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Yamanashi University

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２００３年１０月１１日（土）－１３日（月）
弘前大学理工学部

編集:　佐藤眞久 (山梨大学)

２００４年1月
山梨大学

Organizing Committee of The Symposium on Ring Theory and Representation Theory

The Symposium on Ring Theory and Representation Theory has been held annually in Japan and the Proceedings have been published by the organizing committee. The first Symposium was organized in 1968 by H. Tominaga, H. Tachikawa, M. Harada and S. Endo. After their retirement, in 1997, a new committee was organized for managing the Symposium. The present members of the committee are Y. Hirano (Okayama Univ.), Y. Iwanaga (Shinshu Univ.), S. Koshitani (Chiba Univ.) and K. Nishida (Shinshu Univ.).

The Proceedings of each Symposium is edited by program organizer. Anyone who wants these Proceedings should ask to the program organizer of each Symposium or one of the committee members.

The Symposium in 2004 will be held at Shinshu University in Matsumoto for Sep. 3-5, and the program will be arranged by H. Asashiba (Osaka City Univ.).

Concerning several information on ring theory group in Japan containing schedules of meetings and symposiums as well as the addresses of members in the group, you should refer the following homepage, which is arranged by M. Sato (Yamanashi Univ.):

http://fuji.cec.yamanashi.ac.jp/~ring/ (Japanese)

http://fuji.cec.yamanashi.ac.jp/~ring/japan/ (English)

Yasuo Iwanaga
Nagano, Japan
December, 2003

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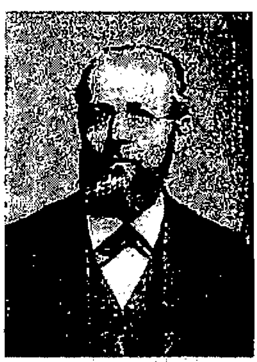
Preface

The year of 2003 is the memorial year in the sense that George Frobenius (1849-1917) published his first paper about so called Frobenius Algebra. In the same year Wright Brother succeeded to fly their air plane Wright Flyer. Also It passed half a century since television programs has been broadcasted in Japan. Comparing the development of ring theory last one century with the development of technology, we must say our development is slower than them. We have great responsibility to develop ring theory in next 100 years faster than any others. We hope this Symposium will play a important roll for this development.

The 36th Symposium on Ring Theory and Representation Theory was held at Hirosaki University on October 11th - 13th, 2003. The symposium and this proceeding are financially supported by Kenji Nishida (Shinshu University) Grant-in-Aid for Scientific Research (B)(1), No.14340007, JSPS. Also Hirosaki University supported the symposium by the request from Professor Kaoru Motose.

We would like to thanks Professors Kenji Nishida, Yasuo Iwanaga, Yasuyuki Hirano and Shigeo Koshitani for their helpful suggestions concerning the symposium. Finally we would like to express our thanks to Professor Kaoru Motose and Katsushi Waki for their great effort and preparation to hold the symposium in Hirosaki University.

Masahisa Sato
Yamanashi
January, 2004



Ferdinand Georg Frobenius (1849-1917)

第 36 回 環論および表現論シンポジウム
プログラム

１０月１１日（土）：

8:50–9:50: 越谷重夫 (千葉大学理学部)
EXAMPLES OF BROUÉ'S ABELIAN DEFECT GROUP CONJECTURE IN REPRESENTATION THEORY OF FINITE GROUPS

10:00–11:00: C.M. Ringel (Bielefeld University)
Foundation of the representation theory of finite dimensional algebras I

11:10–12:10: 伊山 修 (姫路工業大学大学院)
Representation dimension and finitistic dimension conjecture I

13:30–14:15:
第 11 番教室: 吉野雄二 (岡山大学理学部)
A generalization of Zwara's theorem on degeneration of modules
第 12 番教室: 齋藤 睦・W.N. Traves (北海道大学大学院理学研究科・U.S. Naval Academy)
Finite Generations of Rings of Differential Operators of Semigroup Algebras

14:25–15:10:
第 11 番教室: 大貫洋介・佐藤眞久 (東京農工大学工学部・山梨大学工学部)
On degenerations of modules over general rings
第 12 番教室: 倉富要輔・張宰薫 (北九州工業高等専門学校・山口大学理工学研究科)
Lifting modules over right perfect rings

15:30–16:15:
第 11 番教室: Guangming Xie・丸林英俊・小林滋・小松弘明 (鳴門教育大学・岡山県立大学)
Non-Commutative Valuation Rings of $K(X,\sigma,\delta)$ over a Division Ring K
第 12 番教室: 横川賢二 (岡山理科大学)
Non-semisimple Hopf algebra of dimension p (using personal computer)

16:25–17:10:
第 11 番教室: 千葉克夫 (新居浜工業高等専門学校)
Valuations on coproducts of skew fields and free fields
第 12 番教室: 岡竜也 (筑波大学大学院数理物質科学研究科)
遺伝的ホップ代数の特徴づけとスーパーアフィン群への応用
(Characterization of hereditary Hopf algebra and its applications to super affine group)

17:20–18:05:
第 11 番教室: 柳川浩二 (大阪大学大学院理学研究科)
BGG correspondence and Aramova-Herzog's theory on exterior algebra
第 12 番教室: 森田純 (筑波大学数学系)
３次元リー代数の包絡環の分解について、およびその応用
(Factorizations of enveloping algebras of three dimensional Lie algebras and their applications)

１０月１２日（日）：

8:50–9:50: 毛利 出 (University of Toledo)
Some applications of Koszul duality

10:00–11:00: C.M. Ringel (Bielefeld University)
Foundation of the representation theory of finite dimensional algebras II

11:10–12:10: 伊山 修 (姫路工業大学大学院)
Representation dimension and finitistic dimension conjecture II

13:30–14:15:
第 11 番教室: 山形邦夫 (東京農工大学工学部)
ガロア被覆を用いたフロベニウス多元環の構造と表現について
(On structures and representations of Frobenius algebras by using Galois coverings)
第 12 番教室: 高橋亮 (岡山大学大学院自然科学研究科)
On modules of G-dimension zero over non-Gorenstein local rings

14:25-15:10:

第 11 番教室: 星野光男・加藤義明 (筑波大学数学系・筑波大学数学研究科)
Tilting complexes associated with a sequence of idempotents
第 12 番教室: 本瀬香 (弘前大学理工学部)
Let's use cyclotomic polynomials in your lectures for your students

15:30–16:15:
第 11 番教室: 脇克志 (弘前大学理工学部)
On calculations of modular irreducible characters with the help of computers
第 12 番教室: 小池寿俊 (沖縄工業高等専門学校)
Self-duality of quasi-Harada rings and locally distributive rings

16:25–17:10:
第 11 番教室: 稗田吉成 (大阪府立工業高等専門学校)
On $S_R(H)$-blocks II
第 12 番教室: Hisaya Tsutsui・平野康之 (Millersville University・岡山大学理学部)
A pair of rings with common ideals

17:20–17:50:
第 11 番教室: 新堂安孝 ()
Finitely Cogenerated Distributive Modules
第 12 番教室: 植松盛夫 (上武大学経営情報学部)
Global Dimension in Serial left Algebras

18:00–: 懇親会

１０月１３日（月）：

8:50–9:50: Andrzej Nowicki (Nicholus Copernicus University)
Derivations of polynomial rings over a field of characteristic zero

10:00–11:00: C.M. Ringel (Bielefeld University)
Foundation of the representation theory of finite dimensional algebras III

11:10–12:10: 若松隆義 (埼玉大学教育学部)
フロベニウス多元環について
(On Frobenius algebras)

The 36th Symposium On Ring Theory and Representation Theory (2003)

Program

October 11, (Sat.):
8:50–9:50: Shigeo Koshitani (Chiba University)
EXAMPLES OF BROUÉ'S ABELIAN DEFECT GROUP CONJECTURE IN REPRESENTATION THEORY OF FINITE GROUPS
10:00–11:00: C.M. Ringel (Bielefeld University)
Foundation of the representation theory of finite dimensional algebras I
11:10–12:10: Osamu Iyama (Himeji Institute of Technology)
Representation dimension and finitistic dimension conjecture I
13:30–14:15:
Room 11: Yuji Yoshino (Okayama University)
A generalization of Zwara's theorem on degeneration of modules
Room 12: Mutsumi Saito, W.N. Traves (Hokkaido University, U.S. Naval Academy)
Finite Generations of Rings of Differential Operators of Semigroup Algebras
14:25–15:10:
Room 11: Yousuke Onuki, Masahisa Sato (Tokyo Univeristy of Agriculture and Techonology, University of Yamanashi)
On degenerations of modules over general rings
Room 12: Yousuke Kutratomi, Chang Chae Hoon (Kita Kyushu National College of Technology,Yamaguchi University)
Lifting modules over right perfect rings
15:30–16:15:
Room 11: Guangming Xie, Hidetoshi Marubayashi, Shigeru Kobayashi, Hiroaki Komatsu (Naruto Kyouiku University, Okayama Prefectural University)
Non-Commutative Valuation Rings of $K(X,\sigma,\delta)$ over a Division Ring K
Room 12: Kenji Yokogawa (Okayama Science University)
Non-semisimple Hopf algebra of dimension p (using personal computer)
16:25–17:10:
Room 11: Katsuo Chiba (Niihama National College of Technology)
Valuations on coproducts of skew fields and free fields
Room 12: Tatsuya Oka (Tsukuba University)
Characterization of hereditary Hopf algebra and its applications to super affine group
17:20–18:05:
Room 11: Koji Yanagawa (Osaka University)
BGG correspondence and Aramova-Herzog's theory on exterior algebra
Room 12: Jun Morita (Tsukuba University)
Factorizations of enveloping algebras of three dimensional Lie algebras and their applications

