\bullet \in \mathsf{K}^b(\operatorname{proj} A)$. Assume that $P^\bullet$ generates $\mathsf{K}^b(\operatorname{proj} A)$ as a triangulated category. Then $\{P^\bullet[i]\}_{i\in\mathbb{Z}}$ is a generating set for $\mathsf{D}(\operatorname{Mod} A)$, i.e. $\bigcap_{i\in\mathbb{Z}} \operatorname{Ker}(\operatorname{Hom}_{\mathsf{D}(\operatorname{Mod} A)}(P^\bullet[i], -)) = \{0\}$.

Throughout the rest of this section, $P^\bullet \in \mathsf{K}^b(\operatorname{proj} A)$ is a tilting complex with $B = \operatorname{End}_{\mathsf{D}(\operatorname{Mod} A)}(P^\bullet)^{\mathrm{op}}$ and

$$F: \mathsf{D}^b(\operatorname{Mod} B) \xrightarrow{\sim} \mathsf{D}^b(\operatorname{Mod} A)$$

is an equivalence of triangulated categories such that $F(B) = P^\bullet$.

Lemma 1.4. *We have $\mathcal{C}(P^\bullet) \subset \mathsf{D}^b(\operatorname{Mod} A)$ and the functor*

$$\operatorname{Hom}_{\mathsf{D}(\operatorname{Mod} A)}(P^\bullet, -): \mathcal{C}(P^\bullet) \to \operatorname{Mod} B$$

is an equivalence of abelian categories whose quasi-inverse is given by the restriction of F to $\operatorname{Mod} B$.

Lemma 1.5. *Let $n \ge 0$ and assume that $P^i = 0$ for $i > 0$ and $i < -n$. Then the following statements hold.*

(1) *Let* $G\colon \mathsf{D}^{\mathrm{b}}(\operatorname{Mod} A) \xrightarrow{\sim} \mathsf{D}^{\mathrm{b}}(\operatorname{Mod} B)$ *be a quasi-inverse of* F. *Then there exists a tilting complex* $Q^\bullet \in \mathsf{K}^{\mathrm{b}}(\operatorname{proj} B)$ *such that* $G(A) \cong Q^\bullet$ *and* $Q^i = 0$ *for* $i > n$ *and* $i < 0$.

(2) $X^\bullet \cong \sigma'_{\geq -n}(\sigma_{\leq 0}(X^\bullet))$ *in* $\mathsf{D}(\operatorname{Mod} A)$ *for all* $X^\bullet \in \mathcal{C}(P^\bullet)$.

Lemma 1.6. *Let* $m, n, d \in \mathbb{Z}$ *with* $d \geq 0$, *and let* $X^\bullet, Y^\bullet \in \mathsf{K}^{\mathrm{b}}(\operatorname{Mod} A)$. *Assume that* $X^p = 0$ *for* $p < m$, *that* $Y^q = 0$ *for* $q > n$ *and that* $\operatorname{Ext}^i_A(X^p, Y^q) = 0$ *for all* $p, q \in \mathbb{Z}$ *and* $i \geq d$. *Then* $\operatorname{Hom}_{\mathsf{D}(\operatorname{Mod} A)}(X^\bullet, Y^\bullet[i]) = 0$ *for* $i \geq d + n - m$.

Now, we generalize [9, Corollary to Proposition 2.4].

Proposition 1.7. *Let* $n+1$ *be the term length of* $P^\bullet$, *where* $n \geq 0$. *Then the following statements hold.*

(1) $\operatorname{l.gl.dim} A - n \leq \operatorname{l.gl.dim} B \leq \operatorname{l.gl.dim} A + n$.

(2) $\operatorname{inj dim}_A A - n \leq \operatorname{inj dim}_B B \leq \operatorname{inj dim}_A A + n$.

Remark 1.8. It would follow from [10, Proposition 6.2] (resp., its dual) that the finiteness of the left global (resp., selfinjective) dimension of a ring is invariant under derived equivalence.

2. Complexes of finite Gorenstein dimension

Throughout this section, we work over a left and right coherent ring A. Note that $\operatorname{mod} A$ is a thick abelian subcategory of $\operatorname{Mod} A$ in the sense of [5].

Definition 2.1. For any module X we denote by

$$\varepsilon_X\colon X \to \operatorname{Hom}_A(\operatorname{Hom}_A(X, A), A),\ x \mapsto (f \mapsto f(x))$$

the usual evaluation map. Then, for any complex $X^\bullet$ we have a functorial homomorphim

$$\varepsilon_{X^\bullet}\colon X^\bullet \to \operatorname{Hom}^\bullet_A(\operatorname{Hom}^\bullet_A(X^\bullet, A), A)$$

such that $\varepsilon_{X^\bullet}{}^n = \varepsilon_{X^n}$ for all $n \in \mathbb{Z}$. Furthermore, for any $X \in \operatorname{Mod} A$ and $M \in \operatorname{Mod} A^{\mathrm{op}}$ we have a bifunctorial isomorphism

$$\theta_{M,X}\colon \operatorname{Hom}_A(M, \operatorname{Hom}_A(X, A)) \xrightarrow{\sim} \operatorname{Hom}_A(X, \operatorname{Hom}_A(M, A))$$

such that $\theta_{M,X}(f) = \operatorname{Hom}_A(f, A) \circ \varepsilon_X$ for all $f \in \operatorname{Hom}_A(M, \operatorname{Hom}_A(X, A))$. Thus, for any $X^\bullet \in \mathsf{K}(\operatorname{Mod} A)$ and $M^\bullet \in \mathsf{K}(\operatorname{Mod} A^{\mathrm{op}})$ we have a bifunctorial isomorphism

$$\operatorname{Hom}^\bullet_A(M^\bullet, \operatorname{Hom}^\bullet_A(X^\bullet, A)) \xrightarrow{\sim} \operatorname{Hom}^\bullet_A(X^\bullet, \operatorname{Hom}^\bullet_A(M^\bullet, A))$$

and hence, applying $\mathrm{H}^0(-)$, we get a bifunctorial isomorphism

$$\operatorname{Hom}_{\mathsf{K}(\operatorname{Mod} A^{\mathrm{op}})}(M^\bullet, \operatorname{Hom}^\bullet_A(X^\bullet, A)) \xrightarrow{\sim} \operatorname{Hom}_{\mathsf{K}(\operatorname{Mod} A)}(X^\bullet, \operatorname{Hom}^\bullet_A(M^\bullet, A)),$$

which we denote by $\theta_{M^\bullet, X^\bullet}$, such that $\theta_{M^\bullet, X^\bullet}(f)^n = \theta_{M^{-n}, X^n}(f^{-n})$ for all $f \in \operatorname{Hom}_{\mathsf{K}(\operatorname{Mod} A^{\mathrm{op}})}(M^\bullet, \operatorname{Hom}^\bullet_A(X^\bullet, A))$ and $n \in \mathbb{Z}$. Then we have

$$\varepsilon_{X^\bullet} = \theta_{\operatorname{Hom}^\bullet_A(X^\bullet, A), X^\bullet}(\mathrm{id}_{\operatorname{Hom}^\bullet_A(X^\bullet, A)}),$$
$$\varepsilon_{M^\bullet} = \theta_{M^\bullet, \operatorname{Hom}^\bullet_A(M^\bullet, A)}{}^{-1}(\mathrm{id}_{\operatorname{Hom}^\bullet_A(M^\bullet, A)})$$

for all $X^\bullet \in \mathsf{K}(\operatorname{Mod} A)$ and $M^\bullet \in \mathsf{K}(\operatorname{Mod} A^{\mathrm{op}})$.

Definition 2.2 ([13, 3]). We denote by $\mathsf{K}(\operatorname{Proj} A)_{\mathrm{L}}$ the full triangulated subcategory of $\mathsf{K}(\operatorname{Mod} A)$ consisting of complexes $X^\bullet$ such that $\operatorname{Hom}_{\mathsf{K}(\operatorname{Mod} A)}(X^\bullet, -)$ vanishes on the acyclic complexes. Note that $\mathsf{K}^-(\operatorname{Proj} A) \subset \mathsf{K}(\operatorname{Proj} A)_{\mathrm{L}}$. According to the dual of [3, Proposition 2.12], we have an equivalence of triangulated categories $\mathsf{K}(\operatorname{Proj} A)_{\mathrm{L}} \xrightarrow{\sim} \mathsf{D}(\operatorname{Mod} A)$ and $\operatorname{Hom}^\bullet_A(-, A)$ has a right derived functor

$$\boldsymbol{R}\operatorname{Hom}^\bullet_A(-, A) \colon \mathsf{D}(\operatorname{Mod} A) \to \mathsf{D}(\operatorname{Mod} A^{\mathrm{op}})$$

such that the canonical homomorphism $\operatorname{Hom}^\bullet_A(P^\bullet, A) \to \boldsymbol{R}\operatorname{Hom}^\bullet_A(P^\bullet, A)$ is an isomorphism for all $P^\bullet \in \mathsf{K}(\operatorname{Proj} A)_{\mathrm{L}}$.

Lemma 2.3. *For any $X^\bullet \in \mathsf{D}(\operatorname{Mod} A)$ and $M^\bullet \in \mathsf{D}(\operatorname{Mod} A^{\mathrm{op}})$, we have a bifunctorial isomorphism*

$$\operatorname{Hom}_{\mathsf{D}(\operatorname{Mod} A^{\mathrm{op}})}(M^\bullet, \boldsymbol{R}\operatorname{Hom}^\bullet_A(X^\bullet, A)) \xrightarrow{\sim} \operatorname{Hom}_{\mathsf{D}(\operatorname{Mod} A)}(X^\bullet, \boldsymbol{R}\operatorname{Hom}^\bullet_A(M^\bullet, A)),$$

which we denote by $\tilde{\theta}_{M^\bullet, X^\bullet}$.

Definition 2.4. We set

$$\eta_{X^\bullet} = \tilde{\theta}_{\boldsymbol{R}\operatorname{Hom}^\bullet_A(X^\bullet, A), X^\bullet}(\mathrm{id}_{\boldsymbol{R}\operatorname{Hom}^\bullet_A(X^\bullet, A)}),$$
$$\eta_{M^\bullet} = \tilde{\theta}_{M^\bullet, \boldsymbol{R}\operatorname{Hom}^\bullet_A(M^\bullet, A)}{}^{-1}(\mathrm{id}_{\boldsymbol{R}\operatorname{Hom}^\bullet_A(M^\bullet, A)})$$

for $X^\bullet \in \mathsf{D}(\operatorname{Mod} A)$ and $M^\bullet \in \mathsf{D}(\operatorname{Mod} A^{\mathrm{op}})$.

Lemma 2.5. *For any $P^\bullet \in \mathsf{K}(\operatorname{Proj} A)_{\mathrm{L}}$, we have a functorial homomorphism*

$$\xi_{P^\bullet} \colon \operatorname{Hom}^\bullet_A(\operatorname{Hom}^\bullet_A(P^\bullet, A), A) \to \boldsymbol{R}\operatorname{Hom}^\bullet_A(\boldsymbol{R}\operatorname{Hom}^\bullet_A(P^\bullet, A), A)$$

such that $\eta_{P^\bullet} = \xi_{P^\bullet} \circ \varepsilon_{P^\bullet}$.

Remark 2.6. For any $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$, since $\operatorname{Hom}^\bullet_A(P^\bullet, A) \in \mathsf{K}^{\mathrm{b}}(\operatorname{proj} A^{\mathrm{op}})$, it follows by Lemma 2.5 that $\eta_{P^\bullet}$ is an isomorphism.

Definition 2.7. A complex $X^\bullet \in \mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)$ is said to have finite Gorenstein dimension provided that $\operatorname{Hom}_{\mathsf{D}(\operatorname{Mod} A)}(X^\bullet[i], -)$ vanishes on $\operatorname{proj} A$ for $i \ll 0$, i.e. $\boldsymbol{R}\operatorname{Hom}^\bullet_A(X^\bullet, A) \in \mathsf{D}^{\mathrm{b}}(\operatorname{mod} A^{\mathrm{op}})$, and that $\eta_{X^\bullet}$ is an isomorphism. We denote by $\mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)_{\mathrm{fGd}}$ the full triangulated subcategory of $\mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)$ consisting of complexes of finite Gorenstein dimension. It then follows by Definition 2.4 that the functors $\boldsymbol{R}\operatorname{Hom}^\bullet_A(-, A)$ induce a duality between $\mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)_{\mathrm{fGd}}$ and $\mathsf{D}^{\mathrm{b}}(\operatorname{mod} A^{\mathrm{op}})_{\mathrm{fGd}}$.

Definition 2.8. A complex $X^\bullet \in \mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)$ is said to have Gorenstein dimension zero provided that $\operatorname{Hom}_{\mathsf{D}(\operatorname{Mod} A)}(X^\bullet[i], -)$ vanishes on $\operatorname{proj} A$ for $i \neq 0$, i.e. $\boldsymbol{R}\operatorname{Hom}^\bullet_A(X^\bullet, A)$ is isomorphic to a module, and that $\eta_{X^\bullet}$ is an isomorphism. We denote by $\mathcal{G}(\operatorname{mod} A)$ the full additive subcategory of $\operatorname{mod} A$ consisting of modules of Gorenstein dimension zero. Note that $\operatorname{proj} A \subset \mathcal{G}(\operatorname{mod} A)$ and that for any $X \in \mathcal{G}(\operatorname{mod} A)$, $\operatorname{proj\,dim} X < \infty$ implies $X \in \operatorname{proj} A$.

Lemma 2.9. *For any $X \in \operatorname{mod} A$, the following statements are equivalent.*

(1) $X \in \mathcal{G}(\operatorname{mod} A)$.
(2) $\operatorname{Ext}_A^i(X, A) = 0 = \operatorname{Ext}_A^i(\operatorname{Hom}_A(\mathrm{Z}'^{-2}(P^\bullet), A), A)$ *for all* $i > 0$, *where* $P^\bullet$ *is a projective resolution of* X *in* $\operatorname{mod} A$.
(3) X *is reflexive, i.e.* ε_X *is an isomorphism, and* $\operatorname{Ext}_A^i(X, A) = 0 = \operatorname{Ext}_A^i(\operatorname{Hom}_A(X, A), A)$ *for all* $i > 0$.

Proposition 2.10. *For any* $X^\bullet \in \mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)$, *the following statements are equivalent.*

(1) $X^\bullet \in \mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)_{\mathrm{fGd}}$.
(2) *For any quasi-isomorphism* $P^\bullet \to X^\bullet$ *with* $P^\bullet \in \mathsf{K}^{-,\mathrm{b}}(\operatorname{proj} A)$, *there exists* $n \in \mathbb{Z}$ *such that the canonical cochain map* $P^\bullet \to \sigma'_{\geq -n}(P^\bullet)$ *is a quasi-isomorphism and* $\sigma'_{\geq -n}(P^\bullet) \in \mathsf{K}^{\mathrm{b}}(\mathcal{G}(\operatorname{mod} A))$.
(3) $X^\bullet \cong Y^\bullet$ *in* $\mathsf{D}(\operatorname{Mod} A)$ *for some* $Y^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{G}(\operatorname{mod} A))$.

3. Derived equivalent coherent rings

In this section, we continue to work over a left and right coherent ring A.

Definition 3.1 ([5]). A complex $X^\bullet \in \mathsf{D}^-(\operatorname{mod} A)$ is said to have finite projective dimension provided that $\operatorname{Hom}_{\mathsf{D}(\operatorname{Mod} A)}(X^\bullet[i], -)$ vanishes on $\operatorname{mod} A$ for $i \ll 0$. We denote by $\mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)_{\mathrm{fpd}}$ the full triangulated subcategory of $\mathsf{D}^-(\operatorname{mod} A)$ consisting of complexes of finite projective dimension. Then we have an equivalence of triangulated categories $\mathsf{K}^-(\operatorname{proj} A) \xrightarrow{\sim} \mathsf{D}^-(\operatorname{mod} A)$, which induces equivalences of full triangulated subcategories

$$\mathsf{K}^{-,\mathrm{b}}(\operatorname{proj} A) \xrightarrow{\sim} \mathsf{D}^{\mathrm{b}}(\operatorname{mod} A), \quad \mathsf{K}^{\mathrm{b}}(\operatorname{proj} A) \xrightarrow{\sim} \mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)_{\mathrm{fpd}}.$$

In particular, $\mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)_{\mathrm{fpd}} \subset \mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)$.

Remark 3.2. The following statements hold.

(1) $\eta_{X^\bullet}$ is an isomorphism for all $X^\bullet \in \mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)_{\mathrm{fpd}}$.
(2) $\mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)_{\mathrm{fpd}} \subset \mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)_{\mathrm{fGd}}$.

Definition 3.3. We denote by $\underline{\mathcal{G}}(\operatorname{mod} A)$ the residue category of $\mathcal{G}(\operatorname{mod} A)$ over the full additive subcategory $\operatorname{proj} A$, and by

$$\mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)/\mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)_{\mathrm{fpd}}, \quad \mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)_{\mathrm{fGd}}/\mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)_{\mathrm{fpd}}$$

the quotient categories of $\mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)$ and $\mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)_{\mathrm{fGd}}$ over the épaisse subcategory $\mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)_{\mathrm{fpd}}$, respectively. Then we have an additive functor

$$F \colon \underline{\mathcal{G}}(\operatorname{mod} A) \to \mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)/\mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)_{\mathrm{fpd}}$$

which makes the following diagram commute

$$\begin{array}{ccc} \mathcal{G}(\operatorname{mod} A) & \xrightarrow{\text{inc.}} & \mathsf{D}^{\mathrm{b}}(\operatorname{mod} A) \\ {\scriptstyle \text{can.}}\downarrow & & \downarrow{\scriptstyle \text{can.}} \\ \underline{\mathcal{G}}(\operatorname{mod} A) & \xrightarrow[F]{} & \mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)/\mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)_{\mathrm{fpd}}. \end{array}$$

Proposition 3.4 ([2]). *The following statements hold.*

(1) $\underline{\mathcal{G}}(\mathrm{mod}\, A)$ *has the structure of a triangulated category with the Heller suspension* Ω^{-1} *as the translation.*

(2) F *is a fully faithful* ∂*-functor and induces an equivalence of triangulated categories* $\underline{\mathcal{G}}(\mathrm{mod}\, A) \cong \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\, A)_{\mathrm{fGd}}/\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\, A)_{\mathrm{fpd}}$.

Proposition 3.5. *The following statements are equivalent.*

(1) $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\, A)_{\mathrm{fGd}}/\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\, A)_{\mathrm{fpd}} = \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\, A)/\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\, A)_{\mathrm{fpd}}$.

(2) $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\, A)_{\mathrm{fGd}} = \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\, A)$.

(3) $\mathrm{mod}\, A \subset \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\, A)_{\mathrm{fGd}}$.

Lemma 3.6. *Let* $n \geq 0$ *be an integer and assume that* $\mathrm{inj\,dim}\,{}_A A \leq n$ *and* $\mathrm{inj\,dim}\, A_A \leq n$. *Then for any projective resolution* $P^\bullet \to X$ *in* $\mathrm{mod}\, A$ *we have* $Z'^{-n}(P^\bullet) \in \mathcal{G}(\mathrm{mod}\, A)$. *In particular,* $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\, A)_{\mathrm{fGd}} = \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\, A)$.

Lemma 3.7. *Assume that* $\mathrm{inj\,dim}\, A_A < \infty$. *Then* $\eta_{X^\bullet}$ *is an isomorphism for all* $X^\bullet \in \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\, A)$. *In particular, for any* $X^\bullet \in \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\, A)$ *the following statements are equivalent.*

(1) $X^\bullet \in \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\, A)_{\mathrm{fGd}}$.

(2) *For any* $Y^\bullet \in \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\, A)_{\mathrm{fpd}}$, $\mathrm{Hom}_{\mathsf{D}(\mathrm{Mod}\, A)}(X^\bullet, Y^\bullet[i]) = 0$ *for* $i \gg 0$.

Theorem 3.8. *Assume that either* $\mathrm{inj\,dim}\,{}_A A < \infty$ *or* $\mathrm{inj\,dim}\, A_A < \infty$. *Then for any left and right coherent ring B derived equivalent to A we have an equivalence of triangulated categories* $\underline{\mathcal{G}}(\mathrm{mod}\, A) \cong \underline{\mathcal{G}}(\mathrm{mod}\, B)$.

Remark 3.9 ([10, Proposition 9.4]). In case A is a noetherian algebra over a commutative noetherian ring R, every ring B derived equivalent to A is a noetherian R-algebra.

4. The case of finite dimensional algebras

To begin with, we recall the following facts.

Remark 4.1. Let A, B be finite dimensional algebras over a field k. Then $\mathrm{inj\,dim}\,_{A\otimes_k B^{\mathrm{op}}} A \otimes_k B^{\mathrm{op}} \leq \mathrm{inj\,dim}\,{}_A A + \mathrm{inj\,dim}\, B_B$.

Remark 4.2. Let A, B be rings, and let V be an A-B-bimodule such that ${}_A V$ is projective and V_B is flat. Then the following statements hold.

(1) $\mathrm{inj\,dim}\,{}_B \mathrm{Hom}_A(V, X) \leq \mathrm{inj\,dim}\, X$ for all $X \in \mathrm{Mod}\, A$.

(2) $\mathrm{proj\,dim}\,{}_A V \otimes_B Y \leq \mathrm{proj\,dim}\, Y$ for all $Y \in \mathrm{Mod}\, B$.

Remark 4.3 ([15]). For a left and right noetherian ring A, if $\mathrm{inj\,dim}\,{}_A A < \infty$ and $\mathrm{inj\,dim}\, A_A < \infty$ then $\mathrm{inj\,dim}\,{}_A A = \mathrm{inj\,dim}\, A_A$.

Theorem 4.4. *Let A, B be derived equivalent finite dimensional algebras over a field k. Assume that* $\mathrm{inj\,dim}\,{}_A A = \mathrm{inj\,dim}\, A_A < \infty$. *Then there exist* $M \in \mathrm{mod}\, A^{\mathrm{op}} \otimes_k B$ *and*

$N \in \operatorname{mod} A \otimes_k B^{op}$ *such that the pair of functors*

$$M \otimes_A - : \operatorname{mod} A \to \operatorname{mod} B, \qquad N \otimes_B - : \operatorname{mod} B \to \operatorname{mod} A$$

induces an equivalence of triangulated categories $\underline{\mathcal{G}}(\operatorname{mod} A) \cong \underline{\mathcal{G}}(\operatorname{mod} B)$.

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Tamely Ramified Dubrovin Crossed Products*

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Abstract

Let V be a commutative valuation domain of arbitrary Krull-dimension, with quotient field F, and let K be a finite Galois extension of F with group G, and S the integral closure of V in K. Suppose one has a 2-cocycle on G which takes values in the group of units of S. Then one can form the crossed product of G over S, $S * G$, which is a V-order in the central simple F-algebra $K * G$. If we assume $S * G$ is a Dubrovin valuation ring of $K * G$, then the main result of this paper is that, given a suitable definition of tameness for central simple algebras, $K * G$ is tamely ramified and defectless over F if and only if K is tamely ramified and defectless over F. We also study the residue structure of $S * G$, as well as its behaviour upon passage to Henselization.

Introduction

This paper is a sequel to [5]. In the case of fields, all valuation-theoretic notions and terminology are as defined in [1]. Let V be a commutative valuation domain with quotient field F, and let K be a finite Galois extension of F with group G. Let S be the integral closure of V in K. Given a two-cocycle which takes values in the group of units of S, we can always form the crossed product $S * G$. This object has been studied in [3, 8], among other places, assuming that V is a DVR. Recently it has been studied in [5], for an arbitrary valuation ring V.

Let Q be a central simple F-algebra, and let B be a Dubrovin valuation ring of Q with centre V. Associated with the pair (B, V), we have, according to [7], the *value group* of B, $\Gamma_B = \mathrm{st}(B)/U(B)$, where $\mathrm{st}(B) = \{x \in U(Q) \mid xBx^{-1} = B\}$ and $U(\cdot)$ denotes the group of multiplicative units of a given ring; the *ramification index* of B over V, $e(B \mid V) = [\Gamma_B : \Gamma_V]$, where Γ_V is the value group of V; and the *residue degree* of B over V, $f(B \mid V) = [\overline{B} : \overline{V}]$. If $\overline{p}$ is the characteristic exponent of $\overline{V}$, that is, $\overline{p} = \max\{\mathrm{char}(\overline{V}), 1\}$, it was shown in [7, Theorem C] that $[Q : F] = e(B \mid V) f(B \mid V)\eta^2\overline{p}^a$ for some positive integer η and non-negative integer a. We say that B is *defectless* over F when $[Q : F] = e(B \mid V) f(B \mid V)$. The number η is called the *extension number* of V to Q, described in [2]. By [7, Theorem F], B is integral over V if and only if $\eta = 1$. We observe that, if $S * G$ is a Dubrovin valuation ring, which is the assumption for a greater part of this paper, then it is integral over V.

*The detailed version of this paper has been submitted for publication elsewhere.

In the commutative case, we have the following situation. We fix once and for all an extension W of V to K. Let $n = [K : F] = | G |$; Γ_W the value group of W; $e = [\Gamma_W : \Gamma_V]$, the ramification index of W over F; $f = [\overline{W} : \overline{V}]$, the residue degree of W over F; and g be the number of extensions of V to K. It is known that $n = efg\overline{p}^d$ in this case, where d is a non-negative integer. Following [1], we say that (K, W) is defectless over (F, V) if $n = efg$, and we say that (K, W) is tamely ramified over (F, V) if char$(\overline{V})$ does not divide e and $\overline{W}$ is separable over $\overline{V}$.

