Proceedings of the 35th Symposium on Ring Theory and Representation Theory

October 12-14, 2002

Okayama, Japan

Edited by

Yasuyuki Hirano

Okayama University

January, 2003

Okayama, Japan

第３５回 環論および表現論シンポジウム報告集

２００２年 １０ 月12 日−１４ 日

岡山大学環境理工学部棟

２００３年1月

岡山大学

Organizing Committee of The Symposium on
Ring Theory and Representation Theory

The Symposium on Ring Theory and Representation Theory has been held annually in Japan and the Proceedings have been published by the organizing committee. The first Symposium was organized in 1968 by H. Tominaga, H. Tachikawa, M. Harada and S. Endo. After their retirement, in 1977, a new committee was organized for managing the Symposium. The present members of the committee are Y. Hirano (Okayama Univ.), Y. Iwanaga (Shinshu Univ.), S. Koshitani (Chiba Univ.) and K. Nishida (Shinshu Univ.).

The Proceedings of each Symposium is edited by program organizer. Anyone who wants these Proceedings should ask the program organier of each Symposium or one of the committee members.

The next Symposium in 2003 will be held in Hirosaki and its program will be arranged by M. Sato.

Concerning several informations on ring theory group in Japan containing schedules of meetings and symposiums, you can see on the following homepage:

http://fuji.cec.yamanashi.ac.jp/~ring/ (Japanese)
http://civil2.cec.yamanashi.ac.jp/~ring/japan/ (English)

which is arranged by M. Sato of Yamanashi University.

Yasuo Iwanaga
Nagano, Japan
December, 2002

CONTENTS

PREFACE

The 35th Symposium on Ring Theory and Representation Theory was held in Okayama on October 12th - 14th, 2002. The symposium and these proceedings are financially suported by Grant-in Aid for Scientific Research (B)(1) from Japan Society for the Promotion of Science through the arrangements by Professor Kenji Nishida of Shinshu University.

This volume consists of twentyfive articles presented at the symposium. It includes a series of lectures by Kenneth R. Goodearl on "Quantized coordinated rings and related noetherian algebras". We would like to thank all speakers and their coauthors for their contributions.

We would like to thank Professors Hisaaki Fujita, Yasuo Iwanaga, Shigeo Koshitani and Kenji Nishida for their helpful suggestions concerning the symposium. Finally we should like to express our gratitude to Professor Ikehata and his students of Okayama University who contributed in the organization of the symposium.

Yasuyuki Hirano
Okayama
January, 2003

List of Lectures (* = speaker)

柳井 忠 (新居浜工業高等専門学校)
ホップ加群の双対性とその応用

和久井 道久 (大阪大学大学院理学研究科)
9 次元以下の半単純でないホップ代数の表現環について

眢本 義人
ネーター環を特徴づける一つの条件とその双対

Kenneth R. Goodearl (University of California at Santa Barbara)
Quantized coordinate rings and related noetherian algebras I, II, III

増岡 彰 (筑波大学数学系)
接合積からのホップ代数入門

土井 幸雄 (岡山大学教育学部)
群環的代数

星野 光男 (筑波大学数学系), 加藤 義明* (筑波大学大学院数学研究科)
An elementary construction of tilting complexes

藤田 尚昌 (筑波大学数学系)
Neat idempotents and tiled orders having large global dimension

長瀬 潤 (大阪市立大学大学院理学研究科)
Algebra homomorphisms and Hochschild cohomology

速水 孝夫*(東京理科大学大学院理学研究科), 真田 克典 (東京理科大学理学部)
Cohomology rings of the generalized quaternion group

奥山 京 (鳥羽商船高等専門学校)
無限アーベル群における mixed 群について

小林 滋*, 丸林 英俊 (鳴門教育大学), Nicolae Popescu (Institute of Mathematics of the Romanian Academy), Guangming Xie (鳴門教育大学)
Total valuation rings of Ore extensions

Miguel A. Ferrero (Universidade Federal do Rio Grande do Sul)
The unitary strongly prime rings and related radicals

Nicolae Popescu (Institute of Mathematics of the Romanian Academy)
Transitive Galois action on plane compacts

荒谷 督司*, 髙橋 亮 (岡山大学大学院自然科学研究科), 吉野 雄二 (岡山大学理学部)
非可換環上の Cohen-Macaulay 次元

髙橋 亮* (岡山大学大学院自然科学研究科), 吉野 雄二 (岡山大学理学部)
Looking at homological dimensions through Frobenius map

吉川 昌慶 (信州大学大学院工学系研究科)
ハミング・アソシエーションスキームのモジュラー隣接代数

宮本 雅彦 (筑波大学数学系)
Symmetric algebra and modular invariance of VOA

水川 裕司* (北海道大学大学院理学研究科), 山田 裕史 (岡山大学理学部)
Kac-Moody Lie 環の基本表現と Schur 函数

Ziqun Lu (魯 自群)(北京大学及び千葉大学理学部)
Monomial modules and endo-monomial modules

本瀬 香 (弘前大学理工学部)
On the nilpotency index of the radical of a group algebra

倉富 要輔 (山口大学理工学研究科)
Direct sums of lifting modules

千葉 克夫 (新居浜工業高等専門学校)
自由体とその付値

中島 晴久 (城西大学理学部)
Equidimensional actions of algebraic tori on graded normal domains

関口 勝右 (国士舘大学工学部)
Extensions of some 2-groups which preserve the irreducibilities of induced characters

HOPF MODULE DUALITY AND ITS APPLICATION

TADASHI YANAI

ABSTRACT. Let H be a pointed Hopf algebra and A a right H-comodule algebra. We see that if A satisfies certain conditions, then A is isomorphic to $\mathrm{Hom}_{A^{\mathrm{co}H}-}(A, A^{\mathrm{co}H})$ in ${}_A\mathcal{M}^H_{A^{\mathrm{co}H}}$ through an appropriate twist of $A^{\mathrm{co}H}$-module and H-comodule structures. This duality induces some properties on generalized integrals in right H-comodule subalgebras of $D\#H$ including D, where D is a left H-module division algebra, and furthermore, a Galois-type correspondence theorem for X-outer actions of finite dimensional pointed Hopf algebras on prime algebras.

1. 序文

V. Kharchenko によるガロア理論は環の自己同型写像や derivation，及びそれらのなす群やリー代数についての様々な興味深い性質を見出している（[K; P, Chapter 7]）．Kharchenko の理論をホップ代数の作用へ拡張することはホップ代数の研究対象のひとつであり，[Mo, Sect. 6.4; Mi1] などにその結果を見ることができる．その発展として，余可換とは限らないホップ代数の作用によるガロア型対応定理の構築という課題が考えられる．目標は，素多元環 R に有限次分裂ホップ代数 H が X-外部的に作用する場合に，不定元のなす部分環 R^H を含む R の rationally complete な部分代数と，R の対称的マルチンデール商環の中心 K を含む $K\#H$ の H-余加群部分代数が 1 対 1 に対応することを証明し，Kharchenko の結果を含むガロア対応の定理を得ることである．この主張は[Y1] で H が Sweedler の 4 次元ホップ代数の場合に正しいことが示され，以降様々な条件の付いたホップ代数で調べられてきた（[Y2; Y3; Y4]）が，S. Westreich の研究[W]，および共同研究[WY] によって大きく前進し，一般の有限次分裂ホップ代数に対して，対応を与える写像は単射になること，基礎体の標数が 0 である場合は全射にもなることが証明できた．これによって標数 0 のときは問題は解決を見た．しかしながら，標数 0 の仮定では，制限リー代数の制限包絡代数という基本的な例がカバーできておらず，Kharchenko が示した derivation のなすある種の制限リー代数のガロア理論（[K, Thm. 4.5.2]）と対応できる部分がないことになる．従って，制限リー代数の制限包絡代数も含むように，任意標数で対応を与える写像の全射性を証明することが課題として残っていた．

1991 *Mathematics Subject Classification.* 16W30.
The detailed version of this paper has been submitted for publication elsewhere.

この度，筑波大の増岡彰氏との共同研究[MY] により，問題の残された部分が，あるホップ加群の双対性を証明する（Theorem 3.2）ことで解決され，当初の目標の形（任意標数）でガロア型対応定理の完成を見ることができた（Theorem 2.4）．証明に使われた双対性は，有限次ホップ代数 H がその双対 H^* と H-ホップ加群として同型になるという事実を，あるホップ加群に拡張したもので，(拡張された）積分の性質（Proposition 3.3）を含むホップ加群の特徴づけに応用できる．ここでは[MY] で得られた結果のうち，ガロア対応の問題と双対性に関して得られた結果，および対応定理の証明について報告する．

2．記号，定義，対応定理

以降，任意標数の体 $\mathbf{k}$ を基礎体として進める．$\otimes$ は $\otimes_{\mathbf{k}}$ を表す．ホップ代数を H で表し，その余積，余単位元，アンチポードをそれぞれ Δ, ε, S と書く．H の極小部分余代数が全て 1 次元空間であるとき，H は分裂 (pointed) であると言う．例えば，群代数や，リー代数の普遍包絡代数，正標数の制限リー代数の制限包絡代数，Sweedler の 4 次元ホップ代数（[Mo, Ex. 1.5.6]），$sl(2)$ の量子包絡代数 $U_q(sl(2))$（[Mo, p.217]）などは分裂ホップ代数の例である．

多元環 A が左 H-加群で $h\cdot(ab)=\sum(h_1\cdot a)(h_2\cdot b)$（$h\in H, a,b\in A$）となるとき，$A$ は左 H-加群代数であると言う．以降，H が多元環に作用すると言えば，多元環が左 H-加群代数となることを意味する．また，$\mathbf{k}$-空間 M に対し，写像 $\rho_M : M\to M\otimes H$ で $(id_M\otimes\Delta)\circ\rho_M(m)=(\rho_M\otimes id_H)\circ\rho_M(m)$，$(id_M\otimes\varepsilon)\circ\rho_M(m)=m\otimes 1$（$m\in M$）を満たすものが存在するとき，$M$ は右 H-余加群であると言う．ρ_M による像を $\rho_M(m)=\sum m_0\otimes m_1$（$m\in M$）と書く．以降，$H$-余加群と言えば右 H-余加群を意味するものとする．多元環 A が $\rho_A : A\to A\otimes H$ により H-余加群で，$\rho_A(ab)=\sum a_0b_0\otimes a_1b_1$（$a,b\in A$），$\rho_A(1)=1\otimes 1$ を満たすとき A は H-余加群代数と言う．

R を素多元環とし，Q をその対称的マルチンデール商環（[Mo, Def.6.4.2; P, Sect. 10]），K を Q の中心とする．この Section では H は有限次分裂ホップ代数で，R は左 H-加群代数であるとする．$R^H:=\{r\in R|h\cdot r=\varepsilon(h)r,\ \forall h\in H\}$ は R の部分環になる．R^H の元を H の作用による不変元 (invariant) と言う．

次のことが知られている．

Proposition 2.1.

(1) K は体になる（[P, Lemma 10.9]）．

(2) H の R への作用は Q にまで拡張できる（[Mo, Prop.6.4.5, Thm. 6.4.6]）．

2.1(2) の事実から Q と H のスマッシュ積代数 $Q\#H$ （skew group ring のホップ代数への拡張，[Mo, Def. 4.1.3]）が定義でき，$Q\simeq Q\#1$，$H\simeq 1\#H$ を通して Q や H は $Q\#H$ に含まれていると考える．$Q\#H$ は $\rho_{Q\#H}=id_Q\otimes\Delta$ で H-余加群代数になる．

$Q\#H$ の部分集合 X,Y に対して $C_X(Y)=\{x\in X|xy=yx,\forall y\in Y\}$ と定義する．一般的には $C_{Q\#H}(R)\supset K$ となるが，例えば $H=\mathbf{k}G$（G は R の自己同型写像のなす有限群）で，1 以外の G の元が Q の外部自己同型写像 (X-outer automorphism) となる場合

は $C_{Q\#H}(R)=K$ となる．そこで（一般の H に対して），$C_{Q\#H}(R)=K$ が成り立つとき，H の作用は X-外部的であると言う（[Mi3, Def.4.4]）．H の作用が X-外部的であるとき，$H\cdot K\subset K$ となることが分かっており（[Mi2, Bemerkung 15.3]），従って $Q\#H$ の H-余加群部分代数であるスマッシュ積代数 $K\#H$ を作ることができる．

$a,x\in Q, h\in H$ に対して，$(a\#h)\cdot x=a(h\cdot x)$ と定義することにより，Q は左 $Q\#H$-加群になる．次の結果は Kharchenko の differential identity with automorphisms の理論（[K, Chapter 2]）のホップ代数の作用への一般化で，対応定理の証明に使われる．

Lemma 2.2 [Mi1, Thm. 4.1, p. 333]. H を素多元環 R に X-外部的に作用している有限次分裂ホップ代数，Q を R の対称的マルチンデール商環とする．$\xi\in Q\#H$ と 0 でない R のイデアル I に対して，$\xi\cdot I=0$ なら，$\xi=0$ となる．

R の部分環 U が，$r\in R$ と U の 0 でないイデアル I に対して $rI\subset U\Rightarrow r\in U$ を満たすとき，U は rationally complete と言う．

次のことが成り立つ．

Proposition 2.3. H を素多元環 R に X-外部的に作用している有限次分裂ホップ代数，Q を R の対称的マルチンデール商環，K を Q の中心とする．

(1) R^H を含む部分集合 $U\subset R$ に対して，$C_{K\#H}(U)$ は K を含む $K\#H$ の H-余加群部分代数．

(2) K を含む部分集合 $A\subset K\#H$ に対して，$C_R(A)$ は R^H を含む R の rationally complete 部分代数．

そこで，次の 2 つの集合を用意する．

- $\mathcal{U}_{R^H/R}$: R^H を含む R の rationally complete 部分代数全体の集合，
- $\mathcal{A}_{K/K\#H}$: K を含む $K\#H$ の H-余加群部分代数全体の集合．

Proposition 2.3 から，写像

$$\Phi: U\mapsto C_{K\#H}(U),\quad \Psi: A\mapsto C_R(A)$$

は $\mathcal{U}_{R^H/R}$ と $\mathcal{A}_{K/K\#H}$ の間に対応を与えていることが分かる．

再び $H=\mathbf{k}G$ の場合は，$\Phi(U)=K\#\mathbf{k}G^U$（G^U は U の元を固定する G の部分群）で，$A\in\mathcal{A}_{K/K\#H}$ は $K\#\mathbf{k}G'$（G' は G の部分群）と書けて $\Psi(A)=R^{G'}$（G' で固定される R の部分環）となるから（[Y1, Sect. 5]），Φ,Ψ による対応は従来のガロア対応と同じものになる．

そこで，この対応が果たして 1 対 1 になっているかということが問題になる．すなわち，任意の $U\in\mathcal{U}_{R^H/R}$ と $A\in\mathcal{A}_{K/K\#H}$ に対して次の等式が成り立つかどうかを考えたい．

$$\Phi\circ\Psi(U)=U. \tag{$*$}$$

$$\Psi\circ\Phi(A)=A. \tag{$**$}$$

これらの等式が示されれば，次の定理が証明されることになる．

Theorem 2.4（対応定理）[MY, Thm. 3.5]. H を素多元環 R に X-外部的に作用している有限次分裂ホップ代数，Q を R の対称的マルチンデール商環，K を Q の中心とする．このとき，写像 Φ, Ψ により，集合 $\mathcal{U}_{R^H/R}$ と $\mathcal{A}_{K/K\#H}$ の間に 1 対 1 の対応が与えられる．

Theorem 2.4 は，$H = \mathbf{k}G$ の場合は Kharchenko によるガロア対応の定理[K, Thm. 3.10.2] と同じになる．また，$H = u(\mathfrak{g})$（有限次制限リー代数 $\mathfrak{g}$ の制限包絡代数）のときは，Kharchenko による differential リー K-代数のガロア型対応定理[K, Thm. 4.5.2] を $K\#\mathfrak{g}$ に当てはめたものと同じになる（[MY, Remark 3.7]）．[W; WY] では $(*)$ は任意標数で成立し（すなわち Φ は単射となる），$(**)$ は $\mathrm{Char}\,\mathbf{k} = 0$ のとき成立することが証明された．そこで，任意標数で $(**)$ が成立する（すなわち Φ が全射となる）ことを示すことが課題となる．その部分は，次の Section でホップ加群の双対性に関する結果を述べたあと，Section 4. で証明する．

3. 双対性，β-フロベニウス拡大，積分

ここでは H は必ずしも有限次分裂とは限らないホップ代数とする．H から基礎体 $\mathbf{k}$ への線形写像全体 $\mathrm{Hom}(H, \mathbf{k})$ を H^* で表す．H-余加群のなすカテゴリーを $\mathcal{M}^H$ と書く．$M \in \mathcal{M}^H$ のとき，$h^* \in H^*, m \in M$ に対して，$h^* \rightharpoonup m = \sum h^*(m_1)m_0$ によって M は左 H^*-加群となる．

以降，A は H-余加群代数であるとする．$M \in \mathcal{M}^H$ が左 A-加群で，$a \in A, m \in M$ に対して $\rho_M(am) = \rho_A(a)\rho_M(m)$ となるとき，$M \in {}_A\mathcal{M}^H$ と表す．$M \in \mathcal{M}_A^H$ も同様に決める．(このように加群と余加群の構造をあわせ持つものを「ホップ加群」と呼ぶことにする.)

$A^{\mathrm{co}H} := \{a \in A | \rho_A(a) = a \otimes 1\}$ は A の H-余加群部分代数になる．$D = A^{\mathrm{co}H}$ とする．$M \in \mathcal{M}^H$ が両側 (A, D)-加群で $M \in {}_A\mathcal{M}^H$ かつ $M \in \mathcal{M}_D^H$ のとき，$M \in {}_A\mathcal{M}_D^H$ と書く．さらに，M から D への右 D-線形写像全体を $\mathrm{Hom}_{-D}(M, D)$ と書く．$M \in {}_D\mathcal{M}_A^H$，$\mathrm{Hom}_{D-}(M, D)$ も同様に定義する．

有限次ホップ代数や分裂ホップ代数のアンチポード S は全単射になる（[Mo, Thm. 2.1.3, Cor. 5.2.11]）．このとき，S の逆写像を $\bar{S}$ で表す．

Proposition 3.1. [MY, Prop. 2.1] H をアンチポードが全単射であるホップ代数，A を H-余加群代数，$D = A^{\mathrm{co}H}$ とする．

(1) $M \in {}_A\mathcal{M}_D^H$ が有限生成射影的 D-加群のとき，$\mathrm{Hom}_{-D}(M, D) \in {}_D\mathcal{M}_A^H$ となる．
(2) $M \in {}_D\mathcal{M}_A^H$ が有限生成射影的 D-加群のとき，$\mathrm{Hom}_{D-}(M, D) \in {}_A\mathcal{M}_D^H$ となる．

証明（概略）．$M \in {}_A\mathcal{M}_D^H$ が有限生成射影的 D-加群とする．$\varphi \in \mathrm{Hom}_{-D}(M, D)$ に対して $(x\varphi a)(m) = x\varphi(am)$（$x \in D, a \in A, m \in M$）と両側加群の構造を決める．さらに (m_i, φ_i)（$m_i \in M, \varphi_i \in \mathrm{Hom}_{-D}(M, D)$）を dual basis として，$\mathrm{Hom}_{-D}(M, D)$ の H-余加群構造を $\varphi \mapsto \sum \varphi((m_i)_0)\varphi_i \otimes S((m_i)_1)$ によって与えることにより，(1) が示される．

$M \in {}_D\mathcal{M}_A^H$ が有限生成射影的 D-加群のとき，同様に dual basis (m'_j, ψ_j) $(m'_j \in M, \psi_j \in \mathrm{Hom}_{D-}(M,D))$ をとり，$\psi \in \mathrm{Hom}_{D-}(M,D)$ に対して $\psi \mapsto \sum \psi_j \psi((m'_j)_0) \otimes \bar{S}((m'_j)_1)$ によって $\mathrm{Hom}_{D-}(M,D)$ の H-余加群構造を与えることで (2) が導ける． □

特に $A \in {}_D\mathcal{M}_A^H$ (resp. ${}_A\mathcal{M}_D^H$) であるから，A が有限生成射影的 D-加群のときは $\mathrm{Hom}_{D-}(A,D) \in {}_A\mathcal{M}_D^H$ (resp. $\mathrm{Hom}_{-D}(A,D) \in {}_D\mathcal{M}_A^H$) となる．

$M \in \mathcal{M}^H$ のとき，H の群的元 g ($\Delta(g) = g \otimes g$, $g \neq 0$ となる元) に対して $m \mapsto \sum m_0 \otimes g m_1$，$m \mapsto \sum m_0 \otimes m_1 g$ によって M に新しい H-余加群構造が与えられる．それらをそれぞれ $M[g_L]$，$M[g_R]$ で表す．また，β が環 D の自己同型写像で M が左 D-加群のとき，$x \cdot m = \beta(x)m$ $(x \in D, m \in M)$ によって M に新しい D-加群構造を導入したものを ${}_\beta M$ で表す．

次の結果は，H-余加群代数のホップ加群としての双対性を与える．

Theorem 3.2. [MY, Thm. 2.2] H を分裂ホップ代数，A を H-余加群代数とし，A が ${}_A\mathcal{M}^H, \mathcal{M}_A^H$ の対象として単純であるとする（このとき，$A^{\mathrm{co}H}$ は斜体になる ([MY, Prop. 3.1(1)])）．A の $D := A^{\mathrm{co}H}$ 上の次元 $\dim_{D-} A, \dim_{-D} A$ のいずれか一方が有限であれば，ある D の自己同型写像 β と，H の群的元 g が存在して

(1) ${}_D\mathcal{M}_A^H$ の対象として ${}_{\beta^{-1}}A \simeq \mathrm{Hom}_{-D}(A,D)[g_L]$,
(2) ${}_A\mathcal{M}_D^H$ の対象として $A_\beta \simeq \mathrm{Hom}_{D-}(A,D)[g_R]$

となる．

証明（概略）．対称性から $\dim_{-D} A < \infty$ として構わない．A_D は有限生成射影的だから $M := \mathrm{Hom}_{-D}(A,D) \in {}_D\mathcal{M}_A^H$ となる．ここで，$\{m \in M | \rho_M(m) = m \otimes g\} \neq 0$ となる群的元 $g \in H$ をとる．(このような g がとれるところに H が分裂という仮定が使われる.) $I_g = \{h^* \in H^* | h^*(g) = 0\}$ とし，$I_g A$ を $h^* \rightharpoonup a$ $(h^* \in I_g, a \in A)$ で生成される A の H^*-部分加群とする．このとき $\dim_{D-} A/I_g A = \dim_{-D} A/I_g A = 1$ となり，両側 (D,D)-加群として $A/I_g \simeq {}_\beta D \simeq D_{\beta^{-1}}$ となる D の自己同型 β の存在が分かる．このようにして得られる g, β が上の定理を満たす． □

上の結果から，Theorem 3.2 の条件が満たされるときは，環拡大 $A^{\mathrm{co}H} \subset A$ は β-フロベニウス拡大になっていることが分かる．

この定理を次のようなホップ加群に応用する．D は斜多元環でホップ代数 H が左から作用しているとする．このとき，スマッシュ積代数 $D\#H$ は $\rho_{D\#H} = id_D \otimes \Delta$ によって H-余加群代数となる．$D \simeq D\#1$ により $D \subset D\#H$ と考える．$\hat{\varepsilon} = id_D \otimes \varepsilon : D\#H \to D$ とする．A を D を含む $D\#H$ の H-余加群部分代数とし，$\hat{\varepsilon}_A = \hat{\varepsilon}|_A$ とする．明らかに $\hat{\varepsilon}_A \in \mathrm{Hom}_{D-}(A,D)$ である．$\int_A^\ell := \{\eta \in A | a\eta = \hat{\varepsilon}(a)\eta,\ \forall a \in A\}$ の元を A の（拡張された）左積分と呼ぶ．($D = \mathbf{k}$ のとき，$\int_H^\ell$ は通常のホップ代数の左積分 ([Mo, Def. 2.1.1]) である.）次の命題はホップ代数の積分についての結果 ([Mo, Thm. 2.1.3]) や,

[Ko, Thm. 2.2] のホップ代数の右余イデアル部分代数の積分に関する考察の一般化で，対応定理証明の鍵となる．

Proposition 3.3. [MY, Prop. 2.4] 有限次分裂ホップ代数 H に対して，D, A を上のように決めるとき，次が成り立つ．

(1) $\int_A^\ell$ は A の 1 次元右 D-部分空間である．

(2) $H^* \rightharpoonup \int_A^\ell = A$ である．

証明（概略）. (1) A は ${}_A\mathcal{M}^H, \mathcal{M}_A^H$ の対象として単純となる（[MY, Prop. 1.3(1)]）ことから Theorem 3.2 が A に適用できる．$a, b \in A$ に対して，$(a\hat{\varepsilon})(b) = \hat{\varepsilon}(ba) = \hat{\varepsilon}(b\hat{\varepsilon}(a)) = (\hat{\varepsilon}(a)\hat{\varepsilon})(b)$ であるから，$a\hat{\varepsilon} = \hat{\varepsilon}(a)\hat{\varepsilon}$. 従って，3.2(2) の同型で $\hat{\varepsilon}_A D \subset \mathrm{Hom}_{D-}(A, D)$ に対応する A の部分空間が $\int_A^\ell$ であることが分かる．（この結果は H が有限次でなくても，$\dim_{D-} A$ か $\dim_{-D} A$ が有限であれば成り立つ.）

(2) Theorem 3.2(2) から，A は $\mathrm{Hom}_{D-}(A, D)[g_R]$ と左 H^*-加群として同型になる．$H^* \rightharpoonup \hat{\varepsilon}_A D = \mathrm{Hom}_{D-}(A, D)$（[MY, Lemma 2.5(2)]）であることと合わせて，求める結果を得る． □

4. 対応定理の証明

Section 3. で得られた結果を応用して対応定理（Theorem 2.4）を証明する．H を有限次分裂ホップ代数，R を H が X-外部的に作用している素多元環，Q を R の対称的マルチンデール商環，K を Q の中心とする．

Section 2. で述べたように，証明は式 (**) を任意標数で証明することで完成する．$A \in \mathcal{A}_{K/K\#H}$ とし，$A' = \Phi \circ \Psi(A)$ とする．A, A' は Proposition 3.3 の結果を満たす．

まず，$\int_A^\ell \subset \int_{A'}^\ell$ となることを示す．$\eta \in \int_A^\ell$ とする．H が分裂であることとマルチンデール商環の性質から，$\eta \cdot I \subset R$ となる R のイデアル $I(\neq 0)$ の存在が分かる（[Mo, p.97, Thm. 6.4.6]）．$r \in I$ とする．$a \in A$ に対して，η が左積分であるから，$a(\eta \cdot r) = \sum a_0 \cdot (\eta \cdot r)\#a_1 = \sum (a_0\eta) \cdot r\#a_1 = \sum (\hat{\varepsilon}(a_0)\eta) \cdot r\#a_1 = (\eta \cdot r)a$ となり，$\eta \cdot r \in \Psi(A)$ を得る．ここで $a' \in A'$ とすると，$(\eta \cdot r)a' = a'(\eta \cdot r) = \sum a'_0 \cdot (\eta \cdot r)\#a'_1$ となる．この等式の左側に $id_Q \otimes \varepsilon$ を施せば $(id_Q \otimes \varepsilon)((\eta \cdot r)a') = (\eta \cdot r)\hat{\varepsilon}(a') = (\hat{\varepsilon}(a')\eta) \cdot r$, 右側に施せば $(id_Q \otimes \varepsilon)(\sum a'_0 \cdot (\eta \cdot r)\#a'_1) = \sum a'_0 \cdot (\eta \cdot r)\varepsilon(a'_1) = a' \cdot (\eta \cdot r) = (a'\eta) \cdot r$ を得るから，$(\hat{\varepsilon}(a')\eta) \cdot r = (a'\eta) \cdot r$ となる．このことから $(\hat{\varepsilon}(a')\eta - a'\eta) \cdot I = 0$ となるので Lemma 2.2 より $\hat{\varepsilon}(a')\eta = a'\eta$ となり，$\eta \in \int_{A'}^\ell$ を得る．

従って，Proposition 3.3(1) から $\int_A^\ell = \int_{A'}^\ell$ が分かり，3.3(2) から $A = H^* \rightharpoonup \int_A^\ell = H^* \rightharpoonup \int_{A'}^\ell = A'$ となって，(**) の等式が成立する．

REFERENCES

[K] V. K. Kharchenko, *Automorphisms and derivations of associative rings*, Kluwer, Dordrecht, 1991.

[Ko] M. Koppinen, *Coideal subalgebras in Hopf algebras: freeness, integrals, smash products*, Comm. Algebra **21** (1993), 427–444.

[Mi1] A. Milinski, *Actions of pointed Hopf algebras on prime algebras*, Comm Algebra **23** (1995), 313-333.

[Mi2] A. Milinski, *Operationen punktierter Hopfalgebren auf primen Algebren* (1995), Ph. D. thesis (München).

[Mi3] A. Milinski, *X-Inner objects for Hopf crossed products*, J. Algebra **185** (1996), 390–408.

[Mo] S. Montgomery, *Hopf algebras and their actions on rings, CBMS Regional Conf. Ser. in Math.*, vol. 82, AMS, Providence, R. I., 1993.

[MY] A. Masuoka and T. Yanai, *Hopf module duality applied to X-outer Galois theory*, submitted.

[P] D. Passman, *Infinite crossed products*, Academic Press, San Diego, 1989.

[W] S. Westreich, *A Galois-type correspondence theory for actions of finite-dimensional pointed Hopf algebras on prime rings*, J. Algebra **219** (1999), 606–624.

[WY] S. Westreich and T. Yanai, *More about a Galois-type correspondence theory*, J. Algebra **246** (2001), 629–640.

[Y1] T. Yanai, *Correspondence theory of Kharchenko and X-outer actions of pointed Hopf algebras*, Comm. Algebra **25** (1997), 1713–1740.

[Y2] T. Yanai, *Kharchenko's Galois theory and actions of Hopf algebras (Japanese)*, Hopf algebras and Quantum groups, Sûrikaisekikenkyûsho Kôkyûroku 997, Kyoto Univ. RIMS, 1997, pp. 17–26.

[Y3] T. Yanai, *Non-commutative Galois theory and actions of Hopf algebras*, Proceedings of the 31st Symposium on Ring Theory and Representation Theory and Japan-Korea Ring Theory and Representation Theory Seminar (1999), 204–208.

[Y4] T. Yanai, *Galois theory for noncommutative rings and actions of pointed Hopf algebras* (1999), thesis (Okayama University).

Niihama National College of Technology
7-1, Yagumo-cho, Niihama, Ehime 792-8580
Japan

ON REPRESENTATION RINGS OF NON-SEMISIMPLE HOPF ALGEBRAS OF LOW DIMENSION

和久井道久 (MICHIHISA WAKUI)

ABSTRACT. By using the classification results on all Hopf algebras of dimension ≤ 11 due to Williams, Masuoka and Ştefan, we investigate structures of representation rings of non-semisimple Hopf algebras of dimension ≤ 9 over an algebraically closed field k of characteristic 0.

Let $\mathrm{Rep}(A)$ denote the Green ring of a finite dimensional Hopf algebra A over k. If A is a non-semisimple Hopf algebra A of dimension ≤ 9 over k, then $\mathrm{Rep}(A)$ is commutative, and the anti-ring homomorphism $* : \mathrm{Rep}(A) \longrightarrow \mathrm{Rep}(A)$ induced from the antipode of A is an involution. We prove this by determining the isomorphism classes of indecomposable modules of such a Hopf algebra.

1. 序論および主結果

近年、「自然数 n を固定したときに、n 次元のホップ代数を同型を法として分類する」という問題に続々と解答が与えられている。特に、標数 0 の代数閉体上で定義された 11 次元以下のホップ代数については、Williams [17]、増岡彰 [9, 10]、Ştefan [15] によって完全な分類結果が得られている。その結果により、標数 0 の代数閉体上で定義された 9 次元以下の半単純でないホップ代数は以下の表に挙げたホップ代数のどれか 1 つに同型であることがわかる。

次元	半単純でないホップ代数	生成元	関係式	ホップ代数構造
4	T_4	g, x	$g^2 = 1,\ x^2 = 0,\ xg = -gx$	$g \in G,\ x \in P_{g,1}$
8	A'_{C_4}	g, x	$g^4 = 1,\ x^2 = 0,\ xg = -gx$	$g \in G,\ x \in P_{g,1}$
	A''_{C_4}	g, x	$g^4 = 1,\ x^2 = g^2 - 1,\ xg = -gx$	$g \in G,\ x \in P_{g,1}$
	A'''_{C_4}	g, x	$g^4 = 1,\ x^2 = 0,\ xg = \omega_4 gx$	$g \in G,\ x \in P_{g^2,1}$
	$A_{C_2 \times C_2}$	g, h, x	$g^2=h^2=1,\ x^2=0,$ $gh=hg,\ gx=-xg,\ hx=-xh$	$g, h \in G,\ x \in P_{g,1}$
	$(A''_{C_4})^*$	g, x	$g^4 = 1,\ x^2 = 0,\ xg = \omega_4 gx$	$\Delta(g)=g\otimes g-2gx\otimes g^3x,$ $x\in P_{g^2,1}$
	A_{C_2}	g, x, y	$g^2=1,\ x^2=y^2=0,$ $gx=-xg,\ gy=-yg,\ xy=-yx$	$g \in G,\ x, y \in P_{g,1}$
9	T_{9,ω_3}	g, x	$g^3 = 1,\ x^3 = 0,\ xg = \omega_3 gx$	$g \in G,\ x \in P_{g,1}$

ここで、$\omega_n \in k$ $(n = 3, 4)$ は 1 の原始 n 乗根を表わし、$G = G(A)$ は A の群的元の全体、$P_{g,h} = P_{g,h}(A)$ は A の (g, h)-歪原始元の全体を表わす：

$$G(A) = \{g \in A \mid g \neq 0,\ \Delta(g) = g \otimes g\}, \quad P_{g,h}(A) = \{x \in A \mid \Delta(x) = x \otimes g + h \otimes x\}$$

お詫びと注意. ホップ代数として $T_{9,\omega_3} \not\cong T_{9,\omega_3^{-1}}$ なので、半単純でない 9 次元のホップ代数の同型類は 2 個ある。講演の中では、半単純でない 9 次元のホップ代数の同型類の個数は 1 としていましたが、これは誤りです。お詫び申し上げます。

このノートでは、上記の表に挙げたホップ代数の直既約加群を具体的に決定することにより、9次元以下の半単純でないホップ代数の表現環の持つ性質について調べる。単に、表現環と聞くと、いわゆる Grothendieck 環 (例えば、[12, 6] 参照) を連想される方がいるかもしれないが、ここでは Green 環の意味で用いている。Green 環を使う理由は、我々が対象とする代数は半単純でなく、Grothendieck 環が "とても小さい" ことにある。

[1]The paper is in final form and no version of it will be published elsewhere.

ホップ代数に対する Green 環の概念は、有限群 G の群環 $k[G]$ に対する Green 環の概念 [4] を自然に拡張することにより得られる。ここで、その定義を述べよう。簡単のため、体 k 上の有限次元ホップ代数 A について考える。有限次元左 A-加群の同型類全体を $\mathfrak{R}(A)$ によって表わす。有限次元左 A-加群 V に対して、その同型類を $[V]$ で書き表わすことにする。このとき、

$$[V]+[W]=[V\oplus W],\quad [V][W]=[V\otimes W]$$

によって定義される和と積に関して、$\mathfrak{R}(A)$ は単位元を持つ半環をなす。ここでは、$V\otimes W$ を A の左作用

$$a\cdot(v\otimes w)=\sum a_{(1)}v\otimes a_{(2)}w,\quad v\in V,\ w\in W, a\in A,\ \ Delta(a)=\sum a_{(1)}\otimes a_{(2)}$$

により、左 A-加群とみなしている。但し、$\Delta: A\longrightarrow A\otimes A$ は A の余積である。また、半環 $\mathfrak{R}(A)$ の単位元は $[k]$ によって与えられる。ここでは、k を A の左作用

$$a\cdot r=\varepsilon(a)r,\quad r\in k,\ a\in A$$

より、左 A-加群とみなしている。但し、$\varepsilon: A\longrightarrow k$ は A の余単位である。

$\mathfrak{R}(A)$ の半加群としての Grothendieck 加群を $\mathrm{Rep}(A)$ と書くことにする。このとき、$\mathfrak{R}(A)$ の上で述べた半環構造から、単位元を持つ環の構造が $\mathrm{Rep}(A)$ に定まる。そればかりではなく、$\mathrm{Rep}(A)$ は、A の対合 $S: A\longrightarrow A$ から誘導される反環準同型 $*: \mathrm{Rep}(A)\longrightarrow \mathrm{Rep}(A)$ を持つ。この反環準同型 $*$ は、有限次元左 A-加群 V の同値類 $[V]$ に対して、$[V^*]$ を対応させる写像として定義される。ここでは、$V^*=\mathrm{Hom}_k(V,k)$ を A の左作用

$$(a\cdot f)(v)=f(S(a)\cdot v),\quad f\in V^*,\ a\in A,\ v\in V$$

によって左 A-加群とみなしている。

以上のように定義される $*$ 付き環 $\mathrm{Rep}(A)$ を A の **Green 環**と呼ぶことにする。A は有限次元なので、$S^n=id_A$ となる自然数 n が存在する [13]。よって、$*: \mathrm{Rep}(A)\longrightarrow \mathrm{Rep}(A)$ は全単射である。また、Krull-Remak-Schmidt-東屋の定理により、$\mathrm{Rep}(A)$ は $\{[V]\mid V$ は有限次元直既約加群$\}$ を基底に持つ自由 $\mathbb{Z}$-加群である。標数 0 の代数閉体上で定義された 9 次元以下の半単純でないホップ代数の Green 環について、次が成り立つ。

定理 1.1. (1) A'''_{C_4} と $(A''_{C_4})^*$ の Green 環は同型である。
(2) T_4, A'_{C_4}, A''_{C_4}, A'''_{C_4}, $A_{C_2\times C_2}$, T_{9,ω_3} の Green 環は、以下の表のような生成元と関係式によって記述される可換環である (但し、$ab=ba$ というタイプの関係式は省略した)。

Green 環	生成元	関係式	$*$-構造	$\mathbb{Z}$-加群としての階数
$\mathrm{Rep}(T_4)$	$\chi,\ \psi$	$\chi^2=1,\ \psi^2=(1+\chi)\psi$	$\chi^*=\chi,\ \psi^*=\chi\psi$	4
$\mathrm{Rep}(A'_{C_4})$	$\chi,\ \psi$	$\chi^4=1,\ \psi^2=(1+\chi^2)\psi$	$\chi^*=\chi^{-1},\ \psi^*=\chi^2\psi$	8
$\mathrm{Rep}(A''_{C_4})$	$\chi,\ \psi,\ \rho$	$\chi^2=1,\ \psi\rho=2\rho,$ $\psi^2=\rho^2=(1+\chi)\psi$	$\chi^*=\chi^{-1},\ \psi^*=\chi\psi,\ \rho^*=\rho$	6
$\mathrm{Rep}(A'''_{C_4})$	$\chi,\ \psi$	$\chi^4=1,\ \psi^2=(1+\chi)\psi$	$\chi^*=\chi^{-1},\ \psi^*=\chi^{-1}\psi$	8
$\mathrm{Rep}(A_{C_2\times C_2})$	$\chi_1,\ \chi_2,\ \psi$	$\chi_1^2=\chi_2^2=1,$ $\psi^2=(1+\chi_1\chi_2)\psi$	$\chi_i^*=\chi_i^{-1}\ (i=1,2),\ \psi^*=\chi_1\chi_2\psi$	8
$\mathrm{Rep}(T_{9,\omega_3})$	$\chi,\ \psi,\ \rho$	$\chi^3=1,\ \psi^2=1+\chi\rho,$ $\rho^2=\rho(1+\chi+\chi^2),\ \rho\psi=\rho(\chi+\chi^2)$	$\chi^*=\chi^{-1},\ \psi^*=\psi,\ \rho^*=\chi^2\rho$	9

(3) Green 環 $\mathrm{Rep}(A_{C_2})$ の $\mathbb{Z}$-加群としての階数は無限である。より詳しくは、任意の自然数 n に対して、n 次元の直既約な左 A_{C_2}-加群が存在する。

お詫びと注意. 上の表から、$\mathrm{Rep}(T_{9,\omega_3})$ と $\mathrm{Rep}(T_{9,\omega_3^{-1}})$ は同型である。一方、T_{9,ω_3} は非自明なコサイクル変形を持たない [2, 8]。したがって、有限次元左 T_{9,ω_3}-加群のなすテンソル圏と有限次元左 $T_{9,\omega_3^{-1}}$-加群のなすテンソル圏とは同値でない [14]。講演とその Abstract では、9 次元以下の半単純でないホップ代数 A については、有限次元左 A-加群のなすテンソル圏が Green 環によって決まると主張しましたが、これは誤りです。お詫び致します。9 を 8 に取り替えると、正しい主張になります [16]。

上の定理から次の系が導かれる ([16] も参照)。

系 1.2. 標数 0 の代数閉体上で定義された 9 次元以下の半単純でないホップ代数 A の Green 環 $\mathrm{Rep}(A)$ は可換であり、かつ、反環準同型 $*: \mathrm{Rep}(A) \longrightarrow \mathrm{Rep}(A)$ はインボリューションである。さらに、A_{C_2} と同型なホップ代数を除いて、標数 0 の代数閉体上で定義された 9 次元以下の半単純でないホップ代数は、有限表現型 (*i.e.* 直既約な有限生成左 A-加群の同型類は有限個) である。

注意 1. 1. 有限群 G の群環 $k[G]$ の双対ホップ代数 $k[G]^*$ について、$\mathrm{Rep}(k[G]^*) \cong \mathbb{Z}[G]$ が成り立つ。したがって、G が非可換ならば、$\mathrm{Rep}(k[G]^*)$ も非可換である。
2. A が準三角ホップ代数 [3] の構造を持てば、その Green 環は可換であり、かつ、反環準同型 $*: \mathrm{Rep}(A) \longrightarrow \mathrm{Rep}(A)$ はインボリューションである。A_{C_2} は準三角ホップ代数の構造を持つ [5, 16] ので、その Green 環は可換であり、かつ、反環準同型 $*$ はインボリューションである。
3. 系の最後の主張は、A_{C_2} 以外のホップ代数の根基がすべて単項である (x で生成される) ことからも従う (例えば、[7, Proposition 54.8 & Theorem 54.12] を参照)。

2. 定理の証明

8 次元以下のホップ代数については、その Green 環の構造はすでに調べている [16] ので、ここでは、9 次元 Taft 代数 T_{9,ω_3} についてその構造を調べる。

命題 2.1. Λ を 9 次元 Taft 代数 $A = T_{9,\omega_3}$ の部分ホップ代数 $k[G(A)]$ の積分とする。(すなわち、Λ を $1+g+g^2+g^3$ の 0 でないスカラー倍とする。) 左正則加群の 3 つの部分加群

$$V_3 := kx^2\Lambda + kx\Lambda + k\Lambda, \quad V_2 = kx^2\Lambda + kx\Lambda, \quad V_1 = kx^2\Lambda$$

を考える。このとき、A の有限次元直既約加群は以下のいずれか 1 つに同型である。
・1 次元直既約加群： $V_1^{\otimes a}$ $(a = 0, 1, 2)$.
・2 次元直既約加群： $V_1^{\otimes a} \otimes V_2$ $(a = 0, 1, 2)$.
・3 次元直既約加群： $V_1^{\otimes a} \otimes V_3$ $(a = 0, 1, 2)$.
但し、$V_1^{\otimes 0} = k$ と約束する。

PROOF. V を直既約な有限次元左 A-加群とする。$\omega = \omega_3$ とおく。$g^3 = 1$ であるから、V は

$$V = V(g;1) \oplus V(g;\omega) \oplus V(g;\omega^2)$$

と g の作用に関する固有空間 $V(g;\omega^a)$, $a = 0,1,2$ の直和に分解される。

$gx = \omega^{-1}xg$ より、$x(V(g;\omega^a)) \subset V(g;\omega^{a-1})$ が成り立つ。$W_1 := x(V(g;1))$ とおき、それの $V(g;\omega^2)$ における (線形) 補空間を W_2 とする：$V(g;\omega^2) = W_1 \oplus W_2$ 。また、

$$U_0 = x(W_1) \cap x(W_2)$$

とおき、それの $V(g;\omega)$ における (線形) 補空間を U_1 とする：$V(g;\omega) = U_0 \oplus U_1$ 。さらに、各 $i = 1,2$ に対して、

$$W_i = \mathrm{Ker}(x|_{W_i}) \oplus W_i'$$

となる部分空間 W_i' と

$$V(g;1) = Z \oplus \mathrm{Ker}(x|_{V(g;1)})$$

となる部分空間 Z を取る。このとき、

$$\begin{cases} W_i' = W_i' \cap x^{-1}(U_0) + W_i' \cap x^{-1}(U_1) \quad (i = 1,2), \\ Z = Z \cap x^{-1}(W_1' \cap x^{-1}(U_0)) + Z \cap x^{-1}(W_1' \cap x^{-1}(U_1)) + Z \cap x^{-1}(\mathrm{Ker}(x|_{W_1})) \end{cases}$$

が成り立つので、

$$X := U_0 + W_1' \cap x^{-1}(U_0) + W_2' \cap x^{-1}(U_0) + Z \cap x^{-1}(W_1' \cap x^{-1}(U_0)),$$
$$Y := U_1 + W_1' \cap x^{-1}(U_1) + W_2' \cap x^{-1}(U_1) + Z \cap x^{-1}(W_1' \cap x^{-1}(U_1)) + \mathrm{Ker}(x|_{W_1}) + \mathrm{Ker}(x|_{W_2})$$
$$+ Z \cap x^{-1}(\mathrm{Ker}(x|_{W_1})) + \mathrm{Ker}(x|_{V(g;1)})$$

とおくと、ベクトル空間として $V = X \oplus Y$ が成り立つ。この直和分解は左 A-加群としての直和分解にもなっている。なぜならば、$x^3 = 0$ により、

$$x(U_0) = 0, \quad x(U_1) \subset Z \cap x^{-1}(\mathrm{Ker}(x|_{W_1})) + \mathrm{Ker}(x|_{V(g;1)})$$

が成り立つからである。V は直既約であったから、$V = X$ または $V = Y$ が成り立つ。

まず、X について考える。$x|_{W_1'\cap x^{-1}(U_0)}: W_1'\cap x^{-1}(U_0) \longrightarrow U_0$, $x|_{W_2'\cap x^{-1}(U_0)}: W_2'\cap x^{-1}(U_0) \longrightarrow U_0$, $x|_{Z\cap x^{-1}(W_1'\cap x^{-1}(U_0))}: Z\cap x^{-1}(W_1'\cap x^{-1}(U_0)) \longrightarrow W_1'\cap x^{-1}(U_0)$ はすべて線形同型写像であるから、$U_0\neq 0$ ならば、g, x の X への作用は次のような 4 次正方行列 (の直和) として表わすことができる。

$$g\longmapsto \begin{pmatrix}1&0&0&0\\0&\omega&0&0\\0&0&\omega&0\\0&0&0&\omega^2\end{pmatrix},\quad x\longmapsto \begin{pmatrix}0&0&0&0\\0&0&0&0\\1&0&0&0\\0&1&1&0\end{pmatrix}$$

これは 3 次元と 1 次元の加群の直和であり、直既約でない。よって、$V=Y$ でなければならず、したがって、$X=0$ である。

$U_0=0$ ゆえ、$x(W_1')$ と $x(W_2')$ は $V(g;\omega)$ の中で直和である。そこで、$V(g;\omega)=x(W_1')\oplus x(W_2')\oplus U_2$ となる部分空間 U_2 をとる。このとき、V は次の 2 つの部分空間 Y_1, Y_2 の直和になる。

$$Y_1 := Z\cap x^{-1}(W_1') + W_1' + x(W_1'),$$
$$Y_2 := Z\cap x^{-1}(\mathrm{Ker}(x|_{W_1})) + \mathrm{Ker}(x|_{V(g;1)}) + \mathrm{Ker}(x|_{W_1}) + W_2 + x(W_2) + U_2$$

実は、Y_1, Y_2 は V の部分 A-加群になっている。Y_1 が V の部分 A-加群であることは

$$Z\cap x^{-1}(W_1') \underset{\cong}{\xrightarrow{x}} W_1' \underset{\cong}{\xrightarrow{x}} x(W_1') \xrightarrow{x} 0$$

となることからわかる。Y_2 が V の部分 A-加群であることは、

$$x^2(W_2)\subset \mathrm{Ker}(x|_{V(g;1)}),\quad x(U_2)\subset Z\cap x^{-1}(\mathrm{Ker}(x|_{W_1})) + \mathrm{Ker}(x|_{V(g;1)})$$

となることからわかる。後者の包含関係は、次のようにして示される。$u_2\in U_2$ に対して $x(u_2)=z_1+z_2+z_3$ ($z_1\in Z\cap x^{-1}(W_1'), z_2\in Z\cap x^{-1}(\mathrm{Ker}(x|_{W_1}))$, $z_3\in \mathrm{Ker}(x|_{V(g;1)})$) と書く。$0=x^3(u_2)=x^2(z_1)$ となるので、$x(z_1)\in \mathrm{Ker}(x|_{W_1})\cap W_1'=\{0\}$ である。よって、$z_1\in Z\cap \mathrm{Ker}(x|_{V(g;1)})=\{0\}$ である。

V は直既約であるから、$V=Y_1$ または $V=Y_2$ でなければならない。

・$V=Y_1$ の場合：$V\cong V_3$ となる。

・$V=Y_2$ の場合：$W_1'=0$ となるので、$W_1=\mathrm{Ker}(x|_{W_1})$ が成り立つ。また、

$$V(g;1)=Z\cap x^{-1}(\mathrm{Ker}(x|_{W_1}))+\mathrm{Ker}(x|_{V(g;1)}),\quad V(g;\omega^2)=\mathrm{Ker}(x|_{W_1})+W_2,\quad V(g;\omega)=x(W_2)+U_2$$

が成り立つ。$Z_1:=x(U_2)\cap \mathrm{Ker}(x|_{V(g;1)})$ とおき、

$$x(U_2)=Z_1\oplus Z_2,\quad \mathrm{Ker}(x|_{V(g;1)})=Z_1\oplus Z_3,\quad V(g;1)=Z_1\oplus Z_2\oplus Z_3\oplus Z_4$$

を満たす部分空間 Z_2, Z_3, Z_4 をとる。さらに $U_2=\mathrm{Ker}(x|_{U_2})\oplus U_2'$ を満たす部分空間 U_2' をとる。

$$U_2'=U_2'\cap x^{-1}(Z_1)+U_2'\cap x^{-1}(Z_2)$$

と書け、$x(Z_4)+x(Z_2)=W_1$ は W_1 の中で直和になっていることから、

$$Y_1' := U_2'\cap x^{-1}(Z_2)+Z_2+x(Z_2)$$
$$Y_2' := \mathrm{Ker}(x|_{U_2})$$
$$Y_3' := Z_4+x(Z_4)$$
$$Y_4' := W_2+U_2'\cap x^{-1}(Z_1)+x(W_2)+\mathrm{Ker}(x|_{V(g;1)})$$

とおくと、$V=Y_1'\oplus Y_2'\oplus Y_3'\oplus Y_4'$ となることがわかる。$x^3=0$ ゆえ、

$$x^2(Z_2)\subset x(W_1)=\{0\},\ x^2(Z_4)\subset x(W_1)=\{0\},\ x^2(W_2)\subset \mathrm{Ker}(x|_{V(g;1)})$$

となる。したがって、各 Y_i' $(i=1,2,3,4)$ は V の部分 A-加群である。V は直既約なので、ある $i=1,2,3,4$ に対して $V=Y_i'$ となる。

・$V=Y_1'$ の場合：$U_2'\cap x^{-1}(Z_2) \underset{\cong}{\xrightarrow{x}} Z_2 \underset{\cong}{\xrightarrow{x}} x(Z_2)$ であるから、$V=V_1\otimes V_3$ であることがわかる。

・$V=Y_2'$ の場合：$V=V_1^{\otimes 2}$ となる。

・$V=Y_3'$ の場合：$V=V_1\otimes V_2$ となる。

・$V = Y_4'$ の場合：$V(g;1) = \mathrm{Ker}(x|_{V(g;1)})$ となる。系列

$$V(g;\omega^2) \xrightarrow{x} V(g;\omega) \xrightarrow{x} V(g;1) \xrightarrow{x} 0$$

について、冒頭部分と同様の考察を行うことにより、V が $V_1^{\otimes 2} \otimes V_3$, $V_1^{\otimes 2} \otimes V_2$, V_2, k, V_1, $V_1^{\otimes 2}$ のいずれかと同型になることがわかる。 □

注意 2. 体 k 上の有限次元代数 A に対して、次の 2 つは同値であることが知られている [1, Theorem A]：

(i) 直既約な有限次元左 A-加群の同型類の個数は有限個である。

(ii) 任意の直既約な左 A-加群は有限次元である。

このことから、任意の直既約な左 $T_{9,\omega}$-加群は有限次元であることがわかり、その結果として、上の命題で挙げた直既約加群のどれか 1 つに同型である。

補題 2.2. 9 次元 Taft 代数 T_{9,ω_3} の Green 環 $\mathrm{Rep}(T_{9,\omega_3})$ は χ, ψ, ρ によって生成され関係式

$$\chi^3 = 1,\ \psi^2 = 1 + \chi\rho,\ \rho^2 = \rho(1+\chi+\chi^2),\ \chi\psi = \psi\chi,\ \chi\rho = \rho\chi,\ \psi\rho = \rho\psi = \rho(\chi+\chi^2)$$

$$\chi^* = \chi^2,\ \psi^* = \psi,\ \rho^* = \chi^2\rho$$

によって記述される。したがって、$\mathrm{Rep}(T_{9,\omega_3})$ は可換であり、$*$ はインボリューションである。

PROOF.

$$e_2 := x^2\Lambda,\ e_1 := x\Lambda,\ e_0 := \Lambda$$

とおく。また、$\omega = \omega_3$ とおく。V_1, V_2, V_3 を命題 2.1 の直既約加群とする。$V_1^{\otimes 3} = k$, $V_1 \otimes V_2 = V_2 \otimes V_1$, $V_1 \otimes V_3 = V_3 \otimes V_1$ であることはすぐにわかる。少し計算すると

$$\begin{aligned}
V_3 \otimes V_2 &= ke_2 \otimes e_2 + ke_2 \otimes e_1 + k(\omega e_1 \otimes e_1 - e_0 \otimes e_2)\\
&\quad + k(-\omega e_2 \otimes e_1 + e_1 \otimes e_2) + k(-\omega^2 e_1 \otimes e_1 - e_0 \otimes e_2) + k(-e_0 \otimes e_1)\\
&\cong (V_1 \otimes V_3) \oplus (V_1^{\otimes 2} \otimes V_3)\\
V_2 \otimes V_3 &= ke_2 \otimes e_2 + ke_2 \otimes e_1 + ke_2 \otimes e_0\\
&\quad + k(-\omega e_2 \otimes e_1 + e_1 \otimes e_2) + k(e_2 \otimes e_0 + e_1 \otimes e_1) + ke_1 \otimes e_0\\
&\cong (V_1 \otimes V_3) \oplus (V_1^{\otimes 2} \otimes V_3)\\
V_2 \otimes V_2 &= k(-\omega e_2 \otimes e_1 + e_1 \otimes e_2) + (ke_2 \otimes e_2 + k(-\omega^2 e_2 \otimes e_1 - e_1 \otimes e_2) + k(-e_1 \otimes e_1))\\
&\cong k \oplus (V_1 \otimes V_3)\\
V_3 \otimes V_3 &= ke_2 \otimes e_2 + ke_2 \otimes e_1 + ke_2 \otimes e_0\\
&\quad + k(-\omega e_2 \otimes e_1 + e_1 \otimes e_2) + k(e_2 \otimes e_0 + e_1 \otimes e_1) + ke_1 \otimes e_0\\
&\quad + k(-\omega e_1 \otimes e_1 + e_0 \otimes e_2 + e_2 \otimes e_0) + k(e_1 \otimes e_0 + e_0 \otimes e_1) + ke_0 \otimes e_0\\
&\cong (V_1 \otimes V_3) \oplus (V_1^{\otimes 2} \otimes V_3) \oplus V_3
\end{aligned}$$

がわかる。よって、命題 2.1 の直既約加群 V_1, V_2, V_3 の同型類をそれぞれ χ, ρ, ψ とおくと、環 $\mathrm{Rep}(T_{9,\omega_3})$ は χ, ψ, ρ によって生成され、補題の関係式によって記述される。

次に、$\mathrm{Rep}(T_{9,\omega})$ の $*$-構造を決定しよう。

$$S(g) = g^2,\quad S(x) = -xg^2$$

であることに注意すると、

$$\begin{aligned}
V_1^* &= V_1^{\otimes 2}\\
V_2^* &= k(-\omega^2 e_1^*) + ke_2^* \cong V_2\\
V_3^* &= ke_0^* + k(-e_1^*) + k(\omega e_2^*) \cong V_1^{\otimes 2} \otimes V_3
\end{aligned}$$

を得る。ここで、V_2^* における $\{e_2^*, e_1^*\}$ は V_2 の基底 $\{e_2, e_1\}$ の双対基底であり、V_3^* における $\{e_2^*, e_1^*, e_0^*\}$ は V_3 の基底 $\{e_2, e_1, e_0\}$ の双対基底である。したがって、$\mathrm{Rep}(T_{9,\omega})$ 上の反環準同型 $*$ は、

$$\chi^* = \chi^2,\ \psi^* = \psi,\ \rho^* = \chi^2\rho$$

によって完全に決定されることがわかる。また、

$$\chi^{**} = (\chi^2)^* = \chi^4 = \chi,\ \psi^{**} = \psi^* = \psi,\ \rho^{**} = \rho^*\chi = \rho$$

が成り立つので、$*$ はインボリューションである。 □

PROOF OF THEOREM 1.1. (1) A'_{C_4} と A''_{C_4} は互いにコサイクル変形である ([11, Proposition 3] または [16, Corollary 1.7] 参照) ことから従う (直接証明することもできる)。

(2) 8 次元以下のホップ代数の Green 環については、[16, Theorem1.5] による。9 次元ホップ代数 T_{9,ω_3} については、命題 2.1 と補題 2.2 による。

(3) は [16, Theorem 1.5] による。 □

謝辞. 当シンポジウムへの講演申し込みを奨めてくださった宇野勝博先生に感謝いたします。また、講演の機会を与えてくださった平野康之先生をはじめとする当シンポジウムの主催者にお礼申し上げます。

参考文献

[1] M. Auslander, *Large modules over Artin algebras*, In *Algebra, topology, and category theory (a collection of papers in honor of Samuel Eilenberg)*, edited by Alex Heller and Myles Tierney, Academic Press, 1976, p.1–17.

[2] 土井幸雄, 竹内光弘, ホップ代数のコサイクル変形, 数理解析研究所講究録 942 (1996) 29–52.

[3] V.G. Drinfel'd, *Quantum groups*. In *Proceedings of the International Congress of Mathematics, Berkeley, CA., 1987*, 798–820.

[4] W. Feit, *The representation theory of finite groups*. North-Holland, Amsterdam, New York, Oxford, 1982.

[5] S. Gelaki, *On pointed ribbon Hopf algebras*, J. of Algebra **181** (1996) 760–786.

[6] Y. Kashina, *Classification of semisimple Hopf algebras of dimension 16*, J. of Algebra **232** (2000) 617–663.

[7] G. Karpilovsky, *Structure of blocks of group algebras*, (Pitman Manographs and survey in Pure and Applied Math.) Longman Scientific & Technical, 1987.

[8] A. Masuoka, *Cleft extensions for a Hopf algebra generated by a nearly primitive element*, Comm. Algebra **22** (1994) 4537–4559.

[9] A. Masuoka, *Semisimple Hopf algebras of dimension 6, 8*, Israel J. Math. **92** (1995) 361–373.

[10] A. Masuoka, *The p^n theorem for semisimple Hopf algebras*, Proc. of Amer. Math. Soc. **124** (1996) 735–737.

[11] A. Masuoka, *Defending the negated Kaplansky's conjecture*, Proc. of Amer. Math. Soc. **129** (2001) 3185—3192.

[12] D. Nikshych, *K_0-rings and twisting of finite dimensional semisimple Hopf algebras*, Comm. Alg. **26** (1998) 321–342; *Erratum: K_0-rings and twisting of finite dimensional semisimple Hopf algebras*, Comm. Algebra **26** (1998) p.1347.

[13] D. E. Radford, *The order of the antipode of a finite dimesional Hopf algebra is finite*, Amer. J. Math. **98** (1976) 333–355.

[14] P. Schauenburg, *Hopf bigalois extensions*, Comm. Algebra **24** (1996) 3797–3825.

[15] D. Ştefan, *Hopf algebras of low dimension*, J. of Algebra **211** (1999) 343–361.

[16] M. Wakui, *Various structures associated to the representation categories of 8-dimensional non-semisimple Hopf algebras*, to appear in Algebras and Representation Theory.

[17] R. Williams, Ph.D. thesis, Florida State University, 1988 (unpublished).

Department of Mathematics
Osaka University
Toyonaka, Osaka 560-0043, Japan.
E-mail: wakui@math.sci.osaka-u.ac.jp

A CHARACTERIZATION OF NOETHERIAN RINGS AND ITS DUAL

YOSHITO YUKIMOTO

ABSTRACT. In this note we characterize right Noetherian rings by direct decomposability of arbitrary right module to an injective module and an i-reduced module, and we also discuss the rings defined dually with respect to the characterization.

1. INTRODUCTION

A ring R is said to be a right H-ring if every right R-module is a direct sum of an injective module and a small module. Right H-rings have various properties, and are studied by many authors: Harada [4], Rayer [6], and Oshiro [5] for example. To characterize rings R by a direct decomposition of an arbitrary R-module into two specific modules is a new way of definition. We show that this type of characterization is possible for right Noetherian rings. The characterization of right Noetherian rings is dualized easily. However the class of right Noetherian rings is so significant that the dual class of it is expected to be important. Therefore we also discuss relations of the dual class and other classes of rings.

2. CHARACTERIZATION OF NOETHERIAN RINGS

A module M is said to be i-reduced if any submodule $\neq 0$ is not injective. We characterize right Noetherian rings by the direct decomposability of an arbitrary right module to an injective module and an i-reduced module.

Theorem 2.1. *For a ring R the following conditions are equivalent:*

(1) *R is right Noetherian;*
(2) *Any right R-module has a maximal injective submodule;*
(3) *Any right R-module is a direct sum of an injective module and an i-reduced module.*

Proof. (1)⇒(2) by Zorn's lemma.
(2)⇒(3). Clear.
(3)⇒(1). It suffices to show that any direct sum $\oplus_{i\in I}E_i$ of injective indecomposable right R-modules E_i is injective([7] Theorem 4.1). We have a decomposition

$$\oplus_{i\in I}E_i = E \oplus F$$

by (3), where E is an injective right R-module and F is an i-reduced right R-module.

The detailed version of this paper will be submitted for publication elsewhere.

By Proposition 25.5 in [1], there is a subset J of I such that

$$\oplus_{i\in I}E_i = E \oplus (\oplus_{i\in J}E_i).$$

Hence we have

$$F \cong \oplus_{i\in J}E_i,$$

which is impossible if the i-reduced module F is nonzero. Therefore $F = 0$ and $\oplus_{i\in I}E_i = E$ is injective. □

A submodule A of a module M is said to be a small submodule of M if, for any submodule B of M, $A + B = M$ implies $B = M$. A module is said to be small if it is a small submodule of some module. A ring R is said to be a right H-ring if every right R-module is a direct sum of an injective module and a small module.

Corollary 2.2. *For a ring R the following conditions are equivalent:*

(1) *R is a right H-ring;*

(2) *R is right Noetherian and any i-reduced right R-module is small.*

3. Dual of the characterization

We consider a dual notion of right Noetherian rings. In the following, a module M is said to be p-reduced if any factor module of M except 0 ($= M/M$) is not projective.

Proposition 3.1. *Let R be a ring. The following conditions are equivalent:*

(1) *Every right R-module is a direct sum of a projective module and a p-reduced module;*

(2) *For any right R-module M, there exists a minimal element in the set*

$$\mathcal{S} = \{X \leq M \mid M/X \text{ is projective}\}.$$

Proof. (1)⇒(2). Let M be a right R-module, and $M = P \oplus Q$, where P is projective and Q is p-reduced. Then $Q \in \mathcal{S}$. Suppose that $X \leq Q$ and M/X is projective. Then Q/X is projective and $Q \cong (Q/X) \oplus X$. Since Q is p-reduced, we have $Q/X = 0$. Hence Q is minimal in $\mathcal{S}$.

(2)⇒(1). Let Q be a minimal element in $\mathcal{S}$. Then $M = (M/Q) \oplus Q$ with a projective module M/Q. By the minimality, Q is p-reduced. □

The condition (1) in Proposition 3.1 is called right N* condition. In the terms, 'N' stands for 'Noetherian' because the condition (1) in Proposition 3.1 is dual to the condition (3) in Theorem 2.1.

A ring R is said to be a right coH-ring if every right R-module is a direct sum of a projective module and a singular module. Every right coH-ring satisfies right N* since any singular right R-module is p-reduced. In particular any quasi-Frobenius ring satisfies right N* condition.

We would like to answer the question what rings with right N* condition are. The following Theorem 3.3 is a partial answer to it.

Lemma 3.2. *If R is a right hereditary ring and satisfies right N* condition, then there exists a minimum element in*

$$\mathcal{S} = \{X \leq M \mid M/X \text{ is projective}\}$$

for any right R-module M.

Proof. By Proposition 3.1 there exist minimal elements in $\mathcal{S}$. Let X_1 be a minimal element in $\mathcal{S}$, and X_2 any element in $\mathcal{S}$. Then $M/X_1 \cap X_2 \to M/X_1 \oplus M/X_2$, $m + X_1 \cap X_2 \mapsto (m+X_1, m+X_2)$ is a monomorphism to a projective module. Hence $X_1 \cap X_2 \in \mathcal{S}$, because R is right hereditary. We have $X_1 \cap X_2 = X_1$, by the minimality of X_1, and $X_1 \leq X_2$. Hence X_1 is a minimum element in $\mathcal{S}$. □

Theorem 3.3. *Let R be a right hereditary ring. Then the following conditions are equivalent:*

(1) *R satisfies right N^* condition;*
(2) *Any direct product of projective right R-modules is projective.*

Proof. (1)⇒(2). Let $\{P_i\}_{i\in I}$ be an arbitrary family of projective right R-modules, $M = \prod_{i\in I} P_i$, and $Q_j = \{(x_i)_{i\in I} \in M | x_j = 0\}$. Then M/Q_j is projective for any $j \in I$. Since R is right hereditary and satisfies right N^*, there exists a minimum element Q in $\{X \leq M |\ M/X \text{ is projective}\}$ by Lemma 3.2. We have $Q \leq Q_j$ for any $j \in I$, and $Q \leq \bigcap_{j\in I} Q_j = 0$. Hence M is projective.

(2)⇒(1). We show that the set $\mathcal{S} = \{X \leq M |\ M/X \text{ is projective}\}$ has a minimal element for any right R-module M. Let $X_1 \geq X_2 \geq X_3 \geq \dots$ be a descending chain in $\mathcal{S}$. The direct product $P = \prod_{i\in \mathbb{N}} M/X_i$ is projective by (2). The module $M/\bigcap_{i\in\mathbb{N}} X_i$ is isomorphic to a submodule of P. Since R is right hereditary, $M/\bigcap_{i\in\mathbb{N}} X_i$ is right projective and $\bigcap_{i\in\mathbb{N}} X_i \in \mathcal{S}$. Hence $\mathcal{S}$ has a minimal element by Zorn's lemma. Therefore R satisfies right N^* condition by Proposition 3.1. □

Remark 3.1. Chase ([2] Theorem 3.3) showed that (2) in Theorem 3.3 is equivalent to the following condition (without the assumption of R being right hereditary):

(2′) R is right perfect and left coherent.

So we can replace (2) in Theorem 3.3 by (2′).

Example 1. There exists a ring which is right hereditary left coherent but not right perfect. For example $\mathbb{Z}$ is such a ring, and it does not satisfy right N^* condition by Theorem 3.3.

Example 2. The ring

$$R = \begin{pmatrix} \mathbb{Q} & \mathbb{R} \\ 0 & \mathbb{Q} \end{pmatrix}$$

is hereditary and semiprimary. Hence R satisfies right and left N^* conditions by Theorem 3.3. On the other hand every right hereditary right coH-ring is Artinian ([3] Theorem 5.23). But R is not Artinian. Hence R is not a right coH-ring.

References

[1] F.W.Anderson, and K.R.Fuller, *Rings and category of modules* 2nd ed., Springer-Verlag, New York 1992.

[2] S.U.Chase, *Direct products of modules*, Trans. Amer. Math. Soc. **97** (1960), 457–473.

[3] K.R.Goodearl, *Ring theory: nonsingular rings and modules*, Marcel Dekker, New York 1976.

[4] M. Harada, *Non-small modules and non-cosmall modules* in Ring Theory: Proceedings of the 1978 Antwerp Conference, F. van Oystaeyen, ed. Marcel Dekker, New York, (1979), 669–689.

[5] K.Oshiro, *Lifting modules, extending modules and their applications to QF-rings*, Hokkaido Math. J. **13** (1984), 310–338.

[6] M. Rayer, *On small and cosmall modules*, Acta Math. Acad. Sci. Hungar. **39**(4) (1982), 389–392.

[7] D.W.Sharpe, and P.Vamos, *Injective modules*, Cambridge Univ. Press, London 1972.