October 12, (San.):
8:50–9:50: Izuru Mori (University of Toledo)
Some applications of Koszul duality
10:00–11:10: C.M. Ringel (Bielefeld University)
Foundation of the representation theory of finite dimensional algebras II
11:10–12:10: Osamu Iyama (Himeji Institute of Technology)
Representation dimension and finitistic dimension conjecture II
13:30–14:15:
Room 11: Kunio Yamagata (Tokyo Univeristy of Agriculture and Techonology)
On structures and representations of Frobenius algebras by using Galois coverings
Room 12: Ryou Takahashi (Okayama University)
On modules of G-dimension zero over non-Gorenstein local rings
14:25-15:10:
Room 11: Mitsuo Hoshino, Yoshiaki Kato (Tsukuba University)
Tilting complexes associated with a sequence of idempotents
Room 12: Kaoru Motose (Hirosaki University)
Let's use cyclotomic polynomials in your lectures for your students
15:30–16:15:
Room 11: Katsushi Waki (Hirosaki University)
On calculations of modular irreducible characters with the help of computers
Room 12: Kazutoshi Koike (Okinawa National College of Technology)
Self-duality of quasi-Harada rings and locally distributive rings
16:25–17:10:
Room 11: Yoshinari Hieda (Osaka prefectural National College of Technology)
On $S_R(H)$-blocks II
Room 12: Hisaya Tsutsui, Yasuyuki Hirano (Millersville University, Okayama University)
A pair of rings with common ideals
17:20–17:50:
Room 11: Yasutaka Shindo ()
Finitely Cogenerated Distributive Modules
Room 12: Morio Uematsu (Joubu University)
Global Dimension in Serial left Algebras
18:00–: Banque

October 13, (Mon.):
8:50–9:50: Andrzej Nowicki (Nicholus Copernicus University)
Derivations of polynomial rings over a field of characteristic zero
10:00–11:00: C.M. Ringel (Bielefeld University)
Foundation of the representation theory of finite dimensional algebras III
11:10–12:10: Takayoshi Wakamatsu (Saitama University)
On Frobenius algebras

[illegible]

EXAMPLES OF BROUÉ'S ABELIAN DEFECT GROUP CONJECTURE IN REPRESENTATION THEORY OF FINITE GROUPS

SHIGEO KOSHITANI (越谷重夫)[1]

ABSTRACT. In representation theory of finite groups, there is a well-known and important conjecture due to M. Broué. He conjectures that, for any prime p, if a p-block A of a finite group G has an abelian defect group P, then A and its Brauer corresponding p-block B of $N_G(P)$ are derived equivalent. We survey in this article that Broué's conjecture holds for non-principal 3-blocks A with elementary abelian defect group P of order 9 for several sporadic simple groups.

0. Introduction

I believe nobody would disagree that in modular representation theory of finite groups Richard Brauer (1901–77) was a real pioneer [13]. He actually gave a nice survey talk in early sixties [5]. I guess many people would agree that now in representation theory of finite groups three of the most important problems should be the following. Namely,

Conjecture 1. Alperin's weight conjecture (1986) [3]

Conjecture 2. Dade's conjecture (1990) [15], [16], [17], [18], [19]

Conjecture 3. Broué's abelian defect group conjecture (1988) [6], [7], [8], [9], [26]

These three conjectures have actually origins which had already appeared in the Brauer's problems in [5]. We anyway need some notation and terminology.

1991 *Mathematics Subject Classification.* primary 20C20, 20C05; secondary 20C34, 20C15.

Key words and phrases. block, Broué's abelian defect group conjecture, modular representation theory, finite group, derived equivalence, Rickard equivalence.

This paper will not be submitted for publication anywher else.

[1]The author was in part supported by the Ministry of Education, Culture, Sports, Science and Technology, Grant-in-Aid for JSPS Fellows 01016, 2002–2003; and the JSPS (Japan Society for Promotion of Science), Grant-in-Aid for Scientific Research C(2) 14540009, 2002–2003.

(0.1) Notation and assumption. Let k be an algebraically field of characteristic $p > 0$, G is always a finite group, and modules here mean always finitely generated right modules unless stated otherwise. Let A be a block algebra of the group algebra kG, namely, A is an indecomposable two-sided ideal of kG which is a direct summand of ${}_{kG}kG_{kG}$. Then, it is well-known that up to G-conjugacy a unique p-subgroup P of G is attached to the block algebra A, which is called a *defect group* of A, see for instance [43]. Now, let $H = N_G(P)$. Then, by Brauer's first main theorem, there uniquely exists a block algebra B of kH, which has the same defect group P and A and B correspond each other via so-called the *Brauer homomorphism*. Since many years ago, it has been predicted the following.

(0.2) Question. How do A and B resemble each other?

As a matter of fact, the above three conjectures are all on this question. In this survey note we concentrate on the third conjecture, that is, Broué's abelian defect group conjecture. Then, what is Broué's abelian defect group conjecture? To state it let's move on to the next section.

1. Broué's abelian defect group conjecture

(1.1) Broué's abelian defect group conjecture ([6, 6.2.Question], [8, 4.9.Conjecture], [56, §9.2.4, Conjecture], [62, §5.2]). We use the notation and assumption in (0.1). Moreover, if the defect group P is abelian, then the block algebras A and B should be derived equivalent (Rickard equivalent), that is to say, the categories $D^b(\text{mod-}A)$ and $D^b(\text{mod-}B)$ are equivalent as triangulated categories, where $D^b(\text{mod-}A)$ is a bounded derived category of a category mod-A of a finitely generated right kG-modules in A.

There are several (not many) cases where Broué's abelian defect group conjecture has been checked. For example, the author proves the next with N. Kunugi by using many initiated results which are done by L. Puig, T. Okuyama, T. Okuyama and K. Waki, N. Kunugi, the author and H. Miyachi and the author and N. Kunugi, see [50], [44], [45], [47], [48], [49], [27], [32], [33], [34], [11]. Namely,

(1.2) Theorem (Koshitani-Kunugi [28]. If A is the principal block algebra with elementary abelian defect group P of order 9, then Broué's abelian defect group conjecture is true.

It should be noted that to get the above result (1.2) the classification of finite simple groups is necessary.

For other results of Broué's abelian defect group conjecture, see the following Web Page done by Jeremy Rickard:

`http://www.maths.bris.ac.uk/~majcr/adgc/adgc.html`

2. Broué's abelian defect group conjecture for non-principal 3-blocks

Since Broué's abelian defect group conjecture (1.1) has been checked for principal block algebras A with elementary abelian defect group $P \cong C_3 \times C_3$ of order 9, one of the next objects might be the case where A is a *non-principal* block algebra with the same defect group $P \cong C_3 \times C_3$. On this direction the author with N. Kunugi and K. Waki proves that Broué's abelian defect group conjecture holds in the following special cases. Namely,

(2.1) Theorem (Koshitani, Kunugi and Waki). Let A be a non-principal block algebra of kG with elementary abelian defect group P of order 9. Furthermore, if G is one of the following four sporadic simple groups, then Broué's abelian defect group conjecture is true:

(i) $G = O'N$ (O'Nan simple group)
(ii) $G = HS$ (Higman-Sims simple group)
(iii) $G = He$ (Held simple group)
(iv) $G = Suz$ (sporadic Suzuki group).

Proof. For entire proofs see our papers [**30**] and [**31**].

(2.2) Final remark. In this article, perhaps the author has not been able to give a sufficient explanation. Therefore, the author would like the readers to consult papers listed below, especially, for instance [**8**], [**13**], [**21**], [**26**], [**39**], [**42**], [**46**], [**55**], [**56**], [**57**], [**58**], [**59**], [**60**], [**61**] and [**62**].

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Representation dimension of artin algebras [1] [2]

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ABSTRACT. We will study the resolution dimension of functorially finite subcategories. The subcategories with the resolution dimension zero correspond to ring epimorphisms, and rejective subcategories correspond to surjective ring morphisms. We will study a chain of rejective subcategories to construct modules with endomorphisms rings of finite global dimension. We apply these result to study a function $r_\Lambda : \mathrm{mod}\,\Lambda \to \mathbb{N}_{\geq 0}$ which is a natural extension of Auslander's representation dimension.

We study functorially finite subcategories [AS1] from the viewpoint of its resolution dimension (§1.1). Typical examples of functorially finite subcategories are given by morphisms of rings (§1.2) and by cotilting modules (§1.3). The subcategories of resolution dimension zero is called bireflective [St], and they correspond to ring epimorphisms (§1.4(1)). We introduce a special class of bireflective subcategories called *rejective subcategories* (§1.4(2)), and they correspond to factor algebras. Recently, rejective subcategories played a crucial role in the study of representation-finite orders [I1,2][Ru1,2]. In §2, we study certain chains of rejective subcategories called *rejective chains* (§2.1), which give a method to construct rings of finite global dimension (§2.1.1). Recently, in [I3,4], rejective chains were applied to give positive answer to Solomon's conjecture on zeta functions of orders [S1,2] and the finiteness problem of the representation dimension of artin algebras [A][Xi1] (§2.2). Moreover, we show that rejective chains give quasi-hereditary algebras (§2.4) of Cline-Parshall-Scott [CPS1,2]. This provide us a categorical approach to quasi-hereditary algebras originally suggested by Dlab-Ringel [DR2,3].