When B is Dubrovin valuation ring of a central simple F-algebra Q, we will therefore say that B is *tamely ramified* over F if $Z(\overline{B})$ is separable over $\overline{V}$ and char$(\overline{V})$ does not divide $e(B|V)$. We do not assume that $\overline{p}$ is co-prime to $[Q : F]$. However, our definition of tameness is stronger than that of [4], which is applicable only to division algebras with invariant valuation rings. The main result of this paper justifies our choice of the definition of tameness. We readily see that any Azumaya algebra over a valuation ring is tamely ramified and defectless, by [6, Proposition 3.2] and [7, Corollary 3.4].

In section 1, we have the main result of the paper, which states that, assuming $S * G$ is a Dubrovin valuation ring, then $K * G$ is tamely ramified and defectless over F if and only if K/F is tamely ramified and defectless. If we assume that K/F is tamely ramified and defectless then, by [5, Theorem 2], $J(S * G) = J(S) * G$ and therefore $\overline{S * G} \cong M_g(\overline{W} * G^Z)$ by [5, Lemma 2], where G^Z is the decomposition group of W over F. Let G^T be the inertia group of W over F. In section 2, we see that when $S * G$ is a Dubrovin and K/F is tamely ramified and defectless, then $\overline{W} * G^Z$ is a generalized crossed product of G^Z/G^T over $\overline{W} * G^T$. A necessary and sufficient condition is given for the generalized crossed product to become a classical crossed product algebra. In section 3, we see that, if (K_h, W_h) is a Henselization of (K, W), and (F_h, V_h) a Henselization of (F, V), then $(S * G) \otimes_V V_h \cong M_g(W_h * G^Z)$. The value function associated with an integral Dubrovin valuation ring described in [6] easily materialize in $(K * G) \otimes_F F_h$ whenever K/F is tamely ramified and defectless and $S * G$ is a Dubrovin.

1 The Main Result

The main result of this paper follows.

Theorem 1 *Suppose $S * G$ is a Dubrovin valuation ring. Then $(K * G, S * G)$ is tamely ramified and defectless over (F, V) if and only if (K, W) is tamely ramified and defectless over (F, V). When this happens, $f(S * G|V) = ef^2g^2$, and $e(S * G|V) = e$.*

We prove the theorem by showing that if $S*G$ is a Dubrovin, then it is tamely ramified and defectless if and only if $\overline{W} * G^T$ is a separable $\overline{W}$-algebra, and the latter holds if and only if $(\overline{p}, | G^T |) = 1$, by a result of Harada in [3]. From this, it follows that $S * G$ is tamely ramified and defectless if and only if K/F is tamely ramified and defectless, by [5, Lemma 1].

We will see later in Section 3 that, with the assumption contained in Theorem 1, we have, in addition, the result that $\Gamma_{S*G} \cong \Gamma_W$. Examples of tamely ramified and defectless Dubrovin crossed products can easily be constructed using [5, Theorem 3]. In Examples

1 and 2 of [5], we encounter Dubrovin crossed products that are not tamely ramified, although they are defectless.

Proposition 1 *Suppose K/F is tamely ramified and defectless. Then the following are equivalent:*

1. *$S * G$ is a Dubrovin valuation ring of $K * G$.*
2. *$W * G^Z$ is a Dubrovin valuation ring of $K * G^Z$.*
3. *$W * G^T$ is a Dubrovin valuation ring of $K * G^T$.*
4. *$\overline{W} * G^Z$ is a simple ring.*
5. *$\overline{W} * G^T$ is a simple ring.*

Another condition equivalent to $S * G$ being a Dubrovin valuation ring will be given in Theorem 3(2).

2 The Residue Structure of $S * G$

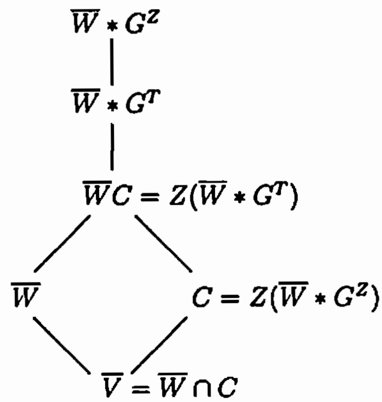
If $S*G$ is a tamely ramified and defectless Dubrovin valuation ring, then $\overline{S * G} \cong M_g(\overline{W} * G^Z)$, and hence, to study the structure of $\overline{S * G}$, one needs only consider $\overline{W} * G^Z$.

Let Q be a central simple F-algebra. It is not always possible to find a maximal subfield of Q which is a Galois extension of F, that is, Q need not be a "classical" crossed product algebra. What is often the case is that there exists a subfield L of Q which is Galois over F but $[L : F]^2 < [Q : F]$. If A is the centralizer of L in Q, and H is the Galois group of L over F, then Q is said to be a *generalized crossed-product of H over A*. In case L is a maximal subfield of Q, we will say that Q is a *classical crossed product algebra.* Recall that if B is a Dubrovin valuation ring of Q with center V, then each $a \in \mathrm{st}(B)$ induces a ring automorhism of $\overline{B}$ via conjugation. In fact, Wadsworth [7, Corollary B] showed that this map induces a surjection $\omega : \Gamma_B/\Gamma_V \mapsto \mathrm{Aut}_{\overline{V}}(Z(\overline{B}))$ (see also [2, Corollary 4.2]). When K/F is tamely ramified and defectless and $S * G$ is a Dubrovin, then $S * G$ is tamely ramified and defectless, by Theorem 1, and hence $Z(\overline{S * G})$ is Galois over $\overline{V}$, by [7, Corollary B].

Theorem 2 *Suppose $S * G$ is a tamely ramified and defectless Dubrovin valuation ring of $K * G$. Let $C = Z(\overline{W} * G^Z)$. Then:*

1. *(a) We have that $\overline{W}C = Z(\overline{W} * G^T)$,*
   (b) Further, $\overline{W}C$ is Galois over both $\overline{W}$ and C,
   (c) $Gal(\overline{W}C/C) \cong Gal(\overline{W}/\overline{V}) \cong G^Z/G^T$ and $Gal(\overline{W}C/\overline{W}) \cong Gal(C/\overline{V})$,
   (d) $\overline{W} * G^Z$ is a generalized crossed product of G^Z/G^T over $\overline{W} * G^T$.

We have the following diagram:



2. *The Wadsworth map, ω, is a bijection if and only if $\overline{W} * G^T$ is commutative. When this happens, $\overline{W} * G^T$ is a maximal subfield of $\overline{W} * G^Z$, it is Galois over C, and $\overline{W} * G^Z$ is a classical crossed product algebra of G^Z/G^T over $\overline{W} * G^T$.*

Remark. From a purely ring-theoretic point of view, the distinction between generalized crossed products and classical crossed products is superfluous in this case: since $G^T \trianglelefteq G^Z$, $\overline{W} * G^Z \cong (\overline{W} * G^T) * (G^Z/G^T)$, and hence $\overline{W} * G^Z$ is always a crossed product of G^Z/G^T over $\overline{W} * G^T$.

The following corollary now follows from Theorem 1, Proposition 1, and Theorem 2.

Corollary 1 *The order $S * G$ is a tamely ramified and defectless Dubrovin valuation ring if and only if $\overline{W} * G^T$ is a simple separable $\overline{W}$-algebra; it is a tamely ramified and defectless Dubrovin valuation ring and the Wadsworth map is a bijection if and only if $\overline{W} * G^T$ is a separable field extension of $\overline{W}$.*

When K/F is tamely ramified and defectless, then $(| G^T |, \overline{p}) = 1$, by [5, Lemma 1]. Therefore, by [1, Corollary 20.10(b), Corollary 20.12, & 20.2], $G^T \cong \Gamma_W/\Gamma_V$, and so G^T is abelian. Thus, if f is the 2-cocycle (not to be confused with the residue degree of W over F), then $\overline{W} * G^T$ is commutative if and only if $f(\sigma, \tau) - f(\tau, \sigma) \in J(W)\ \forall \sigma, \tau \in G^T$. But this characterization of the commutativity of $\overline{W} * G^T$ is hardly illuminating; indeed, an example of a tamely ramified and defectless $S * G$ with $\overline{W} * G^T$ non-commutative is unknown to us, and may well not exist! However, when V is a DVR or, more generally, when G^T is cyclic, then $\overline{W} * G^T$ is commutative. This is the essence of the following proposition.

Proposition 2 *Suppose $S * G$ is a tamely ramified and defectless Dubrovin valuation ring of $K * G$. When $e(B \mid V)$ is square-free, or when $S * G$ is finitely generated over V, then the Wadsworth map is a bijection.*

To prove this proposition, one shows that, in either case, G^T is cyclic.

Remarks. It appears that the inertia group, G^T, plays a critical role in the behaviour of $S * G$. To start with, when $\overline{W} * G^T$ is a simple ring, then $S * G$ is a Dubrovin valuation ring. The converse is false: in both Examples 1 and 2 of [5], $G = G^T$ and $J(S * G) \supset J(S) * G$ and hence $\overline{W} * G^T$ $(= \overline{S} * G = S * G/(J(S) * G))$ is not a simple ring.

Further, when $\overline{W} * G^T$ is $\overline{W}$-separable, then $S * G$ is semihereditary, from the proof of [5, Theorem 2]. When $\overline{W} * G^T$ is a simple separable $\overline{W}$-algebra, then $S * G$ is a tamely ramified and defectless Dubrovin valuation ring and conversely. If $S * G$ is a tamely ramified and defectless Dubrovin valuation ring and G^T is cyclic or, more generally, when $\overline{W} * G^T$ is a separable field extension of $\overline{W}$, then the Wadsworth map is a bijection.

3 The Henselization of $S * G$

Let B be an integral Dubrovin valuation ring of a central simple F-algebra Q. In [6], we encounter a *value function* $\Phi : Q \mapsto \Gamma_B \cup \{\infty\}$ associated with B, which is a surjection, and has the following defining properties: for all $x, y \in Q$, we have

1. $\Phi(x) = \infty$ if and only if $x = 0$,
2. $\Phi(x + y) \geq \min\{\Phi(x), \Phi(y)\}$,
3. $\Phi(xy) \geq \Phi(x) + \Phi(y)$,
4. $B = \{x \in Q \mid \Phi(x) \geq 0\}$ and $J(B) = \{x \in Q \mid \Phi(x) > 0\}$,
5. $\Phi(Q) = \Phi(\mathrm{st}(\Phi)) \cup \{\infty\}$, where $\mathrm{st}(\Phi) = \{x \in U(Q) \mid \Phi(x^{-1}) = -\Phi(x)\}$.

Let (K_h, W_h) be a Henselization of (K, W) (see [1, §17] for definition). Their value groups are the same, that is, $\Gamma_{W_h} = \Gamma_W$. We let ϕ be a valuation on K_h corresponding to W_h. Let (F_h, V_h) be the unique Henselization of (F, V) contained in (K_h, W_h) [1, Theorem 17.11]. By [1, 17.16 & Theorem 17.11], we see that $K_h = KF_h$. Note that $W \cap (K \cap F_h)$ is indecomposed in K, since V_h is indecomposed in K_h. Hence, by [1, Theorem 15.7], we have $F^Z \subseteq K \cap F_h$. But since K/F is a finite Galois extension, [1, Them. 17.7] implies $[K : F] = [K_h : F_h]g$, hence $[K \cap F_h : F] = g$, and so we must have $F^Z = K \cap F_h$. The Galois extension K_h/F_h has therefore group G^Z.

Any $\sigma \in G$ can be considered as a ring automorphism on $K \otimes_F F_h$ via the action $\sigma(k \otimes u) = \sigma(k) \otimes u$, for $k \in K, u \in F_h$. Also, if f is the 2-cocycle, then $f(\sigma, \tau)$ can be identified with $f(\sigma, \tau) \otimes 1 \in U(S \otimes_V V_h)$, for all $\sigma, \tau \in G$. The restriction of σ to $S \otimes_V V_h$ is again an automorphism. Therefore there is a canonical F_h-algebra isomorphism from $(K * G) \otimes_F F_h$ to $(K \otimes_F F_h) * G$ mapping $kx_\sigma \otimes u$ to $(k \otimes u)x_\sigma$, which restricts to an isomorphism between $(S * G) \otimes_V V_h$ and $(S \otimes_V V_h) * G$.

Theorem 3 *We have*

1. $(K * G) \otimes_F F_h \cong M_g(K_h * G^Z)$ *and* $(S * G) \otimes_V V_h \cong M_g(W_h * G^Z)$,
2. *the order* $S * G$ *is a Dubrovin valuation ring of* $K * G$ *if and only if* $W_h * G^Z$ *is a Dubrovin valuation ring of the central simple* F_h*-algebra* $K_h * G^Z$,

3. *if $S * G$ is a tamely ramified and defectless Dubrovin valuation ring of $K * G$, then the map Φ from $K_h * G^Z$ to $\Gamma_W \cup \{\infty\}$ given by $\Phi(\sum_{\sigma \in G^Z} k_\sigma x_\sigma) = \min_{\sigma \in G^Z}\{\phi(k_\sigma)\}$ is a value function corresponding to the Dubrovin valuation ring $W_h * G^Z$.*

Using the above theorem and the ideas in [6], we easily obtain the following corollary.

Corollary 2 *Suppose $S * G$ is a tamely ramified and defectless Dubrovin valuation ring of $K * G$. Then we have $\Gamma_{S*G} \cong \Gamma_W$. If, in addition, the Wadsworth map is a bijection, then $Gal(Z(\overline{S * G})/\overline{V}) \cong G^T$.*

Just as we have been able to define tame central simple algebras, we now define inertial central simple algebras. In the commutative case, W is said to be inertial over F if $[K : F] = f$ and $\overline{W}$ is separable over $\overline{V}$. Therefore, given an arbitrary Dubrovin valuation ring B of Q, we will say that B is *inertial* over F if $[Q : F] = f(B \mid V)$ and $Z(\overline{B})$ is separable over $\overline{V}$, following the terminology used if Q were a division algebra with B as an invariant valuation ring.

In the case when V is Henselian and Q is a division ring with B as an invariant valuation ring, it was shown in [4, §2] that B is inertial over F if and only if it is Azumaya over V. We now easily generalize this result. We hasten to point out that [4] is a far-reaching account of division algebras over Henselian fields.

Proposition 3 *A Dubrovin valuation ring B of Q is inertial over F if and only if it is Azumaya over V.*

The strategy is to show that the statement holds when V is Henselian, using results in [4], and, for the general case, we pass on to $B \otimes_V V_h$.

Recall that W is said to be unramified over F if $e = 1$ and $\overline{W}$ is separable over $\overline{V}$. By [5, Theorem 3], we immediately have the following.

Corollary 3 *Let K/F be an arbitrary Galois extension.*

1. *The V-order $S * G$ is a Dubrovin valuation ring inertial over F if and only if K/F is unramified and defectless.*

2. *If V is Henselian, then $S * G$ is a Dubrovin valuation ring inertial over F if and only if K/F is an inertial extension.*

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BLOCK VARIETIES AND INVARIANCE PROPERTIES

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Abstract. In this report[1], we first give the short explanation for the three notions introduced by M. Linckelmann [10], [11]. Secondly, we show the two basic relations between the block variety $V_{G,b}(U)$ introduced by Linckelmann in [11] and the well-known Carlson's module variety $V_G(U)$ in [4], for a bounded complex U consisting of modules over the block algebra kGb. That is, there is a finite surjective map $V_{G,b}(U) \to V_G(U)$, and conversely there is also a finite surjective map $V_D(iU) \to V_{G,b}(U)$, where D is a defect group of b and i is a souce idempotent of b. Finally, we give the explanation for the invariance properties of block varieties under splendid stable and derived equivalences, and further Green correspondences.

1. 準備

この節では Linckelmann [10], [11] の概要を述べる，定義などの詳しい解説に関しては [17] を参照してください．G を finite group, R を commutative ring とする．G の cohomology ring から RG の Hochschild cohomology ring への diagonal induction functor によって導かれるよく知られた embedding $\delta_G : H^*(G,R) \to HH^*(RG)$ がある．この時 H を G の subgroup, $t_{H,G}$ を有限群の cohomology における transfer とすると，下の図式を可換にするような底辺の linear map を定義したくなるが Linckelmann は自然に symmetric R-algebras の場合に拡張できる形の定義を与えた；A, B は symmetric R-algebra (i.e., R 上 projective であり，$A \cong A^*$ as $A-A$-bimodule). $X = \{ X_n \}$ を bounded complex of $A-B$-bimodules X_n で X_n は left A-module, right B-module として projective とする (以後 bounded perfect complex of $A-B$-bimodules と呼ぶことにする)．この時，transfer map associated with X と呼ばれる linear map $t_X : HH^*(B) \to HH^*(A)$ が定義されて，特に $RG-RH$-bimodule $X = (RG)_H$ に対して下の図式は可換となる．

$$\begin{array}{ccc} H^n(H,R) & \xrightarrow{t_{H,G}} & H^n(G,R) \\ \delta_H \downarrow & & \downarrow \delta_G \\ HH^n(RH) & \xrightarrow[t_{(RG)_H}]{} & HH^n(RG) \end{array}$$

さらに，この transfer map t_X は次の定理 1.1 で述べる特性をもつ．t_X の degree 0 component $t_X^0 : Z(B) \cong HH^0(B) \to HH^0(A) \cong Z(A)$ による 1_B の像 $t_X^0(1_B) = \pi_X$ と置く．また，$HH^*(A)$ における X-stable と呼ばれる元全体から成る subalgebra を $HH_X^*(A)$ と

[1]The detailed version of this paper will be submitted for publication elsewhere.

置く．X の R-dual X^* に対しても同様にして $\pi_{X^*}, HH^*_{X^*}(B)$ を定める．

定理 1.1.([10, 3.6]) π_X が $Z(A)$ において可逆とする．この時 $T_X = \pi_X^{-1} t_X$ とおくことにより，$T_X : HH^*_{X^*}(B) \to HH^*_X(A)$ は R-algebra homomorphism となる．さらに π_{X^*} も $Z(B)$ において可逆なら T_X, T_{X^*} により同型が与えられる．

以後の block 理論全般に関しては [16] を参考にして下さい．k を algebraically closed field of characteristic $p > 0$ とする．b を kG の block idempotent, D を b の defect group (i.e., $\mathrm{Br}^G_P(b) \neq 0$ となる p-subgp. P のうちで maximal なもの，ここで $\mathrm{Br}^G_P : (kG)^P \to kC_G(P)$ は Brauer hom.). i を b の source idempotent (i.e., $(kGb)^D$ の primitive idempotent s.t. $\mathrm{Br}^G_D(i) \neq 0$), e_D を $\mathrm{Br}^G_D(i)$ ($\mathrm{Br}^G_D(i)$ は primitive) を含む $kC_G(D)$ の block とする．この時，Brauer pair (D, e_D) は maximal b-Brauer pair となり，D の任意の subgroup Q に対し $(Q, e_Q) \leq (D, e_D)$ となるただ 1 つの e_Q が存在する (e_Q は $Br^G_Q(i)e_Q = Br^G_Q(i)$ となる $C_G(Q)$ のただ 1 つの block). 定理 1.1 の重要な例として次のものがある．

定理 1.2.([10, 5.6], [11, 5.2])
(1) $kGb - kD$-bimodule $X = kGi$ に対して π_{kGi} と $\pi_{(kGi)^*} = \pi_{ikG}$ ($ikG \cong (kGi)^*$ as $kD - kGb$-bimodules) は可逆となり，$T_{kGi} : HH^*_{ikG}(kD) \cong HH^*_{kGi}(kGb)$ as k-algebras が成り立つ．
(2) G, H を finite groups. b, c を kG, kH の blocks とする．bounded perfect complex X of $kGb - kHc$-bimodules が次の条件をみたすとする ($\simeq$ は homotopy equivalence).

$X \otimes_{kHc} X^* \simeq kGb \oplus U_b$ 　　U_b : bounded complex of proj. $kGb - kGb$-bimodules

$X^* \otimes_{kGb} X \simeq kHc \oplus U_c$ 　　U_c : bounded complex of proj. $kHc - kHc$-bimodules

この時，π_X, π_{X^*} は可逆となり，$T_X : HH^*_{X^*}(kHc) \cong HH^*_X(kGb)$ as k-algebras が成り立つ．

注意 1.3. $kGb - ikGi$-bimodule kGi とその dual $(kGi)^*$ は block algebra kGb とその source algebra $ikGi$ の森田同値を与える．また，derived category の森田理論の言葉で言うと，定理 1.2 (2) の条件をみたすような bounded perfect complex X は $U_b = U_c = \{ 0 \}$ のとき two sided split endomorphism tilting complex のことであり，X が bimodule のときは kGb と kHc は stable equivalence of Morita type の関係にあるということである [7].

次に block b の cohomology ring について見ていく．$Q \leq D$ (i.e., D の subgroup Q) に対し，$E_G((Q, e_Q), (D, e_D)) = \{ \varphi : Q \to D \mid \varphi$ は x-conjugation s.t. ${}^x(Q, e_Q) \leq (D, e_D)$ $(x \in G) \}$ とおく．${}^x(Q, e_Q) \leq (D, e_D) \Leftrightarrow {}^xQ \leq D$, ${}^xe_Q = e_{{}^xQ}$ (${}^xQ$ は Q の cojugate subgp. の意味). さらに，D_γ を (D, e_D) と対応する defect pointed group とする (γ は i を含む point).

定義 1.4.([10, 5.1]) D_γ と associate される block b の cohomology ring を次の様に定義する．$H^\bullet(G,b,D_\gamma) = \{\ [\zeta] \in H^\bullet(D,k) \mid \tilde{\varphi} \circ \mathrm{res}^D_Q([\zeta]) = \mathrm{res}^D_{Q^x}([\zeta])$ for any $Q \le D$ and any $\varphi \in E_G((Q,e_Q),(D,e_D))$ (i.e. Brauer category $Br_{\le(D,e_D)}$ の任意の morphism φ [16]) $\}$. ここで，$\tilde{\varphi}$ は φ によって導かれる conjugation map $H^\bullet(Q,k) \to H^\bullet({}^xQ,k)$ である．

b が principal block の場合，D は G の p-sylow subgroup であるから $\mathrm{res}^G_D : H^\bullet(G,k) \cong \{$ stable elements of $H^\bullet(D,k)\ \}$ ([5, 10.1]). ここで $[\zeta]$ が stable とは，定義 1.4 において Q が sylow intersection ${}^{x^{-1}}D \cap D$ のときに，φ がこの x による conjugate ${}^{x^{-1}}D \cap D \to D \cap {}^xD$ の場合だけで条件式が成り立つことを要求する．ゆえに $\mathrm{Image}(\mathrm{res}^G_D) \supseteq H^\bullet(G,b,D_\gamma)$. 逆に $\mathrm{Image}(\mathrm{res}^G_D) \subseteq H^\bullet(G,b,D_\gamma)$ は明らかであるから $H^\bullet(G,k) \cong H^\bullet(G,b,D_\gamma)$ となる．

2. Linckelmann's varieties と Carlson's varieties

任意の bounded complex U of kGb-modules に対して次の k-algebra homomorphisms から成る可換図式が得られる (α_U の定義は [11] 参照).

$$
\begin{array}{ccccc}
H^\bullet(G,k) & \xrightarrow{\ \delta_G\ } & HH^\bullet(kG) & \xrightarrow{\ \alpha_U\ } & \mathrm{Ext}^\bullet_{kG}(U,U) \\
\Big\downarrow{\scriptstyle \mathrm{res}^G_D} & & \Big\downarrow{\scriptstyle proj.} & & \Big\| \\
H^\bullet(G,b,D_\gamma) & \longrightarrow & HH^\bullet(kGb) & \xrightarrow{\ \alpha_U\ } & \mathrm{Ext}^\bullet_{kGb}(U,U) \\
\Big\downarrow{\scriptstyle \delta_D} & \nearrow{\scriptstyle \cong\ T_{kGi}} & & & \\
HH^\bullet_{ikG}(kD) & & & &
\end{array}
$$

定義 2.1([11, 4.1]) bounded complex U of kGb-modules に対して，$\alpha_U \circ T_{kGi} \circ \delta_D$ の kernel を $I^\bullet_{G,b,D_\gamma}(U)$ とする．U の block variety を $V_{G,b}(U) =$ maximal ideal spectrum of $H^\bullet(G,b,D_\gamma)/I^\bullet_{G,b,D_\gamma}(U)$ と定義する．

$\alpha_U \circ \delta_G$ の kernel を $I^\bullet_G(U)$ とすると maximal ideal spectrum of $H^\bullet(G,k)/I^\bullet_G(U)$ は U の Carlson's variety $V_G(U)$ のことであるから上の可換図式より次が言える．

定理 2.2.([11, 4.4]) $I^\bullet_G(U) = (\mathrm{res}^G_D)^{-1}(I^\bullet_{G,b,D_\gamma}(U))$ が成り立つ．$H^\bullet(G,b,D_\gamma)$ は $\mathrm{Image}(\mathrm{res}^G_D)$ 上有限生成であるから，$(\mathrm{res}^G_D)^* : V_{G,b}(U) \to V_G(U)$ は finite surjective affine map となる．さらに，$\dim V_{G,b}(U) = \dim V_G(U)$ も成り立つ．

Linckelmann [11, 5.1] をもとにして次の２つの可換図式を得ることができる．ここで，$\mathrm{Ext}^\bullet\_(\_,\_)$ を $\mathrm{Ext}^\bullet\_(\_)$ と略記した．また $kGi \otimes_{kD} \_, ikG \otimes_{kGb} \_$ の定義は [11] 参照．この時，Ext-group に出てくる modules の対応は kD-modules V の category と kGb-modules U の category の間の kGi-induction, ikG-restriction と呼んでよいものである．

$$\begin{array}{ccc} HH^{\bullet}_{kGi}(kGb) & \xrightarrow{\alpha_{kGi\otimes_{kD}V}} & \mathrm{Ext}^{\bullet}_{kGb}(kGi\otimes_{kD}V) \\ {\scriptstyle T_{ikG}}\uparrow & & \uparrow{\scriptstyle kGi\otimes_{kD}-} \\ HH_{ikG}(kD) & \xrightarrow[\alpha_V]{} & \mathrm{Ext}^{\bullet}_{kD}(V) \end{array} \qquad \begin{array}{ccc} HH^{\bullet}_{kGi}(kGb) & \xrightarrow{\alpha_U} & \mathrm{Ext}^{\bullet}_{kGb}(U) \\ {\scriptstyle T_{ikG}}\downarrow & & \downarrow{\scriptstyle ikG\otimes_{kGb}-} \\ HH_{ikG}(kD) & \xrightarrow[\alpha_{iU}]{} & \mathrm{Ext}^{\bullet}_{kD}(iU) \end{array}$$

上の可換図式に $\delta_D : H^{\bullet}(G,b,D_{\gamma}) \to HH^{\bullet}_{ikG}(kD)$ を合成することにより次の可換図式が得られる．ここで，ι は包含写像，top horizontal maps は左の図から順に $\alpha_{kGi\otimes_{kD}V} \circ T_{kGi} \circ \delta_D$ と $\alpha_U \circ T_{kGi} \circ \delta_D$，さらに bottom horizontal maps は $\alpha_V \circ \delta_D$ と $\alpha_{iU} \circ \delta_D$ である．

$$\begin{array}{ccc} H^{\bullet}(G,b,D_{\gamma}) & \longrightarrow & \mathrm{Ext}^{\bullet}_{kGb}(kGi\otimes_{kD}V) \\ {\scriptstyle \iota}\downarrow & & \uparrow{\scriptstyle kGi\otimes_{kD}-} \\ H^{\bullet}(D,k) & \longrightarrow & \mathrm{Ext}^{\bullet}_{kD}(V) \end{array} \qquad \begin{array}{ccc} H^{\bullet}(G,b,D_{\gamma}) & \longrightarrow & \mathrm{Ext}^{\bullet}_{kGb}(U) \\ {\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle ikG\otimes_{kGb}-} \\ H^{\bullet}(D,k) & \longrightarrow & \mathrm{Ext}^{\bullet}_{kD}(iU) \end{array}$$

ところで，次のことが成り立つ [6, 1.1].

$$U \text{ is a direct summand of } kGi\otimes_{kD} iU \text{ as } kGb-\text{modules}$$

ゆえに，左の図式で $V = iU$ とするとき $kGi\otimes_{kD} V$ を U と置き換えることができる．Linckelmann's variety は top horizontal map の kernel で定まり，Carlson's variety は bottom horizontal map の kernel で定まるので次のことが示される (Linckelmann も同じ結果を得ている [12]).