Isonokami 2-13-22, Kishiwada, Osaka 596-0001 JAPAN

QUANTIZED COORDINATE RINGS AND RELATED NOETHERIAN ALGEBRAS

K. R. GOODEARL

ABSTRACT. This paper contains a survey of some ring-theoretic aspects of quantized coordinate rings, with primary focus on the prime and primitive spectra. For these algebras, the overall structure of the prime spectrum is governed by a partition into strata determined by the action of a suitable group of automorphisms of the algebra. We discuss this stratification in detail, as well as its use in determining the primitive spectrum - under suitable conditions, the primitive ideals are precisely those prime ideals which are maximal within their strata. The discussion then turns to the global structure of the primitive spectra of quantized coordinate rings, and to the conjecture that these spectra are topological quotients of the corresponding classical affine varieties. We describe the solution to the conjecture for quantized coordinate rings of full affine spaces and (somewhat more generally) affine toric varieties. The final part of the paper is devoted to the quantized coordinate ring of $n \times n$ matrices. We mention parallels between this algebra and the classical coordinate ring, such as the primeness of quantum analogs of determinantal ideals. Finally, we describe recent work which determined, for the 3×3 case, all prime ideals invariant under the group of winding automorphisms governing the stratification mentioned above.

INTRODUCTION

First, a *caveat* concerning the title: This survey is not designed to be either an introduction to or a discussion of quantum groups. Rather, we present some of the ring theory that has arisen in studying the structure of certain algebras found among quantum groups. Here we only give a few words of background, and later we present some representative examples. An introduction to the general theory of quantum groups can be found in many books; as a small sample, we mention [3, 6, 28, 29].

The term 'quantized coordinate rings' refers to certain algebras that, loosely speaking, are deformations of the classical coordinate rings of affine algebraic varieties or algebraic groups. These algebras are typically not commutative, but they turn out to have many other properties analogous to the classical case - for example, they are noetherian, and most of the ones that have been introduced to date are integral domains, with finite global dimension. To take the most basic case, recall that the classical coordinate ring of affine n-space over a field k is just a polynomial ring in n indeterminates over k. Thus, a 'quantized' coordinate ring of affine n-space should be some type of noncommutative polynomial ring in n indeterminates, such as an n-fold iterated skew polynomial extension of k. For the canonical examples, see Section 1.1.

This is an expository paper, based on work published elsewhere.

These notes are arranged in three parts, which focus on prime ideals, primitive ideals, and the quantized coordinate rings of matrices, respectively. Most of the material in Parts I and II is excerpted from [3], where the reader can find a much more detailed development. The aim of Part III is to illustrate how the general picture developed in the first two parts applies to a particularly interesting quantized coordinate ring; the discussion is taken partly from [3] and partly from the recent paper [15].

Throughout, we work over a base field k, and our parameters will be elements of $k^\times$, that is, nonzero scalars from k. The characteristic of k may be arbitrary, and for many results, it does not matter whether or not k is algebraically closed. We will often concentrate on the so-called *generic* case, meaning that our parameters are not roots of unity, but when not specified, the parameters may be arbitrary. The key difference is that when sufficiently many parameters are roots of unity, quantized coordinate rings are finitely generated modules over their centers, and their study proceeds via the theory of rings with polynomial identity. Our aim here is to concentrate on the non-PI case, which requires very different tools (some not yet invented).

I. Prime Ideals

In Part I, we concentrate on prime ideals in quantized coordinate rings and related algebras, more precisely, on ways to organize the *prime spectrum* – the set $\operatorname{spec} A$ of prime ideals in an algebra A. We view $\operatorname{spec} A$ not just as a set, but as a topological space, equipped with the standard Zariski topology.

In order to have available a few examples with which to illustrate the results and techniques, we begin by presenting some of the standard quantized coordinate rings. For a survey of most of the known types, see [10].

1.1. Some quantized coordinate rings. Let $q \in k^\times$. The *quantized coordinate ring of the xy-plane* with parameter q is the k-algebra

$$\mathcal{O}_q(k^2) \stackrel{\text{def}}{=} k\langle x, y \mid xy = qyx \rangle.$$

This algebra is often called a *quantum plane* for short.

Quantized coordinate rings for higher-dimensional spaces are defined similarly, except that more choices of parameters are allowed. Let $\mathbf{q} = (q_{ij})$ be a *multiplicatively antisymmetric* $n \times n$ matrix over k, meaning that $q_{ii} = 1$ and $q_{ji} = q_{ij}^{-1}$ for all i, j. The *quantized coordinate ring of affine n-space* with parameter matrix $\mathbf{q}$ is the k-algebra

$$\mathcal{O}_{\mathbf{q}}(k^n) \stackrel{\text{def}}{=} k\langle x_1, \dots, x_n \mid x_i x_j = q_{ij} x_j x_i \text{ for all } i, j \rangle.$$

There is a single-parameter version of this algebra, defined for $q \in k^\times$ as follows:

$$\mathcal{O}_q(k^n) \stackrel{\text{def}}{=} k\langle x_1, \dots, x_n \mid x_i x_j = q x_j x_i \text{ for all } i < j \rangle.$$

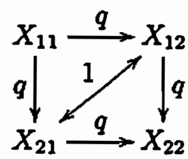
This is the special case of $\mathcal{O}_{\mathbf{q}}(k^n)$ for which the matrix $\mathbf{q}$ has the form

$$\begin{pmatrix} 1 & q & q & \cdots & q & q \\ q^{-1} & 1 & q & \cdots & q & q \\ q^{-1} & q^{-1} & 1 & \cdots & q & q \\ & \vdots & & & \vdots & \\ q^{-1} & q^{-1} & q^{-1} & \cdots & 1 & q \\ q^{-1} & q^{-1} & q^{-1} & \cdots & q^{-1} & 1 \end{pmatrix}.$$

The *quantized coordinate ring of* 2×2 *matrices* with parameter q is the k-algebra $\mathcal{O}_q(M_2(k))$ given by four generators $X_{11}, X_{12}, X_{21}, X_{22}$ and six relations

$$\begin{aligned} X_{11}X_{12} &= qX_{12}X_{11} & X_{12}X_{22} &= qX_{22}X_{12} \\ X_{11}X_{21} &= qX_{21}X_{11} & X_{21}X_{22} &= qX_{22}X_{21} \\ X_{12}X_{21} &= X_{21}X_{12} & X_{11}X_{22} - X_{22}X_{11} &= (q - q^{-1})X_{12}X_{21}. \end{aligned}$$

The first five relations, which are all of the form $xy = ryx$ for generators x and y and scalars r, can be summarized in the following mnemonic diagram:



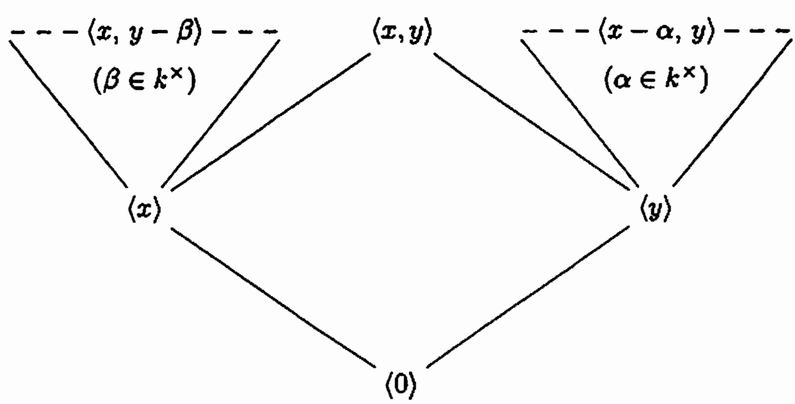
The element $D_q \stackrel{\text{def}}{=} X_{11}X_{22} - qX_{12}X_{21}$ in $\mathcal{O}_q(M_2(k))$ is called the (2×2) *quantum determinant*; as is easily checked, D_q lies in the center of $\mathcal{O}_q(M_2(k))$. The *quantized coordinate rings of* $GL_2(k)$ *and* $SL_2(k)$ are the algebras

$$\mathcal{O}_q(GL_2(k)) \stackrel{\text{def}}{=} \mathcal{O}_q(M_2(k))[D_q^{-1}] \qquad \text{and} \qquad \mathcal{O}_q(SL_2(k)) \stackrel{\text{def}}{=} \mathcal{O}_q(M_2(k))/\langle D_q - 1\rangle.$$

Analogous algebras $\mathcal{O}_q(M_n(k))$, $\mathcal{O}_q(SL_n(k))$, and $\mathcal{O}_q(GL_n(k))$ have been defined for arbitrary n, but we shall not give their definitions until later (Section 3.1).

A general principle from the study of quantum phenomena in physics, which holds equally well in mathematical studies of quantum algebras, is that *quantization destroys symmetry*, meaning that a quantized version of a classical system (physical or mathematical) tends to be more rigid, with less symmetry. We illustrate this principle with coordinate rings of the plane k^2. For instance, the classical coordinate ring $k[x, y]$ has a huge supply of prime ideals, but the quantized coordinate ring has far fewer, as the following example shows. The same can also be said for automorphisms, as we shall see shortly.

1.2. Example. When k is algebraically closed and q is not a root of unity, the prime spectrum of $\mathcal{O}_q(k^2)$ can be displayed as follows:



Another difference in symmetry between the classical and quantized coordinate rings of the plane is found in the automorphisms of these algebras. As an algebraic variety, the plane is completely homogeneous, in that any point can be moved to any other point by a translation. These translations induce automorphisms of $k[x,y]$ of the following form: For any scalars $a,b \in k$, there is a k-algebra automorphism of $k[x,y]$ such that $x \mapsto x+a$ and $y \mapsto y+b$. In the quantum case, however, $\mathcal{O}_q(k^2)$ has no such automorphisms except the identity (corresponding to $a=b=0$). Fortunately, $\mathcal{O}_q(k^2)$ is not bereft of automorphisms – there are multiplicative analogs of the translation automorphisms, mapping x and y to scalar multiples of themselves. In fact, all of our standard examples have a supply of automorphisms of this type, as follows.

1.3. Some automorphisms. We define some families of k-algebra automorphisms on the quantized coordinate rings discussed above. Each family of automorphisms is parametrized by tuples of nonzero scalars, i.e., by elements from one of the multiplicative groups $(k^\times)^r$.

For $(\alpha,\beta) \in (k^\times)^2$, there is an automorphism of $\mathcal{O}_q(k^2)$ such that $x \mapsto \alpha x$ and $y \mapsto \beta y$.

For $(\alpha_1,\dots,\alpha_n) \in (k^\times)^n$, there is an automorphism of $\mathcal{O}_{\mathbf{q}}(k^n)$ such that $x_i \mapsto \alpha_i x_i$ for all i.

For $(\alpha_1,\alpha_2,\beta_1,\beta_2) \in (k^\times)^4$, there are automorphisms of $\mathcal{O}_q(M_2(k))$ and $\mathcal{O}_q(GL_2(k))$ such that $X_{ij} \mapsto \alpha_i\beta_j X_{ij}$ for all i,j. In other words,

$$\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \longmapsto \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix}.$$

The automorphisms above do not all carry over to $\mathcal{O}_q(SL_2(k))$ – we must restrict attention to those which fix the quantum determinant. For $(\alpha,\beta) \in (k^\times)^2$, there is an automorphism of $\mathcal{O}_q(SL_2(k))$ such that

$$\overline{X}_{ij} \longmapsto \alpha^{3-2i}\beta^{3-2j}\,\overline{X}_{ij}$$

for all i,j, that is,

$$\begin{pmatrix} \overline{X}_{11} & \overline{X}_{12} \\ \overline{X}_{21} & \overline{X}_{22} \end{pmatrix} \longmapsto \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} \overline{X}_{11} & \overline{X}_{12} \\ \overline{X}_{21} & \overline{X}_{22} \end{pmatrix} \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}.$$

1.4. Example. The homogeneity of the plane in the classical case carries over to its coordinate ring in the following way – if M_1 and M_2 are any maximal ideals of $k[x,y]$ of codimension 1 (these are the maximal ideals corresponding to points in the plane with coordinates in k), there is an automorphism ϕ of $k[x,y]$ such that $\phi(M_1)=M_2$. Thus, if k is algebraically closed, the maximal ideals of $k[x,y]$ form a single orbit with respect to the automorphisms of this algebra.

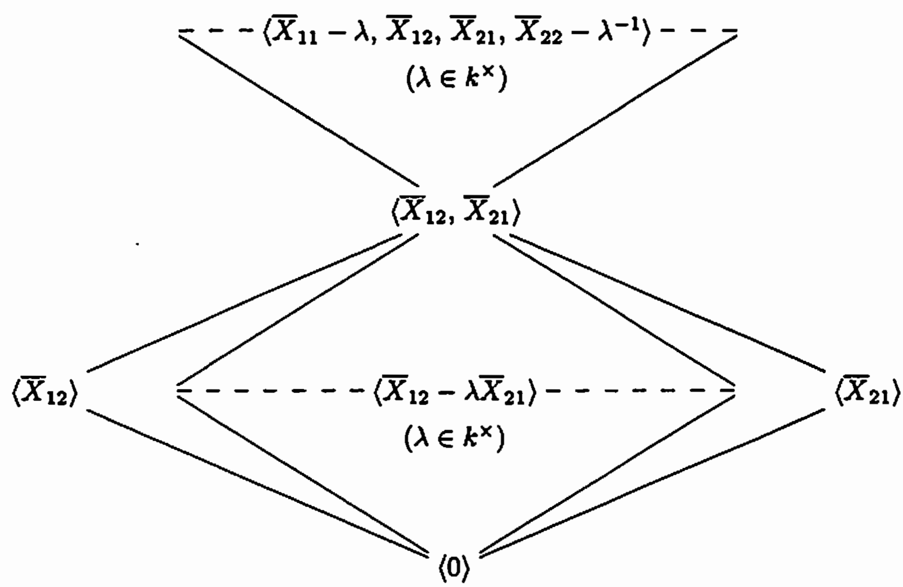
While $\mathcal{O}_q(k^2)$ does not have enough automorphisms to map any maximal ideal onto any other, there are still relatively large orbits. Assuming that k is algebraically closed and q is not a root of unity, the maximal ideals of $\mathcal{O}_q(k^2)$ can be seen in Example 1.2. Using just the automorphisms defined in (1.3), there are three orbits of maximal ideals:

$$\{\langle x,\, y-\beta\rangle \mid \beta \in k^\times\} \qquad \{\langle x,\, y\rangle\} \qquad \{\langle x-\alpha,\, y\rangle \mid \alpha \in k^\times\}.$$

Note that each of the orbits above intersects to a prime ideal which is stable under these automorphisms. The maximal ideals together with these orbit-intersections account for all but one prime ideal of $\mathcal{O}_q(k^2)$; for completeness, note that the remaining prime, namely $\langle 0 \rangle$, is also stable under the automorphisms.

A similar pattern can be observed in $\mathcal{O}_q(SL_2(k))$, as follows.

1.5. Example. When k is algebraically closed and q is not a root of unity, the prime spectrum of $\mathcal{O}_q(SL_2(k))$ can be displayed as shown below:



In this case, the maximal ideals form a single orbit under the automorphisms described in (1.3), while the prime ideals of height 1 form three orbits. The two remaining primes can be described as intersections of infinite orbits.

Patterns analogous to those discussed in Examples 1.4 and 1.5 have been found in all the other quantized coordinate rings introduced so far, assuming that k is algebraically closed and the parameters are generic. With some modifications, the picture can be expanded to include arbitrary infinite base fields. There is some interesting ring theory which explains and predicts this behavior, and our main goal in Part I is to present this theory. Some of the concepts used to describe the picture only involve an arbitrary group of automorphisms of a ring, but the key results hold when the group is an *algebraic torus*, that is, a product of copies of the multiplicative group $k^\times$. The two examples above exhibit orbits of prime ideals which intersect to stable prime ideals, which hints at the importance of such orbit intersections. This hint leads to the key idea – to group prime ideals according to the intersections of their orbits with respect to a specific group of automorphisms.

We begin with arbitrary actions of groups on rings. Whenever we refer to a group acting on a ring, we shall assume that it is acting by means of ring automorphisms (rather than just by permutations or by invertible linear transformations, for instance); similarly, actions on algebras are assumed to be actions by algebra automorphisms.

1.6. H-prime ideals. Let A be a ring, and let H be a group acting on A (by automorphisms). Thus, we are given a homomorphism $\phi : H \to \operatorname{Aut} A$, and we abbreviate $\phi(h)(a)$ to $h(a)$ for $h \in H$ and $a \in A$. (Many authors write $h.a$ for $\phi(h)(a)$.) For any ideal $P \lhd A$, set

$$(P : H) \stackrel{\text{def}}{=} \bigcap_{h \in H} h(P),$$

the largest H-stable ideal of A contained in P.

By restricting the usual definition of a prime ideal to H-stable ideals, we obtain the concept of an *H-prime* ideal of A, namely any proper H-stable ideal J of A such that $I_1 I_2 \not\subseteq J$ for all H-stable ideals $I_1, I_2 \not\subseteq J$. In parallel with the notation $\operatorname{spec} A$, we write $H\text{-spec}\, A$ to denote the set of all H-prime ideals of A. For example, if q is not a root of unity, $k = \overline{k}$, and $H = (k^\times)^2$ acts as in (1.3), then

$$H\text{-spec}\, \mathcal{O}_q(k^2) = \{ \langle x, y \rangle, \langle x \rangle, \langle y \rangle, \langle 0 \rangle \}.$$

1.7. Lemma. *Let H be a group acting on a ring A.*

(a) *If P is any prime ideal of A, then $(P : H)$ is an H-prime ideal of A.*

(b) *Now assume that A is noetherian. Then a proper ideal J of A is H-prime if and only if J equals the intersection of some finite H-orbit of prime ideals.*

In particular, it follows that all H-prime ideals of A are semiprime in this case.

Proof. (a) Easy.

(b) E.g., see [3, Lemma II.1.10]. □

1.8. H-stratifications. Let H be a group acting on a ring A. For each H-prime ideal J of A, let

$$\operatorname{spec}_J A \stackrel{\text{def}}{=} \{P \in \operatorname{spec} A \mid (P : H) = J\}.$$

This set is called *the H-stratum of* $\operatorname{spec} A$ *corresponding to J*. In view of Lemma 1.7(a),

$$\operatorname{spec} A = \bigsqcup_{J \in H\text{-spec}\, A} \operatorname{spec}_J A,$$

a partition that we call the *H-stratification* of $\operatorname{spec} A$.

The H-stratifications just defined have similar properties to the stratifications used in algebraic geometry, as follows.

1.9. Lemma. *Let H be a group acting on a ring A.*

(a) *The closure of each H-stratum in* $\operatorname{spec} A$ *is a union of H-strata.*

(b) *If H-spec A is finite, then each H-stratum is locally closed in* $\operatorname{spec} A$.

Proof. [10, Lemma 3.4]. □

The stratification setup so far is extremely general, and we cannot expect to prove much about it without specializing to cases with additional hypotheses. One key specialization is to assume that H is an *affine algebraic group* over k, by which we just mean that H is isomorphic to a Zariski-closed subgroup of $GL_n(k)$ for some n. Thus, H is an affine algebraic variety as well as a group, and the group operations are morphisms of varieties. We will not need much at all of the general theory of algebraic groups, since we will concentrate on one of the simplest kind, namely algebraic tori. To see that a torus $(k^\times)^r$ is an algebraic group, note that it is isomorphic to the subgroup of $GL_{r+1}(k)$ consisting of matrices (a_{ij}) satisfying the equations $a_{ij} = 0$ for $i \neq j$ and $a_{11}a_{22}\cdots a_{r+1,r+1} = 1$.

1.10. Rational actions. Let A be a k-algebra, and H a group acting on A. (As noted above, in this situation we assume that H acts on A via k-algebra automorphisms.) Moreover, let us assume that H is an algebraic group over k.

The action of H on A is said to be *rational* provided A is a directed union of finite dimensional H-stable k-subspaces V_i such that the restriction maps $H \longrightarrow GL(V_i)$ are morphisms (of algebraic groups), i.e., group homomorphisms which are also morphisms of varieties. Fortunately for our purposes, the theory of algebraic groups provides a nice criterion that allows us to see quite easily when an action of a torus is rational, as follows.

1.11. Rational characters. Suppose that H is an algebraic torus. Recall that a *character* of H (with respect to the base field k) is any group homomorphism $H \to k^\times$. Characters appear whenever H acts on a k-algebra A: If $x \in A$ is an H-eigenvector (i.e., a simultaneous eigenvector for the actions of all the automorphisms from H), then there is a character ϕ of H such that $h(x) = \phi(h)x$ for all $h \in H$. Of course, ϕ is then called the *H-eigenvalue* of x.

A character of H is called *rational* if it is also a morphism of varieties. Let $X(H)$ denote the set of all rational characters of H; this is an abelian group under pointwise multiplication, and it is easily described. Namely, if $H = (k^\times)^r$, then $X(H)$ is a free abelian group in which the r coordinate projections $(k^\times)^r \to k^\times$ form a basis.

1.12. Theorem. *Let H be a torus acting on a k-algebra A, and assume that k is infinite. The action of H on A is rational if and only if*

(a) *The action is semisimple (i.e., A is spanned by H-eigenvectors); and*

(b) *The H-eigenvalues for the H-eigenvectors in A are <u>rational</u> characters.*

Proof. [34, Chapter 5, Corollary to Theorem 36]. □

From a ring-theoretic point of view, conditions (a) and (b) of Theorem 1.12 are the natural and useful conditions. Thus, we could take them as our definition of a rational action of a torus, if desired.

The next lemma illustrates one useful aspect of having a rational action. (Recall that in general, an H-prime ideal in a noetherian ring need only be semiprime.)

1.13. Lemma. *Suppose that H is a torus, acting rationally on a noetherian k-algebra A. Then every H-prime ideal of A is prime.*

Proof. If J is an H-prime ideal of A, then Lemma 1.7(b) implies that $A = (P : H)$ for some prime ideal P whose H-orbit is finite. Hence, the stabilizer subgroup $\mathrm{Stab}_H(P)$ has finite index in H. Since H acts rationally, the map $H \to \operatorname{spec} A$ given by $h \mapsto h(P)$ is continuous with respect to the Zariski topologies on H and $\operatorname{spec} A$ [3, Lemma II.2.8].

Consequently, the set $V = \{h \in H \mid h(P) \supseteq P\}$ is closed in H. (We cannot say immediately that $\mathrm{Stab}_H(P)$ is closed in H because $\{P\}$ need not be closed in $\operatorname{spec} A$.) However, because A is noetherian, any automorphism ϕ of A for which $\phi(P) \supseteq P$ must map P onto itself. Hence, $V = \mathrm{Stab}_H(P)$, and thus $\mathrm{Stab}_H(P)$ is indeed closed in H.

Now H is the disjoint union of the cosets of $\mathrm{Stab}_H(P)$. There are only finitely many cosets, and they are all closed. However, as a variety H is irreducible (because its coordinate ring is a Laurent polynomial ring, hence a domain), so it cannot be a finite union of proper closed subsets. Thus $\mathrm{Stab}_H(P) = P$, that is, the H-orbit of P consists of P alone. Therefore $J = P$, proving that J is prime. □

We can now present a general theorem which provides a picture of the structure of the H-stratification in our current setting. Recall that a *regular* element in a noetherian ring is any non-zero-divisor. We write $\operatorname{Fract} R$ to denote the Goldie quotient ring of a semiprime noetherian ring R, and $Z(R)$ for the center of a ring R.

1.14. Stratification Theorem. [17, 10] *Let A be a noetherian k-algebra, with k infinite, and let $H = (k^\times)^r$ be a torus acting rationally on A. For $J \in H\text{-}\operatorname{spec} A$, let $\mathcal{E}_J$ denote the set of all regular H-eigenvectors in A/J.*

(a) *$\mathcal{E}_J$ is a denominator set, and the localization $A_J \stackrel{\text{def}}{=} (A/J)[\mathcal{E}_J^{-1}]$ is an H-simple ring (with respect to the induced H-action).*

(b) *$\operatorname{spec}_J A$ is homeomorphic to $\operatorname{spec} A_J$ via localization and contraction, and $\operatorname{spec} A_J$ is homeomorphic to $\operatorname{spec} Z(A_J)$ via contraction and extension.*

(c) *$Z(A_J)$ is a Laurent polynomial ring of the form $K_J[z_1^{\pm 1}, \dots, z_{n(J)}^{\pm 1}]$, with $n(J) \le r$, over the fixed field $K_J \stackrel{\text{def}}{=} Z(A_J)^H = Z(\operatorname{Fract} A/J)^H$.*

Proof. [3, Chapter II.3]. □

Of course, the theorem above does not say much if the H-strata are very small and there are many of them. For instance, in the extreme case the H-strata might be singletons, in which case the theorem is trivial. To get the most information out of this picture, we would like there to be only finitely many H-strata, so that the H-stratification breaks up the prime spectrum into relatively large sets. Many quantized coordinate rings are iterated skew polynomial extensions of k, and the following theorem can be applied to those algebras.

1.15. Theorem. [17, 3] *Let A be an iterated skew polynomial algebra*

$$k[x_1][x_2; \tau_2, \delta_2] \cdots [x_n; \tau_n, \delta_n],$$

and let H be a group acting on A, such that $x_1, \dots, x_n$ are H-eigenvectors. Assume that there exist $h_1, \dots, h_n \in H$ such that:

(a) *$h_i(x_j) = \tau_i(x_j)$ for $i > j$; and*

(b) *The h_i-eigenvalue of x_i is not a root of unity for any i.*

Then A has at most 2^n H-prime ideals. Moreover, if H is a torus acting rationally on A, then for each $J \in H\text{-}\operatorname{spec} A$, the field K_J (from part (c) *of the Stratification Theorem) equals k.*

Proof. [3, Theorems II.5.12 and II.6.4]. □

One of the tools involved in proving the Stratification Theorem is an equivalence between rational $(k^\times)^r$-actions and $\mathbb{Z}^r$-gradings, part of which we now sketch. Since this is intended to be applied to the H-prime factor algebras A/J, we work with an algebra called B rather than A.

1.16. Actions versus gradings. Suppose that B is a noetherian k-algebra, with k infinite, and that a torus $H = (k^\times)^r$ acts rationally on B. Because of Theorem 1.12,

$$B = \bigoplus_{g \in X(H)} B_g,$$

where B_g denotes the H-eigenspace of B with eigenvalue g. Since H acts by automorphisms, $B_g B_{g'} \subseteq B_{gg'}$ for all $g, g' \in G$, that is, B is graded by the group $X(H) \cong \mathbb{Z}^r$. (Conversely, any grading of a k-algebra by $\mathbb{Z}^r$ corresponds to a rational action of $(k^\times)^r$ on the algebra.) Problems concerning the H-action translate into problems concerning the grading in the following way:

H-eigenvectors	$\longleftrightarrow$	homogeneous elements
H-stable ideals	$\longleftrightarrow$	homogeneous ideals
H-prime ideals	$\longleftrightarrow$	graded-prime ideals.

To prove part (a) of the Stratification Theorem, we need to be able to localize an H-prime ring B with respect to its regular H-eigenvectors and obtain an H-simple ring. Translating to the graded case, we need to localize a graded-prime ring with respect to its homogeneous regular elements and obtain a graded-simple ring. In other words, what is required is a version of Goldie's Theorem for the setting of graded rings. This cannot be obtained in general – there are easy examples of commutative, noetherian, semiprime $\mathbb{Z}$-graded rings where the localization with respect to all homogeneous regular elements is not graded-simple. For our present purposes, it suffices to consider prime graded rings, for which the following theorem is available.

1.17. Graded Goldie Theorem. *Let G be an abelian group, and let R be a G-graded, graded-prime, right graded-Goldie ring. Let $\mathcal{E}$ be the set of all homogeneous regular elements in R. Then $\mathcal{E}$ is a right denominator set, and $R[\mathcal{E}^{-1}]$ is a graded-simple, graded-artinian ring.*

Proof. [**19**, Theorem 1]. □

Theorem 1.17 moves us to the setting of graded-simple rings, and the prime ideals in such rings can be analyzed as follows.

1.18. Proposition. *Let G be an abelian group, and let R be a G-graded, graded-simple ring.*

(a) spec R *is homeomorphic to* spec $Z(R)$ *via contraction and extension.*

(b) *If $G \cong \mathbb{Z}^r$, then $Z(R)$ is a Laurent polynomial ring, in at most r indeterminates, over the field $Z(R)_1$ (the identity component of $Z(R)$).*

Proof. [**3**, Lemma II.3.7 and Proposition II.3.8]. □

Let us conclude Part I by presenting an open problem.

1.19. Problem. Suppose that A is a noetherian k-algebra, and that a torus $H = (k^\times)^r$ acts rationally on A. Find conditions which imply that A has only finitely many H-primes. These conditions should be

- Reasonably easy to verify; and
- Satisfied by all the standard examples.

In other words, we would like to have a theorem which we can apply to quantized coordinate rings without masses of long calculations. In seeking such a theorem, a warning is in order: When the parameters are roots of unity, quantized coordinate rings usually have infinitely many H-primes. Thus, whatever hypotheses might be used in a solution to this problem will have to correspond to the generic situation when applied to quantized coordinate rings.

II. Primitive Ideals

We now concentrate on primitive ideals as opposed to general prime ideals, and on ways to organize the *primitive spectrum* of a ring A. This set, denoted $\operatorname{prim} A$, is the set of all left primitive ideals of A. We view $\operatorname{prim} A$ as a topological space equipped with the Zariski topology, so that $\operatorname{prim} A$ is a subspace of $\operatorname{spec} A$.

The question whether the left primitive ideals and the right primitive ideals coincide in a noetherian ring remains open. To avoid this problem, we shall use the term *primitive ideal* to refer only to *left* primitive ideals.

In a classical coordinate ring over an algebraically closed field, the maximal ideals correspond to points of the underlying variety. A naive geometric analogy in the non-commutative world would be to view the maximal ideals in a ring as points of a 'non-commutative variety'. However, experience in ring theory teaches us that there are too few maximal ideals in general to hold sufficient information. Further, the influence of representation theory leads us to study the primitive ideals, as one key to irreducible representations (i.e., simple modules). Let us consider our simplest example, $\mathcal{O}_q(k^2)$.

2.1. Example. Assume that k is algebraically closed and q is not a root of unity. The prime ideals of $\mathcal{O}_q(k^2)$ are displayed in Example 1.2. Observe that the only maximal ideals are the ideals

$$\langle x-\alpha,\, y\rangle \qquad \text{and} \qquad \langle x,\, y-\beta\rangle,$$

for $\alpha, \beta \in k$. Comparing these with the maximal ideals in the classical coordinate ring $\mathcal{O}(k^2)$, we see that the maximal ideals of $\mathcal{O}_q(k^2)$ correspond only to points on the x- and y-axes of k^2. From this point of view, the remainder of the xy-plane has been 'lost'.

As suggested above, let us widen our view to include all the primitive ideals. In $\mathcal{O}_q(k^2)$, there is one non-maximal primitive ideal, namely $\langle 0\rangle$. Thus, comparing k^2 with $\operatorname{prim} \mathcal{O}_q(k^2)$, we can now say that the points on the x- and y-axes correspond precisely to the maximal ideals of $\mathcal{O}_q(k^2)$, while all other points of k^2 correspond (not bijectively, of course) to the zero ideal. We could say that the off-axis part of k^2 has 'collapsed' to a single point. Later, we shall elaborate this point of view further.

Since the primitive ideals of an algebra A are (by definition) the annihilators of the simple A-modules, it would seem that to determine these primitive ideals, we should find all the simple A-modules and then calculate their annihilators. However, to find all the simple modules over an infinite dimensional algebra is usually an impossible problem. As a substitute, Dixmier promulgated the following program for enveloping algebras of Lie algebras: Find the primitive ideals first, and then for each primitive ideal, find at least one simple module having that annihilator. In order to carry out this program, we must be able to detect the primitive ideals without knowing the simple modules in advance, and so some criterion other then the definition is required. Since all primitive ideals are prime, the question becomes, how can we tell which prime ideals are primitive? Dixmier developed two criteria, one of which is purely algebraic and one of which is phrased topologically, as follows.

2.2. Rational and locally closed primes. Let P be a prime ideal in a noetherian k-algebra A. First, we say that P is *rational* if and only if $Z(\operatorname{Fract} A/P)$ is algebraic over k. Secondly, we say that P is *locally closed* provided P is a locally closed point in $\operatorname{spec} A$, i.e., the singleton set $\{P\}$ is closed in some neighborhood of P. This condition may be rephrased as follows: P is locally closed if and only if

$$\bigcap\{Q \in \operatorname{spec} A \mid Q \supsetneq P\} \supsetneq P.$$

Thus, P is locally closed if and only if in the prime ring R/P, the intersection of all nonzero prime ideals is nonzero. Prime rings with the latter property are sometimes called *G-rings*, in which case locally closed primes are called *G-ideals*.

2.3. Theorem. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field of characteristic zero. Then*

$$\begin{aligned} \operatorname{prim} U(\mathfrak{g}) &= \{\text{locally closed prime ideals of } U(\mathfrak{g})\} \\ &= \{\text{rational prime ideals of } U(\mathfrak{g})\} \end{aligned}$$

Proof. This theorem was originally proved by Dixmier [9] and Moeglin [32] assuming an algebraically closed base field. Their result was extended to the non-algebraically closed case by Irving and Small [26]. □

2.4. The Dixmier-Moeglin equivalence. We say that an algebra A satisfies the *Dixmier-Moeglin equivalence* if the conclusion of Theorem 2.3 holds in A, that is, the primitive, locally closed, and rational prime ideals of A all coincide.

There are some relations among these three types of prime ideals which hold under fairly general hypotheses. One such hypothesis is the following adaptation of Hilbert's Nullstellensatz to noncommutative noetherian algebras.

2.5. The noncommutative Nullstellensatz. A k-algebra A is said to satisfy the *Nullstellensatz over k* if and only if

(a) The Jacobson radical of every factor ring of A is nil; and
(b) $\operatorname{End}_A(M)$ is algebraic over k for all simple A-modules M.

If A is noetherian, condition (a) is equivalent to A being a Jacobson ring, i.e., $J(A/P) = 0$ for all $P \in \operatorname{spec} A$.

The Nullstellensatz is essentially automatic if the field is large enough. In particular:

2.6. Proposition. [1] *If k is uncountable, then every countably generated k-algebra satisfies the Nullstellensatz over k.*

Proof. [31, Corollary 9.1.8]. □

For algebras over countable fields, the following theorem is often useful.

2.7. Theorem. *Suppose that a k-algebra A has subalgebras $A_0 = k \subseteq A_1 \subseteq \cdots \subseteq A_t = A$ such that for all $i > 0$, either A_i is a finitely generated A_{i-1}-module on each side, or A_i is a homomorphic image of a skew polynomial ring $A_{i-1}[x_i; \tau_i, \delta_i]$. Then A satisfies the Nullstellensatz over k.*

Proof. This is a special case of [31, Theorem 9.4.21]. □

For prime ideals in a noetherian algebra satisfying the Nullstellensatz, the following general implications are known:

$$\text{locally closed} \Longrightarrow \text{primitive} \Longrightarrow \text{rational}$$

[3, Lemma II.7.15]. Closing the loop (i.e., proving that 'rational $\Longrightarrow$ locally closed') is usually the most difficult part of establishing that an algebra satisfies the Dixmier-Moeglin equivalence. In the situation of the Stratification Theorem, it is advantageous to bring the torus action into the loop – this helps in the proofs, and supplies an additional criterion for primitivity, namely the condition that a prime ideal be a maximal element of its H-stratum. Then, two implications need to be proved to close the loop, namely

$$\text{rational} \Longrightarrow \text{maximal in stratum} \Longrightarrow \text{locally closed},$$

but the second is quite easy. The precise theorem is as follows.

2.8. Theorem. [17] *Let A be a noetherian k-algebra with k infinite, and let $H = (k^\times)^r$ be a torus acting rationally on A. Assume that H-spec A is finite, and that A satisfies the Nullstellensatz over k. Then*

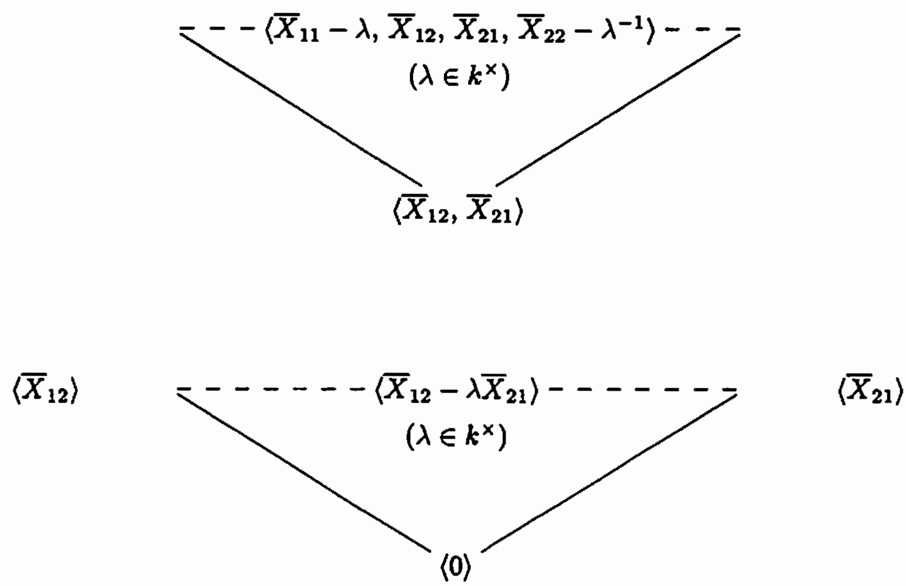
$$\begin{aligned} \operatorname{prim} A &= \{\text{locally closed prime ideals of } A\} \\ &= \{\text{rational prime ideals of } A\} \\ &= \bigsqcup_{J \in H\text{-spec}\, A} \{\text{maximal elements of } \operatorname{spec}_J A\}. \end{aligned}$$

Moreover, if k is algebraically closed, the H-orbits in prim A *coincide with the H-strata* $\operatorname{prim}_J A \overset{\text{def}}{=} (\operatorname{prim} A) \cap (\operatorname{spec}_J A)$.

Proof. [3, Theorems II.8.4 and II.8.14]. □

Of the three criteria for primitivity given in this theorem, the third is typically easiest to apply. Here is an illustration.

2.9. Example. Let us return to our second basic example, $\mathcal{O}_q(SL_2(k))$, assuming that k is algebraically closed and q is not a root of unity. Recall from (1.3) the rational action of $H = (k^\times)^2$ on this algebra. The prime spectrum of $\mathcal{O}_q(SL_2(k))$ was displayed in Example 1.5, and the action of H on these prime ideals is easy to determine. In particular, there are only four H-prime ideals in $\mathcal{O}_q(SL_2(k))$, namely $\langle \overline{X}_{12}, \overline{X}_{21} \rangle$, $\langle \overline{X}_{12} \rangle$, $\langle \overline{X}_{21} \rangle$, and $\langle 0 \rangle$. Thus, there are four H-strata in $\operatorname{spec} \mathcal{O}_q(SL_2(k))$, which we display as follows.



The Nullstellensatz holds for $\mathcal{O}_q(SL_2(k))$ by Theorem 2.7. Hence, Theorem 2.8 implies that $\operatorname{prim} \mathcal{O}_q(SL_2(k))$ consists of all prime ideals except for $\langle \overline{X}_{12}, \overline{X}_{21} \rangle$ and $\langle 0 \rangle$. Moreover, there are precisely four H-orbits in $\operatorname{prim} \mathcal{O}_q(SL_2(k))$:

$$\{ \langle \overline{X}_{11} - \lambda, \overline{X}_{12}, \overline{X}_{21}, \overline{X}_{22} - \lambda^{-1} \rangle \mid \lambda \in k^\times \}$$

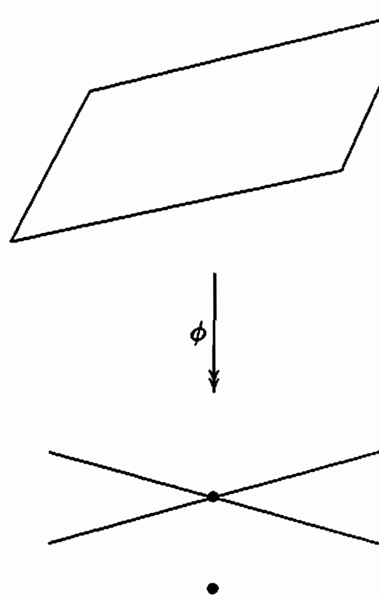
$$\{ \langle \overline{X}_{12} \rangle \} \qquad \{ \langle \overline{X}_{12} - \lambda \overline{X}_{21} \rangle \mid \lambda \in k^\times \} \qquad \{ \langle \overline{X}_{21} \rangle \}$$

Now that we have access to finding the primitive ideals in quantized coordinate rings, let us turn to the global problem – trying to understand the primitive spectrum of such an algebra A as a whole. We would like $\operatorname{prim} A$ to reflect some kind of 'noncommutative geometry'. Since there is as yet no indication of what might play the role of regular functions on $\operatorname{prim} A$, we focus for now on the topological structure of this space.

For the remainder of Part II, assume that k is algebraically closed.

2.10. Problem. Let V be an affine variety over k, with classical coordinate ring $\mathcal{O}(V)$, and suppose that A is some quantized coordinate ring of V. Since $\max \mathcal{O}(V) \approx V$, we may view $\operatorname{prim} A$ as a 'quantization of V'. Then the problem arises: How are $\operatorname{prim} A$ and V related?

2.11. Example. Assume that q is not a root of unity. In Example 2.1, we suggested that prim $\mathcal{O}_q(k^2)$ could be viewed as the union of the x-axis, the y-axis, and one other point obtained from collapsing the rest of the xy-plane. This leads to a map ϕ from k^2 onto prim $\mathcal{O}_q(k^2)$, as in the following sketch:



To describe this map more precisely, recall that prim $\mathcal{O}_q(k^2)$ consists of the maximal ideals $\langle x-\alpha,\, y\rangle$ and $\langle x,\, y-\beta\rangle$, for $\alpha, \beta \in k$, together with $\langle 0\rangle$. Thus, ϕ is given as follows:

$$\begin{aligned} (\alpha,0) &\longmapsto \langle x-\alpha,\, y\rangle \\ (0,\beta) &\longmapsto \langle x,\, y-\beta\rangle \\ \text{other } (\alpha,\beta) &\longmapsto \langle 0\rangle\ . \end{aligned}$$

It is easy to check that ϕ is continuous. In fact, the topology on prim $\mathcal{O}_q(k^2)$ equals the quotient topology induced by ϕ. Thus, prim $\mathcal{O}_q(k^2)$ is a topological quotient of k^2.

2.12. Conjecture. *If an algebra A is one of the 'standard' quantized coordinate rings of an affine variety V, then* prim A *is a topological quotient of V.*

This conjecture is known to hold in several cases:

(1) $A = \mathcal{O}_q(SL_2(k))$, when q is not a root of unity. We invite the reader to try this as an exercise.

(2) $A = \mathcal{O}_{\mathbf{q}}((k^\times)^n) \stackrel{\text{def}}{=} \mathcal{O}_{\mathbf{q}}(k^n)[x_1^{-1}, \ldots, x_n^{-1}]$, for arbitrary q. This follows from work of De Concini-Kac-Procesi [8], Hodges [20], Vancliff [36], Brown-Goodearl [2], Goodearl-Letzter [16], and others.

(3) $A = \mathcal{O}_{\mathbf{q}}(k^n)$ and a more general type of algebra known as a 'quantum toric variety' (which we will describe below), when the subgroup of $k^\times$ generated by the entries of $\mathbf{q}$ does not contain -1. This is work of Goodearl and Letzter **[18]**.

(4) $A = \mathcal{O}_q(\mathfrak{sp}\, k^4)$, the single-parameter quantized coordinate ring of symplectic 4-space, when q is not a root of unity. This algebra was first defined in **[35]**; for somewhat simpler presentations see **[33]** and **[24]**. The topological quotient here was established by Horton **[23**, Theorem 7.9**]**.

2.13. Quantum tori. Given a multiplicatively antisymmetric $n \times n$ matrix $\mathbf{q} = (q_{ij})$ over k, the corresponding *quantized coordinate ring of* $(k^\times)^n$ is the k-algebra

$$\mathcal{O}_{\mathbf{q}}((k^\times)^n) \stackrel{\text{def}}{=} k\langle x_1^{\pm1}, \dots, x_n^{\pm1} \mid x_i x_j = q_{ij} x_j x_i \text{ for all } i, j \rangle.$$

The torus $H = (k^\times)^n$ acts rationally on the algebra $A = \mathcal{O}_{\mathbf{q}}((k^\times)^n)$ in the same way as it acts on $\mathcal{O}_{\mathbf{q}}(k^n)$. As is easily checked, $\langle 0 \rangle$ is the only H-prime in A, and $\operatorname{prim} A$ is a single H-stratum as well as a single H-orbit. Hence, for any primitive ideal P, there is a bijection

$$H/\operatorname{Stab}_H(P) \longleftrightarrow \operatorname{prim} A.$$

This bijection is a homeomorphism, assuming that $H/\operatorname{Stab}_H(P)$ is given the quotient topology. Thus, the fact that $\operatorname{prim} A$ is a topological quotient of H is easily established in this case.

2.14. Quantum affine spaces. Now let $A = \mathcal{O}_{\mathbf{q}}(k^n)$, and recall that $H = (k^\times)^n$ acts rationally on A by k-algebra automorphisms such that $(\alpha_1, \dots, \alpha_n).x_i = \alpha_i x_i$ for $(\alpha_1, \dots, \alpha_n) \in H$ and $i = 1, \dots, n$. Let W be the collection of subsets of $\{1, \dots, n\}$. There is a bijection

$$\begin{aligned} W &\longrightarrow H\text{-spec}\, A \\ w &\longmapsto J_w \stackrel{\text{def}}{=} \langle x_i \mid i \in w \rangle. \end{aligned}$$

Thus, the H-stratifications of $\operatorname{spec} A$ and $\operatorname{prim} A$, and the localizations of A appearing in the Stratification Theorem, are indexed by the H-primes J_w. To simplify notation, we re-index using W. In particular, we write

$$\begin{aligned} \operatorname{prim}_w A &\stackrel{\text{def}}{=} \operatorname{prim}_{J_w} A = \{P \in \operatorname{prim} A \mid x_i \in P \iff i \in w\} \\ A_w &\stackrel{\text{def}}{=} A_{J_w} = (A/J_w)[x_j^{-1} \mid j \notin w] \end{aligned}$$

for $w \in W$. Note that each A_w is a quantum torus.

The torus H acts on $\mathcal{O}(k^n)$ exactly as it does on A; this action is induced from the action of H on k^n by the rule

$$(\alpha_1, \dots, \alpha_n).(a_1, \dots, a_n) \stackrel{\text{def}}{=} (\alpha_1^{-1} a_1, \dots, \alpha_n^{-1} a_n).$$

There are 2^n H-orbits in k^n, which we index by W as follows:

$$(k^n)_w \stackrel{\text{def}}{=} \{(a_1, \dots, a_n) \in k^n \mid a_i = 0 \iff i \in w\}$$

for $w \in W$. We may note that $(k^n)_w$ is isomorphic to a torus of rank $n - |w|$. The results discussed in (2.13) imply that $\text{prim}_w A$ is a topological quotient of $(k^n)_w$ for each w. Thus, the problem is to patch individual topological quotient maps $(k^n)_w \twoheadrightarrow \text{prim}_w A$ together, to obtain a topological quotient map $k^n \twoheadrightarrow \text{prim} A$. Our solution to this problem requires a small technical condition, phrased in terms of $\langle q_{ij} \rangle$, the subgroup of $k^\times$ generated by the entries q_{ij} of $\mathbf{q}$.

2.15. Theorem. *Assume that either $-1 \notin \langle q_{ij} \rangle$ or $\text{char}\, k = 2$. Then there exist compatible, H-equivariant topological quotient maps*

$$k^n \twoheadrightarrow \text{prim}\, \mathcal{O}_{\mathbf{q}}(k^n) \qquad \text{and} \qquad \text{spec}\, \mathcal{O}(k^n) \twoheadrightarrow \text{spec}\, \mathcal{O}_{\mathbf{q}}(k^n)$$

such that for $w \in W$, the inverse image of $\text{prim}_w \mathcal{O}_{\mathbf{q}}(k^n)$ is $(k^n)_w$. Moreover, the fibres over points in $\text{prim}_w \mathcal{O}_{\mathbf{q}}(k^n)$ are G_w-orbits in $(k^n)_w$ for certain subgroups $G_w \subseteq H$.

Proof. [**18**, Theorem 4.11; **11**, Theorem 3.5]. These papers also describe how to calculate the subgroups G_w. □

To illustrate this theorem, we use a single parameter quantum affine 3-space, this being the simplest case in which the topological quotient map differs from what one might naively write down.

2.16. Example. Choose a non-root of unity $q \in k^\times$, and let $A = \mathcal{O}_q(k^3)$. Then the entries q_{ij} in $\mathbf{q}$ consist of q, q^{-1}, and 1, and so the group $\langle q_{ij} \rangle$ is infinite cyclic. In particular, $-1 \notin \langle q_{ij} \rangle$ unless $\text{char}\, k = 2$. Now let p be one of the square roots of q in $k^\times$. The topological quotient map $k^3 \twoheadrightarrow \text{prim} A$ given by Theorem 2.15 can be described as shown below, where all $\lambda_i \in k^\times$.

$$(0,0,0) \longmapsto \langle x_1, x_2, x_3 \rangle$$

$$(\lambda_1,0,0) \longmapsto \langle x_1 - \lambda_1, x_2, x_3 \rangle \qquad (\lambda_1,\lambda_2,0) \longmapsto \langle x_3 \rangle$$

$$(0,\lambda_2,0) \longmapsto \langle x_1, x_2 - \lambda_2, x_3 \rangle \qquad (\lambda_1,0,\lambda_3) \longmapsto \langle x_2 \rangle$$

$$(0,0,\lambda_3) \longmapsto \langle x_1, x_2, x_3 - \lambda_3 \rangle \qquad (0,\lambda_2,\lambda_3) \longmapsto \langle x_1 \rangle$$

$$(\lambda_1,\lambda_2,\lambda_3) \longmapsto \langle \lambda_2 x_1 x_3 - p\lambda_1\lambda_3 x_2 \rangle\,.$$

Note the appearance of p in the final line – without that factor, the resulting map from k^3 to $\text{prim} A$ will still be surjective, but not Zariski-continuous.

2.17. We indicate one basic mechanism from the proof of Theorem 2.15. Given a multiplicatively antisymmetric $n \times n$ matrix $\mathbf{q}$, we write parallel, coordinate-free descriptions of the algebras $R = \mathcal{O}(k^n)$ and $A = \mathcal{O}_{\mathbf{q}}(k^n)$ as follows. Namely, $R = k(\mathbb{Z}^+)^n$, a semigroup algebra, and $A = k^c(\mathbb{Z}^+)^n$, a twisted semigroup algebra for a suitable cocycle $c : \mathbb{Z}^n \times \mathbb{Z}^n \longrightarrow k^\times$. There are many choices of c; we just need to have $c(\epsilon_i, \epsilon_j)c(\epsilon_j, \epsilon_i)^{-1} = q_{ij}$ for all i, j, where $\epsilon_1, \dots, \epsilon_n$ is the standard basis for $\mathbb{Z}^n$. Now both R and A have bases identified with $(\mathbb{Z}^+)^n$, and so there is a vector space isomorphism $\Phi_c : A \longrightarrow R$ which is the identity on $(\mathbb{Z}^+)^n$. Similarly, Φ_c extends to a vector space isomorphism from the group algebra $k\mathbb{Z}^n$ onto the twisted group algebra $k^c\mathbb{Z}^n$, and so for each $w \in W$ we obtain a vector space isomorphism Φ_c from A_w onto a subalgebra R_w of $k\mathbb{Z}^n$. The key to Theorem 2.15 is to choose c so that the above maps behave well:

2.18. Key Lemma. *There is a choice of cocycle c such that Φ_c yields* k-algebra *maps $Z(A_w) \longrightarrow R_w$ for all w.*

For this choice of c, the topological quotient maps $\max R \twoheadrightarrow \operatorname{prim} A$ *and* $\operatorname{spec} R \twoheadrightarrow \operatorname{spec} A$ *can be described by the rule*

$$Q \longmapsto \big(\text{the largest ideal of } A \text{ contained in } \Phi_c^{-1}(Q)\big).$$

Proof. The first statement follows from [**18**, (4.2), (4.6–4.8), (3.5)], while the second is [**11**, Lemma 3.6]. □

For a more precise description of this map in terms of operations within the commutative algebra R, see [**18**, **11**].

Since the method just sketched is based on twisting the polynomial ring $k(\mathbb{Z}^+)^n$ by a cocycle, it readily extends to a somewhat more general class of algebras twisted by cocycles.

2.19. Cocycle twists. Suppose that G is a group, and that R is a G-graded k-algebra. Let $c : G \times G \to k^\times$ be a 2-cocycle, normalized so that $c(1,1) = 1$ (or $c(0,0) = 1$, in case G is written additively). The *twist of R by c* is a k-algebra based on the same G-graded vector space as R, but with a new multiplication $*$ defined on homogeneous elements as follows: $r * s \overset{\text{def}}{=} c(\alpha,\beta)rs$ for $r \in R_\alpha$ and $s \in R_\beta$.

Now specialize to the case that R is a commutative affine G-graded k-algebra, and A is the twist of R by a 2-cocycle c. Then R is generated by finitely many homogeneous elements, say $y_1, \dots, y_n$, of degrees $\alpha_1, \dots, \alpha_n$. The algebra A is generated by the same elements $y_1, \dots, y_n$, and $y_i * y_j = q_{ij} y_j * y_i$ for all i, j, where $q_{ij} = c(\alpha_i, \alpha_j) c(\alpha_j, \alpha_i)^{-1}$. Consequently, $A \cong \mathcal{O}_{\mathbf{q}}(k^n)/I$ for $\mathbf{q} = (q_{ij})$ and some ideal I.

In particular, if $G = \mathbb{Z}^n$ and $\dim R_\alpha = 1$ for all $\alpha \in G$, then R is the coordinate ring of an affine toric variety V, and we regard A as a quantized coordinate ring of V. This case was studied by Ingalls [**25**], who introduced the term *quantum toric variety* to describe the resulting algebras A.

The constructions behind Theorem 2.15 adapt well to factor algebras $\mathcal{O}_{\mathbf{q}}(k^n)/I$, and that theorem extends to the cocyle twisted setting as follows.

2.20. Theorem. *Let G be a torsionfree abelian group, and let R be a commutative, affine, G-graded k-algebra. Let A be the twist of R by a 2-cocycle $c : G \times G \to k^\times$. Assume that $-1 \notin \langle \text{image}(c) \rangle \subseteq k^\times$, or that* $\operatorname{char} k = 2$.

Then there exist compatible topological quotient maps

$$\max R \twoheadrightarrow \operatorname{prim} A \qquad \text{and} \qquad \operatorname{spec} R \twoheadrightarrow \operatorname{spec} A,$$

which are equivariant with respect to the action of a suitable torus.

Proof. [**18**, Theorem 6.3; **11**, Theorem 4.5]. □

It is not clear whether the hypothesis concerning -1 can be removed from Theorems 2.15 and 2.20. We end Part II by putting an extreme case forward as an open problem.

2.21. Problem. Assume that char $k \neq 2$. Consider the single parameter algebras

$$\mathcal{O}_{-1}(k^n) = k\langle x_1, \dots, x_n \mid x_i x_j = -x_j x_i \text{ for all } i \neq j \rangle.$$

The methods used to prove Theorem 2.15 still work for $\mathcal{O}_{-1}(k^2)$ and $\mathcal{O}_{-1}(k^3)$. These methods break down for $\mathcal{O}_{-1}(k^4)$, but extensive ad hoc calculations lead to a Zariski-continuous surjection $k^4 \twoheadrightarrow \operatorname{prim} \mathcal{O}_{-1}(k^4)$; in higher dimensions, the problem is completely open. Thus, we ask:

For $n \geq 4$, is the space $\operatorname{prim} \mathcal{O}_{-1}(k^n)$ a topological quotient of k^n?

III. Quantum Matrices

The focus on topological quotients in Part II was chosen to emphasize one way in which the quantized coordinate ring of a variety can be geometrically similar to the classical coordinate ring. We can also ask about algebraic similarities, of which there are many – chain conditions, homological conditions, etc. In fact, there exist much tighter similarities – many classical theorems have surprisingly close quantum analogs, once they are properly rephrased. We illustrate this principle by discussing quantum matrices, that is, the quantized coordinate rings of varieties of matrices. The 2×2 case was presented in (1.1); we now give the general definition.

3.1. Generators and relations. Let n be a positive integer and $q \in k^\times$. The *quantized coordinate ring of $n \times n$ matrices* with parameter q is the k-algebra with generators X_{ij} for $i, j = 1, \dots, n$ such that for all $i < l$ and $j < m$, the generators $X_{ij}, X_{im}, X_{lj}, X_{lm}$ satisfy the defining relations for $\mathcal{O}_q(M_2(k))$. As in (1.1), five of these relations can be summarized in the following mnemonic diagram:

$$\begin{array}{ccc} X_{ij} & \xrightarrow{\;q\;} & X_{im} \\ {\scriptstyle q}\big\downarrow & \overset{1}{\nearrow\!\!\swarrow} & \big\downarrow{\scriptstyle q} \\ X_{lj} & \xrightarrow{\;q\;} & X_{lm} \end{array}$$

The remaining relation is $X_{ij}X_{lm} - X_{lm}X_{ij} = (q - q^{-1})X_{im}X_{lj}$.

The $n \times n$ quantum determinant is modelled on the usual determinant, but with powers of -1 replaced by powers of $-q$. More precisely, the $n \times n$ *quantum determinant* is the element

$$D_q \overset{\text{def}}{=} \sum_{\pi \in S_n} (-q)^{\ell(\pi)} X_{1,\pi(1)} X_{2,\pi(2)} \cdots X_{n,\pi(n)} \quad \in \quad \mathcal{O}_q(M_n(k)),$$

where S_n denotes the symmetric group and $\ell(\pi)$, the *length* of a permutation π, is the minimum length for an expression of π as a product of simple transpositions $(i, i+1)$. It is known that D_q lies in the center of $\mathcal{O}_q(M_n(k))$. Hence, one defines quantized coordinate rings $\mathcal{O}_q(GL_n(k)) \overset{\text{def}}{=} \mathcal{O}_q(M_n(k))[D_q^{-1}]$ and $\mathcal{O}_q(SL_n(k)) \overset{\text{def}}{=} \mathcal{O}_q(M_n(k))/\langle D_q - 1 \rangle$ as before.

The algebra $\mathcal{O}_q(M_n(k))$ is a bialgebra with comultiplication and counit maps

$$\Delta : \mathcal{O}_q(M_n(k)) \longrightarrow \mathcal{O}_q(M_n(k)) \otimes \mathcal{O}_q(M_n(k)) \qquad \text{and} \qquad \varepsilon : \mathcal{O}_q(M_n(k)) \longrightarrow k$$

such that $\Delta(X_{ij}) = \sum_{l=1}^{n} X_{il} \otimes X_{lj}$ and $\varepsilon(X_{ij}) = \delta_{ij}$ for all i, j. In particular, $\Delta(D_q) = D_q \otimes D_q$ and $\varepsilon(D_q) = 1$.

All this structure is exactly parallel to the classical case, which we get if $q = 1$. Much interesting geometry has resulted from viewing sets of matrices of a given size as algebraic varieties and focusing on constructs from linear algebra as geometric processes. One such line leads to determinantal ideals, as follows.

3.2. Classical determinantal ideals. Let $t \leq n$ be positive integers, and consider the variety

$$V_t \stackrel{\text{def}}{=} \{n \times n \text{ matrices of rank } < t\},$$

the closed subvariety of the affine space $M_n(k)$ defined by the vanishing of all $t \times t$ minors. From linear algebra, V_t is the image of the matrix multiplication map

$$M_{n,t-1}(k) \times M_{t-1,n}(k) \longrightarrow M_n(k).$$

Since $M_{n,t-1}(k)$ and $M_{t-1,n}(k)$ are irreducible varieties, it follows that V_t is irreducible.

Let $I_t \lhd \mathcal{O}(M_n(k))$ be the ideal of polynomial functions vanishing on V_t, so that $\mathcal{O}(M_n(k))/I_t = \mathcal{O}(V_t)$. On the geometric side, V_t is defined by the vanishing of all $t \times t$ minors. However, this only tells us that I_t equals the radical of the ideal generated by the $t \times t$ minors. It is a classical theorem that these minors actually generate this ideal:

3.3. Theorem. *I_t equals the ideal of $\mathcal{O}(M_n(t))$ generated by all $t \times t$ minors.*

Proof. See, e.g., [4, 7]. □

3.4. Corollary. *The set of all $t \times t$ minors in $\mathcal{O}(M_n(k))$ generates a prime ideal.* □

In the quantum world, there is no variety V_t, and so we cannot ask for a direct analog of Theorem 3.3. However, there are analogs of minors, which means that we can look for an analog of Corollary 3.4.

3.5. Quantum minors. Let $I, J \subseteq \{1, \ldots, n\}$ be index sets with $|I| = |J| = t$. We may write the elements of these sets in ascending order, say $I = \{i_1 < \cdots < i_t\}$ and $J = \{j_1 < \cdots < j_t\}$ for short. There is a natural k-algebra embedding $\phi_{I,J} : \mathcal{O}_q(M_t(k)) \to \mathcal{O}_q(M_n(k))$ such that $\phi_{I,J}(X_{lm}) = X_{i_l j_m}$ for all l, m. The *quantum minor with index sets I and J* is the element

$$[I|J] \stackrel{\text{def}}{=} \phi_{I,J}(D_q^{t\times t}) \quad \in \quad \mathcal{O}_q(M_n(k)),$$

where $D_q^{t\times t}$ denotes the quantum determinant in $\mathcal{O}_q(M_t(k))$.

3.6. Theorem. *The ideal I_t of $\mathcal{O}_q(M_n(k))$ generated by all $t \times t$ quantum minors is completely prime, i.e., $\mathcal{O}_q(M_n(k))/I_t$ is an integral domain.*

Proof. [13, Theorem 2.5]. □

Although many steps in the proof of the classical result have no analogs in the quantum case, one part of the classical pattern does carry over, as we now summarize.

3.7. As noted above, V_t is the image of the multiplication map

$$\mu : M_{n,t-1}(k) \times M_{t-1,n}(k) \longrightarrow M_n(k).$$

Hence, the ideal I_t is the kernel of the comorphism

$$\mu^* : \mathcal{O}(M_n) \longrightarrow \mathcal{O}(M_{n,t-1} \times M_{t-1,n}).$$

We may identify $\mathcal{O}(M_{n,t-1} \times M_{t-1,n})$ with $\mathcal{O}(M_{n,t-1}) \otimes \mathcal{O}(M_{t-1,n})$, which allows us to describe μ^* as the composition of the maps

$$\mathcal{O}(M_n) \xrightarrow{\Delta} \mathcal{O}(M_n) \otimes \mathcal{O}(M_n) \xrightarrow{\text{quo}\otimes\text{quo}} \mathcal{O}(M_{n,t-1}) \otimes \mathcal{O}(M_{t-1,n}).$$

3.8. A quantum analog. Quantized coordinate rings for the rectangular matrix varieties $M_{n,t-1}(k)$ and $M_{t-1,n}(k)$ may be defined as the subalgebras of $\mathcal{O}_q(M_n(k))$ generated by those X_{ij} with $j < t$ (respectively, $i < t$). There are natural k-algebra retractions of $\mathcal{O}_q(M_n(k))$ onto these subalgebras, and so

$$\mathcal{O}_q(M_{n,t-1}(k)) \cong \mathcal{O}_q(M_n(k))/\langle X_{ij} \mid j \geq t\rangle$$
$$\mathcal{O}_q(M_{t-1,n}(k)) \cong \mathcal{O}_q(M_n(k))/\langle X_{ij} \mid i \geq t\rangle.$$

Thus, the quantum analog of the comorphism μ^* in (3.7) is the k-algebra map .

$$\mu_q^* \stackrel{\text{def}}{=} \mathcal{O}_q(M_n) \xrightarrow{\Delta} \mathcal{O}_q(M_n) \otimes \mathcal{O}_q(M_n) \xrightarrow{\text{quo}\otimes\text{quo}} \mathcal{O}_q(M_{n,t-1}) \otimes \mathcal{O}_q(M_{t-1,n}).$$

It is easy to check that $\mathcal{O}_q(M_{n,t-1}(k)) \otimes \mathcal{O}_q(M_{t-1,n}(k))$ is an iterated skew polynomial algebra over k, and therefore a domain. Thus, to prove that the ideal I_t of $\mathcal{O}_q(M_n(k))$ is completely prime, one just has to show that $I_t = \ker(\mu_q^*)$. This is the heart of the proof of Theorem 3.6.

To understand quantum analogs of other geometric aspects of matrices, and also to understand the quantum matrix algebra better, we would like to know its prime and primitive ideals. We approach this problem via the Stratification Theorem, as discussed in Parts I and II.

For the remainder of Part III, assume that q is not a root of unity, and set $A = \mathcal{O}_q(M_n(k))$.

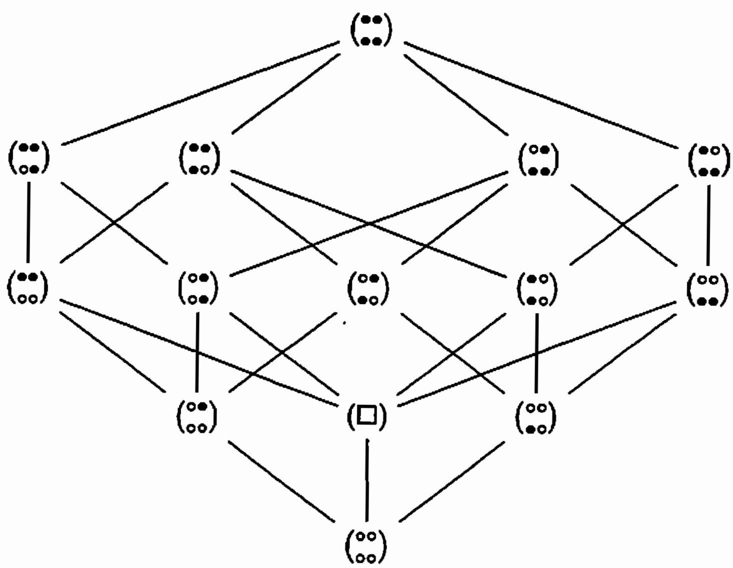
3.9. Problem. In parallel with the 2×2 case discussed in (1.3), the torus $H = (k^\times)^{2n}$ acts on A by k-algebra automorphisms so that

$$(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n).X_{ij} = \alpha_i \beta_j X_{ij}$$

for all i, j. These automorphisms are called 'winding automorphisms', because they arise from the bialgebra structure on A in a manner analogous to the classical winding automorphisms on enveloping algebras of Lie algebras. According to Theorem 1.15, there are at most 2^{n^2} H-prime ideals in A. The basic problem is:

Determine the H-primes of A.

3.10. Example. The 2×2 case of Problem 3.9 is easily solved – there are exactly 14 H-prime ideals in $\mathcal{O}_q(M_2(k))$, as displayed in the following diagram. Each 2×2 pattern here is shorthand for a set of generators of an ideal – a bullet • in position (i, j) corresponds to a generator X_{ij}; a circle ∘ in a given position is a placeholder; and the square □ denotes the 2×2 quantum determinant.



3.11. Certain types of H-primes in A are already known. For convenient labelling, we carry over the term 'rank' from ordinary matrices to the quantum case, as follows. We define the *rank* of a prime ideal P in A to be the minimum r such that P contains all $(r + 1) \times (r + 1)$ quantum minors.

The H-primes of A of rank n are those which do not contain the quantum determinant D_q. By localization, these correspond to the H-primes of $\mathcal{O}_q(GL_n(k))$, and it is known that those, in turn, correspond to the H-primes of $\mathcal{O}_q(SL_n(k))$. The latter can be determined using results of Hodges and Levasseur [21, 22]. In particular, $\mathcal{O}_q(SL_n(k))$ has $(n!)^2$ H-primes, parametrized by $S_n \times S_n$, and it follows from work of Joseph [27, Théorème 3] that each of these H-primes is generated by a set of quantum minors. We conclude that back in A, the H-primes of rank n are generated – up to localization at the powers of D_q – by sets of quantum minors.

At the other extreme, the H-primes of rank at most 1 are the H-primes of A which contain all 2×2 quantum minors. These were determined by Goodearl and Lenagan [12, Proposition 3.4]. There are $(2^n - 1)^2 + 1$ such H-primes, all having the form

$$\langle [I|J] \mid |I| = |J| = 2 \rangle + \langle X_{ij} \mid i \in R \rangle + \langle X_{ij} \mid j \in C \rangle$$

for $R, C \subseteq \{1, \ldots, n\}$. For these H-primes, we have generating sets consisting of quantum minors, since each X_{ij} is a 1×1 quantum minor.

3.12. Conjecture. *Every H-prime of A is generated by a set of quantum minors.*

It is easily seen that the conjecture holds when $n = 2$, in view of (3.10). Cauchon proved that there are enough quantum minors to separate the H-primes in A: For any H-primes $P \subsetneq Q$, there is a quantum minor in $Q \backslash P$ [**5**, Proposition 6.2.2 and Théorème 6.3.1]. The 3×3 case of the conjecture has been established by Goodearl and Lenagan [**15**, Theorem 7.4], and the $n \times n$ case, assuming that $k = \mathbb{C}$ and q is transcendental over $\mathbb{Q}$, has been proved by Launois [**30**, Théorème 3.7.2]. We shall display the solution to the 3×3 case below.