The representation dimension of artin algebras Λ was introduced by M. Auslander [A] as a homological invariant to measure how far an artin algebra is from being representation-finite. We introduce a function r_Λ, which is given by the resolution dimension of certain subcategories (§3.1.2). The value $r_\Lambda(\Lambda \oplus D\Lambda)$ equals to Auslander's representation dimension, and would be quite natural in Auslander's philosophy, the homological approach to the representation theory. Our function r_Λ would give us much more information. For example, although $\mathrm{rep.dim}\,\Lambda$ does not distinguish tame hereditary algebras and wild hereditary algebras (§3.2.2(2)), the supremum $|r_\Lambda|$ of r_Λ determines the representation type of hereditary algebras (§3.3.2). This is an application of Rouquier's result [R] on exterior algebras (§3.6.2), which is proved in §4. We show that the value of $r_\Lambda(\Lambda)$ is closely related to the reflexive-finiteness of Λ (§3.2.1).

All results in §1–3 were explained and proved in [I7].

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[2] The detailed version of this paper will be submitted elsewhere.

0.1 Notations In this paper, any module is assumed to be a left module. For a ring Λ, we denote by J_Λ the Jacobson radical of Λ, and by $\mathrm{mod}\,\Lambda$ (resp. $\mathrm{pr}\,\Lambda$) the category of finitely generated (resp. finitely generated projective) Λ-modules.

Let $\mathcal{C}$ be an additive category, $\mathcal{C}(X,Y) := \mathrm{Hom}_{\mathcal{C}}(X,Y)$, and fg the composition of $f \in \mathcal{C}(X,Y)$ and $g \in \mathcal{C}(Y,Z)$. We denote by $J_{\mathcal{C}}$ the Jacobson radical of $\mathcal{C}$, and by $\mathrm{ind}\,\mathcal{C}$ the set of isoclasses of indecomposable objects in $\mathcal{C}$. In §1–§3, *any subcategory is assumed to be full and closed under isomorphisms, direct sums and direct summands.* For a collection S of objects in $\mathcal{C}$, we denote by $\mathrm{add}\,S$ the smallest subcategory of $\mathcal{C}$ containing S, and by $[S]$ the ideal of $\mathcal{C}$ consisting of morphisms which factor through some object in S. We call $X \in \mathcal{C}$ an *additive generator* of $\mathcal{C}$ if $\mathrm{add}\,X = \mathcal{C}$.

A $\mathcal{C}$*-module* is a contravariant additive functors from $\mathcal{C}$ to the category of abelian groups. We denote by $\mathrm{Mod}\,\mathcal{C}$ the category of $\mathcal{C}$-modules, where $(\mathrm{Mod}\,\mathcal{C})(M,M')$ consists of the natural transformations from M to M'. Then $\mathrm{Mod}\,\mathcal{C}$ forms an abelian category. By Yoneda's Lemma, $\mathcal{C}(\ ,X)$ is a projective object in $\mathrm{Mod}\,\mathcal{C}$. We call $M \in \mathrm{Mod}\,\mathcal{C}$ *finitely presented* if there exists an exact sequence $\mathcal{C}(\ ,Y) \to \mathcal{C}(\ ,X) \to M \to 0$. We denote by $\mathrm{mod}\,\mathcal{C}$ the category of finitely presented $\mathcal{C}$-modules.

1 Functorially finite subcategory

1.1 Definition Let Λ be an artin algebra and $\mathcal{C}' \subseteq \mathcal{C}$ subcategories of $\mathrm{mod}\,\Lambda$.

(1)[AS1] We call $f \in \mathcal{C}(Y,X)$ a *right* $\mathcal{C}'$*-approximation* of X if $Y \in \mathcal{C}'$ and $\mathcal{C}(\ ,Y) \xrightarrow{\cdot f} \mathcal{C}(\ ,X) \to 0$ is exact on $\mathcal{C}'$, or equivalently, $\mathcal{C}(\ ,Y) \xrightarrow{\cdot f} [\mathcal{C}'](\ ,X) \to 0$ is exact on $\mathcal{C}$ (§0.1). We call $\mathcal{C}'$ a *contravariantly finite* subcategory of $\mathcal{C}$ if any $X \in \mathcal{C}$ has a right $\mathcal{C}'$-approximation. Dually, a *left* $\mathcal{C}'$*-approximation* and a *covariantly finite* subcategory are defined. We call $\mathcal{C}'$ *functorially finite* if it is contravariantly and covariantly finite.

(2) A *right* $\mathcal{C}'$*-resolution* of $X \in \mathcal{C}$ is a complex $\cdots \to Y_2 \xrightarrow{f_2} Y_1 \xrightarrow{f_1} Y_0 \xrightarrow{f_0} X$ in $\mathcal{C}$ such that $Y_i \in \mathcal{C}'$ and $\cdots \to \mathcal{C}(\ ,Y_2) \xrightarrow{\cdot f_2} \mathcal{C}(\ ,Y_1) \xrightarrow{\cdot f_1} \mathcal{C}(\ ,Y_0) \xrightarrow{\cdot f_0} \mathcal{C}(\ ,X) \to 0$ is exact on $\mathcal{C}'$. We write $\mathcal{C}'\text{-resol.dim}\,X \leq n$ if X has a right $\mathcal{C}'$-resolution with $Y_{n+1} = 0$, write $\mathcal{C}'\text{-resol.dim}\,\mathcal{C} \leq n$ if $\mathcal{C}'\text{-resol.dim}\,X \leq n$ holds for any $X \in \mathcal{C}$.[3] We call them *right resolution dimension*. Dually, we define a *left* $\mathcal{C}'$*-resolution, left resolution dimension,* $\mathcal{C}'^{op}\text{-resol.dim}\,X$ and $\mathcal{C}'^{op}\text{-resol.dim}\,\mathcal{C}^{op}$.

1.1.1 Proposition *For a functorially finite subcategory $\mathcal{C}'$ of $\mathcal{C} := \mathrm{mod}\,\Lambda$, $0 \leq \mathrm{gl.dim}(\mathrm{mod}\,\mathcal{C}') - \mathcal{C}'\text{-resol.dim}\,\mathcal{C} \leq 2$, $0 \leq \mathrm{gl.dim}(\mathrm{mod}\,\mathcal{C}'^{op}) - \mathcal{C}'^{op}\text{-resol.dim}\,\mathcal{C}^{op} \leq 2$ and $\mathrm{gl.dim}(\mathrm{mod}\,\mathcal{C}') = \mathrm{gl.dim}(\mathrm{mod}\,\mathcal{C}'^{op})$ hold.*

1.2 Example Let $\phi : \Lambda \to \Gamma$ be a morphism of artin algebras. We denote by $\phi^* : \mathrm{mod}\,\Gamma \to \mathrm{mod}\,\Lambda$ the natural induced functor. Then ϕ^* has a left adjoint $\Gamma\otimes_\Lambda\ : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Gamma$ and a right adjoint $\mathrm{Hom}_\Lambda(\Gamma,\ ) : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Gamma$.

(1) Define a (Λ,Λ)-module C_ϕ by an exact seqeuence $\Lambda \xrightarrow{\phi} \Gamma \xrightarrow{a} C_\phi \to 0$. We

[3] We borrowed the notation $\mathcal{C}'\text{-resol.dim}\,X$ in [AB]. Sikko [Si] denote $\mathcal{C}'\text{-resol.dim}\,(\mathrm{mod}\,\Lambda)$ by $\mathrm{gl.dim}(\mathcal{C}',\Lambda)$.

have the associated complexes below given by $d_0^+ = d_0^- = \phi$, $d_i^+(x_1 \otimes x_2 \otimes \cdots \otimes x_i) := 1 \otimes a(x_1) \otimes x_2 \otimes \cdots \otimes x_i$ and $d_i^-(x_1 \otimes \cdots \otimes x_{i-1} \otimes x_i) := x_1 \otimes \cdots \otimes x_{i-1} \otimes a(x_i) \otimes 1$ for $i > 0$.

$$\mathbf{X}_\phi^+ : \quad 0 \to \Lambda \xrightarrow{d_0^+} \Gamma \xrightarrow{d_1^+} \Gamma \otimes_\Lambda C_\phi \xrightarrow{d_2^+} \Gamma \otimes_\Lambda C_\phi \otimes_\Lambda C_\phi \xrightarrow{d_3^+} \Gamma \otimes_\Lambda C_\phi \otimes_\Lambda C_\phi \otimes_\Lambda C_\phi \to \cdots$$

$$\mathbf{X}_\phi^- : \quad 0 \to \Lambda \xrightarrow{d_0^-} \Gamma \xrightarrow{d_1^-} C_\phi \otimes_\Lambda \Gamma \xrightarrow{d_2^-} C_\phi \otimes_\Lambda C_\phi \otimes_\Lambda \Gamma \xrightarrow{d_3^-} C_\phi \otimes_\Lambda C_\phi \otimes_\Lambda C_\phi \otimes_\Lambda \Gamma \to \cdots$$