定理 2.3. D を block b の defect group, $i \in \gamma$ を b の source idempotent とする．この時，$I^{\bullet}_{G,b,D_{\gamma}}(U) = H^{\bullet}(G,b,D_{\gamma}) \cap I^{\bullet}_{D}(iU)$ が成り立つ．$H^{\bullet}(D,k)$ は $H^{\bullet}(G,b,D_{\gamma})$ 上有限生成であるから，$\iota^{\bullet}$ を包含写像 ι から導かれる affine map とすると $\iota^{\bullet} : V_D(iU) \to V_{G,b}(U)$ は finite surjective となる．さらに，$\dim V_D(iU) = \dim V_{G,b}(U)$ も成り立つ．

3. Invariance properties of varieties

定理 3.1.([11, 5.5]) G, H を finite group. b, c は kG, kH の blocks で共通の defect group D をもつ．i, j を b, c の source idempotents とし，i, j と associate される maximal b-Brauer pair, maximal c-Brauer pair を $(D, e_D), (D, f_D)$ とするとき，$E_G((Q,e_Q),(D,e_D)) = E_H((Q,f_Q),(D,f_D))$ for any $Q \leq D$ (ゆえに，b と c の Brauer category は同値) と仮定する．以上の設定のもとで次の条件をみたす bounded complex X of $kGb - kHc$-bimodules が存在すると仮定する．

(i) $X \otimes_{kHc} X^{*} \simeq kGb \oplus U_b$ $\quad U_b$: bounded complex of proj. $kGb - kGb$-bimodules
$\quad X^{*} \otimes_{kGb} X \simeq kHc \oplus U_c$ $\quad U_c$: bounded complex of proj. $kHc - kHc$-bimodules

(ii) $M_{n,t}$ を X の component X_n の indecomposable direct summand とする．任意の n, t において，$M_{n,t}$ は direct summand of $kGi \otimes_{kQ} jkH$ for some $Q \leq D$.

このとき，任意の bounded complex V of kHc-modules に対して，$V_{H,c}(V) \cong V_{G,b}(X \otimes_{kHc} V)$ が成り立つ．

定理の条件に関連して block algebra 上の derived category の森田理論について見ていく．blocks b と c が共通の defect group をもつ条件のもとで，上の定理の条件 (i) (ただし $U_b = U_c = \{ 0 \}$), (ii) をみたす bounded complex X は splendid tilting complex と呼ばれる．(ii) は Linckelmann の条件で，この概念を導入した J. Rickard は本質的には principal blocks b, c の場合を考えているようで (ii) の条件は $M_{n,t} \mid kG \otimes_{kD} kH$ とした ([8], [14])．以後，便宜上 Rickard によるものを splendid Rickard complex, Linckelmann によるものを splendid Linckelmann complex と呼ぶことにする．次に Brauer construction の notation を導入する．Q を finite group G の p-subgroup, M を kG-module とする．このとき，

$$M(Q) = M^Q/(\Sigma_{R \lneq Q} Tr_R^Q(M^R)) \quad (kN_G(Q)\text{-module とみなせる})$$

ここで M^Q は M の Q-fix points (M^R も同様), Tr_R^Q は relative trace. 特に，M が $kGb - kHc$-bimodule ($k(G \times H)$-module とみなせる) で b, c が共通の defect group D をもつとき，D の subgroup Q に対して $M(\Delta Q)$, $\Delta Q = \{ (q, q) \}_{q \in Q}$, は $kC_G(Q) - kC_H(Q)$-bimodule とみなせる ($C_G(Q) \times C_H(Q) \leq N_{G \times H}(\Delta Q)$ による)．

定理 3.2.([8, 1.1], [14, 4.1])
(1) b, c は定理 3.1 の設定をみたす principal blocks とする．X が kGb と kHc に対する splendid Rickard complex ならば，任意の subgroup $Q \leq D$ に対して $X(\Delta Q)$ は $kC_G(Q)$ と $kC_H(Q)$ の principal blocks に対する splendid Rickard complex となる．

(2) b, c は定理 3.1 の設定 (ただし，$E_G((Q, e_Q), (R, e_R)) = E_H((Q, f_Q), (R, f_R))$ for ${}^\forall Q, {}^\forall R \leq D$) をみたす任意の blocks とする．$X$ が kGb と kHc に対する splendid Linckelmann complex ならば，任意の subgroup $Q \leq D$ に対して $e_Q X(\Delta Q) f_Q$ は $kC_G(Q)e_Q$ と $kC_H(Q)e_f$ に対する splendid Rickard complex となる．

注意 3.3.
(1) (a) b, c が principal blocks のとき，splendid Rickard complex は自動的に splendid Linckelmann complex になる ([8, 1.2])．実際 indecomposable $kGb - kHc$-bimodule M に対して，$M \mid kG \otimes_{kD} kH \Rightarrow M \mid kGi \otimes_{kQ} jkH$ (i, j は b, c の source idempotent で ΔQ は M の vertex)．
(b) R. Rouquier によって次のことが注意されている [15, 5.6]; principal blocks b, c に対する定理 3.1 の設定のもとで，X は Rickard の意味での定理 3.1 の条件 (ii) $M_{n,t} \mid kG \otimes_{kD} kH$ をみたすとする．この時，X が principal blocks b, c に対して条件 (i) をみたすことと，任意の non-trivial subgroup $Q \leq D$ おいて $X(\Delta Q)$ が $kC_G(Q)$ と $kC_H(Q)$ の principal blocks に対して $U_b = U_c = \{ 0 \}$ での条件 (i) をみたすこと，すなわち two sided split endomorphism tilting complex となることは同値である．

(2) 一般に，$e_Q X(\Delta Q) f_Q$ が splendid Linckelmann complex であることを示すことはできない．そこで principal block の場合に成り立つ isotypy との関係 (Rickard [14, 6.3]) の一般の blocks への拡張はうまくいかない．すなわち，kGb と kHc に対する splendid Linckelmann complex が存在しても blocks b, c の間の isotypy [2, 4.6] の存在を Rickard の証明にそった形では示すことができない．

定理 3.1 において X が bimodule (すなわち degree 0 のみの complex) で条件 (i), (ii) をみたすとき，Linckelmann [9] は kGb と kHc は splendid stable equivalence の関係にあると呼んでいる (条件 (ii) は block algebra に関する条件で，条件 (i) のみの場合が通常 stable equivalence of Morita type と呼ばれるものである)．以下このことについて説明を加える．stable equivalence of Morita type に関して次のことが知られている．[3, 6.3], [14, 4.1] ではもっと一般的な形で示されているがここでは principal blocks の場合のみ記す．

定理 3.4.([3, 6.3], [14, 4.1], [15, 5.6]) b, c を定理 3.1 の設定をみたす principal blocks とする．$kGb - kHc$-bimodule $M = \oplus M_i$, ここで $M_i \mid kG \otimes_{kD} kH$ に対して，次は同値．
(1) M は kGb, kHc の間の stable equivalence of Morita type を与える．
(2) 任意の non-trivial subgroup $Q \le D$ に対し，$M(\Delta Q)$ は $kC_G(Q), kC_H(Q)$ の principal block algebras の間の通常の Morita equivalence を与える．

Linckelmann は定理 3.2 (2) と同様に pointed group theory を用いて定理 3.4 を (M が indecomposable，設定と (1) より vertex ΔD という条件がつくが) 拡張かつ改良している．

定理 3.5.([8, 3.1]) b, c は kG, kH の blocks で共通の defect group D をもつ．i, j を b, c の source idempotents とする．indecomposable $kGb - kHc$-bimodule M such that $M \mid kGi \otimes_{kD} jkH$ に対して，次は同値．
(1) M は kGb, kHc の間の stable equivalence of Morita type を与える．
(2) 任意の non-trivial subgroup $Q \le D$ に対し，$e_Q M(\Delta Q) f_Q$ は $kC_G(Q)e_Q, kC_H(Q)f_Q$ の間の通常の Morita equivalence を与える．さらに，$E_G((Q, e_Q), (R, e_R)) = E_H((Q, f_Q), (R, f_R))$ for $^\forall Q, ^\forall R \le D$ となる．

block algebras の間の森田型 stable equivalence の具体的な場合として，定理 3.4 をもとにした derived equivalent blocks に関する"奥山の方法"の出発点に位置する設定がある ([13, 3.5])．ここでは 定理 3.5 をもとにして拡張した設定で，次の定理 3.1 の系を与える．

系 3.6. b を kG の任意の block, D を b の defect group (abelian とは仮定しない)．$N = N_G(D)$ とおき，kN の block b_0 を b の Brauer 対応子とする．M を kGb の $(G \times G, \Delta D, G \times N)$ に関する Green 対応子とするとき，$M \otimes_{kNb_0} \_ = b\mathrm{Ind}_N^G \_$ と $M^* \otimes_{kGb} \_ = b_0 \mathrm{Res}_N^G \_$ に

より kNb_0 と kGb が stable equivalence of Morita type の関係にあると仮定する．このとき，この functor から導かれる { indecomposable non-proj. kGb-modules } と {indecomposable non-proj. kNb_0-modules } の間の 1 対 1 対応 (Green 対応を含む) において，その block varieties は不変である．

証明. M を $kGb_{|G\times N}$ の indecomposable direct summand で $kNb_0 \mid M_{|N\times N}$ となるものとする．このとき，M は vertex ΔD となるただ 1 つの $kGb_{|G\times N}$ の ind. direct summand であり，kGb さらに kNb_0 の Green 対応子であることがわかる．また trivial source をもつこともわかるので $M \mid kG \otimes_{kD} kN$ as $kG - kN$-modules. さらに $bMb_0 = M$ より，$M \mid kGb \otimes_{kD} b_0kN$ となる．M が indecomposable より primitive idempotents $i \in (kGb)^D, j \in (kNb_0)^D$ で $M \mid kGi \otimes_{kD} jkN$ となるものがとれるが，もし $Br^G_D(i) = 0$ なら [8, 2.6] より $kGi \mid kGi \otimes_{kR} kD$ as $kGb - kD$-modules となる subgroup $R \lneq D$ が存在する．よって $M \mid kGi \otimes_{kR} jkN$，すなわち $M \mid kG \otimes_{kR} kN = Ind^{G\times N}_{\Delta R}(k)$ となり，これは M の vertex が ΔD に反する．j に関しても同様であり，i, j は b, b_0 の source idempotents であることが分かる．そこで，もし M が stable equivalence of Morita type を与えるならば定理 3.5 より $E_G((Q, e_Q), (D, e_D)) = E_H((Q, f_Q), (D, f_D))$ for any $Q \leq D$ となり，定理 3.1 が適用できる．

さて，M が stable eq. of Morita type を与えることより，indecomposable non projective kNb_0-module V に対し，$M \otimes_{kNb_0} V \cong f(V) \oplus$ (proj. kGb-module), $f(V)$ は indecomposable non proj. kGb-module とできるが，定理 3.1 を適用して $V_{N,b_0}(V) \cong V_{G,b}(M \otimes_{kNb_0} V) = V_{G,b}(f(V)) \cup V_{G,b}(\text{proj. } kGb\text{-module})$ (等号は [6, 3.3]) を得る．ところで，block variety $V_{G,b}(U)$ は homogeneous variety であるから定理 2.2 と 定理 2.3 および [1, II, 5.7.2] を用いて U : projective kGb-module $\Leftrightarrow$ $V_{G,b}(U) = \{ 0 \}$ であることが分かり，系 3.6 は示される．

系 3.6 では stable eq. of Morita type によって与えられる Green 対応のもとで varieties が不変であることを述べたが，上で述べたようにこの場合対応する blocks の Brauer categories は定理 3.1 の条件を自動的にみたしている．そこで対応する blocks の Brauer catrgories が 定理 3.1 の条件 をみたす場合において varieties の不変性について考察した．ここでは defect group を abelian とする．証明は定理 2.3 と，blocks の間の Clifford 理論をもちいる．

命題 3.7. b を abelian defect group D をもつ kG の block, b_0 をその Brauer 対応子とする．vertex D をもつ indecomposable kGb-module M に対して，L を $(G, D, N_G(D))$ に関する M の Green 対応子とするとき，$V_{G,b}(M) \cong V_{N_G(D),b_0}(L)$ となる．

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Auslander-Reiten components and projective modules for finite p-groups

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ABSTRACT. Let G be a finite p-group and $\mathcal{O}$ a complete discrete valuation ring of characteristic 0 with the maximal ideal (π) and the residue field $k = \mathcal{O}/(\pi)$ of characteristic $p > 0$. Let Δ be the connected component of the Auslander-Reiten quiver $\Gamma(\mathcal{O}G)$ containing the projective $\mathcal{O}G$-lattice $\mathcal{O}G$. Suppose that $\mathcal{O}G$ is of infinite representation type. Suppose further that $(\pi) \supsetneqq (2)$ in the case where $p = 2$ and G is the Klein four group. Then the tree class of the stable part Δ_s of Δ is A_∞.

Gは有限群とする．pはGの位数$|G|$を割り切るある素数とし，$(K, \mathcal{O}, k)$はp-モジュラー系とする．すなわち，$\mathcal{O}$は標数0の完備離散付値環で，その極大イデアルの生成元をπとしたとき，剰余体$k = \mathcal{O}/(\pi)$は標数pの体であり，Kは$\mathcal{O}$の商体であるとする．

ここではRで$\mathcal{O}$またはkを表し，RG-加群といえばR-上自由で有限生成なものを意味するものとする．特に$\mathcal{O}G$-加群とは$\mathcal{O}G$-latticeを意味し，射影的／入射的については$\mathcal{O}G$-latticeのなすカテゴリーで考えることにする．

次が報告したい主結果である．

定理 Gがp-群で$\mathcal{O}G$が無限表現型とする．ただし$p = 2$でGがKlein four groupのときは$(\pi) \supsetneqq (2)$も仮定する．このとき射影加群$\mathcal{O}G$を含むAuslander-Reiten componentのstable partのtree classはA_∞である．

群Gがp-群ならば群環$\mathcal{O}G$はlocalであり，そのため$\mathcal{O}G$自身が唯一の直既約な射影的$\mathcal{O}G$-加群であることを注意しておく．また$p = 2$でGがKlein four groupのときは，射影加群$\mathcal{O}G$を含むAuslander-Reiten componentのtree classは$\tilde{D}_4$であることが知られている[D]．

The detailed version of this paper has been submitted for publication elsewhere.

この報告集では，§1で群環の Auslander-Reiten quiver について今までわかっていることを大まかにまとめ，§2で上の主結果の証明のカギとなる事実を説明したい．また§3では完備離散付値環 $\mathcal{O}$ 上の群環の表現型についてある注意をしたい．

群環の Auslander-Reiten 理論については，Benson[B]，Erdmann[E1]，Roggenkamp [R2] に詳しい説明があります．

§1 群環の Auslander-Reiten quiver

$\Gamma(RG)$ で群環 RG の Auslander-Reiten quiver を，$\Gamma_s(RG)$ で RG の stable Auslander-Reiten quiver を表すことにする．ところで Auslander-Reiten quiver $\Gamma(RG)$ とは，“点”の集合としては直既約 RG-加群の同型類 $[M]$ を考えて，直既約 RG-加群 M, N に対して M から N への“既約写像”が存在する時に $[M] \to [N]$ のように“矢”を書くことによって得られる有向グラフのことである．(RG-準同型写像 $f : M \to N$ が既約写像とは，本質的な分解ができないときを云う．即ち次の条件をみたすとき：f は split-mono でも split-epi でもなく，もし $f = hg$ と合成写像に分解されれば，g が split-mono かまたは h が split-epi である．) また，stable Auslander-Reiten quiver とは，Auslander-Reiten quiver から射影加群／入射加群に対応する点と，それらの点につながる矢を取り除いて得られるグラフである．

$\Gamma_s(RG)$ の連結成分（以後 AR component）のグラフとしての形状については次の structure theorem が知られている．

定理 (Riedmann)　AR component Θ に対して，tree class T と，その translation quiver $\mathbb{Z}T$ の自己同型からなる admissible group Π が定まって，$\Theta \cong \mathbb{Z}T/\Pi$ となる．

一般に，群環の tree class について，Webb により次が示された．

定理 (Webb[W])　群環 RG の AR component の tree class は，Dynkin diagram かまたは Euclidean diagram かもしくは A_∞, B_∞, C_∞, D_∞, A_∞^∞ のどれかである．

k 上の群多元環 kG の場合（いわゆるモジュラー表現）では，Erdmann によって次の事実が証明された．

定理 (Erdmann[E2])　群環 kG の block B が wild 表現型であれば，B の AR component の tree class は A_∞ : •—•—•—$\cdots$ である．

ここで block B が wild 表現型であるとは，直既約加群が無限に存在し，それらをうまくパラメタライズして分類することが期待できないときに云う．群環の場

合には，block B が wild 表現型であることは，B の不足群 $\delta(B)$ が cyclic, dihedral, semidihedral, generalized quaternion のいずれでもないということと同値である [E1].

モジュラー表現の場合の AR component の形が Erdmann によってわかったので，今度は整数表現の場合を知りたい.

R に G が自明に作用するとき（即ち，$x \in R$, $g \in G$ に対して $xg = g$ のとき），この RG-加群を自明な加群と云い，R_G で表す．自明な加群 R_G は有限群の表現において注視すべき加群であるが，$\mathcal{O}_G$ を含む AR component については次のことがいえる.

定理 (Inoue-Kawata[IK])　G は p-群で群環 $\mathcal{O}G$ は無限表現型であるとする．但し $p = 2$ で G が Klein four group のときは $(\pi) \supsetneqq (2)$ も仮定する．このとき，自明な $\mathcal{O}G$-加群 $\mathcal{O}_G$ を含む AR component の tree class は A_∞ である.

$\mathcal{O}$ 上の表現においても Erdmann の定理に相当することが成り立つであろうか？まだ一般の AR component についてはわかっておらず，上のような特別な例しか知られていない.

§2　主定理の証明における注意:　射影加群と Auslaner-Reiten 列

冒頭に述べた主定理の証明の方針は，Webb の定理から，A_∞ 以外の tree class の可能性を一つずつ取り除いていくことによる．細かな証明は省略して，それよりもここでは，証明のカギとなるところの，Auslander-Reiten quiver $\Gamma(\mathcal{O}G)$ のなかで射影加群 $\mathcal{O}G$ の周囲がどうなっているかを説明したい.（その結果として特に射影加群 $\mathcal{O}G$ を含む AR component の tree class は A_∞^∞ ではないことがわかる.）

Auslander-Reiten quiver とは，大雑把にいうといわゆる " Auslander-Reiten 列 " を貼りあわせたものである．すなわち，Auslander-Reiten quiver の一つ一つの " 網目 " が Aulsander-Reiten 列に対応している．ここで，RG-加群の完全列

$$\mathcal{E} : 0 \to Z \to Y \xrightarrow{f} X \to 0$$

は次の 3 条件をみたすときに，Auslander-Reiten 列（または almost split 列）と云う：

（1）X と Z は直既約；
（2）$\mathcal{E}$ は分裂していない；
（3）任意の split-epi でない準同型写像 $g : W \to X$ に対し，ある準同型写像 $h : W \to Y$ が存在して $g = fh$ が成り立つ.

アルティン環や order の表現論において，任意の射影的でない直既約加群 X に対し，X で終わるような Auslander-Reiten 列が一意的に存在することが知られている．一意性から，X で終わる Auslander-Reiten 列の始めの項を τX と表し，この τ を Auslander-Reiten translation と呼ぶ．群環の場合，$R=\mathcal{O}$ のときは $\tau=\Omega$ であり，$R=k$ のときは $\tau=\Omega^2$ であることが知られている．ここで Ω は Heller 作用素，すなわち X の projective cover P_X の kernel である（$0\to\Omega X\to P_X\to X\to 0$）([AR], [R1])．この Auslander-Reiten 列と Auslander-Reiten quiver は次の事実によって関連づけられる．

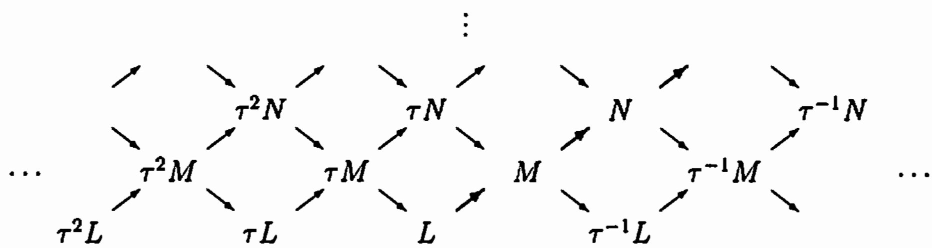
命題　M, N がともに射影的でない直既約 RG-加群のとき，次は同値：

（1）M から N への既約写像が存在する；

（2）M から始まる Auslander-Reiten 列の中間項の直和因子として N が現れる；

（3）N で終わる Auslander-Reiten 列の中間項の直和因子として M が現れる．

群環において頻出する $\mathbb{Z}A_\infty$ 型の AR component を例にとって見てみよう．



この AR component においては網目がそれぞれ Aulsander-Reiten 列に相当している．上の図で M に注目すると完全列 $0\to\tau M\to L\oplus\tau N\to M\to 0$ が Auslander-Reiten 列であり，また L に注目すると完全列 $0\to\tau L\to\tau M\to L\to 0$ が Auslander-Reiten 列である．特に，直既約加群 L が AR component の end(端) に位置することは，L で終わる Auslander-Reiten 列の中間項の projective-free part が直既約であることと同値である．

ところで，G が p-群で群環 $\mathcal{O}G$ が無限表現型のときは，射影加群 $\mathcal{O}G$ が現れる AR 列は次の様になっている [K2]：

$$0\to J(\mathcal{O}G)\to M\oplus\mathcal{O}G\to I(\mathcal{O}G)\to 0$$

ここで，$J(\mathcal{O}G)$ は $\mathcal{O}G$ の radical $\pi\mathcal{O}G+\sum_{g\in G}\mathcal{O}(g-1)$ を表す．G が p-群の場合には，$J(\mathcal{O}G)$ が decomposable であるための必要十分条件は，$|G|=p$ かつ $(\pi)=(p)$ である [HR].（実は一般の有限群 G では，直既約な射影的 $\mathcal{O}G$-lattice P が無限表現型の block に属するなら，P の radical は直既約である [K1].）また $\mathcal{O}G/\pi\mathcal{O}G\cong kG$

は simple socle を持つので，$\mathcal{O}G$ を真に含むような $KG = K \otimes_{\mathcal{O}} \mathcal{O}G$ の $\mathcal{O}G$-部分加群のなかで極小なものが存在するが，それを $I(\mathcal{O}G)$ と書く．また M は，$J(\mathcal{O}G)$ を含むような $I(\mathcal{O}G)$ の極大 $\mathcal{O}G$-部分加群のなかで射影的ではない唯一のものである(下図):

$K \otimes_{\mathcal{O}} \mathcal{O}G$

$I(\mathcal{O}G)$: unique minimal overmodule of $\mathcal{O}G$ in KG

$\mathcal{O}G$ M : unique lattice s.t. $J(\mathcal{O}G) \subsetneqq M \subsetneqq I(\mathcal{O}G)$ and M : non-proj.