Cauchon's and Launois's results are existence theorems – they do not provide descriptions of which sets of quantum minors generate H-primes. Such descriptions are needed not only for completeness, but also to get full benefit from the stratification, e.g., to determine the prime and primitive ideals in each H-stratum via Theorems 1.14 and 2.8. Thus, we accompany Conjecture 3.12 with the following problems.

3.13. Problems. If $J \in H\text{-spec}\, A$, then Theorems 1.14 and 1.15 tell us that the center of the localization $A_J = (A/J)[\mathcal{E}_J^{-1}]$ is a Laurent polynomial ring $k[z_1^{\pm 1}, \dots, z_{n(J)}^{\pm 1}]$.

(a) Assuming J is generated by a set $\mathcal{M}$ of quantum minors, find a formula for $n(J)$ in terms of $\mathcal{M}$.

(b) Find explicit descriptions of the indeterminates $z_i \in A_J$.

We now summarize the solution to the 3×3 case of Conjecture 3.12 given in [**15**].

3.14. As in (3.11), we may divide up the H-primes in $\mathcal{O}_q(M_3(k))$ according to their ranks. Those of ranks 0, 1, and 3 were known earlier, while the ones of rank 2 were first determined in [**15**]. The numerical count is as follows:

rank 0 :	1
rank 1 :	49
rank 2 :	144
rank 3 :	36
total :	230.

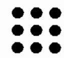
The determination of these H-primes was done partly by *ad hoc* methods, which are unlikely to work in the general case. In particular, Cauchon has given a formula for the total number of H-primes in $\mathcal{O}_q(M_n(k))$ [**5**, Théorème 3.2.2 and Proposition 3.3.2], which shows that $\mathcal{O}_q(M_4(k))$ has 6902 H-primes!

3.15. The H-primes in $\mathcal{O}_q(M_3(k))$ can be displayed as in the following diagrams, where each 3×3 pattern represents a set of generators for an H-prime. As in (3.10), circles are placeholders and bullets represent generators X_{ij}. This time, squares and rectangles represent 2×2 quantum minors whose row and column index sets correspond to the edges. Finally, the diamond that appears in four patterns represents the 3×3 quantum determinant. Below are samples showing the ideals corresponding to two patterns.

 $\longleftrightarrow$ $\langle X_{12}, X_{13}, [23|12], [23|13], [23|23] \rangle$

 $\longleftrightarrow$ $\langle [123|123], X_{31} \rangle$.

The case of rank 0 is trivial – there is only one H-prime of this rank, corresponding to the following pattern:



The 49 H-primes of rank 1 correspond to the patterns in Figure A below.

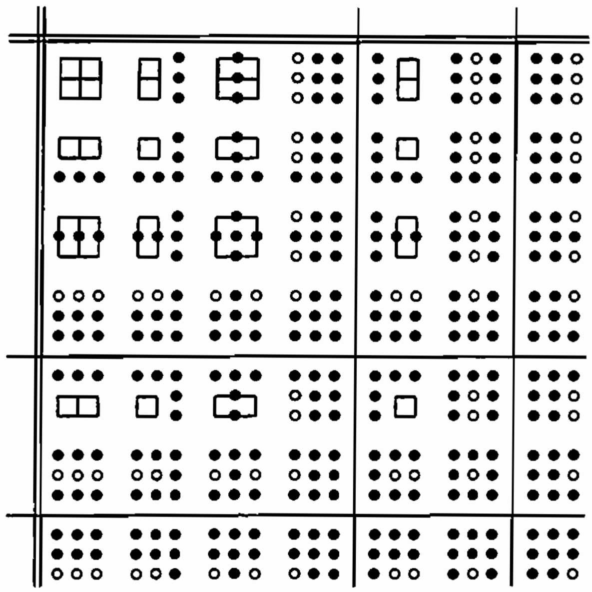


Figure A

As indicated in (3.11), the H-primes of maximal rank were known, up to localization, from the results of [21, 22, 27]. The sets of quantum minors which generate the corresponding H-primes in $\mathcal{O}_q(SL_3(k))$ also, as it turns out, generate the H-primes of rank 3 in $\mathcal{O}_q(M_3(k))$. These 36 ideals correspond to the patterns in Figure B.

The 144 patterns for generating sets of the H-primes of rank 2 in $\mathcal{O}_q(M_3(k))$ are given in our final display, Figure C. The procedure used in [15] to determine these H-primes involved three steps. First, some general theory developed in [14] provided a reduction mechanism relating the H-primes in $\mathcal{O}_q(M_n(k))$ to pairs of H-primes from smaller quantum matrix algebras. Consequently, we could find the H-primes in $\mathcal{O}_q(M_3(k))$ from the (known) H-primes in $\mathcal{O}_q(M_2(k))$, but only as kernels of certain algebra homomorphisms. This process also gave precise counts for the number of H-primes of each rank. In the second step, the information from Step 1 was used to determine the quantum minors contained in each H-prime, thus yielding at least potential sets of generators. Finally, the third step consisted of proving that each set of quantum minors appearing in Step 2 does generate an H-prime, and that the resulting H-primes are distinct. Since that yielded a list of the correct number of H-primes, we were done.

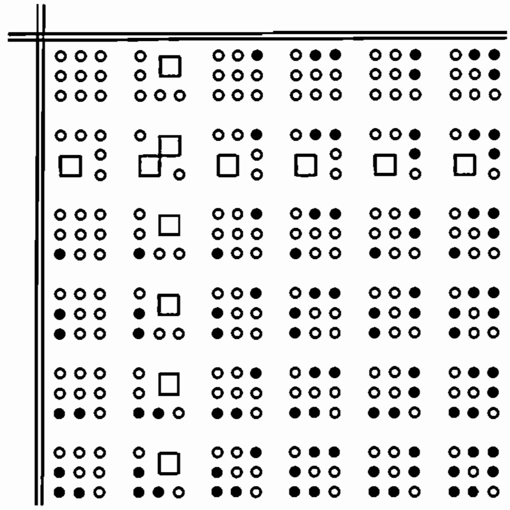


Figure B

It may be useful to break the general problem down into steps of similar type, as follows.

3.16. Problems. We return to the general $n \times n$ situation, keeping the field k arbitrary but still requiring q to be a non-root of unity.

(a) Which sets $\mathcal{M}$ of quantum minors in $\mathcal{O}_q(M_n(k))$ generate prime ideals?

(b) Develop general theorems to prove that suitable ideals of the form $\langle \mathcal{M} \rangle$ are prime.

(c) Find combinatorial data to parametrize the sets $\mathcal{M}$ in (a).

We conclude by stating a result which illustrates one pattern which solutions to the above problems might take. Part (a) is an easy exercise involving the relations among the X_{ij}, part (b) can be proved by showing that $\mathcal{O}_q(M_n(k))/\langle \mathcal{X} \rangle$ is an iterated skew polynomial extension of k, and part (c) is another easy exercise.

3.17. Sample result. (a) *If P is an H-prime ideal of $\mathcal{O}_q(M_n(k))$, then the set $\mathcal{X} = P \cap \{X_{ij} \mid i,j = 1,\ldots,n\}$ satisfies the following condition:*

(*) *If $X_{ij} \in \mathcal{X}$, then either $X_{lm} \in \mathcal{X}$ for all $l \geq i$ and $m \leq j$, or else $X_{lm} \in \mathcal{X}$ for all $l \leq i$ and $m \geq j$.*

(b) *If $\mathcal{X}$ is any subset of $\{X_{ij} \mid i,j = 1,\ldots,n\}$ which satisfies (*), then $\mathcal{X}$ generates an H-prime ideal of $\mathcal{O}_q(M_n(k))$, and $\langle \mathcal{X} \rangle \cap \{X_{ij} \mid i,j = 1,\ldots,n\} = \mathcal{X}$.*

(c) *Given subsets $I, J \subseteq \{1,\ldots,n\}$ and nondecreasing functions $f : \{1,\ldots,n\} \setminus J \to \{2,\ldots,n+1\} \setminus I$ and $g : \{1,\ldots,n\} \setminus I \to \{2,\ldots,n+1\} \setminus J$, the set*

$$\mathcal{X}(I,J,f,g) \stackrel{\text{def}}{=} \big\{X_{ij} \bigm| i \in I\big\} \cup \big\{X_{ij} \bigm| i \notin I;\ j \notin J;\ i \geq f(j)\big\}$$
$$\cup \big\{X_{ij} \bigm| j \in J\big\} \cup \big\{X_{ij} \bigm| i \notin I;\ j \notin J;\ j \geq g(i)\big\}$$

satisfies (*). Conversely, any subset $\mathcal{X} \subseteq \{X_{ij} \mid i,j = 1,\ldots,n\}$ which satisfies (*) equals $\mathcal{X}(I,J,f,g)$ for some I, J, f, g. $\square$

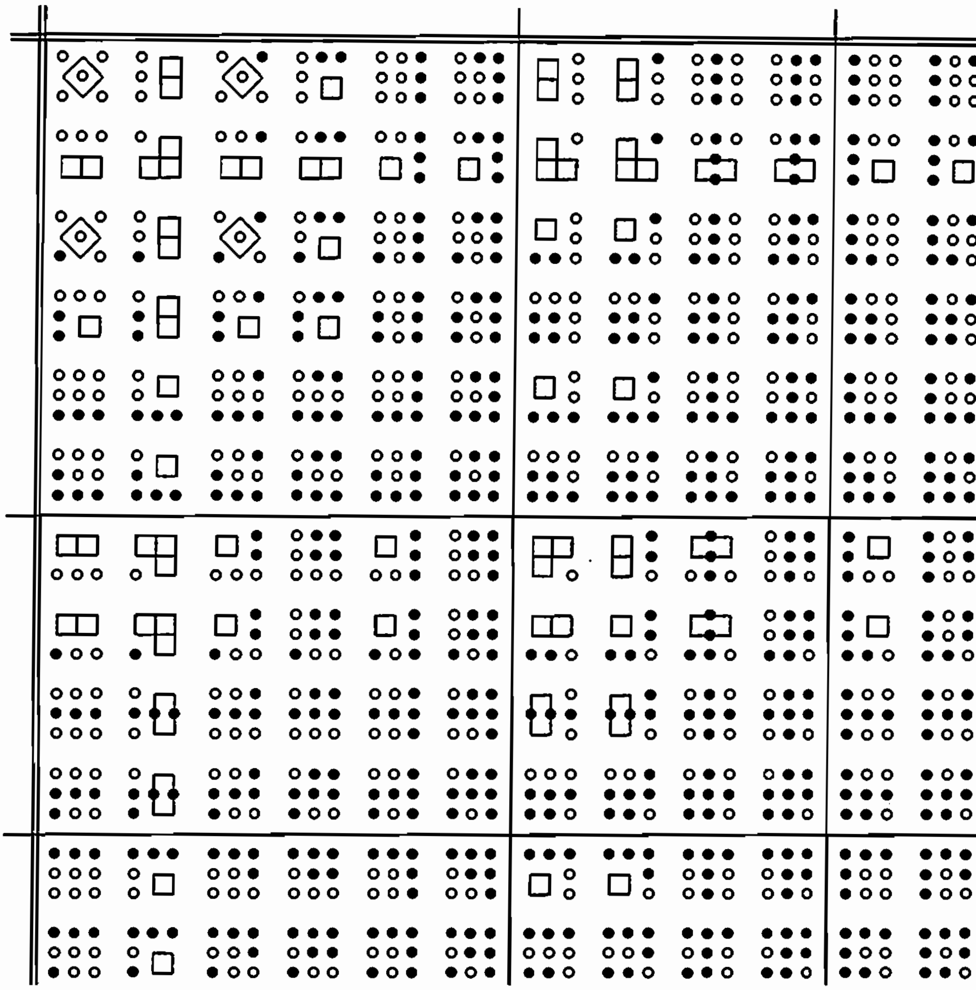


Figure C

References

1. A. S. Amitsur, *Algebras over infinite fields*, Proc. Amer. Math. Soc. 7 (1956), 35-48.
2. K. A. Brown and K. R. Goodearl, *Prime spectra of quantum semisimple groups*, Trans. Amer. Math. Soc. 348 (1996), 2465–2502.
3. ———, *Lectures on Algebraic Quantum Groups*, Advanced Courses in Mathematics CRM Barcelona, Birkhäuser, Basel, 2002.
4. W. Bruns and U. Vetter, *Determinantal Rings*, Lecture Notes in Math. 1327, Springer-Verlag, Berlin, 1988.
5. G. Cauchon, *Spectre premier de $O_q(M_n(k))$. Image canonique et séparation normale*, J. Algebra (to appear).
6. V. Chari and A. Pressley, *A Guide to Quantum Groups*, Cambridge Univ. Press, Cambridge, 1994.
7. C. De Concini, D. Eisenbud, and C. Procesi, *Young diagrams and determinantal varieties*, Invent.

Math. **56** (1980), 129-165.

8. C. De Concini, V. Kac, and C. Procesi, *Some remarkable degenerations of quantum groups*, Comm. Math. Phys. **157** (1993), 405–427.
9. J. Dixmier, *Idéaux primitifs dans les algèbres enveloppantes*, J. Algebra **48** (1977), 96–112.
10. K. R. Goodearl, *Prime spectra of quantized coordinate rings*, in Interactions between Ring Theory and Representations of Algebras (Murcia 1998) (F. Van Oystaeyen and M. Saorín, eds.), Dekker, New York, 2000, pp. 205-237.
11. ______, *Quantized primitive ideal spaces as quotients of affine algebraic varieties*, in Quantum Groups and Lie Theory (A. Pressley, ed.), London Math. Soc. Lecture Note Series 290, Cambridge Univ. Press, Cambridge, 2001, pp. 130-148.
12. K. R. Goodearl and T. H. Lenagan, *Prime ideals in certain quantum determinantal rings*, in Interactions between Ring Theory and Representations of Algebras (Murcia 1998) (F. Van Oystaeyen and M. Saorín, eds.), Dekker, New York, 2000, pp. 239-251.
13. ______, *Quantum determinantal ideals*, Duke Math. J. **103** (2000), 165-190.
14. ______, *Prime ideals invariant under winding automorphisms in quantum matrices*, Internat. J. Math. **13** (2002), 497-532.
15. ______, *Winding-invariant prime ideals in quantum* 3×3 *matrices*, J. Algebra (to appear).
16. K. R. Goodearl and E. S. Letzter, *Prime and primitive spectra of multiparameter quantum affine spaces*, in Trends in Ring Theory (Miskolc, 1996) (V. Dlab and L. Marki, eds.), Canad. Math. Soc. Conf. Proc. Series 22 (1998), 39-58.
17. ______, *The Dixmier-Moeglin equivalence in quantum coordinate rings and quantized Weyl algebras*, Trans. Amer. Math. Soc. **352** (2000), 1381-1403.
18. ______, *Quantum n-space as a quotient of classical n-space*, Trans. Amer. Math. Soc. **352** (2000), 5855-5876.
19. K. R. Goodearl and J. T. Stafford, *The graded version of Goldie's Theorem*, in Algebra and its Applications (Athens, Ohio, 1999) (D. V. Huynh, S. K. Jain, and S. R. López-Permouth, Eds.), Contemp. Math. **259** (2000), 237-240.
20. T. J. Hodges, *Quantum tori and Poisson tori*, Unpublished notes (1994).
21. T. J. Hodges and T. Levasseur, *Primitive ideals of* $\mathbf{C}_q[SL(3)]$, Comm. Math. Phys. **156** (1993), 581-605.
22. ______, *Primitive ideals of* $\mathbf{C}_q[SL(n)]$, J. Algebra **168** (1994), 455-468.
23. K. L. Horton, *Prime spectra of iterated skew polynomial rings of quantized coordinate type*, Ph.D. Dissertation (2002) University of California at Santa Barbara.
24. ______, *The prime and primitive spectra of multiparameter quantum symplectic and Euclidean spaces*, Communic. in Algebra (to appear).
25. C. Ingalls, *Quantum toric varieties* (to appear).
26. R. S. Irving and L. W. Small, *On the characterization of primitive ideals in enveloping algebras*, Math. Z. **173** (1980), 217–221.
27. A. Joseph, *Sur les ideaux génériques de l'algèbre des fonctions sur un groupe quantique*, C. R. Acad. Sci. Paris, Sér. I **321** (1995), 135-140.
28. C. Kassel, *Quantum Groups*, Grad. Texts in Math. 155, Springer-Verlag, New York, 1995.
29. A. Klimyk and K. Schmüdgen, *Quantum Groups and their Representations*, Springer-Verlag, Berlin, 1997.
30. S. Launois, *Les idéaux premiers invariants de* $O_q(\mathcal{M}_{m,p}(\mathbb{C}))$, Preprint (2002).
31. J. C. McConnell and J. C. Robson, *Noncommutative Noetherian Rings*, Wiley-Interscience, Chichester-New York, 1987; Reprinted with corrections, Grad. Studies in Math. 30, Amer. Math. Soc., Providence, 2001.
32. C. Moeglin, *Idéaux primitifs des algèbres enveloppantes*, J. Math. Pures Appl. **59** (1980), 265–336.
33. I. M. Musson, *Ring theoretic properties of the coordinate rings of quantum symplectic and Euclidean space*, in Ring Theory, Proc. Biennial Ohio State–Denison Conf., 1992 (S. K. Jain and S. T. Rizvi, eds.), World Scientific, Singapore, 1993, pp. 248-258.
34. D. G. Northcott, *Affine Sets and Affine Groups*, London Math. Soc. Lecture Note Series 39, Cambridge Univ. Press, Cambridge, 1980.

35. N. Yu. Reshetikhin, L. A. Takhtadjan, and L. D. Faddeev, *Quantization of Lie groups and Lie algebras*, Leningrad Math. J. 1 (1990), 193-225.
36. M. Vancliff, *Primitive and Poisson spectra of twists of polynomial rings*, Algebras and Representation Theory 2 (1999), 269-285.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SANTA BARBARA, CA 93106, USA
E-mail address: goodearl@math.ucsb.edu

An Introduction to Hopf Algebras via Crossed Products

Akira Masuoka

Throughout we work over a fixed ground field k.

It is interesting to see that parallel results hold for various kinds of algebraic systems. Let us first recall the following two theorems which are familiar to algebraists in general.

THEOREM(Wedderburn-Malcev). If A is a finite-dimensional algebra such that $A/\mathrm{Rad}\, A$ is a separable algebra, then the quotient morphism $A \longrightarrow A/\mathrm{Rad}\, A$ splits.

THEOREM(Levi). If $\mathfrak{g}$ is a finite-dimensional Lie algebra in characteristic zero, then the quotient morphism $\mathfrak{g} \longrightarrow \mathfrak{g}/\mathrm{Rad}\,\mathfrak{g}$ splits.

We have further the following parallel result on affine algebraic groups.

THEOREM(Chevalley-Mostow). Let G be an affine algebraic group in characteristic zero, and let G_u denote its unipotent radical, i.e., the largest closed normal unipotent subgroup. Then, G/G_u is linearly reductive and the quotient morphism $G \longrightarrow G/G_u$ splits, so that G is isomorphic to a semidirect product $G_u \rtimes G/G_u$.

The category of affine algebraic groups forms a full subcategory of the category of affine group schemes. The latter is in turn anti-isomorphic to the category of commutative Hopf algebras. Therefore one sees that the last theorem is generalized by the following.

The detailed version [M] of this article has been submitted for publication in the Banach Centre Publications.

THEOREM(Takeuchi [T]). Let H be a commutative Hopf algebra in characteristic zero. Then its coradical R (i.e., the largest cosemisimple subcoalgebra) is a Hopf subalgebra and the Hopf algebra inclusion $R \hookrightarrow H$ splits, so that H is isomorphic to $\overline{H} \rtimes R$, where $\overline{H} = H/HR^+$; this denotes the tensor-product algebra $\overline{H} \otimes R$ endowed with the coalgebra structure of semidirect coproduct which arises from a certain R-comodule coalgebra structure $\overline{H} \longrightarrow R \otimes \overline{H}$ on $\overline{H}$.

The original proof of the 'splitting' part of this theorem is not so easy; Abe gave up providing it in his textbook [A], only stating the theorem (Theorem 4.6.1).

If we remove from Takeuchi's theorem the assumption that H is commutative, what happens? To answer this, it seems natural to assume that the coradical R in H is a Hopf subalgebra. All pointed Hopf algebras H, including the quantized universal envelopes $U_q(\mathfrak{g})$, satisfy this assumption, since then R is spanned by the group of grouplikes in H. As our main result, we answer the question as follows.

THEOREM [M, Theorem 3.1]. Let H be a Hopf algebra in arbitrary characteristic. Suppose that the coradical R in H is a Hopf subalgebra. Then the inclusion $R \hookrightarrow H$ splits as a right R-module coalgebra map, so that H is isomorphic to the semidirect coproduct $H/HR^+ \rtimes R$ just as a right R-module coalgebra.

This gives a simple proof of the 'splitting' part of Takeuchi's theorem, and also of Sullivan's theorem, an analogous result in positive characteristic; see [M, Section 3]. Other applications are given in [MY; CDMM].

The idea which proves our theorem given above is indeed simple; it is related to Hopf crossed coproducts. To explain the idea, we choose, however, to work with more familiar, group crossed products.

Let G be a (discrete) group. A G-graded algebra $A = \bigoplus_{g \in G} A_g$ is called a G-crossed product, if each component A_g contains a unit, u_g, in A. Then, $A_g = Bu_g$ $(= u_g B)$, if we set $B = A_1$, the neutral component. Moreover, the product in A is described by

the weak action $g \rightharpoonup b = u_g b u_g^{-1}$ $(\in B)$ together with

the 2-cocycle $\sigma(g, h) = u_g u_h u_{gh}^{-1}$ $(\in B^\times$:= units in $B)$

so that

$$(bu_g)(cu_h) = b(g \rightharpoonup c)\, \sigma(g, h) u_{gh},$$

where $b, c \in B$, $g, h \in G$. The algebra $\bigoplus_{g \in G} Bu_g$ with this product is denoted by $B \rtimes_\sigma G$. If σ is trivial, this equals the familiar semidirect product $B \rtimes G$. The associativity of product require $\rightharpoonup$ and σ to satisfy

$$(*) \quad \begin{aligned} (g \rightharpoonup (h \rightharpoonup b))\, \sigma(g, h) &= \sigma(g, h)(gh \rightharpoonup b), \\ \sigma(g, h)\, \sigma(gh, l) &= (g \rightharpoonup \sigma(h, l))\, \sigma(g, hl), \end{aligned}$$

where $b \in B$, $g, h, l \in G$. Here the order of product cannot be changed. Therefore the action $\rightharpoonup$ is said to be 'weak', and σ should be called a 'non-abelian' 2-cocycle.

Suppose $A = B \rtimes_\sigma G$, a G-croosed product. Every G-graded ideal in A is the left (or equally right) ideal AI $(= IA)$ generated by some ideal I in B such that $G \rightharpoonup I \subset I$. The quotient G-graded algebra $\bar{A} := A/AI$ equals the G-crossed product $\bar{B} \rtimes_{\bar{\sigma}} G$, where $\bar{B} = B/I$, constructed from the induced weak action $G \curvearrowright \bar{B}$ together with the 2-cocycle $\bar{\sigma} : G \times G \xrightarrow{\sigma} B^\times \longrightarrow \bar{B}^\times$.

LEMMA. Suppose that I is nilpotent and $\mathrm{Ext}^2_{kG}(k, M) = 0$ for all left kG-modules M, where k is regarded as a trivial left kG-module. If $\bar{A} \cong \bar{B} \rtimes G$, then $A \cong B \rtimes G$.

Proof (Sketch). By induction we may suppose $I^2 = 0$. By re-choosing the basis u_g, we may suppose $\bar{\sigma}$ is trivial. Then we have $\sigma_H : G \times G \longrightarrow I$ such that $\sigma(g, h) = 1 + \sigma_H(g, h)$. Now, (*) reduces to

$$g \rightharpoonup (h \rightharpoonup a) = gh \rightharpoonup a,$$

$$\sigma_H(g, h) + \sigma_H(gh, l) = g \rightharpoonup \sigma_H(h, l) + \sigma_H(g, hl),$$

where $a \in I$, $g, h, l \in G$. Thus, I is a left kG-module and σ_H is a 2-cocycle in the standard complex, possibly called the Hochschild complex, for computing $\mathrm{Ext}^{\cdot}_{kG}(k, I)$; the subscript H in σ_H represents 'Hochschild'. Since we suppose $\mathrm{Ext}^2 = 0$, there exists $\nu_H : G \longrightarrow I$ such that $\partial\nu_H = \sigma_H$. If we set $\nu = 1 + \nu_H : G \longrightarrow B^{\times}$, it follows that $b \rtimes g \longmapsto b\,\nu(g)^{-1}u_g$ gives an isomorphism $B \rtimes G \cong A$. ∎

The idea can be referred to so as 'approximating the non-abelian cohomology related to crossed products by the abelian, Hochschild cohomology'; see the title of the article [M].

The lemma above is generalized to algebras of Hopf crossed product, and then dualized to coalgebras of Hopf crossed coproduct. The result immediately proves the last theorem; see [M, Section 4]. The theorem can be thus regarded as vanishing of a sort of non-abelian cohomology in dimension 2. The same idea as above also proves vanishing of such an abelian cohomology in higher dimensions that describes Hopf algebra extensions, in dimension 2; see [M, Corollary 5.4].

References

[A] E.Abe, 'Hopf Algebras', Combridge Univ. Press, 1980.

[CDMM] C.Calinescu, S.Dascalescu, C.Menini and A.Masuoka, Quantum lines over non-cocommtative cosemisimple Hopf algebras (tentative), in preparation.

[M] A.Masuoka, Hopf cohomology vanishing via approximation by Hochschild cohomology, submitted.

[MY] A.Masuoka and T.Yanai, Hopf module duality applied to X-outer actions, J. Algebra, to appear.

[T] M.Takeuchi, On semi-direct product decomposition of affine groups over a field of characteristic 0, Tôhoku Math. Journ. 24(1972) 453-456.

Institute of Mathematics, University of Tsukuba
Ibaraki 305-8571, Japan
E-mail: akira@math.tsukuba.ac.jp

GROUP-LIKE ALGEBRAS

YUKIO DOI

1. Summary

Using the viewpoint of bi-Frobenius algebras we introduce the notion of group-like algebras[1]. The concept generalizes Bose-Mesner algebras of (non-commutative) association schemes and character algebras.

We begin to recall the definition of a bi-Frobenius algebra which is a generalization of a finite-dimensional Hopf algebra. It was recently introduced by Doi-Takeuchi [DT].

1. Definition. [DT] Let H be a finite dimensional algebra and coalgebra over a field k, $\phi \in H^* = \mathrm{Hom}(H,k)$, $t \in H$. Define a map $S : H \to H$ by $S(h) = \sum \phi(t_{(1)}h)t_{(2)}$. Then the 4-tuple (H, ϕ, t, S) is a *bi-Frobenius algebra* (or *bF algebra*) if
(BF1) $\varepsilon(hh') = \varepsilon(h)\varepsilon(h')$, $(\forall h, h' \in H)$ and $\varepsilon(1) = 1$.
(BF2) $\Delta(1) = 1 \otimes 1$.
(BF3) $\phi \leftharpoonup H = H^*$, where $(\phi \leftharpoonup h)(h') := \phi(hh')$.
(BF4) $t \leftharpoonup H^* = H$, where $t \leftharpoonup f := \sum f(t_{(1)})t_{(2)}$.
(BF5) $S(hh') = S(h')S(h)$.
(BF6) $\Delta(S(h)) = \sum S(h_{(2)}) \otimes S(h_{(1)})$.

2. Basic properties. ([DT], [D]) Let (H, ϕ, t, S) be a bi-Frobenius algebra. Then
(2.1) S is a bijection, in particular $S(1) = 1$ and $\varepsilon(S(h)) = \varepsilon(h)$. Conversely the bijectivity of S implies the conditions (BF3) and (BF4). That is,
(BF5,6) + " the bijectivity of S " $\Rightarrow$ (BF3,4).
(2.2) Denote by $\overline{S}$ the composite inverse of S. Then

$$\sum \overline{S}(t_{(2)})\phi(t_{(1)}h) = h = \sum \phi(h\overline{S}(t_{(2)}))t_{(1)}, \ \forall h \in H.$$

(2.3) $\sum h\overline{S}(t_{(2)}) \otimes t_{(1)} = \sum \overline{S}(t_{(2)}) \otimes t_{(1)}h$, $\forall h \in H$. In particular, if $v(H) := \sum \overline{S}(t_{(2)})t_{(1)}$ $(\in Z(H))$ is invertible, then H is separable as an algebra.
(2.4) $\sum \phi(xy_{(1)})S(y_{(2)}) = \sum \phi(x_{(1)}y)x_{(2)}$, $\forall x, y \in H$.
(2.5) t is a right integral in H, i.e., $th = t\varepsilon(h)$, $\forall h \in H$.
(2.6) ϕ is a right integral in H^*, i.e., $\sum \phi(h_{(1)})h_{(2)} = \phi(h)1$, $\forall h \in H$.

3. New results. Let (H, ϕ, t, S) be a bi-Frobenius algebra. Assume that $v(H)$ is an invertible element. Then we have
(3.1) $\varepsilon(t) \neq 0$ and $S(t) = t$.
(3.2) Define $\mu : H \to H$, $\mu(h) = v(H)^{-1} \sum \overline{S}(t_{(2)})ht_{(1)}$. Then $\mu(H) = Z(H)$.
(3.3) If $S^2 = id$, then
(i) $\mu(xy) = \mu(yx)$ $(x, y \in H)$, $H = Z(H) \oplus [H, H]$.

[1]The detailed version of this paper has been submitted for publication elsewhere.

(ii) (Orthogonality of Characters) Let $\overline{k} = k$ and $\chi_0, \cdots, \chi_l$ be the complete set of irreducible characters of H. Then

$$\sum \chi_i(v(H)^{-1}\overline{S}(t_{(2)}))\chi_j(t_{(1)}) = \delta_{ij}.$$

4. Proposition. Let A be a finite dimensional (non-commutative) algebra over a field k and $\mathbf{B} = \{b_0 = 1_A, b_1, \ldots, b_d\}$ a k-basis of A. Let $\varepsilon : A \to k$ be an algebra map and $S : A \to A$ an anti-algebra automorphism such that

(i) $\varepsilon \circ S = \varepsilon$, (ii) For all i, $\varepsilon(b_i) \neq 0$,

(iii) $S(b_i) \in \mathbf{B}$ (then i^* is defined by $b_{i^*} = S(b_i)$).

Define $\phi \in A^* = \mathrm{Hom}(A, k)$ and $t \in A$ by

$$\phi(b_i) = \delta_{i0}, \quad t := b_0 + b_1 + \cdots + b_d$$

and regard A as a coalgebra via $\Delta(b_i) = \frac{1}{\varepsilon(b_i)}\, b_i \otimes b_i$.

Then (A, ϕ, t, S) becomes a bF algebra if and only if

(iv) For all i, j, $p_{ij}^0 = \delta_{ij^*}\varepsilon(b_i)$,

here p_{ij}^k denotes the structure constant for $\mathbf{B}$, i.e., $b_i b_j = \sum_{k=0}^{d} p_{ij}^k b_k$.

In this case, we have that $S^2 = id$.

This result suggests the following definition, which is a non-commutative analogue of Kawada's character algebras (cf. [BI]).

5. Definition. A *group-like algebra* (or *generalized group algebra*) is a 4-tuple $(A, \varepsilon, \mathbf{B}, S)$, where A is a finite dimensional algebra over a field k, $\varepsilon : A \to k$ an algebra map, $\mathbf{B} = \{b_0 = 1, b_1, \ldots, b_d\}$ a k-basis of A,

$S : \mathbf{B} \to \mathbf{B}, b_i \mapsto b_{i^*}$ an involution ($i^{**} = i$) satisfying the following conditions:

(G1) $\varepsilon(b_{i^*}) = \varepsilon(b_i) \neq 0$, $\forall i$,

(G2) $p_{ij}^k = p_{j^* i^*}^{k^*}$, $\forall i, j, k$, here p_{ij}^k denotes the structure constant for $\mathbf{B}$

(G3) $p_{ij}^0 = \delta_{ij^*}\varepsilon(b_i)$, $\forall i, j$.

We say that H is *symmetric* if $S = id$. In this case, it is clearly a commutative algebra.

6. Basic properties. Let $(A, \varepsilon, \mathbf{B}, S)$ be a group-like algebra. Then

(6.1) A is a symmetric algebra, since $\phi(b_i b_j) = p_{ij}^0 = p_{ji}^0 = \phi(b_j b_i)$.

(6.2) $\quad p_{0i}^k + p_{1i}^k + \cdots + p_{di}^k = \varepsilon(b_i), \qquad p_{i0}^k + p_{i1}^k + \cdots + p_{id}^k = \varepsilon(b_i).$

(6.3) $\quad p_{ij}^k \varepsilon(b_k) = p_{kj^*}^i \varepsilon(b_i).$

7. An example. Let $\mathrm{char}(k) \neq 2$ and $q \neq 0 \in k$.

	1	b_1	b_2	b_3	b_4	b_5
1	1	b_1	b_2	b_3	b_4	b_5
b_1	b_1	$\frac{q-1}{2}b_1 + \frac{q+1}{2}b_2$	$q + \frac{q-1}{2}(b_1 + b_2)$	b_5	$qb_3 + \frac{q-1}{2}(b_4 + b_5)$	$\frac{q+1}{2}b_4 + \frac{q-1}{2}b_5$
b_2	b_2	$q + \frac{q-1}{2}(b_1 + b_2)$	$\frac{q+1}{2}b_1 + \frac{q-1}{2}b_2$	b_4	$\frac{q-1}{2}b_4 + \frac{q+1}{2}b_5$	$qb_3 + \frac{q-1}{2}(b_4 + b_5)$
b_3	b_3	b_4	b_5	1	b_1	b_2
b_4	b_4	$\frac{q-1}{2}b_4 + \frac{q+1}{2}b_5$	$qb_3 + \frac{q-1}{2}(b_4 + b_5)$	b_2	$q + \frac{q-1}{2}(b_1 + b_2)$	$\frac{q+1}{2}b_1 + \frac{q-1}{2}b_2$
b_5	b_5	$qb_3 + \frac{q-1}{2}(b_4 + b_5)$	$\frac{q+1}{2}b_4 + \frac{q-1}{2}b_5$	b_1	$\frac{q-1}{2}b_1 + \frac{q+1}{2}b_2$	$q + \frac{q-1}{2}(b_1 + b_2)$

where $S(b_1) = b_2$, $S(b_i) = b_i$ $(i = 3, 4, 5)$, $\varepsilon(b_j) = q$ $(j = 1, 2, 4, 5)$ and $\varepsilon(b_3) = 1$.

2. 補足

2.1. フロベニウス代数の基本. A を体 k 上の代数とすると，その双対空間 $A^* = \mathrm{Hom}(A,k)$ は次の作用 $\rightharpoonup$, $\leftharpoonup$ により両側 A 加群となる.

$$(x \rightharpoonup f)(y) = f(yx), \quad (f \leftharpoonup x)(y) = f(xy)$$

$(x, y \in A,\ f \in A^*)$. 有限次元の代数 A は右 A 加群同型射 $\theta : A \to A^*$ が存在するときフロベニウス代数であるという. 左 A 加群同型射 $\theta' : A \to A^*$ の存在と同値になる (自然同型 $A \simeq A^{**}$ と随伴 $\theta^* : A^{**} \simeq A^*$ の合成を θ' とすればよい). $\phi := \theta(1)$ とおくと，θ が A 準同型だから

$$\theta(a) = \phi \leftharpoonup a \quad (\theta'(a) = a \rightharpoonup \phi), \qquad \forall a \in A$$

となる. また A, A^* が同次元より，フロベニウス代数の定義は A から A^* への右 A 加群全射または右 A 加群単射 θ の存在と同値になる. つまり $\{\phi \leftharpoonup a \mid a \in A\} = A^*$ または非退化な $\phi \in A^*$ の存在と言い換えられる. ここで ϕ が非退化とは「$\phi(ax) = 0\ (\forall x \in A) \Rightarrow a = 0$」がなりたつこと.

k-同型 $A \otimes A \overset{id \otimes \theta}{\to} A \otimes A^* \simeq End(A)$ で id_A に対応する $A \otimes A$ の元 $\sum x_i \otimes y_i$ をフロベニウス代数 (A, ϕ) の双対基底と呼ぶ. 次の性質は重要である：

$$\sum x_i \phi(y_i a) = a = \sum \phi(a x_i) y_i \quad (\forall a \in A) \tag{1}$$

$$\sum a x_i \otimes y_i = \sum x_i \otimes y_i a, \quad (\forall a \in A) \tag{2}$$

(証：(1) の最初の等式は定義から明らか. 後半は任意の $f \in A^*$ に対し $\theta(\sum f(x_i) y_i) = f$ がなりたつことからでる. $\theta^{-1}(f) = \sum f(x_i) y_i$ で $\theta^{-1}\theta = id$ を書き直せばよい. (2) を示すためには任意の $f \in A^*$ に対し $\sum f(a x_i) y_i = \sum f(x_i) y_i a$ をいえばよい. $f = \phi \leftharpoonup b$ としてよく，このとき両辺は (1) よりともに ba となる.)

(2) より $v(A) := \sum x_i y_i$ は A の中心に属す. もしこれが可逆元なら $\sum v(A)^{-1} x_i \otimes y_i$ が separable idempotent となるから，A は分離的代数となる.

2.2. 積分. ひきつづき (A, ϕ) をフロベニウス代数とし，代数射 $\varepsilon : A \to k$ を一つ固定する.

$$I_r(A, \varepsilon) := \{t \in A \mid ta = t\varepsilon(a),\ \forall a \in A\}$$

を (A, ε) の右積分空間といい，そのゼロでない元を A の (ε に関する) 右積分という. もっと一般に任意の右 A 加群 V に対し，不変部分加群

$$V^A := \{v \in V \mid va = v\varepsilon(a),\ \forall a \in A\}$$

が定義でき，$A^A = I_r(A, \varepsilon)$ となる. A のフロベニウス性と $(A^*)^A = k\varepsilon$ であることから，$\phi \leftharpoonup t = \varepsilon$ なる A の元 t が唯一つ定まり $I_r(A) = kt$ となる (積分の一意性).

2.3. フロベニウス余代数. 以上の議論を余代数の上で展開する. C を体 k 上の (有限次元) 余代数とし，その余積を $\Delta : C \to C \otimes C$, $c \mapsto \sum c_{(1)} \otimes c_{(2)}$，余単位射を ε で表す. 双対空間 C^* $=\mathrm{Hom}(C, k)$ は積

$$(f \cdot g)(c) := \sum f(c_{(1)}) g(c_{(2)}), \quad f,\ g \in C^*$$

により代数となり (単位元は ε)，C は次の作用で両側 C^* 加群となる.

$$f \rightharpoonup c = \sum c_{(1)} f(c_{(2)}), \quad c \leftharpoonup f = \sum f(c_{(1)}) c_{(2)}$$

有限次元余代数 C は右 C^* 加群同型射 $\kappa: C^* \to C$ が存在するときフロベニウス余代数という．C^* がフロベニウス代数であることと同値である．$t := \kappa(\varepsilon)$ とおくと，$\kappa(f) = t \leftharpoonup f = \sum f(t_{(1)})t_{(2)}$ となる．

(C,t) をフロベニウス余代数とし，さらに C の元 1_C で $\Delta(1_C) = 1_C \otimes 1_C$ かつ $\varepsilon(1_C) = 1$ をみたすものが与えられたとする．このとき $t \leftharpoonup \phi = 1_C$ をみたす $\phi \in C^*$ が唯一つ定まり，性質

$$\sum \phi(c_{(1)})c_{(2)} = \phi(c)1_C, \quad \forall c \in C$$

をもつ (C^* における右積分)．この性質をみたす ϕ 全体は C^* の 1 次元部分空間を作る．

2.4. Basic properties (2.1)-(2.6) の証明. Summary の定義 1(bF algebra) で，(BF3) は (H,ϕ) がフロベニウス代数であること，(BF4) は (H,t) がフロベニウス余代数であることをいっている．写像 $S: H \to H$, $S(h) = \sum \phi(t_{(1)}h)t_{(2)}$ は同型 $\theta': H \simeq H^*$, $h \mapsto h \rightharpoonup \phi$ と $\kappa: H^* \simeq H$, $f \mapsto t \leftharpoonup f$ の合成と一致することから，S の全単射性がでる．したがって (BF5) より $S(1) = 1$ つまり $\sum \phi(t_{(1)})t_{(2)} = 1$ が導かれた．これは $t \leftharpoonup \phi = 1$ を意味し，上の 2.3 より basic properties の (2.6) が示せた．また S の全単射と (BF6) から $\varepsilon(S(h)) = \varepsilon(h)$, $\forall h \in H$ がなりたち，$S(h) = \sum \phi(t_{(1)}h)t_{(2)}$ の両辺に ε をほどこして $\phi(th) = \varepsilon(h)$ を得る．つまり $\phi \leftharpoonup t = \varepsilon$. よって 2.2 より，$t$ が H の右積分となり (2.5) が示せた．S の逆写像を $\overline{S}$ で表すと，$S(h) = \sum \phi(t_{(1)}h)t_{(2)}$ より

$$h = \sum \overline{S}(t_{(2)})\phi(t_{(1)}h)$$

となり，$\sum \overline{S}(t_{(2)}) \otimes t_{(1)}$ はフロベニウス代数 (H,ϕ) に対する 2 重基底となる．したがってフロベニウス代数の基本から (2.2), (2.3) がでる．(2.4) は (2.3) の双対である．最後に (2.1) の後半部分を示す．$S(h) = t \leftharpoonup (h \rightharpoonup \phi)$ より $h = t \leftharpoonup (\overline{S}(h) \rightharpoonup \phi)$. これから $H = t \leftharpoonup H^*$ がわかり，(H,t) がフロベニウス余代数となる．次に

$$\phi \leftharpoonup (\sum f(\overline{S}(t_{(2)})t_{(1)}) = f$$

が任意の $f \in H^*$ に対して成立する (直接元を代入することによって確かめられる)．これは $\phi \leftharpoonup H = H^*$ を意味し，(H,ϕ) がフロベニウス代数となる．

New results (3.1)-(3.3) の証明については省略.

2.5. BF algebra の例. 1) 有限群 G に対する群環 kG は

$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1 \quad (g \in G)$$

として余代数になる．$\phi(g) = \delta_{1g}$, $t = \sum_{g \in G} g$, $S(g) = g^{-1}$ とおく．このとき $S(x) = \sum_{g \in G} \phi(gx)g$ が kG の任意の元 x に対してなりたつ．S は明らかに全単射で条件 (BF5),(BF6) をみたす．したがって Basic properties (2.1) の後半部分の結果から，(kG,ϕ,t,S) は bF algebra になる．

2) 同じ論法で一般の有限次元ホップ代数 H も bF algebra になることがわかる．ϕ, t をそれぞれ H^*, H の右積分で $\phi(t) = 1$ をみたすものとし，S は通常の antipode とする．よく知られているように S は全単射で (BF5), (BF6) をみたす．さらに等式 $S(h) = \sum \phi(t_{(1)}h)t_{(2)}$ が示せる．よって (H,ϕ,t,S) は bF algebra となる．

3) $H = k[X]/(X^4)$ (as algebras) とする．余代数構造を次で定義する ($x = \overline{X}$).

$$\Delta(1) = 1\otimes 1,\ \Delta(x) = 1\otimes x + x\otimes 1,\ \Delta(x^2) = 1\otimes x^2 + x^2\otimes 1,$$

$$\Delta(x^3) = 1\otimes x^3 + x\otimes x^2 + x^2\otimes x + x^3\otimes 1,\quad \varepsilon(1) = 1,\ \varepsilon(x) = \varepsilon(x^2) = \varepsilon(x^3) = 0.$$

x^2 だけ $\Delta(x^2) = 1\otimes x^2 + x\otimes x + x^2\otimes 1$ と変更しても bF algebra になる．これらはホップ代数ではない．

4) Summary の定義 5 の group-like algebra $(A, \varepsilon, \mathbf{B} = \{b_0 = 1, b_1, \ldots, b_d\}, S)$ が bF algebra になることを (例1,2 と同じ論法で) 説明する．まず余代数構造を $\Delta(b_i) = \frac{1}{\varepsilon(b_i)} b_i \otimes b_i$ で与える．最初に与えられた ε が余単位射になる．また $\varepsilon(b_{i^*}) = \varepsilon(b_i)$ から S が (BF6) をみたす (定義から S は全単射で BF5 もみたす)．$\phi \in A^*$, $t \in A$ を $\phi(b_i) = \delta_{i0}$ $t := b_0 + b_1 + \ldots + b_d$ で定義すとき，

$$\sum \phi(t_{(1)} b_j) t_{(2)} = \sum_{i=0}^{d} \frac{1}{\varepsilon(b_i)} \phi(b_i b_j) b_i = \sum_{i=0}^{d} \frac{p_{ij}^0}{\varepsilon(b_i)} b_i = b_{j^*} = S(b_j)$$

だから，(A, ϕ, t, S) は bF algebra となる．

2.6. Basic properties (6.2), (6.3) について. Group-like algebra は bF algebra だから，basic properties (2.1)-(2.6) をみたす．(6.2) の前半は $t = b_0 + b_1 + \cdots + b_d$ が右積分であることを表している．一般に t が右積分のとき $S(t)$ は左積分になるが，このケースでは $S(t) = t$ だから t は左積分でもある．これが (6.2) の第2式である．

(6.3) の証．$\phi((b_i b_j) b_k) = \phi(p_{ij}^0 b_k + p_{ij}^1 b_1 b_k + \cdots + p_{ij}^d b_d b_k) = p_{ij}^{k^*} \varepsilon(b_{k^*})$ かつ $\phi(b_i(b_j b_k)) = \phi(p_{jk}^0 b_i + p_{jk}^1 b_i b_1 + \cdots + p_{jk}^d b_i b_d) = p_{jk}^{i^*} \varepsilon(b_i)$ だから

$$p_{ij}^{k^*} \varepsilon(b_{k^*}) = p_{jk}^{i^*} \overset{G2}{=} p_{k^* j^*}^{i} \varepsilon(b_i)$$

を得る．最後に k^* と k を入れ替えれば (6.3) を得る．

2.7. Group-like algebra の例. アソシエーション・スキームに付随する隣接代数 (Bose-Mesner algebra) は自然な見方で複素数体 $\mathbb{C}$ 上の group-like algebra となる．この場合，構造定数 p_{ij}^k はすべて非負整数値をとる．以下一般の体 k 上で考える．

1) 2 次元の group-like algebra は次の形に限る：

$A_q(2)$	1	b
1	1	b
b	b	$q + (q-1)b$

$\varepsilon(b) = q (\neq 0) \in k$ で $S = id$. $v := v(A_q(2)) = 2 + \frac{q-1}{q} b$ である．$\varepsilon(t) = 1 + q \neq 0$ なら v は可逆 ($v^{-1} = \frac{(q^2+1)-(q-1)b}{(q+1)^2}$) で半単純. $q = -1$ なら半単純でない．

2) 3 次元で $S \neq id$ の場合は次の形 (ただし char$(k) \neq 2$ とする)：

$A_q(3)$	1	b_1	b_2
1	1	b_1	b_2
b_1	b_1	$\frac{q-1}{2} b_1 + \frac{q+1}{2} b_2$	$q + \frac{q-1}{2}(b_1 + b_2)$
b_2	b_2	$q + \frac{q-1}{2}(b_1 + b_2)$	$\frac{q+1}{2} b_1 + \frac{q-1}{2} b_2$

where $S(b_1) = b_2$ and $q := \varepsilon(b_1) = \varepsilon(b_2) \neq 0$. $v = v(A_q(3)) = 3 + \frac{q-1}{q}(b_1 + b_2)$.
もし $\varepsilon(t) = 2q + 1 \neq 0$ なら v は可逆で

$$v^{-1} = \frac{2q^2 + 1 + (1-q)(b_1 + b_2)}{(2q+1)^2}.$$

Proof. $S \neq id$ だから $S(b_1) = b_2$ である. (G1), (G3) より

$$q := \varepsilon(b_1) = \varepsilon(b_2) \neq 0,\ p_{11}^0 = p_{22}^0 = 0,\ p_{12}^0 = p_{21}^0 = q$$

また (G2) より

$$p_{11}^1 = p_{22}^2,\ p_{12}^1 = p_{12}^2,\ p_{21}^1 = p_{21}^2,\ p_{22}^1 = p_{11}^2$$

さらに (6.3) より

$$p_{12}^2 = p_{21}^1,\ p_{11}^1 = p_{12}^1$$

したがって積は可換で, 乗積表は次の形：

	1	b_1	b_2
1	1	b_1	b_2
b_1	b_1	$\alpha b_1 + \beta b_2$	$q + \alpha(b_1 + b_2)$
b_2	b_2	$q + \alpha(b_1 + b_2)$	$\beta b_1 + \alpha b_2$

次に積分条件 (6.2) から $q = 1 + 2\alpha = \alpha + \beta$. よって $\mathrm{char}(k) \neq 2$ なら $\alpha = \frac{q-1}{2}$, $\beta = \frac{q+1}{2}$, $\mathrm{char}(k) = 2$ なら $q = 1$, $\beta = 1 + \alpha$ となり上の表を得る. $(b_1 b_1)b_2 = b_1(b_1 b_2)$, $(b_1 b_2)b_2 = b_1(b_2 b_2)$ は直接計算で確かめられる. 他のケースは可換性より明らか. したがってこの積は結合律をみたす. □

3) 3 次元で $S = id$ の場合は次の形に限る：

$A_{p,q}^{\beta}(3)$	1	b_1	b_2
1	1	b_1	b_2
b_1	b_1	$p + (p-1-\beta q)b_1 + \beta p b_2$	$\beta q b_1 + (p - \beta p)b_2$
b_2	b_2	$\beta q b_1 + (p - \beta p)b_2$	$q + (q - \beta q)b_1 + (q - 1 - p + \beta p)b_2$

where $\varepsilon(b_1) = p$ and $\varepsilon(b_2) = q$ and $\beta \in k$. $v(A_{p,q}^{\beta}(3))$ は

$$1 + \frac{b_1 b_1}{p} + \frac{b_2 b_2}{q} = 3 + \frac{2p - 1 - \beta(p+q)}{p} b_1 + \frac{q - p - 1 + \beta(p+q)}{q} b_2.$$

これが可逆元なる条件は次の値が 0 でないこと：

$$\frac{(p+q+1)^2}{pq} \cdot \{(\beta^2 - \beta)(p+q)^2 + \beta(q+1)^2 + (1-\beta)(p+1)^2\}$$

References

[BI] E. Bannai and T. Ito, " Algebraic Combinatorics I: Association Schemes ", Benjamin-Cummings, Menlo Park CA, 1984.

[D] Y. Doi, *Substructures of bi-Frobenius algebras*, J. Algebra (to appear)

[DT] Y. Doi and M. Takeuchi, *BiFrobenius algebras*, Contemporary Mathematics 267 (2000), 67-97.

[Z] P.-H. Zieschang, "An Algebraic Approach to Association Schemes", (Lecture Notes in Mathematics; 1628), Springer, 1996.

Faculty of Education, Okayama University
Okayama 700-8530, Japan
e-mail: ydoi@cc.okayama-u.ac.jp

AN ELEMENTARY CONSTRUCTION OF TILTING COMPLEXES

MITSUO HOSHINO AND YOSHIAKI KATO

ABSTRACT. Let A be an artin algebra and $e \in A$ an idempotent with $\mathrm{add}(eA_A) = \mathrm{add}(D(_AAe))$. Then a projective resolution of Ae_{eAe} gives rise to tilting complexes $\{P(l)^\bullet\}_{l\geq 1}$ for A, where $P(l)^\bullet$ is of term length $l+1$. In particular, if A is selfinjective, then $\mathrm{End}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P(l)^\bullet)$ is selfinjective and has the same Nakayama permutation as A. In case A is a finite dimensional algebra over a field and eAe is a Nakayama algebra, a projective resolution of eAe over the enveloping algebra of eAe gives rise to two-sided tilting complexes $\{T(2l)^\bullet\}_{l\geq 1}$ for A, where $T(2l)^\bullet$ is of term length $2l+1$. In particular, if eAe is of Loewy length two, then we get tilting complexes $\{T(l)^\bullet\}_{l\geq 1}$ for A, where $T(l)^\bullet$ is of term length $l+1$.

This note is a summary of our paper ([HK2]).

The notions of tilting complexes and two-sided tilting complexes were introduced by Rickard [Ri1, Ri3]. After that, derived equivalences between selfinjective algebras have been studied by many people (see e.g. [Br], [Ok] [Ri2], [Ri4], [Ro] and their references). The notion of tilting complexes is a generalization of that of tilting modules (see e.g. [CPS], [Ha], [HR], [Mi]). In the case of a selfinjective algebra A, a tilting module T_A is just a projective generator and thus $\mathrm{End}_A(T_A)$ is Morita equivalent to A. On the other hand, there have been known several examples of derived equivalent selfinjective algebras which are not Morita equivalent. Especially, Rickard [Ri2] showed that the Brauer tree algebras with the same numerical invariants are derived equivalent to each other. This generalizes the earlier work of Gabriel and Riedtmann [GR] that the Brauer tree algebras with the same numerical invariants are stably equivalent to each other, since derived equivalent selfinjective algebras are stably equivalent ([KV], [Ri2]). Recently, Okuyama [Ok] introduced a method of constructing tilting complexes associated with idempotents over symmetric algebras. Also, Rouquier and Zimmermann [RZ] gave an example of two-sided tilting complexes associated with local idempotents over symmetric algebras. In this note, we develop these constructions and provide a systematic method of constructing tilting complexes and two-sided tilting complexes over selfinjective algebras (cf. [HK1]).

Let A be a ring and $e \in A$ an idempotent. For any finite projective resolution $f : Q^\bullet \to (1-e)Ae$ in Mod-eAe and for any $l \geq 1$, we construct a complex $P(l)^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ of term length $l+1$ and show that $P(l)^\bullet$ is a tilting complex if and only if $\mathrm{Ext}^i_A(A/AeA, eA) = 0$ for $0 \leq i < l$ (Theorem 2.3). In particular, if A is a selfinjective artin algebra and if $\mathrm{add}(eA_A) = \mathrm{add}(D(_AAe))$, then $\mathrm{End}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P(l)^\bullet)$ is also a selfinjective artin algebra whose Nakayama permutation coincides with that of A (Theorem 3.7). Next, we deal with the case of A being a finite dimensional algebra over a field k. For any finite projective resolution $f : S^\bullet \to eAe$ in Mod-$(eAe)^e$ and for any $l \geq 1$, we construct a complex $T(l)^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathrm{Mod}\text{-}A^e)$ of term length $l+1$ and show that if $\mathrm{add}(eA_A) = \mathrm{add}(D(_AAe))$ and if $\mathrm{add}(\mathrm{Z}^{-l+1}(S^\bullet)_{eAe}) = \mathcal{P}_{eAe}$, as a complex of right A-modules $T(l)^\bullet$ is a tilting complex (Proposition 4.2). Furthermore, if $eA_A \cong D(_AAe)$ and if $\mathrm{Z}^{-l+1}(S^\bullet)$ is faithfully balanced,

The detailed version of this paper will appear elsewhere.

then $T(l)^\bullet$ is a two-sided tilting complex (Theorem 5.4). Finally, as applications, we deal with the case where $eA_A \cong D({}_AAe)$ and eAe is a Nakayama algebra. We show that as a complex of right A-modules $T(l)^\bullet$ is a tilting complex for all $l \geq 1$ and that $T(2l)^\bullet$ is a two-sided tilting complex for all $l \geq 1$ (Proposition 6.2). In particular, if eAe is of Loewy length two, then $T(l)^\bullet$ is a two-sided tilting complex for all $l \geq 1$ (Proposition 6.3). Furthermore, we provide decompositions of these two-sided tilting complexes in the derived Picard group (Remark 4.5 and Propositions 6.2 and 6.3).

Throughout this note, rings are associative rings with identity and modules are unitary modules. Unless otherwise stated, modules are right modules. For a ring A, we denote by A^{op} the opposite ring of A and consider left A-modules as A^{op}-modules. In case A is a finite dimensional algebra over a field k, we denote by A^{e} the enveloping algebra $A^{\mathrm{op}} \otimes_k A$ and consider A-A-bimodules as A^{e}-modules. Sometimes, we use the notation X_A (resp., ${}_AX$) to stress that the module X considered is a right (resp., left) A-module. We denote by Mod-A the category of A-modules and by $\mathcal{P}_A$ the full additive subcategory of Mod-A consisting of finitely generated projective modules. For an object X in an additive category $\mathcal{A}$, we denote by $\mathrm{add}(X)$ the full additive subcategory of $\mathcal{A}$ consisting of objects isomorphic to direct summands of finite direct sums of copies of X. For an additive category $\mathcal{A}$, we denote by $\mathsf{K}(\mathcal{A})$ the homotopy category of cochain complexes over $\mathcal{A}$ and by $\mathsf{K}^-(\mathcal{A})$, $\mathsf{K}^{\mathrm{b}}(\mathcal{A})$ the full subcategories of $\mathsf{K}(\mathcal{A})$ consisting of bounded above and bounded complexes, respectively. In case $\mathcal{A}$ is an abelian category, we denote by $\mathsf{D}(\mathcal{A})$ the derived category of cochain complexes over $\mathcal{A}$. Also, we denote by $\mathrm{B}^i(X^\bullet)$, $\mathrm{Z}^i(X^\bullet)$, $\mathrm{Z}'^i(X^\bullet)$ and $\mathrm{H}^i(X^\bullet)$ the i-th boundary, the i-th cycle, the i-th cocycle and the i-th cohomology of a complex $X^\bullet$, respectively. We refer to [RD], [Ve] and [BN] for basic results in the theory of derived categories and to [Ri1, Ri3] for definitions and basic results in the theory of tilting complexes.

1. Preliminaries

Throughout this note, A is a ring and $e \in A$ is an idempotent. We identify Mod-(A/AeA) with the full subcategory of Mod-A consisting of $X \in$ Mod-A with $Xe = 0$. In this section, we collect several basic facts which we need in later sections.

Lemma 1.1. *For any $l \geq 1$ the following statements are equivalent.*
(1) $\mathrm{Ext}_A^i(A/AeA, eA) = 0$ *for* $0 \leq i < l$.
(2) $\mathrm{Ext}_A^i(-, eA)$ *vanishes on* Mod-(A/AeA) *for* $0 \leq i < l$.

Remark 1.2. For any $X \in$ Mod-A we have functorial isomorphisms

$$\mu_X : X \otimes_A Ae \xrightarrow{\sim} Xe,\ x \otimes a \mapsto xa, \quad \varepsilon_X : Xe \xrightarrow{\sim} \mathrm{Hom}_A(eA, X),\ x \mapsto (a \mapsto xa).$$

Remark 1.3. The functor $- \otimes_A Ae$: Mod-$A \to$ Mod-eAe is exact and has a fully faithful left adjoint $- \otimes_{eAe} eA$: Mod-$eAe \to$ Mod-A. Furthermore, these functors induce an equivalence $\mathrm{add}(eA_A) \xrightarrow{\sim} \mathcal{P}_{eAe}$.

Definition 1.4. For any $X \in$ Mod-A and $M \in$ Mod-eAe we have a bifunctorial isomorphism

$$\theta_{M,X} : \mathrm{Hom}_{eAe}(M, X \otimes_A Ae) \xrightarrow{\sim} \mathrm{Hom}_A(M \otimes_{eAe} eA, X)$$

such that $\theta_{M,X}(f)(m \otimes a) = \mu_X(f(m))a$ for $f \in \mathrm{Hom}_{eAe}(M, X \otimes_A Ae)$, $m \in M$ and $a \in eA$. Thus for any $X^\bullet \in \mathsf{K}(\text{Mod-}A)$ and $M^\bullet \in \mathsf{K}(\text{Mod-}eAe)$ we have a bifunctorial isomorphism

$$\mathrm{Hom}^\bullet_{eAe}(M^\bullet, X^\bullet \otimes^\bullet_A Ae) \xrightarrow{\sim} \mathrm{Hom}^\bullet_A(M^\bullet \otimes^\bullet_{eAe} eA, X^\bullet)$$

and, by applying $\mathrm{H}^0(-)$, we get a bifunctorial isomorphism

$$\tilde{\theta}_{M^\bullet,X^\bullet} : \mathrm{Hom}_{\mathsf{K}(\text{Mod-}eAe)}(M^\bullet, X^\bullet \otimes^\bullet_A Ae) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{K}(\text{Mod-}A)}(M^\bullet \otimes^\bullet_{eAe} eA, X^\bullet).$$

We set

$$\xi_{X^\bullet} = \tilde{\theta}_{X^\bullet \otimes^\bullet_A Ae, X^\bullet}(\mathrm{id}_{X^\bullet \otimes^\bullet_A Ae}) : X^\bullet \otimes^\bullet_A Ae \otimes^\bullet_{eAe} eA \to X^\bullet,$$
$$\zeta_{M^\bullet} = \tilde{\theta}_{M^\bullet, M^\bullet \otimes^\bullet_{eAe} eA}{}^{-1}(\mathrm{id}_{M^\bullet \otimes^\bullet_{eAe} eA}) : M^\bullet \to M^\bullet \otimes^\bullet_{eAe} eA \otimes^\bullet_A Ae$$

for $X^\bullet \in \mathsf{K}(\text{Mod-}A)$ and $M^\bullet \in \mathsf{K}(\text{Mod-}eAe)$, respectively.

Remark 1.5. The following statements hold.
(1) $\xi_{X^\bullet}$ is an isomorphism for all $X^\bullet \in \mathsf{K}(\mathrm{add}(eA_A))$.
(2) $\zeta_{M^\bullet}$ is an isomorphism for all $M^\bullet \in \mathsf{K}(\text{Mod-}eAe)$.

Lemma 1.6 (Auslander). *For any $f : P \to X$ in* Mod-A *with P projective, the following statements are equivalent.*
(1) *The induced epimorphism $\bar{f} : P \to \mathrm{Im}\, f$, $x \mapsto f(x)$ is a projective cover.*
(2) *f is right minimal, i.e., every $h \in \mathrm{End}_A(P)$ with $f \circ h = f$ is an automorphism.*

Lemma 1.7. *For any $X \in$* Mod-A *the following statements hold.*
(1) *For any $f : Q \to X \otimes_A Ae$ in* Mod-eAe *with $Q \in \mathcal{P}_{eAe}$, if f is right minimal then so is $\theta_{Q,X}(f)$.*
(2) *For any $g : P \to X$ in* Mod-A *with $P \in \mathrm{add}(eA_A)$, if g is right minimal then so is $g \otimes_A Ae$.*

Lemma 1.8. *Let A be a noetherian algebra over a complete commutative noetherian local ring. Then A is semiperfect, i.e., $\mathcal{P}_A$ is a Krull-Schmidt category.*

Lemma 1.9. *Let A be a finite dimensional algebra over a field k. Then for any $V \in$* Mod-A^e *the following statements hold.*
(1) *If $V_A \cong A_A$ and $_AV \cong {}_AA$, then V is faithfully balanced.*
(2) *If $V_A \in \mathcal{P}_A$ and $_AV \in \mathcal{P}_{A^{\mathrm{op}}}$, and if V is faithfully balanced, then V is a two-sided tilting complex.*

2. General case

The next lemma will play a key role in our argument below.

Lemma 2.1. *Let $\mathcal{A}$ be an additive category and P an object of $\mathcal{A}$. Let $l \geq 1$ and $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{A})$ with $P^i \in \mathrm{add}(P)$ for $-l \leq i < 0$ and with $P^i = 0$ for $i > 0$ and $i < -l$. Then the following statements hold.*

(1) *If $\mathrm{H}^i(\mathrm{Hom}^\bullet_{\mathcal{A}}(P, P^\bullet)) = 0$ for $i \neq -l$, then $\mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(P^\bullet, P^\bullet[i]) = 0$ for $i > 0$.*

(2) *If $\mathrm{H}^i(\mathrm{Hom}^\bullet_{\mathcal{A}}(P^\bullet, P)) = 0$ for $i \neq l$, then $\mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(P^\bullet, P^\bullet[i]) = 0$ for $i < 0$.*

Definition 2.2 ([RD]). For a complex $X^\bullet$ and $n \in \mathbb{Z}$, we define the following truncations

$$\tau_{\leq n}(X^\bullet): \quad \cdots \to X^{n-2} \to X^{n-1} \to X^n \to 0 \to \cdots,$$
$$\tau_{\geq n}(X^\bullet): \quad \cdots \to 0 \to X^n \to X^{n+1} \to X^{n+2} \to \cdots.$$

Theorem 2.3. *Assume $(1-e)Ae_{eAe}$ admits a projective resolution $f: Q^\bullet \to (1-e)Ae$ with $Q^\bullet \in \mathsf{K}^-(\mathcal{P}_{eAe})$. Denote by $P^\bullet$ the mapping cone of $\tilde{\theta}_{Q^\bullet,(1-e)A}(f)$. Let $l \geq 1$ and set $P(l)^\bullet = eA[l] \oplus \tau_{\geq -l}(P^\bullet)$. Then the following statements hold.*

(1) *$P(l)^\bullet$ is a tilting complex if and only if $\mathrm{Ext}^i_A(A/AeA, eA) = 0$ for $0 \leq i < l$.*

(2) *Assume $\mathrm{Ext}^i_A(A/AeA, eA) = 0$ for $0 \leq i < l$. Then $\mathrm{add}(P(l)^\bullet)$ does not depend on the choice of f whenever A is a noetherian algebra over a complete commutative noetherian local ring.*

3. The case of selfinjective artin algebras

In this section, A is an artin algebra over a commutative artin ring R and $\{e_1, \ldots, e_n\}$ is a basic set of orthogonal local idempotents in A. Set $I = \{1, \ldots, n\}$ and $I_0 = \{i \in I \mid e_i \in AeA\}$. Let E be an injective envelope of the R-module $R/\mathrm{rad}\, R$. We set $D = \mathrm{Hom}_R(-, E)$ and $\nu = D \circ \mathrm{Hom}_A(-, A)$.

Remark 3.1. For any $i \in I$ the following statements are equivalent.

(1) $i \in I_0$.

(2) $e_i(A/AeA) = 0$.

(3) $(A/AeA)e_i = 0$.

Definition 3.2. Let $l \geq 1$. For $i \in I_0$ we set $P_i(l)^\bullet = e_iA[l]$. For $i \in I - I_0$, let $f_i : Q_i^\bullet \to e_iAe$ be a minimal projective resolution in Mod-eAe, $P_i^\bullet$ the mapping cone of $\tilde{\theta}_{Q_i^\bullet, e_iA}(f_i)$ and $P_i(l)^\bullet = \tau_{\geq -l}(P_i^\bullet)$. We set $P_0(l)^\bullet = \bigoplus_{i \in I} P_i(l)^\bullet$.

Remark 3.3. Let $i \in I - I_0$. Then $P_i^0 = e_iA$ and $P_i^r = Q_i^{r+1} \otimes_{eAe} eA \in \mathrm{add}(eA_A)$ for all $r < 0$. Also, by Remark 1.3 and Lemma 1.7(1) $d^r_{P_i}$ is right minimal for all $r < 0$.

Proposition 3.4. *Assume $\mathrm{add}(eA_A) = \mathrm{add}(D({}_AAe))$. Then $P_0(l)^\bullet$ is a tilting complex for all $l \geq 1$.*

Definition 3.5. Assume A is selfinjective. Then we have a permutation σ of I, called the Nakayama permutation, such that $\nu(e_iA) \cong e_{\sigma(i)}A$ for all $i \in I$.

Remark 3.6. Assume A is selfinjective. Then the following statements are equivalent.

(1) $\mathrm{add}(eA_A) = \mathrm{add}(D({}_AAe))$.
(2) $\mathrm{add}(eA_A)$ is stable under ν.
(3) I_0 is stable under σ.

Theorem 3.7. *Assume A is selfinjective and $\mathrm{add}(eA_A) = \mathrm{add}(D({}_AAe))$. Then for any $l \geq 1$ we have*

$$D\mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P_i(l)^\bullet, P_0(l)^\bullet) \cong \mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P_0(l)^\bullet, P_{\sigma(i)}(l)^\bullet)$$

for all $i \in I$, i.e., $\mathrm{End}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P_0(l)^\bullet)$ is a selfinjective artin R-algebra whose Nakayama permutation coincides with σ.

4. Complexes of bimodules

Throughout the rest of this note, A is a finite dimensional algebra over a field k and $D = \mathrm{Hom}_k(-, k)$.

In this section, we assume $d = \dim_k eAe \geq 2$.

Definition 4.1. Applying Definition 1.4 to the idempotent $e \otimes e \in A^e$, for any $X^\bullet \in \mathsf{K}(\mathrm{Mod}\text{-}A^e)$ and $M^\bullet \in \mathsf{K}(\mathrm{Mod}\text{-}(eAe)^e)$ we have a bifunctorial isomorphism

$$\mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}(eAe)^e)}(M^\bullet, eA \otimes^\bullet_A X^\bullet \otimes^\bullet_A Ae) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A^e)}(Ae \otimes^\bullet_{eAe} M^\bullet \otimes^\bullet_{eAe} eA, X^\bullet),$$

which we denote by $\bar{\eta}_{M^\bullet, X^\bullet}$.