(2) Define a (Γ,Γ)-module D_ϕ by an exact seqeuence $0 \to D_\phi \xrightarrow{b} \Gamma \otimes_\Lambda \Gamma \xrightarrow{c} \Gamma \to 0$, where c is the multiplication map. We have the associated complexes below given by $e_0^+ = e_0^- = c$, $e_i^+(x_0 \otimes x_1 \otimes \cdots \otimes x_i) := b(x_0 x_1) \otimes \cdots \otimes x_i \in \Gamma \otimes_\Lambda \Gamma \otimes_\Gamma D_\phi^{\otimes i-1} = \Gamma \otimes_\Lambda D_\phi^{\otimes i-1}$ and $e_i^-(x_0 \otimes \cdots \otimes x_{i-1} \otimes x_i) := x_0 \otimes \cdots \otimes b(x_{i-1}x_i) \in D_\phi^{\otimes i-1} \otimes_\Gamma \Gamma \otimes_\Lambda \Gamma = D_\phi^{\otimes i-1} \otimes_\Lambda \Gamma$ for $i > 0$.

$$\mathbf{Y}_\phi^+ : \quad \cdots \to \Gamma \otimes_\Lambda D_\phi \otimes_\Gamma D_\phi \otimes_\Gamma D_\phi \xrightarrow{e_3^+} \Gamma \otimes_\Lambda D_\phi \otimes_\Gamma D_\phi \xrightarrow{e_2^+} \Gamma \otimes_\Lambda D_\phi \xrightarrow{e_1^+} \Gamma \otimes_\Lambda \Gamma \xrightarrow{e_0^+} \Gamma \to 0$$

$$\mathbf{Y}_\phi^- : \quad \cdots \to D_\phi \otimes_\Gamma D_\phi \otimes_\Gamma D_\phi \otimes_\Lambda \Gamma \xrightarrow{e_3^-} D_\phi \otimes_\Gamma D_\phi \otimes_\Lambda \Gamma \xrightarrow{e_2^-} D_\phi \otimes_\Lambda \Gamma \xrightarrow{e_1^-} \Gamma \otimes_\Lambda \Gamma \xrightarrow{e_0^-} \Gamma \to 0$$

1.2.1 Theorem *(1)* $\mathcal{X}_\phi := \operatorname{add}\phi^*(\operatorname{mod}\Gamma)$ *is a functorially finite subcategory of* $\operatorname{mod}\Lambda$. *A left and right* $\mathcal{X}_\phi$*-resolutions of* $X \in \operatorname{mod}\Lambda$ *are given by* $\mathbf{X}_\phi^+ \otimes_\Lambda X$ *and* $\operatorname{Hom}_\Lambda(\mathbf{X}_\phi^-, X)$ *respectively.*

(2) $\mathcal{Y}_\phi^+ := \operatorname{add}\operatorname{Hom}_\Lambda(\Gamma, \operatorname{mod}\Lambda)$ *is a covariantly finite subcategory of* $\operatorname{mod}\Gamma$. *A left* $\mathcal{Y}_\phi^+$*-resolutions of* $Y \in \operatorname{mod}\Gamma$ *is given by an exact sequence* $\operatorname{Hom}_\Gamma(\mathbf{Y}_\phi^+, Y)$.

(3) $\mathcal{Y}_\phi^- := \operatorname{add}\Gamma \otimes_\Lambda (\operatorname{mod}\Lambda)$ *is a contravariantly finite subcategory of* $\operatorname{mod}\Gamma$. *A right* $\mathcal{Y}_\phi^-$*-resolutions of* $Y \in \operatorname{mod}\Gamma$ *is given by an exact sequence* $\mathbf{Y}_\phi^- \otimes_\Gamma Y$.

1.3 Example Another well-known example of functorially finite subcategories is given by Auslander-Buchweitz theory [AB] and its application to the cotilting theory in [AR1]. Let Λ be an artin algebra and T a cotilting Λ-module with $\operatorname{id}_\Lambda T \leq n$ [M].

1.3.1 Theorem $\mathcal{X}_T := \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}_\Lambda^i(X,T) = 0$ *for any* $i > 0\}$ *is a functorially finite subcategory of* $\operatorname{mod}\Lambda$ *with* $\mathcal{X}_T$-resol.dim $(\operatorname{mod}\Lambda) = n$, $\mathcal{X}_T^{op}$-resol.dim $(\operatorname{mod}\Lambda^{op}) = \max\{n-2, 0\}$ *and* $n \leq \operatorname{gl.dim}(\operatorname{mod}\mathcal{X}_T) \leq \max\{n, 2\}$.

1.4 Definition Let Λ be an artin algebra and $\mathcal{C}' \subseteq \mathcal{C}$ subcategories of $\operatorname{mod}\Lambda$.

(1) We call $\mathcal{C}'$ a *coreflective* (resp. *reflective*) subcategory of $\mathcal{C}$ if $\mathcal{C}'$-resol.dim $\mathcal{C} = 0$ (resp. $\mathcal{C}'^{op}$-resol.dim $\mathcal{C}^{op} = 0$) holds. This coincides with the usual definition [St], namely $\mathcal{C}'$ is coreflective (resp. reflective) if and only if the inclusion functor $\mathcal{C}' \to \mathcal{C}$ has a right (resp. left) adjoint [HS]. In this case, the counit ϵ^- (resp. unit ϵ^+) gives a right (resp. left) $\mathcal{C}'$-approximation. We call $\mathcal{C}'$ *bireflective* if it is reflective and coreflective.

(2) We call $\mathcal{C}'$ a *right* (resp. *left*) *rejective* subcategory of $\mathcal{C}$ [I2,4] if the following equivalent conditions are satisfied.

(i) Any $X \in \mathcal{C}$ has a monic right (resp. epic left) $\mathcal{C}'$-approximation.

(ii) The inclusion functor $\mathcal{C}' \to \mathcal{C}$ has a right (resp. left) adjoint with a counit ϵ^- (resp. unit ϵ^+) such that ϵ_X^- is monic (resp. ϵ_X^+ is epic) for any $X \in \mathcal{C}$.

We call $\mathcal{C}'$ *rejective* if it is left and right rejective. Any right (resp. left) rejective subcategories are coreflective (resp. reflective), but the converse does not hold in general.

1.4.1 A morphism $\phi : \Lambda \to \Gamma$ of artin algebras is called a *ring epimorphism* if it is epic in the category of rings. This is equivalent to that the functor $\phi^* : \operatorname{mod}\Gamma \to \operatorname{mod}\Lambda$ is full. For example, the inclusion $\begin{pmatrix} k & k \\ 0 & k \end{pmatrix} \subset M_2(k)$ is a ring epimorphism.

Theorem *Let* Λ *be an artin algebra and* $\mathcal{C}$ *a subcategory of* $\operatorname{mod}\Lambda$.

(1) The conditions below are equivalent, and there exists a bijection between rejective subcategories of $\operatorname{mod}\Lambda$ *and factor algebras of* Λ.

(i) $\mathcal{C}$ *is a rejective subcategory of* $\operatorname{mod}\Lambda$.

(ii) $\mathcal{C} = \mathcal{X}_\phi := \phi^*(\operatorname{mod}\Gamma)$ *for a surjective ring morphism* $\phi : \Lambda \to \Gamma$.

(iii) $\mathcal{C}$ *is closed under submodules and factor modules.*

(2) The conditions below are equivalent, and there exists a bijection between bireflective subcategories of $\operatorname{mod}\Lambda$ *and ring epimorphisms from* Λ *to artin algebras (cf. [GD]).*

(i) $\mathcal{C}$ *is a bireflective subcategory of* $\operatorname{mod}\Lambda$.

(ii) $\mathcal{C} = \mathcal{X}_\phi := \phi^*(\operatorname{mod}\Gamma)$ *for a ring epimorphism* $\phi : \Lambda \to \Gamma$ *between artin algebras.*

(iii) $\mathcal{C}$ *is functorially finite, and closed under kernels and cokernels.*

1.5 If $\phi : \Lambda \to \Gamma$ is not a ring epimorphism, it seems to be difficult to study the behavior of $\mathcal{X}_\phi$ in general. But an interesting example is given by radical embeddings [EHIS], which also appeared in [N].

Theorem *Let* $\Lambda \overset{\phi}{\subset} \Gamma$ *be artin algebras with* $J_\Lambda = J_\Gamma$.

(1) $\mathcal{X}_\phi := \operatorname{add}\phi^*(\operatorname{mod}\Gamma)$ *satisfies* $\mathcal{X}_\phi$-resol.dim$(\operatorname{mod}\Lambda) \leq 1$ *and* $\mathcal{X}_\phi^{op}$-resol.dim$(\operatorname{mod}\Lambda^{op}) \leq 1$.

(2) $\mathcal{Y}_\phi^+ := \operatorname{add}\operatorname{Hom}_\Lambda(\Gamma, \operatorname{mod}\Lambda)$ *and* $\mathcal{Y}_\phi^- := \operatorname{add}\Gamma \otimes_\Lambda (\operatorname{mod}\Lambda)$ *coincide with* $\operatorname{mod}\Gamma$.