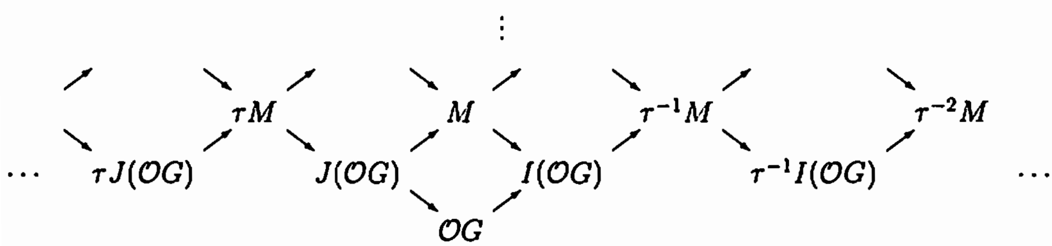
$J(\mathcal{O}G)$: unique maximal submodule of $\mathcal{O}G$

さて，M の直既約性について，次がわかった．

命題 [K2]　G が p-群で $\mathcal{O}G$ が無限表現型のとき，ただし $p = 2$ で G が Klein four group のときは $(\pi) \supsetneqq (2)$ も仮定する．このとき M は直既約である．

注意 ([IK], [K2])　M は直既約でないときもある：　実際，$|G| = p^2$ で $(\pi) = (p)$，または $|G| = p$ で $(\pi^2) = (p)$ ならば，自明な加群 $\mathcal{O}_G$ が M の直和因子として現れる．

従って，射影加群 $\mathcal{O}G$ の周辺の AR quiver の様子は，次の様になっている：



命題 [K]　G が p-群で $\mathcal{O}G$ が無限表現型のとき，ただし $p = 2$ で G が Klein four group のときは $(\pi) \supsetneqq (2)$ も仮定する．このとき $J(\mathcal{O}G)$ は AR component の end に位置する．特に，射影加群 $\mathcal{O}G$ をふくむ AR component には end があるので，その tree class は A_∞^∞ ではない．

このようにして A_∞^∞ の可能性を消したが，Webb の定理に現れる他の tree class の可能性を一つずつ消して行くことによって (詳細は [K2] を参照して下さい)，主定理の証明は完成される．

§3　無限表現型の群環

完備離散付置環 $\mathcal{O}$ 上の群環 $\mathcal{O}G$ の表現型について，Heller-Reiner は次を示した．

定理 [Heller-Reiner]　p-群 G が位数 p^3 以上なら群環 $\mathcal{O}G$ は無限表現型である．

この節では上の事実に対して，Auslander-Reiten 理論による別証明を与えたい．

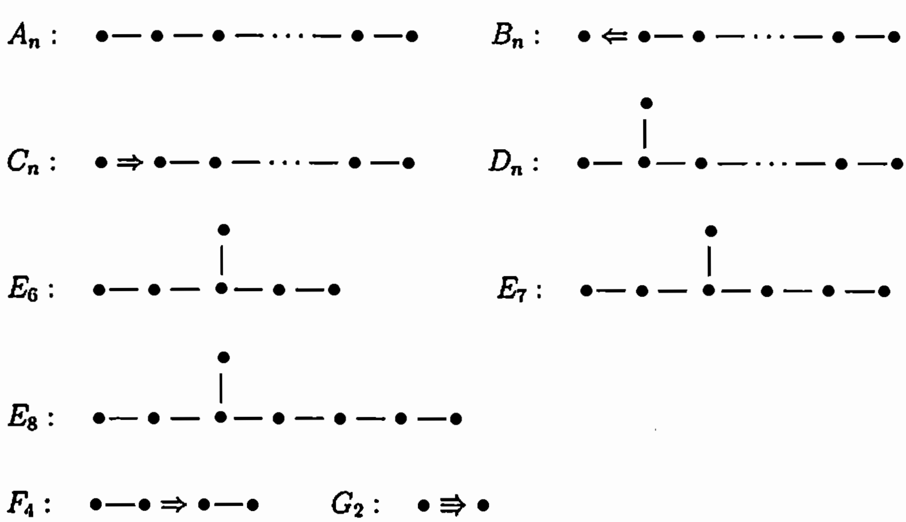
直既約 RG-加群 M について，ある自然数 n があって $\tau^n M \cong M$ （τ は Auslander-Reiten translation ）となるとき，M を τ に関して周期的であると云う．次の結果は Happel-Preiser-Ringel による．

定理 [Happel-Preiser-Ringel]　AR component Θ が Auslander-Reiten translation τ に関して周期的な直既約加群を含むとする．このとき，

（1）Θ が無限個の直既約加群を含めば，Θ は tube である．

（2）Θ が有限個の直既約加群からなれば（従って有限表現型ならば）Θ の tree class は Dynkin である．(とくに，Θ の end に位置するような直既約加群からなる τ-軌道の個数は高々 3 個である.)

注意　Dynkin diagrams は次の通り：



よって Happel-Preiser-Ringel の結果から，もし Auslander-Reiten component の end に位置するような直既約加群からなる τ-軌道が 4 個以上存在することがいえれば，それは無限表現型であるとわかる．

H は群 G の部分群とする．自明な RH-加群 R_H の誘導加群 $R_H\uparrow^G := R_H \otimes_{RH} RG$ を置換加群と云い，また置換加群の直既約因子を自明なソースを持つ加群と云う．また，直既約 RG-加群 M に対し，G の部分群からなる次のような集合

$$\{H \leq G \mid M \text{ はある } RH\text{-加群 } W \text{ の誘導加群 } W\uparrow^G \text{ の直和因子に同型}\}$$

には極小なものが（共役を除いて）一意的に存在するが，この部分群を M のヴァーテックスと云う．ヴァーテックスやソースについて詳しくは Nagao-Tsushima の本 [NT] を見て下さい．

自明なソースを持つ $\mathcal{O}G$-加群の Auslander-Reiten 列について次がわかった．

命題　G は（p-群とは限らない）有限群とし，X は自明なソースを持つ $\mathcal{O}G$-加群で射影的ではないとする．このとき X で終わる Auslander-Reiten 列の中間項は直既約である．とくに X は AR component の end に位置する．

今から Happel-Preiser-Ringel の定理を利用して，Heller-Reiner の定理の別証明をしよう．

p-群 G は巡回群かもしくは 4 元数形の 2 群の場合だけを考えればよい：実際，そうでなければ，群環 $\mathcal{O}G$ は無限表現型（例えば自明な加群 $\mathcal{O}_G$ は Ω-周期的ではない）となる．従って，すべての $\mathcal{O}G$-加群は Ω-周期的である．群環 $\mathcal{O}G$ において $\tau = \Omega$ なので Happel-Preiser-Ringel の定理から AR component は，もし無限表現型なら tube であり，また，もし有限表現型ならばその tree class は Dynkin である．

H_1, H_2 を G の部分群で位数がそれぞれ p, p^2 のものとする．G が p-群なので Green の定理から，$\mathcal{O}_G$，$\mathcal{O}_{H_1}\uparrow^G$，$\mathcal{O}_{H_2}\uparrow^G$ は直既約であり，また上の命題から，これらの自明なソースをもつ加群は AR component の end に位置し，更に（$\mathcal{O}_G$, $\mathcal{O}_{H_1}\uparrow^G$, $\mathcal{O}_{H_2}\uparrow^G$ のヴァーテックスはそれぞれ G, H_1, H_2 と互いに異なり，直既約加群のヴァーテックスは $\tau(=\Omega)$ を作用させても変わらないので）互いに異なる τ-軌道に属する．また§2 の命題から $\mathcal{O}G$ の radical $J(\mathcal{O}G)$ も AR component の end に位置している．いま $J(\mathcal{O}G)/\pi J(\mathcal{O}G) \cong k_G \oplus \Omega k_G$ であるが，自明なソースを持つ $\mathcal{O}G$-加群（およびそれに Ω を作用させてできる $\mathcal{O}G$-加群達）を mod π で reduction しても直既約なので，$J(\mathcal{O}G)$ は自明なソースを持つ加群とは別の τ-軌道に属することがわかる．これらのことから AR component の end に位置するような直既約加群からなる τ-軌道が 4 個以上存在することがいえた．よって上の注意から群環 $\mathcal{O}G$ は無限表現型であるとわかる．

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Non-Commutative Valuation Rings of Skew Polynomial Quotient Rings[1]

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Let Q be a simple Artinian ring with automorphism σ and let $Q(X,\sigma)$ be the quotient ring of the skew polynomial ring $Q[X,\sigma]$. In this note, we discuss some special non-commutative valuation rings of $Q(X,\sigma)$ which are obtained by the pullback. For any ring R, we use the following notation : $J(R)$ is the Jacobson radical of R, $U(R)$ is the unit group of R, $Z(R)$ is the center of R, $\mathbf{N}$ is the set of all natural numbers and $\mathbf{Z}$ is the ring of integers. Let us start with the following definition of non-commutative valuation rings.

DEFINITION. (1) A subring R of a division ring D is called a *total* valuation ring if, for any non-zero $d \in D$, either $d \in R$ or $d^{-1} \in R$.
(2) A total valuation ring R of D is called *invariant* if $dRd^{-1} = R$ for any non-zero $d \in D$.

If R is an invariant valuation ring of D, then $\Gamma_R = U(D)/U(R)$ is a totally ordered group(not necessarily commutative) in the following definition ; $aU(R)+bU(R) = abU(R)$ and $aU(R) \geq bU(R)$ if $aR \subseteq bR$ for any non-zero $a, b \in D$. The natural map $v : D \longrightarrow \Gamma_R$ satisfies the following
(i) $v(ab) = v(a) + v(b)$
(ii) $v(a+b) \geq min\{v(a), v(b)\}$
for any non-zero $a, b \in D$.

Conversely, let G be a totally ordered group and let $v : D \longrightarrow G$ be a map satisfying (i) and (ii). Then $R = \{d \in D \mid v(d) \geq 0\}$ is an invariant valuation ring. So the invariant valuation rings are completely determined by the totally ordered groups and also we know that invariant valuation rings are the same as ones defined by Schilling [S], who initiated non-commutative valuation rings. However these non-commutative valuation rings have the following two disadvantages.
(a)These are not defined in a simple Artinian ring.
(b)Suppose that D is a finite dimension over $Z(D)$ and let V be a valuation ring of $Z(D)$. Then there does not necessarily exist a total valuation ring R of D lying over V, i.e., $R \cap Z(D) = V$ (see [V]).

In 1984, Dubrovin defined a non-commutative valuation ring R of a simple Artinian ring Q by using the concept of places which is similar to commutative case (see [ZS], [D_1]) and he showed that R is equivalent the following two conditions
(α) R is local, i.e., $R/J(R)$ is a simple Artinian ring.
(β) R is Prüfer, i.e., every finitely generated one-sided ideal is projective and a generator.
In [D_1] and [D_2], he obtained so many significant properties of R which looked like genuine non-commutative valuation rings. Nowadays we call R satisfying (α) and (β) ***Dubrovin*** valuation ring.

[1]A part of this note has been submitted to the Journal of Algebra and its Application and the other part will be submitted for publication elsewhere

For simplicity, a little while, let V be a valuation ring of a field F with $\sigma \in Aut(F)$ and δ, a left σ-derivation, i.e., $\delta(a+b) = \delta(a) + \delta(b)$ and $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$ for any $a, b \in F$. The set $F[X, \sigma, \delta]$ of all polynomials over F in an indeterminate X is a ring in the following definition ; $Xa = \sigma(a)X + \delta(a)$ for any $a \in F$. Since $F[X, \sigma, \delta]$ is a principal ideal domain ([R_0]), it has the quotient field denoted by $F(X, \sigma, \delta)$.

PROBLEM. Find out all non-commutative valuation rings of $F(X, \sigma, \delta)$ lying over V.

In the case $\sigma = 1$, $\delta = 0$, the problem was initiated by MacLane [M], 1936. After MacLane so many ring theorists have been involved in this problem. However the problem has not been solved completely, though there has been a great progress. This note is concerned with $\delta = 0$, σ, non-trivial. We firstly study the trivial case, namely, $V = F$. By the definition of total valuation rings, we know that, for any total valuation ring R of $F(X, \sigma)$, we have either $X \in R$ or $X^{-1} \in R$ so that either $R \supseteq F[X, \sigma]$ or $F[X^{-1}, \sigma]$. So the following are very useful to find out total valuation rings of $F(X, \sigma)$.

PROPOSITION 1 [R_0]. Let F be a field with $\sigma \in Aut(F)$ and let $F_0 = \{a \in F \mid \sigma(a) = a\}$. Then
(1) If $o(\sigma)$, the order of σ, is infinite, then $F_0 = Z(F[X, \sigma])$ and $Spec(F[X, \sigma]) = \{ XF[X, \sigma], (0) \}$.
(2) If $o(\sigma) = n < \infty$, then $Z(F[X, \sigma]) = F_0[X^n]$ and $Spec(F[X, \sigma]) = \{XF[X, \sigma], (0)\} \cup \{p(X^n)F[X, \sigma] \mid F_0[X] \ni p(X)$ and $p(X)$ is irreducible over F $\}$.

Let P be a prime ideal of a ring R. If $C(P) = \{c \in R \mid c$ is regular mod P $\}$ is a regular Ore set of R, then we say that P is *localizable* and denote by R_P the localization of R at P.

Set $P_X = XF[X, \sigma], P_{X^{-1}} = X^{-1}F[X^{-1}, \sigma]$. Moreover in the case $o(\sigma) = n < \infty$, set $\mathcal{P} = \{P = p(X^n)F[X, \sigma] \mid P \in Spec(F[X, \sigma])$ different from $XF[X, \sigma]$ and $F[X, \sigma]/P$ is a division ring $\}$. Since $F[X, \sigma]$ is a principal ideal ring, any prime ideal of $F[X, \sigma]$ is localizable. We denote by $R_X = F[X, \sigma]_{P_X}, R_{X^{-1}} = F[X^{-1}, \sigma]_{P_{X^{-1}}}$ and $R_P = F[X, \sigma]_P, P \in \mathcal{P}$. Under those notation, using Proposition 1, we have the following

THEOREM 2. Let F be a field with $\sigma \in Aut(F)$.
(1) If $o(\sigma) = \infty$, then R_X and $R_{X^{-1}}$ are only total valuation rings of $F(X, \sigma)$ containing F and $L = R_X \cap R_{X^{-1}}$ is a principal and duo ring. Here by duo rings we mean any one-sided ideal is two-sided.
(2) If $o(\sigma) = n < \infty$, then $\{R_X, R_{X^{-1}}, R_P \mid P \in \mathcal{P}\}$ are only total valuation rings of $F(X, \sigma)$ containing F and $L = R_X \cap R_{X^{-1}} \cap \cap R_P (P \in \mathcal{P})$ is a principal and duo ring.

Let us study the nontrivial case. It is very difficult to find out all no-commutative valuation rings lying over V. In this note, we only discuss by using Morandi's and Krull's methods.

Let S be a Dubrovin valuation ring of a simple Artinian ring Q and let $\varphi : S \longrightarrow \bar{S} = S/J(S)$ be the natural homomorphism. Suppose that $\mathcal{R}$ is an order in the simple Artinian ring $\bar{S}$. Set $R_{(1)} = \varphi^{-1}(\mathcal{R})$, the complete converse image of $\mathcal{R}$ by φ. In [M_0], he proved that $R_{(1)}$ is Prüfer if and only if $\mathcal{R}$ is Prüfer. We will apply this to the following case ; let $\sigma \in Aut(Q)$ and let R be an order in Q. Since $Q[X,\sigma]$ is a principal ideal ring, $P_X = XQ[X,\sigma]$ is localizable and $T = Q[X,\sigma]_{P_X}$ is a Noehterian Dubrovin valuation ring. Let $\varphi : T \longrightarrow \bar{T} = T/J(T) \cong Q$ and set $R_{(1)} = \varphi^{-1}(R) = R + J(T) = R + XT$ which is lying over R.

THEOREM 3. (1) R is a Dubrovin valuation ring if and only if so is $R_{(1)}$. In this case, $Spec(R_{(1)}) = \{pP_{(1)}, XT \mid p \in spec(R)\}$.
(2) Suppose that Q is a division ring.
(a) R is total if and only if so is $R_{(1)}$.
(b) R is invariant if and only if so is $R_{(1)}$

In $[BS_2]$, they obtained (2)(a) in Theorem 3 by using some properties of skew formal power series rings. Next we shall introduce Krull's method. For simplicity, let V be a valuation ring of a field F with $\sigma \in Aut(V)$. Then $J(V)[X,\sigma]$ is localizable and $R^{(1)} = V[X,\sigma]_{J(V)[X,\sigma]}$ is a total valuation ring of $F(X,\sigma)$ with $R^{(1)} \cap F = V$ (see $[BS_1]$). σ naturally induces an automorphism $\bar{\sigma}$ of $\bar{V} = V/J(V)$ so that we can consider the skew polynomial ring $\bar{V}[X,\bar{\sigma}]$ and its quotient division ring $\bar{V}(X,\bar{\sigma})$. Let $\psi : R^{(1)} \longrightarrow \bar{V}(X,\bar{\sigma})$ be the natural homomorphism given by $\psi(f(X)c(X)^{-1}) = \bar{f(X)}\bar{c(X)}^{-1}$, where $f(X), c(X) \in V[X,\sigma], c(X) \notin J(V)[X,\sigma]$ and $\bar{f(X)} = \bar{a}_0 + \bar{a}_1 X + ... + \bar{a}_n X^n$ for $f(X) = a_0 + a_1 X + ... + a_n X^n$. As in Theorem 2, let $\mathcal{P} = \{\bar{P} = p(\bar{X}^n)\bar{V}[X,\bar{\sigma}] \in Spec(\bar{V}[X,\bar{\sigma}]) \mid \bar{V}[X,\bar{\sigma}]/\bar{P}$ is a division ring $\}$. Set $S = \psi^{-1}(\bar{V}[X,\bar{\sigma}]) = V[X,\sigma] + J(V)R^{(1)}$, Prüfer by Morandi, $P_X = \psi^{-1}(X\bar{V}[X,\bar{\sigma}]) = XV[X,\sigma] + J(V)R^{(1)}$ and $P = \psi^{-1}(\bar{P}) = p(X^n)V[X,\sigma] + J(V)R^{(1)}$. Then it is not difficult to prove that P_X and P are localizable and we have the following

THEOREM 4. Let V be a valuation ring of a field F with $\sigma \in Aut(V)$. Then $R^{(1)}, S_X = S_{P_X}$ and S_P are total valuation rings lying over V. Moreover $R^{(1)} \supset S_X$ and $R^{(1)} \supset S_P$.

Let R be a Dubrovin valuation ring of a simple Artinian ring Q. In [W], Wadsworth defined the *value group* Γ_R as follows ; $st(R) = \{q \in Q \mid qR = Rq\}$, the *stabilizer* of R under the action of $U(Q)$ and $\Gamma_R = st(R)/U(R)$. Γ_R is a totally ordered group in the following definitions ; $aU(R) \cdot bU(R) = abU(R)$ and $aU(R) \geq bU(R)$ iff $aR \subseteq bR$ for any $a, b \in st(R)$. Now coming back to $R_{(1)}$, we classify the given automorphism σ of a field F based on the valuation ring V in order to study $st(R_{(1)})$ and $\Gamma_{R_{(1)}}$.

The following lemma is elementary but useful.

LEMMA 5. Assume that Q is a division ring and R is a total valuation ring. Then
(1) $st(R) = st(R_{(1)}) \cap Q$. In particular, R is invariant iff $st(R_{(1)}) \supseteq U(Q)$.
(2) Let $f(X) = f_0 + f_1 + ... + f_n X^n \in Q[X,\sigma]$ with $f_0 \neq 0$. Then $f(X)R_{(1)} = f_0 R_{(1)}$

By using Lemma 5 and elementary calculations of polynomials, we have the following

THEOREM 6. Let F be a field with $\sigma \in Aut(F)$ and let V be a valuation ring of F.

Then
(1) If $\sigma(V) = V$, i.e., $\sigma \in Aut(V)$, then $st(R_{(1)}) = U(F(X,\sigma))$ and $\Gamma_{R_{(1)}} = \{qX^n U(R_{(1)}) \mid 0 \neq q \in F, n \in Z\} = \Gamma_V(\bar{X})$ with $\bar{X}\bar{q} = \bar{\sigma(q)}\bar{X}$, where $\bar{X} = XU(R_{(1)})$ and $\bar{q} = qU(R_{(1)})$.
(2) If $\sigma^n(V)$ is not contained in V for any $n \in \mathrm{N}$, then $st(R_{(1)}) = \{q, f(X)c(X)^{-1} \mid 0 \neq q \in F, f(X), c(X) \in F[X,\sigma]$ with $f(0) \neq 0$ and $c(0) \neq 0\}$ and $\Gamma_{R_{(1)}} = \Gamma_V$.
(3) If $\sigma(V) \subset V$, then $st(R_{(1)}) = \{q, f(X)c(X)^{-1} \mid 0 \neq q \in F, f(X), c(X) \in F[X,\sigma]$ with $0 \neq f(0), 0 \neq c(0)\}$ and $\Gamma_{R(1)} = \Gamma_V$.
(4) If $\sigma^j(V)$ is not contained in V for any j, $1 \leq j \leq n-1$ and $\sigma^n(V) = V$ for some $n \in \mathrm{N}$, then $st(R_{(1)}) = \{q, X^{nl}f(X)c(X)^{-1} \mid 0 \neq q \in F, f(X), c(X) \in F[X,\sigma]$ with $0 \neq f(0), 0 \neq c(0)$ and $l \in \mathrm{Z}\}$ and $\Gamma_{R_{(1)}} = \Gamma_V(\bar{X^n})$ with $\bar{X^n}\bar{q} = \bar{\sigma^n(q)}\bar{X^n}$, where $\bar{X^n} = X^n U(R_{(1)})$ and $\bar{q} = qU(R_{(1)})$.
(5) If $\sigma^j(V)$ is not contained in V for any j, $1 \leq j \leq n-1$ and $\sigma^n(V) \subset V$ for some $n \in \mathrm{N}$, then $st(R_{(1)}) = \{q, f(X)c(X)^{-1} \mid 0 \neq q \in Q, f(X), c(X) \in F[X,\sigma]$ with $0 \neq f(0), 0 \neq c(0)\}$ and $\Gamma_{R_{(1)}} = \Gamma_V$.

Since $Xa = \sigma(a)X$ for any $a \in F$, we see that σ is extended to the automorphism of $F(X,\sigma)$ which is obtained by the conjugation of X. We denote it the same symbol σ. The following is easy to prove.

LEMMA 7. The following are equivalent :
(1) $\sigma(R_{(1)}) = R_{(1)}$.
(2) $X \in st(R_{(1)})$.
(3) $\sigma(V) = V$.

From Theorem 6 and Lemma 7, we have the following remark.

REMARK. Let V be a valuation ring of a field F with $\sigma \in Aut(F)$, then
(1) $R_{(1)}$ is invariant iff $\sigma \in Aut(V)$.
(2) $R_{(1)}$ is not invariant iff σ is one of (2) $\sim$ (5) in Theorem 6.
(3) If $\sigma \in Aut(V)$, then $\Gamma_{R_{(1)}}$ is Abelian iff $aV = \sigma(a)V$ for any $a \in F$.

Let R be a Dubrovin valuation ring of a simple Artinian ring Q with finite dimension over its center. It has been shown that Γ_R is Abelian by Dubrovin $[D_2]$. In the case Q is of infinite dimension over its center, Remark(3) shows that there exist invariant valuation rings R and S such that Γ_R is Abelian and Γ_S is non-Abelian.

REMARK. We can construct valued groups (F, V) with $\sigma \in Aut(F)$ satisfying (1) $\sim$ (5) in Theorem 6 and also Γ_R is Abelian, Γ_S is non Abelian, where R and S are valuation rings of $F(X,\sigma)$.

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Cohen-Macaulay isolated singularities with a dualizing module

Kenji Nishida

This is a summary of my paper [7]. Let $(R, \mathfrak{m})$ be a commutative Gorenstein complete local ring of $\dim R = d$. Let Λ be a module finite R-algebra and a Cohen-Macaulay isolated singularity [5], i.e., Λ satisfies $\mathrm{gl}\dim \Lambda_p = \dim \Lambda_p$ for all $p \in \mathrm{Supp}_R\Lambda - \{\mathfrak{m}\}$ and is a Cohen-Macaulay R-module of $\dim_R \Lambda = d$.