Proposition 4.2. *Let $f : S^\bullet \to eAe$ be a projective resolution in $\mathrm{Mod}\text{-}(eAe)^e$ with $S^\bullet \in \mathsf{K}^-(\mathcal{P}_{(eAe)^e})$. Denote by $T^\bullet$ the mapping cone of $\bar{\eta}_{S^\bullet, A}(f)$ and set $T(l)^\bullet = \tau_{\geq -l}(T^\bullet)$ for $l \geq 1$. Assume $\mathrm{add}(eA_A) = \mathrm{add}(D({}_AAe))$. Let $l \geq 1$ and assume $\mathrm{add}(\mathrm{Z}^{-l+1}(S^\bullet)_{eAe}) = \mathcal{P}_{eAe}$. Then as a complex of A-modules $T(l)^\bullet$ is a tilting complex.*

Corollary 4.3. *Let $f : S^\bullet \to eAe$ be the standard free resolution in $\mathrm{Mod}\text{-}(eAe)^e$, i.e., the mapping cone of f is the standard complex of eAe in the sense of* [CE, Chapter IX]. *Denote by $T^\bullet$ the mapping cone of $\bar{\eta}_{S^\bullet, A}(f)$. Assume $\mathrm{add}(eA_A) = \mathrm{add}(D({}_AAe))$. Then as a complex of A-modules $T(l)^\bullet = \tau_{\geq -l}(T^\bullet)$ is a tilting complex for all $l \geq 1$.*

Remark 4.4. Assume in Corollary 4.3 that $eA_A \cong D({}_AAe)$. For $j \geq 0$ we set $s_j(t) = (t^{j+1} + (-1)^j)/(t+1)$, a polynomial of degree j. Then it is not difficult to see that for any $l \geq 1$ we have

$$\begin{aligned} T(l)^\bullet \otimes^\bullet_A \mathrm{Hom}^\bullet_A(T(l)^\bullet, A_A) &\cong A \bigoplus (Ae \otimes_k eA)^{(s)} \\ &\cong \mathrm{Hom}^\bullet_A(T(l)^\bullet, {}_AA) \otimes^\bullet_A T(l)^\bullet \end{aligned}$$

in $\mathsf{K}(\mathrm{Mod}\text{-}A^e)$, where $s = s_{l-1}(d)(s_l(d) + (-1)^l)$. Thus $T(l)^\bullet$ is a two-sided tilting complex if and only if $l = 1$ and $d = 2$. However, even if $l \geq 2$ or $d \geq 3$, it is possible for $\mathrm{End}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(T(l)^\bullet)$ to be Morita equivalent to A (cf. Sections 5, 6 below).

Remark 4.5. Consider the case where $e = e^{(1)} + e^{(2)}$ with the $e^{(i)}$ idempotents in eAe such that $e^{(1)}Ae^{(2)} = 0 = e^{(2)}Ae^{(1)}$. Let $f_i : S_i^\bullet \to e^{(i)}Ae^{(i)}$ be a projective resolution in Mod-$(e^{(i)}Ae^{(i)})^e$ with $S_i^\bullet \in \mathsf{K}^-(\mathcal{P}_{(e^{(i)}Ae^{(i)})^e})$ for $i = 1, 2$ and set $f = \mathrm{diag}(f_1,\ f_2)$: $S_1^\bullet \oplus S_2^\bullet \to e^{(1)}Ae^{(1)} \oplus e^{(2)}Ae^{(2)}$. For $i = 1, 2$ we denote by $T_i^\bullet$ the mapping cone of $\tilde{\eta}^{(i)}_{S_i^\bullet, A}(f_i)$, where $\tilde{\eta}^{(i)}$ denotes the bifunctorial isomorphism obtained by replacing e with $e^{(i)}$ in Definition 4.1. Also, we denote by $T^\bullet$ the mapping cone of $\tilde{\eta}_{S_1^\bullet \oplus S_2^\bullet, A}(f)$. For $l \geq 1$, we set $T_i(l)^\bullet = \tau_{\geq -l}(T_i^\bullet)$ for $i = 1, 2$ and $T(l)^\bullet = \tau_{\geq -l}(T^\bullet)$. Then for any $l \geq 1$ we have

$$T_1(l)^\bullet \otimes_A^\bullet T_2(l)^\bullet \cong T(l)^\bullet \cong T_2(l)^\bullet \otimes_A^\bullet T_1(l)^\bullet$$

in $\mathsf{K}(\text{Mod-}A^e)$.

5. Two-sided tilting complexes

In this section, we assume $eA_A \cong D({}_AAe)$. We set $(-)^* = \mathrm{Hom}_{eAe}(-,\ eAe_{eAe})$.

Lemma 5.1. *The following statements hold.*
(1) $D(Ae \otimes_k eA) \cong Ae \otimes_k eA$ *in* Mod-A^e.
(2) $eAe \cong D(eAe)$ *in* Mod-eAe. *In particular,* eAe *and hence* $(eAe)^e$ *are selfinjective.*
(3) $eA \cong (Ae)^*$ *in* Mod-$((eAe)^{op} \otimes_k A)$.

Lemma 5.2. *For any* $V \in \mathcal{P}_{(eAe)^e}$ *the following statements hold.*
(1) $V^* \in \mathcal{P}_{(eAe)^e}$.
(2) $Ae \otimes_{eAe} V \otimes_{eAe} eA \in \mathrm{add}({}_AAe \otimes_k eA_A)$.
(3) $\mathrm{Hom}_A(Ae \otimes_{eAe} V \otimes_{eAe} eA,\ A_A) \cong Ae \otimes_{eAe} V^* \otimes_{eAe} eA$ *in* Mod-A^e.

Lemma 5.3. *For any* $V_1,\ V_2 \in \mathcal{P}_{(eAe)^e}$ *we have* $V_1 \otimes_{eAe} V_2^* \in \mathcal{P}_{(eAe)^e}$.
In the following, we fix a projective resolution $f : S^\bullet \to eAe$ in Mod-$(eAe)^e$ with $S^\bullet \in \mathsf{K}^-(\mathcal{P}_{(eAe)^e})$. We denote by $T^\bullet$ the mapping cone of $\tilde{\eta}_{S^\bullet, A}(f)$ and set $T(l)^\bullet = \tau_{\geq -l}(T^\bullet)$ for $l \geq 1$.

Theorem 5.4. *Let* $l \geq 1$ *and assume* $\mathrm{Z}^{-l+1}(S^\bullet)$ *is faithfully balanced. Then* $T(l)^\bullet$ *is a two-sided tilting complex.*

Proposition 5.5. *Let* $m \geq 1$ *and assume* $S^\bullet \cong \tau_{\leq -m}(S^\bullet)[-m]$ *as complexes of* $(eAe)^e$-*modules. Then for any* $l \geq 1$ *we have isomorphisms in* $\mathsf{K}(\text{Mod-}A^e)$

$$T(m)^\bullet \otimes_A^\bullet T(l)^\bullet \cong T(m+l)^\bullet \cong T(l)^\bullet \otimes_A^\bullet T(m)^\bullet.$$

6. Applications

In this section, we assume $e = \sum_{i \in I_0} e_i$ with the notation in Section 3 and $eA_A \cong D({}_AAe)$. We set $J = \mathrm{rad}\, A$ and assume $\dim_k e_iAe/e_iJe = \dim_k e_iJe/e_iJ^2e = 1$ and $\dim_k e_iAe = d \geq 2$ for all $i \in I_0$. Then eAe is a selfinjective Nakayama algebra and $e \otimes e = \sum_{i,j \in I_0} e_i \otimes e_j$ with the $e_i \otimes e_j$ orthogonal local idempotents in $(eAe)^e$. Note that we do not exclude the case of e being a local idempotent. Also, eAe may fail to be connected.

There exists a permutation σ of I_0 such that $e_iJe/e_iJ^2e \cong e_{\sigma(i)}Ae/e_{\sigma(i)}Je$ in Mod-eAe for all $i \in I_0$. For any $i \in I_0$ we fix $w_i \in e_iJe_{\sigma(i)} - e_iJ^2e_{\sigma(i)}$. Then for each $i \in I_0$ we have a k-basis $\{e_i, w_i, \cdots, w_iw_{\sigma(i)}\cdots w_{\sigma^{d-2}(i)}\}$ for e_iAe. For $l \geq 0$ we set

$$S^{-l} = \bigoplus_{i \in I_0} eAe_i \otimes_k e_{\gamma(i)}Ae,$$

where $\gamma = \sigma^{rd}$ if $l = 2r$ and $\gamma = \sigma^{rd+1}$ if $l = 2r+1$. We set

$$f : S^0 \to eAe,\ u \otimes v \mapsto uv.$$

Let $r \geq 0$ and $\rho = \sigma^{rd}$. We define homomorphisms in Mod-$(eAe)^e$

$$d_S^{-2r-1} : S^{-2r-1} \to S^{-2r}, \qquad d_S^{-2r-2} : S^{-2r-2} \to S^{-2r-1}$$

by $d_S^{-2r-1}(e_i \otimes e_{\sigma\rho(i)}) = w_i \otimes e_{\sigma\rho(i)} - e_i \otimes w_{\rho(i)}$ and

$$\begin{aligned} d_S^{-2r-2}(e_i \otimes e_{\sigma^d\rho(i)}) &= e_i \otimes w_{\sigma\rho(i)} \cdots w_{\sigma^{d-1}\rho(i)} \\ &+ \sum_{j=1}^{d-2} w_i \cdots w_{\sigma^{j-1}(i)} \otimes w_{\sigma^{j+1}\rho(i)} \cdots w_{\sigma^{d-1}\rho(i)} \\ &+ w_i \cdots w_{\sigma^{d-2}(i)} \otimes e_{\sigma^d\rho(i)} \end{aligned}$$

for all $i \in I_0$, respectively.

Lemma 6.1. *We have a projective resolution $f : S^\bullet \to eAe$ in* Mod-$(eAe)^e$.

In the following, we denote by $T^\bullet$ the mapping cone of $\bar{\eta}_{S^\bullet,A}(f)$ and set $T(l)^\bullet = \tau_{\geq -l}(T^\bullet)$ for $l \geq 1$.

Proposition 6.2. *The following statements hold.*

(1) *As a complex of A-modules $T(l)^\bullet$ is a tilting complex for all $l \geq 1$.*

(2) *$T(2l)^\bullet$ is a two-sided tilting complex for all $l \geq 1$.*

(3) *Let m be the exponent of σ^d. Then*

$$T(2m)^\bullet \otimes^\bullet_A T(l)^\bullet \cong T(2m+l)^\bullet \cong T(l)^\bullet \otimes^\bullet_A T(2m)^\bullet$$

in $\mathsf{K}(\text{Mod-}A^e)$ *for all $l \geq 1$.*

Proposition 6.3. *Assume $d = 2$. Then the following statements hold.*

(1) *$T(l)^\bullet$ is a two-sided tilting complex for all $l \geq 1$.*

(2) *Let m' be the exponent of σ. Then*

$$T(m')^\bullet \otimes^\bullet_A T(l)^\bullet \cong T(m'+l)^\bullet \cong T(l)^\bullet \otimes^\bullet_A T(m')^\bullet.$$

in $\mathsf{K}(\text{Mod-}A^e)$ *for all $l \geq 1$.*

References

[BN] M. Bökstedt and A. Neeman, Homotopy limits in triangulated categories, Compositio Math. 86 (1993) 209-234.

[Br] M. Broué, Rickard equivalences and block theory, Groups '93 Galway /St Andrews I (C. M. Campbell et al., eds.), London Math. Soc. Lecture Note Ser., vol. 211, Cambridge Univ. Press, Cambridge, 1995, pp. 58-79.

[CE] H. Cartan and S. Eilenberg, Homological algebra, Princeton Univ. Press, 1956.

[CPS] E. Cline, B. Parshall and L. Scott, Derived categories and Morita theory, J. Algebra 104 (1986), 397-409.

[GR] P. Gabriel and C. Riedtmann, Group representations without groups, Comment. Math. Helv. 54 (1979) 240-287.

[Ha] D. Happel, Triangulated categories in the representation theory of finite dimensional algebras, London Math. Soc. Lecture Note Ser., vol. 119, Cambridge Univ. Press, Cambridge, 1988.

[HR] D. Happel and C. M. Ringel, Tilted Algebras, Trans. AMS 274 (1982), 399-443.

[RD] R. Hartshorne, Residues and duality, Lecture Notes in Mathematics, vol. 20, Springer, Berlin, 1966.

[HK1] M. Hoshino and Y. Kato, Tilting complexes defined by idempotents, Comm. Algebra 30 (2002), 83-100.

[HK2] M. Hoshino and Y. Kato, An elementary construction of tilting complexes, J. Pure Appl. Algebra (to appear)

[KV] B. Keller and D. Vossieck, Sous les catégories dérivées, C. R. Acad. Sci. Paris Sér. I Math. 305 (1987) 225-228.

[Mi] Y. Miyashita, Tilting modules of finite projective dimension, Math. Z. **193** (1986), 113-146.

[Ok] T. Okuyama, Some examples of derived equivalent blocks of finite groups, preprint.

[Ri1] J. Rickard, Morita theory for derived categories, J. London Math. Soc. (2) 39 (1989) 436-456.

[Ri2] J. Rickard, Derived categories and stable equivalence, J. Pure Appl. Algebra 61 (1989) 303-317.

[Ri3] J. Rickard, Derived equivalences as derived functors, J. London Math. Soc. (2) 43 (1991) 37-48.

[Ri4] J. Rickard, Splendid equivalences: Derived categories and permutation modules, Proc. London Math. Soc. (3) 72 (1996), 331-358.

[Ro] R. Rouquier, From stable equivalences to Rickard equivalences for blocks with cyclic defect, Groups '93 Galway /St Andrews II (C. M. Campbell et al., eds.), London Math. Soc. Lecture Note Ser., vol. 212, Cambridge Univ. Press, Cambridge, 1995, pp. 512-523.

[RZ] R. Rouquier and A. Zimmermann, Picard groups for derived module categories, preprint.

[Ve] J. L. Verdier, Catégories dérivées, état 0, Lecture Notes in Mathematics, vol. 569, Springer, Berlin, 1977, pp. 262-311.

M. Hoshino
Institute of Mathematics
University of Tsukuba
Ibaraki, 305-8571, Japan
E-mail address: hoshino@math.tsukuba.ac.jp

Y. Kato
Institute of Mathematics
University of Tsukuba
Ibaraki, 305-8571, Japan
E-mail address: ykato@math.tsukuba.ac.jp

NEAT IDEMPOTENTS AND TILED ORDERS HAVING LARGE GLOBAL DIMENSION

HISAAKI FUJITA

Let D be a discrete valuation ring with quotient field K. It is known by Jategaonkar [7] that for a fixed integer $n \geq 2$, there are only finitely many tiled D-orders of finite global dimension in the full $n \times n$ matrix ring $\mathrm{M}_n(K)$. But it is not known even what is the maximum global dimension. Neat idempotents are introduced and studied by Ágoston, Dlab and Wakamatsu [1] for finite dimensional algebras in connection with quasi-hereditary algebras. In this paper[1] we report some results obtained by applying an idea of neat idempotents to tiled D-orders having large global dimension.

Let e_n be a primitive idempotent of a semiperfect Noetherian ring R with Jacobson radical J. Then e_n is said to be *neat* if $\mathrm{Ext}^i_R(S_n, S_n) = 0$ for all $i \geq 1$ where S_n is the simple right R-module e_nR/e_nJ. We explain some properties of neat idempotents and relationships between R and eRe where $e = 1 - e_n$.

In [5], Jansen and Odenthal found a tiled D-order having large global dimension. Namely, for each even integer $N \geq 8$, they constructed tiled D-orders JO_N in $\mathrm{M}_N(K)$ whose global dimension is $2N-8$. As a main application of neat idempotents, we improve their example. Namely, starting from $n = 6$, we construct tiled D-orders Γ_n in $\mathrm{M}_n(K)$ inductively, and we show that $\mathrm{gl.dim}\Gamma_6 = \mathrm{gl.dim}\Gamma_7 = 5$ and $\mathrm{gl.dim}\Gamma_n = 2n - 8$ for *all* $n \geq 8$. For any even $N \geq 8$, Γ_N is isomorphic to the example of Jansen and Odenthal.

We now recall some facts on tiled D-orders having finite global dimension. In his study of global dimension of orders ([10], [11]), Tarsy found a tiled D-order having global dimension $n-1$, and among other things, he conjectured that if Λ is a D-order in $\mathrm{M}_n(K)$, then $\mathrm{gl.dim}\Lambda \leq n-1$. As a strategy to prove Tarsy's conjecture, Jategaonkar [7] conjectured that if Λ is a tiled D-orders of finite global dimension, then there exists a primitive idempotent e_n in Λ such that $e_n\Lambda e$ or $e\Lambda e_n$ is $e\Lambda e$-projective where $e = 1 - e_n$. In some special cases, both conjectures were settled by some authors. (See [6], [7], [8], [2], and [3].) However in [8], Kirkman and Kuzmanovich found a counterexample to Jategaonkar's conjecture. A counterexample to Tarsy's conjecture was also found in [3] by providing a tiled D-order in $\mathrm{M}_n(K)$ of global dimension n for all $n \geq 6$. It had been expected to find tiled D-orders in $\mathrm{M}_n(K)$ having finite global dimension larger than n. In [9], Rump found a tiled D-order R_8 in $\mathrm{M}_8(K)$ having global dimension 9 from an idea of σ-posets. On the other hand, Jansen and Odenthal found the example mentioned above.

In Section 1 we state some properties of neat idempotents in semiperfect Noetherian rings. In Section 2 we describe how to construct the tiled D-order Γ_n. Its global dimension can be computed using results in Section 1. In Section 3 we give another two tiled D-orders having relatively large global dimension. In Section 4, two questions on tiled D-orders of finite global dimension are posed, one of which can be considered as an improved version of Jategaonkar's conjecture above.

---

[1]The detailed version of this paper has been submitted for publication elsewhere.

1. Neat idempotents in semiperfect Noetherian rings

Let R be a basic semiperfect Noetherian ring with Jacobson radical J. Let $e_1, \dots, e_n$ be orthogonal primitive idempotents of R with $1 = e_1 + \cdots + e_n$. Put $S_n = e_nR/e_nJ$, $e = 1 - e_n$ and $I = ReR$. Then e_n is said to be *neat* if $\mathrm{Ext}^i_R(S_n, S_n) = 0$ for all $i \geq 1$.

The following proposition is a slight modification of Proposition 1 in [1].

Proposition 1. *The following statements are equivalent for a primitive idempotent e_n.*

(1) e_n is neat.

(2) Let

$$\cdots \to P_i \to \cdots \to P_1 \to e_nJ \to 0$$

be a minimal projective resolution of e_nJ. Then for each $i \geq 1$, $P_i \in \mathrm{add}(eR)$.

(3) $e_nJe \otimes_{eRe} eR \cong e_nJ$ by the evaluation map and $\mathrm{Tor}^{eRe}_i(e_nJe, eR) = 0$ for all $i \geq 1$.

(4) $Re \otimes_{eRe} eR \cong I, e_nJe_n = e_nIe_n$ and $\mathrm{Tor}^{eRe}_i(Re, eR) = 0$ for all $i \geq 1$.

By (4) of Proposition 1, the notion of a neat idempotent is left-right symmetric. As an immediate consequence, we have the following corollary.

Corollary 2. *If e_n is a neat idempotent then $\mathrm{pd}_R(e_nJ) = \mathrm{pd}_{eRe}(e_nRe)$ and $\mathrm{pd}_R(Je_n) = \mathrm{pd}_{eRe}(eRe_n)$.*

Next, we give a converse of Corollary 2, using a projective complex considered in [4]. We need the following lemma.

Lemma 3. *$I = ReR$ is a maximal ideal if and only if $\mathrm{Ext}^1_R(S_n, S_n) = 0$.*

Proposition 4. *Suppose that $\mathrm{Ext}^1_R(S_n, S_n) = 0$ and $\mathrm{pd}_R(e_nJ) = s < \infty$. Then e_n is neat if and only if $\mathrm{pd}_{eRe}(e_nRe) \leq s$.*

Remark. In some examples, we can easily compute projective dimensions of e_nJ and e_nRe even if their minimal projective resolutions are too complicated. So, Proposition 4 gives a useful criterion for neat idempotents in such examples.

We used a projective complex in the proof of Proposition 4. Its homology group can be characterized as follows.

Lemma 5. *Let X be a finitely generated right R-module. Put $L_0 = XI$ and let $0 \to K_1 \to P_0 \to L_0 \to 0$ be a short exact sequence with P_0 a projective cover of L_0. For $i \geq 1$, inductively, put $L_i = K_iI$ and let $0 \to K_{i+1} \to P_i \to L_i \to 0$ be a short exact sequence with P_i a projective cover of L_i. Then $K_{i+1}/L_{i+1} \cong \mathrm{Tor}^{eRe}_i(Xe, eR)$ for $i \geq 1$ and $K_1/L_1 \cong \mathrm{Ker}(Xe \otimes_{eRe} eR \to X)$.*

The following proposition is a refinement of Proposition 2.6 in [8]. We can computate $\mathrm{gl.dim}\Gamma_n$ explicitly, using this proposition.

Proposition 6. *Suppose that e_nRe (eRe_n) is isomorphic to a right (left) ideal of eRe. Suppose that $\mathrm{Ext}^1_R(S_n, S_n) = 0$, $\mathrm{gl.dim}\, eRe = r + 1 < \infty$ and $\mathrm{pd}_R(e_nJ) = s < \infty$. Put $t = \mathrm{pd}_{eRe}(eJe_n)$. Then the following statements hold.*

(1) If $s + t > r$ then $\mathrm{gl.dim}R = s + t + 2$.

(2) If $s + t < r$ then $\mathrm{gl.dim}R = r + 1 = \mathrm{gl.dim}\, eRe$.

(3) If $s + t = r$ then $\mathrm{gl.dim}R \leq r + 2$.

Therefore if e_n is neat then $\mathrm{gl.dim}R \leq 2r + 2$.

Remark. Proposition 2.2 of [8] shows that gl.dim$eRe \le$ gl.dim$R$+pd$_{eRe}(e_n Re)$. Hence if e_n is neat in R then gl.dim$eRe \le 2 \cdot$ gl.dim$R - 1$. (See Proposition 2 in [1] too.)

We need the following two facts in the induction step to compute gl.dimΓ_n.

Corollary 7. *Suppose that* $\mathrm{Ext}^1_R(S_n, S_n) = 0$ *and* $\mathrm{pd}_R(e_n J) = s < \infty$. *Let* X *be a finitely generated right* R*-module with* $\mathrm{pd}_{eRe}(Xe) = m < \infty$. *Suppose that there exists* ℓ $(1 \le \ell \le m)$ *such that* $\mathrm{Tor}^{eRe}_i(Xe, eR) = 0$ *if* $i \ge \ell$ *and* $\mathrm{Tor}^{eRe}_{\ell-1}(Xe, eR) \ne 0$ *if* $\ell \ge 2$ *and that* $m < s + \ell$. *Then* $\mathrm{pd}_R X = s + \ell + 1$.

Lemma 8. *Suppose that* e_n *is neat in* R. *Then for any right* R*-module* X,

$$\mathrm{Tor}^{eRe}_i(Xe, eR) \cong \mathrm{Tor}^R_i(X, Je_n) \quad \textit{for all } i \ge 1.$$

2. The inductive construction of Γ_n

Let $n (\ge 2)$ be an integer, and let λ_{ij} $(1 \le i, j \le n)$ be non-negative integers satisfying

$$\lambda_{ik} + \lambda_{kj} \ge \lambda_{ij}, \quad \lambda_{ii} = 0 \quad \text{for all } i, j, k \ (1 \le i, j, k \le n)$$

and

$$\lambda_{ij} + \lambda_{ji} > 0 \quad \text{for all } i, j \ (1 \le i, j \le n, i \ne j).$$

Then $\Lambda = (\pi^{\lambda_{ij}} D)$ is a D-order in the full matrix ring $\mathrm{M}_n(K)$. Such a D-order Λ is called *tiled.* In what follows, we abbreviate $\Lambda = (\pi^{\lambda_{ij}} D)$ as $\Lambda = (\lambda_{ij})$.

Let $\Lambda = (\lambda_{ij})$ be a tiled D-order in $\mathrm{M}_n(K)$. Then Λ is a basic, semiperfect Noetherian ring of Krull dimension one. The matrix units $e_1 = e_{11}, \ldots, e_n = e_{nn}$ are primitive orthogonal idempotents of Λ with $1 = e_1 + \cdots + e_n$. Let J be the Jacobson radical of Λ, which is given by replacing all diagonal entries D of Λ by πD.

The valued quiver $Q(\Lambda) = (Q(\Lambda)_0, Q(\Lambda)_1, v)$ of Λ is defined as follows. (See [12].) $Q(\Lambda)_0 = \{1, \ldots, n\}$ is the set of vertices. $Q(\Lambda)_1$ is the set of arrows defined by

$$\alpha : i \to j \in Q(\Lambda)_1 \quad \text{if } \lambda_{jk} + \lambda_{ki} > \lambda_{ji} \text{ for all } k \ (1 \le k \le n, k \ne i, j).$$

The map v from $Q(\Lambda)_1$ to non-negative integers is defined by

$$v(\alpha) = \begin{cases} \lambda_{ji} & (i \ne j) \\ 1 & (i = j) \end{cases}$$

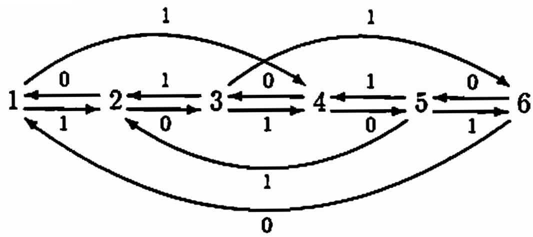
for any $\alpha : i \to j \in Q(\Lambda)_1$.

Λ can be recovered by $Q(\Lambda)$. Namely, for each i, j $(1 \le i, j \le n, i \ne j)$,

$$\lambda_{ij} = \min\{v(p) \mid p \text{ is a path from } j \text{ to } i \text{ in } Q(\Lambda)\}$$

where $v(p)$ is the sum of values of all arrows appearing in p. Note that for any path p from j to i in $Q(\Lambda)$ with $v(p) = \lambda_{ij}$, vertices appearing in p are distinct each other.

Construction of Γ_N: Let Γ_6 be the tiled D-order in $\mathrm{M}_6(K)$ having the following valued quiver:



Let $N = 2n(\geq 6)$ be an even integer. As an induction hypothesis, we assume that $\Gamma_N = (\gamma_{ij})$ is a tiled D-order in $M_N(K)$ with the following property:

$$(*) \begin{cases} Q(\Gamma_N) \text{ has arrows } i \to i+1,\ i+1 \to i\ (1 \leq i \leq N-1), \\ 1 \to N-2,\ 3 \to N,\ N \to 1, \\ N-1 \to 2,\ N \to 5,\ N-4 \to 1,\ 4 \to N-1; \\ \text{for each } \alpha : i \to j \in Q(\Gamma_N)_1, \text{ if } i \text{ is even then } j \text{ is odd and } v(\alpha) = 0, \\ \text{if } i \text{ is odd then } j \text{ is even and } v(\alpha) = 1 \end{cases}$$

Note that Γ_6 has this property.

Step of Γ_{N+1}: We make a new valued quiver Q' by adding a new vertex $N+1$ and four valued arrows $N \xrightarrow{0} N+1$, $N+1 \xrightarrow{1} N$, $2 \xrightarrow{0} N+1$ and $N+1 \xrightarrow{1} 4$ to the valued quiver $Q(\Gamma_N)$. Then for any i, j $(1 \leq i, j \leq N)$, put

$$\gamma_{i,N+1} = \min\{v(p) \mid p \text{ is a path from } N+1 \text{ to } i \text{ in } Q'\}$$
$$\gamma_{N+1,j} = \min\{v(p) \mid p \text{ is a path from } j \text{ to } N+1 \text{ in } Q'\}$$

and put $\Gamma_{N+1} = (\gamma_{ij})_{1 \leq i,j \leq N+1}$ where $\gamma_{N+1,N+1} = 0$.

Then Γ_{N-1} is a tiled D-order in $M_{N+1}(K)$ with $Q(\Gamma_{N+1}) = Q'$.

Step of Γ_{N+2}: We make a new valued quiver Q'' by adding a new vertex 0 and five valued arrows $0 \xrightarrow{0} 1$, $1 \xrightarrow{1} 0$, $0 \xrightarrow{0} N-1$, $N-3 \xrightarrow{1} 0$ and $N+1 \xrightarrow{1} 0$ to the valued quiver $Q(\Gamma_{N+1})$. Then for any i, j $(1 \leq i, j \leq N+1)$, put

$$\gamma_{i0} = \min\{v(p) \mid p \text{ is a path from } 0 \text{ to } i \text{ in } Q''\}$$
$$\gamma_{0j} = \min\{v(p) \mid p \text{ is a path from } j \text{ to } 0 \text{ in } Q''\}$$

and put $\Gamma_{N+2} = (\gamma_{ij})_{0 \leq i,j \leq N+1}$ where $\gamma_{0,0} = 0$.

Then Γ_{N+2} is a tiled D-order in $M_{N+2}(K)$ with $Q(\Gamma_{N+2}) = Q''$.

We shift the names of vertices from $0, 1, \ldots, N+1$ to $1, 2, \ldots, N+2$, respectively. Let u be a diagonal matrix in $M_{N+2}(K)$ with the (i,i)-entry π if i is odd and 1 otherwise. Then $u\Gamma_{N+2}u^{-1}$ is a tiled D-order with the property $(*)$. Thus, we have constructed Γ_N by induction.

For even $N \geq 8$, one can verify that $\Gamma_N \cong JO_N$ by inner automorphism given by a permutation and change of values.

We note that primitive idempotents coresppondiing to new vertices in the inductive construction of Γ_N are neat.

We compute gl.dimΓ_6 = 5 first. Using Proposition 6 (2), we obtain that gl.dimΓ_6 = gl.dimΓ_7 = 5. Then using results of neat idempotents and Proposition 6 (1), we show that gl.dim$\Gamma_n = 2n - 8$ inductively.

3. Another tiled orders having large global dimension

In [9], Rump found a tiled D-order R_8 in $M_8(K)$ of global dimension 9, which is larger than gl.dim$JO_8 = 8$. R_8 is also a modification of Example 2.5 in [3] by means of σ-posets. (See [9].) The following example may be a natural extension of Example 2.5 in [3] in this direction.

Example 1. Let $N = 2n (\geq 6)$ be an even integer. Let $Q = (Q_0, Q_1, v)$ be the valued quiver such that $Q_0 = \{1, 2, \ldots, N\}$ is the set of vertices, Q_1 is the set of the following $6n - 5$ arrows

$$2k-1 \to 2k, 2k \to 2k-1 \ (1 \leq k \leq n)$$

$$2k+1 \to 2k, 2k \to 2k+1, 2k+2 \to 2k-1 \ (1 \leq k \leq n-1)$$

$$2k-1 \to 2k+4 \ (1 \leq k \leq n-2)$$

and that for $\alpha : i \to j \in Q_1$, $v(\alpha) = 1$ (if i is odd) and 0 (if i is even). Let Λ_N be the tiled D-order defined by Q. Then gl.dim$\Lambda_N = 3n - 3$.

Example 2. Let

$$\Lambda = \begin{pmatrix} 0 & 1 & 1 & 2 & 2 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

be a tiled D-order in $M_8(K)$. Then gl.dim$\Lambda = 10$.

By experiments, we guess that inductive extension of Example 2 exceeds Γ_N in global dimension.

4. Remarks

As pointed out in Example 4 of [1], there is a path algebra A of finite global dimension with no neat primitive idempotent. However, in the class of tiled D-orders, we do not know such examples. We note that Proposition 4 is a useful criterion for neat idempotents in tiled D-orders of finite global dimension.

Question 1. Does any tiled D-order of finite global dimension have a neat primitive idempotent ?

Question 1 can be considered as an improved version of Jategaonkar's conjecture. If Question 1 is true, using Proposition 6 and its remark, we can show that $3 \cdot 2^{n-5}$ is a upper bound of finite global dimensions of tiled D-orders in $M_n(K)$ for $n \geq 6$. Using computer, we have verified the upper bound is 6 when $n = 6$.

For a tiled D-order $\Lambda = (\lambda_{ij})$ in $M_n(K)$, put $d(\Lambda) = \sum_{1 \le i,j \le n} \lambda_{ij}$. We call $d(\Lambda)$ *depth* of Λ. It is known that Λ is hereditary if and only if $d(\Lambda) = \frac{1}{2}n(n-1)$, which is the smallest depth among tiled D-orders in $M_n(K)$.

Question 2. If gl.dim$\Lambda < \infty$, then $d(\Lambda) \le \frac{1}{6}(n+1)n(n-1)$?

Let Ω_n be the tiled D-order in $M_n(K)$ given by the following valued quiver

$$1 \underset{1}{\overset{0}{\rightleftarrows}} 2 \underset{1}{\overset{0}{\rightleftarrows}} 3 \underset{1}{\overset{0}{\rightleftarrows}} \cdots \underset{1}{\overset{0}{\rightleftarrows}} n-1 \underset{1}{\overset{0}{\rightleftarrows}} n.$$

Then gl.dim$\Omega_n = 2$ and $d(\Omega_n) = \frac{1}{6}(n+1)n(n-1)$.

If Question 1 is true then one can show that Ω_n is a unique (up to isomorphism) basic tiled D-order in $M_n(K)$ of finite global dimension with the largest depth.

References

[1] I. Ágoston, V. Dlab and T. Wakamatsu, *Neat algebras*, Comm. in Algebra **19(2)** (1991), 433–442.

[2] J. A. De La Peña and A. Raggi-Cárdenas, *On the global dimension of algebras over regular local rings*, Illinois J. Math. **32** (3) (1988), 520–533.

[3] H. Fujita, *Tiled orders of finite global dimension*, Trans. Amer. Math. Soc. **322** (1990), 329–341. Erratum: Trans. Amer. Math. Soc. **327** (1991), 919–920.

[4] H. Fujita, *A construction of tilting modules associated with a simple module*, J. Algebra **145** (1992), 224–230.

[5] W. S. Jansen and C. J. Odenthal, *A tiled order having large global dimension*, J. Algebra **192** (1997), 572–591.

[6] V. A. Jategaonkar, *Global dimension of triangular orders over a discrete valuation ring*, Proc. Amer. Math. Soc. **38** (1973), 8–14.

[7] V. A. Jategaonkar, *Global dimensions of tiled orders over a discrete valuation ring*, Trans. Amer. Math. Soc. **196** (1974), 313–330.

[8] E. Kirkman and J. Kuzmanovich, *Global dimensions of a class of tiled orders*, J. Algebra **127** (1989), 57–72.

[9] W. Rump, *Discrete posets, cell complexes, and the global dimension of tiled orders*, Comm. Algebra **24(1)** (1996), 55–107.

[10] R. B. Tarsy, *Global dimension of orders*, Trans. Amer. Math. Soc. **151** (1970), 335–340.

[11] R. B. Tarsy, *Global dimension of triangular orders*, Proc. Amer. Math. Soc. **28(2)** (1971), 423–426.

[12] A. Wiedemann and K. W. Roggenkamp, *Path orders of global dimension two*, J. Algebra **80** (1983), 113–133.

Institute of Mathematics
University of Tsukuba
Tsukuba, Ibaraki 305-8571 JAPAN
E-mail: fujita@math.tsukuba.ac.jp

ALGEBRA HOMOMORPHISMS AND HOCHSCHILD COHOMOLOGY

H. NAGASE
(長瀬 潤)

ABSTRACT. Let A and B be finite dimensional algebras and $f : B \to A$ an algebra homomorphism preserving the identity. We consider a relationship among the algebra homomorphism f, the canonical functor $f^* : \operatorname{mod} A \to \operatorname{mod} B$, the Hochschild cohomology induced from f and the non-commutative differential module $\Omega_B A$.

1. 序章

Drozd ([5]) により, 代数的閉体上の有限次元代数は, tame と wild と呼ばれるクラスに分けられることが示されている. これらのクラスの正確な定義は２章で与えられるが, 代数が tame であるとは, 任意の n に対して, n-次元直既約加群が有限個の 1-パラメーターで分類されるときを言い, 代数が wild であるとは, n 次元直既約加群を分類するパラメーターの次元が n の増加と共に複雑に増加するため, 分類に望みが持てないときを言う.

Crawley-Boevey は, [2] において, tame 代数上の有限次元直既約加群が各次元ごとに有限個を除いて, τ-invariant であることを示し, その逆が成り立つことを予想した. ここで, τ は Auslander-Reiten translation DTr ([1] 参照) であり, 加群 X が *τ-invariant* であるとは, $X \cong \tau X$ であるときを言う.τ-invariant 加群が含まれる AR-quiver の形が知られていることから, この予想は AR-quiver の様子で tame 代数を特徴付けるもでである. また, この予想は tame 代数を特徴付けるものであるが, 対偶, 「wild 代数は, ある次元に無限個の τ-variant 加群を持つ.」を考えることで, wild 代数の特徴付けとみなす. ここで, 加群 X が *τ-variant* であるとは, $X \ncong \tau X$ であるときを言う. 便宜上, ある次元に無限個の τ-variant 加群を持つ代数を *τ-wild* と呼ぶことにすると, この予想は, 任意の wild 代数が τ-wild であることを示すことである. 一方, wild 代数の定義より, 任意の wild 代数 B に対して, wild hereditary 代数 A と, 代数の写像 $f : B \to A$ が存在して, f より導かれる関手 $f^* : \operatorname{mod} A \to \operatorname{mod} B$ が embedding となる. embedding の定義は２章で与えられる. そして, de la Pena ([4]) の結果の特別な場合として, wild hereditary 代数は τ-wild であることが知られているので, f^* が τ-wildness を保存する為の条件に興味が持たれる. その条件の一つに, 積写像 $A \otimes_B A \to A$ の kernel $\Omega_B A$ が現われる. 実際 [8] において, $\Omega_B A$ が射影的 A-A-両側加群のとき, f^* が τ-wildness を保存することが示されている. このことから, $\Omega_B A$ の射影性に興味が移るが, この報告では, $\Omega_B A$ の射影性, 非可換環の写像の smoothness と "relative" Hochschild cohomology の関係が示される. smoothness と "relative" Hochschild cohomology の定義はそれぞれ４章と５章で与えられる.

2. 準備

この報告を通して, k を代数的閉体とする. 代数と言えば, k-代数を意味し, 断りのないかぎり, k 上有限次元とする. また, 代数の間の写像は, 単位元を保存するものとする. $\operatorname{mod} A$ で有限次元右 A-加群の圏を表す. 関手 $F : \operatorname{mod} A \to \operatorname{mod} B$ が *embedding* であるとは,

The detailed version of this paper will be submitted for publication elsewhere.

合代数を表す. 代数 A が *wild* であるとは, $k\langle x,y\rangle$-A-両側加群 M が存在して, (1) M は $k\langle x,y\rangle$ 上, 有限ランクの自由加群, (2) 関手 $-\otimes_{k\langle x,y\rangle} M : \operatorname{mod} k\langle x,y\rangle \to \operatorname{mod} A$ が embedding, の2つの条件を満たすときを言う. wild 代数上の有限次元直既約加群の分類には望みがないとされているが, その理由の一つとして, 任意の有限生成 k-代数 R に対し, embedding $\operatorname{mod} R \to \operatorname{mod} k\langle x,y\rangle$ が存在することが挙げられる.

この報告では tame 代数について議論することはないが, wild 代数との比較のため, 定義を与えておく. 代数 A が *tame* であるとは, 任意の自然数 n に対して, 有限個の $k[x]$-A-両側加群 $M_{n,1},\ldots,M_{n,i_n}$ で, 左 $k[x]$-加群として, ランク n の自由加群となるものが存在して, 任意の n-次元直既約 A-加群が $k[x]/(x-c)\otimes_{k[x]} M_{n,j}$ $(c\in k,\ j\in\{1,\ldots,i_n\})$ の形で得られるときを言う. つまり, tame 代数上の n-次元直既約加群は, i_n 個の 1-パラメーター c によって分類される. 有限次元代数ではないが, tame 代数の典型的な例として, 1 変数多項式環が挙げられる.

積写像 $m : A\otimes_B A \to A$ $(m(a\otimes b)=ab)$ の kernel を $\Omega_B A$ と書き, 非可換微分加群と呼ぶ. この定義は, 可換環論における微分加群の定義とは, 違うものであるが, 次に説明する $\Omega_B A$ の特徴付は, 可換環における微分加群の特徴付け ([7] 参照) と類似していることから, この報告では, $\Omega_B A$ を非可換微分加群と呼ぶことにする. $\Omega_B A$ の特徴付は次で与えられる. 任意の A-A-両側加群 M に対して, $\operatorname{Der}_B(A,M)$ で A から M への B-derivation 全体の集合をあらわす. このとき, 同型 $\operatorname{Hom}_{A-A}(\Omega_B A, M)\cong \operatorname{Der}_B(A,M)$ $(f\mapsto fd)$ を得る. ここで, $d : A\to\Omega_B A$ $(a\mapsto a\otimes 1-1\otimes a)$. この同型写像は M に関して自然で, $\Omega_B A$ の特徴付けを与えている.

3. 予想と非可換微分加群

ここでは, [8] の結果を用いて, Crawley-Boevey の予想と非可換微分加群の関係について説明する. 序章で説明したように, Crawley-Boevey の予想を示すことは, 任意の wild 代数が τ-wild であることを示すことである. 任意に wild 代数 B をとってくると wild 代数の定義より, embedding $F : \operatorname{mod} k\langle x,y\rangle \to \operatorname{mod} B$ が存在する. 任意の代数 C に対して, embedding $\operatorname{mod} C\to \operatorname{mod} k\langle x,y\rangle$ が存在することが知られているので, embedding $G : \operatorname{mod} C \to \operatorname{mod} B$ が存在することになる. 特に, C として wild hereditary 代数をとってくる. [4] より, wild hereditary は τ-wild であることが知られているので, いつ関手 G が τ-wildness を保存するかに興味が持たれる. 関手 G は exact かつ faithful であることから, C-B-両側加群 $G(C)$ は, C-加群として, 射影的かつ generater になっている. よって, C と $\operatorname{End}_C(G(C))$ は森田同値になっている. その関手を $G' : \operatorname{mod} C\to \operatorname{mod}\operatorname{End}_C(G(C))$ とおく. また, $G(C)$ の C-B-両側加群としての構造から, 代数の写像 $f : B\to \operatorname{End}_C(G(C))$ が存在する. このとき, G' と f から導かれる関手 $f^* : \operatorname{mod}\operatorname{End}_C(G(C))\to \operatorname{mod} B$ の合成 $f^*G' : \operatorname{mod} C\to\operatorname{mod} B$ は $G=f^*G'$ を満たす. 森田同値は τ-wildness を保存することから, f^* が τ-wildness を保存すれば, G も保存することになる. よって, 代数の写像 $f : B\to A$ より導かれる関手 $f^* : \operatorname{mod} A\to\operatorname{mod} B$ が, いつ, τ-wildness を保存するかに興味が移る. そこで, 次ぎの結果を得る ([8] 参照).

命題 3.1. 任意の代数の写像 $f : B\to A$ に対して, 関手 $f^* : \operatorname{mod} A\to\operatorname{mod} B$ が *embedding* であるとする. このとき, 非可換微分加群 $\Omega_B A$ が射影的 A-A-両側加群であれば, f^* は τ-*wildness* を保存する.

上の命題において, 非可換微分加群の射影性は必ずしも必要ではないが, 非可換微分加群が射影的になる例が幾つか存在し, 射影性について考察することに興味が持たれる. そこで, 次の章ではその射影性と非可換環の写像の smoothness との関係について考察する.

[3] において, Cuntz と Quillen は可換環における smoothness の定義 ([7] 参照) を非可換環に適用し, その環を quasi-free と呼んで扱っている. 一方, [6] において, Le Bruyn は D.Quillen の名前の頭文字をとって, quasi-free のことを q-smooth と呼んでいる. ここでは混乱の恐れはないと思われるので, 区別をせずに smooth と呼ぶことにする. 以下に, 写像の smooth の定義を与える. 代数の写像 $f : B \to A$ が *smooth* であるとは, 任意の有限次元とは限らない代数の写像の全射 $s : C \to D$ で $(\mathrm{Ker}\, s)^2 = 0$ となるもの, そして, 任意の代数の写像 $g : B \to C$ と $h : A \to D$ に対して, $hf = sg$ が成り立つとき, 代数の写像 $t : A \to C$ が存在して, $g = tf$ かつ $h = st$ が成り立つときを言う. つまり, B-代数の写像 h と 全射 s に対して, B-代数の写像 t が存在して, $h = st$ が成り立つときを考えている.

可換環論においては, 微分加群と smoothness の間の関係がいくつか知られている ([7] 参照) . そこで, この章では, 可換環論でのアイデアをもとに, 非可換微分加群の射影性と smoothness との関係を示す. この関係を示す為に, 3つの補題を用意する. 次の補題は, 微分加群と非可換微分加群の特徴付けの類似性から, [7] の Theorem 28.4 の証明と同じ方針で示される.

補題 4.1. 代数の写像 $f : B \to A$ において, $k \to A$ が *smooth* であれば, 次が同値 :

(1) $f : B \to A$ が *smooth* ;
(2) 任意の A-A-両側加群 M に対して, $\mathrm{Der}_k(A, M) \to \mathrm{Der}_k(B, M)$ が全射 ;
(3) $\alpha : A \otimes_B \Omega_k B \otimes_B A \to \Omega_k A \quad (\alpha(a \otimes b \otimes a') = aba')$ が分裂単射.

次の補題は, 上の補題の (3) における写像 α の kernel と cokernel を考えたものである. 証明は [3] を参照.

補題 4.2. 代数の写像 $f : B \to A$ と, A-A-両側加群の写像 $\alpha : A \otimes_B \Omega_k B \otimes_B A \to \Omega_k A$ $(\alpha(a \otimes b \otimes a') = aba')$ に対して, 次の *exact sequence* を得る.

$$0 \to \mathrm{Tor}_1^B(A, A) \to A \otimes_B \Omega_k B \otimes_B A \xrightarrow{\alpha} \Omega_k A \to \Omega_B A \to 0$$

次の補題の証明も [3] を参照.

補題 4.3. 代数 A において, 次が同値 :

(1) $k \to A$ が *smooth* ;
(2) A が *hereditary* ;
(3) $\Omega_k A$ が射影的 A-A-両側加群.

以上の3つの補題より, 直ちに, 次の命題が示される.

命題 4.4. 代数の写像 $f : B \to A$ において, A が *hereditary* であれば, 次が同値 :

(1) $f : B \to A$ が *smooth* ;
(2) $\mathrm{Tor}_1^B(A, A) = 0$ かつ $\Omega_B A$ が射影的 A-A-両側加群.

この命題をふまえて, 次の章では非可換微分加群と relative Hochschild cohomology 関係について考察する.

5. 非可換微分加群と RELATIVE HOCOSCHILD COHOMOLOGY

この章では, 代数 A に対して, $A^e = A \otimes_k A^{\mathrm{op}}$ と置き, A-A-両側加群を A^e-加群とみなし, $\Omega_k A$ を ΩA と略す. 自然数 n と A^e-加群 M に対して, A の n 番目 M 係数 Hochschild cohomology $\mathrm{H}^n(A, M)$ を $\mathrm{Ext}^n_{A^e}(A, M)$ で定義する. このとき, 代数の写像 $f : B \to A$ は長完全列

$$\cdots \to \mathrm{H}^n(A, M) \to \mathrm{H}^n(B, M) \to \mathrm{H}^n(f, M) \to \mathrm{H}^{n+1}(A, M) \to \cdots$$

を導く. この報告では, $\mathrm{H}^n(f, M)$ を n 番目 M 係数 relative Hochschild cohomology と呼ぶことにする.

以下で, relative Hochschild cohomology と非可換微分加群の関係を見る為に, 次の命題を用意する.

命題 5.1. 代数の写像 $f: B \to A$ と A^e-加群 M に対して, A が *hereditary* であるとき, 次の完全列が存在する.

$$0 \to \mathrm{Ext}^1_{A^e}(\Omega_B A, M) \to \mathrm{H}^1(f, M) \to \mathrm{Hom}_{A^e}(\mathrm{Tor}_1^B(A, A), M) \to \mathrm{Ext}^2_{A^e}(\Omega_B A, M) \to \mathrm{H}^2(f, M).$$

証明 B^e-加群の写像 $h: B \to A^e \otimes_{B^e} B, (b \mapsto 1 \otimes b)$ と A^e-加群の同型 $A^e \otimes_{B^e} B \cong A \otimes_B A$ により, 単射 $h_1: \mathrm{Ext}^1_{A^e}(A \otimes_B A, M) \to \mathrm{Ext}^1_{B^e}(B, M) = H^1(B, M)$ が導かれる. h_1 は単射 $h_2: \mathrm{Ext}^1_{A^e}(\Omega_B A, M) \to H^1(f, M)$ を導き, 次の図式を可換にする.

$$\begin{array}{ccccccc}
\mathrm{Ext}^1_{A^e}(A, M) & \longrightarrow & \mathrm{Ext}^1_{A^e}(A \otimes_B A, M) & \longrightarrow & \mathrm{Ext}^1_{A^e}(\Omega_B A, M) & \longrightarrow & \mathrm{Ext}^2_{A^e}(A, M) = 0 \\
\| & & \downarrow h_1 & & \downarrow h_2 & & \| \\
\mathrm{H}^1(A, M) & \longrightarrow & \mathrm{H}^1(B, M) & \longrightarrow & \mathrm{H}^1(f, M) & \longrightarrow & \mathrm{H}^2(A, M) = 0
\end{array}$$

$\mathrm{Ext}^2_{A^e}(A, M) = \mathrm{H}^2(A, M) = 0$ は $\mathrm{Ext}^2_{A^e}(A, M) \cong \mathrm{Ext}^1_{A^e}(\Omega A, M)$ と補題 4.3 より導かれる. この図式より, $\mathrm{Cok}\, h_2 \cong \mathrm{Cok}\, h_1$ を得る. 一方, 可換図式,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega B & \longrightarrow & B \otimes B & \longrightarrow & B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow h & & \\
0 & \longrightarrow & K & \longrightarrow & A \otimes A & \longrightarrow & A \otimes_B A & \longrightarrow & 0
\end{array}$$

より, 次の可換図式

$$\begin{array}{ccccccc}
\mathrm{Hom}_{A^e}(A \otimes A, M) & \longrightarrow & \mathrm{Hom}_{A^e}(K, M) & \longrightarrow & \mathrm{Ext}^1_{A^e}(A \otimes_B A, M) & \longrightarrow & 0 \\
\| & & \downarrow h_3 & & \downarrow h_1 & & \\
\mathrm{Hom}_{B^e}(B \otimes B, M) & \longrightarrow & \mathrm{Hom}_{B^e}(\Omega B, M) & \longrightarrow & \mathrm{Ext}^1_{B^e}(B, M) & \longrightarrow & 0
\end{array}$$

を得る. よって, $\mathrm{Cok}\, h_1 \cong \mathrm{Cok}\, h_3$ が成り立つ. また, 補題 4.2 より, 短完全列

$$0 \to \mathrm{Tor}_1^B(A, A) \to A \otimes_B \Omega_k B \otimes_B A \to \mathrm{Im}\, \alpha \to 0$$

が存在するが, この短完全列と adjointness を使って, 次の完全列

$$0 \to \mathrm{Cok}\, h_3 \to \mathrm{Hom}_{A^e}(\mathrm{Tor}_1^B(A, A), M) \to \mathrm{Ext}^1_{A^e}(\mathrm{Im}\, \alpha, M) \to \mathrm{Ext}^1_{A^e}(A \otimes_B \Omega B \otimes_B A, M)$$

を得る. 最後に, 短完全列 $0 \to \mathrm{Im}\alpha \to \Omega A \to \Omega_B A \to 0$ と ΩA が射影的 A^e-加群であること (補題 4.3) より, $\mathrm{Ext}^1_{A^e}(\mathrm{Im}\, \alpha, M) \cong \mathrm{Ext}^2_{A^e}(\Omega_B A, M)$ が言え, 写像 $\Omega_k B \to A \otimes_B \Omega B \otimes_B A$ $(b \mapsto 1 \otimes b \otimes 1)$ から導かれる $\mathrm{Ext}^1_{A^e}(A \otimes_B \Omega B \otimes_B A, M) \to \mathrm{Ext}^1_{B^e}(\Omega B, M) \cong \mathrm{Ext}^2_{B^e}(B, M) \cong \mathrm{H}^2(f, M)$ が単射であることから命題の完全列が得られる.

□

上の命題 5.1 と命題 4.4 より, 次の結果が導かれる.

定理 5.2. 代数の写像 $f: B \to A$ に対して, A が *hereditary* のとき, 次が同値 :

(1) $f: B \to A$ が *smooth* ;
(2) $\mathrm{Tor}_1^B(A, A) = 0$ かつ $\Omega_B A$ が射影的 A^e-加群 ;
(3) 任意の A^e-加群 M に対して, $\mathrm{H}^1(f, M) = 0$.

3章で説明したように, Crawley-Boevey の予想に関しては, A が hereditary であるときの非可換微分加群 $\Omega_B A$ が射影的 A^e-加群になることに興味があったので, 命題 4.4 を使って, 次の系を考えることができる.

系 5.3. 代数の写像 $f : B \to A$ に対して, A が *hereditary* のとき, 次が同値：

(1) $\Omega_B A$ が射影的 A^e-加群；

(2) $\Omega_B A$ が射影的片側 A-加群であり, 任意の単純 A^e-加群 M に対し, $\dim \mathrm{H}^1(f, M) = \dim \mathrm{Hom}_{A^e}(\mathrm{Tor}_1^B(A, A), M)$.

証明 任意の A^e-加群 X に対し, X が射影的 A^e-加群であることと, X が射影的右 A-加群かつ任意の単純左 A-加群 M に対して, $X \otimes_A M$ が射影的左 A-加群であることが同値である事を使う.

□

参考文献

[1] Auslander, M., Reiten, I. and Smalo, O.: *Representation Theory of Artin Algebras*, Cambridge studies in advanced mathematics **36**, Cambridge Univ. Press, 1995.

[2] Crawley-Boevey, W. W.: *On tame algebras and bocses*, Proc. London Math. Soc. (3) **56**, 1988, 451–483.

[3] Cuntz, J., Quillen, D.: *Algebra extentions and nonsingularity*, J. Amer. Math. Soc. **8**, 1995, 251–289.

[4] de la Peña, J.A.: *On the dimension of the module-varieties of tame and wild algebras*, Comm. in Alg. **19(6)**, 1991, 1795–1807.

[5] Drozd, Yu. A.: *Tame and wild matrix problems*, in "Representations and Quadratic Forms", Institute of Mathematics, Academy of Sciences, Ukrainian SSR, Kiev, 1979, 39–74; English transl., Amer. Math. Soc. Transl. **128**, 1986, 31–55.

[6] Le Bruyn, L.: *Noncommutative geometry @n*, **http://xxx.lanl.gov/math.AG/9904171**, 1999, 14pp.

[7] Matsumura, H.: *Commutative ring theory*, Cambridge studies in advanced mathematics **8**, Cambridge Univ. Press, 1980.

[8] Nagase, H.: *non-strictly wild algebras*, J. London Math. Soc., to appear.

[9] Stenström, S.: *Ring of Quotients*, Die Grundlehren der mathematischen Wissenschaften **217**, Springer-verlag, 1975.

Department of Mathematics
Osaka City University
3-3-138 Sugimoto, Sumiyoshi-ku,
Osaka, 558-8585, Japan
E-mail: nagase@sci.osaka-cu.ac.jp

Cohomology Rings of the Generalized Quaternion Group †

Takao Hayami
Katsunori Sanada

1 Introduction

Let $Q_t = \langle x, y | x^{2t} = 1, x^t = y^2, yxy^{-1} = x^{-1} \rangle$ be the generalized quaternion group of order $4t$ for any positive integer $t \geq 2$. We set $\Lambda = \mathbb{Z}Q_t$. It is well known that there exists a Λ-free resolution (Y, δ) of $\mathbb{Z}$ of period 4. Our aim is to determine cohomology rings of generalized quaternion groups by means of the periodic resolution (Y, δ) and a diagonal approximation (Δ_Y) on (Y, δ) (see [Ha] and [HaSa]).

In Section 2, we will give initial parts of chain transformations in both directions lifting the identity map on $\mathbb{Z}$ between (Y, δ) and the standard resolution for Q_t (Propositions 1 and 2). These chain transformations will be used to give a diagonal approximation on the periodic resolution.

In Section 3, we describe some main results of this note. In Section 3.1, we consider the cohomology ring $H^*(Q_{2^r}, {}_\psi\Gamma)$ of the generalized quaternion group Q_{2^r} of order 2^{r+2} with coefficients in an order ${}_\psi\Gamma$. If we put $e = (1 - x^{2^r})/2 \in \mathbb{Q}Q_{2^r}$, then e is a central idempotent of $\mathbb{Q}Q_{2^r}$. We set $\zeta = xe$, $i = x^{2^{r-1}}e$, $j = ye$ and $K = \mathbb{Q}(\zeta + \zeta^{-1})$. Then $\mathbb{Q}Q_{2^r}e$ is the quaternion algebra over K. In the following we set $R = \mathbb{Z}[\zeta + \zeta^{-1}]$, the ring of integers of K, then $\Gamma := \Lambda e (= \mathbb{Z}Q_{2^r}e)$ is an R-order of $\mathbb{Q}Q_{2^r}e$. Let ${}_\psi\Gamma$ denote Γ regarded as a Q_{2^r}-module using a ring homomorphism $\psi : \Lambda \to \Gamma^e; x \mapsto \zeta \otimes (\zeta^{-1})^\circ, y \mapsto j \otimes (j^{-1})^\circ$. We determine the ring structure of the cohomology $H^*(Q_{2^r}, {}_\psi\Gamma) := \bigoplus_{n \geq 0} H^n(Q_{2^r}, {}_\psi\Gamma)$ of Q_{2^r} for $r \geq 2$ (Theorem 1). The case of $r = 1$ is known in [Sa]. In Section 3.2, we give an explicit description of the cohomology ring $H^*(Q_t, \mathbb{Z}) := \bigoplus_{n \geq 0} H^n(Q_t, \mathbb{Z})$ for arbitrary generalized quaternion groups Q_t of order $4t$ for $t \geq 2$ (Theorem 2). In fact, although $H^*(Q_{2^r}, \mathbb{Z})$ is well known (see [T] for example), it seems that the precise description of $H^*(Q_t, \mathbb{Z})$ is not given in any literature. In Section 3.3, we determine the ring structure of the Hochschild cohomology $HH^*(\Lambda) := \bigoplus_{n \geq 0} H^n(\Lambda, \Lambda)$ using a ring isomorphism

$$HH^*(\Lambda) \xrightarrow{\sim} H^*(Q_t, {}_\varphi\Lambda) := \bigoplus_{n \geq 0} H^n(Q_t, {}_\varphi\Lambda)$$

and calculating the cup product in $H^*(Q_t, {}_\varphi\Lambda)$ (Theorem 3). In the above, ${}_\varphi\Lambda$ denotes Λ regarded as a Q_t-module by conjugation.

2 Resolutions of Q_t and chain transformations

Let Q_t denote the generalized quaternion group of order $4t$ for any positive integer $t \geq 2$: $Q_t = \langle x, y | x^{2t} = 1, x^t = y^2, yxy^{-1} = x^{-1} \rangle$. We set $\Lambda = \mathbb{Z}Q_t$. Then the following

†The detail version of this note has appeared in Comm. Algebra and SUT J. Math.

periodic Λ-free resolution of $\mathbb{Z}$ of period 4 is well known (see [CaE, Chapter XII, Section 7], [T, Chapter 3, Periodicity]):

$$\begin{aligned}(Y,\delta):\quad & \cdots \to \Lambda^2 \xrightarrow{\delta_1} \Lambda \xrightarrow{\delta_4} \Lambda \xrightarrow{\delta_3} \Lambda^2 \xrightarrow{\delta_2} \Lambda^2 \xrightarrow{\delta_1} \Lambda \xrightarrow{\epsilon} \mathbb{Z} \to 0,\\ & \delta_1(c_1,c_2) = c_1(x-1)+c_2(y-1),\\ & \delta_2(c_1,c_2) = (c_1L + c_2(xy+1), -c_1(y+1)+c_2(x-1)),\\ & \delta_3(c) = (c(x-1), -c(xy-1)),\\ & \delta_4(c) = cN,\end{aligned}$$

where L denotes $x^{t-1}+x^{t-2}+\cdots+1$ ($\in \Lambda$), Λ^2 denotes the direct sum $\Lambda \oplus \Lambda$ and N denotes $\sum_{w\in Q_t} w$ ($\in \Lambda$). In the following, we set $\delta_{4k+i} = \delta_i$ for any integer $k \geq 0$ and $1 \leq i \leq 4$ since (Y,δ) is periodic of period 4.

Let (X,d) be the standard resolution of Q_t. In this section, we will give initial parts of chain transformations v and u in both directions lifting the identity map on $\mathbb{Z}$ between the resolutions (Y,δ) and (X,d). These chain transformations are used to give a diagonal approximation $(\Delta_Y)_{p,q} := u_p \otimes u_q \cdot \Delta_{p,q} \cdot v_{p+q}$ on (Y,δ) in Section 3.

We introduce the notation $*$ for basis elements in X_i ($i \geq 0$) as follows:

$$\begin{aligned}\sigma_0[\sigma_1] * \sigma_2[\cdot] &:= \sigma_0[\sigma_1\sigma_2] \ (\in (X_G)_1),\\ \sigma_0[\sigma_1] * \sigma_2[\sigma_3|\ldots|\sigma_i] &:= \sigma_0[\sigma_1\sigma_2|\sigma_3|\ldots|\sigma_i] \ (\in (X_G)_{i-1})\end{aligned}$$

for $\sigma_0, \sigma_1, \ldots, \sigma_i \in Q_t$.

Proposition 1. *A chain transformation $v_n : Y_n \to X_n$ ($n \geq 0$) lifting the identity map on $\mathbb{Z}$ is given inductively as follows:*

$$\begin{aligned}& v_0(1) = [\cdot];\\ & v_{4k+1}(1,0) = [x] * v_{4k}(1), \quad v_{4k+1}(0,1) = [y] * v_{4k}(1);\\ & v_{4k+2}(1,0) = [L-1] * v_{4k+1}(1,0) - [y] * v_{4k+1}(0,1),\\ & v_{4k+2}(0,1) = [x] * v_{4k+1}(0,1) + [xy] * v_{4k+1}(1,0);\\ & v_{4k+3}(1) = [x] * v_{4k+2}(1,0) - [xy] * v_{4k+2}(0,1);\\ & v_{4k+4}(1) = [N] * v_{4k+3}(1) \qquad \text{for} \quad k \geq 0.\end{aligned}$$

Proof. It suffices to show that the equation $d_n v_n = v_{n-1}\delta_n$ holds for any $n \geq 1$ and this is easily proved by induction on k. □

Next, for any integer $\iota \geq 0$ and $0 \leq \lambda, \mu < 2t$, we set

$$L_\iota = \begin{cases} x^{\iota-1} + x^{\iota-2} + \cdots + 1 & (\iota \geq 1)\\ 0 & (\iota = 0),\end{cases} \qquad P_\iota = Lxy - L_\iota(xy+1),$$

$$a_{\lambda,\mu} = \begin{cases} 1 & (\lambda+\mu \geq 2t)\\ 0 & (\lambda+\mu < 2t),\end{cases} \quad b_{\lambda,\mu} = \begin{cases} 0 & (\lambda \geq \mu)\\ -1 & (\lambda < \mu),\end{cases} \quad c_{\lambda,\mu} = \begin{cases} 1 & (\lambda-\mu \geq t)\\ 0 & (-t \leq \lambda-\mu < t)\\ -1 & (\lambda-\mu < -t).\end{cases}$$

and furthermore we set

$$d^{0,q}_{\lambda,\mu} = a_{\lambda,\mu} \quad (\text{for } q = 0,1), \quad d^{1,q}_{\lambda,\mu} = \begin{cases} b_{\lambda,\mu} & (\text{for } q = 0) \\ c_{\lambda,\mu} & (\text{for } q = 1). \end{cases}$$

Proposition 2. *We can define a chain transformation $u : X \to Y$ whose initial part $u_n : X_n \to Y_n \quad (0 \le n \le 3)$ is as follows:*

$$\begin{aligned}
&u_0 : [\cdot] \mapsto 1;\\
&u_1 : [x^i y^p] \mapsto (L_i, px^i);\\
&u_2 : [x^i y^p | x^j y^q] \mapsto px^{i-j}(-q, L_j) + d^{p,q}_{i,j}\left(1 - x^t y, Lxy\right);\\
&u_3 : [x^i | x^j y^p | x^k y^q] \mapsto d^{p,q}_{j,k} L_i \left(x^{t+1} y + 1\right)\\
&\qquad [x^i y | x^j | x^k y^q] \mapsto a_{j,k} P_i\\
&\qquad [x^i y | x^j y | x^k] \mapsto -x^{i-j} L_k + b_{j,k} P_i\\
&\qquad [x^i y | x^j y | x^k y] \mapsto (c_{j,k} - 1) P_i + x^{i-j} L_k xy - x^{i-j} L_j (xy + 1);
\end{aligned}$$

where $0 \le i, j, k < 2t$, $p = 0, 1$ and $q = 0, 1$.

Proof. It suffices to show that the equation $\delta_n u_n = u_{n-1} d_n$ holds for $n = 1, 2$ and 3. In fact, for any integer $n \ge 4$, we can define u_n inductively. The proof is straightforward but it is complicated. □

3 Cohomology rings of generalized quaternion groups

In this section, we will determine some cohomology rings of generalized quaternion groups by means of the periodic resolution (Y, δ) and a diagonal approximation (Δ_Y) on (Y, δ).

3.1 Cohomology ring with coefficient in an order

Let Q_{2^r} denote the generalized quaternion group of order 2^{r+2} for any positive integer r: $Q_{2^r} = \langle x, y | x^{2^{r+1}} = 1, x^{2^r} = y^2, yxy^{-1} = x^{-1}\rangle$. In this subsection, we determine the cohomology ring $H^\bullet(Q_{2^r}, {}_\psi\Gamma)$ of Q_{2^r} with coefficient in an order ${}_\psi\Gamma$. We set $e = (1 - x^{2^r})/2 \in \mathbb{Q}Q_{2^r}$ and denote xe by ζ, a primitive 2^{r+1}-th root of e. Then e is a central idempotent of $\mathbb{Q}Q_{2^r}$ and $\mathbb{Q}Q_{2^r}e$ is the quaternion algebra over the field $K = \mathbb{Q}(\zeta + \zeta^{-1})$ with identity e, that is, $\mathbb{Q}Q_{2^r}e = K \oplus Ki \oplus Kj \oplus Kij$ where we set $i = x^{2^{r-1}}e$ and $j = ye$. In the following, we set $R = \mathbb{Z}[\zeta + \zeta^{-1}]$, the ring of integers of K, and $\Lambda = \mathbb{Z}G$, then $\Gamma = \Lambda e (= \mathbb{Z}[\zeta, j] = R \oplus R\zeta \oplus Rj \oplus R\zeta j)$ is an R-order of $\mathbb{Q}Q_{2^r}e$. Let ${}_\psi\Gamma$ denote Γ regarded as a Q_{2^r}-module using a ring homomorphism $\psi : \Lambda \to \Gamma^e; x \mapsto \zeta \otimes (\zeta^{-1})^\circ, y \mapsto j \otimes (j^{-1})^\circ$. Note that $(\zeta + \zeta^{-1})^2$ divides 2 in R (see [HaSa, Lemma 1]). Thus it follows that $2e/(\zeta + \zeta^{-1})$ is in R and in the following we denote this expression by η.

Applying the functor $\mathrm{Hom}_\Lambda(-, {}_\psi\Gamma)$ to the periodic resolution (Y, δ) in Section 2, we have the following complex which gives $H^n(Q_{2^r}, {}_\psi\Gamma)$, where we identify $\mathrm{Hom}_\Lambda(Y_0, {}_\psi\Gamma)$

with Γ, $\mathrm{Hom}_\Lambda(Y_1, {}_\psi\Gamma)$ with $\Gamma^2 := \Gamma \oplus \Gamma$ and so on:

$$\begin{aligned}
&(\mathrm{Hom}_\Lambda(Y, {}_\psi\Gamma), \delta^\#) : 0 \to \Gamma \xrightarrow{\delta_1^\#} \Gamma^2 \xrightarrow{\delta_2^\#} \Gamma^2 \xrightarrow{\delta_3^\#} \Gamma \xrightarrow{\delta_4^\#} \Gamma \to \cdots, \\
&\quad \delta_1^\#(\gamma) = ((x-1)\gamma, (y-1)\gamma), \\
&\quad \delta_2^\#(\gamma_1, \gamma_2) = (L\gamma_1 - (y+1)\gamma_2, (xy+1)\gamma_1 + (x-1)\gamma_2), \\
&\quad \delta_3^\#(\gamma_1, \gamma_2) = (x-1)\gamma_1 - (xy-1)\gamma_2, \\
&\quad \delta_4^\#(\gamma) = N\gamma.
\end{aligned}$$

We note that $x\gamma = \zeta\gamma\zeta^{-1}$ and $y\gamma = j\gamma j^{-1}$. So we have $x\zeta = \zeta$, $xj = \zeta^2 j$, $y\zeta = \zeta^{-1}j$ and $yj = j$. In particular, $Lj = 0$ holds because $\zeta^{2^r} = -e$.