(3) ϕ^* *induces a full faithful functor* $\mathcal{C}' := \operatorname{mod}\Gamma/[\operatorname{mod}\Gamma/J_\Gamma] \to \mathcal{C} := \operatorname{mod}\Lambda/[\operatorname{mod}\Lambda/J_\Lambda]$, *and* $\mathcal{C}'$ *forms a rejective subcategory of* $\mathcal{C}$.

2 Rejective chain

Let Λ be an artin algebra and $\mathcal{C}' \subseteq \mathcal{C}$ subcategories of $\operatorname{mod}\Lambda$. We call $\mathcal{C}$ *semisimple* if $J_\mathcal{C} = 0$ holds. We call $\mathcal{C}'$ a *cosemisimple* subcategory of $\mathcal{C}$ if $\mathcal{C}/[\mathcal{C}']$ is semisimple, namely, any non-invertible morphism in $\mathcal{C}$ between indecomposable objects factor through an object in $\mathcal{C}'$.

2.1 Definition Let Λ be an artin algebra and $0 = \mathcal{C}_m \subseteq \mathcal{C}_{m-1} \subseteq \cdots \subseteq \mathcal{C}_0 = \mathcal{C}$ a chain of subcategories of $\operatorname{mod}\Lambda$. We call it a *left* (resp. *right*) *rejective chain* of *length* m if $\mathcal{C}_{n+1}$ is a cosemisimple left (resp. right) rejective subcategory of $\mathcal{C}_n$ for any n $(0 \leq n < m)$. In this case, let ϵ_n^+ (resp. ϵ_n^-) be the unit (resp. counit) of the natural inclusion $\mathcal{C}_{n+1} \to \mathcal{C}_n$. We call the chain above Λ-*total* if $\epsilon_{n,X}^+$ (resp. $\epsilon_{n,X}^-$) is epic (resp. monic) in $\operatorname{mod}\Lambda$ for any $X \in \mathcal{C}_n$ and n $(0 \leq n < m)$. The following theorem plays an important role in §3.

2.1.1 Theorem *Let Λ be an artin algebra and $\mathcal{C}$ a subcategory of* $\mathrm{mod}\,\Lambda$ *with* $\#\,\mathrm{ind}\,\mathcal{C} < \infty$. *Assume that $\mathcal{C}$ has a Λ-total left (resp. right) rejective chain of length $m > 0$. Then* $\mathcal{C}$-resol.dim$\,(\mathrm{mod}\,\Lambda) < m$ *(resp.* $\mathcal{C}^{op}$-resol.dim$\,(\mathrm{mod}\,\Lambda^{op}) < m$*) and* gl.dim$(\mathrm{mod}\,\mathcal{C}) \leq m$ *hold.*

2.1.2 Example Let Λ be an artin algebra and $m := \mathrm{LL}(\Lambda)$ the Loewy length of Λ. Put $\mathcal{C}_n := \mathrm{add} \bigoplus_{i=0}^{m-n} \Lambda/J_\Lambda^i$. Then $0 = \mathcal{C}_m \subseteq \mathcal{C}_{m-1} \subseteq \cdots \subseteq \mathcal{C}_0$ gives a Λ-total left rejective chain. Put $\Gamma := \mathrm{End}_\Lambda(\bigoplus_{i=0}^{m} \Lambda/J_\Lambda^i)$. By 2.1.1, we obtain Auslander's result gl.dim $\Gamma \leq m$ [A]. We will show in 2.4.1 that Γ is quasi-hereditary, the result of Dlab-Ringel [DR2].

2.2 Generalizing the construction of the preprojective partition of Auslander-Smalo [AS2], we can prove that certain kind of subcategories has a left (resp. right) rejective chain. The author applied 2.2.1 to prove Solomon's conjecture on zeta functions on orders [I3], and 2.2.2 to prove the finiteness of the representation dimension of artin algebras [I4] (see §3.1.1). See [I5] for their unified approach.

2.2.1 Theorem *Let Λ be an artin algebra. For $n \in \mathbb{N}$, put $\mathcal{C}^{(n)} := \mathrm{add}\{X \in \mathrm{ind}(\mathrm{mod}\,\Lambda) \mid \mathrm{length}_\Lambda X < n\}$. If $\#\,\mathrm{ind}\,\mathcal{C}^{(n)} < \infty$, then $\mathcal{C}^{(n)}$ has a Λ-total left rejective chain and a Λ-total right rejective chain.*

2.2.2 Theorem *Let Λ be an artin algebra and $M_0 := M \in \mathrm{mod}\,\Lambda$. Put $M_{n+1} := M_n J_{\mathrm{End}_\Lambda(M_n)}$ (resp. $M_{n+1} := D((DM_n)J_{\mathrm{End}_\Lambda(DM_n)})$) inductively. Take $m > 0$ such that $M_m = 0$. Then $\mathcal{C}_n := \mathrm{add} \bigoplus_{i=n}^{m} M_i$ gives a Λ-total right (resp. left) rejective chain $0 = \mathcal{C}_m \subseteq \mathcal{C}_{m-1} \subseteq \cdots \subseteq \mathcal{C}_0$.*

2.3 Let us recall quasi-hereditary algebras of Cline-Parshall-Scott [CPS1,2]. A two-sided ideal I of an artin algebra Γ is called *heredity* if $I^2 = I$, $IJ_\Gamma I = 0$ and $I \in \mathrm{pr}\,\Gamma$. This condition is left-right symmetric since the last condition is equivalent to $I \in \mathrm{pr}\,\Gamma^{op}$. An artin algebra Γ is called *quasi-hereditary* if it has a *heredity chain*, which is a chain $0 = I_m \subseteq I_{m-1} \subseteq \cdots \subseteq I_0 = \Gamma$ of ideals of Γ such that I_{n-1}/I_n is a heredity ideal of Γ/I_n for any n $(0 < n \leq m)$. Let us recall the following theorem in [CPS1,2].

2.3.1 Theorem *Let Γ be an artin algebra. If I is a heredity ideal of Γ, then* $0 \leq \mathrm{gl.dim}\,\Gamma - \mathrm{gl.dim}\,\Gamma/I \leq 2$ *holds. Consequently, if Γ is a quasi-hereditary algebra with a chain $0 = I_m \subseteq I_{m-1} \subseteq \cdots \subseteq I_0 = \Gamma$, then* gl.dim $\Gamma \leq 2m - 2$ *holds.*

2.4 There exists a bijection between equivalence classes of Krull-Schmidt categories $\mathcal{C}$ with additive generators M and Morita-equivalence classes of semiperfect rings Γ, which is given by $\mathcal{C} \mapsto \mathcal{C}(M, M)$ and the converse is given by $\Gamma \mapsto \mathrm{pr}\,\Gamma$. In this case, the set of subcategories $\mathcal{C}'$ of $\mathcal{C}$ and the set of idempotent ideals I of Γ correspond bijectively by $\mathcal{C}' \mapsto I := [\mathcal{C}'](M, M)$. In particular, we have a bijection between semisimple rejective subcategories $\mathcal{C}'$ of $\mathcal{C}$ and heredity ideals I of Γ. A heredity chain corresponds to a chain $0 = \mathcal{C}_m \subseteq \mathcal{C}_{m-1} \subseteq \cdots \subseteq \mathcal{C}_0 = \mathcal{C}$ of subcategories of $\mathcal{C}$ such that $\mathcal{C}_n\,/[\mathcal{C}_{n+1}]$ is a semisimple rejective subcategory of $\mathcal{C}/[\mathcal{C}_{n+1}]$ for any n $(0 \leq n < m)$. The following theorem gives a categorical approach to quasi-hereditary algebras.

2.4.1 Theorem *Any Γ-total right (resp. left) rejective chain is a heredity chain.*

3 Representation Dimension

3.1 Definition Let Λ be an artin algebra and $M \in \operatorname{mod}\Lambda$. Put

$$g_\Lambda(M) := \operatorname{gl.dim}\operatorname{End}_\Lambda(M) \text{ and } r_\Lambda(M) := \inf\{g_\Lambda(M \oplus N) \mid N \in \operatorname{mod}\Lambda\}.$$

By 3.1.1 below, r_Λ gives a function $\operatorname{mod}\Lambda \to \mathbb{N}_{\geq 0}$. Put $|r_\Lambda| := \sup\{r_\Lambda(M) \mid M \in \operatorname{mod}\Lambda\}$. We call $\operatorname{rep.dim}\Lambda := r_\Lambda(\Lambda \oplus D\Lambda)$ the *representation dimension* of Λ, introduced by Auslander [A].

3.1.1 Theorem *$r_\Lambda(M) < \infty$ holds for any $M \in \operatorname{mod}\Lambda$.*

PROOF Immediate from 2.1.1 and 2.2.2.∎

3.1.2 The following theorem gives a method to calculate r_Λ [A][EHIS] (cf. 1.1.1).

Theorem *Let Λ be an artin algebra, $M \in \operatorname{mod}\Lambda$ and $\mathcal{C} := \operatorname{add}M$. If $\Lambda \oplus D\Lambda \in \mathcal{C}$, then* $\mathcal{C}$-resol.dim $(\operatorname{mod}\Lambda) = \mathcal{C}^{op}$-resol.dim $(\operatorname{mod}\Lambda^{op}) = \max\{g_\Lambda(M) - 2, 0\}$.