We set $\mathrm{latt}\,\Lambda$ to be a category of all left Λ-lattices. By definition, $M \in \mathrm{latt}\,\Lambda$ if and only if $M \in \mathrm{mod}\,\Lambda$ and M is a Cohen-Macaulay R-module of $\dim_R M = d$. We denote the duality between $\mathrm{latt}\,\Lambda$ and $\mathrm{latt}\,\Lambda^{op}$ by D; $DM = \mathrm{Hom}_R(M, R)$ for $M \in \mathrm{latt}\,\Lambda$ or $\mathrm{latt}\,\Lambda^{op}$.

We define a dualizing module. A Λ-Λ-bimodule $\omega \in \mathrm{latt}\,\Lambda$ is called a *dualizing module* if it satisfies the following (d0)-(d3):

(d0) $\Lambda \cong (\mathrm{End}_\Lambda \omega)^{op}$, $\Lambda \cong \mathrm{End}_{\Lambda^{op}} \omega$ naturally,

(d1) $\mathrm{id}_\Lambda \omega = \mathrm{id}_{\Lambda^{op}} \omega = e$ for some integer $e < \infty$,

(d2) $\mathrm{Ext}^i_\Lambda(\omega, \omega) = \mathrm{Ext}^i_{\Lambda^{op}}(\omega, \omega) = 0$ for $i > 0$,

(d3) $\mathrm{Ext}^d_\Lambda(S, D(\omega_\Lambda)) \neq 0$ for each simple Λ-module S and $\mathrm{Ext}^d_{\Lambda^{op}}(S', D({}_\Lambda\omega)) \neq 0$ for each simple Λ^{op}-module S'.

Following Foxby [4], [3] we define the two subcategories $\mathcal{A}$ and $\mathcal{B}$ in mod Λ as follows. A Λ-module M is in $\mathcal{A}$ if and only if

A1) $\mathrm{Tor}^\Lambda_i(\omega, M) = 0$ for $i > 0$,

A2) $\mathrm{Ext}^i_\Lambda(\omega, \omega \otimes_\Lambda M) = 0$ for $i > 0$,

A3) the canonical homomorphism $\varphi_M : M \to \mathrm{Hom}_\Lambda(\omega, \omega \otimes_\Lambda M)$, defined by $\varphi_M(m)(x) = x \otimes m (m \in M,\ x \in \omega)$, is an isomorphism.

A Λ-module N is in $\mathcal{B}$ if and only if

B1) $\mathrm{Ext}^i_\Lambda(\omega, N) = 0$ for $i > 0$,

B2) $\mathrm{Tor}^\Lambda_i(\omega, \mathrm{Hom}_\Lambda(\omega, N)) = 0$ for $i > 0$,

B3) the canonical homomorphism $\psi_N : \omega \otimes_\Lambda \mathrm{Hom}_\Lambda(\omega, N) \to N$, defined by $\psi_N(x \otimes f) = f(x)$ $(x \in \omega,\ f \in \mathrm{Hom}_\Lambda(\omega, N))$, is an isomorphism.

1. Lemma. Let M be a nonzero finitely generated Λ-module. Then $\omega \otimes_\Lambda M \neq 0$ and $\mathrm{Hom}_\Lambda(\omega, M) \neq 0$.

Proof. Let $P \to \omega \to 0$ (respectively, $P' \to \omega \to 0$) be a projective cover of the Λ(respectively, Λ^{op})-module ω. Then every simple Λ-module S (respectively, simple Λ^{op}-module S') is in P/JP (respectively, $P'/P'J$). Hence the same facts hold for $\omega/J\omega$ and $\omega/\omega J$.

Let $\dim R > 0$. Let $x \in \mathfrak{m}$ be M-regular. Applying $\omega \otimes_\Lambda -$ to an exact sequence $0 \to M \xrightarrow{x} M \to M/xM \to 0$, we get an exact sequence

$$\omega \otimes_\Lambda M \xrightarrow{x} \omega \otimes_\Lambda M \to \omega \otimes_\Lambda M/xM \to 0.$$

Thus, in order to prove $\omega \otimes_\Lambda M \neq 0$, it suffices to show that $\omega \otimes_\Lambda M/xM \neq 0$. Hence we can assume that $\mathrm{depth}_R M = 0$. Thus $\mathrm{depth}_R \mathrm{Hom}_R(M, M) = 0$, and then $\mathrm{soc}_{\Lambda^{op}} \mathrm{Hom}_R(M, M) \neq 0$. Therefore, $\mathrm{Hom}_R(\omega \otimes_\Lambda M, M) \cong \mathrm{Hom}_{\Lambda^{op}}(\omega, \mathrm{Hom}_R(M, M)) \neq 0$ by the first paragraph. Hence we have $\omega \otimes_\Lambda M \neq 0$.

This note is not in final version. A detailed version is submitted for publication elsewhere.

Next, we show that $\mathrm{Hom}_\Lambda(\omega, M) \neq 0$ holds. Suppose that $\mathrm{Ass}_R M = \{\mathfrak{m}\}$. Then $\mathrm{depth}_R M = 0$, so $\mathrm{Hom}_\Lambda(\omega, M) \neq 0$. We assume that there exists $p \in \mathrm{Ass}_R M$ such that $p \neq \mathfrak{m}$. Since Λ is an isolated singularity, ω_p is a projective Λ_p-module and Λ_p^{op}-module. Since $\Lambda_p = \mathrm{End}_{\Lambda_p^{op}} \omega_p$, ω_p is a generator over Λ_p. Hence $\mathrm{Hom}_{\Lambda_p}(\omega_p, M_p) \neq 0$, so that $\mathrm{Hom}_\Lambda(\omega, M) \neq 0$.

The proof of the following 2 and 3 are almost the same as those in [4] using the above Lemma.

2. Theorem. (cf. [4], Proposition 1.4) *Let $M \in \mathrm{mod}\,\Lambda$. Then the following hold.*
(1) $M \in \mathcal{A}$ if and only if $\omega \otimes_\Lambda M \in \mathcal{B}$.
(2) $M \in \mathcal{B}$ if and only if $\mathrm{Hom}_\Lambda(\omega, M) \in \mathcal{A}$.

3. Lemma. (cf. [4], Lemma 1.3) *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be exact in mod Λ. If M_i and M_j $(1 \leq i \neq j \leq 3)$ are in $\mathcal{A}$ (respectively, $\mathcal{B}$), then the rest is also in $\mathcal{A}$ (respectively, $\mathcal{B}$).*

Let $M^\dagger = \mathrm{Hom}_\Lambda(M, \Lambda)$ or $\mathrm{Hom}_{\Lambda^{op}}(M, \Lambda)$ for $M \in \mathrm{mod}\,\Lambda$ or $\mathrm{mod}\,\Lambda^{op}$. Recall that a Λ-module M has *Gorenstein dimension* zero, denoted by $\text{G-dim}_\Lambda M = 0$, if $M^{\dagger\dagger} \cong M$ and $\mathrm{Ext}^i_\Lambda(M, \Lambda) = \mathrm{Ext}^i_{\Lambda^{op}}(M^\dagger, \Lambda) = 0$ for $i > 0$, and has Gorenstein dimension less than or equal to $k < \infty$, denoted by $\text{G-dim}_\Lambda M \leq k$, if there exists an exact sequence $0 \to G_n \to \cdots \to G_0 \to M \to 0$ with $\text{G-dim}_\Lambda G_i = 0$ for $0 \leq i \leq n$. We put $\mathcal{G}_0 := \{M \in \mathrm{mod}\,\Lambda : \text{G-dim}_\Lambda M = 0\}$ and $\mathcal{G} := \{M \in \mathrm{mod}\,\Lambda : \text{G-dim}_\Lambda M < \infty\}$.

4. Theorem. (cf. [4], Proposition 2.5) *It follows that $\mathcal{G}_0 \subset \mathcal{A}$.*

We need a lemma to prove Theorem 4.

Lemma. *Let $M \in \mathrm{mod}\,\Lambda$ and assume that $\mathrm{Tor}_i^\Lambda(\omega, M) = 0$ for $i > 0$. Then $\mathrm{Ext}^i_\Lambda(M, \Lambda) = 0$ for $i > 0$ if and only if $\mathrm{Ext}^i_\Lambda(\omega \otimes_\Lambda M, \omega) = 0$ for $i > 0$.*

Proof. Let $\cdots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$ be a projective resolution of M. Then $\cdots \to \omega \otimes_\Lambda P_1 \to \omega \otimes_\Lambda P_0 \to \omega \otimes_\Lambda M \to 0$ is exact by assumption. Thus we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Hom}_\Lambda(\omega \otimes_\Lambda M, \omega) & \to & \mathrm{Hom}_\Lambda(\omega \otimes_\Lambda P_0, \omega) & \to & \mathrm{Hom}_\Lambda(\omega \otimes_\Lambda P_1, \omega) & \to & \cdots \\ & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \\ 0 & \to & \mathrm{Hom}_\Lambda(M, \Lambda) & \xrightarrow{\tilde{f}_0} & \mathrm{Hom}_\Lambda(P_0, \Lambda) & \xrightarrow{\tilde{f}_1} & \mathrm{Hom}_\Lambda(P_1, \Lambda) & \to & \cdots, \end{array}$$

where $\tilde{f}_i = \mathrm{Hom}(f_i, id)$ $(i = 1, 2)$ and $\mathrm{Hom}_\Lambda(\omega \otimes_\Lambda M, \omega) \cong \mathrm{Hom}_\Lambda(M, \mathrm{Hom}_{\Lambda^{op}}(\omega, \omega)) \cong \mathrm{Hom}_\Lambda(M, \Lambda)$ and so on. Since $\mathrm{Ext}^i_\Lambda(\omega \otimes_\Lambda P_j, \omega) = 0$ for $i > 0$, $j \geq 0$, we have that the first row is exact if and only if $\mathrm{Ext}^i_\Lambda(\omega \otimes_\Lambda M, \omega) = 0$ for $i > 0$. Hence the assertion of Lemma directly follows.

The proof of Theorem 4. We put $L^* = \mathrm{Hom}_\Lambda(L, \omega)$ for $L \in \mathrm{mod}\,\Lambda$, similarly for $L \in \mathrm{mod}\,\Lambda^{op}$. Let $M \in \mathcal{G}_0$. Let I be an injective Λ-module. Then $\mathrm{Tor}_i^\Lambda(I, M) \cong \mathrm{Hom}_\Lambda(\mathrm{Ext}^i_\Lambda(M, \Lambda), I) = 0$ for $i > 0$. Since ω has finite injective dimension, this implies that $\mathrm{Tor}_i^\Lambda(\omega, M) = 0$ for $i > 0$. By Lemma, $\mathrm{Ext}^i_\Lambda(\omega \otimes_\Lambda M, \omega) = 0$ for $i > 0$, so $\omega \otimes_\Lambda M \in \mathcal{C}(\Lambda)$. Thus $\omega \otimes_\Lambda M = (\omega \otimes_\Lambda M)^{**}$. The right hand side is isomorphic to $\mathrm{Hom}_{\Lambda^{op}}(\mathrm{Hom}_\Lambda(M, \mathrm{Hom}_{\Lambda^{op}}(\omega, \omega)), \omega) = M^{\dagger *}$, so that $\omega \otimes_\Lambda M \cong M^{\dagger *}$.

We prove A2), that is, $\mathrm{Ext}^i_\Lambda(\omega, \omega \otimes_\Lambda M) = 0$ for $i > 0$. By the above argument, it suffices to show that $\mathrm{Ext}_\Lambda(\omega, M^{\dagger *}) = 0$. Let $0 \to N \to P \to M^\dagger \to 0$ be a projective cover of $M^\dagger$ in $\mathrm{mod}\,\Lambda^{op}$. Since $\text{G-dim}_{\Lambda^{op}} M^\dagger = 0$, we have $\text{G-dim}_{\Lambda^{op}} N = 0$. Thus $P^* \cong \omega \otimes_\Lambda P^\dagger$

and $N^* \cong \omega \otimes_\Lambda N^\dagger$. Since G-dim$_\Lambda N^\dagger = 0$, we have $\mathrm{Tor}_i^\Lambda(\omega, N^\dagger) = 0$. Therefore, an exact sequence $0 \to M \to P^\dagger \to N^\dagger \to 0$ provides an exact sequence

$$(1)\ 0 \to M^{\dagger *} \to P^* \to N^* \to 0.$$

From (1), we get a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Hom}_\Lambda(\omega, M^{\dagger *}) & \to & \mathrm{Hom}_\Lambda(\omega, P^*) & \to & \mathrm{Hom}_\Lambda(\omega, N^*) & \to & \mathrm{Ext}^1_\Lambda(\omega, M^{\dagger *}) & \to & 0 \\ & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow & & & & \\ 0 & \to & M & \longrightarrow & P^\dagger & \longrightarrow & N^\dagger & \to & 0 & & \end{array}$$

where $\mathrm{Hom}_\Lambda(\omega, M^{\dagger *}) \cong \mathrm{Hom}_{\Lambda^{op}}(M^\dagger, \mathrm{Hom}_\Lambda(\omega,\omega)) \cong M^{\dagger\dagger} \cong M$ and so on. Therefore, $\mathrm{Ext}^1_\Lambda(\omega, M^{\dagger *}) = 0$. Since G-dim$_\Lambda N^\dagger = 0$, we have $\mathrm{Ext}^1_\Lambda(\omega, N^*) = 0$, so that $\mathrm{Ext}^2_\Lambda(\omega, M^{\dagger *}) = \mathrm{Ext}^1_\Lambda(\omega, N^*) = 0$. The long exact sequence obtained from (1) by applying $\mathrm{Hom}_\Lambda(\omega, -)$ enables us to proceed this argument. Thus we get $\mathrm{Ext}^i_\Lambda(\omega, M^{\dagger *}) = 0$ for $i > 0$.

To complete the proof, we must show that φ_M is an isomorphism. Let $\cdots \to P_1 \to P_0 \to M \to 0$ be a projective resolution of M and $N_i := \Omega^i M$ $(i = 1, 2, \dots)$ an i-th sygygy of M. Then we have an exact sequence $\cdots \to \omega \otimes_\Lambda P_1 \to \omega \otimes_\Lambda P_0 \to \omega \otimes_\Lambda M \to 0$. Since G-dim$_\Lambda N_i = 0$ for $i > 0$, we see that $\mathrm{Ext}^i_\Lambda(\omega, \omega \otimes_\Lambda N_j) = 0$ for $i > 0, j > 0$ by the previous paragraph. Thus we get a commutative diagram with exact rows

$$\begin{array}{ccccccc} \mathrm{Hom}_\Lambda(\omega, \omega \otimes_\Lambda P_1) & \to & \mathrm{Hom}_\Lambda(\omega, \omega \otimes_\Lambda P_0) & \to & \mathrm{Hom}_\Lambda(\omega, \omega \otimes_\Lambda M) & \to & 0 \\ \varphi_{P_1}\uparrow & & \varphi_{P_0}\uparrow & & \varphi_M\uparrow & & \\ P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0. \end{array}$$

Since φ_{P_i} $(i = 0, 1)$ are isomorphisms, φ_M is also an isomorphism. Hence $M \in \mathcal{A}$.

5. Corollary. *We have $\mathcal{G} \subset \mathcal{A}$.*

Proof. This follows directly from Lemma 3 and Theorem 4.

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On Nakayama rings

KAZUAKI NONOMURA

Let R be an artinian ring. Then we call R a right Nakayama ring (or a right serial ring) if every indecomposable projective right R-module is uniserial and we call R a right co-Nakayama ring (or a right co-serial ring) if every indecomposable injective right R-module is uniserial. Moreover we call R a right QF-3 ring if the injective hull of R_R is projective or equivalently if the projective cover of every injective right R-module is injective.

K. R. Fuller [4] has shown that a ring R is a right Nakayama and right co-Nakayama ring if and only if it is a two-sided Nakayama ring. K. Oshiro [9] has shown that a ring R is a right Nakayama QF-3 ring if and only if it is a two-sided Nakayama ring. We shall show that a right co-Nakayama QF-3 ring is a two-sided Nakayama ring.

We call an artinian ring R a right co-Harada ring if for any essential extension of any indecomposable projective right R-module is indecomposable projective (see [7], [8] and [9] for co-Harada rings). We shall also give a necessary and sufficient condition for a right co-Harada ring to be a two-sided Nakayama ring.

Throughout this paper, we assume that R stands for a semiprimary ring with identity unless otherwise stated and all R-modules are unitary. M_R (resp. ${}_RM$) is used to denote that M is a right (resp. left) R-module. Let M be an R-module. We use $E(M)$, $J(M)$ and $S(M)$ to denote its injective hull, Jacobson radical and socle, respectively. By J we denote the Jacobson radical of R. $L \leq M$ (resp. $L < M$) means L is a submodule of M (resp. $L \leq M$ and $L \neq M$). We denote the set of primitive idempotents of R by $\mathrm{Pi}(R)$ and the composition length of M by $|M|$.

We call R a right QF-3 ring if $E(R_R)$ is projective and a ring is called a QF-3 ring if it is both left and right QF-3. It is well-known that a one-sided artinian ring which is either left QF-3 or right QF-3 is QF-3.

The following result is used in this paper. It was proved for right artinian rings by Fuller [4].

Theorem (Colby and Rutter [3, Theorem 1.3]). For a right perfect ring R, the following conditions are equivalent;

(1) An injective hull of every projective right R-module is projective;

(2) A projective cover of every injective right R-module is injective.

The detailed version of this paper will be submitted for publication elsewhere.

Moreover in this case, R is a QF-3 and semiprimary ring.

We call a module M local if $J(M)$ is a small maximal submodule of M, and M colocal if $S(M)$ is an essential simple submodule of M.

In this paper, we consider the following condition;

(∗) Let M_1 and M_2 be right R-modules. For any simple sub-factor modules N_i/K_i of M_i with $N_1/K_1 \cong N_2/K_2$, where $M_i \geq N_i > K_i$ $(i = 1, 2)$, any isomorphism $\theta : N_1/K_1 \to N_2/K_2$ can be lifted to either a morphism $\varphi : M_1 \to M_2$ or a morphism $\varphi' : M_2 \to M_1$.

Lemma 1. *Let M be a finite direct sum $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ of right R-modules $M_1, M_2, \cdots, M_n$. Assume that M_i and M_j satisfy the condition (∗) for any $i \neq j \in \{1, \cdots, n\}$. Then for any local submodule L of M, there exists a decomposition $M = M_1' \oplus M_2' \oplus \cdots \oplus M_n'$ of M such that $M_i' \cong M_i$ for each $i (1 \leq i \leq n)$, and $L \leq M_k'$ for some k $(1 \leq k \leq n)$.*

An R-module M is called uniserial if any two submodules of M are comparable. Since R is a semiprimary ring, every uniserial R-module has finite length. A ring R is a right Nakayama ring (resp. a right co-Nakayama ring) if every indecomposable projective (resp. injective) right R-module is uniserial and R is called a Nakayama ring (resp. a co-Nakayama ring) if it is both left and right Nakayama (resp. co-Nakayama).

Lemma 2. *Let R be a right* QF-3 *and right co-Nakayama ring. Then any uniserial injective right R-modules M_1 and M_2 satisfy the condition (∗).*

Theorem 3. *Let R be a right* QF-3 *and right co-Nakayama ring. Then R is a Nakayama ring.*

Proof. Let eR be a local projective module with $e \in \mathrm{Pi}(R)$. Then since $E(eR)$ is projective by assumption, $E(eR)$ is finitely generated by (see e.g. [10, Lemma 6.1]) and so eR is a submodule of a direct sum $E(eR) = E_1 \oplus E_2 \oplus \cdots \oplus E_n$ of some injective uniserial modules E_i $(i = 1, 2, \cdots, n)$, so by Lemmas 1 and 2, eR is a uniserial module. Hence R is a right Nakayama ring. Therefore R is a Nakayama ring by [4, Theorem 5.4]. □

Remark. Let $u \in fR$ and $v \in gR$ for $f, g \in \mathrm{Pi}(R)$. If $uR \to vR$ $(uc \mapsto vc; c \in R)$ can be extended to either a left multiplication map $fR \to gR$ $(u \mapsto au = v;\ a \in R)$ or $gR \to fR$ $(v \mapsto bv = u;\ b \in R)$, then $Ru \leq Rv$ or $Ru \geq Rv$.

A submodule N of M is called a waist in M if either $N \leq X$ or $N \geq X$ is satisfied for any submodule X of M. As is easily seen, if R is right Nakayama QF-3, then gR is a waist in $E(gR)$ for any $g \in \mathrm{Pi}(R)$. For an element x of R and a right ideal K of R, $r_K(x)$ denotes the annihilator right ideal: $r_K(x) = \{a \in K \mid xa = 0\}$. Note that for each $e \in \mathrm{Pi}(R)$ and $I \leq R_R$, we have $r_{eR}(\ell_R(eI)) = r_R(\ell_R(eI))$ since $\ell_R(eI) + R(1-e) = \ell_R(eI)$.

Lemma 4. *Let R be a ring with e, f and g in* Pi(R) *and $u = fue \in fRe$, $v = gve \cdot\in gRe$ and assume that fR_R is a colocal module and gR_R is a waist in $E(gR_R)$. If $r_{eR}(u) \leq r_{eR}(v)$ holds, then either $Ru \leq Rv$ or $Ru \geq Rv$ holds.*

In order tha we characterize Nakayama rings anew, we consider the following condition;

($\sharp$) For any elements u and v of R with $u = fue$ and $v = gve$, where e, f and $g \in \mathrm{Pi}(R)$, either $r_{eR}(u) \leq r_{eR}(v)$ or $r_{eR}(u) \geq r_{eR}(v)$ holds.

Let $e \in \mathrm{Pi}(R)$. Since R is semiprimary, in case ${}_RRe$ is not uniserial, we have some elements $u = fue \in Je$ - J^2e and $v = gve \in Je$ - J^2e for $f, g \in \mathrm{Pi}(R)$ such that $Je/J^2e \geq R\bar{u} \oplus R\bar{v}$, where $\bar{u} = u + J^2e$ and $\bar{v} = v + J^2e$. Then by Lemma 4 and Remark, we have the following theorem.

Theorem 5. *If R satisfies ($\sharp$) and gR_R is a waist in $E(gR_R)$ for each $g \in$ Pi(R), then R is Nakayama.*

The ring $\mathbb{Z}$ of integers satisfies ($\sharp$) but is not Nakayama. In the following, we give another example.

Example. Let $\mathbb{M}_3(\mathrm{K})$ be the matrix algebra over a field K and R be the following subalgebra of $\mathbb{M}_3(\mathrm{K})$;

$$R = \left\{ \begin{bmatrix} a & 0 & 0 \\ x & b & 0 \\ y & z & b \end{bmatrix} \mid a,\ b,\ x,\ y,\ z \in \mathrm{K} \right\}$$

Then R is a left Nakayama ring but is not a right Nakayama ring. On the other hand, R satisfies ($\sharp$). We denote (i, j)-matrix units of $\mathbb{M}_3(\mathrm{K})$ by e_{ij} and put $e = e_{11}$ and $f = e_{22} + e_{33}$. Then eR is a simple module. Put $fI = \mathrm{K}e_{31} + \mathrm{K}e_{32}$. Then as is easily checked, $fR > fI > 0$ are the annihilator submodules of fR.

Let R be a right Nakayama ring and assume that R is either a right co-Nakayama or a right QF-3 ring. Then for each $e \in \mathrm{Pi}(R)$, $E(eR)$ is uniserial since eR is colocal. Hence by Theorem 5, we have the following corollary, which was obtained by Fuller [4, Theorem 5.4] and Oshiro [9, Theorem 6.1]).

Corollary 6. *Let R be a right Nakayama ring. If R is either a right co-Nakayama or right* QF-3 *ring, then R is a Nakayama ring.*

Summarizing results on characterizations of Nakayama rings, we have;

Theorem 7. *The following conditions are equivalent;*

(1) *R is Nakayama.*

(2) *R is right Nakayama and right co-Nakayama.*

(3) *R is* QF-3 *and right Nakayama.*

(4) *R is* QF-3 *and right co-Nakayama.*

Proof. (1) $\Leftrightarrow$ (2) and (1) $\Rightarrow$ (3) and (4). By Fuller [4, Theorem 5.4], [1, Theorem 32.2] and Harada [6, Theorem 6] (or Corollary 6).

(3) $\Rightarrow$ (1). By Oshiro [6, Theorem 6.1] (or Corollary 6).

(4) $\Rightarrow$ (2). By Theorem 3. □

We call R a right QF-2 ring if eR is colocal for any $e \in \mathrm{Pi}(R)$. A ring is called QF-2 if it is left and right QF-2. Dually, we call R a right QF-2* ring if $E(eR/eJ)$ is local for any $e \in \mathrm{Pi}(R)$. A ring is called QF-2* if it is left and right QF-2*. Any right Nakayama (resp. right co-Nakayama) ring is clearly right QF-2 (resp. right QF-2*). In Theorem 7, we can replace conditions "QF-3, right co-Nakayama or right Nakayama" by conditions "left QF-2, QF-2* or right QF-2" as follows;

Proposition 8. *The following hold.*

(1) *If R is a right Nakayama left* QF-2 *ring, then R is a Nakayama ring.*

(2) *If R is either a right Nakayama right* QF-2* *ring or a right co-Nakayama right* QF-2 *ring, then R is a Nakayama ring.*

Proof. (1) Since R is a two-sided QF-2 ring, R is QF-3 (e.g. see [1, Theorem 31.7]) and by Theorem 7, R is a Nakayama ring.