Proposition 3. *The module structure of $H^n(Q_{2^r}, {}_\psi\Gamma)$ is represented by the form of the subquotient of the complex $\mathrm{Hom}_\Lambda(Y, {}_\psi\Gamma)$ as follows:*

$$H^n(Q_{2^r}, {}_\psi\Gamma) = \begin{cases} R & \text{for } n = 0 \\ R/2^{r+1}(\zeta+\zeta^{-1}) & \text{for } n \equiv 0 \mod 4,\ n \neq 0 \\ R(\zeta j - \eta j, 0)/(\zeta+\zeta^{-1}) \oplus R(0, e-\eta\zeta)/(\zeta+\zeta^{-1}) & \\ \qquad \oplus R(j - \eta\zeta j, j - \eta\zeta j)/(\zeta+\zeta^{-1}) & \text{for } n \equiv 1 \mod 4 \\ R(2^{r-1}\eta\zeta, e)/(\zeta+\zeta^{-1}) \oplus R(e,0)/(\zeta+\zeta^{-1}) & \\ \qquad \oplus R(\zeta, 0)/2^r\eta \oplus R(j,j)/(\zeta+\zeta^{-1}) & \\ \qquad \oplus R(0, \zeta j)/(\zeta+\zeta^{-1}) & \text{for } n \equiv 2 \mod 4 \\ R(e-\eta\zeta)/(\zeta+\zeta^{-1}) \oplus Rj/(\zeta+\zeta^{-1})(e-\eta^2) & \\ \qquad \oplus R(\zeta j - \eta j)/(\zeta+\zeta^{-1}) & \text{for } n \equiv 3 \mod 4. \end{cases}$$

In the above, M/s denotes the quotient module M/sM for a R-module M and an element $s \in R$.

Next, we calculate the products of the generators $A = (\zeta j - \eta j, 0)$, $B = (0, e - \eta\zeta)$ and $C = (j - \eta\zeta j, j - \eta\zeta j)$ of $H^1(Q_{2^r}, {}_\psi\Gamma)$ using the diagonal approximation $\Delta_Y = u_p \otimes u_q \cdot \Delta_{p,q} \cdot v_{p+q}$ on (Y, δ), which is given by direct calculations. These are obtained as the composition of the following homomorphisms on the cochain level:

$$\begin{aligned}
\Gamma^2 \otimes \Gamma^2 &\xrightarrow{\alpha_2^{-1} \otimes \alpha_2^{-1}} \mathrm{Hom}_\Lambda(Y_1, {}_\psi\Gamma) \otimes \mathrm{Hom}_\Lambda(Y_1, {}_\psi\Gamma) \\
&\xrightarrow{(\Delta_Y)_{1,1}^\#} \mathrm{Hom}_\Lambda(Y_2, {}_\psi\Gamma \otimes {}_\psi\Gamma) \\
&\xrightarrow{\text{natural}} \mathrm{Hom}_\Lambda(Y_2, {}_\psi\Gamma) \\
&\xrightarrow{\alpha_2} \Gamma^2,
\end{aligned}$$

where α_1 denotes the isomorphism $\mathrm{Hom}_\Lambda(Y_1, {}_\psi\Gamma) \xrightarrow{\sim} \Gamma^2$ Then the following equations hold in $H^2(Q_{2^r}, {}_\psi\Gamma)$:

$$A^2 = (2^{r-1}\eta\zeta, e), \quad B^2 = (e, 0), \quad C^2 = (2^{r-1}\eta\zeta, e) + (e, 0),$$

$$AB = BA = (0, \zeta j), \quad AC = CA = 2^{r-1}\eta^2(\zeta, 0), \quad BC = CB = (j, j).$$

Note that the generators of $H^2(Q_{2^r}, {}_\psi\Gamma)$ except $(\zeta, 0)$ are generated by the products of A, B and C, and the equation $A^2 + B^2 + C^2 = 0$ holds in $H^2(Q_{2^r}, {}_\psi\Gamma)$. In the following, we put $D = (\zeta, 0)$, which is a generator of $H^2(Q_{2^r}, {}_\psi\Gamma)$, and then we have $AC = 2^{r-1}\eta^2 D$. Similarly, we will compute the cup products of A, B, C and D etc. Then the following equations hold in $H^3(Q_{2^r}, {}_\psi\Gamma)$:

$$A^2C = AC^2 = B^3 = ABC = BD = DB = 0,$$
$$A^2B = BC^2 = e - \eta\zeta, \quad C^3 = B^2C = AD = DA = (e - \eta^2)j,$$
$$A^3 = AB^2 = CD = DC = \zeta j - \eta j.$$

If $r = 2$, by the above, the generators of $H^3(Q_4, {}_\psi\Gamma)$ are generated by the products of A, B, C and D. If $r > 2$, the generators of $H^3(Q_{2^r}, {}_\psi\Gamma)$ except j are generated by the products of A, B, C and D. In the following, we put $E = j$, which is a generator of $H^3(Q_{2^r}, {}_\psi\Gamma)$, and then we have $C^3 = (e - \eta^2)E$. Then the following equations hold in $H^4(Q_{2^r}, {}_\psi\Gamma)$:

$$A^4 (= A^2B^2 = B^2C^2 = C^4 = ACD) = CE = EC = 2^{r+1}e,$$
$$D^2 = \left(\zeta + \zeta^{-1}\right)^2 - 4e, \ AE = EA = BE = EB = 0.$$

In the following, we put $F = e$ which is the generator of $H^4(Q_{2^r}, {}_\psi\Gamma)$, and then we have $A^4 = 2^{r+1}F$ and $D^2 = ((\zeta+\zeta^{-1})^2 - 4)F$. Since $\mathbb{Z}$ is a Q_{2^r}-direct summand of ${}_\psi\Gamma$ using the embedding map $\mathbb{Z} \to {}_\psi\Gamma$ by $1 \mapsto e$, we have the following monomorphism of the complete cohomology rings:

$$\hat{H}^*(Q_{2^r}, \mathbb{Z}) := \bigoplus_{r \in \mathbb{Z}} \hat{H}^r(Q_{2^r}, \mathbb{Z}) \to \hat{H}^*(Q_{2^r}, {}_\psi\Gamma) := \bigoplus_{r \in \mathbb{Z}} \hat{H}^r(Q_{2^r}, {}_\psi\Gamma).$$

Since F above which is an element of $R/\left(2^{r+1}\left(\zeta + \zeta^{-1}\right)\right)$ in $H^4(Q_{2^r}, {}_\psi\Gamma)$ is the image of an element of order 2^{r+2} in $H^4(Q_{2^r}, \mathbb{Z})$, invertible in $\hat{H}^*(Q_{2^r}, \mathbb{Z})$, by the above map, it follows that F is also an invertible element in $\hat{H}^*(Q_{2^r}, {}_\psi\Gamma)$. Moreover, the equations $DE = ED = (0, 0)$ hold in $H^5(Q_{2^r}, {}_\psi\Gamma)$ and the equation $E^2 = (0, 0)$ holds in $H^6(Q_{2^r}, {}_\psi\Gamma)$. By summarizing Proposition 3 and the above equations we have the following theorem:

Theorem 1. *If $r = 2$, the cohomology ring $H^*(Q_4, {}_\psi\Gamma)$ is isomorphic to*

$$R[A, B, C, D, F]/(\sqrt{2}A, \sqrt{2}B, \sqrt{2}C, 4\sqrt{2}D, 8\sqrt{2}F, A^2 + B^2 + C^2, AC - 4D,$$
$$A^2C, AC^2, B^3, ABC, BD, A^4 - 8F, D^2 + 2F),$$

and if $r > 2$ the cohomology ring $H^*(Q_{2^r}, {}_\psi\Gamma)$ is isomorphic to

$$R[A, B, C, D, E, F]/((\zeta + \zeta^{-1})A, (\zeta + \zeta^{-1})B, (\zeta + \zeta^{-1})C,$$
$$2^r\eta D, (e - \eta^2)(\zeta + \zeta^{-1})E, 2^{r+1}(\zeta + \zeta^{-1})F,$$
$$A^2 + B^2 + C^2, AC - 2^{r-1}\eta^2 D,$$
$$A^2C, AC^2, B^3, ABC, BD, A^4 - 2^{r+1}F,$$
$$D^2 + \left(4 - (\zeta + \zeta^{-1})^2\right)F, DE, E^2),$$

where $R = \mathbb{Z}[\zeta + \zeta^{-1}]$, $\deg A = \deg B = \deg C = 1$, $\deg D = 2$, $\deg E = 3$ and $\deg F = 4$.

3.2 Integral cohomology ring $H^*(Q_t, \mathbb{Z})$

In this subsection, we determine the ring structure of the cohomology $H^*(Q_t, \mathbb{Z})$ of the generalized quaternion group Q_t by the similar method in Section 3.1. In fact, although $H^*(Q_{2^r}, \mathbb{Z})$ is well known (see [T] for example), it seems that the precise description of $H^*(Q_t, \mathbb{Z})$ is not given in any literature.

Theorem 2. *A precise description of the cohomology ring $H^*(Q_t, \mathbb{Z})$ for $t \geq 2$ is given as follows:*

$$H^*(Q_t, \mathbb{Z}) = \begin{cases} \mathbb{Z}[A, B, C]/(2A, 2B, 4tC, A^2, B^2 - 2tC, AB - 2tC) & (t \equiv 0 \mod 4) \\ \mathbb{Z}[A, B, C]/(2A, 2B, 4tC, A^2, B^2, AB - 2tC) & (t \equiv 2 \mod 4) \\ \mathbb{Z}[X, Y]/(4X, 4tY, X^2 - tY) & (t \equiv 1 \mod 4) \\ \mathbb{Z}[X, Y]/(4X, 4tY, X^2 + tY) & (t \equiv 3 \mod 4), \end{cases}$$

where $\deg A = \deg B = \deg X = 2$ *and* $\deg C = \deg Y = 4$.

Proof. (i) If t is even, we calculate the products of the generators $A = (1, 0)$, $B = (0, 1)$ of $H^2(Q_t, \mathbb{Z})$. In fact, we have $A^2 = 0, B^2 = -(t^2 + 2t)$ and $AB (= BA) = 2t$ in $H^4(Q_t, \mathbb{Z})$. (ii) If t is odd, we have $X^2 = t^2$ in $H^4(Q_t, \mathbb{Z})$ for the generator $X = ((t-1)/2, 1)$ of $H^2(Q_t, \mathbb{Z})$. □

3.3 Hochschild cohomology ring $HH^*(\Lambda)$

Let R be a commutative ring. We set $\Lambda = RG$ for a finite group G. If G is an abelian group, Holm [Hol] and Cibils and Solotar [CiSo] prove the following ring isomorphism exists:

$$HH^*(RG) \simeq RG \otimes_R H^*(G, R).$$

If G is a non abelian group, it seems more difficult to investigate the ring structure of $HH^*(RG)$. As for the additive structure of the Hochschild cohomology, it was well known that $HH^n(RG)$ is isomorphic to the direct sum of the ordinary group cohomology of the centralizers of representatives of the conjugacy classes of G (see [B, Theorem 2.11.2], [SiW, Section 4]):

$$HH^*(RG) \simeq \bigoplus_j H^*(G_j, R).$$

However, Siegel and Witherspoon [SiW] define a new product on $\bigoplus_j H^*(G_j, R)$, making the above additive isomorphism multiplicative. Besides, they calculate the Hochschild cohomology rings of $\mathbb{F}_3 S_3, \mathbb{F}_2 A_4$, $\mathbb{F}_2 D_{2^n}$ using this new product. In the following, we calculate the ring structure $HH^*(\mathbb{Z}Q_t)$ for arbitrary generalized quaternion group Q_t using a ring isomorphism $HH^*(\Lambda) \xrightarrow{\sim} H^*(Q_t, {}_\varphi\Lambda)$ and calculating the ordinary cup product in $H^*(Q_t, {}_\varphi\Lambda)$ above by the method different from [SiW].

In fact, although the module structure of $H^n(Q_t, {}_\varphi\Lambda)$ is easily obtained by its additive decomposition, we need the particular generators to determine the ring structure of

$H^*(Q_t, {}_\varphi\Lambda)$ by the method similar to Sections 3.1 and 3.2. By calculating the products of the generators using the diagonal approximation Δ_Y, we have the following theorem (see [Ha]):

Theorem 3. *Let Q_t be the generalized quaternion group of order 4t. We set $\Lambda = \mathbb{Z}Q_t$.*

(i) *If t is even, the Hochschild cohomology ring $H^*(Q_t, {}_\varphi\Lambda)(\simeq HH^*(\Lambda))$ is commutative, generated by the elements*

$$A_0,\ B_0,\ (C_i)_0,\ D_0,\ E_0 \in H^0(Q_t, {}_\varphi\Lambda),$$
$$(A_\alpha)_2,\ (A_\beta)_2,\ (B_\alpha)_2,\ (B_\beta)_2,\ (C_i)_2,\ D_2,\ E_2 \in H^2(Q_t, {}_\varphi\Lambda),$$
$$A_4 \in H^4(Q_t, {}_\varphi\Lambda),$$

for $i = 1, 2, \ldots, t-1$, where A_0 is the identity element. The relations are given by [Ha, Section 3.1].

(ii) *If t is odd, the Hochschild cohomology ring $H^*(Q_t, {}_\varphi\Lambda)(\simeq HH^*(\Lambda))$ is commutative, generated by the elements*

$$A_0,\ B_0,\ (C_i)_0,\ D_0,\ E_0 \in H^0(Q_t, {}_\varphi\Lambda),$$
$$A_2,\ B_2,\ (C_i)_2,\ D_2,\ E_2 \in H^2(Q_t, {}_\varphi\Lambda),$$
$$A_4 \in H^4(Q_t, {}_\varphi\Lambda)$$

for $i = 1, 2, \ldots, t-1$, where A_0 is the identity element. The relations are given by [Ha, Section 3.2].

References

[B] D. J. Benson, *Representations and cohomology II: cohomology of groups and modules*, Cambridge University Press, Cambridge, 1991.

[CaE] H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton University Press, Princeton. NJ., 1956.

[CiSo] C. Cibils and A. Solotar, *Hochschild cohomology algebra of abelian groups*, Arch. Math. **68** (1997), 17–21.

[Ha] T. Hayami, *Hochschild cohomology ring of the integral group ring of the generalized quaternion group*, SUT J. of Math. **38** (2002), 83–126.

[Hol] T. Holm, *The Hochschild cohomology ring of a modular group algebra: the commutative case*, Comm. Algebra **24** (1996), 1957–1969.

[HaSa] T. Hayami and K. Sanada, *Cohomology ring of the generalized quaternion group with coefficients in an order*, Comm. Algebra **30** (2002), 3611–3628.

[Sa] K. Sanada, *Remarks on cohomology rings of the quaternion group and the quaternion algebra*, SUT J. of Math. **31** (1995), 85–92.

[SiW] S. F. Siegel and S. J. Witherspoon, *The Hochschild cohomology ring of a group algebra*, Proc. London Math. Soc. (3) **79** (1999), 131–157.

[T] C. B. Thomas, *Characteristic classes and the cohomology of finite groups*, Cambridge University Press, Cambridge, 1986.

Takao Hayami
Department of Mathematics, Science University of Tokyo
Wakamiya-cho 26, Shinjuku-ku, Tokyo 162-0827, Japan
E-mail: hayami@minserver.ma.kagu.sut.ac.jp

Katsunori Sanada
Department of Mathematics, Science University of Tokyo
Wakamiya-cho 26, Shinjuku-ku, Tokyo 162-0827, Japan
E-mail: sanada@rs.kagu.tus.ac.jp

MIXED GROUPS IN ABELIAN GROUP THEORY

TAKASHI OKUYAMA

ABSTRACT. In this note, we introduce mixed groups in Abelian Group Theory. First we give notation and basics. Next we recall [2, Vol.2 p.186 Example 2]. Using this example, we show an entrance of mixed groups in Abelian Group theory.

1. INTRODUCTION

In 1917, Levi constructed non-splitting abelian groups. After Baer partially solved the splitting problem, numerous authors have studied many variations of the splitting problem. Furthermore, the splitting problem has been investigated for modules by various authors. Stratton solved the splitting problem for mixed groups of torsion-free rank 1 in [10] and studied the splitting problem for torsion-free finite rank modules over discrete valuation rings in [11]. Using the concept of purifiable subgroups, we also characterized the abelian groups of finite torsion-free rank that are splitting.

Let G be an abelian group and T the maximal torsion subgroup of G. Then there exists a subgroup A that is maximal with respect to the property of being disjoint from T. The subgroup A is called a *T-high subgroup* of G. Suppose that the subgroup A is purifiable in G. Then there exists a pure hull H of A in G such that $G = H \oplus T'$ where T' is a subgroup of T. Then the subgroup H has a property that H_p is bounded for every prime p and H is an ADE group. If H is torsion-free, then the group G is splitting. So we think that this way is useful to characterize splitting mixed groups.

On the other hand, the group G has a property that G/A is torsion. So we can consider that the group G is an extension of the torsion-free group A by the torsion group G/A. We use this way to try to characterize mixed groups.

All groups considered are abelian mixed groups. The terminologies and notations not expressly introduced here follow the usage of [2]. Throughout this note, $\mathbf{Z}$ denotes a set of integers, $\mathbf{P}$ a set of prime integers, and $p \in \mathbf{P}$.

1991 *Mathematics Subject Classification.* 20K21, 20K27.
This is the final version.

2. Notation and Basics

2.1. Splitting. Let G be a group. If every element of G is of finite order, then G is a *torsion group*, while G is torsion-free if all its elements, except for 0, are of infinite order. Mixed groups contain both nonzero elements of finite order and elements of infinite order.

Proposition 2.1. [2, Theorem 1.1] *The set T of all elements of finite order in a group G is a subgroup of G. Then T is a torsion group and the quotient group G/T is torsion-free. Hence T is the maximal torsion subgroup of G.*

Proof. Since $0 \in T$, T is not empty. If $a, b \in T$, i.e., $ma = 0$ and $nb = 0$ for some $m, n \in \mathbf{Z}$, then $mn(a - b) = 0$, and so $a - b \in T$. Hence T is a subgroup of G. We show that G/T is torsion-free. Suppose that $c + T \in G/T$ such that $l(c + T) \in T$ for some $l \in \mathbf{Z}$. Then $lc \in T$ and $c \in T$. Hence $c + T = T$ is the zero of G/T. By the previous argument, it is easy to see that T is the maximal torsion subgroup of G. □

Definition 2.2. *Let G be a group and T the maximal torsion subgroup of G. G is said to be splitting if T is a direct summand of G; i.e. $G = T \oplus F$ for some torsion-free subgroup F of G.*

2.2. Socle. Let

$$G[p] = \{g \in G \mid pg = 0\}.$$

$G[p]$ is called a *p-socle* of G. This is an elementary group in the sense that every element has a square-free order.

2.3. Pure subgroup.

Definition 2.3. *A subgroup A of a group G is said to be neat in G if, for every $p \in \mathbf{P}$,*

$$A \cap pG = pA.$$

Moreover, A is said to be pure in G if, for every $n \in \mathbf{Z}$,

$$A \cap nG = nA.$$

2.4. N–high Subgroups.

Definition 2.4. *Let N be a subgroup of a group G. Then a subgroup A of G is said to be N-high in G if A is maximal with respect to the property of being disjoint from N.*

The existence of N–high subgroups are guaranteed by Zorn's lemma. Combining the results in [3] and [1], we obtain the following characterization of N–high subgroups of groups.

Proposition 2.5. *Let N be a subgroup of a group G. Then a subgroup A of G is N-high in G if and only if*

1. $A \cap N = 0$,
2. *A is neat in G,*
3. $G[p] = A[p] \oplus N[p]$ *for every* $p \in \mathbf{P}$, *and*
4. $G/(A \oplus N)$ *is torsion.*

Corollary 2.6. *A torsion-free subgroup A of a group G is T-high in G if and only if*

1. *A is neat in G and*
2. *G/A is torsion.*

2.5. **Almost-dense.**

Definition 2.7. *A subgroup A of G is said to be almost-dense in G if, for every pure subgroup K of G containing A, the maximal torsion subgroup of G/K is divisible.*

The following is a characterization of almost-dense subgroups.

Proposition 2.8. *The following properties are equivalent:*

1. *A is almost-dense in G;*
2. *For all integers $n \geqq 0$ and all primes p,*

$$A + p^{n+1}G \supseteq p^n G[p].$$

2.6. **Purifiable subgroups.**

Definition 2.9. *A is said to be purifiable in G if, among the pure subgroups of G containing A, there exists a minimal one. Such a minimal pure subgroup is called a pure hull of A.*

Not all subgroups are purifiable in a given group.

Proposition 2.10. *Let G be a group and A a subgroup of G. Suppose that A is purifiable in G. Let H be a pure subgroup of G containing A. Then H is a pure hull of A in G if and only if the following three conditions are satisfied:*

1. *A is almost-dense in H;*
2. *H/A is torsion;*
3. *For every $p \in \mathbf{P}$, there exists a nonnegative integer m_p such that*

$$p^{m_p} H[p] \subseteq A.$$

2.7. **ADE groups.**

Definition 2.11. *Let A be a torsion-free group. A group X is said to be an almost-dense extension group (ADE group) of A if A is almost-dense and $T(X)$-high in X where $T(X)$ is the maximal torsion subgroup of X. Such a subgroup A is called a moho subgroup of X.*

3. An Example

We recall [2, Vol.2 p.186 Example 2].

Example 3.1. *Let $p_1, p_2, \dots, p_i, \dots$ be diffrent primes, and define*

$$T = \oplus_{i=1}^{\infty} \langle b_i \rangle \text{ with } o(b_i) = p_i.$$

Then T is the maximal torsion subgroup of $\Pi_{i=1}^{\infty} \langle b_i \rangle$. Consider $a_0 = (b_1, \dots, b_i, \dots) \in \Pi_{i=1}^{\infty} \langle b_i \rangle$. For $i \neq j$, the equation $p_j x = b_i$ is uniquely solvable in $\langle b_i \rangle$, thus $\Pi_{i=1}^{\infty} \langle b_i \rangle$ contains unique elements $a_i (i = 1, 2, \dots)$ such that a_i has 0 for its ith coordinate and satisfies

$$p_i a_i = (b_1, \dots, b_{i-1}, 0, b_{i+1}, \dots) = a_0 - b_i. \tag{3.2}$$

Let

$$G = \langle T, a_i \mid i \geqq 1 \rangle.$$

As for this group G, we have the following properties.

Property 3.3. *T is the maximal torsion subgroup of G.*

Property 3.4. *G is not splitting.*

Proof. Suppose that G is splitting. Then $G = T \oplus F$ for some torsion-free subgroup F of G. for every $i \geqq 0$, we can write

$$b_i = t_i + f_i$$

where $t_i \in T$ and $f_i \in F$ and

$$a_0 = t_0 + f_0$$

where $t_0 \in T$ and $f_0 \in F$. By (3.2), for every $i \geqq 1$, we have

$$p_i t_i + p_i f_i = p_i b_i = a_0 - b_i = (t_0 - b_i) + f_i.$$

Equating the T-coordinates, we have

$$p_i t_i = t_0 - b_i$$

for all $i \geqq 1$. Then there exists $p_j \in \mathbf{P}$ such that $(p_j, o(t_0)) = 1$. The prime p_j satisfies $p_j t = t_0$ for some $t \in T$. Then $p_j(t - t_j) = b_j$. This contradicts the choice of b_j. Hence G is not splitting. □

Property 3.5. *Let $A = \langle a_0 \rangle$. Then the following hold.*

1. *A is T-high in G.*
2. *G is a pure hull of A in G.*
3. *G is an ADE group with A as a moho subgroup.*

Proof. (1) It is immediate that G/A is torsion and $T \cap A = 0$. Let $x \in G$ such that $p_i x \in A \setminus p_i A$. Then $p_i x = \alpha a_0 = \alpha(b_i + p_i a_i)$ for some integer α with $(\alpha, p_i) = 1$ and so $\alpha b_i = p_i(x - a_i) \in p_i G$. This contradicts the choice of b_i. Hence A is neat in G. By Corollary 2.6, A

is T-high in G.
(2) By Property 3.3 and (3.2), A is almost-dense in G. It is immediate that G/A is torsion and $0 = p_i G[p] \subset A$. Hence, by Proposition 2.10, G is a pure hull of A.
(3) By (1) and (2), A is T-high and almost-dense in G. By Definition 2.11, G is an ADE group with A as a moho subgroup.

□

4. A PROSPECT IN THE FUTURE

First we give a useful lemma.

Lemma 4.1. *Let H be a pure subgroup of a group G containing some T-high subgroup of G. If, for each prime p, U_p is a subgroup of G such that $G_p = H_p \oplus U_p$, then $G = H \oplus U$ where $U = \oplus_p U_p$.*

Proof. Let $ng \in H \oplus U$ with $g \in G$ and $n \in \mathbf{Z}$. Then we have $mng \in H$ for some integer m. Since H is pure in G, there exists $h \in H$ such that $mng = mnh$. Then $g - h \in T \subset H \oplus U$ and so $H \oplus U$ is pure in G. Since $H \oplus U$ is essential in G, $G = H \oplus U$. □

Theorem 4.2. *Let G be a group and T the maximal torsion subgroup of G. Let A be a T-high subgroup of G. Suppose that A is purifiable in G. If H is a pure hull of A in G, then*

$$G = H \oplus T'$$

where H is an ADE group with A as a moho subgroup and T' is a subgroup of T.

Proof. Let H be a pure hull of A in G. By Proposition 2.10(3), for every $p \in \mathbf{P}$, there exists a nonnegative integer m_p such that $p^{m_p}H \subseteq A$. Since A is torsion-free, we have $p^{m_p}H \subseteq A \cap T = 0$. Hence the maximal p-subgroup H_p of H is bounded. Note that H_p is pure in G. By [2, Theorem 27.5], H_p is a direct summand of the maximal p-subgroup G_p of G. Hence $G_p = H_p \oplus K_p$ for some subgroup K_p of G_p. By Lemma 4.1, $G = H \oplus K$ where $K = \oplus_p K_p$.

By Proposition 2.10(1), A is almost-dense in G. Hence H is an ADE group with A as a moho subgroup. □

In Theorem 4.2, if the subgroup H is torsion-free, then G is splitting. So the concept of purifiable subgroups is useful to characterize splitting mixed groups.

Definition 4.3. *Let G be a group and T the maximal torsion subgroup of G. Let L be a maximal independent system of G/T. The cardinality of L is called the torsion-free rank of G.*

Let G be a group of torsion-free rank 1 all of whose maximal p-subgroups are cyclic. Then, by [6, Theorem 5.2], all subgroups of G are purifiable in G. Using Theorem 4.2, we characterized the group G in [8].

Now we pose the following problems.

Problem 4.4. *Which subgroup of a group G is purifiable in G ?*

We already characterized purifiable torsion-free finite rank subgroups in [7] and [9]. Using these results, we characterized splitting mixed groups of finite torsion-free rank.

Problem 4.5. *Study ADE groups.*

We studied ADE groups G of torsion-free rank 1 all of whose maximal p-subgroups are cyclic in [5].

As for groups of torsion-free rank 1, a characterization theorem of countable mixed groups of torsion-free rank 1 was established in [2, Theorem 104.3].

However, in general, [2, Theorem 104.3] is not true for arbitrary mixed groups of torsion-free rank 1, because Megibben presented a counterexample in [4].

References

[1] K. Benabdallah and J. Irwin. On N-High Subgroups of Abelian Groups, *Bull. Soc. Math. France*, 96:337-346, 1968.

[2] L. Fuchs. *Infinite Abelian Groups, Vol. I, II.* Academic Press. 1970 and 1973.

[3] J. Irwin and E. A. Walker. On N-High Subgroups of Abelian Groups, *Pacific J. Math.*, 11(4):1363-1374, 1961.

[4] C. Megibben. On mixed groups of torsion-free rank one, *Illinois J. Math.*, 11:134-144, 1967.

[5] T. Okuyama. On Almost-Dense Extension Groups of Torsion-Free Groups, *J. Algebra*, 202:202-228, 1998.

[6] T. Okuyama. On Purifiable Subgroups in Arbitrary Abelian Groups, *Comm. Algebra*, 28(1):121-139, 2000.

[7] T. Okuyama. On Purifiable Torsion-Free Rank-One Subgroups, *Hokkaido Math. J.*, 30(2):373-404, 2001.

[8] T. Okuyama. On Locally Cyclic Abelian Groups of Torsion-Free Rank 1, *Kyushu J. Math.*, 55(2):301-320, 2001.

[9] T. Okuyama. Splitting Mixed Groups of Finite Torsion-Free Rank. *Preprint.*

[10] A. E. Stratton. On the Splitting of Rank-One Abelian Groups, *J. Algebra*, 19(2):254-260, 1971.

[11] A. E. Stratton. A Splitting Theorem for Mixed Abelian Groups, *Instituto Nazionale di Mathematica Symposia Mathematica*, 13:109-125, 1974.

Department of Mathematics, Toba National College of Maritime Technology, 1-1, Ikegami-cho, Toba-shi, Mie-ken, 517-8501, Japan

E-mail address: okuyamat@toba-cmt.ac.jp

TOTAL VALUATION RINGS OF ORE EXTENSIONS [1]

Guangming Xie, Shigeru Kobayashi
Hidetoshi Marubayashi, Nicolae Popescu

ABSTRACT. We considered extensions of a total valuation ring V of a skew field K to the Ore extension $K(X;\sigma,\delta)$ for an endomorphism σ of K and a σ-derivation δ. It is shown that there exists an extension R of V with $\overline{X}$ is transcendental over $V/J(V)$ if and only if (σ,δ) is compatible, where $\overline{X} = [X + J(R^{(1)})]$. In the case V is invariant, it is established that there is an invariant extension R of V in $K(X;\sigma,\delta)$ such that $\overline{X}$ is transcendental if and only if $\sigma(a)V = aV$ and $\delta(a) \in aV$ for all $a \in K$.

1. INTRODUCTION

Let σ be an endomorphism of a skew field K. A (left) σ-derivation of K is any additive map $\delta : K \to K$ such that $\delta(ab) = \sigma(a)\delta(b)+\delta(a)b$ for all $a, b \in K$. Then there exists a ring S, containing K as a subring, such that S is a free left K-module with a basis of the form $1, X, X^2, \cdots$, and $Xa = \sigma(a)X + \delta(a)$ for all $a \in K$ (cf. [GW], Proposition 1.10). The ring S is denoted $K[X;\sigma,\delta]$ and is called a skew polynomial ring of K. It is known that the ring $K[X;\sigma,\delta]$ is a principal left ideal domain (cf. [GW], Theorem 1.11), so that $K[X;\sigma,\delta]$ is a left Ore domain. We denote $K(X;\sigma,\delta)$ as the quotient division ring of $K[X;\sigma,\delta]$. This is the corresponding Ore extension of K. We say that the pair (K, V) is a *valued skew field* if K is a skew field with the subring V such that $a \in K\backslash V$ implies $a^{-1} \in V$, i.e., V is a *total valuation* ring of K. We consider the extensions of V in $K(X;\sigma,\delta)$, i.e., the total valuation ring R of $K(X;\sigma,\delta)$ with $R \cap K = V$. Let R be an extension of V in $K(X;\sigma,\delta)$ and $J(V)$ the Jacobson radical of V and $J(R)$ be the Jacobson radical of R. Then since $J(V) = J(R) \cap K$, $V/J(V)$ is a subring of $R/J(R)$. If $\pi_V : V \to V/J(V)$ is the canonical map, one put $\pi_V(a) = \bar{a}$ for all $a \in V$, and also $\pi_R : R \to R/J(R)$. An element $\bar{f}$ in $R/J(R)$ is called (left) *transcendental* over $V/J(V)$ if for any natural number n, and any elements $\overline{a_0}, \overline{a_1}, \cdots, \overline{a_n} \in V/J(V)$, $\overline{a_0} + \overline{a_1}\bar{f} + \cdots + \overline{a_n}\bar{f}^n = \bar{0}$ implies $\overline{a_i} = \bar{0}$ for all i ($i = 0, \cdots, n$). In [BT], they considered the conditions on σ, δ that $\sigma(V) \subseteq V$, $\delta(V) \subseteq V$ and (i) $\sigma(r)$ is in $J(V)$ if and only if r is in $J(V)$ for r in V, (ii) $\delta(J(V))$ is contained in

[1] The detailed version of this paper has been submitted for publication elswhere

$J(V)$, which is called compatible with $J(V)$. By using Lemma 3.2 of [BS], we can know the condition (i) is equivalent to the condition that $\sigma(J(V)) \subseteq J(V)$. So in this paper, we use the compatibility, as follows, (σ, δ) is called *compatible* with V if $\sigma(V) \subseteq V$, $\sigma(J(V)) \subseteq J(V)$, and $\delta(V) \subseteq V$, $\delta(J(V)) \subseteq J(V)$ in order to characterize the existence of an extension of V in which $\overline{X}$ is transcendental over $V/J(V)$. In the case, V is (right and left) chain ring, in [BT], they have shown that if (σ, δ) is compatible with V, then $J(V)[X;\sigma,\delta]$ is localizable and the ring of quotients $R^{(1)} = V[X;\sigma,\delta]_{J(V)[X;\sigma,\delta]}$ is a chain ring. Since it is easy to see that V is a total valuation ring is equivalent to that V is a chain ring, if V is a total valuation ring and (σ, δ) is compatible with V, then $R^{(1)} = V[X;\sigma,\delta]_{J(V)[X;\sigma,\delta]}$ is a total valuation ring with $R^{(1)} \cap K = V$ and $\overline{X}$ is transcendental over $V/J(V)$ (Proposition 2.1).

We shall show that there exists a total valuation ring R of $K(X;\sigma,\delta)$ which is an extension of V and $\overline{X}$ is transcendental over $V/J(V)$ if and only if (σ, δ) is compatible with V (Theorem 2.2). We shall show that R above is equal to $R^{(1)}$. In the case V is invariant, it is shown that there exists an invariant valuation ring R of $K(X;\sigma,\delta)$ which is an extension of V and $\overline{X}$ is transcendental over $V/J(V)$ if and only if $\delta(a) \in aV$ and $\sigma(a)V = aV$ for any $a \in K$.

2. Characterization of $R^{(1)}$

Let (K, V) be a valued skew field and σ an endmorphism of K, and δ a σ-derivation. Recall that (σ, δ) is called compatible with V if $\sigma(V) \subseteq V$, $\sigma(J(V)) \subseteq J(V)$, and $\delta(V) \subseteq V$, $\delta(J(V)) \subseteq J(V)$. If V is compatible with V, then Theorem 1 of [BT] shows that the ring of quotients $R^{(1)} = V[X;\sigma,\delta]_{J(V)[X;\sigma,\delta]}$ exists and is a total valuation ring. To show that R is an extension of V, let $\alpha \in R^{(1)} \cap K$ and assume that α is not contained in V. Then α^{-1} is contained in $J(V)$. Since $J(V)R^{(1)} = J(R^{(1)})$, it follows that $\alpha^{-1} \in J(V) \subseteq J(V)R^{(1)} = J(R^{(1)})$. Hence $\alpha \notin R^{(1)}$, which is a contradiction. This implies that $R^{(1)} \cap K \subseteq V$. Since the converse inclusion is clear, so we obtain that $R^{(1)} \cap K = V$. Since (σ, δ) is compatible, we can consider the division ring $V/J(V)(\overline{X};\overline{\sigma},\overline{\delta})$, where $\overline{\sigma} \in \mathrm{End}(V/J(V))$ is defined by $\overline{\sigma}(\overline{a}) = \overline{\sigma(a)}$ and $\overline{\sigma}$-derivation $\overline{\delta}$ is defined by $\overline{\delta}(\overline{a}) = \overline{\delta(a)}$ for all $\overline{a} \in V/J(V)$, and natural surjective homomorphism $\varphi : R^{(1)} \rightarrow V/J(V)(\overline{X};\overline{\sigma},\overline{\delta})$ is naturally defined by

$$\varphi(g^{-1}f) = \overline{g}^{-1}\overline{f}$$

Clearly $\ker\varphi = J(R^{(1)})$, that is, $R^{(1)}/J(R^{(1)}) \cong V/J(V)(\overline{X};\overline{\sigma},\overline{\delta})$. In particular, $\overline{X}$ is transcendental over $V/J(V)$. So we get the following.

Proposition 2.1. *If (σ,δ) is compatible with V. Then $R^{(1)}$ is an extension of V and $\overline{X}$ is transcendental over $V/J(V)$.*

We shall give a characterization of $R^{(1)}$ as follows.

Theorem 2.2. *Let V be a total valuation ring of K. Then the following conditions are equivalent.*

(1) *There exists a total valuation ring R of $K(X;\sigma,\delta)$ with $R\cap K = V$ and $\overline{X} \in R/J(R)$ is transcendental over $V/J(V)$.*

(2) *(σ,δ) is compatible with V.*
If R satisfies the equivalent conditions, then $R = R^{(1)}$.

Next we consider in the case that V is *invariant*, that is, $dVd^{-1} = V$ for all $0 \neq d \in K$. If V is invariant, then we can define the value group of V as $\Gamma_V = U(K)/U(V)$, where $U(K)$ and $U(V)$ denote the set of units of K and V respectively. Γ_V become a totally ordered group by $d_1U(V) \leq d_2U(V)$ if and only if $d_1V \supseteq d_2V$ for any $d_1, d_2 \in U(K)$. The mapping $v : U(K) \longrightarrow \Gamma_V$ defined by $d \longrightarrow dU(V)$, where $d \in K$, satisfies the following conditions;

(2.1) For any $a, b \in K, v(ab) = v(a) + v(b)$,

where we use an additive notations for Γ_V.

(2.2) $v(a+b) \geq \min\{v(a), v(b)\}$ if $a+b \neq 0$.

v is called a *valuation* on K.

Theorem 2.3. *Let V be a total valuation ring. Then the following conditions are equivalent.*

(1) *There exists an invariant valuation ring R of $K(X;\sigma,\delta)$ with $R\cap K = V$ and $\overline{X} \in R/J(R)$ is transcendental over $V/J(V)$.*

(2) *V is an invariant valuation ring, and $\delta(a) \in aV$ and $\sigma(a)V = aV$ for all a in K.*

(3) *V is an invariant valuation ring with valuation v, and there is a valuation ω on $K(X;\sigma,\delta)$ such that $\omega(f) = \min\{v(a_i)\}$, where $f = a_0 + a_1X + \cdots + a_nX^n \in K[X;\sigma,\delta]$*

In the following examples, we shall give examples of valuation rings V of a field K such that (σ,δ) is compatible with V. But V 's do not satisfy the conditions (2) in Theorem 2.3 so that $R^{(1)}$ is a total valuation ring of $K(X;\sigma,\delta)$ but not invariant.

Example 2.4. Let F be a field and $K = F(t)$ be the rational function field over F with the endmorphism σ defined by $\sigma(a) = a$ for all $a \in F$ and $\sigma(t) = t^2$. We define σ-derivation δ by $\delta(\alpha) = \sigma(\alpha) - \alpha$ for all $\alpha \in K$. Let v be the t-adic valuation of K and V be the valuation ring of v, then for any $\alpha \in V$, $v(\sigma(\alpha)) = 2v(\alpha)$ and $v(\delta(\alpha)) \geq v(\alpha)$, hence (σ, δ) is compatible with V and $\delta(\alpha) \in \alpha V$ for all $\alpha \in V$. On the other hand, $v(\sigma(t)) = v(t^2) = 2 > 1 = v(t)$. This implies that $\sigma(t)V \neq tV$.

Example 2.5. Let F be a field and G be the abelian group $\sum_{i \in \mathbb{N}} \oplus \mathbb{Z}_i$, where $\mathbb{Z}_i = \mathbb{Z}$, ordered lexicographically, and let $K = F(X_i \mid i \in \mathbb{N})$ be the field of rational functions over F in indeterminates X_i. We define a valuation on K with value group G as follows,

$$\begin{aligned} v(a) &= 0 \ (a \in F) \\ v(X_i) &= g_i = (0, 0, \cdots, \overset{i}{1}, \cdots) \in G \\ v(f) &= \min\{m_i g_{i_1} + \cdots + m_n g_{i_n}\}, \\ \text{where } f &= \sum a_{i_1 \cdots i_n} X_{i_1}^{m_1} \cdots X_{i_n}^{m_n} \\ v(g^{-1} f) &= -v(g) + v(f). \end{aligned}$$

We define $\sigma \in \mathrm{End}(K)$ by $\sigma(a) = a$ for all $a \in F$ and $\sigma(X_i) = X_{i+1}$ and a σ-derivation δ by $\delta(\alpha) = \sigma(\alpha) - \alpha$ for all $\alpha \in K$. Let $f = \sum a_{i_1 \cdots i_\ell} X_{i_1}^{m_1} \cdots X_{i_n}^{m_\ell}$ and $g = \sum b_{j_1 \cdots j_k} X_{j_1}^{n_1} \cdots X_{j_k}^{n_k}$ and let $\alpha = g^{-1} f \in K$. Suppose that $v(f) = m_1 g_{i_1} + \cdots + m_\ell g_{i_\ell}$ and $v(g) = n_1 g_{j_1} + \cdots + n_k g_{j_k}$. Then

$$v(\alpha) = v(f) - v(g) = (m_1 g_{i_1} + \cdots + m_\ell g_{i_\ell}) - (n_1 g_{j_1} + \cdots + n_k g_{j_k}).$$

Hence

$$v(\sigma(\alpha)) = (m_1 g_{i_1+1} + \cdots + m_\ell g_{i_\ell+1}) - (n_1 g_{j_1+1} + \cdots + n_k g_{j_k+1}).$$

So it is clear that $v(\alpha) \geq 0$ if and only if $v(\sigma(\alpha)) \geq 0$ and $v(\alpha) > 0$ if and only if $v(\sigma(\alpha)) > 0$. Let V be the valuation ring of v. Then we have that $\sigma(V) \subseteq V$ and $\sigma(J(V)) \subseteq J(V)$. Since $\delta(\alpha) = \sigma(\alpha) - \alpha$, we also have that $\delta(V) \subseteq V$ and $\delta(J(V)) \subseteq J(V)$. On the other hand, since $v(\sigma(X_1)) = v(X_2) = g_2 < g_1 = v(X_1)$, it follows that $X_1 V \subsetneq X_2 V = v(\sigma(X_1))V$. Further since $\delta(X_1) = \sigma(X_1) - X_1 = X_2 - X_1$,

$$v(\delta(X_1)) = v(X_2 - X_1) = \min\{v(X_1), v(X_2)\} = g_2 < g_1 = v(X_1).$$

This shows that $\sigma(X_1)V \neq X_1 V$ and $\delta(X_1) \notin X_1 V$.

References

[BT] H. H. Brungs and G. Törner *Extensions of chain rings*, Math. Z. 185 (1984), 93–104.

[BS] H. H. Brungs and M. Schröder *Valuation rings in Ore extensions*, Journal of Algebra. **235** (2001), 665–680.

[GW] K. R. Goodearl and R. B. Warfield, *An introduction to Noncommutative Noetherian Rings*, London Math. Soc. 1989.

[MMU] H. Marubayashi, H. Miyamoto and A. Ueda *Valuation rings and semi-hereditary orders*, K-Monographs in Math., 3, Kluwer Academic Publishers, 1997.

Department of Mathematics,
Naruto University of Education
Takashima, Naruto, 772-8502, Japan

Department of Mathematics,
Naruto University of Education
Takashima, Naruto, 772-8502, Japan
E-mail address: skoba@naruto-u.ac.jp

Department of Mathematics,
Naruto University of Education
Takashima, Naruto, 772-8502, Japan
E-mail address: marubaya@naruto-u.ac.jp

Institute of Mathematics of the Romanian Academy
P.O. Box 1-764, Ro-70700
Bucharest Romania
E-mail address: nipopesc@stoilow.imar.ro

Unitary Strongly Prime Rings and Ideals

Miguel Ferrero*

Abstract

A *unitary strongly prime* ring is defined as a prime ring whose central closure is simple with identity element. The class of unitary strongly prime rings is a special class of rings and the corresponding radical is called the *unitary strongly prime radical*. In this paper we give a survey including several recent results on unitary strongly prime rings and applications to the study of R-disjoint maximal ideals of polynomial rings over R in a finite number of indeterminates. Also, some open questions concerning the Brown-McCoy radical are posed.

Introduction

A left module M over the ring R is called *strongly prime* if for any non-zero $x, y \in M$ there exists a finite set of elements $\{r_1, \dots, r_n\} \subseteq R$ such that $Ann_R\{r_1x, \dots, r_nx\} \subseteq Ann_R\{y\}$, where $n = n(x, y)$ and $Ann_R(S)$ denotes the annihilator of S in R. This definition was first given by J. Beachy in 1975 [1].

Taking $M = R$ in the above definition, the notion of *left strongly prime rings* is obtained. Left strongly prime rings were first studied by D. Handelman and J. Lawrence [10]. Later on several authors studied left (right) strongly prime rings and ideals and the left (right) strongly prime radical (see, for example [3, 9, 17, 18, 7]). In particular, in the last two quoted papers examples of rings which are strongly prime only on one side were given. Thus this notion of strongly prime rings is not symmetric.

Symmetric strongly prime rings are defined in ([23], Chap. 35). The multiplication ring $M(R)$ of R is defined as the subring of $End_{\mathbb{Z}}R$, acting from the left on R, generated as a ring by all the left and right multiplications l_a and r_b, where $a, b \in R^\#$, and $l_a x = ax$, $r_b x = xb$, for $x \in R$, where $R^\#$ denotes the ring obtained from R by adjoining an identity. So each $\lambda \in M(R)$ is of the form $\lambda = \sum_k l_{a_k} r_{b_k}$, where $a_k, b_k \in R^\#$, and $\lambda x = \sum_k a_k x b_k$, $x \in R$. In this way R is a left module over $M(R)$. Then a (*symmetric*) *strongly prime ring* is defined as a ring which is strongly prime when is considered as a module over $M(R)$.

For rings with identity element, (symmetric) strongly prime rings and ideals were first studied by A. Kăucikas and R. Wisbauer [15]. The notion seems to be not so useful for rings without identity element. Then in [8] we adapted the definition to rings without

[0]1991 Mathematics Subject Classification: 16N40, 16N60, 16N80.

This paper is a survey including recent results on the subject.

*Partially supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq., Brasil.)

identity element and define the notion of unitary strongly prime rings. A unitary strongly prime ring is defined as a prime ring whose central closure is simple with identity. If R has an identity our definition is equivalent to the one used in [15].

The purpose of this paper is to survey results on unitary strongly prime rings and ideals which appear in the papers mentioned above. We also present some new results that we learned from A. Kaučikas in a meeting that took place at Banach International Center of Mathematics (Warsaw, June of 2002).

Throughout this paper rings are associative but do not necessarily have an identity element. For a ring R, $Z(R)$ denotes the center of R.

1 Prerequisites

Let R be a semiprime ring. The self-injective hull of R considered as (R, R)-bimodule, endowed with a canonical ring structure, is called the *central closure* of R (see [23], Sect. 32). Equivalently, the central closure of R may be considered as the subring of the Martindale right (left) ring of quotients $Q = Q_r(R)$ (or $Q_l(R)$) of R generated by R and the center $C(R)$ of Q, which is called the *extended centroid* of R.

Throughout this paper, for a prime ring R we denote by $C(R)$ (or just C) the extended centroid of R and by RC the central closure of R. As a basic property we recall that for any ideal I of R we have $C(R) = C(I)$ ([2], Corollary 2.1.12 and Proposition 2.2.2).

Assume that $\phi : R \to S$ is a monomorphism of rings. Then S becomes a canonical R-bimodule. In this paper we say that ϕ is a *centred monomorphism* if there exists a surjective ring homomorphism $\Phi : R\langle X\rangle \to S$ such that $\Phi\,|_R = \phi$, where $R\langle X\rangle$ denotes a free ring over R in X, a set of indeterminates. When this is the case we also say that S is a *centred extension* of R

If R has an identity element, then the definition agrees with the usual definition: we may consider $\Phi((X)) \subseteq S$ as a set of R-centralizing generators, where (X) denotes the monoid generated by the set X (cf. [6], [15]).

For basic notions and terminology on radicals we refer the reader to [4] and [21].

Let $\mathcal{A}$ be a class of rings such that every non-zero ideal of a ring in $\mathcal{A}$ can be homomorphically mapped onto some non-zero ring of $\mathcal{A}$. Then $\mathcal{A}$ determines a so called *upper radical* property, which we denote by $\mathcal{A}$ again. Thus the rings in $\mathcal{A}$ are all semisimple rings with respect to this upper radical, and $\mathcal{A}$ is the largest radical for which this happens.

Recall that a class of prime rings $\mathcal{A}$ is said to be a *special class* if for any non-zero ideal I of a ring R, I belongs to $\mathcal{A}$ if and only if R is in $\mathcal{A}$.

Any special class of rings $\mathcal{A}$ determines an upper radical. This radical contains the prime radical and is *hereditary*, i.e., for any ring R and ideal I of R, the $\mathcal{A}$-radical of I is equal to the intersection $\mathcal{A}(R) \cap I$, where $\mathcal{A}(R)$ denotes the $\mathcal{A}$-radical of R. Moreover, $\mathcal{A}(R)$ is equal to the intersection of all ideals P of R such that $R/P \in \mathcal{A}$ ([4], Ch. 7).

Assume that R is prime. We will consider the ring obtained from R by adjoining an identity, defined as usual in the following way ([11], 2.17, Ex. 5): Consider R as an algebra over the ring of integers $\mathbb{Z}$ and put $T = R \oplus \mathbb{Z}$ with the operations:

$$(a,n)+(b,m)=(a+b,n+m) \text{ and } (a,n)(b,m)=(ab+ma+nb,nm),$$

for $(a,n),(b,m) \in T$. The *natural extension* of R to a ring with identity $R^{\#}$ is defined as the ring $T/Ann_T(R)$, where $Ann_T(R) = \{t \in T \mid Rt = 0\}$ is an ideal of T. Since R is prime, $Ann_T(R) \cap R = 0$ and hence we may consider $R \subseteq R^{\#}$. It follows that $R^{\#}$ is prime with unit and R is an essential ideal of $R^{\#}$.

As we said in the introduction, the multiplication ring $M(R)$ of R is defined as the subring of $End_{\mathbb{Z}}R$, acting from the left on R, generated as a ring by all the left and right multiplications l_a and r_b, where $a, b \in R^{\#}$, and $l_a x = ax$, $r_b x = xb$, for $x \in R$. So each $\lambda \in M(R)$ is of the form $\lambda = \sum_k l_{a_k} r_{b_k}$, where $a_k, b_k \in R^{\#}$, and $\lambda x = \sum_k a_k x b_k$, $x \in R$. Thus R is a left $M(R)$-module and, in particular, sending $\lambda \in M(R)$ to $\lambda 1 = \sum_k a_k b_k$ we have a projection from $M(R)$ to R which is a left $M(R)$-homomorphism, where 1 denotes the identity of $R^{\#}$.

2 Unitary Strongly Prime Rings and Ideals

The definition of strongly prime rings given in ([23], Chap. 35) is not the same as the one we want to use here. In fact, in this book, a module M over a ring R is said to be strongly prime if it is subgenerated by each of its non-zero submodules. This definition agrees with the definition given by J. Beachy in [1]. Then a ring R is said to be (*symmetric*) *strongly prime* if R is a left strongly prime module over the multiplication ring $M(R)$. As a consequence, a ring R is strongly prime if it is prime and the central closure RC is a simple ring. Thus if R has not an identity element, then R can be a strongly prime ring even if RC has not an identity element (e. g., a simple ring without identity element has these properties).

The definition used in [15] for rings with identity element is just the same as in [23]. An element $a \in R$ is said to be a symmetric zero divisor if for any finite subset $\{a_1, ..., a_n\} \subseteq (a)$, $Ann_{M(R)}\{a_1, ..., a_n\} \not\subseteq Ann_{M(R)}\{1\}$, where (a) denotes the ideal generated by a ([15], Section 2). We denote by $zd(R)$ the set of symmetric zero divisors of R.

The following result was proved in ([15], Theorem 2.1).

2.1 Theorem. *Let R be a ring with identity element. The following conditions are equivalent:*

(*i*) *R is a strongly prime ring;*

(*ii*) *$zd(R) = 0$;*

(*iii*) *R is prime and the central closure RC of R is a simple ring;*

(*iv*) *for any non-zero $a, b \in R$, there exist $\lambda_1, \ldots, \lambda_n \in M(R)$ such that $Ann_{M(R)}\{\lambda_1 a, ..., \lambda_n a\} \subseteq Ann_{M(R)}\{b\}$;*

(*v*) *for any non-zero $a \in R$, there exist $a_1, \ldots, a_n \in (a)$ such that*

$$\sum_i x_i a_k y_i = 0, \text{ for all } 1 \le k \le n, \text{ implies } \sum_i x_i y_i = 0;$$

(*vi*) *there exists a centred monomorphism $\phi : R \to K$, where K is a simple ring;*

(*vii*) *there exists a centred monomorphism* $\phi : R \to S$, *where the ring* S *has the following property: for each non-zero ideal* I *of* R, *its extension* SIS *is equal to* S.

For rings without identity we have to modify slightly the definition in order to have a more useful notion. The following was given in ([8], Section 2).

2.2 Definition. *A prime ring* R *is said to be unitary strongly prime (u-strongly prime, for short) if* RC *is a simple ring with identity element.*

The class of unitary strongly prime rings is a nice class of rings. In fact, denote by $\mathcal{S}$ the class of all u-strongly prime rings and by $\mathcal{S}'$ the class of all strongly prime rings. In ([8], Section 2) we proved that $\mathcal{S}$ is a special class of rings and the class $\mathcal{S}'$ is not special. Moreover we have the following

2.3 Proposition. *The class* $\mathcal{S}$ *is the largest special class of rings* $\mathcal{A}$ *which is contained in* $\mathcal{S}'$ *and satisfies the property: if* $R \in \mathcal{A}$, *then* $RC \in \mathcal{A}$.

A finite subset $A = \{a_1, \dots, a_n\} \subseteq R$ is said to be an insulator, if

$$Ann_{M(R)}\{a_1, \dots, a_n\} \subseteq Ann_{M(R)}\{1_{R^\#}\},$$

i.e., if $\lambda a_1 = \dots = \lambda a_n = 0$, implies that $\lambda 1 = 0$, for $\lambda \in M(R)$. By Proposition 2.6 of [15], a finite subset $A = \{a_1, \dots, a_n\}$ of a prime ring R is an insulator if and only if $1 \in RC$, where $C = C(R^\#) = C(R)$, i.e., there exist $c_1, \dots, c_n \in C$ such that $a_1c_1 + \dots a_nc_n = 1$.

The connection between Theorem 2.1 of [15] and our definition of u-strongly prime rings is given by the following

2.4 Theorem. ([*8*], 2.4) *For any ring* R *the following conditions are equivalent:*

(*i*) R *is u-strongly prime;*

(*ii*) R *is prime and* $R^\#$ *is strongly prime;*

(*iii*) *there exists a centred monomorphism* $\phi : R \to S$, *where* S *is a simple ring with identity;*

(*iv*) *there exists a centred monomorphism* $\phi : R \to S$, *where* S *is a ring with identity with the property: for each non-zero ideal* I *of* R, *its extension* SIS *is equal to* S;

(*v*) R *is prime and any non-zero ideal of* R *contains an insulator.*

An ideal P of a ring R is said to be *u-strongly prime* if the factor ring R/P is a u-strongly prime ring [15]. U-strongly prime ideals have a nice behaviour concerning centred extensions. In fact, we have the following

2.5 Lemma. *For a centred monomorphism* $\phi : R \to S$ *of rings we have:*

(*i*) *If* P *is a u-strongly prime ideal of* S, *then* $\phi^{-1}(P)$ *is a u-strongly prime ideal of* R.

(*ii*) *If* I *is a u-strongly prime ideal of* R *and* P *is an ideal of* S *which is maximal with respect to the condition* $\phi^{-1}(P) = I$, *then* P *is a u-strongly prime ideal of* S.

As a consequence of the above, if S is a u-strongly prime ring and a centred extension of R, then R is also u-strongly prime.

A strongly semiprime ring R can be defined as a semiprime ring R such that any essential ideal of R contains an insulator ([15], Sect. 2). There are other interesting results on strongly prime rings and strongly semiprime rings which are proved in ([15], Sect. 2). For example:

(1) A ring R is strongly prime if and only if its multiplication ring $M(R)$ is strongly prime. In this case their extended centroids are canonically isomorphic, and the central closure of $M(R)$ is isomorphic to $RC \otimes_C (RC)^0$.

(2) If R is a strongly prime ring and S is Morita equivalent to R, then S is strongly prime and the extended centroids of R and S are isomorphic.

(3) If R is strongly semiprime, then the canonical map $RC \otimes_R RC \to RC$ is an isomorphism and RC is flat as a left and right R-module.

(4) Let R be a strongly semiprime ring and let $\mathcal{F}$ be the set of all right ideals of R containing an insulator. Then $\mathcal{F}$ is a symmetric Gabriel filter and the corresponding localization is perfect. Also, the central closure RC is canonically isomorphic to que quotient ring of R with respect to $\mathcal{F}$.

3 The Unitary Strongly Prime Radical

For the rest of the paper $\mathcal{S}$ denotes the class of all u-strongly prime rings as well as the upper radical determined by the class $\mathcal{S}$ [4]. By Proposition 2.3, the radical $\mathcal{S}$ is a special radical and for every ring R, $\mathcal{S}(R)$ is equal to the intersection of all ideals P of R such that $R/P \in \mathcal{S}$. This radical is called the unitary strongly prime radical of R and was introduced in [15] for rings with identity element and in [8] for any ring.

Recall that the Brown-McCoy radical $U(R)$ of R is defined as the intersection of all ideals M of R such that R/M is a simple ring with identity [4]. Since every simple ring with identity is in $\mathcal{S}$, the u-strongly prime radical is contained in the Brown-McCoy radical.

Also, the Levitzki radical $L(R)$ of R is the largest locally nilpotent ideal of R. By ([15], Theorem 3.3) the unitary strongly prime radical contains $L(R)$.

A ring R is said to have a large center if any non-zero ideal of R has non-zero intersection with $Z(R)$. Let $\mathcal{P}$ be the class of all non-zero prime rings with large center. The class $\mathcal{P}$ is also a special class of rings [20]. Let R be a ring in $\mathcal{P}$. Then R is prime and any non-zero ideal I of R contains a central element c. Thus $\{c\}$ is an insulator in I and by Proposition 2.4 R is in $\mathcal{S}$. It follows that $\mathcal{P} \subseteq \mathcal{S}$ and, in particular, for any ring T, $\mathcal{S}(T) \subseteq \mathcal{I}(T)$, where $\mathcal{I}$ is the upper radical determined by the class $\mathcal{P}$.

In [16] Krempa proved that the Brown-McCoy radical of a polynomial ring $R[x]$ in one indeterminate x is equal to $(U(R[x]) \cap R)[x]$. Similar result also holds for any set X of either commuting or non-commuting indeterminates: $U(R[X]) = (U(R[X]) \cap R)[X]$ ([12], 1.6 and [20], Corollary 13).

The ideal $U(R[x]) \cap R$ can be completely described. In fact, by Corollary 4 of [20] $U(R[x]) = \mathcal{I}(R)[x]$. But it is still not known what is exactly $U(R[X])$, when X is any set of indeterminates. In Section 5 we will give some information concerning this question.

We can give a precise description of the u-strongly prime radical of a polynomial and a free ring ([8], 3.3):

3.1 Theorem. *Let R be a ring and X any set of either commuting or non-commuting indeterminates. Then $\mathcal{S}(R[X]) = \mathcal{S}(R)[X]$.*

4 Maximal Ideals of Polynomial Rings

For any ring R and cardinal number α we denote by $R[X_\alpha]$ the polynomial ring over R in a set X_α of α commuting indeterminates.

Given a ring R, the *pseudo-radical* $ps(R)$ of R is defined as the intersection of all non-zero prime ideals of R. It was proved in ([5], Corollary 2.2) that if R is a ring with identity and there exists a maximal ideal of $R[x]$ which is R-disjoint, then $ps(R)$ is non-zero. More generally, for rings with identity it was proved in Corollary 2 of [20] that $R[x]$ contains a maximal ideal which is R-disjoint if and only if $R \in \mathcal{P}$ and $ps(R)$ is non-zero, where $\mathcal{P}$ is the class of prime rings with large center, as in Section 3. We extended this result in [8], as we will see in this section.

Fist, Corollary 2.2 of [5] has been extended to polynomial rings in several commuting indeterminates and without identity. We have the following

4.1 Proposition. ([8], 4.3) *Assume that $n \geq 1$ is a natural number and there exists a maximal ideal M of $R[X_n]$ which is R-disjoint. Then $ps(R) \neq 0$.*

Note that if there exists an ideal M of $R[X_n]$ such that $R[X_n]/M$ is a simple ring with identity and $M \cap R = 0$, then R is u-strongly prime and $ps(R) \neq 0$ (Propositions 2.4 and 4.1). This type of u-strongly prime rings are very important in the study of maximal ideals and the Brown-McCoy radical of polynomial rings.

The subclass of $\mathcal{S}$ consisting of all u-strongly prime rings R with $ps(R) \neq 0$ will be denoted by $\mathcal{S}_1$ and we put $\mathcal{S}_2 = \mathcal{S} \setminus \mathcal{S}_1$. If R is a prime ring and I is a non-zero ideal of R, then it is easy to see that $ps(R) \neq 0$ if and only if $ps(I) \neq 0$. Using this fact it can be proved that both classes $\mathcal{S}_1$ and $\mathcal{S}_2$ are special classes of rings. However, while the class $\mathcal{S}_1$ is relevant in the computation of the u-strongly prime radical and the Brown-McCoy radical of polynomial rings, the class $\mathcal{S}_2$ can be ignored. In fact, we have the following

4.2 Proposition. ([8], 4.7) *Any ring in $\mathcal{S}_2$ is a sub-direct product of rings from $\mathcal{S}_1$. In particular, the u-strongly prime radical of any ring R is equal to the intersection of all ideals P of R with $R/P \in \mathcal{S}_1$.*

The classification of u-strongly prime rings induces a partition of $\mathcal{P}$ into subclasses $\mathcal{P}_1$ and $\mathcal{P}_2$ in an obvious way, i.e., $R \in \mathcal{P}_1$ if and only if $R \in \mathcal{P} \cap \mathcal{S}_1$.

The following result is an extension of Corollary 2 of [20]. By factoring out the ideal P, it gives a complete description of ideals P of R such that there exists an ideal M of $R[x]$ with $R[x]/M$ a simple ring with an identity and $M \cap R = P$.

4.3 Theorem. ([8], 4.8) *For any ring R, the following conditions are equivalent:*
(i) There exists an R-disjoint ideal M of $R[x]$ *such that* $R[x]/M$ *is simple with identity.*
(ii) $R \in \mathcal{P}_1$.
(iii) R is prime and $ps(R) \cap Z(R) \neq 0$.
(iv) $R \in \mathcal{S}$ *and there exists* $c \in C$ *such that* $RC = R[c]$.

As a consequence of the above theorem we obtain the result corresponding to Proposition 4.2 for the upper radical defined by the class $\mathcal{P}$: any ring in $\mathcal{P}_2$ is a sub-direct product of rings from $\mathcal{P}_1$. In particular, for any ring R the ideal $\mathcal{I}(R)$ is equal to the intersection of all the ideals P of R with $R/P \in \mathcal{P}_1$.

For more than one indeterminate we obtained the following extension of the above. The result gives a characterization for rings in $\mathcal{S}_1$.

4.4 Theorem. ([8], 4.12) *For any ring R, the following conditions are equivalent:*
(i) There exist $n \geq 1$ *and an R-disjoint ideal M of* $R[X_n]$ *such that* $R[X_n]/M$ *is simple with identity.*
(ii) $R \in \mathcal{S}_1$.
(iii) R is prime and $ps(R)$ *contains an insulator.*
(iv) $R \in \mathcal{S}$ *and for some* $m \geq 1$ *there exist* $c_1, \ldots, c_m \in C$ *such that* $RC = R[c_1, \ldots, c_m]$.

As a particular case of Theorem 4.4 we have that if R is simple without identity, then $R[X_n]$ is a Brown-McCoy radical ring, for any $n \geq 1$. This gives an extension of a result which was already known for one indeterminate ([20], Corollary 3).

4.5 Remark. The main problem of Theorem 4.4 is that we cannot assure that the numbers n and m which appear in the statement are always equal and compare this with the number of elements of an insulator contained in $ps(R)$. From the proof of the theorem we can see that if $ps(R)$ contains an insulator subset with m elements, then there exists an R-disjoint ideal M of $R[X_n]$ such that $R[X_n]/M$ is simple with identity and RC can be obtained by adjoining n elements of C to R, for some $n \leq m$, but we do not know whether the converse also holds. This is an interesting question which is related to some open problems we will see in the next section.

5 Brown-McCoy radical of polynomial rings

As we said in Section 3, for any cardinal α there exists an ideal $\mathcal{I}_\alpha = \mathcal{I}_\alpha(R)$ of R such that the Brown-McCoy radical $U(R[X_\alpha])$ is equal to $\mathcal{I}_\alpha[X_\alpha]$. If we consider a single indeterminate x, then the ideal $\mathcal{I}_1$ coincides with the ideal $\mathcal{I}(R)$ defined in [20] and already mentioned before.

For $\beta \geq \alpha$ we have $\mathcal{I}_\beta \subseteq \mathcal{I}_\alpha$ since every ideal M of $R[X_\alpha]$ such that $R[X_\alpha]/M$ is a simple ring with identity can easily be extended to an ideal M' of $R[X_\beta]$ such that the factor ring $R[X_\beta]/M'$ is also a simple ring with identity. Also, since the Brown-McCoy radical of any ring contains the u-strongly prime radical, it follows from Theorem 3.1 that $\mathcal{S}(R) \subseteq \mathcal{I}_\alpha$, for any cardinal α.

Thus for any ring R and cardinal number α we have

$$I_1 \supseteq I_2 \supseteq \ldots \supseteq I_\alpha \supseteq \mathcal{S}(R),$$

We cannot give an answer to the following

Question 1. *Is there a ring R for which the above sequence is not constant?*

The following positive result was proved in ([8], 5.1 and 5.4). It gives an extension of a result which is well-known for commutative rings.

5.1 Theorem. *Assume that R is a ring and X is a set of either commuting or non-commuting indeterminates. Then we have $U(R[X]) = \mathcal{S}(R[X])(= \mathcal{S}(R)[X])$, provided that R is a PI ring or X is an infinite set.*

From the above theorem it follows that the sequence of Question 1 is constant when R is a PI ring and for any infinite cardinal α we have $I_\alpha = \mathcal{S}(R)$.

Any prime PI ring has large center and is always u-strongly prime. The question of whether a u-strongly prime ring has always large center was raised by K. Beidar (private communication). He conjectured that this not true at least for u-strongly prime rings with zero pseudo radical. But the question seems to be open until now.

A positive answer to the question on whether a u-strongly prime ring with non-zero pseudo radical has always large center would imply that the theorem above will be true for any ring, and our Question 1 will have a negative answer. Moreover, in this case we will have $\mathcal{I}_n(R) = \mathcal{S}(R)$, for any ring R and $n \geq 1$.

Actually, to prove that the last relation holds it would be enough to give a positive answer to the following

Question 2. *Is it true that if $R \in \mathcal{S}_1$, then $ps(R)$ contains a non-zero central element?*

A precise description of the Brown-McCoy radical of $R[X_n]$ will be obtained if we could compare the numbers appearing in Theorem 4.4. In fact, if we could show that there exist $n \geq 1$ and an ideal M of $R[X_n]$ which is R-disjoint and such that $R[X_n]/M$ is simple with identity if and only if R is prime and $ps(R)$ has an insulator with n elements, then we would obtain that the Brown-McCoy radical of $R[X_n]$ will be equal to $I_n[X_n]$, where I_n is equal to the intersection of all prime ideals P of R such that $ps(R/P)$ contains an insulators of cardinality n.

If R is a nil ring, then the Brown-McCoy radical of $R[x]$ is a Brown-McCoy radical ring ([20], Corollary 3, (ii)). It is not known whether a polynomial ring in two or more indeterminates over a nil ring R must be Brown-McCoy radical ([20], Question 1, (a)). On the other hand, it is also an open problem whether the upper nil radical of a ring is contained in the strongly prime radical ([15], Problem). As it has been proved in ([8], 5.5) these two questions are related:

5.2 Proposition. *The following conditions are equivalent:*

(i) For any ring R, the upper nil radical is contained in the u-strongly prime radical of R.

(ii) If R is a nil ring, then a polynomial ring over R in any finite number of commuting indeterminates is a Brown-McCoy radical ring.

Thus the following is also an open problem:

Question 3. *Are the equivalent conditions of Proposition 5.2 true?*

6 Some additional results

In the rest of the paper we assume that R has an identity element. Note that in this case u-strongly prime rings (ideals) are just strongly prime rings (ideals). The results that we will present in the following were shown to the author by A. Kaučikas. Some of them are contained in papers by him ([13], [14]) and others are not published yet.

Assume that $\phi : R \to S$ is a centred homomophism. Following [13] we say that ϕ is an integral homomorphism if for any $\{s_1, \ldots, s_n\} \subseteq S$ there exists a subring $A \subseteq S$ which is generated as an R-module by a finite set of R-centralizing elements and $\{s_1, \ldots, s_n\} \subseteq A$. In this case, when R is a subring of S and ϕ is the inclusion, we say that S is an integral extension of R.

In [13], Kaučikas proved that if $R \subseteq S$ is a centred integral extension of R, then for any prime ideal p of R there exists a prime ideal P of S lying over p, i.e, with $P \cap R = p$. It follows, in particular, that the same result holds for strongly prime ideals. This result was somehow extended later on in another paper by the same author.

First, Theorem 2.2 of [14] shows that for a ring R, an ideal of R which is maximal among ideals which do not contain an insulator is strongly prime. This result implies that if $R \subseteq S$ is a centred extension and p is an ideal of R which is maximal among ideals which do not contain an insulator, then $p^e \cap R = p$, where $p^e = SpS$ is the extension of p to S.

Using this it is possible to show the following

6.1 Theorem. *Assume that $\phi_i : R \to S_i$ are centred homomorphism. Then the tensor product of $S_1 \otimes_R \ldots \otimes_R S_n$ is non-zero if and only if there exist ideals $I \lhd R$ and $I_j \lhd S_j$, $j = 1, \ldots, n$, with $\phi_j^{-1}(I_j) = I$, for all j.*

As a corollary it follows that the tensor product of centred monomorphisms is non-zero.