3.2 Immediately, we obtain Auslander's theorem [A] below, which gives the reason why we call $\operatorname{rep.dim}\Lambda$ the representation dimension. Notice that it is easily checked that $\operatorname{rep.dim}\Lambda \leq 1$ and $|r_\Lambda| \leq 1$ occurs only when Λ is semisimple.

Theorem *Let Λ be an artin algebra. Then* $\operatorname{rep.dim}\Lambda \leq 2$ *if and only if* $|r_\Lambda| \leq 2$ *if and only if Λ is representation-finite.*

PROOF Fix $M \in \operatorname{mod}\Lambda$ and assume that $\mathcal{C} := \operatorname{add}M$ satisfies $\Lambda \oplus D\Lambda \in \mathcal{C}$. By 3.1.2, $g_\Lambda(M) \leq 2$ is equivalent to $\mathcal{C}$-resol.dim $(\operatorname{mod}\Lambda) = 0$ which means that, for any $X \in \operatorname{mod}\Lambda$, there exists a morphism $Y \xrightarrow{f} X$ such that $Y \in \mathcal{C}$ and $\operatorname{Hom}_\Lambda(\ ,Y) \xrightarrow{\cdot f} \operatorname{Hom}_\Lambda(\ ,X)$ is an isomorphism on $\mathcal{C}$. This is equivalent to that f is an isomorphism by $\Lambda \in \mathcal{C}$. Thus $g_\Lambda(M) \leq 2$ is equivalent to $\mathcal{C} = \operatorname{mod}\Lambda$. Thus we obtain the assertion.∎

3.2.1 Theorem *If Λ is 1-Gorenstein artin algebra [AR2][FGR], then $r_\Lambda(\Lambda) \leq 2$ if and only if Λ is reflexive-finite, i.e. Λ has only finitely many isoclasses of indecomposable reflexive modules.*

3.2.2 Example Let Λ be an artin algebra with the Loewy length $\operatorname{LL}(\Lambda)$.

(1) $r_\Lambda(\Lambda) \leq g_\Lambda(\bigoplus_{i=0}^{\operatorname{LL}(\Lambda)} \Lambda/J_\Lambda^i) \leq \operatorname{LL}(\Lambda)$ holds by 2.1.2. In particular, $\operatorname{rep.dim}\Lambda \leq \operatorname{LL}(\Lambda)$ holds if Λ is selfinjective [A].

(2) If Λ is hereditary, then one can check that $\operatorname{rep.dim}\Lambda \leq g_\Lambda(\Lambda \oplus D\Lambda) \leq 3$ holds [A]. On the other hand, if $J_\Lambda^2 = 0$, then Λ is stably equivalent to a hereditary algebra [ARS], and $\operatorname{rep.dim}\Lambda \leq 3$ holds by 3.4(2) below [A][X]. This can be proved directly by $\operatorname{rep.dim}\Lambda \leq g_\Lambda(\Lambda \oplus \Lambda/J_\Lambda \oplus D\Lambda) \leq 3$.

(3) As we shall see in 3.7, algebras with representation dimension at most 3 form an interesting class. Several classes of algebras are known to have the representation dimension at most 3 [Xi1][CP][H1,2][BHS]. The following theorem of Erdmann-Holm-Schröer and the author [EHIS] also gives such algebras.

3.2.3 Theorem *Let* $\Lambda \overset{\phi}{\subset} \Gamma$ *be artin algebras. Assume that* Γ *is representation-finite and* J_Λ *is an ideal (resp. left ideal, right ideal) of* Γ. *Then* rep.dim $\Lambda \leq 3$ *(resp.* $r_\Lambda(\Lambda) \leq 3$, $r_\Lambda(D\Lambda) \leq 3$*) holds.*

3.2.4 Example In [EHIS], the representation dimension of a special biserial algebras is shown to be at most 3 by applying 3.2.3. Now let us consider the representation dimension of a clannish algebra A over a field k [CB]. Then there exists a Bäckström $k[[t]]$-order Λ with a hereditary overorder Γ and an ideal I of Γ such that $J_\Lambda = J_\Gamma$ and $A = \Lambda/I$ by [I6;1.3(3)]. Since $B := \Gamma/I$ is a cyclic Nakayama algebra with $J_A = J_B \subset A \subset B$, we obtain that B is representation-finite and rep.dim $A \leq 3$ by 3.2.3.

3.3 Tame-Wild dichotomy and r_Λ By famous Drozd's Theorem [D], finite dimensional algebras over an algebraically closed field is divided into two classes, tame and wild. Any representation-finite algebra is tame. A certain class of wild algebras called *controlled wild* algebras was introduced in [Ha]. Ringel conjectures that any wild algebra is controlled wild. We will prove the following theorem in 3.6.2.

Theorem *If* Λ *is a controlled wild algebra, then* $|r_\Lambda| = \infty$.

3.3.1 Question We conjecture that any wild algebra satifies $|r_\Lambda| = \infty$. This follows from Ringel's conjecture. On the other hand, does any tame algebra satify $|r_\Lambda| < \infty$? As the theorem below shows, this is true for hereditary algebras. Also, it is an interesting question whether any tame algebra satifies rep.dim $\Lambda \leq 3$ or not [BHS]. These questions can be regarded as a part of the study of tame algebras in terms of endomorphism rings.

3.3.2 Theorem *Let* Λ *be a finite dimensional hereditary algebra. Then the value of* $|r_\Lambda|$ *is given as follows.*

associated valued quiver [DR1]	Dynkin	extended Dynkin	else
$\lvert r_\Lambda \rvert$	≤ 2	3	∞

3.4 Let Λ and Γ be artin algebras. We say that Λ is *finitely equivalent* to Γ if there exists subcategories $\mathcal{X}$ and $\mathcal{X}'$ of mod Λ and mod Γ respectively such that $\#\,\mathrm{ind}\,\mathcal{X} < \infty$, $\#\,\mathrm{ind}\,\mathcal{X}' < \infty$ and mod $\Lambda/[\mathcal{X}]$ is equivalent to mod $\Gamma/[\mathcal{X}']$. Especially, when $\mathcal{X} = \mathrm{pr}\,\Lambda$ and $\mathcal{X}' = \mathrm{pr}\,\Gamma$, we say that Λ is *stably equivalent* to Γ [ARS]. Xiangqian's result (2) below is proved by an application of Auslander-Reiten theory. We need relative homological algebra of Auslander-Solberg to prove (1).

Theorem *Assume that* Λ *and* Γ *are not representation-finite.*
(1) If Λ *is finitely equivalent to* Γ, *then* $|r_\Lambda| = |r_\Gamma|$.
(2)[X] If Λ *is stably equivalent to* Γ, *then* rep.dim Λ = rep.dim Γ.

3.5 Definition We shall introduce two homological invariant of Λ which is closely related to the function r_Λ. Let Λ and Γ be artin algebras.

(1) We write $\Lambda \preceq \Gamma$ if there exists $P \in \mathrm{pr}\,\Gamma$ such that $\mathrm{End}_\Gamma(P)$ is Morita-equivalent to Λ. Obviously, $\preceq$ gives a partial order on the set of Morita-equivalence classes of artin algebras. Define the *expanded dimension* of Λ by $\mathrm{exp.dim}\,\Lambda := \inf\{\mathrm{gl.dim}\,\Gamma \mid \Lambda \preceq \Gamma\}$. This concept first appeared in Auslander's observation in [A] such that $\mathrm{exp.dim}\,\Lambda < \infty$ by 2.1.2. For $n = 0, 1$, $\mathrm{exp.dim}\,\Lambda = n$ if and only if $r_\Lambda(\Lambda) = n$ if and only if $\mathrm{gl.dim}\,\Lambda = n$.

(2) Let $\mathcal{C}$ be a subcategory of $\mathrm{mod}\,\Lambda$. We define the *weak resolution dimension* $\mathrm{wresol.dim}\,\mathcal{C}$ as the minimal number $n \geq 0$ which satisfies equivalent conditions below (cf. §1.1). ($\mathrm{wresol.dim}(\mathrm{mod}\,\Lambda) + 2$ coincides with $\mathrm{rwrep.dim}\,\Lambda$ in [R].)

(i) There exists $M \in \mathrm{mod}\,\Lambda$ such that, for any $X \in \mathcal{C}$, there exists an exact sequence $0 \to M_n \to \cdots \to M_0 \to Y \to 0$ with $M_i \in \mathrm{add}\,M$, $Y \in \mathcal{C}$ and $X \in \mathrm{add}\,Y$.

(ii) There exists $M \in \mathrm{mod}\,\Lambda$ such that, for any $X \in \mathcal{C}$, there exists an exact sequence $0 \to Y \to M_0 \to \cdots \to M_n \to 0$ with $M_i \in \mathrm{add}\,M$, $Y \in \mathcal{C}$ and $X \in \mathrm{add}\,Y$.

3.5.1 Let Λ be an artin algebra. One can easily check the following facts:

(1) $\mathrm{exp.dim}\,\mathrm{End}_\Lambda(X) \leq r_\Lambda(X)$ holds for any $X \in \mathrm{mod}\,\Lambda$.

(2) $\mathrm{wresol.dim}(\mathrm{mod}\,\Lambda) \leq \mathrm{exp.dim}\,\Lambda \leq r_\Lambda(\Lambda) \leq \min\{\mathrm{gl.dim}\,\Lambda, \mathrm{rep.dim}\,\Lambda, \mathrm{LL}(\Lambda)\}$.