(2) If R is a right Nakayama and right QF-2* ring, then any indecomposable injective right R-module E is local, so E is uniserial. Hence R is right co-Nakayama and by Theorem 7, R is a Nakayama ring. The other assertion is shown by the dual argument. □

We call an artinian ring R a right co-Harada ring if for any essential extension of any indecomposable projective right R-module is indecomposable projective. As is easily seen, if R is a right co-Harada ring, eR is a waist in $E(eR)$ for each $e \in \mathrm{Pi}(R)$. Hence by Theorem 5, we have;

Theorem 9. *R is a right co-Harada ring satisfying the condition (♯) if and only if R is a Nakayama ring.*

We denote the right global dimension of R by $\mathrm{r.gl.dim}\, R$. In the following, we give another proof to the result [2, Theorem 7] obtained by Baba.

Corollary 10 ([2, Theorem 7]). *Let R be a right co-Harada ring with $\mathrm{r.gl.dim}\, R \leq 2$. Then R is a Nakayama ring.*

Proof. Let $u = fue$ be an element of R, where $e, f \in \mathrm{Pi}(R)$. Then for a left multiplication $\hat{u} : eR \to fR$ $(ec \to uc; c \in R)$ by u, $r_{eR}(u) = \mathrm{Ker}\, \hat{u}$ is projective. Since R is right co-Harada, $r_{eR}(u)$ is a waist in $E(r_{eR}(u)_R)$ $(= E(eR_R))$, which shows the assertion by Theorem 9. □

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On the construction of stable equivalence functor not of Morita type

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Throughout this paper, let K be a field and an algebra means a finite dimensional selfinjective K-algebra with an identity. Moreover, we assume that an algebra has no semisimple algebra summand.

We shall study the condition to be a triangle functor for a stable equivalence functor. Let $\mathcal{C}$ and $\mathcal{D}$ be triangulated categories. We say that a functor $F : \mathcal{C} \to \mathcal{D}$ is equivalent if F is an equivalence functor as additive categories, and a functor $F : \mathcal{C} \to \mathcal{D}$ is triangle equivalent if F is an equivalence functor as triangulated categories, that is, $FT_{\mathcal{C}} \simeq T_{\mathcal{D}}F$ and $FX \xrightarrow{Fu} FY \xrightarrow{Fv} FZ \xrightarrow{Fw} T_{\mathcal{D}}FX$ is a triangle in $\mathcal{D}$ whenever $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T_{\mathcal{C}}X$ is a triangle in $\mathcal{C}$. Happel [3] showed that a stable category is regarded as a triangulated category. In Section 1, we shall show that for a stable equivalence functor $\Phi : \underline{\mathrm{mod}}\, A \to \underline{\mathrm{mod}}\, B$, Φ is a triangle functor if and only if Φ commutes with the syzygy functors of A and B, namely Φ commutes with the translation functors of A and B.

In Section 2, we study the stable equivalence functor $\Phi_p : \underline{\mathrm{mod}}\, A \to \underline{\mathrm{mod}}\, B$ introduced in [6]. By the definition of Φ_p, $\Phi_p(S)$ is isomorphic to $S' \oplus P$ for each simple A-module S, where S' is a simple B-module and P is a projective B-module. So, if Φ_p is not induced by a Morita equivalence $\mathrm{mod}\, A \to \mathrm{mod}\, B$, then Φ_p is not of Morita type [5].

1 A stable category

Let $\mathcal{A}$ be an abelian category. We denote by $\mathrm{C}(\mathcal{A})$ the category of (cochain) complexes $X^\bullet = (X^n, d_X^n)$ in $\mathcal{A}$;

$$\cdots \longrightarrow X^{n-1} \xrightarrow{d_X^{n-1}} X^n \xrightarrow{d_X^n} X^{n+1} \longrightarrow \cdots$$

with $d_X^{n-1} d_X^n = 0$ for all integer n. The residue category of $\mathrm{C}(\mathcal{A})$ by the homotopy relation is called the homotopy category of $\mathcal{A}$, denoted by $\mathrm{K}(\mathcal{A})$. We denote by $\mathrm{K}^-(\mathcal{A})$ or $\mathrm{K}^b(\mathcal{A})$ the full subcategory of $\mathrm{K}(\mathcal{A})$ consisting of bounded above complexes or bounded complexes, respectively. For $* =$ nothing, $-$ or b, $\mathrm{K}^{*,\phi}(\mathcal{A})$ is the full subcategory of $\mathrm{K}^*(\mathcal{A})$ consisting of acyclic complexes $X^\bullet$, that is, $\mathrm{Im}\, d_X^{n-1} = \mathrm{Ker}\, d_X^n$ for all integer n. A homotopy category $\mathrm{K}^*(\mathcal{A})$ is considered as triangulated category. The shift functor $T : \mathrm{K}^*(\mathcal{A}) \to \mathrm{K}^*(\mathcal{A})$ with

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The detailed version of this paper will be submitted for publication elsewhere.

$T(X^n, d_X^n) = (X^{n+1}, d_X^{n+1})$ is the translation functor. For any morphism $f^\bullet : X^\bullet \to Y^\bullet$ in $\mathrm{K}^\bullet(\mathcal{A})$, its mapping cone is defined by

$$C(f^\bullet) := \left(Y^\bullet \oplus TX^\bullet, \begin{bmatrix} d_Y & 0 \\ Tf & d_{TX} \end{bmatrix} \right),$$

then it induces the triangle $X^\bullet \xrightarrow{f^\bullet} Y^\bullet \xrightarrow{\begin{bmatrix}1 & 0\end{bmatrix}} C(f^\bullet) \xrightarrow{\begin{bmatrix}0 & 1\end{bmatrix}^t} TX^\bullet$ in $\mathrm{K}^\bullet(\mathcal{A})$.

Next, let A be a K-algebra. We denote by $\iota_X : X \to I_X$ and $\pi_X : P_X \to X$ the the injective hull and projective cover of X, respectively. The stable category $\underline{\mathrm{mod}}\, A$ of A is defined as follows; the objects of $\underline{\mathrm{mod}}\, A$ are the same objects of $\mathrm{mod}\, A$. For A-modules X, Y, a morphism from X to Y in $\underline{\mathrm{mod}}\, A$ is given by its residue class in $\mathrm{Hom}_A(X, Y)/\mathrm{Proj}_A(X, Y)$, where $\mathrm{Proj}_A(X, Y)$ is the subset of $\mathrm{Hom}_A(X, Y)$ consisting of morphisms which factor through projective A-modules.

Happel showed that the stable category $\underline{\mathrm{mod}}\, A$ is considered as a triangulated category [3]. In fact, the translation functor of $\underline{\mathrm{mod}}\, A$ is given by the inverse of the syzygy functor Ω. For each morphism $\underline{f} : X \to Y$ in $\underline{\mathrm{mod}}\, A$, the mapping cone $C(\underline{f})$ of $\underline{f}$ is given by the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & I_X & \longrightarrow & \Omega^{-1}X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow & & \| & & \\ 0 & \longrightarrow & Y & \xrightarrow{g} & C(\underline{f}) & \xrightarrow{h} & \Omega^{-1}X & \longrightarrow & 0, \end{array}$$

then it induces the triangle $X \xrightarrow{\underline{f}} Y \xrightarrow{\underline{g}} Z \xrightarrow{\underline{h}} \Omega^{-1}X$ in $\underline{\mathrm{mod}}\, A$.

We denote by $\mathrm{K}^{-,b}(\mathcal{P}_A)$ the full subcategory of $\mathrm{K}^-(\mathcal{P}_A)$ consisting of a complexes $X^\bullet$ with bounded cohomology i.e., $\mathrm{Im}\, d_X^{n-1} = \mathrm{Ker}\, d_X^n$ for $n \ll 0$. Keller-Vossieck and Rickard proved that a stable category of A is triangulated equivalent to the quotient category $\mathrm{K}^{-,b}(\mathcal{P}_A)/\mathrm{K}^b(\mathcal{P}_A)$ [4][8]. We can obtain the similar statement.

Proposition 1.1. *[7] For an algebra A, a stable category of A is triangle equivalent to $\mathrm{K}^\phi(\mathcal{P}_A)$.*

In fact, for any object $P^\bullet$ in $\mathrm{K}^\phi(\mathcal{P}_A)$, there exists an A-module X without projective direct summand such that $P^\bullet$ is isomorphic to $P_X^\bullet$ in $\mathrm{K}^\phi(\mathcal{P}_A)$, which is defined by the injective resolution $0 \to X \xrightarrow{\iota_X} P_X^1 \xrightarrow{d_X^1} P_X^2 \to \cdots$, the projective resolution $\cdots \to P_X^{-1} \xrightarrow{d_X^{-1}} P_X^0 \xrightarrow{\pi_X} X \to 0$ and $d_X^0 = \pi_X \iota_X$. Then, a complex $P^\bullet$ in $\mathrm{K}^{-,b}(\mathcal{P}_A)/\mathrm{K}^b(\mathcal{P}_A)$ corresponds to the complex $T^{-n}P_{\mathrm{Im}\, d_P^n}$ in $\mathrm{K}^\phi(\mathcal{P}_A)$ for $n \ll 0$. The detail of proof is referred in [7].

We need the next lemma in order to prove the later theorem.

Lemma 1.2. *Let X be an A-module without projective direct summand, $P_X = \oplus_i Ae_i$ and $P_{\Omega^{-1}X} = \oplus_j Ae_j$ be decompositions of indecomposable projective. Then the composition map $\pi_X \iota_X$ is of the form $(r_{ij})_{ij} : \oplus_i Ae_i \to \oplus_j Ae_j$, where r_{ij} is the right multiplication map of an element in $\mathrm{rad}(e_i Ae_j)$ for each i, j.*

Proof. We may naturally identify a right multiplication map from Ae_i to Ae_j with an element in $e_i Ae_j$. Assume that r_{ij} is not contained in $\mathrm{rad}(e_i Ae_j)$. Then $(r_{ij})_j : Ae_i \to \oplus_j Ae_j$ is a monomorphism, thus the induced map $(r_{ij})_j \pi_X : Ae_i \to X$ is also monomorphism, contradiction. □

By Lemma 1.2, we may assume that the differential d_P^n of $P^\bullet$ in $\mathrm{K}^\phi(\mathcal{P}_A)$ is of the matrix $(r_{i_n i_{n+1}})_{i_n i_{n+1}}$ with $r_{i_n i_{n+1}} \in \mathrm{rad}(e_{i_n} A e_{i_{n+1}})$ for any integer n.

Theorem 1.3. *Let $\Phi : \underline{\mathrm{mod}}\, A \to \underline{\mathrm{mod}}\, B$ be a stable equivalence functor. If $\Omega_B \Phi \simeq \Phi \Omega_A$ holds, then Φ is a triangle functor i.e., $\underline{\mathrm{mod}}\, A$ and $\underline{\mathrm{mod}}\, B$ are triangle equivalent.*

Proof. By Proposition 1.1, we obtain the triangle equivalent functor $F_A : \underline{\mathrm{mod}}\, A \to \mathrm{K}^\phi(\mathcal{P}_A)$ with $F_A(M) = P_M^\bullet$ for any A-module M without projective direct summand, $F_B : \underline{\mathrm{mod}}\, B \to \mathrm{K}^\phi(\mathcal{P}_B)$ is defined similarly. Assume that $\Omega_B \Phi \simeq \Phi \Omega_A$. It is sufficient to show that $\Phi' := F_B \Phi F_A^{-1} : \mathrm{K}^\phi(\mathcal{P}_A) \to \mathrm{K}^\phi(\mathcal{P}_B)$ is a triangle functor. It is clear that $T_B \Phi' \simeq \Phi' T_A$. Next, let $X^\bullet \xrightarrow{f^\bullet} Y^\bullet \xrightarrow{\begin{bmatrix}1 & 0\end{bmatrix}} C(f^\bullet) \xrightarrow{\begin{bmatrix}0 & 1\end{bmatrix}^t} TX^\bullet$ be a triangle in $\mathrm{K}^\phi(\mathcal{P}_A)$. We consider the differential of the mapping cone $C(\Phi'(f^\bullet))$. Then we obtain

$$\begin{bmatrix} d_{\Phi'(Y)} & 0 \\ T_B \Phi'(f^\bullet) & d_{T_B \Phi'(X)} \end{bmatrix} = \begin{bmatrix} d_{\Phi'(Y)} & 0 \\ \Phi' T_A(f^\bullet) & d_{\Phi' T_A(X)} \end{bmatrix} = \begin{bmatrix} \Phi' d_Y & 0 \\ \Phi' T_A(f^\bullet) & \Phi' d_{T_A(X)} \end{bmatrix}.$$

Since Φ' is a K-linear and commutes with finite direct sums, we have $\Phi' C(f^\bullet) \simeq C(\Phi'(f^\bullet))$ for any morphism $f^\bullet$ in $\mathrm{K}^\phi(\mathcal{P}_A)$. Consequently, $\Phi'(X^\bullet) \xrightarrow{\Phi'(f^\bullet)} \Phi'(Y^\bullet) \xrightarrow{\begin{bmatrix}1 & 0\end{bmatrix}} C(\Phi'(f^\bullet)) \xrightarrow{\begin{bmatrix}0 & 1\end{bmatrix}^t} T\Phi'(X^\bullet)$ is a triangle in $\mathrm{K}^\phi(\mathcal{P}_B)$. □

In [1, Chapter X], it was studied the conditions to follow $\Omega_B \Phi \simeq \Phi \Omega_A$ for a stable equivalence functor $\Phi : \underline{\mathrm{mod}}\, A \to \underline{\mathrm{mod}}\, B$, and it was proved that $\Omega_B \Phi \simeq \Phi \Omega_A$ if A and B are both symmetric algebras. In fact, since it was proved $(\tau_B^{-1} \Omega_B)\Phi \simeq \Phi(\tau_A^{-1} \Omega_A)$ in [1], we obtain that $\Omega_B \Phi \simeq \Phi \Omega_A$ if and only if $\mathcal{N}_B \Phi \simeq \Phi \mathcal{N}_A$, where τ_A, τ_B are the Auslander-Reiten translations of A, B and $\mathcal{N}_A, \mathcal{N}_B$ are the Nakayama functors of A, B, respectively. Therefore we easily obtain the following corollary.

Corollary 1.4. *Let $\Phi : \underline{\mathrm{mod}}\, A \to \underline{\mathrm{mod}}\, B$ be a stable equivalence functor. If $\mathcal{N}_B \Phi \simeq \Phi \mathcal{N}_A$ holds, then Φ is a triangle functor. In particular, if A and B are symmetric, then any equivalence functor $\Phi : \underline{\mathrm{mod}}\, A \to \underline{\mathrm{mod}}\, B$ is a triangle functor.*

2 The stable equivalence not of Morita type

In this section, we assume that an algbera has no algebra summand of Loewy length 2. First, we recall how to construct the stable equivalence functor induced by a socle equivalence, which is introduced in [6]. For an algebra A, $\bar{a}$ is denoted by the residue element of $a \in A$ in $A/\operatorname{soc} A$. For an A-module M and an algebra automorphism f of A, ${}_f M$ is an A-module by changing the operation of A, that is, $a \cdot m = f(a)m$ for each $a \in A$ and $m \in M$. Similarly, N_f is defined for a right A-module N.

Let ν_A be the Nakayama automorphism of A, and $\bar{\nu}_A$ be the automorphism of $A/\operatorname{soc} A$ defined by $\bar{\nu}_A(\bar{a}) = \overline{\nu_A(a)}$ for $a \in A$. Since an algebra is selfinjective, there is an A-bimodule isomorphism $\varphi_A : A \to D(A)_{\nu_A}$. We may consider the $A/\operatorname{soc} A$-bimodule isomorphism $\varphi'_A : \operatorname{rad} A \to D(A/\operatorname{soc} A)_{\bar{\nu}_A}$ and $\varphi''_A : \operatorname{rad} A/\operatorname{soc} A \to D(\operatorname{rad} A/\operatorname{soc} A)_{\bar{\nu}_A}$

defined by $\{\varphi'_A(a)\}(\bar{c}) := \{\varphi_A(a)\}(c)$ and $\{\varphi''_A(\bar{a})\}(\bar{b}) := \{\varphi_A(a)\}(b)$ for any $a, b \in \operatorname{rad} A$ and $c \in A$ (see [6]).

Let A and B be socle equivalent algebras, that is, there is an algebra isomorphism $p : A/\operatorname{soc} A \xrightarrow{\sim} B/\operatorname{soc} B$. Let $\{e_i\}_{i=1}^n$ be a complete set of orthogonal primitive idempotents of A. For each element $a \in A$, we choose a representative $\tilde{p}(a)$ of the residue class $p(\bar{a})$, and define a map $\tilde{p} : A \to B$. We may assume that $\{\tilde{p}(e_i)\}_{i=1}^n$ is a complete set of orthogonal primitive idempotents of B without loss of generality, and set $e_{p(i)} := \tilde{p}(e_i)$. Now, we set $p_n = \bar{\nu}_A^{n-1} p \bar{\nu}_B^{-n+1}$ which is an isomorphism from $A/\operatorname{soc} A$ to $B/\operatorname{soc} B$, and we set $A/\operatorname{soc} A$-bimodule isomorphism $\psi_p := \varphi'_A \operatorname{D}(p^{-1}) {\varphi'_B}^{-1} : \operatorname{rad} A \to {}_{p_0}(\operatorname{rad} B)_p$. Note that $\psi_p(arb) = \tilde{p_0}(a)\psi_p(r)\tilde{p}(b)$ for any $a, b \in A$ and $r \in \operatorname{rad} A$.

Lemma 2.1. *Let p and q be algebra isomorphisms from $A/\operatorname{soc} A$ to $B/\operatorname{soc} B$. Then $\psi_p = \psi_q$ if and only if $p = q$.*

Proof. It is easy to prove because φ'_A and φ'_B are bijections. □

Proposition 2.2. *The following conditions are equivalent for an algebra isomorphism $p : A/\operatorname{soc} A \xrightarrow{\sim} B/\operatorname{soc} B$.*

1. *There are regular maps $\varphi_A(1_A)$ of A and $\varphi_B(1_B)$ of B, respectively, such that $\{\varphi_A(1_A)\}(ab) = \{\varphi_B(1_B)\}(\tilde{p}(a)\tilde{p}(b))$ for all $a, b \in \operatorname{rad} A$.*
2. *$\overline{\psi_p(a)} = p(\bar{a})$ for all $a \in \operatorname{rad} A$.*
3. *The following diagram commutes;*

$$\begin{array}{ccc} \operatorname{rad} A/\operatorname{soc} A & \xrightarrow{p} & \operatorname{rad} B/\operatorname{soc} B \\ {\scriptstyle \varphi''_A}\downarrow & & \downarrow{\scriptstyle \varphi''_B} \\ \operatorname{D}(\operatorname{rad} A/\operatorname{soc} A) & \xrightarrow{\operatorname{D}(p^{-1})} & \operatorname{D}(\operatorname{rad} B/\operatorname{soc} B)\,. \end{array}$$

Proof. It is clear that $2 \iff 3$ by the definition of ψ_p, and $1 \implies 2$ was proved in [6]. Assume that the condition 3 is satisfied. Let $a, b \in \operatorname{rad} A$. Then we obtain the following equation

$$\begin{aligned} \{\varphi_A(1_A)\}(ab) = &= \left\{\varphi''_A(\bar{b})\right\}(\bar{a}) = \left\{\operatorname{D}(p)\varphi''_B p(\bar{b})\right\}(\bar{a}) \\ &= \left\{\varphi''_B(p(\bar{b}))\right\}(p(\bar{a})) = \{\varphi_B(1_B)\}(\tilde{p}(a)\tilde{p}(b)). \end{aligned}$$

□

Now, the functor $G_p : \operatorname{K}^\phi(\mathcal{P}_A) \to \operatorname{K}^\phi(\mathcal{P}_B)$ is defined as follows. For each object $P^\bullet = (\oplus_{i_n} Ae_{i_n}, (r_{i_n i_{n+1}})_{i_n i_{n+1}})$, we give

$$G_p(P^\bullet) = \left(\oplus_{i_n} Be_{p_n(i_n)}, \left(\psi_{p_n}(r_{i_n i_{n+1}})\right)_{i_n i_{n+1}}\right).$$

For a morphism $f^\bullet$ from $(\oplus Ae_{i_n}, (r_{i_n i_{n+1}}))$ to $(\oplus Ae_{j_n}, (t_{j_n j_{n+1}}))$, f^n is of the form $(f_{i_n j_n})_{i_n j_n}$ for some $f_{i_n j_n} \in e_{i_n} A e_{j_n}$. A morphism $G_p(f^\bullet)$ is defined by

$$G_p(f^\bullet)^n = (\tilde{p}_n(f_{i_n j_n}))_{i_n j_n}$$

for each integer n. We denote by F_p the functor from $\underline{\operatorname{mod}}\, A$ to $\underline{\operatorname{mod}}\, B$ which is induced by $G_p : \operatorname{K}^\phi(\mathcal{P}_A) \to \operatorname{K}^\phi(\mathcal{P}_B)$.

Proposition 2.3. *If an algebra isomorphism $p : A/\operatorname{soc} A \xrightarrow{\sim} B/\operatorname{soc} B$ satisfies the equivalent conditions of Proposition 2.2, then F_p is an equivalence functor, namely $\underline{\mathrm{mod}}\, A$ and $\underline{\mathrm{mod}}\, B$ are equivalent.*

Proof. See [6]. □

Proposition 2.4. *Let A and B be selfinjective algebras with an algebra isomorphism $p : A/\operatorname{soc} A \to B/\operatorname{soc} B$. Assume that there is a stable equivalence functor such that it is given as Proposition 2.3. Then the following are equivalent;*

1. $F_p\Omega_A = \Omega_B F_p$.
2. $\bar{\nu}_A p = p\bar{\nu}_B$.
3. *$F_p : \underline{\mathrm{mod}}\, A \to \underline{\mathrm{mod}}\, B$ is a triangulated functor.*

Proof. It follows that $2 \Longrightarrow 3 \Longleftrightarrow 1$ by Proposition 2.3 and Theorem 1.3. Assume that the condition 1 is satisfied. Then $GT_A = T_BG$ i.e., $\psi_{p_n} = \psi_{p_{n+1}}$. We have $p_n = p_{n+1}$ by Lemma 2.1. □

In [2], Broué introduced the stable equivalence of Morita type in order to study the stable equivalence which is induced by a derived equivalence. Let M be a B-A-bimodule and N an A-B-bimodule such that M, N induce a stable equivalence of Morita type [2], that is, M, N are projective both as left and right modules, and if $N \otimes_B M \simeq A \oplus X$ as A-A-bimodules and $M \otimes_A N \simeq B \oplus Y$ as B-B-bimodules, where X is a projective A-A-bimodule and Y is a projective B-B-bimodule. Note that M, N induce Morita equivalent if and only if both X and Y are zero. In particular, if either X or Y are zero, then so is the rest one. Linckelmann proved the following proposition which characterizes the stable equivalence to induced Morita equivalent.

Proposition 2.5. *[5] Let A and B be algebras. Let M be a B-A-bimodule and N an A-B-bimodule such that M, N induce a stable equivalence functor of Morita type. Then the functor $M \otimes_A - : \mathrm{mod}\, A \to \mathrm{mod}\, B$ is an equivalence if and only if for any simple A-modules, the B-module $M \otimes_A S$ is also simple.*

Proof. Note that we may consider $N \simeq N' \oplus N''$, where N' is an indecomposable nonprojective A-B-bimodule and N'' is a projective A-B-bimodule, moreover $N' \otimes_B S'$ is indecomposable nonprojective A-module for every simple B-module S'.

Assume that $M \otimes_A S$ is simple B-module for any simple A-module S. Then we have $N \otimes_B M \otimes_A S \simeq (A \oplus X) \otimes_A S \simeq S \oplus (X \otimes_A S)$. By above assertion, $X \otimes_A S = 0$, so $X = 0$. Also, we have $M \simeq M \otimes_A (N \otimes_B M) \simeq (B \oplus Y) \otimes_B M \simeq M \oplus (Y \otimes_B M)$, so $Y = 0$. □

Now, let F_p be the functor constructed as above for an algebra isomorphism $p : A/\operatorname{soc} A \to B/\operatorname{soc} B$. Then $F_p(S)$ is simple B-module for any simple A-module S. In fact, p induces the bijection $A/\operatorname{rad} A \to B/\operatorname{rad} B$ because algebras have no semisimple algebra summand. The restriction map of ψ_p to $\operatorname{soc} A$ is the bijection $\operatorname{soc} A \to \operatorname{soc} B$. Therefore, if $\mathrm{mod}\, A$ and $\mathrm{mod}\, B$ are not Morita equivalent, then F_p is the stable equivalence not of Morita type.

In [7], we gave the example constructed by the different way. Let $p : B/\operatorname{soc} B \to A/\operatorname{soc} A$ be an algebra isomorphism. We obtain the equivalence functor $F'_p : \operatorname{mod}(A/\operatorname{soc} A) \to \operatorname{mod}(B/\operatorname{soc} B)$ defined by $F'_p(M) = {}_pM$ for each $A/\operatorname{soc} A$-module M. Moreover, we consider the correspondence $\Psi_p : \underline{\operatorname{mod}}\, A \to \underline{\operatorname{mod}}\, B$ defined by $\Psi_p(M) = F'_p(M)$ and $\Psi_p(\underline{f}) = \underline{F'_p(f)}$ for each $A/\operatorname{soc} A$-module M and morphism f in $\underline{\operatorname{mod}}\, A$. Then it follows that Ψ_p is a well-defined equivalence functor if and only if for every indecomposable projective A-module P, there is a nonzero composite morphism $\rho\omega$ factoring through P such that $\underline{F'_p(\rho\omega)} = 0$, $\rho : \operatorname{rad} P \to \operatorname{rad} P/\operatorname{soc} P$ is an epimorphism and $\omega : \operatorname{rad} P/\operatorname{soc} P \to P/\operatorname{soc} P$ is a monomorphism. Note that Ψ_p is also stable equivalence not of Morita type.