Finally, we include some results that Kaučikas explained to me in a recent meeting and which are not published yet.

6.2 Definition. *The ring R is said to be a geometric Jacobson ring if for any n and maximal ideal M of $R[X_n]$, $M \cap R$ is a maximal ideal of R.*

A Brown-McCoy ring is defined as a ring R such that any prime ideal of R is an intersection of maximal ideals. It is well-known that R is a Brown-McCoy ring if and only if for any maximal ideal M of $R[x]$, $M \cap R$ is a maximal ideal of R. If R is a Brown-McCoy ring, then $R[x]$ is also a Brown-McCoy ring ([22]). It follows that if R is a Brown-McCoy ring, then R is a geometric Jacobson ring.

The following result was proved by Kaučikas. We can give a short proof based on the results in [8].

6.3 Theorem. *A ring R is a geometric Jacobson ring if and only if any strongly prime ideal of R is an intersection of maximal ideals.*

In fact, assume that R is a geometric Jacobson ring and p is a strongly prime ideal of R. Then we have two possibilities. If the pseudo radical of R/p is non-zero, then there exists a maximal ideal M of $R[X_n]$, for some n, such that $M \cap R = p$, by Theorem 4.4. Hence p is a maximal ideal of R. In the second case, p is equal to the intersection of all prime ideals p_i such that the pseudo radical of R/p_i is non-zero, by Proposition 4.2. Thus p is an intersection of maximal ideals.

Conversely, if any strongly prime ideal is an intersection of maximal ideals, then for a maximal ideal M of $R[X_n]$ the ideal $p = M \cap R$ of R is strongly prime and $ps(R/p) \neq 0$. Since p must be an intersection of maximal ideals, then p itself has to be maximal.

We finish the paper with the following result proved also by Kaučikas.

6.4 Theorem. *If R is a geometric Jacobson ring and S is a centred integral extension of R, then S is also a geometric Jacobson ring.*

Acknowledgement I would like to thank Prof. Hidetosi Marubayashi from Natuto University of Education for the arrangements and support given to me. It made possible my participation in the 35th Symposium on Ring Theory and Representation Theory, Okayama, October of 2002.

References

[1] J. A. Beachy, Some aspects of noncommutative localization. Noncommutative Ring Theory, Lectures Notes in Math. Vol. 545, Springer-Verlag, 1975, Berlin, 2-31.

[2] K. I. Beidar, W. S. Martindale III and A. V. Mikhalev, Rings with generalized identities, Marcel Dekker, New York 1996.

[3] F. Cedó, Strongly prime power series rings, Comm. Algebra 25 (7) (1997), 2237-2242.

[4] N. Divinsky, Rings and radicals, Allen and Unwin, London, 1965.

[5] M. Ferrero, Prime and maximal ideals in polynomial rings, Glasgow Math. J. 37 (1995), 351-362.

[6] M. Ferrero, Centred bimodules over prime rings: closed submodules and applications to ring extensions, J. Algebra 172 (1995), 470-505.

[7] M. Ferrero and J. Matczuk, Strongly primeness and singular ideals of skew polynomial rings, Math. J. Okayama Univ. 42 (2000), 11-17.

[8] M. Ferrero and R. Wisbauer, Unitary strongly prime rings and related radicals, J. Pure Appl. Algebra, to appear.

[9] N. Groenewald and G. Heyman, Certain classes of ideals in group rings II, Comm. Algebra 9 (1981), 137-148.

[10] D. Handelman and J. Lawrence, Strongly prime rings, Trans. Amer. Math. Soc. 211 (1975), 209-223.

[11] N. Jacobson, Basic Algebra I, W.H. Freeman, New York, 1985 (Second edition).

[12] E. Jespers and E. R. Puczyłowski, The Jacobson and Brown-McCoy radicals of rings graded by free groups, Comm. Algebra 19 (2) (1991), 551-558.

[13] A. Kaučikas, On centred and integral homomorphisms, Lietuvos Matematikos Rinkinys, 37-3 (1997), 355-358.

[14] A. Kaučikas, Insulating sets and tensor products in Procesi category, Leituvos matematiku draugijos mokslo darbai, IV tomas (2000), 1-4.

[15] A. Kaučikas and R. Wisbauer, On strongly prime rings and ideals, Comm. Algebra 28 (11) (2000), 5461-5473.

[16] J. Krempa, On radical properties of polynomial rings, Bull. Acad. Polon. Sci., Ser. Sci. Math. Astronom. Phys 29 (1972), 545-548.

[17] C. Năstăsescu and F. Van Oystaeyen, The strongly prime radical of graded rings, Bull. Soc. Math. Belgique, Ser. B, 36 (1984), 243-251.

[18] M. M. Parmenter, D. S. Passman and P. N. Stewart, The strongly prime radical of crossed products, Comm. Algebra 12 (9) (1984), 1099-1113.

[19] E. R. Puczyłowski, Behaviour of radical properties of rings under some algebraic constructions, Proc. Radical Theory, Eger (Hungary), Coll. Math. János Bolyai (1982), 449-480.

[20] E. R. Puczyłowski and A. Smoktunowicz, On maximal ideals and the Brown-McCoy radical of polynomial rings, Comm. Algebra 26 (8) (1998), 2473-2484.

[21] F. A. Szsz, Radicals of Rings, John Wiley and Sons, New York, 1881.

[22] J. F. Watters, The Brown-McCoy radical and Jacobson rings, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. 27 (1976), 91-99.

[23] R. Wisbauer, Modules and algebras: Bimodule structure and group actions on algebras, Addison Wesley Longman, Harlow, 1996.

Instituto de Matemática,
Universidade Federal do Rio Grande do Sul,
91509-900, Porto Alegre, Brazil
e-mail: ferrero@mat.ufrgs.br

GALOIS ACTION ON PLANE COMPACTS

NICOLAE POPESCU♣

1. INTRODUCTION

Let $\mathbb{Q}$ be the field of rational numbers and let $\mathbb{C}$ be the field of complex numbers. Denote by $\overline{\mathbb{Q}}$ the field of algebraic numbers, i.e. the algebraic closure of $\overline{\mathbb{Q}}$ in $\mathbb{C}$ and $G = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, the Galois group of all $\mathbb{Q}$-automorphisms of $\overline{\mathbb{Q}}$. As usual, if $x \in \overline{\mathbb{Q}}$, denote $O(x)$ the set of all conjugates of x over $\mathbb{Q}$. Then $O(x)$ is a finite set and the mapping $G \times O(x) \to O(x)$ defined by: $(\sigma, y) \rightsquigarrow \sigma(y)$ gives an action of G on the set $O(x)$. If one endow G with so-called Krull topology (see [Ar]), end $O(x)$ with discrete topology, then for any $y \in O(x)$, the mapping $G \to O(x)$ defined by $\sigma \rightsquigarrow \sigma(y)$ gives a continuous and onto mapping. In this way G acts continuous and transitive on $O(x)$, for any $x \in \overline{\mathbb{Q}}$.

A such action is natural and rised in time many interesting questions on the structure of group G. Moreover one says that the group G hide almost all Mathematics, and it study is far to be accomplished !

One can put the question if there are another subsets M of $\mathbb{C}$ such that G acts transitively and continuous on M and to describe it. Precisely, let M be a subset of $\mathbb{C}$, endowed with the induced topology. On says that there exists a *transitive Galois action* on M if there exists a mapping:

$$G \times M \to M,\ (\sigma, x) \rightsquigarrow \sigma x$$

such that

i) If e denote the neutral element of G, then $ex = x$ for all $x \in M$.

ii) $\sigma(\tau x) = (\sigma\tau)x$ for all $\sigma, \tau \in G,\ x \in M$.

iii) The action is transitive, i.e. for any $x \in M$ the set $\{\sigma(x)\}_{x \in G}$ (the orbit of x) is just M.

iv) For any $x \in M$, the mapping $G \to M$, $\sigma \rightsquigarrow \sigma(x)$ is continuous.

By iv) there result that the set M must be a compact subset of $\mathbb{C}$.

---

[1]This is an expository paper on the recent results of the author.

In what follows one try to indicate and describe a wide class of compact subsets M of $\mathbb{C}$ such that G acts transitive and continuous on M. This is possible using some results on the so-called "spectral completion of $\overline{\mathbb{Q}}$"(see [PPP], [PPZ1]- [PPZ4]).

2. THE SPECTRAL COMPLETION OF ALGEBRAIC NUMBERS

1. *Notations.* As usual denote $\mathbb{Q}$ the field of rational numbers, $\mathbb{R}$ the field of real numbers, and $\mathbb{C}$ the field of complex numbers. $\overline{\mathbb{Q}}$ will denote the field of algebraic numbers, i.e. the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Also $G = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, denote the absolute Galois group, i.e. the group of all automrphisms of $\overline{\mathbb{Q}}$ over $\mathbb{Q}$. Usually one endow G with "Krull topology" (see [Ar]). If e is the neutral element of G, then by $\bar{e}$ denote the "conjugate automorphism" of $\overline{\mathbb{Q}}$, $\bar{e}(x) = \bar{x}$, the complex conjugate. For any $x \in \mathbb{C}$, denote $|x|$ the usual "module" of the complex number x. The topology considered on $\mathbb{C}$ and induced on all its subsets is defined as usual by "module".

Endow $\overline{\mathbb{Q}}$ with induced topology. Then the only automorphism of $\overline{\mathbb{Q}}$ which is continuous is $\bar{e}$, the complex conjugation.

2. *The spectral norm on* $\overline{\mathbb{Q}}$. According to [PPP], for any $x \in \overline{\mathbb{Q}}$ let us denote:

$$||x|| = \sup\{|\sigma(x)|, \sigma \in G\},$$

the "spectral norm" of x. Then one has:

$$||x + y|| \leq ||x|| + ||y||, \; x, y \in \overline{\mathbb{Q}},$$

$$||xy|| \leq ||x|| \cdot ||y||,$$

$$||x|| = 0 \text{ if and only if } x = 0$$

In this way $(\overline{\mathbb{Q}}, ||\cdot||)$ become a $\mathbb{Q}-$normed algebra. It is easy to see that for any $x \in \overline{\mathbb{Q}}$ and any $\sigma \in G$, one has: $||x|| = ||\sigma(x)||$. This shows that any automorphism of $\overline{\mathbb{Q}}$ is continuous with respect to the spectral norm.

3. *The completion of* $(\overline{\mathbb{Q}}, ||\cdot||)$. Denote by $\tilde{\overline{\mathbb{Q}}}$, the completion of $\overline{\mathbb{Q}}$ (defined as usual) with respect to $||\cdot||$. Also denote by the same symbol "$||\cdot||$" the natural extension of spectral norm to $\tilde{\overline{\mathbb{Q}}}$.Then one define a normed ring $(\tilde{\overline{\mathbb{Q}}}, ||\cdot||)$, which is in a natural way a normed $\mathbb{R}-$algebra.

Let $x \in \tilde{\overline{\mathbb{Q}}}$. Then $x = \lim_{\|\cdot\|} x_n$, where $\{x_n\}_{n\geq 0}$ is a Cauchy sequence of algebraic numbers. By the definition of spectral norm there result that for any $\sigma \in G$, the sequence $\{\sigma(x_n)\}_{n\geq 0}$ is also Cauchy with respect to $\|\cdot\|$. Then let us denote:

$$\tilde{\sigma}(x) = \lim_{\|\cdot\|} \sigma(x_n).$$

It is not difficult to see that the mapping $x \rightsquigarrow \tilde{\sigma}(x)$ defines a continuous automorphism of $\mathbb{R}-$algebra $\tilde{\overline{\mathbb{Q}}}$. Moreover one has:

Theorem 1. *The mapping $\sigma \rightsquigarrow \overline{\sigma}$ defines an isomorphism between $G = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and the group of all continuous $\mathbb{R}-$automorphism of $\tilde{\overline{\mathbb{Q}}}$ denoted*

$$\mathrm{Gal}_{\mathrm{cont}}(\tilde{\overline{\mathbb{Q}}}/\mathbb{R}) = \tilde{G}.$$

Usually we shall identify G to $\tilde{G}$ via described isomorphism and also for $\sigma \in G$ denote $\sigma = \tilde{\sigma} \in \tilde{G}$.

4. *The orbit and pseudo-orbit of an element of* $\tilde{\overline{\mathbb{Q}}}$. Let $x \in \tilde{\overline{\mathbb{Q}}}$. Then $x = \lim_{\|\cdot\|} x_n$, where $\{x_n\}_n$ is a Cauchy sequence of $\overline{\mathbb{Q}}$. For any $\sigma \in G$, denote $\sigma(x) = \lim_{\|\cdot\|} \sigma(x_n)$. Denote

$$O(x) = \{\sigma(x), \sigma \in G\}$$

the orbit of x with respect to G. Since the sequence $\{x_n\}_n$ is Cauchy with respect to the spectral module, then result that for any $\tau \in G$, the sequence $\{\tau(x_n)\}_n$ is also Cauchy with respect to usual module $|\cdot|$ of complex numbers. Then let us denote

$$x_\tau = \lim_{|\cdot|} \tau(x_n) \in \mathbb{C}.$$

Then to any element $x \in \tilde{\overline{\mathbb{Q}}}$, one can assign the set

$$C(x) = \{x_\sigma | \sigma \in G\}$$

of complex numbers, called the *pseudo-orbit* of x.

Theorem 2. *Let $x \in \tilde{\overline{\mathbb{Q}}}$. Endow $O(x)$ with the induced topology of $\tilde{\overline{\mathbb{Q}}}$ and $C(x)$ with the topology induced by the complex numbers. Then the both maps:*

$$G \to O(x),\ \sigma \rightsquigarrow \sigma(x),$$

$$G \to C(x),\ \sigma \rightsquigarrow x_\sigma$$

are continuous. Then $O(x)$ is a compact subset of $\tilde{\overline{\mathbb{Q}}}$ and $C(x)$ a compact and symmetric (with respect Ox-axis) subset of $\mathbb{C}$. Moreover one has $\|x\| = \sup\{|x_\sigma|, \sigma \in G\}$.

The set $\{C(x) | x \in \tilde{\overline{\mathbb{Q}}}\}$ covers almost all compact subsets of $\mathbb{C}$.

Theorem 3. *(see* [PPZ1], *Theorem 1.10). For any compact and symmetric subset M of $\mathbb{C}$, there exists at least an element $x \in \tilde{\overline{\mathbb{Q}}}$ such that $M = C(x)$.*

3. Topological generic elements

1. *Topological generic elements.* For any $x \in \tilde{\overline{\mathbb{Q}}}$, denote $\widetilde{\mathbb{Q}[x]}$ the closure in $\tilde{\overline{\mathbb{Q}}}$ of the ring $\mathbb{Q}[x]$. One has $\widetilde{\mathbb{Q}[x]} = \widetilde{\mathbb{R}[x]}$. One says that a closed $\mathbb{R}-$subalgebra A of $\tilde{\overline{\mathbb{Q}}}$ has a *topological generic element* if there exists $x \in \tilde{\overline{\mathbb{Q}}}$ such that $A = \widetilde{\mathbb{Q}[x]}$.

Let $\mathbb{Q} \subseteq L \subseteq \overline{\mathbb{Q}}$ be an intermediate subfield. Denote $\tilde{L}$ the completion (closure) of L with respect to the spectral norm. If $L = \mathbb{Q}[x]$, for some $x \in \overline{\mathbb{Q}}$, the is not hard to see that $\widetilde{\mathbb{Q}[x]} = \tilde{L} = \mathbb{R}[x]$, i.e x is just a topological generic element of $\tilde{L}$. Moreover one has the general result:

Theorem 4. *(see* [PPZ1], *Theorem 2.1). For any intermediate subfield $\mathbb{Q} \subseteq L \subseteq \tilde{\overline{\mathbb{Q}}}$, the $\mathbb{R}-$subalgebra $\tilde{L}$ of $\tilde{\overline{\mathbb{Q}}}$, has a topological generic element, i.e.*

$$\tilde{L} = \widetilde{\mathbb{Q}[x]}$$

for a suitable element $x \in \tilde{\overline{\mathbb{Q}}}$. Particularly there exists $x \in \tilde{\overline{\mathbb{Q}}}$ such that $\tilde{\overline{\mathbb{Q}}} = \widetilde{\mathbb{Q}[x]}$.

(There is a forthcoming paper result that any closed subalgebra A of $\tilde{\overline{\mathbb{Q}}}$ has a topological generic element.)

2. *Condition (H).* Let $x \in \tilde{\overline{\mathbb{Q}}}$. For any two elements $\sigma, \tau \in G$, one has $\sigma(x)_\tau = x_{\tau\sigma}$. One says that x verify the *condition H* if for any three elements σ, τ, χ of G, the equality $x_{\sigma\chi} = x_{\tau\chi}$, implies $x_\sigma = x_\tau$.

Theorem 5. *(* [PPZ2], *Theorems 2.4, 2.6, 2.7). Let $x \in \tilde{\overline{\mathbb{Q}}}$. Denote $H(x) = \{\sigma \in G | \sigma(x) = x\}$. Then $O(x)$ the orbit of x can be canonically identified with $G/H(x) = \{\sigma H(x) | \sigma \in G\}$. by the map: $\sigma \rightsquigarrow \sigma(x)$.*

1) The topology of $O(x)$, induced by $\tilde{\overline{\mathbb{Q}}}$, and the topology of $G/H(x)$ (the quotient topology) are coincident.

2) The map $G/H(x) \to C(x)$, defined by $\sigma H(x) \rightsquigarrow x_\sigma$ is a homomorphism if and only if x has property (H).

3) Let L be a subfield of $\overline{\mathbb{Q}}$, and x a topological generic element of $\tilde{L}$ (see 4). Then x has property (H).

4) Let x be an element of $\tilde{\overline{\mathbb{Q}}}$ which has property (H), and L the subfield of $\overline{\mathbb{Q}}$ fixed by $H(x)$. Then $\tilde{L} = \widetilde{\mathbb{Q}[x]}$, and x is a topological generic element of $\tilde{L}$.

4. Transitive Galois action on plane compacts

Proposition 1. *Let $x \in \tilde{\overline{\mathbb{Q}}}$ be an element with the property (H). Then the map:*

$$G \times C(x) \to C(x),\ (\sigma, x_\tau) \rightsquigarrow x_{\tau\sigma}$$

is a transitive Galois action on the pseudo-orbit $C(x)$ and

$$H(x) = \{\sigma \in G | x_\sigma = x_e,\ e \text{ being the neutral element of } G\}.$$

Particularly, if L is a subfield of $\overline{\mathbb{Q}}$, and x a topological generic element of $\tilde{L}$, then the map $\sigma \cdot x_\tau = x_{\tau\sigma}$ gives a transitive Galois action on $C(x)$. The next result shows that on $C(x)$ where x is an element of $\tilde{\overline{\mathbb{Q}}}$ with property (H), the transitive Galois action above defined is unique.

Theorem 6. *Let x and y be two elements of $\tilde{\overline{\mathbb{Q}}}$ such that x has property (H) and $C(x) = C(y)$. Then there exists $\sigma \in G$ such that $y = \sigma^{-1}(x)$.*

2. *Subsets of $\mathbb{C}$ with uniform covering.*

Definition. Let M be a compact subsets of $\mathbb{C}$. One says that M has a *uniform covering* (or that M is a *Cantor compact subset*) if there exists:

1) a sequence of positive real numbers $\varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_n > \cdots$ whose limit is zero.

2) a sequence of nonnegative integers, $N_1 = 1 < N_1 < \cdots < N_n < \cdots$, with $k_n = N_{k+1}/N_k$ an integer for any $k \geq 1$, such that for any pair (ε_n, N_n), $n \geq 1$, one can find a disjoint reduced covering of M with compact subset of $\mathbb{C}$. $M_{n1}, \ldots, M_{nN_n}$ with diameter $\leq \varepsilon_n$. Moreover these coverings have the following property: for all $1 \leq i \leq N_n$ any M_{ni}, contains the same number h_n of subsets $M_{n+1,j}$, $1 \leq j \leq N_{n+1}$.

If M is a symmetric (with respect the Ox-axis) one says that it has a uniform covering if $M_+ = \{z = x + iy \in M, y \geq 0\}$ has a uniform covering $\{M_{ni}\}_{n,i}$ like above, and $M_- = \{z = x + iy \in M, y \leq 0\}$ has the uniform covering $\{\overline{e}(M_{ni})\}_{ni}$, the conjugate of the first.

The classical Cantor set, or plane compacts obtained in the same way, are compacts with uniform covering. In fact a compact with uniform covering is a projective limit of finite sets and the topology of projective limit is coincident with topology induced by $\mathbb{C}$. Moreover these compacts are totally disconnected.

3. Now one can give the main result:

Theorem 7. *([PPZ2], Theorems 3.1, 3.2, 3.3) Let M be a compact subset of $\mathbb{C}$. The following assertions are equivalents:*

a) There exists a transitive Galois action on M.

b) There exists an element $x \in \tilde{\overline{\mathbb{Q}}}$ with property (H) such that M is coincident with the pseudo-orbit of x $M = C(x)$.

c) The set M has a symmetric uniform covering.

d) There exists a subfield L of $\overline{\mathbb{Q}}$ and a topological generic element x of $\overline{L}$ such that $M = C(x)$.

Moreover if

$$(\sigma, a) \rightsquigarrow \sigma a,$$

$$(\sigma, a) \rightsquigarrow \sigma * a$$

are two transitive Galois action on M, then there exists an element $\tau \in G$ such that

$$\sigma * a = \sigma\tau \cdot a,$$

for all $\sigma \in G$ and $a \in M$.

Let $M = \{x_1, \cdots, x_n\}$ be a finite set of complex numbers with just n elements.

Can be proved (see [PPZ5], [P1]) that there exists a transitive Galois action on M if and only if there exists an algebraic number α, such that $O(\alpha) = \{\alpha_1 = \alpha, \ldots, \alpha_n\}$ contains just n elements (i.e. α has degree n over $\mathbb{Q}$), and a polynomial $f(X) \in \mathbb{R}[X]$ of degree $n-1$, such that

$$x_i = f(\alpha_i),\ 1 \leq i \leq n.$$

In this case, $x = f(\alpha)$ is a generator of $\mathbb{R}[\alpha]$, i.e. $\mathbb{R}[\alpha] = \mathbb{R}[x]$, and $O(x) = M$.

The finite sets of $\mathbb{C}$, endowed with a transitive Galois action, are closely related with finite extensions of $\mathbb{Q}$. Also, the infinite subsets of $\mathbb{C}$ endowed with transitive Galois action, correspond somewhat to the set of "conjugates" for topological generic elements of infinite (algebraic) extensions of $\mathbb{Q}$.

4. An example. Denote $\mathbb{Z}_2$ the ring of 2-adic numbers and let $\mathbb{Z}_2^*$ the group of unit elements of it. Any element of $\mathbb{Z}_2$ can be represented uniquely as an infinite sum $x = a_0+2a_1+2^2a_2+\cdots$ where a_i is an integer and $0 \leq a_i \leq 1$. Then the element x belongs to $\mathbb{Z}_2^*$ if and only if $a_0 = 1$. Hence $\mathbb{Z}_2^* = 1 + 2\mathbb{Z}_2$. Denote K the classical Cantor set and let

$$h : \mathbb{Z}_2^* \to K$$

the map which assign to $x = 1 + 2a_1 + 2^2a_2 + \cdots$ the number $h(x) = \sum_{i\geq 1} \frac{2a_i}{3^i}$. It is easy to see that h is a bijective map, and so one can define on K a group structure such that h is an isomorphism of groups. Moreover, if one endow $\mathbb{Z}_2^*$ with the natural topology induced by $\mathbb{Z}_2$, and K by the topology induced by $\mathbb{C}$, then h is also a homeomorphism.

Now let $k : G \to \mathbb{Z}_2^*$ the so-called 2-cyclotomic character (see [W]). One know that k is a surjective continuous homomorphism to $G/Ker\, h$ (endowed with quotient topology) and $\mathbb{Z}_2^*$ (endowed to natural topology). In this way one can see that the composite map hk gives a transitive Galois action on K.

5. Analytic function associated to a transitive Galois action

Assume M is a compact subset of $\mathbb{C}$ and there exists a transitive Galois action on M. This mean (see 7 and 5) that there exists a homeomorphism $f : G/H \to M$ where H is a suitable closed subgroup of G (here G/H is endowed to quotient topology and M with topology induced by $\mathbb{C}$).

Since G/H is endowed with the Haar measure (induced in a canonical way by the Haar measure on G), then also M can be equipped (via f) by a Haar measure, denoted χ. We remark that since M is totally disconnected, then the Lebesgue measure induced by $\mathbb{C}$ is necessary zero. However the above Haar measure χ is never zero, even M is finite. It is happen that for the case of classical Cantor set the Haar measure it is coincident with so-called Haussdorf measure, but generally these measure are different.

Then one can consider the function

$$F(M,z) = \exp\Big(\int_M (z-x)d\chi(x)\Big)$$

where z is a parameter. Can be show (see [PPZ3] and [PPZ4]) that $F(M,z)$ is an analytic function in $\mathbb{C}\cup\{\infty\}\setminus M$. However it can be extended with zero by continuity, in all the points of M. However $F(M,z)$ cannot be extended by analycity in no points of M. If $M = O(x)$, where $x \in \overline{\mathbb{Q}}$, and $F(z)$ is the minimal polynomial of x, then $F(O(x),z) = F(z)^{1/n}$ where $n = \deg F(z)$.

References

[Ar] Artin E., *Algebraic Numbers and Algebraic Functions*. Gordon and Breach, Science Publishers. N.Y.. London, Paris (1967).

[PPP] Pasol, V., Popescu, A. and Popescu, N., *Spectral norm on valued fields*, Math. Z. **238**(2001), 101-114.

[PPV] Popescu, E.L., Popescu, N. and Vraciu, C. *Completion of the spectral extension of p-adic valuation*, Rev. Roumanie Math. Pures Appl., Thome **46**, no 6(2001), 805-817.

[PPZ1] Popescu, A., Popescu, N. and Zaharescu A. *On the spectral norm of algebraic numbers*. (To appear in Math. Nach.)

[PPZ2] Popescu, A., Popescu, N. and Zaharescu A. *Galois structure on the Plane Compacts*. (To appear.)

[PPZ3] Popescu, A., Popescu, N. and Zaharescu A. *Transcendental divisors and their critical functions*. (To appear.)

[PPZ4] Popescu, A., Popescu, N. and Zaharescu A. *Trace series on* $\tilde{\overline{\mathbb{Q}}}_K$. (To appear.)

[PPZ5] Popescu, A., Popescu, N. and Zaharescu A. *Galois theory on* $\tilde{\overline{\mathbb{Q}}}$. (To appear.)

[P1] Popescu, E.L., *A generalization of Hensel's Lemma*, Rev. Roumanie Math. Pures Appl., Thome **38**, no 5(1993), 802-805.

[P2] Popescu, E.L., *O-basis and fundamental basis of local fields*, Rev. Roumanie Math. Pures Appl., Thome 45, no 4(2000), 671-680.

[V] Vraciu, C. *On the residual transcendental extension of a valuation on a field* K *to* $K(X,\sigma)$. (To appear.)

[W] Washington, L.C., *Introduction to Cyclotomic Fields*. Springer, 1982.

♣ Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, Bucharest 70700, ROMANIA

E-mail: nipopesc@stoilow.imar.ro

COHEN-MACAULAY DIMENSIONS OVER NON-COMMUTATIVE RINGS [1]

TOKUJI ARAYA, RYO TAKAHASHI AND YUJI YOSHINO

The Cohen-Macaulay dimension for a module over a commutative local ring has been defined by A.A.Gerko. That is a homological invariant sharing many properties with projective dimension and Gorenstein dimension. The main purpose of this paper is to extend the notion of Cohen-Macaulay dimension for modules to that for bounded complexes over non-commutative noetherian rings. We try to persue it in the most general context as possible as we can.

The key role will be played by semi-dualizing bimodules, and we shall show that a semi-dualizing bimodule yields a duality between subcategories of the derived categories.

§1 $\mathcal{A}(C)$-dimensions for modules

Throughout the present paper, we assume that R (resp. S) is a left (resp. right) noetherian ring. Let R-mod (resp. mod-S) denote the category of finitely generated left R-modules (resp. finitely generated right S-modules). When we say simply an R-module (resp. an S-module), we mean a finitely generated left R-module (resp. a finitely generated right S-module).

In this section, we shall define the notion of $\mathcal{A}(C)$-dimension of a module, and study its properties. For this purpose, we begin with defining semi-dualizing bimodules.

Definition 1.1 We call an (R, S)-bimodule C a *semi-dualizing bimodule* if the following conditions hold.

(1) The right homothety morphism $S \to \mathrm{Hom}_R(C, C)$ is a bijection.

(2) The left homothety morphism $R \to \mathrm{Hom}_S(C, C)$ is a bijection.

(3) $\mathrm{Ext}^i_R(C, C) = \mathrm{Ext}^i_S(C, C) = 0$ for all $i > 0$.

In the rest of this section, C always denotes a semi-dualizing (R, S)-bimodule.

Definition 1.2 We say that an R-module M is *C-reflexive* if the following conditions are satisfied.

(1) $\mathrm{Ext}^i_R(M, C) = 0$ for all $i > 0$.

(2) $\mathrm{Ext}^i_S(\mathrm{Hom}_R(M, C), C) = 0$ for all $i > 0$.

(3) The natural morphism $M \to \mathrm{Hom}_S(\mathrm{Hom}_R(M, C), C)$ is a bijection.

One can of course consider the same for right S-modules by symmetry.

[1]We wrote the more detailed contents of this paper in [1] and it is contributing them to Journal of algebra.

Definition 1.3 If the following conditions hold for $N \in$ mod-S, we say that N is C-reflexive.

(1) $\mathrm{Ext}^i_S(N, C) = 0$ for all $i > 0$.

(2) $\mathrm{Ext}^i_R(\mathrm{Hom}_S(N, C), C) = 0$ for all $i > 0$.

(3) The natural morphism $N \to \mathrm{Hom}_R(\mathrm{Hom}_S(N, C), C)$ is a bijection.

Example 1.4 (1) Both of the ring R and the semi-dualizing module C are C-reflexive R-modules. Similarly, S and C are C-reflexive S-modules.

(2) Let M be an R-module. If G-dimension of M is 0, then M is R-reflexive in this sense.

We remarks that C-reflexive modules have following properties.

Lemma 1.5 (1) *Let $0 \to L_1 \to L_2 \to L_3 \to 0$ be a short exact sequence either in R-mod or in mod-S. Assume that L_3 is C-reflexive. Then, L_1 is C-reflexive if and only if so is L_2.*

(2) *If L is a C-reflexive module, then so is any direct summand of L. In particular, any projective module is C-reflexive.*

(3) *The functors $\mathrm{Hom}_R(-, C)$ and $\mathrm{Hom}_S(-, C)$ yield a duality between the full subcategory of R-mod consisting of all C-reflexive R-modules and the full subcategory of mod-S consisting of all C-reflexive S-modules.*

(4) *The following conditions are equivalent for $M \in R$-mod (resp. $M \in$ mod-S) and $n \in \mathbb{Z}$.*

(i) There exists an exact sequence $0 \to X_n \to X_{n-1} \to \cdots \to X_0 \to M \to 0$ such that each X_i is a C-reflexive module.

(ii) For any projective resolution $P_\bullet$: $\cdots \to P_{m+1} \to P_m \to \cdots \to P_0 \to M \to 0$ of M and for any $m \geq n$, we have that $\mathrm{Coker}(P_{m+1} \to P_m)$ is a C-reflexive module.

(iii) For any exact sequence $\cdots \to X_{m+1} \to X_m \to \cdots \to X_0 \to M \to 0$ with each X_i being C-reflexive, and for any $m \geq n$, we have that $\mathrm{Coker}(X_{m+1} \to X_m)$ is a C-reflexive module.

Imitating the way of defining the G-dimension in [2], we make the following definition.

Definition 1.6 For $M \in R$-mod, we define the ${}_R\mathcal{A}(C)$-dimension of M by

$$ {}_R\mathcal{A}(C)\text{-dim } M = \inf \left\{ n \;\middle|\; \begin{array}{l} \text{there exists an exact sequence of finite length} \\ 0 \to X_n \to X_{n-1} \to \cdots \to X_0 \to M \to 0, \\ \text{where each } X_i \text{ is a } C\text{-reflexive } R\text{-module.} \end{array} \right\} $$

Here we should note that we adopt the ordinary convention that $\inf \emptyset = +\infty$.

Theorem 1.7 *If ${}_R\mathcal{A}(C)$-dim $M < \infty$ for a module $M \in R$-mod, then*

$$ {}_R\mathcal{A}(C)\text{-dim } M = \sup\{\, n \mid \mathrm{Ext}^n_R(M,C) \neq 0 \,\}. $$

First of all we should notice that in the case $R = S = C$, the ${}_R\mathcal{A}(R)$-dimension is the same as the G-dimension.

Furthermore, we are able to see that the ${}_R\mathcal{A}(C)$-dimension extends the Cohen-Macaulay dimension over a commutative ring R. More precisely, suppose that R and S are commutative local rings. Note that if there is a semi-dualizing (R,S)-bimodule, then R must be isomorphic to S. Thus semi-dualizing bimodules are nothing but semi-dualizing R-modules in this case, and the definition of the Cohen-Macaulay dimension of an R-module M is

$$ \text{CM-dim}M = \inf\left\{ {}_{R'}\mathcal{A}(C')\text{-dim } M \;\middle|\; \begin{array}{l} R' \text{ is faithfully flat over } R. \\ C' \text{ is a semi-dualizing } R'\text{-module.} \end{array} \right\}. $$

§2 $\mathcal{A}(C)$-dimensions for complexes

Again in this section, we assume that R (resp. S) is a left (resp. right) noetherian ring. We denote by $\mathfrak{D}^b(R\text{-mod})$ (resp. $\mathfrak{D}^b(\text{mod-}S)$) the derived category of R-mod (resp. mod-S) consisting of complexes with bounded finite homologies.

For a complex $M^\bullet$ we always write it as

$$ \cdots \to M^{n-1} \xrightarrow{\partial^n_M} M^n \xrightarrow{\partial^{n+1}_M} M^{n+1} \xrightarrow{\partial^{n+2}_M} M^{n+2} \to \cdots, $$

and the shifted complex $M^\bullet[m]$ is the complex with $M^\bullet[m]^n = M^{m+n}$.

According to Foxby [5], we define the *supremum*, the *infimum* and the *amplitude* of a complex $M^\bullet$ as follows;

$$ \begin{cases} s(M^\bullet) = \sup\{\, n \mid \mathrm{H}^n(M^\bullet) \neq 0 \,\}, \\ i(M^\bullet) = \inf\{\, n \mid \mathrm{H}^n(M^\bullet) \neq 0 \,\}, \\ a(M^\bullet) = s(M^\bullet) - i(M^\bullet). \end{cases} \tag{2.1} $$

Note that $M^\bullet \cong 0$ iff $s(M^\bullet) = -\infty$ iff $i(M^\bullet) = +\infty$ iff $a(M^\bullet) = -\infty$.

Suppose in the following that $M^\bullet \not\cong 0$. A complex $M^\bullet$ is called bounded if $s(M^\bullet) < \infty$ and $i(M^\bullet) > -\infty$ (hence $a(M^\bullet) < \infty$). And $\mathfrak{D}^b(R\text{-mod})$ is, by definition, consisting of bounded complexes with finitely generated homologies. Thus, whenever $M^\bullet \in \mathfrak{D}^b(R\text{-mod})$, we have

$$ -\infty < i(M^\bullet) \leq s(M^\bullet) < +\infty. $$

and $a(M^\bullet)$ is a non-negative integer.

We remark that the category R-mod can be identified with the full subcategory of $\mathfrak{D}^b(R\text{-mod})$ consisting of all the complexes $M^\bullet \in \mathfrak{D}^b(R\text{-mod})$ with $s(M^\bullet) = i(M^\bullet) = a(M^\bullet) = 0$ or otherwise $M^\bullet \cong 0$. Through this identification we always think of R-mod as the full subcategory of $\mathfrak{D}^b(R\text{-mod})$.

Now we fix a semi-dualizing (R,S)-bimodule C. Associated to it, we can consider the following subcategory of $\mathfrak{D}^b(R\text{-mod})$.

Definition 2.1 For a semi-dualizing (R, S)-bimodule C, we denote by ${}_R\mathcal{A}(C)$ the full subcategory of $\mathfrak{D}^b(R\text{-mod})$ consisting of all complexes $M^\bullet$ that satisfy the following two conditions.

(1) $\mathbf{R}\mathrm{Hom}_R(M^\bullet, C) \in \mathfrak{D}^b(\text{mod-}S)$.

(2) The natural morphism $M^\bullet \to \mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_R(M^\bullet, C), C)$ is an isomorphism in $\mathfrak{D}^b(R\text{-mod})$.

If R is a left and right noetherian ring and if $R = S = C$, then we should note from the papers of Avramov-Foxby [3, (4.1.7)] and Yassemi [7, (2.7)] that ${}_R\mathcal{A}(R) = \{ M^\bullet \in \mathfrak{D}^b(R\text{-mod}) \mid \text{G-dim } M^\bullet < \infty\}$.

First of all we should notice the following fact.

Lemma 2.2 *Let C be a semi-dualizing (R, S)-bimodule. Then the subcategory ${}_R\mathcal{A}(C)$ of $\mathfrak{D}^b(R\text{-mod})$ is a triangulated subcategory which contains R, and is closed under direct summands. In particular, ${}_R\mathcal{A}(C)$ contains all projective R-modules.*

The following lemma says that R-modules in ${}_R\mathcal{A}(C)$ form the subcategory of modules of finite ${}_R\mathcal{A}(C)$-dimension.

Lemma 2.3 *Let M be an R-module. Then the following two conditions are equivalent.*

(1) ${}_R\mathcal{A}(C)\text{-dim } M < \infty$,

(2) $M \in {}_R\mathcal{A}(C)$.

Recall from Theorem 1.7 that if an R-module M has finite ${}_R\mathcal{A}(C)$-dimension, then we have ${}_R\mathcal{A}(C)\text{-dim } M = s(\mathbf{R}\mathrm{Hom}_R(M, C))$. Therefore it will be reasonable to make the following definition.

Definition 2.4 Let C be a semi-dualizing (R, S)-bimodule and let $M^\bullet$ be a complex in $\mathfrak{D}^b(R\text{-mod})$. We define the ${}_R\mathcal{A}(C)$-dimension of $M^\bullet$ to be

$$\begin{cases} {}_R\mathcal{A}(C)\text{-dim } M^\bullet = s(\mathbf{R}\mathrm{Hom}_R(M^\bullet, C)) & \text{if } \ M^\bullet \in {}_R\mathcal{A}(C), \\ {}_R\mathcal{A}(C)\text{-dim } M^\bullet = +\infty & \text{if } \ M^\bullet \notin {}_R\mathcal{A}(C). \end{cases}$$

Note that this definition is comptatible with that of ${}_R\mathcal{A}(C)$-dimension for R-modules in §1.

Also in the category $\mathfrak{D}^b(\text{mod-}S)$, we can construct the notion similar to that in $\mathfrak{D}^b(R\text{-mod})$.

Definition 2.5 Let C be a semi-dualizing (R, S)-bimodule. We denote by $\mathcal{A}_S(C)$ the full subcategory of $\mathfrak{D}^b(\text{mod-}S)$ consisting of all complexes $N^\bullet$ that satisfy the following two conditions.

(1) $\mathbf{R}\mathrm{Hom}_S(N^\bullet, C) \in \mathfrak{D}^b(R\text{-mod})$.

(2) The natural morphism $N^\bullet \to \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_S(N^\bullet, C), C)$ is an isomorphism in $\mathfrak{D}^b(\text{mod-}S)$.

Definition 2.6 Let C be a semi-dualizing (R,S)-bimodule and let $N^\bullet$ be a complex in $\mathfrak{D}^b(\text{mod-}S)$. We define the $\mathcal{A}_S(C)$-dimension of $N^\bullet$ to be

$$\begin{cases} \mathcal{A}_S(C)\text{-dim } N^\bullet = s(\mathrm{RHom}_S(N^\bullet, C)) & \text{if } N^\bullet \in \mathcal{A}_S(C), \\ \mathcal{A}_S(C)\text{-dim } N^\bullet = +\infty & \text{if } N^\bullet \notin \mathcal{A}_S(C). \end{cases}$$

Note that all the properties concerning ${}_R\mathcal{A}(C)$ and ${}_R\mathcal{A}(C)$-dimension hold true for $\mathcal{A}_S(C)$ and $\mathcal{A}_S(C)$-dimension by symmetry.

Theorem 2.7 *Let C be a semi-dualizing (R,S)-bimodule as above. Then the functors* $\mathrm{RHom}_R(-,C)$ *and* $\mathrm{RHom}_S(-,C)$ *yield a duality between the categories ${}_R\mathcal{A}(C)$ and $\mathcal{A}_S(C)$.*

References

[1] T.Araya, R.Takahashi and Y.Yoshino, *Homological invariants associated to semi-dualizing complexes*, preprint

[2] M.Auslander and M.Bridger, *Stable module theory*, Mem. Amer. Math. Soc. 94 (1969).

[3] L.Avramov and H.B.Foxby, *Ring homomorphisms and finite Gorenstein dimension*, Proc. London Math. Soc. (3) **75** (1997), no. 2, 241–270.

[4] L.W.Christensen, *Semi-dualizing complexes and their Auslander categories*, Trans. Amer. Math. Soc. **353** (2001), 1839–1883.

[5] H.B.Foxby, *Bounded complexes of flat modules*, J. Pure Appl. Algebra **15** (1979), no. 2, 149–172.

[6] A.A.Gerko, *On homological dimensions*, Mat. Sb. (N.S.) **192** (2001), no. 8, 79–94 [Russian]; [English translation: Sb. Math. **192** (2001), no. 7-8, to appear].

[7] S.Yassemi, *G-dimension*, Math. Scand. **77** (1995), no. 2, 161–174.

TOKUJI ARAYA
GRADUATE SCHOOL OF NATURAL SCIENCE AND TECHNOLOGY,
OKAYAMA UNIVERSITY, OKAYAMA 700-8530, JAPAN
E-mail address : araya@math.okayama-u.ac.jp

RYO TAKAHASHI
GRADUATE SCHOOL OF NATURAL SCIENCE AND TECHNOLOGY,
OKAYAMA UNIVERSITY, OKAYAMA 700-8530, JAPAN
E-mail address : takahasi@math.okayama-u.ac.jp

YUJI YOSHINO
FACULTY OF SCIENCE, OKAYAMA UNIVERSITY,
OKAYAMA 700-8530, JAPAN
E-mail address : yoshino@math.okayama-u.ac.jp

LOOKING AT HOMOLOGICAL DIMENSIONS THROUGH FROBENIUS MAP

RYO TAKAHASHI AND YUJI YOSHINO

1. Introduction

Throughout this note, we assume that all rings are commutative and noetherian. Projective dimension and Gorenstein dimension (abbr. G-dimension) have played important roles in the classification of modules and rings. Recently, complete intersection dimension (abbr. CI-dimension) and Cohen-Macaulay dimension (abbr. CM-dimension) have been introduced by Avramov-Gasharov-Peeva [2] and Gerko [6], respectively. These dimensions are called *homological dimensions*, and share many properties with each other. Among them, the following properties are especially important.
(A) They satisfy the Auslander-Buchsbaum-type equalities.
(B) All of the finitely generated modules over a regular (resp. complete intersection, Gorenstein, Cohen-Macaulay) local ring are of finite projective (resp. CI-, G-, CM-) dimension, and a local ring is a regular (resp. complete intersection, Gorenstein, Cohen-Macaulay) ring if the projective (resp. CI-, G-, CM-) dimension of its residue class field is finite.
(C) A finitely generated module of finite projective (resp. CI-, G-) dimension has finite CI- (resp. G-, CM-) dimension.

Let R be a local ring of prime characteristic p, and let $f : R \to R$ be the Frobenius map on R, that is, $f(a) = a^p$ for $a \in R$. For an integer e, we denote by $f^e : R \to R$ the e-th power of f, hence $f^e(a) = a^{p^e}$ for $a \in R$. The R-algebra ${}^eR$ is nothing but R as a ring and its R-algebra structure is given through f^e. The ring R is said to be *F-finite* if ${}^1R$ is a finitely generated R-module.

In the rest of this note, we assume that R is always an F-finite local ring of prime characteristic p with unique maximal ideal $\mathfrak{m}$ and residue class field $k = R/\mathfrak{m}$.

Kunz [8] has proved that R is regular if and only if ${}^eR$ is R-flat for some $e > 0$. Since we assume that R is F-finite, this result can be described in terms of projective dimension.

Kunz' Theorem. *The following conditions are equivalent.*

1. (1) *R is a regular ring.*
2. (2) $\mathrm{pd}_R{}^eR < \infty$ *for every* $e > 0$.
3. (3) $\mathrm{pd}_R{}^eR < \infty$ *for some* $e > 0$.

We can prove similar theorems for other homological dimensions. Let $\nu(R)$ denote the minimum integer n satisfying $\mathrm{H}^0_{\mathfrak{m}}(R/\boldsymbol{x}R) \cap \mathfrak{m}^n(R/\boldsymbol{x}R) = 0$ for some maximal R-regular sequence $\boldsymbol{x}$. The following theorems hold.

Theorem 1.1. *Suppose that k is a perfect field. Then the following conditions are equivalent.*

This is not in a final form. This note is a summary of the paper [10].

(1) *R is a Cohen-Macaulay ring.*
(2) CM-$\dim_R {}^eR < \infty$ *for every* $e > 0$.
(3) CM-$\dim_R {}^eR < \infty$ *for some* $e > 0$ *with* $p^e \geq \nu(R)$.

Theorem 1.2. *The following conditions are equivalent.*

(1) *R is a Gorenstein ring.*
(2) G-$\dim_R {}^eR < \infty$ *for every* $e > 0$.
(3) G-$\dim_R {}^eR < \infty$ *for some* $e > 0$.

Theorem 1.3. *The following conditions are equivalent.*

(1) *R is a complete intersection.*
(2) CI-$\dim_R {}^eR < \infty$ *for every* $e > 0$.
(3) CI-$\dim_R {}^eR < \infty$ *for some* $e > 0$ *with* $p^e \geq \nu(R)$.

2. Proofs of Theorems

In this section, we shall give the proofs of Theorem 1.1, 1.2, and 1.3.

Herzog [7, Satz 5.2] has proved that a finitely generated R-module M has finite injective dimension if (and only if) $\mathrm{Ext}^i_R({}^eR, M) = 0$ for any $i > 0$ and infinitely many $e > 0$. By utilizing the method of his proof, we can state his result in a slightly more general setting as follows.

Lemma 2.1 (Herzog). *Let e be an integer with* $p^e \geq \nu(R)$, *and let M be a finitely generated R-module. Suppose that* $\mathrm{Ext}^i_R({}^eR, M) = 0$ *for any* $i \gg 0$. *Then M has finite injective dimension.*

Lemma 2.2. *Suppose that R is complete and contains a field K. Then for any perfect field L that is an extension of K, there is an isomorphism*

$${}^e(R\widehat{\otimes}_K L) \cong {}^eR\widehat{\otimes}_K L$$

of $(R\widehat{\otimes}_K L)$*-algebras.*

Proof. First of all, note that $R\widehat{\otimes}_K L \cong \varprojlim_n (R/\mathfrak{m}^n \otimes_K L)$ by definition. Replacing R by $R/\mathfrak{m}^n$, we may assume that R is artinian, and it will suffice to prove that ${}^e(R \otimes_K L) \cong {}^eR \otimes_K L$. We define a map $\phi : R \times L \to {}^eR \otimes_K L$ by $\phi(x, z) = x \otimes z^{p^{-e}}$, which is well-defined because L is a perfect field . Since this is K-bilinear, it induces a K-linear map $\Phi : R \otimes_K L \to {}^eR \otimes_K L$ such that $\Phi(x \otimes z) = \phi(x, z)$. Now define a mapping $\alpha : {}^e(R \otimes_K L) \to {}^eR \otimes_K L$ by $\alpha(x \otimes z) = \Phi(x \otimes z) = x \otimes z^{p^{-e}}$, and we can show that the map is an $(R \otimes_K L)$-algebra homomorphism. In a similar way, we can define the inverse map $\beta : {}^eR \otimes_K L \to {}^e(R \otimes_K L)$ where $\beta(x \otimes z) = x \otimes z^{p^e}$. □

The following proposition is a key to prove Theorem 1.1.

Proposition 2.3. *Let* ϕ *be a faithfully flat homomorphism from R to a local ring* $(S, \mathfrak{n}, l)$ *with artinian closed fiber. Suppose that there is a non-zero finitely generated S-module C and an integer e with* $p^e \geq \nu(R)$ *such that* $\mathrm{Ext}^i_S({}^eR \otimes_R S, C) = 0$ *for* $i \gg 0$. *Then R is Cohen-Macaualy.*

Proof. Sice $\nu(\widehat{R}) \leq \nu(R)$, replacing ϕ by $\widehat{\phi} : \widehat{R} \to \widehat{S}$, we may assume that both R and S are complete. Thus R and S admit the coefficient fields K and L, respectively. Since K is perfect, we can choose L such that $\phi(K) \subseteq L$. Let us denote by $\overline{L}$ (resp. $\overline{l}$) the algebraic closure of the field L (resp. l), and set $R' = R\widehat{\otimes}_K\overline{L}$ and $S' = S\widehat{\otimes}_L\overline{L}$. We easily see that $(R', \mathfrak{m}R', \bar{l})$ and $(S', \mathfrak{n}S', \bar{l})$ are (noetherian) complete local rings, and are faithfully flat over R and S, respectively. Note also that $\nu(R') \leq \nu(R)$.

We claim that S' is faithfully flat over R'. For this, let $F_\bullet$ be an R-free resolution of k. Then $F_\bullet \otimes_R R'$ is an R'-free resolution of $k \otimes_R R' \cong \bar{l}$. Hence we have

$$\mathrm{Tor}_1^{R'}(\bar{l}, S') \cong \mathrm{H}_1((F_\bullet \otimes_R R') \otimes_{R'} S') \cong \mathrm{H}_1(F_\bullet \otimes_R S') \cong \mathrm{Tor}_1^R(k, S') = 0,$$

as the composite $R \to S \to S'$ is a flat homomorphism. Now applying the local criterion of flatness, we see that S' is faithfully flat over R'.

On the other hand, Lemma 2.2 implies that ${}^eR' \cong {}^eR\widehat{\otimes}_K\overline{L} \cong {}^eR \otimes_R R'$. Hence we obtain ${}^eR' \otimes_{R'} S' \cong ({}^eR \otimes_R S) \otimes_S S'$, and consequently,

$$\mathrm{RHom}_{S'}({}^eR' \otimes_{R'} S',\ C \otimes_S S') \cong \mathrm{RHom}_S({}^eR \otimes_R S,\ C) \otimes_S S'.$$

Thus, replace ϕ and C by $\phi' : R' \to S'$ and $C \otimes_S S'$ respectively, and we may assume that R and S have the common residue field.

Then, since $S/\mathfrak{m}S$ is artinian, we see that $S/\mathfrak{m}S$ is a finitely generated R-module, and so is S as S is separated in $\mathfrak{m}$-adic topology (cf. [9, Theorem 8.4]). Therefore the S-module C is also finitely generated over R. Since $\mathrm{Ext}_R^i({}^eR, C) = 0$ for $i \gg 0$ by the assumption, Lemma 2.1 yields that C has finite injective dimension over R. As a result, R is Cohen-Macaulay (cf. [3, Remark 9.6.4]). □

Now we can prove Theorem 1.1.

Proof of Theorem 1.1.
(1) ⇒ (2): This follows from [6, Theorem 3.9].
(2) ⇒ (3): This is trivial.
(3) ⇒ (1): By the Auslander-Buchsbaum-type formula [6, Theorem 3.8], we have $\text{CM-dim}_R{}^eR = \operatorname{depth} R - \operatorname{depth}_R{}^eR = 0$. Hence there exists a local flat extension $\phi : R \to S$ together with a semi-dualizing S-module C such that ${}^eR \otimes_R S$ is C-reflexive. (For the details of semi-dualizing modules, see [4].) Taking a minimal prime ideal $\mathfrak{p}$ of $\mathfrak{m}S$, and replacing S and C by $S_\mathfrak{p}$ and $C_\mathfrak{p}$ respectively, we may assume that the closed fiber of ϕ is artinian. Since we have from the definition that $\mathrm{Ext}_S^i({}^eR \otimes_R S, C) = 0$ for any $i > 0$, we can apply Proposition 2.3 to get that R is Cohen-Macaulay. □

Of course there are several missing cases in Theorem 1.1 which we cannot prove at this moment. Firstly, we do not know if the theorem is still true or not without the assumption that the residue field is perfect. Secondly, we hope but cannot prove that the condition that $\text{CM-dim}_R{}^1R < \infty$ already implies that R is Cohen-Macaulay.

Next, we will prove Theorem 1.2. For this, we need the following lemma.

Lemma 2.4. *If* $\text{G-dim}_R{}^eR < \infty$ *for some integer* e, *then* $\text{G-dim}_R{}^{2e}R < \infty$.

Proof. We have $\text{G-dim}_R{}^eR = \operatorname{depth} R - \operatorname{depth}_R{}^eR = 0$, and hence $\mathrm{Hom}_R({}^eR, R) \cong \mathrm{RHom}_R({}^eR, R)$. Now denote by C the module $\mathrm{Hom}_R({}^eR, R)$, and we see from [11, Theorem 2.7] that

$$\begin{aligned} \mathrm{RHom}_{{}^eR}(C, C) &\cong \mathrm{RHom}_{{}^eR}(\mathrm{RHom}_R({}^eR, R), \mathrm{RHom}_R({}^eR, R)) \\ &\cong \mathrm{RHom}_R(\mathrm{RHom}_R({}^eR, R), R) \\ &\cong {}^eR. \end{aligned}$$

Therefore C is a semi-dualizing ${}^eR$-module.

We would like to show that C is isomorphic to ${}^eR$ as an ${}^eR$-module. For an R-module M, denote by $\mu_R(M)$ the minimum number of generators of M and by $r_R(M)$ the type of M, i.e. $\mu_R(M) = \dim_k(M \otimes_R k)$ and $r_R(M) = \dim_k \mathrm{Ext}^t_R(k, M)$ with $t = \mathrm{depth}_R M$. To show that $C \cong {}^eR$, let ${}^ek$ denote the residue field of ${}^eR$, and put $t = \mathrm{depth}\, R$. Since $\mathbf{R}\mathrm{Hom}_{{}^eR}({}^ek, C) \cong \mathbf{R}\mathrm{Hom}_{{}^eR}({}^ek, \mathbf{R}\mathrm{Hom}_R({}^eR, R)) \cong \mathbf{R}\mathrm{Hom}_R({}^ek, R)$, we have

$$\mathrm{Ext}^t_{{}^eR}({}^ek, C) \cong \mathrm{Ext}^t_R({}^ek, R).$$

Note that $\mathrm{depth}_{{}^eR} C = \mathrm{depth}\, {}^eR = t$. Hence comparing the k-dimension of the both sides of the above isomorphism, we have $r_{{}^eR}(C) \cdot \dim_k {}^ek = \dim_k \mathrm{Ext}^t_{{}^eR}({}^ek, C) = \dim_k \mathrm{Ext}^t_R({}^ek, R) = r_R(R) \cdot \dim_k {}^ek$. Therefore we obtain $r_{{}^eR}(C) = r_R(R) = r_{{}^eR}({}^eR)$. On the other hand, since C is a semi-dualizing ${}^eR$-module, it is easy to see that $\mu_{{}^eR}(C) \cdot r_{{}^eR}(C) = r_{{}^eR}({}^eR)$. It follows from this that $\mu_{{}^eR}(C) = 1$, that is, C is a cyclic ${}^eR$-module. But since every semi-dualizing module is faithful, we have $C \cong {}^eR$ as desired.

Since we have an isomorphism $\mathbf{R}\mathrm{Hom}_R({}^eR, R) \cong {}^eR$, we should note that there is an ismorphism

$$\mathbf{R}\mathrm{Hom}_{{}^eR}(X, {}^eR) \cong \mathbf{R}\mathrm{Hom}_R(X, R)$$

for any bounded complex X of finitely generated ${}^eR$-modules. In fact, $\mathbf{R}\mathrm{Hom}_{{}^eR}(X, {}^eR) \cong \mathbf{R}\mathrm{Hom}_{{}^eR}(X, \mathbf{R}\mathrm{Hom}_R({}^eR, R)) \cong \mathbf{R}\mathrm{Hom}_R(X, R)$. Thus we have an isomorphism

$$\mathbf{R}\mathrm{Hom}_{{}^eR}(\mathbf{R}\mathrm{Hom}_{{}^eR}({}^{2e}R, {}^eR), {}^eR) \cong \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R({}^{2e}R, R), R).$$

Noting that $\text{G-dim}_{{}^eR} {}^{2e}R = \text{G-dim}_{{}^eR} {}^e({}^eR) = \text{G-dim}_R {}^eR < \infty$, we see that the left hand side is isomorphic to ${}^{2e}R$. It follows from this that $\text{G-dim}_R {}^{2e}R < \infty$, and the proof is completed. □

Finally, we shall prove Theorem 1.3.

Proof of Theorem 1.3.
(1) ⇒ (2): This follows from [2, (1.3)].
(2) ⇒ (3): This is obvious.
(3) ⇒ (1): Since $\text{G-dim}_R {}^eR \le \text{CI-dim}_R {}^eR < \infty$, it follows from Theorem 1.2 that R is Gorenstein, in particular, it is a Cohen-Macaulay ring. Thus, from the definition of $\nu(R)$, we see that there is an R-sequence $\boldsymbol{x} = x_1, x_2, \cdots, x_d$ where $d = \dim R$ such that $\mathfrak{m}^{[p^e]}(R/\boldsymbol{x}R) = 0$.

In general, if $x \in R$ is a non-zero divisor on R, then there is an exact sequence of ${}^eR$-modules $0 \to {}^eR \to {}^eR \to {}^e(R/xR) \to 0$. Regarding this as an exact sequence of R-modules, we can show that $\text{CI-dim}_R {}^e(R/xR) < \infty$. Then it follows from [2, (1.12)] that $\text{CI-dim}_{R/xR} {}^e(R/xR) < \infty$.

By a successive use of this, we see that $\text{CI-dim}_{R/\boldsymbol{x}R} {}^e(R/\boldsymbol{x}R) < \infty$. Therefore, replacing R by $R/\boldsymbol{x}R$, we may assume that R is artinian and $\mathfrak{m}^{[p^e]} = 0$.

Note that the elements in the maximal ideal $\mathfrak{m}$ act trivially on ${}^eR$, hence the R-module ${}^eR$ is actually an $R/\mathfrak{m}$-module of finite CI-dimension over R. Therefore we have that $\text{CI-dim}_R R/\mathfrak{m} < \infty$. Then it follows from [2, (1.3)] that R is a complete intersection, and the proof is finished. □

Comparing this theorem with Theorem 1.2, we have an enough reason to make a conjecture that the condition $\text{CI-dim}_R {}^1R < \infty$ for an F-finite local ring R would imply the complete intersection property of R.

References

[1] M.Auslander and M.Bridger, *Stable module theory*, Memoirs of the American Mathematical Society, **94**, 1969.

[2] L.Avramov, V.Gasharov and I.Peeva, Complete Intersection dimension, *Inst. Hautes Études Sci. Publ. Math.* **86** (1997), 67-114.

[3] W.Bruns and J.Herzog, *Cohen-Macaulay rings, revised version*, Cambridge University Press, 1998.

[4] L.W.Christensen, Semi-dualizing complexes and their Auslander categories, *Trans. Amer. Math. Soc.* **353** (2001), no. 5, 1839-1883.

[5] H.-B.Foxby, Gorenstein modules and related modules, *Math. Scand.* **31** (1972), 267-284 (1973).

[6] A.A.Gerko, On homological dimensions, *Mat. Sb.* **192** (2001), no. 8, 79-94 [in Russian]; Translation in *Sb. Math.* **192** (2001), no. 7-8, 1165-1179.

[7] J.Herzog, Ringe der Charakteristik p und Frobeniusfunktoren, *Math. Z.* **140** (1974), 67-78.

[8] E.Kunz, Characterizations of regular local rings of characteristic p, *Amer. J. Math.* **91** (1969), 772-784.

[9] H.Matsumura, *Commutative ring theory*, Cambridge University Press, 1986.

[10] R.Takahashi and Y.Yoshino, Characterizing Cohen-Macaulay local rings by Frobenius maps, preprint.

[11] S.Yassemi, G-dimension, *Math. Scand.* **77** (1995), no. 2, 161-174.

Graduate School of Natural Science and Technology, Okayama University, Okayama 700-8530, Japan

E-mail address: takahasi@math.okayama-u.ac.jp

Faculty of Science, Okayama University, Okayama 700-8530, Japan

E-mail address: yoshino@math.okayama-u.ac.jp

Modular adjacency algebras of the Hamming association schemes [1]

MASAYOSHI YOSHIKAWA

Abstract

The adjacency algebra of an association scheme is defined over an arbitrary field. This is always semisimple over a field of characteristic 0, but not semisimple over a field of prime characteristic p, in general. The structure of the adjacency algebra over a field of prime characteristic was not studied enough before now. Therefore, we considered the structure of the modular adjacency algebra of the Hamming scheme $H(n,q)$, that is the one of the most basic and important association schemes.

We will decide the structure of the adjacency algebra of $H(n,q)$ over any field for any n and q, and describe the algebra as a factor algebra of a polynomial ring.

§1Hamming schemes For the definitions, refer to [2].
The Hamming scheme $H(n,q)$ is P-polynomial scheme, and

$$B_1 = \begin{Bmatrix} * & 1 & \cdots & i & \cdots & n \\ 0 & q-2 & \cdots & i(q-2) & \cdots & n(q-2) \\ n(q-1) & (n-1)(q-1) & \cdots & (n-i)(q-1) & \cdots & * \end{Bmatrix}.$$

and the intersection number is

$$p_{ijk} = \sum_{s=0}^{n-k} \binom{k}{k-i+s} \binom{i-s}{k-j+s} \binom{n-k}{s} (q-1)^s (q-2)^{i+j-k-2s}.$$

Since the intersection numbers are the structure constants of the adjacency algebra, if we consider over a field of characteristic p, we may consider the intersection numbers in modulo p. For any prime p such that $p \nmid q$, the adjacency algebra of $H(n,q)$ over a field of characteristic p is semisimple (see [2, Theorem 2.3], [1, Theorem 1.1] and [5, Theorem 4.2]). For each prime p, the prime field $\mathbb{F}_p$ of characteristic p is a splitting field for the adjacency algebra of $H(n,p)$ over $\mathbb{F}_p$ (see [4, Theorem 3.4, Corollary 3.5]). For all prime

[1] I will send to Journal of Algebraic Combinatorics

p such that $p \mid q$, $\mathbb{F}_p H(n,p) \cong \mathbb{F}_p H(n,q)$ because $p_{ijk}^{(n,p)} \equiv p_{ijk}^{(n,q)} \pmod p$. Therefore it is enough to decide the structure of $\mathbb{F}_p H(n,p)$ for all prime p, for deciding the structure of the modular adjacency algebra of any $H(n,q)$ over any field. Thus we fix a prime p and set $H(n) := H(n,p)$.

§2 $H(p^r - 1)$

Since the size of the adjacency matrix of $H(n)$ is p^n, the adjacency algebra of $H(n)$ over a field of characteristic p is local and the unique irreducible representation is $A_i \mapsto p_{i\,i\,0}$ (see [4, Theorem 3.4, Corollary 3.5]). So the prime field $\mathbb{F}_p$ of characteristic p is a splitting field for the adjacency algebra of $H(n)$ over $\mathbb{F}_p$.

Since we consider the adjacency algebras only over $\mathbb{F}_p$, we set $\mathfrak{A}_n := \mathbb{F}_p H(n)$.

By the definition,

$$B_1^{(p^r-1)} = \begin{pmatrix} B_1^{(p-1)} & & & \\ & B_1^{(p-1)} & & \\ & & \ddots & \\ & & & B_1^{(p-1)} \end{pmatrix},$$

therefore if we set $A_i^{(p-1)} = v_i(A_1^{(p-1)})$, it follows that for $0 \leq \alpha \leq p-1$,

$$A_{pi+\alpha}^{(p^r-1)} = v_\alpha(A_1^{(p^r-1)})A_{pi}^{(p^r-1)}.$$

Then since any $c_i^{(p-1)} \not\equiv 0 \pmod p$, we can define v_α over $\mathbb{F}_p$ for $0 \leq \alpha \leq p-1$. For calculating $B_{pi+\alpha}^{(p^r-1)}$, we prepare the following theorem and corollary.

Theorem 2.1. (**Lucas' theorem** [3, Theorem 3.4.1]) *Let p be prime, and let*

$$m = a_0 + a_1 p + \cdots + a_k p^k,$$
$$n = b_0 + b_1 p + \cdots + b_k p^k,$$

where $0 \leq a_i, b_i < p$ for $i = 0, 1, \ldots, k-1$. Then

$$\binom{m}{n} \equiv \prod_{i=0}^{k} \binom{a_i}{b_i} \pmod{\mathrm{p}}.$$

Corollary 2.2. *We assume the same condition for theorem* **2.1** *and $0 \leq \alpha, \beta < p$. Then*

$$\binom{pm+\alpha}{pn+\beta} \equiv \binom{m}{n}\binom{\alpha}{\beta} \pmod{\mathrm{p}}.$$

Now we want to know $B_{pi+\alpha}^{(p^r-1)}$ that is the coefficients of $A_{pi+\alpha}^{(p^r-1)}A_{pj+\beta}^{(p^r-1)}$. But it is enough that we investigate $A_{pi}^{(p^r-1)}A_{pj}^{(p^r-1)}$, i.e. $p_{pi\ pj\ k}^{(p^r-1)}$ because we know $v_\alpha(A_1^{(p^r-1)})v_\beta(A_1^{(p^r-1)})$.

We assume that $k = k' + pk''$ and $s = s' + ps''$ where $0 \le k', s' < p$. Then by Corollary 2.2, it follows that

$$0 < s' < p - k' \Rightarrow \binom{k}{k - pi + s} \equiv 0 \pmod p,$$

$$p - 1 - k' < s' < p \Rightarrow \binom{p^r - 1 - k}{s} \equiv 0 \pmod p,$$

and if $s' = 0$,

$$k' \neq 0 \Rightarrow \binom{pi - s}{k - pj + s} \equiv 0 \pmod p.$$

Therefore it follows that if $k = pk''$,

$$\begin{aligned} p_{pi\ pj\ k}^{(p^r-1)} &= \sum_{s=0}^{p^r-1-k} \binom{k}{k - pi + s}\binom{pi - s}{k - pj + s}\binom{p^r - 1 - k}{s} \\ &\qquad\qquad \times (p-1)^s(p-2)^{pi+pj-k-2s} \\ &\equiv p_{ijk''}^{(p^{r-1}-1)} \pmod p, \end{aligned}$$

and if $p \nmid k$, $p_{pi\ pj\ k}^{(p^r-1)} \equiv 0 \pmod p$.

Thus

$$\begin{aligned} A_{pi+\alpha}^{(p^r-1)}A_{pj+\beta}^{(p^r-1)} &= v_\alpha(A_1^{(p^r-1)})v_\beta(A_1^{(p^r-1)})A_{pi}^{(p^r-1)}A_{pj}^{(p^r-1)} \\ &\equiv \sum_{k=0}^{p^{r-1}-1}\sum_{\gamma=0}^{p-1} p_{ijk}^{(p^{r-1}-1)}p_{\alpha\beta\gamma}^{(p-1)}A_{pk+\gamma}^{(p^r-1)}. \end{aligned}$$

By the above argument, it follows that

$$B_{pi+\alpha}^{(p^r-1)} = B_i^{(p^{r-1}-1)} \otimes B_\alpha^{(p-1)}.$$

Repeating the same argument, we know that for all non-negative integer m such that $0 \le m \le p^r - 1$ and $m = m_0p^0 + m_1p^1 + \cdots + m_{r-1}p^{r-1}$,

$$B_m^{(p^r-1)} = B_{m_{r-1}}^{(p-1)} \otimes B_{m_{r-2}}^{(p-1)} \otimes \cdots \otimes B_{m_0}^{(p-1)}.$$

From this fact, we obtain that

$$\mathfrak{A}_{p^r-1} \cong \overbrace{\mathfrak{A}_{p-1} \otimes \mathfrak{A}_{p-1} \otimes \cdots \otimes \mathfrak{A}_{p-1}}^{r}.$$

Theorem 2.3. $\mathfrak{A}_{p-1} \cong \mathbb{F}_p C_p$

Therefore the following theorem holds.

Theorem 2.4. *For all positive integer r, $\mathfrak{A}_{p^r-1}$ is isomorphic to the group algebra of the elementary abelian group of order p^r over $\mathbb{F}_p$.*

§3The structure of $\mathfrak{A}_n$

In the previous section, we considered the structure of $\mathfrak{A}_{p^r-1}$. To determine the structure of $\mathfrak{A}_n$, in general, we construct an algebra homomorphism $\mathfrak{A}_{n+1} \to \mathfrak{A}_n$.

We obtain that $A_i^{(n+1)} = I \otimes A_i^{(n)} + K \otimes A_{i-1}^{(n)}$ by indexing with a nice order, where I is the $p \times p$ identity matrix, K is the $p \times p$ matrix such that the diagonal entries are 0 and the others 1, $A_{-1}^{(n)} = A_{n+1}^{(n)} = O$. This means that $\mathfrak{A}_{n+1}$ is a subalgebra of $\mathfrak{A}_1 \otimes \mathfrak{A}_n$. The unique irreducible character of $\mathfrak{A}_1$ is $A_0^{(1)} \mapsto 1, A_1^{(1)} \mapsto -1$.