(3) $\mathrm{wresol.dim}\,\mathrm{add}\,\Omega^2(\mathrm{mod}\,\Lambda) \leq \max\{\mathrm{exp.dim}\,\Lambda - 2, 0\}$.

(4) If a morphism $\phi : \Lambda \to \Gamma$ of artin algebras satisfies $\mathrm{add}\,\phi^*(\mathrm{mod}\,\Gamma) = \mathrm{mod}\,\Lambda$, then $\mathrm{wresol.dim}(\mathrm{mod}\,\Lambda) \leq \mathrm{wresol.dim}(\mathrm{mod}\,\Gamma)$.

3.6 Rouquier's theorem below [R;6.10,6.9] gave the first example of algebras with representation dimension greater than 3. It follows from 4.2 and 4.3 below, which will be proved in §4.6. Consequently, $\mathrm{exp.dim}\,\Lambda$ and $r_\Lambda(\Lambda)$ is also $n+1$ by 3.5.1(2)(3).

Theorem *Let k be a field and $\Lambda = \bigwedge(k^n)$ the exterior algebra with $n > 0$. Then* $\mathrm{wresol.dim}(\mathrm{mod}\,\Lambda) + 2 = \mathrm{rep.dim}\,\Lambda = n + 1$.

3.6.1 Example We can obtain many artin algebras with large representation dimension by using 3.6. Again let $\Lambda = \bigwedge(k^n)$ be the exterior algebra with $n > 0$.

(1) An artin algebra Γ with $\Lambda \preceq \Gamma$ satisfies $n + 1 = \mathrm{exp.dim}\,\Lambda \leq \mathrm{exp.dim}\,\Gamma \leq \mathrm{rep.dim}\,\Gamma$ by 3.5.1(2).

(2) If $\phi : \Lambda \to \Gamma$ is a split monomorphism of artin algebras, then $n-1 = \mathrm{wresol.dim}(\mathrm{mod}\,\Lambda) \leq \mathrm{wresol.dim}(\mathrm{mod}\,\Gamma) \leq \mathrm{rep.dim}\,\Gamma$ by 3.5.1(4).

3.6.2 Proof of 3.3 For any artin algebra Γ, there exists $M \in \mathrm{mod}\,\Lambda$ and an ideal I of $\mathrm{End}_\Lambda(M)$ such that $\mathrm{End}_\Lambda(M) = \Gamma \oplus I$ [Ha;2.3]. Considering the case Γ is the exterior algebra over n-dimensional vector space, we obtain $n-1 \leq \mathrm{wresol.dim}(\mathrm{mod}\,\mathrm{End}_\Lambda(M)) \leq \mathrm{exp.dim}\,\mathrm{End}_\Lambda(M) \leq r_\Lambda(M) \leq |r_\Lambda|$ by 3.6.1(2) and 3.5.1(1)(2). Thus $|r_\Lambda| = \infty$ holds.∎

3.7 Finitistic dimension conjecture Let Λ be an artin algebra and $\mathrm{fin.dim}\,\Lambda := \sup\{\mathrm{pd}\,X \mid X \in \mathrm{mod}\,\Lambda,\ \mathrm{pd}\,X < \infty\}$ the *finitistic dimension* of Λ [B]. The finitistic dimension conjecture (FDC) asserts that $\mathrm{fin.dim}\,\Lambda < \infty$ holds for any artin algebra Λ.

We refer to [Z] for known results and the relationship to other homological conjecture. Recently, Igusa-Todorov [IT] introduced a function ψ_Λ and applied it to prove (FDC) for artin algebras with $\operatorname{rep.dim}\Lambda \le 3$. We refer [EHIS] and [Xi2,3] for approach to (FDC) using Igusa-Todorov's theorem.

3.7.1 Lemma *Let Λ be an artin algebra. Then there exists a function $\psi_\Lambda : \operatorname{mod}\Lambda \to \mathbb{N}_{\ge 0}$ with the following properties.*

(i) If $\operatorname{pd} X < \infty$, then $\psi_\Lambda(X) = \operatorname{pd} X$.

(ii) $\operatorname{add} X \subseteq \operatorname{add} Y$ implies $\psi_\Lambda(X) \le \psi_\Lambda(Y)$.

(iii) If $0 \to X \to Y \to Z \to 0$ is exact with $\operatorname{pd} Z < \infty$, then $\operatorname{pd} Z \le \psi_\Lambda(X \oplus Y) + 1$.

3.7.2 Theorem *Let Λ be an artin algebra. If $\operatorname{wresol.dim} \operatorname{add} \Omega^n(\operatorname{mod}\Lambda) \le 1$ holds for some n, then $\operatorname{fin.dim}\Lambda < \infty$. Thus $\operatorname{rep.dim}\Lambda \le 3$ (resp. $r_\Lambda(\Lambda) \le 3$, $\operatorname{exp.dim}\Lambda \le 3$) implies $\operatorname{fin.dim}\Lambda < \infty$.*

Proof By 3.5.1(2)(3), we only have to show the former assertion. Let $M \in \operatorname{mod}\Lambda$ be in 3.5(2)(i). For any $X \in \operatorname{mod}\Lambda$ with $\operatorname{pd} X < \infty$, take an exact sequence $0 \to M_1 \to M_0 \to Y \to 0$ with $M_i \in \operatorname{add} M$ and $\Omega^n X \in \operatorname{add} Y$. Then $\operatorname{pd} X \le \operatorname{pd} Y + n \le \psi_\Lambda(M_1 \oplus M_0) + n + 1 = \psi_\Lambda(M) + n + 1$ holds by 3.7.1(ii)(iii).∎

4 Dimensions of triangulated categories

In this section, we shall give a proof of Rouquier's theorem 3.6.

4.1 Definition [R;3.1] Let $\mathcal{T}$ be a triangulated category, and $\mathcal{C}$ and $\mathcal{C}'$ subcategories of $\mathcal{T}$. We denote by $\mathcal{C} * \mathcal{C}'$ the subcategory of $\mathcal{T}$ consisting of X such that there exists a triangle $Y \to X \to Y' \to \cdots$ such that $Y \in \mathcal{C}$ and $Y' \in \mathcal{C}'$. We denote by $\langle\mathcal{C}\rangle$ the smallest full subcategory of $\mathcal{T}$ containing $\mathcal{C}$ and closed under finite direct sums, direct summands and shifts. Put $\mathcal{C} \diamond \mathcal{C}' := \langle \mathcal{C} * \mathcal{C}' \rangle$. Put $\langle\mathcal{C}\rangle_0 := 0$, $\langle\mathcal{C}\rangle_i := \langle\mathcal{C}\rangle_{i-1} \diamond \langle\mathcal{C}\rangle$ for $i > 0$ and $\langle\mathcal{C}\rangle_\infty := \bigcup_{i\ge 0}\langle\mathcal{C}\rangle_i$. We sometimes denote $\langle\mathcal{C}\rangle_i$ by $\langle\mathcal{C}\rangle_{\mathcal{T},i}$. The *dimension* $\dim\mathcal{T}$ of $\mathcal{T}$ is the minimal integer $d \ge 0$ such that there exists $M \in \mathcal{T}$ with $\mathcal{T} = \langle M\rangle_{d+1}$.

4.1.1 Example This concept is motivated by examples in algebraic geometry. Let k be a field and X a scheme over k.

(1) If X is smooth quasi-projective, then $\dim D^b(\operatorname{coh} X) \le 2\dim X$ [R;5.8].

(2) If X is smooth affine and of finite type, then $\dim D^b(\operatorname{coh} X) = \dim X$ [R;5.37].

(3) If X is separated and of finite type, then $\dim D^b(\operatorname{coh} X) < \infty$ [R;5.38]. This is due to Kontsevich, Bondal and Van den Bergh for non-singular case [BV].

4.2 Let Λ be a selfinjective artin algebra. Then the stable category $\underline{\operatorname{mod}}\Lambda$ [ARS] forms a triangulated category. The following proposition gives the relationship between the dimension of stable categories and the representation dimension.

Proposition [R;6.9] $\dim \underline{\operatorname{mod}}\Lambda \le \operatorname{wresol.dim}(\operatorname{mod}\Lambda) \le \operatorname{rep.dim}\Lambda - 2 \le \operatorname{LL}(\Lambda) - 2$.

Proof Since Λ is selfinjective, $\Omega^2(\operatorname{mod}\Lambda) = \operatorname{mod}\Lambda$ holds. By 3.5.1(2)(3), it suffices to show the left inequality. Put $n := \operatorname{wresol.dim}(\operatorname{mod}\Lambda)$ and take $M \in \operatorname{mod}\Lambda$ in

3.5(2)(i). For any $X \in \operatorname{mod}\Lambda$, there exists an exact sequence $0 \to M_n \to \cdots \to M_0 \to Y \to 0$ with $M_i \in \operatorname{add} M$ and $X \in \operatorname{add} Y$. Put $Y_0 := Y$ and $Y_n := M_n$. Then we have an exact sequence $0 \to Y_i \to M_{i-1} \to Y_{i-1} \to 0$ with $M_{i-1} \in \langle M \rangle$ $(0 < i \leq n)$. Inductively, $Y_i \in \langle M \rangle_{n-i+1}$ holds for any i. Thus $Y \in \langle M \rangle_{n+1}$ and $\langle M \rangle_{n+1} = \underline{\operatorname{mod}}\Lambda$ hold.▮

4.3 Theorem [R;6.10] *Let* $\Lambda := \wedge(k^n)$ *be the exterior algebra with* $n > 0$. *Then* $\dim \underline{\operatorname{mod}}\Lambda = n - 1$.