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Lattice-finite rings and their Auslander orders

Wolfgang Rump

In this article we give a condensed presentation of some recent results on lattice-finite noetherian rings[1]. For a left noetherian ring R, we define an *R-lattice* as a finitely generated left R-module without simple submodules. We call R *lattice-finite* if there are only finitely many isomorphism classes of indecomposable R-lattices. When R is an order over a Dedekind domain, these concepts coincide with the usual ones. There have been two reasons for studying that generalization. First, when "representation-finiteness" is added in the equation

$$\text{artinian algebras} : \text{classical orders} = \text{artinian rings} : \boxed{\;?\;},$$

the free space can be filled with "lattice-finite noetherian rings". The second reason and original motivation arose from a study of Iyama's papers on τ-categories [2, 3, 4]. For representation-finite orders Λ over a complete discrete valuation domain, the structure of the τ-category Λ-lat of Λ-lattices is described there. Roughly speaking, a τ-category is an additive category with (a generalized version of) Auslander-Reiten sequences. By a reduction to its essential core, the connection between representation-finite Λ and Λ-lat boils down to a relationship between lattice-finite noetherian rings and a natural class of τ-categories, adding a new aspect to the conception of non-commutative curves.

The first three sections of this article are devoted to the global theory of lattice-finite noetherian rings. We prove that every such ring R is an order in a semisimple ring. For R-lattices over an arbitrary noetherian ring R, all Krull dimensions occur except 0. The restriction to lattice-finite rings R, however, implies that $\operatorname{Kdim} R \leqslant 1$ (Theorem 1). Following Auslander, the category of R-lattices can be replaced by the endomorphism ring $A(R)$ of an additive generator in R-lat. Then we can prove that $A(R)$ is again an order in a semisimple ring (Theorem 2), thereby providing a justification for calling $A(R)$ the *Auslander order* of R.

In contrast to classical orders, lattice-finite noetherian rings R need not be fully bounded. For example, R may be simple with ${}_RR$ non-artinian. It is easy to see that every principal ideal domain R has exactly one isomorphism class of indecomposable R-lattices. This does not hold for principal left ideal domains R with non-principal right ideals. (They have R-lattices which cannot be embedded into a free R-lattice!)

[1]§§4-5 contain results of [11] which has been submitted for publication, while an extension of §§1-3 will be submitted for publication elsewhere.

For a subring R of a ring R' we call R' a *left overorder* of R if ${}_RR'$ has no simple submodules, and the left conductor of R' in R has finite index in R' as a left R-module. When R is left noetherian, a left overorder R' is determined by a corresponding full embedding R'-lat $\hookrightarrow$ R-lat into the category of R-lattices. Therefore, it follows that every lattice-finite left noetherian ring R has a maximal left overorder R'. When R is left and right noetherian, we show that R' is an Asano order (Theorem 3). This means that ideals which contain a regular element are invertible. To prove this, we show that lattice-finite noetherian rings R have an idempotent ideal $I(R)$ which provides an obstruction to invertibility. The obstruction vanishes when R is maximal.

For a lattice-finite noetherian ring R, a necessary condition for the existence of almost split sequences in R-lat is that the endomorphism rings of indecomposable R-lattices are local. In this case a maximal left overorder R' of R is semiperfect. We prove that the converse holds: If R has a semiperfect maximal left overorder R', then R-lat is a strict τ-category (Theorem 5). Then $R' \cong \mathrm{M}_{n_1}(\Omega_1) \times \cdots \times \mathrm{M}_{n_s}(\Omega_s)$ with (non-commutative) discrete valuation domains Ω_i. In the commutative case, for example, almost split sequences exist for the local rings R of a simple plane curve singularity, but not for the local ring of the singularity given by Newton's curve $y^2 = x^3 + x^2$.

Some different characterizations for the existence of almost split sequences in R-lat are provided by Theorem 4, in particular, two criteria in terms of irreducible R-lattices. Another criterion states that any decreasing sequence $E_0 \supsetneq E_1 \supsetneq E_2 \supsetneq \cdots$ of indecomposable R-lattices is *divergent*, that is, for each indecomposable full sublattice E of E_0, there exists some $i \in \mathbb{N}$ with $E_i \subset E$. We do not know whether this property holds for all (not necessarily representation-finite) classical orders.

1 Left orders

Let R be a left noetherian ring. By R-**mod** we denote the category of finitely generated left R-modules. A module $E \in R$-**mod** with no simple submodules is said to be an *R-lattice* [11]. The full subcategory R-lat of R-lattices in R-**mod** is integral and almost abelian in the sense of [8]. For any additive category $\mathcal{A}$, let $\mathrm{ind}\,\mathcal{A}$ denote a fixed class of objects representating the isomorphism classes of indecomposable objects in $\mathcal{A}$. Monic and epic morphisms in $\mathcal{A}$ are said to be *regular*. If an inclusion $E \hookrightarrow F$ of R-modules is regular in R-lat, we call F an *overlattice* of E. In case ind(R-lat) is finite, the ring R is said to be *lattice-finite*.

For example, if Λ is a D-order [6] in a finite dimensional algebra A, where D is a Dedekind domain, the familiar concept of Λ-lattice coincides with our general notion. In this classical case it is well-known that Λ cannot be lattice-finite unless A is semisimple. In general, we do not even know in advance that a lattice-finite ring R is an order. Our first result (Proposition 1) shows that R is a left order in a semisimple ring.

Lemma 1. *Let R be a lattice-finite left noetherian ring. For every R-lattice $E \neq 0$ there are only finitely many isomorphism-classes of submodules or overlattices of E. Moreover, there exists a submodule F of E with a non-invertible regular endomorphism $F \to F$ in* R-**lat**.

Proof. Since R is left noetherian, the uniform dimension d of E is finite. Therefore, the number of indecomposable direct summands in a decomposition of a submodule or an overlattice of E is bounded by d. This proves the first statement. Since $E \neq 0$, there is an infinite sequence of submodules $E \supsetneq E_1 \supsetneq E_2 \supsetneq \cdots$ with regular embeddings $E_{i+1} \hookrightarrow E_i$ in R-lat. Hence $E_i \cong E_j$ for some $i < j$. Therefore, the submodule E_i admits a non-invertible regular morphism $E_i \xrightarrow{\sim} E_j \hookrightarrow E_i$. □

Proposition 1. *Every lattice-finite left noetherian ring R with ${}_RR \in R$-*lat *is a left order in a semisimple ring.*

Proof. By Goldie's theorem it suffices to prove that R is semiprime. Suppose that there is an ideal $N \neq 0$ with $N^2 = 0$. By Lemma 1 there is a submodule N_0 of N with a non-invertible regular endomorphism $r\colon N_0 \to N_0$. Let Q be the injective envelope of ${}_RR$. Since r is an essential monomorphism, it can be factored into $r\colon N_0 \hookrightarrow N_1 \xrightarrow{\sim} N_0$ with an R-submodule N_1 of Q. So we get an infinite sequence $N_0 \subsetneq N_1 \subsetneq N_2 \subsetneq \cdots$ of submodules N_i of ${}_RQ$ with $N_i \cong N_0$ such that N_i/N_0 is of finite length. We set $R_i := R + N_i$. The noetherian property of R_i implies that the length of R_i/N is unbounded. On the other hand, $N_i \cong N_0 \subset N$ implies $NN_i = 0$ and thus $NR_i = N$ for all $i \in \mathbb{N}$. Therefore, the R-lattices R_i belong to infinitely many isomorphism-classes, a contradiction to Lemma 1. □

2 The Auslander order

Recall that the *Krull dimension* $\operatorname{Kdim} M$ of a noetherian module $M \neq 0$ (in the sense of Gabriel-Rentschler, see [1]) is defined as the deviation of the poset of submodules of M. Here the *deviation* $\operatorname{dev} \Omega$ of a poset Ω is defined recursively as follows. If Ω is artinian, then $\operatorname{dev} \Omega := 0$. If α is an ordinal number with $\operatorname{dev} \Omega \not< \alpha$, then $\operatorname{dev} \Omega := \alpha$ if for each descending sequence $a_0 \geqslant a_1 \geqslant \cdots$ in Ω, almost all intervals $[a_{i+1}, a_i]$ have a deviation $< \alpha$. For a left noetherian ring $R \neq 0$ we set $\operatorname{Kdim} R := \operatorname{Kdim}({}_RR)$. The *uniform dimension* of a module M will be denoted by $\operatorname{udim} M$.

Let R be a left noetherian ring. We call an R-lattice E *irreducible* if $E \neq 0$, and the composition of each pair of non-zero morphisms $A \to E \to B$ in R-lat is non-zero. Equivalently, this says that for each submodule F of E with $E/F \in R$-lat, either $F = 0$ or $F = E$. Since R is left noetherian, every R-lattice E admits a descending chain $E = E_0 \supsetneq E_1 \supsetneq E_2 \supsetneq \cdots$ of submodules such that E_i/E_{i+1} are irreducible R-lattices. If

$E_n = 0$ for some $n \in \mathbb{N}$, the smallest such n does not depend on the particular choice of the sequence (E_i). Then $\rho(E) := n$ will be called the *rational length* of E. If the sequence (E_i) is infinite, we set $\rho(E) := \infty$. The following proposition is easy to verify.

Proposition 2. *Let R be a left noetherian ring. An R-lattice E satisfies* $\operatorname{Kdim} E \leqslant 1$ *if and only if* $\rho(E) < \infty$.

Obviously, the inequality $\operatorname{udim} E \leqslant \rho(E)$ holds for any R-lattice E. Under the assumption that R is also right noetherian, we can show that equality holds in case R is lattice-finite. The following proposition is crucial.

Proposition 3. *Let R be a lattice-finite noetherian ring. Every essential submodule of an R-lattice E has finite index in E.*

In particular, Proposition 3 shows that for a lattice-finite noetherian ring R, every uniform R-lattice is irreducible. As an immediate consequence, we get

Theorem 1. *Let R be a lattice-finite noetherian ring. Then* $\operatorname{udim} E = \rho(E)$ *holds for any R-lattice E. In particular,* $\operatorname{Kdim} R \leqslant 1$.

For a lattice-finite left noetherian ring R, the category $\mathcal{A} := R$-**lat** is tantamount to the ring $A(R) := \operatorname{End}_{\mathcal{A}}(\bigoplus \operatorname{ind} \mathcal{A})^{\mathrm{op}}$. (Note that every R-lattice admits a decomposition into finitely many indecomposable R-lattices.) For classical orders R, the ring $A(R)$ coincides with the Auslander order. In general, even if R is not lattice-finite, the category R-**lat** is localizable with respect to regular morphisms. But this does not automatically imply that $A(R)$ is an order. Using results of [8], the following theorem can be obtained as a consequence of Theorem 1.

Theorem 2. *Let R be a lattice-finite noetherian ring. Then $A(R)$ is an order in a semisimple ring $\widetilde{A}(R)$. If Q is the quotient ring of R/I, where I denotes the sum of all length-finite left ideals of R, then $\widetilde{A}(R)$ is Morita equivalent to Q.*

3 Asano orders

Let R be a subring of a ring R'. We call R' a *left overorder* [11] of R if ${}_RR' \in R$-**lat** and $(R : R')_r := \{a \in R \mid R'a \subset R\}$ has finite index in ${}_RR'$. By [11], Proposition 4, R is left noetherian if and only if R' is so, and R is a left order in a given semisimple ring Q if and only if R' is a left order in Q. For a left overorder R' of a left noetherian ring R, it follows from [11], Proposition 5, that R'-lat is a full subcategory of R-lat. A left noetherian ring R will be called *maximal* if ${}_RR \in R$-lat, and R has no proper left overorder.

In contrast to classical orders, a maximal noetherian ring R need not be lattice-finite. Moreover, there may be maximal ideals of R having infinite index as a left ideal. When R is lattice-finite, however, it can be shown that R is an Asano order. Recall that an order R in a semisimple ring is said to be an *Asano order* [7] if every ideal I of R which contains a regular element is invertible. In view of the relationship between left overorders of R and full subcategories of R-lat ([11], Proposition 5), the following proposition is obvious.

Proposition 4. *Every lattice-finite left noetherian ring R with ${}_RR \in R$-lat has a maximal left overorder.*

For two-sided noetherian rings we have

Theorem 3. *Every maximal lattice-finite noetherian ring R is an Asano order.*

Outline of proof. The first step is to show that up to isomorphism, there are only finitely many simple left R-modules $S_1, \ldots, S_n$ with $\mathrm{Ext}^1_R(S_i, R) = 0$. In fact, let $E_i \hookrightarrow R \twoheadrightarrow S_i$ be short exact sequences. Then $S_i \not\cong S_j$ implies that $E_i \not\cong E_j$. So there cannot be infinitely many S_i. Now let I be any left ideal of finite index in R such that the composition factors of R/I are all isomorphic to some S_i with $i \in \{1, \ldots, n\}$. Then $\mathrm{Ext}^1_R(R/I, R) = 0$. With $(\ )^\bullet := \mathrm{Hom}_R(-, R)$ this implies $I^{\bullet\bullet} = R$. Since R is lattice-finite, the length of $I^{\bullet\bullet}/I$ must be bounded for all such I. Consequently, there exists a minimal left ideal I with the mentioned property. The minimality of I then implies that $I^2 = I$. Since I is of finite index in R, we get $I^m R \subset I$ for some $m \in \mathbb{N}$. Hence I is an ideal.

By Lemma 1, the length $l_E := E/IE$, where E runs through all left ideals of R, has a finite maximum l. It can be shown that $l_E = l_F = l$ implies $l_{E+F} = l$, and that $l_B \leqslant l_A + l_C$ holds for every short exact sequence $A \rightarrowtail B \twoheadrightarrow C$ of R-lattices.

Suppose that $I \neq R$. Since I contains a regular element, there exists a left ideal $L \subset I$ with $l_L = l$. Let E be the sum of all these L. Then $l_E = l$, and E is essential in I. Moreover, using Propositions 1 and 3, it can be shown that E is an ideal of R. Since $IE \neq E$, we have $E \neq I$. For a composition series $E \subsetneq E' \subsetneq \cdots \subsetneq I$, let $S := E'/E$ be the first composition factor. Then $ES = 0$. Hence there exists a maximal ideal M of R with $E \subset M$ and $MS = 0$. On the other hand, $IS = 0$ would imply $IE' = I^2E' \subset IE$, and thus $l_{E'} > l_E = l$. Therefore, S does not occur as a composition factor in R/I. In particular, this gives $I \not\subset M$. By the maximality of R, we have $M \subset MM^\bullet \subset R$ and $M \subset M^\bullet M \subset R$. We show that M is invertible. If not, we would have $MM^\bullet = M$ or $M^\bullet M = M$. In either case, we get $M^\bullet = R$ by the maximality of R. But this would imply $\mathrm{Ext}^1_R(R/M, R) = 0$, a contradiction to $\mathrm{Ext}^1_R(S, R) \neq 0$. So M is invertible.

Next it can be shown that $l_{M^iE} \leqslant l_{M^jE}$ holds for $i \leqslant j$. Since R is lattice-finite, there are integers $i < j$ with $M^iE \cong M^jE$. Hence $l_{M^iE} = l_E = l$ for all $i \in \mathbb{Z}$. Furthermore, it can be proved that $M^{-1}E \subset I$. But $ME' \subset E$ implies $E \subsetneq M^{-1}E$, a contradiction to the maximality of $E \subset I$.

Thus we have shown that $I = R$. Let M be any maximal ideal of R such that ${}_RM$ is essential in ${}_RR$. By Proposition 3, this implies that ${}_RM$ has finite index in ${}_RR$. By the same argument as above, we infer that M is invertible. Now it follows (see [10], Proposition 5) that every ideal of R which is essential as a left ideal is invertible. Hence R is an Asano order. □

4 The semiperfect case

Let $\mathcal{A}$ be a *Krull-Schmidt category*, i. e. an additive category such that every object of $\mathcal{A}$ is a finite direct sum of objects with local endomorphism rings. Then the ideal $\operatorname{Rad}\mathcal{A}$ of $\mathcal{A}$ generated by the non-invertible morphisms between indecomposable objects is said to be the *radical* of $\mathcal{A}$. A morphism $f\colon A \to B$ in $\mathcal{A}$ is called *right almost split* if $f \in \operatorname{Rad}\mathcal{A}$, and every $A' \to B$ in $\operatorname{Rad}\mathcal{A}$ factors through f. *Left almost split* morphisms are defined in a dual way. A sequence

$$\tau A \overset{v}{\rightarrowtail} \vartheta A \overset{u}{\longrightarrow} A$$

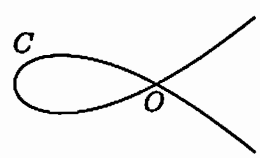
of morphisms in $\mathcal{A}$ is said to be *right almost split* (for A) if u and v are right resp. left almost split morphisms, and $v = \ker u$. Note that we do not assume that A is indecomposable. Up to isomorphism, a right almost split sequence for A is uniquely determined by A. *Left almost split sequences* $B \longrightarrow \vartheta^- B \twoheadrightarrow \tau^- B$ (for B) are defined dually. Following Iyama [2], we call $\mathcal{A}$ a *strict τ-category* if right and left almost split sequences exist for all objects of $\mathcal{A}$. By [2], 2.3, this implies that every right almost split sequence for an indecomposable object A with $\tau A \neq 0$ is left almost split (with τA indecomposable).

Let R' be a left noetherian Asano order in a semisimple ring. If R' is semiperfect, then $\operatorname{Rad} R'$ is invertible. By [9], Proposition 1.6, and [10], Proposition 2, this implies that

$$R' \cong \mathrm{M}_{n_1}(\Omega_1) \times \cdots \times \mathrm{M}_{n_s}(\Omega_s)$$

with (non-commutative) discrete valuation domains Ω_i. Thus $|\operatorname{ind}(R'\text{-lat})| = s$.

Now let R be a lattice-finite noetherian ring with ${}_RR \in R$-lat. By Proposition 4 and Theorem 3, R has an Asano left overorder R'. If R is semiperfect, however, R' need not be so. For example, let R be the local ring of the plane curve C: $y^2 = x^3 + x^2$



at the singularity $(0,0)$. Using the parameter $t = \frac{y}{x}$, we can regard R as a subring $\{f \in R' \mid f(1) = f(-1)\}$ of $R' := \{\frac{p}{q} \in \mathbb{R}(t) \mid q(\pm 1) \neq 0\}$. Then R' is the unique maximal left overorder of R. Here we have ind(R'-**lat**) $= \{R'\}$ and ind(R-**lat**) $= \{R, R'\}$. Whereas R is local, R' has two maximal ideals $R'(t \pm 1)$ corresponding to the points of the normalization $\mathbb{A}^1$ of C lying over the singularity $(0,0)$. Thus R' is not semiperfect.

The example shows, in particular, that there are semiperfect lattice-finite noetherian rings R for which the Auslander order $A(R)$ is not semiperfect. In this case, R-**lat** cannot have Auslander-Reiten sequences. Our next result (Theorem 4) gives a criterion for R to have a semiperfect Asano left overorder.

Lemma 2. *For a lattice-finite left noetherian semilocal ring R, the Jacobson radical* Rad : R-**lat** $\to$ R-**lat** *has a right adjoint* Rad°: R-**lat** $\to$ R-**lat**. *For each R-lattice C there exists an integer $n = n(C) \in \mathbb{N}$ such that for any short exact sequence $A \overset{v}{\rightarrowtail} B \twoheadrightarrow C$ in* R-**lat**, *the inclusion $A \hookrightarrow (\mathrm{Rad}^\circ)^n A$ factors through v.*

Proof. The first statement says that every R-lattice E has a greatest overlattice F with $\mathrm{Rad}\, F \subset E$. This follows by Lemma 1. The second statement is proved by means of a projective presentation of C. □

We call a strictly decreasing sequence (a_i) in a poset Ω *divergent* if for each $a \in \Omega$ there is an index $i \in \mathbb{N}$ with $a_i \leqslant a$. If every strictly decreasing sequence in Ω is divergent, Ω itself is said to be *lower divergent*. For a left noetherian ring R, we call an R-lattice E *divergent* if E has a local endomorphism ring with radical J such that $JE \neq E$. When R is an order in a semisimple ring Q, and $M \in Q$-**mod**, we denote the set of finitely generated R-submodules E of M with $QE = M$ by $\mathcal{L}(M)$. Then $\mathcal{L}(M)$ is a lattice with respect to $+$ and $\cap$.

Theorem 4. *Let R be a lattice-finite noetherian semilocal ring with ${}_RR \in R$-**lat**, and let Q be the quotient ring of R. The following are equivalent:*

(a) *R has a semiperfect maximal left overorder R'.*
(b) *For any simple Q-module S, the poset $\mathcal{L}(S)$ is lower divergent.*
(c) *For every $M \in Q$-**mod**, the poset of indecomposable R-lattices in $\mathcal{L}(M)$ is lower divergent.*
(d) *Every indecomposable R-lattice is divergent.*
(e) *Every irreducible R-lattice is divergent.*

Proof. (a) $\Rightarrow$ (b): With $J := \mathrm{Rad}\, R'$ we have $J^m \subset R$ for some $m \in \mathbb{N}$. For a chain $E_0 \supsetneq E_1 \supsetneq \cdots$ in $\mathcal{L}(S)$ this gives gives $J^m E_i \subset E_i \subset R'E_i$. Since the R'-lattices in $\mathcal{L}(S)$ form a chain, (b) follows.

(a) $\Rightarrow$ (c): If a sequence $E_0 \supsetneq E_1 \supsetneq \cdots$ in $\mathcal{L}(M)$ does not diverge, there is an $E \in \mathcal{L}(M)$ with $E_i \not\subset E$ for all i. Then we find a projection p: $M \twoheadrightarrow Q$ with $p(E_i) \not\subset p(E)$ for all i. By virtue of (b), the sequence $(p(E_i))$ must be stationary. If the sequence

$(E_i \cap \operatorname{Ker} p)$ in $\mathcal{L}(\operatorname{Ker} p)$ does not diverge, we can repeat the argument. By induction, we find an epimorphism q: $M \twoheadrightarrow N$ and an integer $k \in \mathbb{N}$ with $q(E_i) = q(E_k)$ for $i \geqslant k$ such that $(E_i \cap \operatorname{Ker} q)$ diverges. Now assume that the E_i are indecomposable. We set $J := \operatorname{Rad} R'$ and $n := n(q(E_k))$ according to Lemma 2. Choose $m \in \mathbb{N}$ with $(\operatorname{Rad}^\circ)^n({}_R R') \subset J^{-m}$ and $J^m \subset R$. Then $E_j \cap \operatorname{Ker} q \subset J^{2m}(E_k \cap \operatorname{Ker} q)$ for some $j \in \mathbb{N}$. So we get $(\operatorname{Rad}^\circ)^n(E_j \cap \operatorname{Ker} q) \subset (\operatorname{Rad}^\circ)^n J^{2m}(E_k \cap \operatorname{Ker} q) \subset J^m(E_k \cap \operatorname{Ker} q) \subset E_k \cap \operatorname{Ker} q$. By Lemma 2, this implies that the short exact sequence $E_k \cap \operatorname{Ker} q \hookrightarrow E_k \twoheadrightarrow q(E_k)$ splits. Since E_k is indecomposable, this gives $q = 0$, whence (E_i) diverges.

(c) $\Rightarrow$ (d): If $E \in \operatorname{ind}(R\text{-}\mathbf{lat})$ is not divergent, there are non-invertible $e_i \in \operatorname{End}_R(E)$ with $e_1(E) + \cdots + e_n(E) = E$. We may assume that e_1 is monic. Further we may assume that $E \in \mathcal{L}(M)$ for some $M \in Q\text{-}\mathbf{mod}$. If e_i is not monic, consider a decomposition $M = Qe_i(E) \oplus C$. As Q is semisimple, we find a Q-linear map f: $M \twoheadrightarrow C$ with $f(E) \in \mathcal{L}(C)$ and $\operatorname{Ker} f \cap \operatorname{Ker} e_i = 0$. Since R is right noetherian, $f(E)$ is compressible in the sense of [5]. Therefore, we find a regular morphism r: $f(E) \to e_1(E) \cap C$ in R-lat. Hence $e_1(E) + e_i(E)$ does not change if we replace e_i by the monomorphism $\binom{e_i}{rf}$: $E \to e_i(E) \oplus (E \cap C) \hookrightarrow E$. So we may assume that the $e_1, \ldots, e_n$ are regular, and $e_1(E) \not\subset e_n(E)$. Since $e_1(E) = e_1e_1(E) + \cdots + e_1e_n(E)$, we infer that $e_1e_i(E) \not\subset e_n(E)$ for some i. By the same argument, $e_1e_ie_j(E) \not\subset e_n(E)$ for some j. Thus we obtain a non-divergent sequence $E \supsetneq e_1(E) \supsetneq e_1e_i(E) \supsetneq e_1e_ie_j(E) \supsetneq \cdots$.