Therefore we can define naturally the mapping f_{n+1} for each positive integer n by

$$f_{n+1} : \mathfrak{A}_{n+1} \to \mathfrak{A}_n$$
$$A_i^{(n+1)} = I \otimes A_i^{(n)} + K \otimes A_{i-1}^{(n)} \mapsto A_i^{(n)} - A_{i-1}^{(n)}.$$

Proposition 3.1. *For each positive integer n, $f_{n+1} : \mathfrak{A}_{n+1} \to \mathfrak{A}_n$ above is an algebra epimorphism.*

By Theorem 2.4 and the algebra isomorphism from the adjacency algebra to the intersection algebra, for all positive integer r, $\mathfrak{A}_{p^r-1}$ is isomorphic to $\mathbb{F}_p(\underbrace{C_p \times C_p \times \cdots \times C_p}_{r})$. Let $x_1, x_2, \ldots, x_r$ be the generators of each C_p starting from the right. Then the element of $\mathfrak{A}_{p^r-1}$ corresponding to x_i by the algebra isomorphism above, is $A_{p^{i-1}}^{(p^r-1)}$.

From the representation theory of the finite group, there exists the algebra isomorphism g from the quotient ring $\mathfrak{P}_r = F_p[X_1, X_2, \ldots, X_r]/\langle X_1^p, \cdots, X_r^p\rangle$ of the polynomial ring of r variables over $\mathbb{F}_p$ to $\mathbb{F}_p(\underbrace{C_p \times C_p \times \cdots \times C_p}_{r})$ by

$g(X_i) = 1 - x_i$. Therefore we can define an algebra isomorphism $s_r : \mathfrak{P}_r \to \mathfrak{A}_{p^r-1}$ by

$$s_r(X_i) = A_0^{(p^r-1)} - A_{p^{i-1}}^{(p^r-1)}.$$

We define a weight function wt on the set of the monomials of $\mathfrak{P}_r$ by

$$wt(X_i) = p^{i-1},\ wt(\prod_j X_j^{k_j}) = \sum_j k_j p^{j-1}.$$

Proposition 3.2. *For all positive integers m such that $1 \leq m \leq p-1$,*

$$(A_0^{(p^r-1)} - A_{p^i}^{(p^r-1)})^m = m! \sum_{n=0}^{m} \binom{m}{n} (-1)^n A_{np^i}^{(p^r-1)}.$$

And if $i \neq j, 0 \leq \alpha, \beta \leq p-1$,

$$A_{\alpha p^i}^{(p^r-1)} A_{\beta p^j}^{(p^r-1)} = A_{\alpha p^i + \beta p^j}^{(p^r-1)}.$$

Let $Y_i = X_{i_0}^{k_0} X_{i_1}^{k_1} \cdots X_{i_s}^{k_s}$ be the monomial of $\mathfrak{P}_r$ such that $wt(Y_i) = i$. Then by the above two equations, the following Proposition holds.

Proposition 3.3.

$$s_r(Y_i) = (\prod_{j=0}^{s} k_j!) \sum_{n=0}^{p^r-1} \binom{i}{n} (-1)^n A_n^{(p^r-1)}.$$

Then the following theorem holds that is the main theorem.

Theorem 3.4. *We set $\mathfrak{P} = \mathbb{F}_p[X_1, X_2, \cdots]/\langle X_1^p, X_2^p \cdots \rangle$, and for all positive integer n, we set*

$$W_n = \langle\ x \mid x \text{ is the monomial of } \mathfrak{P} \text{ such that } wt(x) > n \rangle.$$

Then it holds that $\mathfrak{P}/W_n \cong \mathfrak{A}_n$ as algebras.

Proof. It is enough that we show that,

$$\mathfrak{P}_r/W_n \cong \mathfrak{A}_n \quad \text{for } n < p^r.$$

Furthermore it is enough that we show that for each positive integer n such that $n \leq p^r - 1$, $Y_n \in \mathrm{Ker} f_n f_{n+1} \cdots f_{p^r-1} s_r$. □

Remark 1 We set for all positive integer n, q,

$$G_{n,q} = (\overbrace{S_q \times S_q \times \cdots \times S_q}^{n}) \rtimes S_n, \ H_{n,q} = (\overbrace{S_{q-1} \times S_{q-1} \times \cdots \times S_{q-1}}^{n}) \rtimes S_n.$$

Let K be a field. Then $KH(n,q)$ and the Hecke algebra $\mathrm{End}_{KG_{n,q}}(1_{H_{n,q}}^{G_{n,q}})$ are isomorphic as algebras (see [2, III.2]). Therefore we also could decide the structure of $\mathrm{End}_{KG_{n,q}}(1_{H_{n,q}}^{G_{n,q}})$. In particular, Theorem **2.4** means that for all positive integer r, if $n = p^r - 1$, the Hecke algebra $\mathrm{End}_{\mathbb{F}_pG_{n,p}}(1_{H_{n,p}}^{G_{n,p}})$ is isomorphic to the group algebra $\mathbb{F}_p(\underbrace{C_p \times C_p \times \cdots \times C_p}_{r})$.

References

[1] Z. Arad, E. Fisman, and M. Muzychuk, Generalized table algebras, Israel J. Math. **144** (1999) 29-60

[2] E. Bannai and T. Ito, "Algebraic Combinatorics. I. Association Schemes," Benjamin-Cummings, Menlo Park, CA, 1984.

[3] P. -J. Cameron, "Combinatorics: topics, techniques, algorithms," Cambridge University Press, 1994.

[4] A. Hanaki, Locality of a modular adjacency algebra of an association scheme of prime power order, to appear in Arch. Math.

[5] A. Hanaki, Semisimplicity of Adjacency Algebras of Association Schemes, J. Alg. **225**, 124-129 (2000).

Department of Mathematical Sciences, Faculty of Science, Shinshu University, Matsumoto 390-8621, Japan
E-mail:yoshi@math.shinshu-u.ac.jp

Symmetric algebra and modular invariance property of trace functions of vertex operator algebra

Masahiko Miyamoto (University of Tsukuba, Dept. of Math.)

1 Introduction

The purpose of my talk is to show an application of classical result in finite dimensional ring theory to vertex operator algebra (conformal field theory.) See [Miyamoto] for further details. In the theory of vertex operator algebra, the one of the most important features is a modular invariance property. This property was first observed in many examples and proved by Zhu if VOA satisfies C_2-cofiniteness condition and is also rational, that is, all modules are completely reducible. In order to prove this property, he introduced Zhu algebra. This result is called Zhu's theory, which is a first theorem about the modular invariance property of trace function on modules. From then, "rationality" has been thought to be a condition for a modular invariance. However, I recently proved a modular invariant property of vertex operator algebra without assuming the rationality, but C_2-cofiniteness. In my proof, a classical result about symmetric algebras (algebra with a symmetric linear function) played an essential role. So I would like to show a relation between symmetric algebra and vertex operator algebra.

In this paper, we will follow my lecture and I added the several definition at the end of this paper.

Introduction

Vertex operator algebra (shortly VOA) is algebraic (mathematical) version of Conformal Field Theory (shortly CFT) in physics and we have to treat infinite dimensional non-associative algebras.

However, the most properties of good CFT are controlled by **finite dimensional ordinary algebras**. Therefore, the theory of finite dimensional algebra plays an essential role in CFT.

Today's talk is one example.

Conformal Field Theory is the fundamental theory for (super) string theory and is a theory on a Riemann surface and so it has a geometrical meanings, but we focus ourselves to algebraic site.

Brief Introduction of VOA

VOA is an infinite dimensional $\mathbb{N}$-graded vector space

$$V = \oplus_{n=0}^{\infty} V_n \qquad \dim V_n < \infty$$

with infinitely many products $\times_m$ ($\forall m \in \mathbb{Z}$) satisfying suitable conditions. We define a weight $|w|$ of $v \in V_i$ is i. For any $m \in \mathbb{Z}$ and $v \in V$, m-th product satisfies

$$v_m (= v \times_m) : V_n \to V_{n+m}$$

shifts grading by m.

Moreover, V has two special elements, $\mathbf{1} \in V_0$ called **Vacuum** and $\omega \in V_2$ called **Virasoro element**. In particular, ω defines a special complex number $c \in \mathbb{C}$ called **central charge**.

Namely, $V = V_0 \oplus V_1 \oplus V_2 \oplus \ldots$. Actually, this decomposition is given by grading operator $L(0)$ as eigenspaces of non-negative integer eigenvalues.
Vacuum looks like an identity and Virasoro element controls grading and differential. Virasoro element is also a generalization of Casimir element. We can see the precise definition of VOA in Appendix. (The correct definition of v_m satisfies $v_m : V_n \to V_{|v|-1-m+n}$, but we don't need this fact in this talk.) The weight originally came from the energy level in conformal field theory.

Similary, we define modules

$$W = W(0) \oplus W(1) \oplus W(2) \oplus \ldots$$

Action of $v \in V, v_n^W : W(m) \to W(m+n)$

$L(0) := o(\omega)$ is a grading operator.

As we mentioned, $o(v)$ is a grade-preserving operator for any $v \in V$. If $W = \oplus_{n=0}^{\infty} W(n)$ is irreducible, then $L(0)$ acts on $W(0)$ as a sclar k for some $k \in \mathbb{C}$ and $L(0)$ acts on $W(m)$ as a scalar $k+m$.

See Appendix for the definition of modules and weak modules.

We define a trace function for a grade-preserving operator $o(v) = v_0$ and a $\mathbb{Z}$-graded module W.

Def. Trace function

$$\begin{array}{lccccccc}
W = & W(0) & \oplus & W(1) & \oplus & W(2) & \oplus\cdots \\
q^{L(0)}: & q^r & & q^{r+1} & & q^{r+2} & \cdots \\
 & o(v)_{|W(0)} & & o(v)_{|W(1)} & & o(v)_{|W(2)} & \cdots \\
 & \mathrm{tr}_{|W(0)}o(v)q^r & & \mathrm{tr}_{|W(1)}o(v)q^{r+1} & & \mathrm{tr}_{|W(2)}o(v)q^{r+2} & \cdots \\
 & \mathrm{tr}_{|W(0)}o(v)q^r & + & \mathrm{tr}_{|W(1)}o(v)q^{r+1} & + & \mathrm{tr}_{|W(2)}o(v)q^{r+2} & +\cdots
\end{array}$$

Comment: Since each homogeneous space $W(n)$ is of finite dimension, $\mathrm{tr}_{|W(n)}o(v)$ is well defined. Originally, we multiply q^k if $L(0)$ acts on $W(n)$ as a scalar k. However, since we will consider general cases, $L(0)$ may not act on $W(n)$ as a scalar and so we need to define $\mathrm{tr}_{W(n)}(o(v)q^{L(0)})$ directly.

Simply, we denote it by $S^W(v,\tau) = \mathrm{tr}_W o(v)q^{L(0)-c/24}$, which is called a trace function, where c is central charge. In particular, $S^W(\mathbf{1},\tau) = q^{r-c/24}\sum_{n=0}^{\infty}\dim W(n)q^n$ is called a **character of** W.

In CFT, these characters play essential roles since it was not difficult to calculate characters of modules when we construct modules.

Modular transformation

For $\theta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$, set

$$S|\theta(v,\tau) = \left(\frac{1}{c\tau+d}\right)^{|v|}S(v,\theta\tau)$$

where $\theta\tau = \frac{a\tau+b}{c\tau+d}$ and $|v|$ denotes weight.

<u>What is this ?</u>

In "Good" Conformal Field Theory, the above should be a linear combinations of meaningful functions.

There are several methods used in CFT without proofs. For example, if V is rational VOA, then Verlinde-formula insists $S|\theta(v,\tau)$ is a linear combination of characters of all modules. In particular, all characters will appear in the trasformed function of character of V by $\tau \to \frac{-1}{\tau}$. So by using modular transformation, they are able to determine all modules.

C_2-cofinite condition(natural condition) gives differential equation. As solutions, we have:
$$S|\theta(v,\tau) = \sum_{t=0}^{p}\sum_{s=0}^{q}\sum_{i=0}^{\infty} C_{t,s,i}(v) q^i q^{r_s} \tau^t$$

C_2-cofiniteness was originally introduced by Zhu as technical reason to get a differential equation. However, the author recently showed in [Miyamoto] that C_2-cofiniteness is an essential condition to define trace functions on every modules. For example, V is C_2-cofinite if and only if every weak module is a direct sum of generalized eigenspaces of $L(0)$, which is a necessary condition to define a trace function on it.

Ordinary Algebras in VOA

V has infinitely many products $\Rightarrow$ we can construct new meaningful products such that some factor space becomes an ordinary algebra. The most important example is Zhu algebra $A(V) = V/O(V)$.

For a V-module $W = W(0) \oplus W(1) \oplus W(2) \oplus ...$, a grade-preserving operator $o(v)$ acts on $W(0)$ (a top module of W),
$$v \in O(V) \subseteq V \Leftrightarrow o(v) = 0 \quad \text{on } W(0)$$
for any modules W. A product $^{\exists}*$ on $A(V)$ s.t. $o(v*u) = o(v)o(u)$ on $W(0)$. Then $A(V) = V/O(V)$ becomes an algebra of these zero modes on top modules.

The precise definitions of $A(V)$ and $O(V)$ are given by different ways. The important property of Zhu algebra is:

Conversely, if T is an $A(V)$-module then there is a V-module $T \oplus {}^{\exists}T(1) \oplus {}^{\exists}T(2) \oplus \cdots$ whose top module is T.

An important result for modular invariance property is:

$C_{t,s,0}(\cdot) : V \to \mathbb{C}$ is a symmetric function of $A(V)$.

By using this result, Zhu showed that if $A(V)$ is semisimple, then $C_{t,s,0}$ is a linear combination of trace functions (by Eilenberg-Nakayama's theorem). Hence $S(v, A\tau)$ is a linear sum of trace functions. Namely,

Theorem(Zhu) If V is C_2-cofinite and $A(V)$ is semisimple, then $< S^W(v,\tau)|W$ irr. V-mod. $>$ is $SL(2,\mathbb{Z})$-invariant.

We will treat the **general case**, that is, $A(V)$ is non-semisimple (**Artin Ring**) and so we need ring theoretic arguments.

As we mentioned $\phi = C_{t,s,0}(\ ) : V \to \mathbb{C}$ is a symmetric function of $A(V)$. Then $A(V)/\mathrm{Rad}\phi$ becomes a symmetric algebra. We will use a result by C.Nesbitt and W.Scott about a symmetric algebra. The symmetric algebra in my talk is not a symmetric tensor algebra. We will give the definition of symmetric algebra.

Def. of Symmetric algebra.
Let A be a finite dimensional algebra/$\mathbb{C} \ni 1$.
A is Frobenius algebra $\Leftrightarrow$ left mod. ${}_AA \cong \mathrm{Hom}_{\mathbb{C}}(A_A, \mathbb{C})$.
Let $R(a)$, $L(a)$: denote right, left regular actions of $a \in A$ on A,
Frobenius algebra $\Leftrightarrow$ $^\exists Q \in Mat(\mathbb{C})$ s.t. $Q^{-1}R(a)Q = L(a)$.

A is **symmetric algebra**
$\Leftrightarrow$ Q is a symmetric matrix.
$\Leftrightarrow$ A has a symmetric map $\phi \in \mathrm{Hom}(A, \mathbb{C})$ s.t. $\mathrm{Rad}\phi = 0$.
$\Leftrightarrow$ A has an associative nondegenerated bilinear form $\langle\ ,\ \rangle$,
where $\mathrm{Rad}\phi = \{a \in A, \phi(Aa) = 0\}$ and "associativity" means $\langle ab, c\rangle = \langle a, bc\rangle$.

A result we will use to explain my method is given by

C.Nesbitt, W.Scott (1943)
(A short proof (Oshima 1952))

A is symmetric algebra $\Leftrightarrow$
its basic algebra is symmetric.

Def. of basic algebra.
Decompose (simple components)

$$A/J(A) = A_1 \oplus \cdots \oplus A_k$$

$$e_1 \quad \cdots \quad e_k$$

primitive idempotents

Set $e = e_1 + \cdots + e_k$
(we may view: idempotent $e \in A$.)

eAe is called a **basic algebra** of A.

Note: Ae is an A-module and $eAe = \mathrm{End}_A(Ae)$. $eAe/J(eAe) \cong \mathbb{C} \oplus \cdots \oplus \mathbb{C}$.

Their result says that

> A is symmetric if and only if eAe is symmetric.

For example,

> $R_m = \left\{ g = \begin{pmatrix} A_g & B_g \\ O & A_g \end{pmatrix} \mid A_g, B_g \in M_{m,m}(\mathbb{C}) \right\}$ is a symmetric algebra with a symmetric linear map $\phi(g) = \mathrm{tr} B_g$.
> Its basic algebra is $P = \left\{ \alpha = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{C} \right\}$ with $\phi(\alpha) = b$.
> A module $R_m e = \mathbb{C}^m \oplus \mathbb{C}^m$ is direct sum of two same modules.

The structure of $R_m e$ as right P-module is important. Right P-module $R_m e$ is a direct sum of two isomorphic right P-modules $\mathbb{C}^m$.

My method is:

> A symmetric map ϕ of Zhu algebra $A(V)$ is given
> $\Rightarrow A = A(V)/\mathrm{Rad}\phi$ is a symmetric algebra.
> $\Rightarrow$ its basic algebra $P = eAe$ is symmetric.
> Then we construct right P-, left V-modules W such that **the basic algebra of** $\mathrm{End}_P(W)$ **is** P.

As pointed out by Iwanaga, $\mathrm{End}_P(W)$ is not a finite dimensional algebra. We always have to consider a filtration $W^n = \oplus_{m=0}^n W(m)$, which is of finite dimension.

We will explain my method by using example R_m. First we have a symmetric algebra R_2 (a factor ring of Zhu algebra), then P is a symmetric algebra (because it is a basic algebra of R_2. Then construct a right P-module $W = C^m \oplus C^m$. Then $\mathrm{End}_P(W) \cong R_m$ is a symmetric algebra (because its basic algebra is P).

> We will consider $V \subseteq \mathrm{End}_P(W)$. We will call such a module W **interlocked with** ϕ.
> Then $\mathrm{End}_P(W)$ has sym. map tr^ϕ.
> We view tr^ϕ as a new kind of trace map on W and define explicitly.
> We will call it pseudo-trace on W

Precisely, V is not contained $\mathrm{End}_P(W)$. $\mathrm{End}_P(W)$ contains v_n for $v \in V$ and $n \in \mathbb{Z}$ and so $\mathrm{End}_P(W^n)$ contains a subring generated by $v^1_{n_1} \cdots v^k_{n_k}$ with $\sum n_j \leq 0$.

Let's me explain pseudo-traces again.

Note For a vector space W and $\alpha \in \mathrm{End}(W)$, the ordinary trace map is given as a trace of matrix representation of $\alpha : W \to W$. On the other hand, if W is a right P-module, $\alpha \in \mathrm{End}_P(W)$ and $W/WJ(P) \cong W\mathrm{soc}(P)$ as V-modules, then α is represented by matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and pseudo-trace of α is given by a trace of submatrix B corresponding to $W/WJ(P) \longrightarrow W\mathrm{soc}(P)$

Using pseudo-trace functions, we have:

$S|\theta(v,\tau)$ becomes a linear sum of pseudo-trace functions
$S^W(v,\tau) = \mathrm{tr}^{\phi}_W o(v) q^{L(0)-c/24}$
of interlocked V-modules W.

Although we started from the ordinary trace functions, we can also start from pseudo-trace functions and then we have the same conclusion. Therefore, we have:

Main Theorem (General case)
If a VOA is C_2-cofinite, then the space spanned by pseudo-trace functions
$\langle S^W(v,\tau) \mid W$ interlocked with a symmetric linear map of $A(V)\rangle$
is $SL(2,\mathbb{Z})$-invariant.

In CFT, the above space is called a conformal block for torus, since $q = e^{2\pi i\tau}$ gives a period of $\tau \to \tau + 1$ and z in $Y(v,z)$ gives another period. The important result is:

The dimension of the above space is finite.
Actually, its dimension is equal to the dimension of the space of symmetric linear maps of Zhu algebra $A(V)$.
In particular, for any irreducible module W and $v \in V$,
$< \mathrm{tr}_W(v,\tau)^{SL(2,\mathbb{Z})} >$
is of finite dimension.

To tell the truth, the second statement are not correct. We have to consider the n-th Zhu algebra $A_n(V)$ for general VOAs and the dimension is equal to the dimension of the space of symmetric linear maps of $A_n(V)$, but we don't have time to explain it and the above statement are true for the most known VOAs.

[Note]
Generalized character was introduced by physicist M. Flohr(1995) in order to to obtain a modular invariant property of characters for some CFT. Namely, he choose a basis of
$< S_W(\tau)^{SL(2,\mathbb{Z})} \mid W$ irreducible modules $>$
for some CFT and call them generalized characters.

Our result says that the true meaning of generalized character should be pseudo-trace function $S^W(1,\tau)$ for an interlocked module W.

There are many unsolved cases.

If g is an automorphism of V, we have g-symmetric function of $A_g(V)$ from g-twisted modules.

$$\phi(ab) = \lambda\mu^{-1}\phi(ba)$$

if $g(a) = \lambda a$, $g(b) = \mu b$ (eigenvalues)

Can we extend it to a g-symmetric (pseudo-trace) function of V? This gives an extension of modular invariance property of orbifold VOA.

Suppose $A(V)$ has a bilinear form. Can we extend it among V-modules ?

These are essentially **problems of finite dimensional algebras.**

2 Appendix

In the definition of vertex operator algebra, you will see many conditions. However, these conditions makes VOA compact so that even VOA is of infinite dimensional vector space, it plays like a finite dimensional algebra.

Definition A *vertex operator algebra* (VOA) is a quadruple $(V, Y, \mathbf{1}, \omega)$, where V is a $\mathbb{Z}_+$-graded vector space $V = \coprod_{n\in\mathbb{Z}_+} V_n$ and

$$\begin{array}{rccc} Y(\cdot,z): & V & \longrightarrow & \mathrm{End}V[[z,z^{-1}]] \\ & v & \longrightarrow & Y(v,z) = \sum_{i\in\mathbb{Z}} v(i)z^{-i-1} \end{array}$$

is a linear map from V to $(\mathrm{End}V)[[z,z^{-1}]]$ and $Y(v,z) = \sum_{n\in\mathbb{Z}} v(n)z^{-n-1}$ is called the *vertex operator* associated to v, and $\mathbf{1}$ and ω are specified elements in V_0 and V_2, respectively, such that the following conditions hold:

($A1$) [Vacuum element]
$Y(\mathbf{1},z)=id_V$;

($A2$) $Y(v,z)\mathbf{1}\in V[[z]]$ and $\lim_{z\to 0}Y(a,z)\mathbf{1}=a$ for any $v\in V$;

($A3$) $v(m)u\in V_{h+k-m-1}$ for $v\in V_h, u\in V_k$;

($A4$) $\dim V_n<\infty$;

($A5$) [Virasoro element]
$L_i=\omega(i+1)$ satisfy the Virasoro algebra relations:
$[L_m,L_n]=(m-n)L_{m+n}+\delta_{m+n,0}\frac{m^3-m}{12}c$,
where c is some constant, which is called central charge of V;

($A6$) [L_{-1}-derivative formula]
$Y(\omega(0)v,z)=Y(L_{-1}v,z)=[L_{-1},Y(v,z)]=\frac{d}{dz}Y(v,z)$ for any $v\in V$;

($A7$) The following *Commutativity* holds:
$(z-w)^N Y(a,z)Y(b,w)=(z-w)^N Y(b,w)Y(a,z)$
for any $a,b\in V$.

Definition 1 *A* **weak module** *for* $(V,Y,\mathbf{1},\omega)$ *is a vector space* M *equipped with a formal power series*

$$Y^M(v,z)=\sum_{n\in\mathbb{Z}}v_n^M z^{-n-1}\in(\mathrm{End}(M))[[z,z^{-1}]]$$

called the module vertex operator of v *for* $v\in V$ *satisfying:*

($W1$) $Y^M(\mathbf{1},z)=1_M$;

($W2$) $Y^M(\omega,z)=\sum L^M(n)z^{-n-1}$ *satisfies:*

($W2.a$) *the Virasoro algebra relations and*

($W2.b$) *the* $L(-1)$*-derivative property:*$Y^M(L(-1)v,z)=\frac{d}{dz}Y^M(v,z)$,

($W3$) *Commutativity:* $(z-w)^N(Y^M(v,z)Y^M(u,w)-Y^M(u,w)Y^M(u,z))=0$

($W4$) *Associativity:* $Y^M(u_nv,z)=Y^M(u,z)_nY^M(v,z)$
for $u,v\in V$ *and* $Y(u,z)=\sum u_nz^{-n-1}$

Definition 2 *A* **module** *for* $(V,Y,\mathbf{1},\omega)$ *is a weak module* (M,Y) *satisfying*

($M1$) M *is an* $\mathbb{N}$*-graded* $M=\oplus_{n\geq 0}M_n$ *and* $\dim M_n<\infty$

($M2$) $L^M(0)$ *acts on* M_n *semisimply and*

($M3$) $v_{|v|-1+i}M_n\subseteq M_{n-i}$.

For ring theorists, the definition of modules looks strange because the infinite direct sum of modules is not a module. This is because VOA comes from CFT in physics and they considered only the set of irreducible modules at first.

Definition 3 *For a VOA V, set $C_2(V) = \langle v_{-2}u \mid v, u \in V\rangle$. We call V C_2-***cofinite** *if $V/C_2(V)$ is of finite dimension.*

$V/C_2(V)$ becomes a Poison algebra with $\bar{v}\cdot\bar{u} = \overline{v_{-1}u}$ and $[\bar{v},\bar{v}] = \overline{v_0u}$, where $\bar{v} = v + C_2(V)$.

References

R.E. Borcherds, Vertex algebras, Kac-Moody algebras, and the Monster, *Proc. Natl. Acad. Sci. USA* **83** (1986),

M. Miyamoto, A modular invariance property of vertex operator algebra satisfying C_2-cofiniteness., math.QA/0209101

C. Nesbitt and W.M. Scott, Some remarks on algebras over an algebraically closed field. *Ann. of Math.*, **44** (1943)

M. Osima, A note on symmetric algebra, *Proc. Japan Acad.* **28** (1952)

Y. Zhu, Modular invariance of characters of vertex operator algebras, *J. Amer. Math. Soc.* **9** (1996),

University of Tsukuba, 1-1-1 Tenoudai, Tsukuba 305, Japan.

PLETHYSM OF SCHUR FUNCTIONS AND THE BASIC REPRESENTATION OF $A_2^{(2)}$

HIROSHI MIZUKAWA AND HIRO-FUMI YAMADA

1. Introduction

We present a formula for Schur functions indexed by rectangular Young diagrams. More precisely, we give an expression of the plethysm $p_2 \circ S_{\square(n,m)}$, where p_2 is the power sum of degree two and $S_{\square(n,m)}$ is the Schur function indexed by the rectangular partition $\square(n,m) = (m^n)$ (Theorem 4.1). Though the formula is described in a combinatorial way, it can be explained naturally from the viewpoint of the basic representation of the affine Lie algebra of type $A_2^{(2)}$. As a merit of our understanding, it becomes clear that the formula gives an explicit expression of a homogeneous polynomial τ-function of a hierarchy of nonlinear differential equations. Proofs and details can be found in [8].

2. Symmetric Functions and Partitions

We denote by P_n the set of all partitions of n, SP_n the set of all strict partitions of n and OP_n the set of those partitions of n whose parts are odd numbers. Let χ_ρ^λ be the irreducible character of the symmetric group S_n, indexed by $\lambda \in P_n$ and evaluated at the conjugacy class ρ, and ζ_ρ^λ be the irreducible *negative* character of the double cover $\tilde{S}_n$ (cf. [3]), indexed by $\lambda \in SP_n$ and evaluated at the conjugacy class ρ. Here we recall symmetric functions of variables $x = (x_1, x_2, \cdots)$ which are discussed in this paper. Let $p_r(x) = \sum_{i\geq 1} x_i^r$ be the power sum symmetric function for $r \geq 1$. The Schur functions are defined as follows:

$$S_\lambda(x) = \sum_{\rho\in P_n} z_\rho^{-1}\chi_\rho^\lambda p_\rho(x).$$

For $\lambda \in SP_n$ define Schur's Q-function and P-function by

$$Q_\lambda(x) = \sum_{\rho\in OP_n} 2^{(l(\lambda)+l(\rho)+\epsilon(\lambda))/2} z_\rho^{-1}\zeta_\rho^\lambda p_\rho(x),$$

$$P_\lambda(x) = 2^{-l(\lambda)}Q_\lambda(x),$$

where

$$\epsilon(\lambda) = \begin{cases} 0 & \text{if } n - l(\lambda) \text{ is even,} \\ 1 & \text{if } n - l(\lambda) \text{ is odd.} \end{cases}$$

For a symmetric function $F(x)$, the *plethysm* $p_r \circ F(x)$ with the r-th power sum p_r is by definition [7, p135]

$$p_r \circ F(x) = F(x^r).$$

Fix λ be a strict partition and r be a positive odd integer. Put

$$t = (r-1)/2.$$

A $(t+1)$-tuple of partitions $(\lambda^{bc(r)}, \lambda^b[0], \ldots, \lambda^b[t])$ is attached to $\lambda \in SP_n$; $\lambda^{bc(r)}$ is the r-bar core of λ and the collection $\lambda^{bq(r)} = (\lambda^b[0], \ldots, \lambda^b[t])$ is the r-bar quotient of λ (cf. [10]).

3. Basic Representation of $A_2^{(2)}$

We discuss the basic representation of the affine Lie algebra of type $A_2^{(2)}$ following [5]. Here the Schur functions, Schur's P and Q-functions are described in terms of the so called Sato variables: $u_j = p_j/j$ $(j \geq 1)$ for S_λ, $s_j = 2p_j/j$ $(j \geq 1,\ odd)$ or $t_j = 2p_j/j$ $(j \geq 1,\ odd)$ for P_λ and Q_λ. We will denote them by $S_\lambda(u)$, $P_\lambda(s)$, $Q_\lambda(t)$, etc. Put $\Gamma = \mathbb{C}[t_j;\ j \geq 1,\ odd]$, whose basis is chosen as $\{P_\lambda;\ \lambda \in SP_n,\ n \in \mathbb{N}\}$. Associated with the Cartan matrix

$$(a_{ij})_{i,j\in\{0,1\}} = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix},$$

the Lie algebra $\mathfrak{g}$ of type $A_2^{(2)}$ is generated by $e_i, f_i, \alpha_i^\vee (i = 0,1)$ and d subject to the relations

$$[\alpha_i^\vee, \alpha_j^\vee] = 0, \qquad [\alpha_i^\vee, e_j] = a_{ij}e_j, \qquad [\alpha_i^\vee, f_j] = -a_{ij}f_j,$$
$$[e_i, f_j] = \delta_{i,j}\alpha_i^\vee, \qquad (\mathrm{ad}e_i)^{1-a_{ij}}e_j = (\mathrm{ad}f_i)^{1-a_{ij}}f_j = 0 \quad (i \neq j),$$

and

$$[d, \alpha_i^\vee] = 0, \qquad [d, e_j] = \delta_{j,0}e_j, \qquad [d, f_j] = -\delta_{j,0}f_j.$$

The Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is spanned by $\alpha_0^\vee, \alpha_1^\vee$ and d. Choose the basis $\{\alpha_0, \alpha_1, \Lambda_0\}$ for the dual space $\mathfrak{h}^*$ of $\mathfrak{h}$ by the pairing

$$< \alpha_i^\vee, \alpha_i > = a_{ij}, \qquad < \alpha_i^\vee, \Lambda_0 >= \delta_{i,0},$$
$$< d, \alpha_j > = \delta_{0,j}, \qquad < d, \Lambda_0 >= 0.$$

The fundamental imaginary root is $\delta = 2\alpha_0 + \alpha_1$.

The basic representation of $\mathfrak{g}$ is by definition the irreducible highest weight $\mathfrak{g}$-module with highest weight Λ_0. The weight system of the basic representation is well known:

$$P(\Lambda_0) = \{\Lambda_0 - p\delta + q\alpha_1\ ;\ p \geq 2q^2,\ p, q \in \frac{1}{2}\mathbb{Z},\ p + q \in \mathbb{Z}\}.$$

A weight Λ on the parabola $\Lambda_0 - 2q^2\delta + q\alpha_1$ is said to be maximal in the sense that $\Lambda + \delta$ is no longer a weight. For any maximal weight Λ, the multiplicity of $\Lambda - n\delta$ $(n \in \mathbb{N})$ is known to be equal to $p(n)$, the number of partitions of n. A construction of the basic representation in *principal* grading is realized on the space $\Gamma^{(3)} = \mathbb{C}[t_j;\ j \geq 1,\ \text{odd},\ j \not\equiv 0(\text{mod}3)]$ ([5]). A P-function $P_\lambda(t)$ is not necessarily contained in $\Gamma^{(3)}$. However, if the strict partition λ is a 3-bar core, then $P_\lambda(t) \in \Gamma^{(3)}$ and in fact $P_\lambda(t)$ is a maximal weight vector. More generally we "kill" the variables t_{3j} $(j \geq 1,\ odd)$ in the P-function $P_\lambda(t)$ and consider the reduced P-function:

$$P_\lambda^{(3)}(t) := P_\lambda(t)|_{t_3 = t_9 = \cdots = 0} \in \Gamma^{(3)}.$$

It is shown in [9] that $P_\lambda^{(3)}(t)$ is a weight vector for any strict partition λ, and that

$$\begin{aligned}&\{P_\lambda^{(3)}(t);\ \lambda \text{ is a strict partition with no part divisible by } 3\}\\ =&\{P_\lambda^{(3)}(t);\ \lambda \text{ is a strict partition with } \lambda^{bq(3)} = (\emptyset, \lambda^b[1])\}\end{aligned}$$

form a weight basis for $\Gamma^{(3)}$. The weight of a reduced P-function with a given strict partition λ is known as follows. Draw the Young diagram λ and fill each cell with 0 or 1 in such a way that, in each row the sequence (010) repeats from the left as long as possible. If k_0 (resp. k_1) is the number of 0's (resp. 1's) written in the Young diagram, then the weight of the corresponding reduced P-function is $\Lambda_0 - k_0\alpha_0 - k_1\alpha_1$. A removable i-node (i=0,1) is a node $\boxed{i}$ of the boundary of λ which can be removed. An indent i-node (i=0,1) is a concave corner on the rim of λ where a node $\boxed{i}$ can be added. The action of $\mathfrak{g}$ to the reduced P-function $P_\lambda^{(3)}(t)$ is described as follows:

$$e_i P_\lambda^{(3)} = \sum_{\mu \in \mathcal{E}_i^1(\lambda)} P_\mu^{(3)},$$

where $\mathcal{E}_i^1(\lambda)$ is the set of the strict partitions which can be obtained by removing a removable i-node from λ, and

$$f_i P_\lambda^{(3)} = \sum_{\mu \in \mathcal{F}_i^1(\lambda)} P_\mu^{(3)},$$

where $\mathcal{F}_i^1(\lambda)$ is the set of the strict partitions which can be obtained by adding an indent i-node to λ. For instance

$$\begin{aligned} e_0 P_{(4,3,1)}^{(3)} &= P_{(4,2,1)}^{(3)} + P_{(4,3)}^{(3)},\\ f_1 P_{(4,3,1)}^{(3)} &= P_{(5,2,1)}^{(3)} + P_{(4,3,2)}^{(3)}.\end{aligned}$$

Another realization of the basic representation is known, one in the homogeneous grading. The isomorphism between principal and homogeneous realizations is given by Leidwanger [6]. Put

$$\mathcal{B} = \mathbb{C}[u_j,\ s_{2j-1};\ j \geq 1].$$

Define the mapping Φ by

$$\begin{aligned} \Phi\ :\ \Gamma &\xrightarrow{\sim} \mathcal{B} \otimes \mathbb{C}[q, q^{-1}],\\ P_\lambda(t) &\longmapsto 2^{p(\lambda)} \bar{\delta}_3(\lambda) P_{\lambda^b[0]}(s) S_{\lambda^b[1]}(u) \otimes q^{m(\lambda)},\end{aligned}$$

where

$$p(\lambda) = \sum_{\lambda_i \not\equiv 0 \pmod 3} \left[\frac{\lambda_i - 1}{3}\right],$$

and $m(\lambda)$ is determined by drawing the 3-bar abacus of λ:

$$\begin{aligned} m(\lambda) = &\text{ (number of beads on the first runner of } \lambda)\\ &- \text{(number of beads on the second runner of } \lambda).\end{aligned}$$

For example

$$\Phi(P_{(7,5,3,1)}(t)) = 8P_{(1)}(s)S_{(2,1,1)}(u) \otimes q.$$

Leidwanger [6] shows that Φ is indeed an isomorphism and that, if we denote by V the subalgebra of $\mathcal{B}$ generated by u_{2j} and $2^{2j-1}u_{2j-1} - s_{2j-1}$ ($j \geq 1$), then

$$\Phi(\Gamma^{(3)}) = V \otimes \mathbb{C}[q, q^{-1}].$$

The representation of $\mathfrak{g}$ on $V \otimes \mathbb{C}[q, q^{-1}]$, which is induced by Φ, is the basic representation in the homogeneous grading. In fact, if we define the degree in $V \otimes \mathbb{C}[q, q^{-1}]$ by

$$\deg f(u, s) \otimes q^m = 2 \deg f(u, s) + m^2,$$

then deg $\Phi(P_\lambda^{(3)})$ is equal to the number of 0-nodes in λ.

4. Rectangular Schur Functions and $A_2^{(2)}$

Let ℓ be a positive integer and $\Lambda_\ell = (3\ell - 2, 3\ell - 5, \cdots, 7, 4, 1)$. Each cell of the Young diagram of Λ_ℓ is supposed to be filled with 0 or 1 as in Section 3. Let $\mathcal{F}_1^m(\Lambda_\ell)$ $(0 \leq m \leq \ell)$ be the set of the strict partitions which are obtained by adding m $\boxed{1}$'s to Λ_ℓ. It is obvious that $|\mathcal{F}_1^m(\Lambda_\ell)| = \binom{\ell}{m}$. We are now ready to state the result in this note.

Theorem 4.1.

(1)
$$\sum_{\mu \in \mathcal{F}_1^m(\Lambda_\ell)} \bar{\delta}_3(\mu) S_{\mu^\flat[1]} = \varepsilon(\ell, m) p_2 \circ S_{\square(\ell-m,m)},$$

where

$$\varepsilon(\ell, m) = \begin{cases} (-1)^{\binom{m}{2}} & (0 \leq m \leq \frac{\ell}{2}) \\ (-1)^{\binom{\ell-m+1}{2}+(\ell-m)m} & (\frac{\ell}{2} \leq m \leq \ell). \end{cases}$$

It is shown in [9] that, in the principal realization of the basic representation of $A_2^{(2)}$, the P-functions $P_{\Lambda_\ell}(t) = P_{\Lambda_\ell}^{(3)}(t)$ $(\ell \geq 1)$ are the maximal weight vectors which allow non-zero action of f_1. As is explained in the previous section, we have

$$\frac{1}{m!} f_1^m P_{\Lambda_\ell}^{(3)} = \sum_{\mu \in \mathcal{F}_1^m(\Lambda_\ell)} P_\mu^{(3)}.$$

The left-hand side of (1) is nothing but the image of $\frac{1}{m!} f_1^m P_{\Lambda_\ell}^{(3)}$ under the Leidwanger isomorphism Φ to the homogeneous realization (dropping $q^{m(\mu)} = q^{\ell - 2m}$). Note that $p(\mu) = \binom{\ell}{2}$ for all $\mu \in \mathcal{F}_1^m(\Lambda_\ell)$. Therefore the formula (1) can be thought of as the homogeneous realization of the weight vectors which are obtained by acting the group SL_2 to a maximal weight vector in the basic representation of $A_2^{(2)}$.

References

[1] C. Carré and B. Leclerc, Splitting the square of a Schur function into its symmetric and antisymmetric parts, J. Algebraic Combin. 4 (1995), no. 3, 201-231.

[2] L. Carini and J. Remmel, Formulas for the expansion of the plethysms $s_2[s_{(a,b)}]$ and $s_2[s_{(n^k)}]$, Discrete Math. 193 (1998), no. 1-3, 225-233.

[3] P. N. Hoffman and J. F. Humphreys, *Projective Representations of the Symmetric Groups*, Oxford, 1992.

[4] T. Ikeda and H. -F. Yamada, Polynomial τ-functions of the NLS-Toda hierarchy and the Virasoro singular vectors, Lett. Math. Phys. (to appear)

[5] V. G. Kac, D. A. Kazhdan, J. Lepowsky and R. L. Wilson, Realization of the basic representations of the Euclidean Lie algebras, Adv. Math. 42 (1981), no. 1, 83-112.

[6] S. Leidwanger, Basic representations of $A_{n-1}^{(1)}$ and $A_{2n}^{(2)}$ and the combinatorics of partitions, Adv. Math. 141 (1999), no. 1, 119-154.
[7] I. G. Macdonald, *Symmetric Functions and Hall Polynomials, 2nd. ed.* , Oxford, 1995.
[8] H. Mizukawa and H.-F. Yamada, Rectangular Schur functions and the basic representation of affine Lie algebras, arXiv:math.CO/0206225.
[9] T. Nakajima and H. -F. Yamada, Schur's Q-functions and twisted affine Lie algebras, Adv. Stud. in Pure Math. 28 (2000), 241-259.
[10] J. B. Olsson, Combinatorics and Representations of Finite Groups, Essen, 1993.

HIROSHI MIZUKAWA, DEPARTMENT OF MATHEMATICS, HOKKAIDO UNIVERSITY, SAPPORO 060-0810, JAPAN
E-mail address: mzh@math.sci.hokudai.ac.jp mzh@math.okayama-u.ac.jp

HIRO-FUMI YAMADA, DEPARTMENT OF MATHEMATICS, OKAYAMA UNIVERSITY, OKAYAMA 700-8530, JAPAN
E-mail address: yamada@math.okayama-u.ac.jp

Monomial Modules and Endo-monomial Modules

Ziqun Lu
School of Mathematical Sciences, Peking University, Beijing 100871, China

0. Notation

Let G be a finite group. Let (K, R, F) be a p-modular system, where R is a complete discrete valuation ring of maximal idea πR, K is the quotient field of R of characteristic 0, and $F = R/\pi R$ is the residual field of R of characteristic $p > 0$. We assume that K and F are big enough.

Here a module means a finitely generated right module. We assume that all RG-modules are R-free. For a subgroup H of G, and for an FG-module X and an FH-module Y, we write X_H for the restriction of X to H and $\mathrm{Ind}_H^G(Y)$ for the induction of Y to G. When $H \triangleleft G$ and Y is an FH-module, we denote $I_G(Y)$ the inertia subgroup of Y in G. If there is no confusion, we often use $\otimes$ instead of $\otimes_R$ and $\otimes_F$. Let V be an RG-module. We denote by V^* the dual module $\mathrm{Hom}_R(V, R)$ of V.

1. Generalized Brauer construction

For a p-subgroup P of G, R. Brauer defined a homomorphism of algebras $\mathrm{Br}_P : Z(FG) \to Z(C_G(P))$ between the centers of the corresponding group algebras. M. Broué generalized Brauer homomorphism from group algebras to modules. If V is an RG-module and P is a subgroup of G, then V^P denotes the set of fixed points of P on V, $\mathrm{tr}_I^P : V^I \to V^P$ denotes the trace map, for $I \leq P$, and

$$\overline{V}(P) = V^P/(\Sigma_{I<H}\mathrm{tr}_I^P(V^I) + \pi V^P)$$

is called the *Brauer construction*. Boltije and Külshammer gave a more generalized construction called generalized Brauer construction. Now we recall from the definition of generalized Brauer construction.

Let P be a p-subgroup of G, and let $\varphi : P \to R^\times$ be a homomorphism. We denote by $R_\varphi = R$ the RP-module

$$1_R \cdot g := \varphi(g)1_R$$

We define

$$V^{(P,\varphi)} := \{v \in V | v \cdot g = \varphi(g)v, \forall g \in P\},$$

and a generalized trace map

$$\begin{aligned} tr_{(I,\psi)}^{(P,\varphi)} : V^{(I,\psi)} &\longrightarrow V^{(P,\varphi)} \\ v &\mapsto \Sigma_{h\in P\backslash I}\varphi(h^{-1})v \cdot h, \end{aligned}$$

[0]The detailed version of this paper will be submitted for publication elsewhere.

where $(I, \psi) \leq (P, \varphi)$, i.e. $I \leq P$, $\psi = \varphi_I$, and a generalized Brauer construction

$$\overline{V}(P, \varphi) := V^{(P,\varphi)}/(\Sigma_{(I,\psi)<(P,\varphi)}\mathrm{tr}_{(I,\psi)}^{(P,\varphi)}(V^{(I,\psi)} + \pi V^{(P,\varphi)}).$$

We denote by $\mathrm{Br}_{(P,\varphi)}$ the connonical map from $V^{(P,\varphi)}$ to $\overline{V}(P, \varphi)$.

Note that if φ is the trivial homomorphism, then $V^{(P,\varphi)} = V^P$ and $\overline{V}(P, \varphi)$ is just the Brauer construction. We use V^P and $\overline{V}(P)$ instead of $V^{(P,1)}$ and $\overline{V}(P, 1)$, respectively.

Proposition 1.1 *Let M be a RG-module. Let P be a p-subgroup of G, and let $\varphi : P \to R^\times$ be a homomorphism. Then*
(a) $M^{(P,\varphi)} \cong (M \otimes R_\varphi^*)^P$ as RP-modules.
(b) $\overline{M}(P, \varphi) \cong \overline{(M \otimes R_\varphi^*)}(P)$ as FP-module.

Proof (a) Let $m \in M^{(P,\varphi)}$. Let $1_{R_\varphi^*}$ be the unitary element of R_φ^*. For $g \in P$, we have

$$(m \otimes 1_{R_\varphi^*}) \cdot g = m \cdot g \otimes 1_{R_\varphi^*} g = \varphi(g) m \otimes \varphi(g^{-1}) 1_{R_\varphi^*} = m \otimes 1_{R_\varphi^*}.$$

Then $m \otimes 1_{R_\varphi^*} \in (M \otimes R_\varphi^*)^P$. Thus $f : m \mapsto m \otimes 1_{R_\varphi^*}$ is a map from $M^{(P,\varphi)}$ to $(M \otimes R_\varphi^*)^P$. It is easy to see that f is a one-to-one map from $M^{(P,\varphi)}$ onto $(M \otimes R_\varphi^*)^P$. We only need to verify that f is a RP-module homomorphism. For $g \in P$, we have

$$f(m \cdot g) = mg \otimes 1_{R_\varphi^*},$$

and

$$f(m) \cdot g = (m \otimes 1_{R_\varphi^*}) \cdot g = mg \otimes 1_{R_\varphi^*}.$$

Thus $f(m \cdot g) = f(m) \cdot g$, as desired.

(b) We define a map from $\bar{M}(P, \varphi)$ to $\overline{(M \otimes R_\varphi^*}(P)$ by

$$f : \bar{m} \mapsto \overline{m \otimes 1_\varphi^*}$$

If $\bar{m} = 0$, then $m \in \Sigma_{(I,\psi)<(P,\varphi)}\mathrm{tr}_{(I,\psi)}^{(P,\varphi)}(V^{(I,\psi)} + \pi V^{(P,\varphi)}$. Thus $m \otimes 1_\varphi^* \in \Sigma_{(I,\psi)<(P,\varphi)}\mathrm{tr}_I^P(V^I + \pi V^P$. Thus $\overline{m \otimes 1_\varphi^*} = 0$. So f is a one-to-one map. It is easy to verify that f is surjective. For $g \in G$, $h(\bar{m} \cdot g) = \overline{mg \otimes 1_\varphi^*} = (\overline{m \otimes 1_\varphi^*}) \cdot g = f(\bar{m}) \cdot g$. Thus f is an FP-module isomorphism. Thus we have the following communicative diagraph.

$$\begin{array}{ccc} M^{(P,\varphi)} \ni m & \mapsto & m \otimes 1_{R_\varphi^*} \in (M \otimes R_\varphi^*)^P \\ \downarrow & & \downarrow \\ \bar{M}(P, \varphi) \ni \bar{m} & \overset{h}{\mapsto} & m \bar{\otimes} 1_{R_\varphi^*} \in \overline{(M \otimes R_\varphi^*)}(P) \end{array}$$

2. Monomial Modules

Let V be a G-module. We call V a *monomial module* if V is a finte direct sum of RG-modules of $\mathrm{Ind}_H^G(W)$, where W is a linear RH-module (or a RH-module of R-rank 1) for some subgroup

H of G. We have a more general notion. A RG-module V is called a *p-monomial module* if V_P is a monomial module for any p-subgroup P of G.

Any indecomposable RG-module is associated with three invariants: a defect group (vertex), a source module, and a defect multiplicity module. We know also that these three invariants parametrize an indecomposable module. In this section, we try to parametrize indecomposable p-monomial modules.

Theorem 2.1(Boltje and Külshammer) *Let V be a monomial RG-module. Let P be a p-subgroup of G. Let $\varphi \in \mathrm{Hom}(P, R^{\times})$. Then $\overline{V}(P,\varphi)$ is a monomial $F\overline{N}_G(P,\varphi)$-module. And $\dim_F \overline{V}(P,\varphi)$ is equivalent to the multiplicity of R_φ occurs as a direct summand in V_P.*

Definition 2.2 Let P be a p-group. Let $\varphi : P \to R^{\times}$ be a homomorphism. We call a RP-monomial module M a (P, φ)-monomial if $M \otimes R_\varphi^*$ is a RP-permutation module.

We have the following equivalent description of (P,φ)-monomial modules.

Proposition 2.3 *Let P be a p-group. Let M be a RP-module. Then the following two statements are equivalent*
(a) M is a (P,φ)-monomial module.
(b) M is a finite direct sum of induced modules $\mathrm{Ind}_Q^P(X)$, where $Q \leq P$ and $X \cong (R_\varphi)|_Q$.

Proposition 2.4 *Let M be a p-permutaion RG-module. Set $A = \mathrm{End}_R$ (M). Then there is a natural action of $\overline{A}(P)$ on $\overline{M}(P)$ and this induces an isomorphism of $F\overline{N}_G(P)$-algebras*

$$\overline{A}(P) \cong \mathrm{End}_F(\overline{M}(P)).$$

We have the following generalization of Proposition 2.4.

Lemma 2.5 *Let P be a p-subgroup of G. Let φ be a homomorphism from P to $R^{\times}$. Let M be a RG-module such that M_P is a (P, φ)-monomial module. Let $A = \mathrm{End}_R(M)$. Then*
(a) M is an endo-permutation RP-module.
(b) There is a natural action of $\overline{A}(P)$ on $\overline{M}(P, \varphi)$, and this induces an $F\overline{N}_G(P, \varphi)$-algebra isomorphism

$$\overline{A} \cong \mathrm{End}_F(\overline{M}(P, \varphi)).$$

Now we can state our main theorem of this section.

Theorem 2.6 *Let M be an p-monomial RG-module and let $A = \mathrm{End}_R$ (M). Let P be a p-subgroup of G. Let γ be a local point in A^P. Denote by $\varphi_\gamma : P \to R^{\times}$ the homomorphism given by the RP-module iM, $i \in \gamma$. Then*
(a) There is a natural action of $\overline{A}(P)$ on $\overline{M}(P,\varphi_\gamma)$ and this induces an $F\overline{N}_G(P)$-algebra isomorphism

$$\overline{A}(P) \cong \bigoplus_{\gamma \in LP(A^P)} \mathrm{End}_F(\overline{M}(P, \varphi_\gamma))$$

(b) The multiplicity algebra of γ is isomorphic to $\mathrm{End}_F(\overline{M}(P, \varphi_\gamma))$, and the mulitiplicity module of γ is a module over the ordinary group algebra $F\overline{N}_G(P, \varphi_\gamma)$ and is isomorphic to $\overline{M}(P, \varphi_\gamma)$.

3. Endo-monomial *RP*-modules

We fix a p-group P in this section. A RP-module is called an *endo-monomial* RP-module if $\mathrm{End}_R(M)$ is a monomial module. We have the following basic properties of endo-monomial modules.

Proposition 3.1 *Let P be a finite P-group. Then*
(a) Any monomial RP-module is an endo-monomial RP-module.
(b) Any direct summand of an endo-monomial module is an endo-monomial module.
(c) If M is an endo-monomial RP-module and Q is a subgroup of P, then M_Q is an endo-monomial RQ-module.
(d) If M and N are endo-monomial modules, then M^* and $M \otimes N$ are endo-monomial modules.
(e) If M is an endo-monomial module, then $\Omega(M)$ and $\Omega^{-1}(M)$ are endo-monomial modules.

Lemma 3.2 *Let M be an indecomposable endo-monomial RP-module with vertex P. Then the trival module R is a direct summand of $M \otimes M^*$. Thus we have* $\mathrm{rank}_R(M)$ *is prime to p.*

We have the following crucial result for endo-monomial modules.

Proposition 3.3 *Let L be a RP-monomial module. Let M and N be two indecomposable direct summand of L with common vertex P. Then $M \cong N \otimes R_\varphi$ for some $\varphi \in$* $\mathrm{Hom}(P, R^\times)$.

Proof Since L is an endo-monomial RP-module, $\mathrm{Hom}_R(M, N)$ is an endo-monomial RP-module. We have

$$\mathrm{Hom}_R(M,N) \cong M^* \otimes N \cong \Sigma_{(Q,\varphi)/P} \oplus m_{(Q,\varphi)} \times \mathrm{Ind}_Q^P(R_\varphi)$$

Hence

$$M \otimes M^* \otimes N \cong \Sigma_{(Q,\varphi)/P} \oplus m_{(Q,\varphi)} \times \mathrm{Ind}_Q^P(M|_Q \otimes R_\varphi)$$

Thus any indecomposable direct summand of $M \otimes M^* \otimes N$ with vertex P must isomorphic to $M \otimes R_\varphi$. By Lemma 3.2, N is a direct summand of $M \otimes M^* \otimes N$ with vertex P. Thus $N \cong M \otimes R_\varphi$. As desired.

Corollary 3.4 *Let M and N be indecomposable endo-monomial RP-modules with vertx P. Then $M \oplus N$ is an endo-monomial RP-module if and only if $M \cong N \otimes R_\varphi$ for some $\varphi \in$* $\mathrm{Hom}(P, R^\times)$.

Proposition 3.5 *Let P be a p-group and let Q be a subgroup of P. Let A and B be monomial RQ-algebras. Then $\overline{A \otimes B}(Q) \cong \Sigma_{\varphi \in \mathrm{Hom}(Q,R^\times)} \oplus \overline{A}(Q,\varphi) \otimes_F \overline{B}(Q,\varphi^{-1})$.*

Theorem 3.6 *Let A be a R-simple monomial P-algebra, and let Q be a subgroup of P. Then the F-algebra $\overline{A}(Q)$ is semisimple if it's non-zero.*

Proposition 3.7 *Let A and B be monomial P-algebras. Then $\overline{A \otimes B}(Q) \cong \Sigma_{\varphi \in \mathrm{Hom}(P,R^\times)} \oplus \bar{A}(P,\varphi) \otimes_F \bar{B}(P,\varphi^{-1})$.*

Proposition 3.8 *Let L be an endo-monomial RP-module. Set $A = \mathrm{End}_R(L)$. We assume that $\bar{A}(P) \neq 0$. Then A^P has unique local point if and only if $\bar{A}(P,\varphi) = 0$ for any $1 \neq \varphi \in$* $\mathrm{Hom}(P, R^\times)$.

Proof By assumption, we have $\overline{\mathrm{End}_R(A)}(P)$ is simple. By Proposition 3.7, we have $\overline{\mathrm{End}_R(A)}($

$P) \cong \bar{A}(P) \otimes \bar{A}^{\bullet}(P)$. Thus $\bar{A}(P)$ is simple. Thus A^P has a unique local point, as desired.

Theorem 3.9(Hartmann) *Let P be an abelian p-group. Then endo-monomial RP-modules are endo-permutation RP-modules.*

Proof Let M be an endo-monomial RP-module. Set $A = \mathrm{End}_R(M)$. Then $\bar{A}(P) \neq 0$ and A^P has a unique local point. Thus $\bar{A}(P, \varphi) = 0$ for any $1 \neq \varphi \in \mathrm{Hom}(P, R^{\times})$ by Proposition 3.8. Let Q be a subgroup of P. Then there exists an indecomposable RQ-module W such that $M|_Q \cong nW$. Thus as Q-algebra A is isomorphic to $n^2\mathrm{End}_R(W)$. As above, for any $\varphi \in \mathrm{Hom}(Q, R^{\times})$, we have $\overline{\mathrm{End}_R(W)}(Q, \varphi) = 0$. Thus M is an endo-permutation module.

If A is an R-simple monomial P-algebra, we have $A \cong \mathrm{End}_R(M)$ for some R-module M. Note that A may not have an interior structure, so that M may not be an endo-monomial module. But $\overline{A}(P) \neq 0$ implies this.

Proposition 3.10 *Let A be an endo-monomial P-algebra with $\overline{A}(P) \neq 0$. Then then there exists an interior P-algebra structure on A inducing the given P-algebra structure.*

We define a *generalized Dade P-algebra* to be an R-simple monomial P-algebra such that $\overline{A}(P) \neq 0$. A generalized Dade P-algebra is called *neutral* if $A \cong \mathrm{End}_R(M)$ for some monomial RP-module M. By Proposition 3.5, if A and B are generalized Dade P-algebra, then $A \otimes B$ is a generalized Dade P-algebra. We now define a equivalence relation on the set of all generalized Dade P-algebras. Two generalizde Dade P-algebras A and B are called *similar* if there exist two neutral generalized Dade P-algebras S and T such that $A \otimes S \cong B \otimes T$.

Analogous to the Dade group, the set of equivalent class of generalizde Dade P-algebras has the structure of an abelian group, given by

$$[A_1] + [A_2] := [A_1 \otimes A_2].$$

The class of neutral generalized Dade P-algebras is the identity element. The inverse element of $[A]$ is $[A^{\bullet}]$.

Acknowledgement

This work was supported by JSPS (Janpan Society for Promotion of Science), Grant-in-Aid for Scientific Research (Tokubetsu Kenkyuin Shorei-hi) 01016, 2001. The author wants to pay his hearty thanks to Prof. Koshitani for providing all kinds of help, and also to people around him such as N. Yoshida, K. Ishikawa and F. Tasaka for their generous help.

Reference

1. R. Boltji and B. Külshammer, A generalized Brauer construction and linear source modules. *Trans. Amer. Math. Soc.*352 (2000), no. 7, 3411-3429.

2. R. Hartmann, Endo-monomial modules over p-groups and their classification in the abelian case, Preprint in Benson's homepage.

On the nilpotency index of the radical of a group algebra

Kaoru Motose

Let $t(G)$ be the nilpotency index of the radical $J(KG)$ of a group algebra KG of a finite p-solvable group G over a field K of characteristic $p > 0$. Then it is well known by D. A. R. Wallace [7] that

$$p^e \geq t(G) \geq e(p-1)+1,$$

where p^e is the order of a Sylow p-subgroup of G.

H. Fukushima [1] characterized a group G of p-length 2 satisfying $t(G) = e(p-1)+1$, see also [4]. Unfortunately, his characterization holds under a condition such that the p'-part $V = O_{p',p}(G)/O_p(G)$ of G is abelian.

In this note, using Dickson near fields, we shall give an explicit example (see Example 1) such that a group G of p-length 2 has the non abelian p'-part V and satisfies $t(G) = e(p-1)+1$. This example will be new and have a contributions in our research. Example 2 is also very interesting because quite different objects (see [3] and [5]) are unified on the ground of Dickson near fields.

Let H be a sharply 2-fold transitive group on $\Delta = \{0, 1, \alpha, \beta, \dots, \gamma\}$ (see [8, p.22]), let $V = H_0$ be a stabilizer of 0 and let U be the set consisting of the identity ε and fixed point-free permutations in H. Then U is an elementary abelian p-subgroup of H with the order p^s (see 1). Let σ be a permutation of order p on Δ satisfying conditions

$$\sigma H \sigma^{-1} \subseteq H, \ \sigma^p = 1, \ \sigma(0) = 0, \text{ and } \sigma(1) = 1.$$

Then it is easy to see $\sigma U \sigma^{-1} \subseteq U$ and $\sigma V \sigma^{-1} \subseteq V$. We set $W = <\sigma>$ and $C_V(\sigma) = \{v \in V \mid \sigma v = v\sigma\}$. Assume that there exists a normal subgroup T of WV contained in V such that V is a semi-direct product of T by $C_V(\sigma)$. We set $G = <W, T, U>$.

The final version of this note will be submitted for publication elsewhere. This note was financially supported by the Grant in-Aid for Scientific Research from Japan Society for the Promotion of Science (Subject No. 1164003).

Now, we can prove the following results.

1. *U is a normal and elementary abelian p-subgroup of H and Δ is a near field of characteristic p with respect to the proper sum and product.*

2. *σ is an automorphism of Δ.*

3. *WT is a Frobenius group with kernel T and complement W.*

4. *$G = TC_G(\sigma)T$.*

5. *$(J(KW)\hat{T}KG)^n \subseteq J(KW)^n\hat{T}KG$, where $\hat{T} = \sum_{t\in T} t$.*

A result 1 is well known. We can see from the result 2 and the classification of finite near fields (see [9]) that Δ is a Dickson near field because Δ has an automorphism of order p where p is the characteristic of Δ.

Theorem. *Let S be a subgroup of V containing T and let p^{s+1} be the order of a Sylow p-subgroup WU of $M = < S, W, U >$. Then $t(M) = (s+1)(p-1)+1$.*

We shall present some examples about Theorem.

Example 1. Let (q, n) be a Dickson pair where p is a prime and $q = p^r$ for a positive integer r. Then (q^p, n) is also a Dickson pair because $q^p \equiv -1 \bmod 4$ if and only if $q \equiv -1 \bmod 4$. Let $F = F_{q^{pn}}$ be a finite field of order q^{pn} and Let $D = D_{q^{pn}}$ be a finite Dickson near field defined by the automorphism $\tau : x \to x^{q^p}$ of F. Then an automorphism $\sigma : x \to x^{q^n}$ of F is also of D by [9, Satz 18] or [6, Theorem 5] because $p^{rn} = q^n \equiv 1 \bmod n$ (see also [6, Theorem 1]).

Let ω be a generator of the multiplicative group $F^{\bullet}$ and we set $a = \omega^n$, $b = \omega$ in $F^{\bullet}$. Then the multiplicative group $D^{\bullet}$ of D has the structure

$$D^{\bullet} = < a,\ b \mid a^m = 1,\ b^n = a^t,\ bab^{-1} = a^{q^p} >,$$

where $m = \frac{q^{pn}-1}{n}$ and $t = \frac{m}{q^p-1}$. Here we use the usual symbol as the product in D for simplicity. Do not confuse with the product in F. We consider some permutations on D.

$$u_c : x \to x + c \text{ for } c \in D, \quad v_c : x \to cx \text{ for } c \in D^{\bullet}.$$

Then we have some relations

$$u_c u_d = u_{d+c},\ v_c v_d = v_{cd},\ v_c u_d v_c^{-1} = u_{cd},\ \sigma u_c \sigma^{-1} = u_{\sigma(c)},\ \sigma v_c \sigma^{-1} = v_{\sigma(c)}$$

on u_c, v_c, σ. We set

$$U = \{u_c \mid c \in D\},\ V = \{v_c \mid c \in D^*\},\ W = < \sigma >,$$

and

$$T = \{v_c \in V \mid c \in < a^{\frac{q^n-1}{n}} >\}.$$

It is easy to see that UV is sharply 2-fold transitive on D, T is normal in WV and the order of T is $\frac{q^{pn}-1}{q^n-1}$ because products of a and x in D are the same in F. On the other hand, the set $C_V(\sigma)$ is equal to $F^*_{q^n}$ as a set and the order of $C_V(\sigma)$ is q^n-1. Since $\frac{q^{pn}-1}{q^n-1}$ and q^n-1 are relatively prime, we have $V = C_V(\sigma)T$, $C_V(\sigma) \cap T = \{\varepsilon\}$. Let S be a subgroup of V containing T and $M = < S, W, U >$. Then $t(M) = (rpn+1)(p-1)+1$ by Theorem, where p^{rpn+1} is the order of a Sylow p-subgroup WU of M.

If we put $D = F$ for the extreme case $n = 1$, we have the same example as in [3].

Example 2. If $(q,n) \neq (3,2)$ and p is not a divisor of r, then D_{q^n} has no automorphisms of order p, and so we consider $D_{q^{pn}}$. But D_{3^2} has an automorphism σ of order 3 and we can consider the affine group $G = < \sigma, V, U >$ over D_{3^2} where D_{3^2} is a Dickson near fields defined by an automorphism $x \to x^3$ of $F_{3^2} = F_3[x]/(x^2+1) = \{s + ti \mid i^2 = -1, s, t \in F_3\}$, σ is defined by $\sigma(s+ti) = s+t+ti$, and the permutation group U, V are defined as in Example 1. This group G is isomorphic to $Qd(3)$, namely, a group defined by semi-direct product of $F_3^{(2)}$ by $SL(2,3)$ using the natural action where $F_3^{(2)}$ is 2-dimensional vector space over F_3 and $SL(2,3)$ is the special linear group over $F_3^{(2)}$. In this case 3^3 is the order of a Sylow 3-subgroup of G and it is known form [5] that $t(G) = 9 > 7 = 3(3-1)+1$.

This observation is very interesting because quite different objects (see [3] and [5]) are unified on the ground of Dickson near fields.

References

1. H. Fukushima, On groups G of p-length 2 whose nilpotency indices of $J(KG)$ are $a(p-1)+1$, Hokkaido Math. J., 20(1991), 523-530.

2. K. Motose, On radicals of group rings of Frobenius groups, Hokkaido Math. J., 3(1974), 23-34.

3. K. Motose, On the nilpotency index of the radical of a group algebra. III, J. London Math. Soc., (2) 25(1982), 39-42.

4. K. Motose, On the nilpotency index of the radical of a group algebra. IV, Math. J. Okayama Univ., 25(1983), 35-42.

5. K. Motose, On the nilpotency index of the radical of a group algebra. V, J. Algebra., 90(1984), 251-258.

6. K. Motose, On finite Dickson near fields, Bull. Fac. Sci. Technol. Hirosaki Univ., 2 (2001), 69-78.

7. D.A.R Wallace, Lower bounds for the radical of the group algebra of a finite p-soluble group, Proc. Edinburgh Math. Soc., (2) 16 (1968/69), 127-134.

8. H. Wielandt, Finite permutation groups, Academic Press, 1964.

9. H. Zassenhaus, Über endliche Fastkörper, Abh. Math. Sem. Univ. Hamburg, 11 (1935/36), 187-220.

Kaoru Motose
Department of Mathematical System Science,
Faculty of Science and Technology,
Hirosaki University, Hirosaki 036-8561, Japan
E-mail address: skm@cc.hirosaki-u.ac.jp

DIRECT SUMS OF LIFTING MODULES

YOSUKE KURATOMI

A right R-module M is said to be an *extending* module, if it satisfies the following property: For any submodule X of M, there exists a direct summand of M which contains X as an essential submodule. Dually, M is said to be a *lifting* module, if it satisfies the dual property: For any submodule X of M, there exists a direct summand of M which is a co-essential submodule of X. For these properties, the following problems are fundamental unsolved problems:

Problem A When is a direct sum of extending modules extending ?

Problem B When is a direct sum of lifting modules lifting ?

Problem A is studied in several papers [1], [2], [3], [4] and [5]. In particular, in [2], we introduced a new concept of relative injectivity (that is, generalized relative injectivity) and using this relative injectivity, we showed the following result:

Theorem I. Let M_1 and M_2 be extending modules and put $M = M_1 \oplus M_2$. Then M is extending for $M = M_1 \oplus M_2$ if and only if M_i is generalized M_j-injective $(i \neq j)$.

This theorem seems to be a nice result on Problem A. For Problem B, it is natural to study a dual result for Theorem I. We can naturally define generalized relative projectivity. However, it is not so trivial to give a proof for a dual result of Theorem I. Recently, in [8], Mohamed and Müller tried to give a proof for a dual result. But, they did not succeed. They gave a proof under a certain assumption.

Now, in my paper [7], I gave a proof for a dual result above. This note in an abstract of this my paper.

Our main theorems are the following:

Result 1 Let R be any ring and let M_1 and M_2 be lifting modules and put $M = M_1 \oplus M_2$. Then M is lifting for $M = M_1 \oplus M_2$ if and only if M_i' is generalized M_j-projective $(i \neq j)$ for any direct summand M_i' of M_i.

Result 2 Let R be any ring and let M_1 and M_2 be lifting modules with the finite internal exchange property and put $M = M_1 \oplus M_2$. Then M is a lifting module with the finite internal exchange property if and only if M_i is generalized M_j-projective $(i \neq j)$.

1. Preliminaries

A submodule S of a module M is said to be a *small* submodule, if $M \neq K + S$ for any proper submodule K of M and we write $S \ll M$ in this case. Let M be a module and let N and K be submodules of M with $K \subseteq N$. K is said to be a *co-essential* submodule of N in M if $N/K \ll M/K$ and we write $K \subseteq_c N$ in this case. Let X be a submodule of M. X is called *co-closed* submodule in M if X has not a proper co-essential submodule in M. X' is called *co-closure* of X in M if X' is a co-closed submodule of M with $X' \subseteq_c X$. $K <_\oplus N$ means that K is a direct summand of N. Let $M = M_1 \oplus M_2$ and let $\varphi : M_1 \to M_2$ be a homomorphism. Put $\langle M_1 \overset{\varphi}{\to} M_2 \rangle = \{m_1 - \varphi(m_1) \mid m_1 \in M_1\}$.