(e) $\Rightarrow$ (a): For irreducible R-lattices E, F we write $E \prec F$ if there exists a surjection $E^n \twoheadrightarrow F$. By (e) this gives a partial ordering on the isomorphism classes of irreducibles. For each simple $S \in \operatorname{ind}(Q\text{-}\mathbf{mod})$ we choose an irreducible $E_S \in \mathcal{L}(S)$ which is maximal with respect to $\prec$. If $E, F \in \mathcal{L}(S)$ are isomorphic to E_S, then the surjection $E \oplus F \twoheadrightarrow E+F$ shows that $E + F \cong E_S$. Hence $E = E + F$ or $F = E + F$. Consequently, the irreducible $E \in \mathcal{L}(S)$ with $E \cong E_S$ form a chain. Now it is easily seen that R has a left overorder R' with $R'\text{-lat} \approx \operatorname{add}\{E_S \mid S \in \operatorname{ind}(Q\text{-}\mathbf{mod}) \text{ simple}\}$. Thus R' is a semiperfect Asano order.

The implication (c) $\Rightarrow$ (d) also yields (b) $\Rightarrow$ (e). Since (d) $\Rightarrow$ (e) is trivial, the proof is complete. □

5 Auslander-Reiten sequences

In this section we prove that the equivalent conditions of Theorem 4 are necessary and sufficient for the existence of almost split sequences in R-lat. The necessity follows from the next proposition which provides a weaker version of condition (d) in Theorem 4.

Proposition 5. *Let R be a lattice-finite noetherian ring. Then R-*lat *is a Krull-Schmidt category if and only if every indecomposable R-lattice is divergent.*

Proof. Let I be the sum of all length-finite left ideals of R. By Proposition 4 and Theorem 3, R/I has an Asano left overorder R', and R' is left noetherian by [11], Proposition 4. If R-lat is a Krull-Schmidt category, then R' is semiperfect. So the equivalence (a) $\Leftrightarrow$ (d) of Theorem 4 completes the proof. $\square$

Recall that an object Q of an additive category with kernels and cokernels is said to be *projective (injective)* if the functor $\mathrm{Hom}_{\mathcal{A}}(Q,-)$ (resp. $\mathrm{Hom}_{\mathcal{A}}(-,Q)$) from $\mathcal{A}$ to the category of abelian groups preserves short exact sequences.

Lemma 3. *Let R be a lattice-finite left noetherian semilocal ring. A morphism $f\colon E \to F$ in R-lat is a kernel if and only if it does not factor through a non-invertible regular morphism $r\colon E \to E'$.*

Proof. Assume first that f is a kernel. If $r\colon E \to E'$ is regular and $f = gr$, then the cokernel c of f satisfies $cgr = 0$. Hence $cg = 0$, and $g = fs$ for some $s\colon E' \to E$. Thus $f(1 - sr) = 0$, and therefore, $sr = 1$. Since r is regular, this implies that r is invertible. Conversely, assume that f does not factor through a non-invertible regular morphism r: $E \to E'$. Let $g\colon H \rightarrowtail E$ be the kernel of f, and let $g'\colon \mathrm{Rad}^\circ H \to \mathrm{Rad}^\circ E$ be the unique extension of g. Then f factors through the regular morphism $E \hookrightarrow E + g'(\mathrm{Rad}^\circ H)$. Hence $g'(\mathrm{Rad}^\circ H) \subset E$, and thus $\mathrm{Rad}^\circ H = H$ since g is a kernel. Consequently, $H = 0$. So f is monic, whence a kernel. $\square$

Lemma 4. *Let R be a left noetherian ring, and let $A \overset{v}{\rightarrowtail} B \overset{u}{\twoheadrightarrow} C$ be a non-split short exact sequence of R-lattices with $\mathrm{End}_{\mathcal{A}}(C)$ local such that every non-split-monic morphism $A \to X$ in R-lat factors through v. Then every non-split-epic morphism $f\colon Y \to C$ in R-lat factors through u.*

Proof. There is a commutative diagram

$$\begin{array}{ccccc} A & \overset{v}{\rightarrowtail} & B & \overset{u}{\twoheadrightarrow} & C \\ \vdots{\scriptstyle e} & & \downarrow{\scriptstyle i} & & \| \\ X & \overset{x}{\rightarrowtail} & B \oplus Y & \overset{(u\ f)}{\twoheadrightarrow} & C \end{array}$$

with $i = \binom{1}{0}$ and $x = \ker(u\ f)$. If e is not split monic, then $e = gv$ for some g. Thus $(i - xg)v = 0$, and we get some $h\colon C \to B \oplus Y$ with $i - xg = hu$. Hence $(1 - (u\ f)h)u = u - (u\ f)i = 0$, and thus $(u\ f)$ is a split epimorphism. Since $\mathrm{End}_{\mathcal{A}}(C)$ is local, this is a contradiction. Therefore, e is split monic. As the left-hand square of the diagram is a pushout, it follows that $(i\ x)\colon B \oplus X \to B \oplus Y$ is a split epimorphism. Therefore, $(u\ f)$ factors through u, whence f factors through u. $\square$

Theorem 5. *For a lattice-finite noetherian ring R, the category R-lat is a strict τ-category if and only if it is a Krull-Schmidt category.*

Proof. Assume that R-lat is a Krull-Schmidt category. For any projectice indecomposable R-lattice P, the R-module $P/\text{Rad}\,P$ is simple. Otherwise, there would be submodules $E, F \subsetneq P$ with $E + F = P$. Then the natural surjection $E \oplus F \twoheadrightarrow P$ would split, a contradiction to the Krull-Schmidt category property. As a consequence, $0 \to \text{Rad}\,P \to P$ is a right almost split sequence. Using Lemma 3, the dual argument shows that every injective R-lattice admits a left almost split sequence $I \to \text{Rad}^{\circ} I \to 0$. Now let $C \in R$-lat be non-projective indecomposable. Then there exists a morphism $C \to C'$ in R-lat which does not factor through some cokernel $C'' \twoheadrightarrow C'$. Taking the pullback, we get a non-split short exact sequence $A \overset{a}{\rightarrowtail} B \overset{b}{\twoheadrightarrow} C$. Let $\mathcal{A}$ be the class of morphisms $f\colon A \to F$ in R-lat which do not factors through a. Note that for any such f, the pushout H of a and f gives rise to a non-split short exact sequence $F \rightarrowtail H \twoheadrightarrow C$. We introduce four binary relations on $\mathcal{A}$. For $e, f \in \mathcal{A}$ we write $e < f$ if there exists a morphism g which is not split monic such that $f = ge$. If g can be chosen as a kernel (a cokernel, a regular morphism) we write $e <_k f$ ($e <_c f$, $e <_r f$). By [8], Proposition 2, these relations are transitive. For any binary relation $\lhd$ on a subset $\mathcal{A}'$ of $\mathcal{A}$ we call $f \in \mathcal{A}'$ *maximal* if there is no $g \in \mathcal{A}'$ with $f \lhd g$. Let $\mathcal{A}_c$ be the set of maps in $\mathcal{A}$ which are maximal with respect to $<_c$. Since R is left noetherian, it follows that for each $e \in \mathcal{A} \setminus \mathcal{A}_c$ there exists some $f \in \mathcal{A}_c$ with $e <_c f$. As $1 \in \mathcal{A}$, we infer that $\mathcal{A}_c \neq \varnothing$. If $f\colon A \twoheadrightarrow A_1 \oplus A_2$ is in $\mathcal{A}_c$ with $A_i \neq 0$, then both components $A \twoheadrightarrow A_1 \oplus A_2 \twoheadrightarrow A_i$ factor through a. Hence f factors through a, a contradiction. Thus for each $f\colon A \twoheadrightarrow F$ in $\mathcal{A}_c$, the R-lattice F is indecomposable. Let $\mathcal{A}_r$ be the set of $f \in \mathcal{A}_c$ which are maximal with respect to $<_r$ in $\mathcal{A}_c$. Then Theorem 4 implies that for every $e \in \mathcal{A}_c \setminus \mathcal{A}_r$ there exists some $f \in \mathcal{A}_r$ with $e <_r f$. Now let $\mathcal{A}_k$ be the set of $f \in \mathcal{A}_r$ which are maximal with respect to $<_k$ in $\mathcal{A}_r$. Since R is lattice-finite, Theorem 1 shows that $\mathcal{A}_k \neq \varnothing$. We claim that every $e \in \mathcal{A}_k$ is maximal with respect to $<$ in $\mathcal{A}$. Suppose that $e < f$, say, $f = ge$. By the above, we can assume $f \in \mathcal{A}_r$. Then $g = drc$ with a cokernel c, a regular morphism r, and a kernel d. Since $e \in \mathcal{A}_c$, it follows that c is invertible. If r is not invertible, then $e <_r rce$. Hence $e \in \mathcal{A}_r$ implies that $rce \notin \mathcal{A}_c$. So we find some $e' \in \mathcal{A}_c$ with $rce <_c e'$. Thus $e <_r e'$, a contradiction. Therefore, g is a kernel. Since $f \in \mathcal{A}_r$, this gives a contradiction. By Lemma 4, we get a right almost split sequence $A' \rightarrowtail B' \twoheadrightarrow C$. By the dual argument, we get for each non-injective indecomposable R-lattice A a right almost split sequence $A \rightarrowtail B \twoheadrightarrow C$ in R-lat. Hence R-lat is a strict τ-category. □

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UNRELATED PAIRS OF MODULES

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Throughout this paper all rings will have non-zero identities and all modules will be unitary modules, A will always denote a ring.

INTRODUCTION

In this paper we introduce a new concept of 'unrelated pair', improve some results of Stephenson and Erdoğdu (see References) and show new results.

Definition 1. Two left A-modules M_1 and M_2 are said to form an *unrelated pair* when the following condition holds:

If there are submodules $N_i \leq L_i \leq {}_AM_i$ $(i = 1, 2)$ *such that* $L_1/N_1 \simeq L_2/N_2$, *then* $L_1 = N_1$ *and* $L_2 = N_2$.

In Section 1 we consider the basic properties of unrelated pairs and give equivalent conditions of unrelated pairs (see Main Theorem 1).

In Section 2 we apply results of Section 1 to obtain extensions of Erdoğdu's results on semidistributive modules and show relations among serial modules and distributive modules, Bezout modules.

Definition 2. Let M be a left A-module. Then we define the following.

(1) ${}_AM$ is said to be distributive if the lattice $\mathrm{Lat}({}_AM) = \{N | N \leq {}_AM\}$ is distributive.
(2) ${}_AM$ is said to be semidistributive if there are distributive modules M_i $(i \in I)$ such that ${}_AM = \oplus_{i \in I} M_i$.
(3) ${}_AM$ is said to be uniserial if the lattice $\mathrm{Lat}({}_AM)$ is a chain.
(4) ${}_AM$ is said to be serial if there are uniserial modules M_i $(i \in I)$ such that ${}_AM = \oplus_{i \in I} M_i$.
(5) ${}_AM$ is said to be a Bezout module if all finitely generated submodules of ${}_AM$ are cyclic.

By results obtained in Section 3, other properties of unrelated pairs (see Main Theorem 2), structures of invariant modules and distributive modules on a commutative ring are shown.

1. UNRELATED PAIRS

Section 1 では、以下の Main Theorem を求めます。これによって unrelated pair の基本的な性質が判明します。

The final version of this paper will be submitted for publication elsewhere.

Main Theorem 1. *Let ${}_AM$ be a direct sum of two left A-modules M_1, M_2. Then the following conditions are equivalent.*

(1) *${}_AM_1$ and ${}_AM_2$ form an unrelated pair.*
(2) *For any submodule $N \leq {}_AM$, there are $N_i \leq {}_AM_i$ $(i = 1, 2)$ such that ${}_AN = N_1 \oplus N_2$.*
(3) *If there are submodules $N \leq X \leq L \leq {}_AM$ such that $X \cap M_2 = N \cap M_2$ and $X + (L \cap M_2) = L$, then $X = (L \cap M_1) \oplus (N \cap M_2)$.*
(4) *If there are modules $N_i \leq L_i \leq {}_AM_i$ $(i = 1, 2)$ and $X \leq {}_AL_1 \oplus L_2$ such that $X\pi \oplus L_2\pi = L_1\pi \oplus L_2\pi$ where π is the natural epimorphism from M to $M/(N_1 \oplus N_2)$, then $X\pi = L_1\pi$.*
(5) *L_1/N_1 are not isomorphic to L_2/N_2 for any simple subfactors L_i/N_i of ${}_AM_i$ $(i = 1, 2)$.*
(6) *$A = \ell_A(m_1) + \ell_A(m_2)$ for any elements $m_1 \in M_1$, $m_2 \in M_2$.*
(7) *$A(m_1 + m_2) = Am_1 \oplus Am_2$ for any elements $m_1 \in M_1$, $m_2 \in M_2$.*
(8) *$A(m_1 + m_2) = (Am_1 \cap A(m_1 + m_2)) \oplus (Am_2 \cap A(m_1 + m_2))$ for any elements $m_1 \in M_1$, $m_2 \in M_2$.*
(9) *$A(m_1 + m_2)\nu_i \leq A(m_1 + m_2)$ for any elements $m_1 \in M_1$, $m_2 \in M_2$ where ν_i is a projection from $M = M_1 \oplus M_2$ to M_i $(i = 1, 2)$.*

まず初めに、unrelated pair に関する簡単な同値関係と、direct summand に関する結果を示します。

Remark 1.1. Let M_1, M_2 be left A-modules. Then the following conditions are equivalent.

(1) ${}_AM_1$ and ${}_AM_2$ form an unrelated pair.
(2) $\mathrm{Hom}_A(L_1/N_1, L_2/N_2) = 0$ for any submodules $N_1 \leq L_1 \leq {}_AM_1$, $N_2 \leq L_2 \leq {}_AM_2$.
(3) $\mathrm{Hom}_A(N_1, M_2/N_2) = 0$ for any submodules $N_1 \leq {}_AM_1$, $N_2 \leq {}_AM_2$.

Lemma 1.1. *Let ${}_AM$ be a direct sum of two left A-modules M_1, M_2 and $N_i \leq L_i \leq {}_AM_i$ be submodules $(i = 1, 2)$, φ be a homomorphism from L_1/N_1 to L_2/N_2. Then the following conditions hold.*

(1) *If φ is non-zero, then there is a submodule $X \leq {}_AL_1 \oplus L_2$ such that $N_1 \oplus N_2 < X$, $X \cap L_2 = N_2$, $X + L_2 = L_1 \oplus L_2$, $X \neq L_1 \oplus N_2$.*
(2) *If φ is an isomorphism, then there is a submodule $X \leq {}_AL_1 \oplus L_2$ such that $X \cap L_i = N_i$ and $X + L_i = L_1 \oplus L_2$ $(i = 1, 2)$.*

この 2 つの事実から以下が求められます。

Proposition 1.2. *Let ${}_AM$ be a direct sum of two left A-modules M_1, M_2. Then the following conditions are equivalent.*

(1) *${}_AM_1$ and ${}_AM_2$ form an unrelated pair.*
(2) *For any submodule $N \leq {}_AM$, there are $N_i \leq {}_AM_i$ $(i = 1, 2)$ such that ${}_AN = N_1 \oplus N_2$.*
(3) *If there are submodules $N \leq X \leq L \leq {}_AM$ such that $X \cap M_2 = N \cap M_2$ and $X + (L \cap M_2) = L$, then $X = (L \cap M_1) \oplus (N \cap M_2)$.*

(4) *If there are submodules* $N \leq X \leq L \leq {}_AM$ *such that* $X \cap M_1 = N \cap M_1$ *and* $(L \cap M_1) + X = L$, *then* $X = (N \cap M_1) \oplus (L \cap M_2)$.

また、同様の証明方法により、次の結果も得ることが出来ます。

Corollary 1.3. *Let* ${}_AM$ *be a direct sum of two left* A*-modules* M_1, M_2. *Then the following conditions are equivalent.*

(1) ${}_AM_1$ *and* ${}_AM_2$ *form an unrelated pair.*
(2) *If there are modules* $N_i \leq L_i \leq {}_AM_i$ $(i = 1, 2)$ *and* $X \leq {}_AL_1 \oplus L_2$ *such that* $X\pi \oplus L_2\pi = L_1\pi \oplus L_2\pi$ *where* π *is the natural epimorphism from* M *to* $M/(N_1 \oplus N_2)$, *then* $X\pi = L_1\pi$.
(3) *If there are modules* $N_i \leq L_i \leq {}_AM_i$ $(i = 1, 2)$ *and* $X \leq {}_AL_1 \oplus L_2$ *such that* $L_1\pi \oplus X\pi = L_1\pi \oplus L_2\pi$ *where* π *is the natural epimorphism from* M *to* $M/(N_1 \oplus N_2)$, *then* $X\pi = L_2\pi$.

これは一般に以下の形で考えられます。

Remark 1.2. Let ${}_AM$ be a left A-module and N be a direct summand of ${}_AM$. Then the following conditions are equivalent.

(1) The direct complement of N in ${}_AM$ is uniquely determined.
(2) $\mathrm{Hom}_A(M/N, N) = 0$.

更に次の結果も示しておきます。

Remark 1.3. Let M be a left A-module and m_1, m_2 be elements of M. If $Am_1 \cap Am_2 = 0$, then the following conditions are equivalent.

(1) $A = \ell_A(m_1) + \ell_A(m_2)$ where $\ell_A(m_i) = \{a \in A \mid am_i = 0\} \leq {}_AA$.
(2) $A(m_1 + m_2) = Am_1 \oplus Am_2$.
(3) $A(m_1 + m_2) = (Am_1 \cap A(m_1 + m_2)) \oplus (Am_2 \cap A(m_1 + m_2))$.
(4) $A(m_1 + m_2)\nu_i \leq A(m_1 + m_2)$ where ν_i is a projection from $Am_1 \oplus Am_2$ to Am_i $(i = 1, 2)$.

以上の結果から、以下を得ることが出来ます。

Proposition 1.4. *Let* ${}_AM$ *be a direct sum of two left* A*-modules* M_1, M_2. *Then the following conditions are equivalent.*

(1) ${}_AM_1$ *and* ${}_AM_2$ *form an unrelated pair.*
(2) L_1/N_1 *are not isomorphic to* L_2/N_2 *for any simple subfactors* L_i/N_i *of* ${}_AM_i$ $(i = 1, 2)$.
(3) $A = \ell_A(m_1) + \ell_A(m_2)$ *for any elements* $m_1 \in M_1$, $m_2 \in M_2$.
(4) $A(m_1 + m_2) = Am_1 \oplus Am_2$ *for any elements* $m_1 \in M_1$, $m_2 \in M_2$.

以上により、Main Theorem 1 が成立することがわかります。また、それにより以下が分かります。

Corollary 1.5. *Let* ${}_AM$ *be a direct sum of two left* A*-modules* M_1, M_2. *If* ${}_AM$ *is invariant* (*every submodule of* ${}_AM$ *is a right* $\mathrm{End}({}_AM)$*-module.*), *then* ${}_AM_1$ *and* ${}_AM_2$ *form an unrelated pair.*

2. Distributive modules

Section 2 では、distributive module と semidistributive module の関係について、unrelated pair を用いていくつかの結果を示します。

Remark 2.1. Let M be a left A-module. Then we have the following conditions.

(1) If ${}_AM$ is a uniserial module, then ${}_AM$ is serial and distributive.
(2) If ${}_AM$ is serial or distributive, then ${}_AM$ is a semidistributive module.
(3) If ${}_AM$ is uniserial, then ${}_AM$ is a Bezout module.

まず初めに、unrelated pair に関する結果の簡単な拡張を求めます。

Remark 2.2. Let ${}_AM$ be a direct sum of left A-modules M_i ($i \in F$, F is a finite set). If M_i and M_j form an unrelated pair for all distinct subscripts $i, j \in F$, then M_k and $\oplus_{i \in F \setminus k} M_i$ form an unrelated pair for any subscript $k \in F$.

これにより、以下が得られます。

Lemma 2.1. *Let ${}_AM$ be a direct sum of left A-modules M_i ($i \in I$). Then the following conditions are equivalent.*

(1) *${}_AM_i$ and ${}_AM_j$ form an unrelated pair for all distinct subscripts $i, j \in I$.*
(2) *For any submodule $N \leq {}_AM$ there are $N_i \leq {}_AM_i$ ($i \in I$) such that ${}_AN = \oplus_{i \in I} N_i$.*

上記の Lemma により、以下の 2 つの結果を得ることが出来ます。

Theorem 2.2. *Let ${}_AM = \oplus_{i \in I} M_i$ be a semidistributive module with distributive modules M_i ($i \in I$). Then the following conditions are equivalent.*

(1) *${}_AM$ is distributive.*
(2) *For all distinct subscripts $i, j \in I$ and for any elements $m_i \in M_i$, $m_j \in M_j$, $Am_i \oplus Am_j$ is distributive.*
(3) *${}_AM_i$ and ${}_AM_j$ form an unrelated pair for all distinct subscripts $i, j \in I$.*

Lemma 2.3. *Let ${}_AM = \oplus_{i \in I} M_i = \oplus_{h \in H} N_h$ be a sum of indecomposable modules M_i, N_h ($i \in I, h \in H$). If ${}_AM_i$ and ${}_AM_j$ form an unrelated pair for all distinct subscripts $i, j \in I$, then there is a bijection φ from H to I such that $N_h = M_{(h)\varphi}$.*

また、上記の結果から、更にいくつかの事実が簡単に求められます。

Proposition 2.4. *Let ${}_AM$ be a serial module. If ${}_AM$ is distributive,then the following conditions hold.*

(1) *${}_AM$ is a Bezout module.*
(2) *The decomposition of ${}_AM$ with respect to uniserial modules is uniquely determined.*

Corollary 2.5. *Let ${}_AM$ be a module with $\ell_A(m) = 0$ for any non-zero element $m \in M$. If ${}_AM$ is a distributive module or a Bezout module, then ${}_AM$ is indecomposable.*

Remark 2.3. Let ${}_AM$ be a serial module with $\ell_A(m) = 0$ for any non-zero element $m \in M$. Then the following conditions are equivalent.

(1) ${}_AM$ is a distributive module.

(2) ${}_AM$ is a Bezout module.
(3) ${}_AM$ is a uniserial module.

3. Finitely generated modules over a commutative ring

Section 3 では、commutative ring 上の finitely generated module に関する結果を示します。

Remark 3.1. Let A be a commutative ring and M_1, M_2 be finitely generated A-modules. Then the following conditions are equivalent.

(1) ${}_AM_1$ and ${}_AM_2$ form an unrelated pair.
(2) $A = \ell_A(M_1) + \ell_A(M_2)$ where $\ell_A(M_i) = \{a \in A |\ aM_i = 0\} \leq {}_AA$

上記 Remark から、以下を求めることが出来ます。

Main Theorem 2. *Let A be a commutative ring, M be a finitely generated A-module and N be a finitely generated submodule of ${}_AM$. If ${}_AN$ and ${}_AM/N$ form an unrelated pair, then the following conditions hold.*

(1) *N is a direct summand of ${}_AM$.*
(2) *The direct complement of N in ${}_AM$ is uniquely determined.*
(3) *There is a cyclic ideal $I \leq {}_AA$ such that $N = IM$.*

また、Main Theorem 2 が non-commutative ring 上では成り立たない例を次にあげます。

Example 1. Let K be a field. And we put A, M, N as follows.

$$A := \begin{pmatrix} K & & & & & \\ K & K & & & & \\ \vdots & \cdots & \ddots & & & \\ K & \cdots & \cdots & K & & \\ \vdots & \cdots & \cdots & \cdots & \ddots & \\ K & \cdots & \cdots & \cdots & \cdots & K \end{pmatrix}, \quad M := \begin{pmatrix} K \\ \vdots \\ K \\ K \\ \vdots \\ K \end{pmatrix}, \quad N := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ K \\ \vdots \\ K \end{pmatrix}$$

Then the following conditions hold.

(1) ${}_AN$ and ${}_AM/N$ form an unrelated pair.
(2) ${}_AN$ is not a direct summand of ${}_AM$.

また、Main Theorem 2 から、簡単に以下の結果を得ることが出来ます。

Remark 3.2. Let A be a commutative ring, M be a finitely generated A-module and N be a finitely generated submodule of ${}_AM$. If ${}_AN$ and ${}_AM/N$ form an unrelated pair, then the following conditions are equivalent.

(1) ${}_AM$ is a distributive module.
(2) ${}_AN$ and ${}_AM/N$ are distributive modules.

Corollary 3.1. *Let A be a commutative ring, M be a finitely generated A-module and N be a finitely generated submodule of ${}_AM$. If ${}_AM$ be distributive or invariant, then the following conditions are equivalent.*

(1) *N has a direct complement in $_AM$*
(2) *$_AN$ and $_AM/N$ form an unrelated pair.*

Corollary 3.2. *Let A be a commutative ring and M be a noetherian A-module. If $_AM$ has a non-zero small submodule, then* $\mathrm{Rad}(_AM)$ *and* $\mathrm{Top}(_AM)$ *don't form an unrelated pair.*

また、更に以下の結果を得ることが出来ます。

Theorem 3.3. *Let A be a commutative ring and M be a finitely generated faithful A-module, $\mathcal{A}$ be a set of idempotents of A and $\mathcal{R}$ be a set of idempotents of $R \doteqdot$* $\mathrm{End}(_AM)$. *If $_AM$ is distributive or invariant, then there is a bijection*

$$\varphi: \mathcal{A} \to \mathcal{R};\ e \mapsto \varepsilon = \pi_e \iota_e$$

where π_e is a projection from $_AM$ to eM and ι_e is an inclusion from eM to $_AM$.

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