The detailed version of this paper has been submitted for publication elsewhere.

Then this is a submodule of M which is called *the graph* with respect to $M_1 \to M_2$. Note that $M = M_1 \oplus M_2 = \langle M_1 \overset{\hat{s}}{\to} M_2 \rangle \oplus M_2$.

A module M has the finite internal exchange property if, for any finite direct sum decomposition $M = M_1 \oplus \cdots \oplus M_n$ and any direct summand X of M, there exists $\overline{M_i} \subseteq M_i$ $(i = 1, \cdots, n)$ such that $M = X \oplus \overline{M_1} \oplus \cdots \oplus \overline{M_n}$.

A module M is said to be a *lifting* module if, for any submodule X, there exists a direct summand X^* of M such that $X^* \subseteq_c X$.

Let $\{M_i \mid i \in I\}$ be a family of modules and let $M = \oplus_I M_i$. M is said to be a *lifting module for* the decomposition $M = \oplus_I M_i$ if, for any submodule X of M, there exist $X^* \subseteq M$ and $\overline{M_i} \subseteq M_i$ $(i \in I)$ such that $X^* \subseteq_c X$ and $M = X^* \oplus (\oplus_I \overline{M_i})$, that is, M is a lifting module and satisfies the internal exchange property in the direct sum $M = \oplus_I M_i$.

2. Generalized Projective

A module A is said to be generalized B-projective (B-cojective) if, for any homomorphism $f : A \to X$ and any epimorphism $g : B \to X$, there exist decompositions $A = A_1 \oplus A_2$, $B = B_1 \oplus B_2$, a homomorphism $h_1 : A_1 \to B_1$ and an epimorphism $h_2 : B_2 \to A_2$ such that $g \circ h_1 = f|_{A_1}$ and $f \circ h_2 = g|_{B_2}$ (cf. [8]). The concept of generalized projective is a dual one of generalized injective (cf. [2]). Note that every B-projective modules is generalized B-projective. A module A is said to be *small B-projective* if, for any epimorphism $g : B \to X$ and any homomorphism $f : A \to X$ with $\mathrm{Im} f \ll X$, there exists a homomorphism $h : A \to B$ such that $g \circ h = f$ (cf. [6]).

Proposition 2.1. (cf. [8]) *Let B^* be a direct summand of B. If A is generalized B-projective, then A is generalized B^*-projective.*

Proposition 2.2. *Let A be a module with the finite internal exchange property and let A^* be a direct summand of A. If A is generalized B-projective, then A^* is generalized B-projective.*

Let X be a submodule of a module M. A submodule Y of M is called a *supplement* of X in M if $M = X + Y$ and $X \cap Y \ll Y$. Note that supplement Y of X in M is co-closed in M. A module M is *weakly supplemented* (*$\oplus$-supplemented*) if, for any submodule X of M, there exists a submodule (direct summand) Y of M such that Y is supplement of X in M. A module M is called *supplemented* if, X contains a supplement of Y in M whenever $M = X + Y$. We note that $\oplus$-supplement modules and supplemented modules are weakly supplemented. Now we consider the following condition:

$(*)$ Any submodule of M has a co-closure in M.

By [9, Proposition 3], we see that any module M over right perfect ring satisfies the condition $(*)$.

Proposition 2.3. (cf. [6]) *A module M is supplemented if and only if M is weakly supplemented with $(*)$.*

By proposition above, we obtain the following:

Proposition 2.4. *Let $M = A \oplus B$ be weakly supplemented with $(*)$ and let A^* be a direct summand of A. If A is generalized B-projective, then A^* is generalized B-projective.*

We do not know whether the proposition above for every module are correct.

3. Main Results

First, we give the following results.

Proposition 3.1. (cf. [8, Theorem 2.8]) *Let M_1 and M_2 be modules and put $M = M_1 \oplus M_2$. If M is lifting for $M = M_1 \oplus M_2$, then M_1 and M_2 are relative generalized projective.*

The following is dual to [2, Theorem 2.15].

Proposition 3.2. *Let $M = \oplus_I M_i$ and let $M_i = M_i' \oplus M_i''$ $(i \in I)$. If M is lifting for $M = \oplus_I M_i$, then $N = \oplus_I M_i'$ is lifting for $N = \oplus_I M_i'$*

Now we give the following without a proof.

Proposition 3.3. *Let M_2 be a lifting module, let M_1 be generalized M_2-projective and put $M = M_1 \oplus M_2$. Then, for any submodule X of M with $M = M_1 + X$, there exist a direct summand X^* of M and a direct summand M_i' of M_i $(i = 1, 2)$ such that $M = X^* \oplus M_1' \oplus M_2'$ and $X^* \subseteq_c X$.*

These results give the following theorem that is one of main results.

Theorem 3.4. *Let M_1 and M_2 be lifting modules and put $M = M_1 \oplus M_2$. Then M is lifting for $M = M_1 \oplus M_2$ if and only if M_i' is generalized M_j-projective for any $M_i' <_\oplus M_i$ $(i \neq j)$.*

Proof. " Only if " part : This is clear from Proposition 3.2 and Proposition 3.1.

" If " part : Let M_1 and M_2 be lifting modules and put $M = M_1 \oplus M_2$. Assume that M_i' is generalized M_j-projective for any direct summand M_i' of M_i $(i \neq j)$. Let X be a submodule of M. Since M_1 is lifting, there exists a decomposition $M_1 = M_1' \oplus M_1''$ such that $M_1' \subseteq_c \pi_{M_1}(X)$. Put $X' = (M_1'' \oplus M_2) \cap X$. Since M_2 is lifting, there exists a decomposition $M_2 = M_2' \oplus M_2''$ such that $M_2' \subseteq_c \pi_{M_2}(X')$. Put $X'' = (M_1'' \oplus M_2'') \cap X$, then we see

$$\pi_{M_i''}(X'') \ll M_i'' \quad (i = 1, 2)$$

and

$$\begin{aligned} M &= \pi_{M_1}(X) + M_1'' + M_2 = X + M_1'' + M_2 \\ &= X + M_1'' + \pi_{M_2}(X') + M_2'' = X + M_1'' + X' + M_2'' \\ &= X + (M_1'' \oplus M_2''). \end{aligned}$$

So we see

$$X'' \subseteq \pi_{M_1''}(X'') \oplus \pi_{M_2''}(X'') \ll M_1'' \oplus M_2''.$$

Now set $K = M_1 \oplus M_2''$ and $L = (X + M_1'') \cap K$. Then $M = K \oplus M_2'$ and $K = M_2'' + L$. Since $M_2' \subseteq \pi_{M_2}(X') \subseteq X + M_1''$, $X + M_1'' = M_2' \oplus L$. Let $\varphi : K \to K/L$ be the canonical epimorphism and put $f = \varphi|_{M_1'}$ and $g = \varphi|_{M_2''}$. As $K = M_2'' + L$, f_2 is an epimorphism. By Lemma 2.1, M_1' is generalized M_2''-projective. Thus there exist decompositions $M_1' = \overline{M_1'} \oplus \overline{\overline{M_1'}}$, $M_2'' = \overline{M_2''} \oplus \overline{\overline{M_2''}}$, a homomorphism $\varphi_1 : \overline{M_1'} \to \overline{\overline{M_2''}}$ and an epimorphism $\varphi_2 : \overline{M_2''} \to \overline{\overline{M_1'}}$ such that $f|_{\overline{M_1'}} = g\varphi_1$ and $g|_{\overline{M_2''}} = f\varphi_2$. Given $x = \overline{m_1'} - \varphi_1(\overline{m_1'}) \in \langle \overline{M_1'} \overset{\varphi_1}{\to} \overline{\overline{M_2''}} \rangle$, then

$$\varphi(x) = \varphi(\overline{m_1'}) - \varphi\varphi_1(\overline{m_1'}) = f(\overline{m_1'}) - g\varphi_1(\overline{m_1'}) = f(\overline{m_1'}) - f(\overline{m_1'}) = 0.$$

So $\langle \overline{M_1'} \xrightarrow{\varphi_1} \overline{\overline{M_2''}} \rangle \subseteq \mathrm{Ker}\varphi = L$. Similarly we get $\langle \overline{M_2''} \xrightarrow{\varphi_2} \overline{\overline{M_1'}} \rangle \subseteq L$, and so $M_1'' \oplus \langle \overline{M_1'} \xrightarrow{\varphi_1} \overline{\overline{M_2''}} \rangle \oplus \langle \overline{M_2''} \xrightarrow{\varphi_2} \overline{\overline{M_1'}} \rangle \subseteq L$. Putting $T = M_1'' \oplus \langle \overline{M_1'} \xrightarrow{\varphi_1} \overline{\overline{M_2''}} \rangle \oplus \langle \overline{M_2''} \xrightarrow{\varphi_2} \overline{\overline{M_1'}} \rangle \oplus M_2'$, we have

$$M = T \oplus \overline{\overline{M_1'}} \oplus \overline{\overline{M_2''}}$$

and

$$T \subseteq M_2' \oplus L = X + M_1''.$$

And we get

$$T = (X + M_1'') \cap T = M_1'' + (T \cap X). \qquad \cdots (*)$$

Now we see $\langle \overline{M_2''} \xrightarrow{\varphi_2} \overline{\overline{M_1'}} \rangle \oplus \overline{\overline{M_1'}} = \langle \overline{M_2''} \xrightarrow{\varphi_2} \overline{\overline{M_1'}} \rangle + \overline{M_2''}$ since φ_2 is an epimorphism, and so

$$M = T \oplus \overline{\overline{M_1'}} \oplus \overline{\overline{M_2''}} = T + \overline{M_2''} + \overline{\overline{M_2''}} = T + M_2'' = (T \cap X) + (M_1'' \oplus M_2'').$$

As $X'' = (M_1'' \oplus M_2'') \cap X \ll M_1'' \oplus M_2''$, we get

$$T \cap X \subseteq_c X.$$

Now define $\alpha : M_1'' \oplus \overline{M_1'} \to \overline{\overline{M_2''}}$, $\beta : M_2' \oplus \overline{M_2''} \to \overline{\overline{M_1'}}$ by $\alpha(m_1'' + \overline{m_1'}) = \varphi_1(\overline{m_1'})$ and $\beta(m_2' + \overline{m_2''}) = \varphi_2(\overline{m_2''})$, respectively. Then

$$T = \langle M_1'' \oplus \overline{M_1'} \xrightarrow{\alpha} \overline{\overline{M_2''}} \rangle \oplus \langle M_2' \oplus \overline{M_1''} \xrightarrow{\beta} \overline{\overline{M_1'}} \rangle.$$

As $(*)$ and $M_1'' \subseteq \langle M_1'' \oplus \overline{M_1'} \xrightarrow{\alpha} \overline{\overline{M_2''}} \rangle$,

$$T = M_1'' + (T \cap X) = \langle M_1'' \oplus \overline{M_1'} \xrightarrow{\alpha} \overline{\overline{M_2''}} \rangle + (T \cap X).$$

By Proposition 3.3, there exist decompositions $M_1'' \oplus \overline{M_1'} = (\overline{M_1'' \oplus \overline{M_1'}}) \oplus (\overline{\overline{M_1'' \oplus \overline{M_1'}}})$ and $M_2' \oplus \overline{M_2''} = (\overline{M_2' \oplus \overline{M_2''}}) \oplus (\overline{\overline{M_2' \oplus \overline{M_2''}}})$ such that $T = T' \oplus \langle \overline{M_1'' \oplus \overline{M_1'}} \xrightarrow{\alpha} \overline{\overline{M_2''}} \rangle \oplus \langle \overline{M_2' \oplus \overline{M_2''}} \xrightarrow{\beta} \overline{\overline{M_1'}} \rangle$ and $T' \subseteq_c T \cap X$. As $T \cap X \subseteq_c X$,

$$T' \subseteq_c X.$$

On the other hand, we see

$$\begin{aligned} M &= T \oplus \overline{\overline{M_1'}} \oplus \overline{\overline{M_2''}} \\ &= T' \oplus \langle \overline{M_1'' \oplus \overline{M_1'}} \xrightarrow{\alpha} \overline{\overline{M_2''}} \rangle \oplus \langle \overline{M_2' \oplus \overline{M_2''}} \xrightarrow{\beta} \overline{\overline{M_1'}} \rangle \oplus \overline{\overline{M_1'}} \oplus \overline{\overline{M_2''}} \\ &= T' \oplus \overline{M_1'' \oplus \overline{M_1'}} \oplus \overline{M_2' \oplus \overline{M_2''}} \oplus \overline{\overline{M_1'}} \oplus \overline{\overline{M_2''}} \\ &= T' \oplus (\overline{M_1'' \oplus \overline{M_1'}} \oplus \overline{\overline{M_1'}}) \oplus (\overline{M_2' \oplus \overline{M_2''}} \oplus \overline{\overline{M_2''}}). \end{aligned}$$

Therefore M is lifting for $M = M_1 \oplus M_2$. □

As immediate consequences of Proposition 2.4 and Theorem 3.4, we obtain the following.

Corollary 3.5. *Let M_1 and M_2 be lifting modules and put $M = M_1 \oplus M_2$. Assume that M satisfies the condition $(*)$. Then M is lifting for $M = M_1 \oplus M_2$ if and only if M_i is generalized M_j-projective $(i \neq j)$.*

The following is immediate from Theorem 3.4 and the proof of [2, Theorem 2.11].

Theorem 3.6. *Let $M_1, \cdots, M_n$ be lifting modules and put $M = M_1 \oplus \cdots \oplus M_n$. Then M is lifting for $M = M_1 \oplus \cdots \oplus M_n$ if and only if M_i' and T are relative generalized projective for any $M_i' <_\oplus M_i$ and any $T <_\oplus (\oplus_{j \neq i} M_j)$.*

As immediate consequences of Theorem 3.6, we obtain the following.

Corollary 3.7. *Let $M_1, \cdots, M_n$ be lifting modules and put $M = M_1 \oplus \cdots \oplus M_n$. If M_i and M_j are relative projective $(i \neq j)$, then M is lifting for $M = M_1 \oplus \cdots \oplus M_n$.*

From [2, Proof of theorem 2.15], Proposition 2.2 and Theorem 3.4, we get the following results.

Theorem 3.8. *Let M_1 and M_2 be lifting modules with the finite internal exchange property and put $M = M_1 \oplus M_2$. Then the following conditions are equivalent.*
(1) *M is lifting with the finite internal exchange property.*
(2) *M is lifting for $M = M_1 \oplus M_2$.*
(3) *M_i is generalized M_j-projective $(i \neq j)$.*

Theorem 3.9. *Let $M_1, \cdots, M_n$ be lifting modules with the finite internal exchange property and put $M = M_1 \oplus \cdots \oplus M_n$. Then the following conditions are equivalent.*
(1) *M is lifting with the finite internal exchange property.*
(2) *M is lifting for $M = M_1 \oplus \cdots \oplus M_n$.*
(3) *M_i and $\oplus_{j \neq i} M_j$ are relative generalized projective.*

As immediate consequences of Theorem 3.9, we obtain the following.

Corollary 3.10. *Let $M_1, \cdots, M_n$ be lifting modules with the finite internal exchange property and put $M = M_1 \oplus \cdots \oplus M_n$. If M_i and M_j are relative projective $(i \neq j)$, then M is lifting with the finite internal exchange property.*

Finally, we can obtain the following.

Theorem 3.11. *Let $M_1, \cdots, M_n$ be hollow modules and put $M = M_1 \oplus \cdots \oplus M_n$. Then M is lifting for $M = M_1 \oplus \cdots \oplus M_n$ if and only if M_i is generalized M_j-projective $(i \neq j)$.*

References

[1] Y. Baba and M. Harada: *On almost M-projectives and almost M-injectives.* Tsukuba J. Math. **14** (1990), 53-69.

[2] K. Hanada, Y. Kuratomi and K. Oshiro: *On direct sums of extending modules and internal exchange property* , Journal of Algebra **250** (2002), 115-133.

[3] M. Harada and K. Oshiro: *On extending property of direct sums of uniform modules*, Osaka J. Math. **7** (1981), 767-785.

[4] A. Harmanci and P.F. Smith: *Finite direct sums of CS-modules.* Houston J. Math. **19** (1993), 523-532.

[5] J. Kado, Y. Kuratomi and K. Oshiro: *CS-property of direct sums of uniform modules*, International Symposium on Ring Theory, Trend Math. (2001), 149-159.

[6] D. Keskin: *On lifting modules.* Comm. Algebra **28** (2000), 3427-3440.

[7] Y. Kuratomi: *On direct sums of extending modules and internal exchange property*, preprint.

[8] S.H. Mohamed and B.J. Müller: *Cojective modules.* preprint.

[9] K. Oshiro: *Semiperfect modules and quasi-semiperfect modules.* Osaka J. Math. **20** (1983), 337-372.

Department of Mathematical Sciences
Yamaguchi University
Yamaguchi 753-8512 JAPAN

Free Fields in Complete Skew Fields and Their Valuations

Katsuo Chiba

Abstract. The main purpose of this paper is to prove the following [4, Theorem 1]: let D be a countable skew field with a countable center C, X a countable set, and let K be a subfield of D which is its own bicentralizer and whose centralizer K' is such that the left K-space KcK' is infinite-dimensional over K, for all $c \in D-\{0\}$. Then (1) if there is a discrete valuation v on D and an element t of K' such that $v(t)>0$, then the completion $\hat{D}$ of D with respect to the topology defined by v contains the free field $D_K(X)$ on the set X, (2) the skew field of Laurent series $D((z))$ in z over D contains $D_K(X)$. The result provides also a new method for constructing valuations of free fields.

A. Lichtman, L. Makar-Limanov, J. Goncalves 等により非可換な斜体の乗法群が非可換な乗法的自由群や自由半群などを含むかどうか，あるいはある中心上無限次元の斜体が自由多元環，自由群環を含むかどうかという問題が研究されている．ここではある条件を満たす斜体が自由体を含むことを示す．自由体とはもともとは Amitsur が論文[1]により斜体の rational identity の研究の過程で発見した自由多元環の universal field of fractions である．自由体は Bergman [2], Cohn [5,6] 等により研究，再構成された．また Cohn による semifir

This is the final version.

等の universal field of fractions は可換環の商環と異なって表現することが難しいが Cohn の matrix localization によると可換環の商環の自然な拡張と考えられ扱いやすい.

命題 1 [5, Corollary 7.5.11]. R を semifir, Σ を R 上の full matrix 全体とすると universal Σ-inverting ring は R の universal field of fractions である.

Cohn に従って, この論文では自由体は次の様に少し一般的に定義する. D を斜体, K を D の部分斜体, X を集合とすると, D-自由環 $D_K<X>$ は semifir であり universal field of fractions をもち, それを $D_K(X)$ と書き自由体と言う, $D=K$ の場合自由体を $K(X)$ と書く. また $D_K<X>$ は semifir だから, 命題 1 より $D_K<X>$ の full matrix 全体 Σ による universal Σ-inverting ring は自由体 $D_K(X)$ である.

主定理を述べる前に次の様に記号を定める. 以下 D を可算斜体, C をその中心で可算集合, X を可算集合, K を D の部分斜体でその bicentralizer が K 自身とする.

定理 1. K の centralizer を K' とし, 任意の $c\in D-\{0\}$ に対して KcK' が無限次元左 K-ベクトル空間とする.

(1)D に離散付値 v があり, $v(K')\neq 0$ とすると, D の v による完備化 $\hat{D}$ は自由体 $D_K(X)$ を含む.

(2)Laurent series からなる斜体 $D((z))$ は自由体 $D_K(X)$ を含む.

定理 1 の証明には次の形の specialization lemma が必要である. これは

Amitsur, Cohn の specialization lemma を一般化したものである.

補題 1[cf. 3, Lemma 6]. D を斜体, C をその中心で無限集合, X を集合, K を D の部分斜体でその bicentralizer が K 自身とし, H を K' の乗法群の非中心的 subnormal subgroup とする. 任意の零でない D の元 c に対して $[KcK':K]=\infty$ とすると, 任意の $D_K<X>$ 上の full matrix は X に適当な H の元を代入すれば D 上の可逆行列になる.

定理 1 の証明

(1), (2)同様な証明であるから, (2)の証明の概略を述べる. D, X は可算だから $D_K<X>$ 上の full matrix も可算である. それを

$$A_1(x_i),\ A_2(x_i),\ A_3(x_i),\ldots\ A_n(x_i),\ldots$$

とする. 補題 1 を使うと, $d_i\in D((z))\ \ (i=1,2,3,\ldots)$ で

$$A_1(d_i),\ A_2(d_i),\ A_3(d_i),\ldots\ A_n(d_i),\ldots$$

が可逆行列になるものが存在することがわかる.

命題 1 より $D_K<X>$ の full matrix 全体 Σ による universal Σ-inverting ring は自由体 $D_K(X)$ であるから $D((z))$ の中で D と $d_i\in D((z))\ \ (i=1,2,3,\ldots)$ で生成された D-field は自由体 $D_K(X)$ と同型になる. (終)

斜体の値群が非可換順序群の付値についての研究は少ない. 定理 1 により, 自由体の付値でその値群が可換とは限らない場合について得られたいくつかの結果を報告する(定理 2, 3). また次の定理 4 により自由体を含む具体的な斜体をあたえる.

定理 2. D の任意の付値は自由体 $D_K(X)$ に拡張できる．もとの付値が可換であれば拡張された付値も可換な付値になっている．

定理 3. k を可換体，$k(X)$を自由体，G を非可換で可算順序群とする．$k(X)$の k-付値でその値群が $Z\times G$ となるものがある．ここで Z は整数全体の加法群で，自然な順序を持ち，$Z\times G$ の順序は辞書式順序とする．

定理 4. k を可換体，σ を k の自己同型で無限位数で k^{σ} が無限集合とする．このとき Laurent series からなる斜体 $k((y;\sigma))$ は k^{σ} 上の自由体 $k^{\sigma}(X)$ を含む．

次に自由体は単純な値群，単純な剰余体の付値を持つことがわかる．

定理 5. k を標数 0 の可換体とする．自由体 $k(X)$は剰余体が k 上の 1 変数関数体の k-離散付値をもつ．

最後に次の問題を提出する．

定理 4 によると Q を有理数体，$k=Q(t)$ を Q 上の一変数関数体 σ を k の自己同型で無限位数とする．例えば $\sigma(t)=t+1$ とする．このとき Laurent series からなる斜体 $k((z;\sigma))$ は Q 上の自由体 $Q(x,y)$ を含む．x，y の $k((z;\sigma))$ の中での具体的な形を求めよ．

References

[1] S. A. Amitsur: Rational identities and applications to algebra and geometry, J.Algebra 3 (1966), 304-359.

[2] G. M. Bergman: Skew fields of noncommutative rational functions, after Amitsur, Seminaire Schutzenberger-Lentin-Nivat, Anne 1969/70, No. 16, Paris, 1970.

[3] K.Chiba: Generalized rational identities of subnormal subgroups of skew fields, Proc. Amer. Math. Soc. 124(1996), 1649-1653.

[4] K. Chiba: Free fields in complete skew fields and their valuations, to appear in J. Algebra.

[5] P.M.Cohn: Free rings and their relations, 2nd ed. Academic Press, 1985.

[6] P. M. Cohn: Skew fields ,Theory of general division rings, Encyclopedia of Mathematics and its Applications, 57, Cambridge University Press , Cambridge, 1995.

Niihama National College of Technology

7-1 Yagumo-cho Niihama 792-8580 Japan

E-mail address: chiba@sci.niihama-nct.ac.jp

EQUIDIMENSIONAL ACTIONS OF ALGEBRAIC TORI ON NORMAL GRADED DOMAINS

HARUHISA NAKAJIMA

ABSTRACT. Let T be an algebraic torus and X an affine conical normal variety over an algebraically closed field K of arbitrary characteristic p. We consider equidimensional and stable regular actions of T on X compatible with the conical structure. Using theory of associated cones (cf. [BK, GM, W2]) and a generalization of R. P. Stanley's criterion (cf. [S, N1]) for a module of relative invariants of finite groups to be free, we show that such actions are almost cofree, i.e., there are finite subgroups N of T such that the actions of T/N on $X//N$ are cofree, especially in the case where $p = 0$.

1. Introduction

In this paper, we suppose that all algebraic varieties are defined over an algebraically closed field K of arbitrary characteristic p. Without specifying, G (resp. T) will always stand for a reductive algebraic group (resp. connected algebraic torus). For an affine variety X, $\mathcal{O}(X)$ denotes the K-algebra of all regular functions on X. When a regular action of G on an affine variety X (abbr. (X, G)) (cf. [GM]) is given, we define $\mathcal{O}(X)^G$ to be the K-subalgebra consisting of all invariants of G in $\mathcal{O}(X)$. An affine variety X is said to be *conical*, if $\mathcal{O}(X)$ is equipped with a positive graduation $\mathcal{O}(X) = \bigoplus_{i \geq 0} \mathcal{O}(X)_i$ over K, and, equivalently, there is a half K^*-action on X with a unique fixed point x_0 satisfying $\lim_{t \to 0} t \cdot x = x_0$ for all $x \in X$. In this case, an action (X, G) is said to be *conical*, if the associated action G preserves the graduation of $\mathcal{O}(X)$. Since $\mathcal{O}(X)^G$ is finitely generated as a K-algebra, we denote by $X//G$ the affine variety associated with $\mathcal{O}(X)^G$, i.e., the algebraic quotient of (X, G) and by $\pi_{X,G}$ the quotient map $X \to X//G$. The action (X, G) is said to be *cofree* (resp. *equidimensional*), if $\mathcal{O}(X)$ is $\mathcal{O}(X)^G$-free (resp. if $\pi_{X,G} : X \to X//G$ is equidimensional). Recall that (X, G) is said to be *stable*, if X contains a non-empty open subset consisting of closed G-orbits. An affine (X, G) is said to be pointed with a base point $x_0 \in X$, if x_0 is G-invariant. In this case, we define the nullcone $\mathcal{N}(X, G)$ to be the affine scheme Spec $(\mathcal{O}(X)/\mathcal{O}(X) \cdot \mathfrak{M}_{x_0}^G)$, where $\mathfrak{M}_{x_0}$ denotes the maximal ideal of x_0.

1991 *Mathematics Subject Classification.* 14L30, 20G05, 13A50, 13B15.

Key words and phrases. invariant theory, algebraic tori, normal domains, divisors.

The author is partially supported by the Grant-in-Aid for Scientific Research (C); the Ministry of Education, Science, Sports and Culture, Japan; Japan Society for the Promotion of Science.

This is an expository paper of the lecture on the recent results of the author.

In [N3], we obtain the two results on affine factorial varieties with actions of algebraic tori under the assumtion that the base fields are of characteristic zero as a generalization of [W1] as follows.

Theorem 1 ([N3]). *Suppose that K is of characteristic zero. Let X be an affine conical factorial variety with a conical stable action of T and let V be a dual space of a minimal homogeneous T-submodule of $\mathcal{O}(X)$ generating $\mathcal{O}(X)$ as a K-algebra. Then the following conditions are equivalent:*

(1) *(X,T) is equidimensional.*
(2) *(X,T) is cofree.*
(3) *(V,T) is cofree.*
(4) *$\mathcal{N}(X,T)$ is a complete intersection and X is defined by T-ivariant polynomial functions on V.*

Let $\mathfrak{X}(G)$ stand for the rational linear character group of G over K which is regarded as an additive group. For any $\chi \in \mathfrak{X}(G)$, set

$$\mathcal{O}(X)_\chi = \{x \in \mathcal{O}(X) \mid \sigma(x) = \chi(\sigma) \cdot x \text{ for any } \sigma \in G\},$$

whose elements are called *semi-invariants* of G relative to χ in $\mathcal{O}(X)$. Clearly $\mathcal{O}(X)_\chi$ is an $\mathcal{O}(X)^G$-module. For a rational G-module U or for an affine G-variety U, set

$$\mathfrak{X}^U(G) = \{\chi \in \mathfrak{X}(G) \mid \mathcal{O}(U)_\chi \neq (0)\},$$

$$\mathfrak{X}_U(G) = \{\chi \in \mathfrak{X}(G) \mid \mathcal{O}(U)_\chi \cdot \mathcal{O}(U)_{-\chi} \neq (0)\}.$$

For a non-pointed X, the next slight modification follows, similarly as in [H], from Theorem 1, Luna's slice theorem and the property of pointed varieties and graded algebras.

Theorem 2 ([N3]). *Suppose that K is of characteristic zero, G is reductive. and X be a smooth affine variety with a regular G-action. If the action of G on X is equidimensional, then, for any point ξ of X/G, $\mathrm{Cl}(\mathcal{O}(X/G)_\xi)$ is isomorphic to a quotient of the abelization of G_x/G_x^o, where x denotes a point in the unique closed orbit in X over ξ.*

The purpose of this lecture is to generalize a part of Theorem 1, which seems to be fundamental in the study of equidimensional actions of non-semisimple reductive algebraic groups, to the case where the variety X is non-factorial normal and the ground field K is of arbitrary characterisitic.

Throughout this paper, let the symbol p denote the characteristic of K and let $\mathbb{Z}_0$ denote the set of non-negative integers.

2. Divisorial modules of semi-invariants

Let $\mathrm{Ht}_1(X)$ denote the set of generic points $\xi \in \mathrm{sp}(X)$, the scheme associated with X, of irreducible closed subvaristies Z of X of codimension one and let $\mathrm{Ht}_1(X,G)$ denote the subset of $\mathrm{Ht}_1(X)$ consisting of ξ such that $\{\pi_{X,G}(\xi)\}^-$ are so in $X//G$. Moreover, when X is normal, let $\mathrm{e}(\xi, \pi_{X,G}(\xi))$ denote the order $v_\xi(f)$ of a zero of a local parameter f of $\pi_{X,G}(\xi)$ along $Z = \{\xi\}^-$, which is called the *reduced ramification index of ξ over $\pi_{X,G}(\xi)$*, where v_ξ stands for the discrete valuation of Y. Denote by $\mathrm{e}_p(\xi, \pi_{X,G}(\xi))$ the p-part of $\mathrm{te}(\xi, \pi_{X,G}(\xi))$ if $p > 0$ and, otherwise, denote $\mathrm{e}(\xi, \pi_{X,G}(\xi))$. For any $\xi \in \mathrm{sp}(X)$, let $D_G(\xi)$ denote the stabilizer of G at ξ and let $I_G(\xi)$ denote the kernel of the canonical homomorphism $D_G(\xi) \to \mathrm{Aut}(k(\xi))$, where $k(\xi)$ denotes the residue class field of $\mathcal{O}_{\mathrm{sp}X,\xi}$. Let $\mathrm{Ht}_1^\infty(X, G)$ denote the set consisting of $\xi \in \mathrm{Ht}_1(X)$ such that $\xi \cap \mathcal{O}(X)^G \neq (0)$.

Proposition 3 ([N4]). *Suppose that G^0 is linearly reductive. Then $I_G(\mathfrak{P})|_X$ is finite for any $\mathfrak{P} \in \mathrm{Ht}^\infty(X, G)$.*

For $n \in \mathbb{N}$, let s be the natural number such that $p^s \| n$ if $p > 0$, or, otherwise, put $s = 0$. Then p^s (resp. $n/p^s \in \mathbb{N}$) is said to be the *p-part* (resp. *p'-part*) of n and $\sharp_{p'}(F)$ denotes the p'-part of the cardinality of a set F. Especially we denote the p'-part $\mathrm{e}(\xi, \pi_{X,H}(\xi))$ by $\mathrm{e}_{p'}(\xi, \pi_{X,H}(\xi))$.

The relation between the reduced remification indices and the inertia groups are studied in [N3].

Theorem 4 ([N4]). *Suppose that G^0 is a torus. Then the following conditions are equivalent:*

(1) *$G = Z_G(G^0)$, or $G/Z_G(G^0)$ is a p-group in the case where $p > 0$.*
(2) *The equalities*

$$\mathrm{e}_{p'}(\xi, \pi_{X,H}(\xi)) = \sharp_{p'}(I_H(\xi)|_X) \qquad (\forall \xi \in \mathrm{Ht}_1(X, H))$$

hold for any closed subgroup H of G containing $Z_G(G^0)$ and for any affine normal variety X with a regular effective stable action of H.

Let us introduce further notations under the circumstances that G^0 is an algebraic torus, $G = Z_G(G^0)$ and the action (X, G) is faithful. For $\mathfrak{p} \in \mathrm{Ht}_1^\infty(X, G)$, we choose $\delta_\mathfrak{p}$ from $\mathfrak{X}(I_G(\mathfrak{p}) \cdot G^0)$ in such a way that

$$< \delta_\mathfrak{p}, \mathfrak{X}^{\mathcal{O}(X)/\mathfrak{p}}(I_G(\mathfrak{p}) \cdot G^0) >= \mathfrak{X}(I_G(\mathfrak{p}) \cdot G^0)$$

and put

$$s_\mathfrak{p}(\chi) = \inf\{r \in \mathbb{Z}_0 \mid \chi|_{I_G(\mathfrak{p})\cdot G^0} \equiv r\delta_\mathfrak{p} \bmod < \mathfrak{X}^{\mathcal{O}(X)/\mathfrak{p}}(I_G(\mathfrak{p}) \cdot G^0) >\},$$

$$D_\chi = \sum_{\mathfrak{p} \in \mathrm{Ht}_1^\infty(X,G)} s_\mathfrak{p}(\chi) \; \mathrm{div}_X(\mathfrak{p}) \in \mathrm{Div}(X).$$

In [N1], we have obtained a criterion $\mathcal{O}(X)_\chi$ to be a free $\mathcal{O}(X)^G$-module of rank one in terms of the special semi-invariant g_χ under the assumption that G is finite over K of arbitrary characteristic. Then, by Theorem 4, we come up with

Theorem 5. *Suppose that G^0 is an algebraic torus and $G = Z_G(G^0)$. Let (X, G) be a stable action of G on an affine normal variety X such that $G \subsetneq \mathrm{Aut}(X)$. Then the following conditions are equivalent for $\chi \in \mathfrak{X}X(G)$:*

(1) *$D_\chi = \mathrm{div}_{\mathbf{R}}(f_\chi)$ for some $f_\chi \in \mathcal{O}(X)_\chi$.*
(2) *$\mathcal{O}(X)_\chi$ is a free $\mathcal{O}(X)^G$-module of rank one.*

This can be extended to in the case where X is factorial and G^0 is linearly reductive.

For a conical affine variety X, we define the *associated cone* of a subset Z of X as follows: If $f \in \mathcal{O}(X)$, let $\mathrm{gr}(f)$ be the leading homogeneous component of f in $\mathcal{O}(X)$ and, if $\mathfrak{I}$ be an ideal of $\mathcal{O}(X)$, let $\mathrm{gr}(\mathfrak{I})$ is the ideal of $\mathcal{O}(X)$ generated by $\{\mathrm{gr}(f) \mid f \in \mathfrak{I}\}$. As a set, the associated cone $\mathcal{C}(S)$ of $Z \subseteq X$ is defined to be the subset of consisting of closed points x on which all functions of $\mathrm{gr}(\mathfrak{I}(S))$ vanish, where $\mathfrak{I}(Z)$ denote the defining ideal of Z in $\mathcal{O}(X)$. If Z is an affine scheme $\mathrm{sp}(Z) = \mathrm{Spec}(\mathcal{O}(X)/\mathfrak{I})$, the schematic structure $\mathrm{sp}(\mathcal{C}(Z)$ on $\mathcal{C}(Z)$ is defined by the ring $\mathcal{O}(X)/\mathrm{gr}(\mathfrak{I})$.

Theorem 6 (W. Borho-H. Kraft [BK]). *Let $G \to GL(V)$ be a finite dimensional representataion of a connected linearly reductive algebraic group G. Suppose that an orbit GP of a point $P \in V$ is semistable. Then we have, as sets,*

$$\overline{K^*GP} \backslash GP = \mathcal{C}(GP) = \overline{K^*GP} \cap \mathcal{N}(V, G)$$

By this and a proof of Theorem 2.5 of [W2], we must have

Lemma 7. *Let (Y, T_1) be a conical action of $T_1 \cong K^*$ on a normal affine conical variety Y. Let Ω be a finite gererating system of $\mathcal{O}(Y)$ as a K-algebra consisting of homogeneous semi-invariants of T_1. Fixing an isomorphism $\nu\colon \mathfrak{X}(T_1) \cong \mathbb{Z}$, we define the subsets; $\Omega_+ = \{x \in \Omega \mid x \in \mathcal{O}(Y)_\chi, \nu(\chi) > 0\}$, $\Omega_- = \{x \in \Omega \mid x \in \mathcal{O}(Y)_\chi, \nu(\chi) < 0\}$. Let $f \in \Omega_+$ and $g \in \Omega_-$ be elements such that $\sqrt{\mathcal{O}(Y)f}$ (resp. $\sqrt{\mathcal{O}(Y)g}$) is maximal in*

$$\{\sqrt{\mathcal{O}(Y)x} \mid x \in \Omega_+ \text{ (resp. } x \in \Omega_-) \}.$$

If (Y, T_1) is equidimensional and stable, then:

(1) *$\mathcal{O}(Y)$ is itntegral over $\mathcal{O}(Y)^{T_1}[f, g]$.*
(2) *If χ is any non-zero rational character of T_1, then there is a $u \in \mathbb{N}$ depending χ such that $(\mathcal{O}(Y)_\chi)^u \subset \mathcal{O}(Y) \cdot f$ or $(\mathcal{O}(Y)_\chi)^u \subset \mathcal{O}(Y) \cdot g$*

Proof (Outline). Since the quotient morphism $\pi_{Y,T_1} : Y \to Y//T_1$ is dominant equidimensional and $Y//T_1$ is normal, the morphism π_{Y,T_1} is open. For a semistable orbit T_1P in $Y \backslash Y^{T_1}$, put

$$U_P = \pi_{Y,T_1}(Y \backslash \overline{K^* \pi_{Y,T_1}^{-1}(\pi_{Y,T_1}(P))}).$$

Then, as $P \notin U_P$ and U_P is open, we have $U_P \cap \overline{K^*P} = \emptyset$. Thus

$$\mathcal{N}(Y, T_1) \subseteq \overline{K^* \pi_{Y,T_1}^{-1}(\pi_{Y,T_1}(P))},$$

and consequently, by Theorem 6, we conclude that, as sets,

$$\mathcal{C}(\pi_{Y,T_1}^{-1}(\pi_{Y,T_1}(P))) = \mathcal{N}(Y, T_1).$$

Let h be an element of Ω_- and suppose $h \notin \sqrt{\mathcal{O}(Y)g}$. Let be a, b, c and d natural neumbers such that $f^a g^b \in \mathcal{O}(Y)^{T_1}$ and $f^c h^d \in \mathcal{O}(Y)^{T_1}$. Put $x := f^a g^b$ and $y := f^c h^d$.

Suppose that $\sqrt{\mathcal{O}(Y)^{T_1}x} \not\subset \sqrt{\mathcal{O}(Y)^{T_1}y}$. Then let $\mathfrak{M}$ be a maximal ideal of $\mathcal{O}(Y)^{T_1}$ satisfying $\mathfrak{M} \not\ni x$, $\mathfrak{M} \ni y$ and let $\mu\colon \mathcal{O}(Y)^{T_1} \to K$ be the K-algebra map associted with $\mathfrak{M}$. Since

$$\mathfrak{M} \cdot \mathcal{O}(Y) \ni h^{a \cdot d}(x^c - \mu(x^c)) - g^{b \cdot c} y^a,$$

we see $\sqrt{\mathcal{O}(Y) \cdot \mathfrak{M}} \ni h$ and, by the equality as above, that $h \in \sqrt{\mathcal{O}(Y) \cdot \mathcal{O}(Y)_+^{T_1}}$.

Suppose $\sqrt{\mathcal{O}(Y)^{T_1}x} \subset \sqrt{\mathcal{O}(Y)^{T_1}y}$. Choose n, m from $\mathbb{N}$ and z from $\mathcal{O}(Y)^{T_1}$ in such a way that $x^n = y^m \cdot z$ and $\sqrt{\mathcal{O}(Y)^{T_1}z} \not\subset \sqrt{\mathcal{O}(Y)^{T_1}x}$. Then

$$f^{a \cdot n} \cdot g^{b \cdot n} = f^{c \cdot m} \cdot h^{d \cdot m} \cdot z.$$

Say $a \cdot n \leq c \cdot m$. Since $g^{b \cdot n} = f^{c \cdot m - a \cdot n} \cdot h^{d \cdot m} \cdot z \in h \cdot \mathcal{O}(Y)$, we have $\sqrt{\mathcal{O}(Y)^{T_1}g} = \sqrt{\mathcal{O}(Y)^{T_1}h}$, which is a contradiction. Thus $a \cdot n - c \cdot m - 1$ is non-negative. Express

$$h^{d \cdot m \cdot a} \cdot z^a = x \cdot f^{a \cdot (a \cdot n - c \cdot m - 1)} \cdot g^{b \cdot (a \cdot n - 1)}$$

and let $\mathfrak{N}$ be a maximal ideal of $\mathcal{O}(Y)^{T_1}$ such that $\mathfrak{N} \not\ni z$, $\mathfrak{N} \ni x$. Let κ be the K-algebra map $\mathcal{O}(Y)^{T_1} \to K$ associated with $\mathfrak{N}$. Then we see

$$\mathfrak{N} \cdot \mathcal{O}(Y) \ni x \cdot f^{a \cdot (a \cdot n - c \cdot m - 1)} \cdot g^{b \cdot (a \cdot n - 1)} - h^{d \cdot m \cdot a} \cdot (z^a - \kappa(z^a)),$$

and it follows that $\sqrt{\mathcal{O}(Y) \cdot \mathcal{O}(Y)_+^{T_1}} \ni h$ from the equality on the associated cone of semistable orbits as above.

We can continue this procedure, and, consequntly, conclude that both Ω_- and Ω_+ are contained in $\sqrt{\mathcal{O}(Y) \cdot (\mathcal{O}(Y)^{T_1}[f, g])_+}$, which implies (1) easily.

Clearly the action (Spm $\mathcal{O}(Y)[f, g]$, T_1) is cofree. Let $\chi \in \mathfrak{X}^X(T)$ and let $v \in \mathcal{O}(Y)_\chi$ be a nonzero element. By (1), we see

$$v^l + w_1 \cdot v^{l-1} + \cdots + w_l = 0$$

for some semi-invariants $w_i \in \mathcal{O}(Y)^{T_1}[f, g]$. Suppose $\nu(\chi) > 0$. For any $\eta \in \mathfrak{X}(T_1)$ such that $\nu(\eta) > 0$, we have

$$(\mathcal{O}(Y)^{T_1}[f, g])_\eta = \mathcal{O}(Y)^{T_1} \cdot f^e \cdot g^t$$

for some $e \in \mathbb{N}$, $t \in \mathbb{Z}_0$. Thus $v^l \in \mathcal{O}(Y) \cdot f$, and then, for a sufficiently large $u \in \mathbb{N}$, we must have $(\mathcal{O}(Y)_\chi)^u \subset \mathcal{O}(Y) \cdot f$. $\square$

3. Stable and equidimensional actions

The action (X,T) is said to be *radially-cofree*, if, for any $\chi \in \mathfrak{X}(T)$ with $\mathcal{O}(X)_\chi \neq \{0\}$, there is a natural number m such that $\mathcal{O}(X)_{n\cdot m\chi}$, $n \in \mathbb{N}$, are free as $\mathcal{O}(X)^T$-modules.

Theorem 8. *Suppose that (X,T) is conical, stable and equidimensional. Then the action (X,T) is radially-cofree.*

Proof (Outline). Let χ be any non-zero linear character of T such that $\mathcal{O}(X)_\chi \neq \{0\}$. We apply the last assertion of Lemma 7 to the conical stable and equidimensional action $(X/\mathrm{Ker}\ \chi,\ T/\mathrm{Ker}\ \chi)$, and then, for any $a \in \mathbb{N}$, we can choose a $u(a) \in \mathbb{N}$ depending on a and a semi-invariant $f \in \mathcal{O}(X)^{\mathrm{Ker}\ \chi}$ of T in such a way that $(\mathcal{O}(X)_{a\chi})^{u(a)} \subset \mathcal{O}(X)^{\mathrm{Ker}\ \chi} \cdot f$. The subgroup $< \chi \mathrm{mod\ Ker}\ \chi >$ is of index p^w $w \in \mathbb{Z}_0$, in the case of $p > 0$, in $\mathfrak{X}(T/\mathrm{Ker}\ \chi) = \mathbb{Z} \cdot \psi$ mod Ker χ for some $\psi \in \mathfrak{X}(T)$. So we may suppose $\chi = v\psi$, where $v = 1$ if $p = 0$, and otherwise $v = p^w$. The element f is regarded as an element of $\mathcal{O}(X)_{s\psi}$ for some $s \in \mathbb{N}$. Let b be any natural number. By Lemma 7, we obtain m, $n \in \mathbb{N}$ and a divisorial integral ideal $\mathfrak{J}$ of $\mathcal{O}(X)$ such that

$$(((\mathcal{O}(X)_{b\cdot s\psi} \cdot \mathcal{O}(X))^\sim)^m)^\sim = f^n \cdot \mathfrak{J}$$

and $\mathfrak{J} \not\subset \sqrt{\mathcal{O}(X)f}$. Since

$$(((\mathcal{O}(X)_{b\cdot s\psi} \cdot \mathcal{O}(X))^\sim)^m)^\sim \subset (\mathcal{O}(X)_{(b\cdot m-n)\cdot s\psi})\mathcal{O}(X))^\sim \cdot f^n,$$

we see $\mathfrak{J} \subset (\mathcal{O}(X)_{(b\cdot m-n)\cdot s\psi} \cdot \mathcal{O}(X))^\sim$. If $b \cdot m > n$, then

$$(\mathcal{O}(X)_{(b\cdot m-n)\cdot s\psi})^{u((b\cdot m-n)\cdot s)} \subset \mathcal{O}(X) \cdot f,$$

and hence $b \cdot m \leq n$ and $(\mathcal{O}(X)_{b\cdot s\chi})^m \cdot \mathcal{O}(X) \subset \mathcal{O}(X) \cdot f^{b\cdot m}$. So we see $(\mathcal{O}(X)_{b\cdot s\psi})^m = f^{b\cdot m} \cdot \mathcal{O}(X)^T$ and, since $\mathcal{O}(X)_{b\cdot s\chi} \ni f^b$ and

$$(((\mathcal{O}(X)_{b\cdot s\psi} \cdot \mathcal{O}(X))^\sim)^m)^\sim = ((\mathcal{O}(X)_{b\cdot s\psi})^m \cdot \mathcal{O}(X))^\sim = \mathcal{O}(X) \cdot f^{b\cdot m},$$

we have $(\mathcal{O}(X)_{b\cdot s\psi}\mathcal{O}(X))^\sim = \mathcal{O}(X) \cdot f^b$. Cosequently we see that $\mathcal{O}(X)_{b\cdot v\cdot s\psi} = \mathcal{O}(X)_{b\cdot s\chi}$ are $\mathcal{O}(X)^T$-free. □

Denote by $\mathcal{R}_X(G)$ the subgroup of G generated by $\cup_{\mathfrak{p}\in \mathrm{Ht}_1(X,G)} I_G(\mathfrak{p})$ say $\mathcal{R}_X(G)$ the *generalized reflection subgroup for* the action (X,G).

Proposition 9 ([N4]). *Suppose that G^0 is linearly reductive and that (X,G) is an effective action on an affine normal X. Set $Y = X//\mathcal{R}_X(G)$. Then:*

(1) *The group $\mathcal{R}_X(G)$ consists of finite members and closed in G.*
(2) $\mathcal{R}_Y(G) = \mathcal{R}_X(G)$.
(3) $I_G(\mathfrak{q})|_Y = \{1\}$ *for* $\mathfrak{q} \in \mathrm{Ht}_1(Y,G)$.

We now study on a conical action of T on an affine normal X. Let Λ denote a generating system $\{f_1, \dots, f_n\}$ of $\mathcal{O}(X)$ consisting of semi-invariants of T and put $\mathrm{supp}_X(f) = \{\mathfrak{p} \in \mathrm{Ht}_1(X,T) \mid \mathrm{v}_\mathfrak{p}(f) > 0\}$. Furtheremore, set

$$\Omega_\Lambda = \{\mathrm{div}_{X//T}(\mathfrak{p} \cap \mathcal{O}(X)^T) \mid \mathfrak{p} \in \cup_i \mathrm{supp}_X(f_i)\},$$

$$\mathrm{Cl}_\Lambda(X//T) = (\mathrm{Div}(X//T) \to \mathrm{Cl}(X//T))(< \Omega_\Lambda >).$$

Then the torsion part of the subgroup $\mathrm{Cl}_\Lambda(X//T)$ of the class group is finite and, if $p = 0$, the p' -part of the order is identified with the order. We say that (X,T) is p-cofree, if there is a subgroup Δ of $\mathfrak{X}(T)$ of index p^n for some $n \in \mathbb{N}$ such that

$$\Delta \subseteq \{\chi \in \mathfrak{X}(T) \mid \mathcal{O}(X)_\chi \cong \mathcal{O}(X)^T\}$$

in the case where $p > 0$. As a matter of convenience, we identify 0-cofreeness (p = 0) with cofreeness.

Now, our main result is

Theorem 10. *Suppose that X is an affine conical normal variety with a conical action of an algebraic torus T such that T can be regarded as a subgroup of* $\mathrm{Aut}(X)$. *If the action (X,T) stable and equidimensional, then we can choose a finite subgroup N of T in such a way that $N \supseteq \mathcal{R}_X(T)$, $\exp(N/\mathcal{R}_X(T)) \leq \sharp_{p'}(\mathrm{tor}(\mathrm{Cl}_\Lambda(X//T))$ and $(X//N, T/N)$ is p-cofree.*

Skech of proof. Let χ be a linear character in $\mathcal{R}_X(T)^\perp \subseteq \mathfrak{X}(G)$ and f be a nonzero homogeneous element in $\mathcal{O}(X)_\chi$. Then, in general,

$$\mathrm{v}_\mathfrak{p}(f) \in \mathrm{e}_{p'}(\mathfrak{p}, \mathfrak{p} \cap \mathcal{O}(X)^T) \cdot \mathbb{Z}_0$$

for any $\mathfrak{p} \in \mathrm{Ht}_1(X,T)$. Since $\mathcal{O}(X)_{s\chi} \cong \mathcal{O}(X)^T$ for some $s \in \mathbb{N}$ (cf. Theorem 8) and $\pi_{X,T} : X \to X//T$ is no-blowing-up of codimension one, there is a pair of effective divisors D_1, D_2 such that

$$D_1 \in \mathbb{Q} \cdot \mathrm{div}_X(\mathcal{Q}(\mathcal{O}(X)^T)^*),$$

$D_2 = D_\chi$ and

$$\mathrm{div}_X(f) = D_1 + D_2.$$

The assertion in this theorem follows from these observations and the Galois descent method of divisor class groups of rings of invariants (cf. [M, N2]). □

Especially in case of $p = 0$, for an effective stable action (X,T), it is equidimensional if and only if there is a finite subgroup N of T such that $(X//N, T/N)$ is cofree.

Finally, we would like to pointed out that Theorem 10, which is regarded as a generalization of Theorem 4.2 of [N3], seems to do not imply Theorem 1 even if K is of characteristic zero, because $|\mathrm{tor}(\mathrm{Cl}_\Lambda(X//T))|$ is not characterized in that theorem.

References

[BK] W. Borho, H. P. Kraft, *Über Bahenen und deren Deformationen bei linearen Aktionen reductiver Gruppen*, Comment. Math. Helvetici **54** (1979), 1-104.

[GM] H. P. Kraft, Geometrische Methoden in der Invariantentheorie, Aspecte der Mathematik D1, Braunschweig-Wiesbaded: Vieweg 1984.

[H] W. H. Hesselink, *Desingularizations of varieties of nullforms*, Invent. Math. **55** (1979), 141-163.

[M] A.R. Magid, *Finite generation of class groups of rings of invariants*, Proc. Amer. Math. Soc. **60**, 45-48 (1976).

[N1] H. Nakajima, *Relative invariants of finite groups*, J. Algebra **79** (1982), 218–234.

[N2] H. Nakajima, *Class groups of localities of rings of invariants of reductive algebraic groups*, Math. Zeit. **182** (1983), 1–15.

[N3] H. Nakajima, *Equidimensional actions of algebraic tori*, Ann. Inst. Fourier **45** (1995), 681–705.

[N4] H. Nakajima, *Reduced ramification indices of quotient morphisms under torus actions*, J. Algebra **242** (2001), 536–549.

[S] R.P. Stanley, *Relative invariants of finite groups generated by pseudo-reflections*, J. Algebra **49** (1977), 134–148.

[W1] D. Wehlau, *A proof of the Popov conjecture for tori*, Proc. of Amer. Math. Soc. **114** (1992), 839-845.

[W2] D. Wehlau, *Equidimensional varieties and associated cones*, J. Algebra (1993), 47-53.

Haruhisa Nakajima
Department of Mathematics
Faculty of Science
Josai University
Keyakidai, Sakado 350-0295
Japan

Extensions and irreducibility of induced characters of some 2-groups

Katsusuke SEKIGUCHI

1. Introduction

For a finite group G , we denote by $\mathrm{Irr}(G)$ the set of complex irreducible characters of G and by $\mathrm{FIrr}(G)$ ($\subset \mathrm{Irr}(G)$) the set of faithful irreducible characters of G.

Let Q_n , D_n , SD_n and C_n denote the generalized quaternion group , the dihedral group of order $2^{n+1}(n \geq 2)$, the semidihedral group of order $2^{n+1}(n \geq 3)$ and the cyclic group of order $2^n(n \geq 0)$,respectively.

As is stated in [4] , these groups have remarkable properties among all 2-groups . Moreover, Yamada and Iida [5] proved the following result:

Let $\mathbf{Q}$ denote the rational field. Let G be a 2-group and χ a complex irreducible character of G. Then there exist subgroups $H \triangleright N$ in G and the complex irreducible character ϕ of H such that $\chi = \phi^G$, $\mathbf{Q}(\chi) = \mathbf{Q}(\phi)$, $N = Ker\phi$ and

$$H/N \cong Q_n \ (n \geq 2), \text{ or } D_n \ (n \geq 3), \text{ or } SD_n \ (n \geq 3), \text{ or } C_n \ (n \geq 0),$$

where $\mathbf{Q}(\chi) = \mathbf{Q}(\chi(g),\ g \in G)$.

In this note, we consider the following problem:

Problem *Let ϕ be a faithful irreducible character of H, where $H = Q_n$ or D_n or SD_n. Determine the 2-group G such that $H \subset G$ and the induced character ϕ^G is also irreducible.*

This prroblem was raised by Yamada and Iida ([4]).

It is well-known that the groups Q_n , D_n and SD_n have faithful irreducible characters. It is also known that they are algebraically conjugate to each other. Hence the irreducibility of ϕ^G , where ϕ is a faithful irreducible character of $H = Q_n$ or D_n or SD_n, does not depend on the particular choice of ϕ, but depends only on these groups.

The detailed version of this paper has been published ([7]).

This problem has been solved in each of the following cases:

(1) When $[G:H]=2$ or 4 ([4]),

(2) When H is a normal subgroup of G ([3]),

for all $H=Q_n$ or D_n or SD_n.

For other results concerning this problem, see [2].

The purpose of this note is to give a complete answer to this problem for all $H=Q_n$ or D_n or SD_n. For details, see [6] and [7].

2. Statements of the results

We use the following notation throught this paper.

- The dihedral group $D_n=\langle a,\ b\rangle$ $(n\geq 2)$ with
  $a^{2^n}=1,\qquad b^2=1,\qquad bab^{-1}=a^{-1}.$
- The generalized quaternion group $Q_n=\langle a,\ b\rangle$ $(n\geq 2)$ with
  $a^{2^n}=1,\qquad b^2=a^{2^{n-1}},\qquad bab^{-1}=a^{-1}.$
- The semidihedral group $SD_n=\langle a,\ b\rangle$ $(n\geq 3)$ with
  $a^{2^n}=1,\qquad b^2=1,\qquad bab^{-1}=a^{-1+2^{n-1}}.$

To state our results, we have to introduce the following groups:

(1) $D(n,m)=\langle a,\ b,\ u_m,\rangle$ $(\triangleright D_n=\langle a,\ b\rangle)$ $(1\leqq m\leqq n-2)$ with
$a^{2^n}=b^2=u_m^{2^m}=1,\ bab^{-1}=a^{-1},\ u_mau_m^{-1}=a^{1+2^{n-m}},$
$u_mb=bu_m.$

(2) $Q(n,m)=\langle a,\ b,\ u_m,\rangle$ $(\triangleright Q_n=\langle a,\ b\rangle)$ $(1\leqq m\leqq n-2)$ with
$a^{2^n}=u_m^{2^m}=1,\ b^2=a^{2^{n-1}},\ bab^{-1}=a^{-1},\ u_mau_m^{-1}=a^{1+2^{n-m}},$
$u_mb=bu_m.$

(3) $D_0(n,1,1)=\langle a,\ b,\ u_1,\ x\rangle$ $(\triangleright D(n,1)=\langle a,\ b,\ u_1\rangle)$ with
$a^{2^n}=b^2=u_1^2=x^2=1,\ bab^{-1}=a^{-1},\ u_1au_1^{-1}=a^{1+2^{n-1}},\ u_1b=bu_1,$
$xax^{-1}=au_1,\ xbx^{-1}=bu_1,\ u_1x=xu_1.$

(4) $Q_0(n,1,1)=\langle a,\ b,\ u_1,\ x\rangle$ $(\triangleright Q(n,1)=\langle a,\ b,\ u_1\rangle)$ with
$a^{2^n}=u_1^2=x^2=1,\ b^2=a^{2^{n-1}},\ bab^{-1}=a^{-1},\ u_1au_1^{-1}=a^{1+2^{n-1}},$
$u_1b=bu_1,\ xax^{-1}=au_1,\ xbx^{-1}=a^{2^{n-1}}bu_1,\ u_1x=xu_1.$

(5) $D(n,m,1) = \langle a,\ b,\ u_m,\ x\rangle$ ($\triangleright D(n,m) = \langle a,\ b,\ u_m\rangle$) ($2 \leqq m \leqq n-3$) with
$a^{2^n} = b^2 = u_m^{2^m} = 1,\ bab^{-1} = a^{-1},\ u_m a u_m^{-1} = a^{1+2^{n-m}}$,
$u_m b = b u_m, xax^{-1} = a^{1+2^{n-m-1}} u_m^{2^{m-1}},\ xbx^{-1} = b u_m^{2^{m-1}},\ x u_m x^{-1} = u_m$,
$x^2 = u_m^{e_m}$, where e_m is an odd integer defined by the relation ,
$(1+2^{n-m})^{e_m} \equiv (1+2^{n-m-1})^2 \pmod{2^n}$.

(6) $Q(n,m,1) = \langle a,\ b,\ u_m,\ x\rangle$ ($\triangleright Q(n,m) = \langle a,\ b,\ u_m\rangle$) ($2 \leqq m \leqq n-3$) with
$a^{2^n} = u_m^{2^m} = 1,\ b^2 = a^{2^{n-1}},\ bab^{-1} = a^{-1},\ u_m a u_m^{-1} = a^{1+2^{n-m}}$,
$u_m b = b u_m, xax^{-1} = a^{1+2^{n-m-1}} u_m^{2^{m-1}},\ xbx^{-1} = b u_m^{2^{m-1}},\ x u_m x^{-1} = u_m$,
$x^2 = u_m^{e_m}$, where e_m is an odd integer defined by the relation ,
$(1+2^{n-m})^{e_m} \equiv (1+2^{n-m-1})^2 \pmod{2^n}$.

REMARK *We can show that the elements $u_m^{e_m}$ defined in* (5) *and* (6) *are uniquely determined, so the groups $D(n,m,1)$ and $Q(n,m,1)$ are uniquely determined for each integers n and m.*

Yamada and Iida ([4]) proved the following

Theorem 0.1 ([4]) (1) *Let $n \geq 4$ and $\phi \in \mathrm{FIrr}(D_n)$. Let G be a 2-group such that $D_n \subset G$ and $[G : D_n] = 2^2$. Suppose that $\phi^G \in \mathrm{Irr}(G)$, Then $G \cong D(n,2)$ or $D_0(n,1,1)$.*

(2) *Let $n \geq 4$ and $\phi \in \mathrm{FIrr}(Q_n)$. Let G be a 2-group such that $Q_n \subset G$ and $[G : Q_n] = 2^2$. Suppose that $\phi^G \in \mathrm{Irr}(G)$, then $G \cong Q(n,2)$ or $Q_0(n,1,1)$.*

(3) *Let $n \geq 4$ and $\phi \in \mathrm{FIrr}(SD_n)$. Let G be a 2-group such that $SD_n \subset G$ and $[G : SD_n] = 2^2$. Suppose that $\phi^G \in \mathrm{Irr}(G)$, Then $G \cong Q(n,2)$ or $Q_0(n,1,1)$ or $D(n,2)$ or $D_0(n,1,1)$.*

REMARK In [4] , they also determined the groups G for the case $[G : H] = 2$, for all $H = Q_n$ or D_n or SD_n.

Further, Iida ([3]) proved the following

Theorem 0.2 ([3]) (1) *Let $\phi \in \mathrm{FIrr}(D_n)$. Let G be a 2-group such that $D_n \subsetneqq G$ and $D_n \lhd G$. Suppose that $\phi^G \in \mathrm{Irr}(G)$, then $G \cong D(n,m)$ for some integer m, $1 \leqq m \leqq n-2$.*

(2) *Let $\phi \in \mathrm{FIrr}(Q_n)$. Let G be a 2-group such that $Q_n \subsetneqq G$ and $Q_n \lhd G$. Suppose that $\phi^G \in \mathrm{Irr}(G)$, then $G \cong Q(n,m)$ for some integer m, $1 \leqq m \leqq n-2$.*

(3) *Let $\phi \in \mathrm{FIrr}(SD_n)$. Let G be a 2-group such that $SD_n \subsetneqq G$ and $SD_n \lhd G$. Suppose that $\phi^G \in \mathrm{Irr}(G)$, then $G \cong Q(n,m)$ or $D(n,m)$ for some integer m, $1 \leqq m \leqq n-2$.*

Our main theorems are the following

Theorem 1 *Let* $\phi \in \mathrm{FIrr}(D_n)$. *Suppose that* G *is a 2-group such that* $D_n \subset G$, $\phi^G \in \mathrm{Irr}(G)$ *and* $[G : D_n] = 2^m$. *Then*

(1) $m \leqq n-2$,

(2) $G \cong D(n,1)$ if $m = 1$.

(3) $G \cong D(n,2)$ *or* $D_0(n,1,1)$ if $m = 2$.

(4) $G \cong D(n,m)$ *or* $D(n,m-1,1)$ if $3 \leqq m \leqq n-2$.

Theorem 2 *Let* $\phi \in \mathrm{FIrr}(Q_n)$. *Suppose that* G *is a 2-group such that* $Q_n \subset G$, $\phi^G \in \mathrm{Irr}(G)$ *and* $[G : Q_n] = 2^m$. *Then*

(1) $m \leqq n-2$,

(2) $G \cong Q(n,1)$ if $m = 1$.

(3) $G \cong Q(n,2)$ *or* $Q_0(n,1,1)$ if $m = 2$.

(4) $G \cong Q(n,m)$ *or* $Q(n,m-1,1)$ if $3 \leqq m \leqq n-2$.

Theorem 3 *Let* $\phi \in \mathrm{FIrr}(SD_n)$. *Suppose that* G *is a 2-group such that* $SD_n \subset G$, $\phi^G \in \mathrm{Irr}(G)$ *and* $[G : SD_n] = 2^m$. *Then*

(1) $m \leqq n-2$,

(2) $G \cong D(n,1)$ *or* $Q(n,1)$ *if* $m = 1$.

(3) $G \cong D(n,2)$ *or* $Q(n,2)$ *or* $D_0(n,1,1)$ *or* $Q_0(n,1,1)$ *if* $m = 2$.

(4) $G \cong D(n,m)$ *or* $Q(n,m)$ *or* $D(n,m-1,1)$ *or* $Q(n,m-1,1)$ *if* $3 \leqq m \leqq n-2$.

3. Sketch of the proof

To prove the theorems, we need some results concerning the criterion of the irreducibility of induced characters.

We denote by $\zeta = \zeta_{2^n}$ a primitive 2^nth root of unity. It is known that, for $H = Q_n$ or D_n, there are $2^{n-1} - 1$ irreducible characters ϕ_ν $(1 \leqq \nu < 2^{n-1})$ of H, which are not linear:

$\phi_\nu(a^i) = \zeta^{\nu i} + \zeta^{-\nu i}, \qquad \phi_\nu(a^i b) = 0 \qquad (1 \leqq i \leqq 2^n).$

For $H = SD_n$, there are $2^{n-1} - 1$ irreducible characters ϕ_ν $(-2^{n-2} \leqq \nu \leqq 2^{n-2}$ for odd $\nu, 1 \leqq \nu < 2^{n-1}$ for even $\nu)$ of H, which are not linear:

$\phi_\nu(a^i) = \zeta^{\nu i} + \zeta^{\nu i(-1+2^{n-1})}, \qquad \phi_\nu(a^i b) = 0 \qquad (1 \leqq i \leqq 2^n).$

Each irreducible character ϕ_ν of Q_n or D_n or SD_n is induced from a linear character η_ν of the maximal normal cyclic subgroup $\langle a \rangle$:

$$\eta_\nu(a^i) = \zeta^{\nu i} \ (1 \leqq i \leqq 2^n).$$

Therefore, for a group $G \supset H = D_n$, or Q_n or SD_n ϕ_ν^G is irreducible if and only if $\eta_\nu^G = (\eta_\nu^H)^G$ is irreducible. For $H = Q_n$ or D_n or SD_n, an irreducible character ϕ_ν of H is faithful if and only if ν is odd. The faithful irreducible characters ϕ_ν of H are algebraically conjugate to each other.

We need the following result (cf [1, p.245])

Proposition 1 *(Criterion for Irreducibility of Induced Characters) Let G be a finite group and H be a subgroup of G. Let ϕ be an irreducible character of H. Then the induced character ϕ^G is irreducible if and only if, $(\phi\ ,\ {}^x\phi\ )_{H\cap {}^xH} = 0$ for all $x \notin H$, where ${}^x\phi$ is the conjugate character of ϕ.*

Using this result, we have the following

Proposition 2 *Let $\langle a\rangle \subset H \subset G$, where $H = D_n$ or Q_n or SD_n and $\langle a\rangle$ is a maximal normal cyclic subgroup of H. Let ϕ be a faithful irreducible character of H. Then the following conditions are equivalent*

(1) *ϕ^G is irreducible.*

(2) *For each $x \in G - \langle a\rangle$, there exists $y \in \langle a\rangle \cap x\langle a\rangle x^{-1}$ such that $xyx^{-1} \neq y$.*

Sketch of the proof of Theorem 1

Let G be a 2-group, satisfying the conditions of Theorem 1. As usual, we denote by $N_G(H)$ the normalizer of H in G for a subgroup H of G.

We define subgroups of G by

$$N_1 = N_G(D_n), \quad \text{and} \quad N_{i+1} = N_G(N_i), \text{ for } \quad i \geq 1,$$

then, of course,

$$D_n \subseteq N_1 \subseteq N_2 \subseteq N_3 \subseteq N_4 \subseteq \cdots \subseteq G.$$

We can show the following claims:

Claim I $\quad N_1 \cong D(n,m)$, *for some integer m*, $1 \leqq m \leqq n-2$ ([3]) .

Claim II $\quad$ Suppose that $N_1 = D(n,m) \subsetneqq G$, then $m \leqq n-3$.

Claim III $\quad$ Suppose that $N_1 = D(n,m) \subsetneqq G$, then $N_2/N_1 = N_2/D(n,m) \cong C_1$.

Claim IV $\quad$ Suppose that $N_1 = D(n,m) \subsetneqq G$. Then,

(1) $N_2 \cong D_0(n,1,1)(\supsetneqq D(n,1))$ if $m = 1$.

(2) $N_2 \cong D(n,m,1)(\supsetneqq D(n,m))$ if $2 \leqq m \leqq n-3$.

Claim V $\quad N_G(N_2) = N_2$.

For the prooofs of Claims II, III, IV and V, see [7].

Proof of Theorem 1 $\quad$ Since G is a 2-group, Claim V means that $G = N_2$. Therefore we have $G = N_1$ or N_2. Hence we can get Theorem 1.

Proofs of Theorems 2 and 3 are essentially the same as that of Theorem 1.

References

[1] C. Curtis and I. Reiner: *Methods of representation theory with applications to finite groups and orders I*. Wiley-Interscience, 1981.

[2] Y.Iida : *Normal extensions and induced characters of 2-groups M_n*. Hokkaido Math. J.30(2001) , 163–176.

[3] Y.Iida: *Normal extensions of a cyclic p-group* . Comm. Algebra 30(2002), 1801–1805.

[4] Y.Iida and T.Yamada: *Extensions and induced characters of quaternion, dihedral and semidihedral groups.* SUT J.Math. 27 (1991), 237–262.

[5] Y.Iida and T.Yamada: *Types of faithful metacyclic 2-groups.* SUT J.Math. 28 (1992), 23–46.

[6] K.Sekiguchi: *Extensions of some 2-groups which preserve the irreducibilities of induced characters.* Osaka J. Math. 37 (2000), 773-788.

[7] K.Sekiguchi: *Extensions and the irreducibilities of induced characters of some 2-groups* . Hokkaido Math. J. 31(2002),79-96.

Department of Civil Engineering
Faculty of Engineering
Kokushikan University
4-28-1 Setagaya Setagaya-Ku
Tokyo154-8515 Japan
e-mail :sekiguch@kokushikan.ac.jp