4.4 Definition Let $\mathcal{A}$ be an abelian category with an autofunctor $\mathbb{F} : \mathcal{A} \to \mathcal{A}$. Define a new category $(\mathcal{A}, \mathbb{F})$ as follows: An object is a apir (M, d) consisting of an object M of $\mathcal{A}$ and $d \in \mathcal{A}(M, \mathbb{F}M)$ such that $d\mathbb{F}(d) = 0$. Put $(\mathcal{A}, \mathbb{F})((M, d), (M', d')) = \{f \in \mathcal{A}(M, M') \mid d\mathbb{F}(f) = fd'\}$. We regard $\mathcal{A}$ as a full subcategory of $(\mathcal{A}, \mathbb{F})$ by $M \mapsto (M, 0)$. We call a morphism f in $(\mathcal{A}, \mathbb{F})$ quasi-isomorphism if the induced map $\operatorname{Ker} d / \operatorname{Im} d \to \operatorname{Ker} d' / \operatorname{Im} d'$ is an isomorphism in $\mathcal{A}$. Now $(\mathcal{A}, \mathbb{F})$ has the structure of exact category, where an exact sequence is a sequence which is a split exact as a sequence in $\mathcal{A}$. Then $(\mathcal{A}, \mathbb{F})$ forms a Frobenius category, and its homotopy category $H(\mathcal{A}, \mathbb{F})$ is defined as usual. We define the derived category $D(\mathcal{A}, \mathbb{F})$ as the localization of $H(\mathcal{A}, \mathbb{F})$ by quasi-isomorphisms. Later we shall use the three categories below.

(1)[R;5.1.3] Let Γ be a k-algebra. Then the category of differential Γ-modules is $\operatorname{diff}\Gamma := (\operatorname{mod}\Gamma, 1)$. Put $D\operatorname{diff}\Gamma := D(\operatorname{mod}\Gamma, 1)$. Then $D\operatorname{diff}\Gamma(M, M') = \prod_{n \geq 0} \operatorname{Ext}^n_\Gamma(M, M')$ holds for any $M = (M, 0)$, $M' = (M', 0) \in D\operatorname{diff}\Gamma$.

(2)[K] Let Γ be a graded k-algebra. Then the category of differential graded Γ-modules is $\operatorname{diffgr}\Gamma := (\operatorname{grmod}\Gamma, \mathbb{F})$, where $\mathbb{F}$ is the shift. Put $D\operatorname{diffgr}\Gamma := D(\operatorname{grmod}\Gamma, \mathbb{F})$.

(3) Let Γ be a commutative graded k-algebra, $X := \operatorname{Proj}\Gamma$ and $\mathcal{O}_x$ the local ring at the point $x \in X$. For the shift $\mathbb{F} : \operatorname{coh} X \to \operatorname{coh} X$, we put $\operatorname{diffcoh} X := (\operatorname{coh} X, \mathbb{F})$ and $D\operatorname{diffcoh} X := D(\operatorname{coh} X, \mathbb{F})$. We have the following commutative diagram:

$$\begin{array}{ccccc}
\operatorname{diffgr}\Gamma & \xrightarrow{\widetilde{(\ )}} & \operatorname{diffcoh} X & \xrightarrow{(\ )_x} & \operatorname{diff}\mathcal{O}_x \\
\downarrow & & \downarrow & & \downarrow \\
D\operatorname{diffgr}\Gamma & \xrightarrow{\widetilde{(\ )}} & D\operatorname{diffcoh} X & \xrightarrow{(\ )_x} & D\operatorname{diff}\mathcal{O}_x
\end{array}$$

4.5 Proposition [R;5.11] *Let* Γ *be a* k*-algebra and* L *a* Γ*-module with* $\operatorname{pd}_\Gamma L \geq n$. *Then* $L \notin \langle \Gamma \rangle_{D\operatorname{diff}\Gamma, n}$.

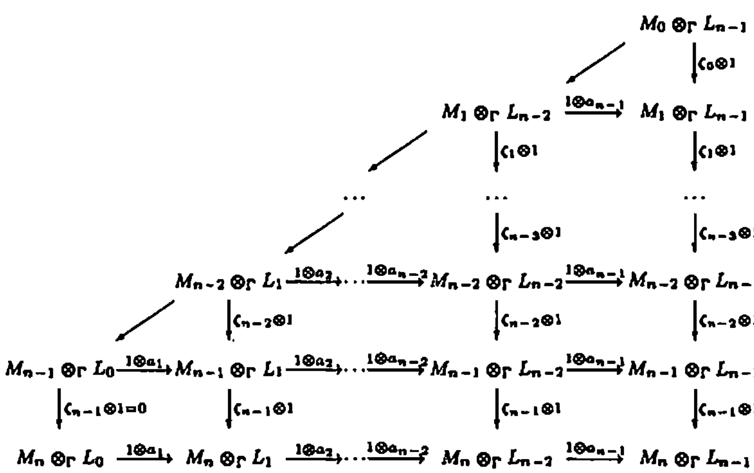
Proof (i)[R;5.10] Put $\Gamma^{en} := \Gamma \otimes_k \Gamma^{op}$. We will show that there exist Γ^{en}-modules $M_0 = \Gamma$, $M_1, \cdots, M_n$ which are projective as left and as right Γ-modules, and $\zeta_i \in \operatorname{Ext}^1_{\Gamma^{en}}(M_i, M_{i+1})$ $(0 \leq i < n)$ such that $(\zeta_0 \cdots \zeta_{n-1}) \otimes_\Gamma 1_L$ is a non-zero element of $\operatorname{Ext}^n_\Gamma(L, M_n \otimes_\Gamma L)$.

Let $\cdots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} \Gamma \to 0$ be a projective resolution of the Γ^{en}-module Γ. Put $M_{i+1} := \operatorname{Ker} f_i$. Then we have an exact sequence $0 \to M_{i+1} \to P_i \to M_i \to 0$, which gives an element $\zeta_i \in \operatorname{Ext}^1_{\Gamma^{en}}(M_i, M_{i+1})$. We have a projective resolution $0 \to M_n \otimes_\Gamma L \to P_{n-1} \otimes_\Gamma L \to \cdots \to P_0 \otimes_\Gamma L \to L \to 0$ of the Γ-module L, and the assertion follows.

(ii) Assume $L \in \langle\Gamma\rangle_n$. Then there exist triangles $L_{j-1} \xrightarrow{a_j} L_j \xrightarrow{b_j} N_j \to \cdots$ $(0 < j < n)$ such that $L_0, N_j \in \langle\Gamma\rangle$ $(0 < j < n)$ and L is a direct summand of L_{n-1}. By 4.4(1), we regard $\zeta_i \otimes 1_N \in \mathrm{Ext}^1_\Gamma(M_i \otimes_\Gamma N, M_{i+1} \otimes_\Gamma N) \subseteq D\,\mathrm{diff}\,\Gamma(M_i \otimes_\Gamma N, M_{i+1} \otimes_\Gamma N)$ for any Γ-module N. Thus we have the following morphism of triangles in $D\,\mathrm{diff}\,\Gamma$:

$$\begin{array}{ccccccc}
M_i \otimes_\Gamma L_{j-1} & \xrightarrow{1_{M_i}\otimes a_j} & M_i \otimes_\Gamma L_j & \xrightarrow{1_{M_i}\otimes b_j} & M_i \otimes_\Gamma N_j & \longrightarrow & \cdots \\
\downarrow{\scriptstyle \zeta_i \otimes 1_{L_{j-1}}} & & \downarrow{\scriptstyle \zeta_i \otimes 1_{L_j}} & & \downarrow{\scriptstyle \zeta_i \otimes 1_{N_j}} & & \\
M_{i+1} \otimes_\Gamma L_{j-1} & \xrightarrow{1_{M_{i+1}}\otimes a_j} & M_{i+1} \otimes_\Gamma L_j & \xrightarrow{1_{M_{i+1}}\otimes b_j} & M_{i+1} \otimes_\Gamma N_j & \longrightarrow & \cdots
\end{array}$$

Since the right-hand side map is zero by $N_j \in \langle\Gamma\rangle$, the morphism $\zeta_i \otimes 1_{L_j}$ factors through $1_{M_{i+1}} \otimes a_j$. Thus we have the following commutative diagram:



Since $\zeta_{n-1} \otimes 1_{L_0} = 0$ holds by $L_0 \in \langle\Gamma\rangle$, the right-hand side composition $(\zeta_0 \cdots \zeta_{n-1}) \otimes 1_{L_{n-1}}$ in the above diagram is zero, a contradiction to (i) and 4.4(1). ∎

4.6 Proof of 4.3 In the rest of this paper, let $\Lambda := \wedge(k^n)$ be the exterior algebra and $\Gamma := k[x_1, \cdots, x_n]$ the polynomial ring. By 4.2 and $\mathrm{LL}(\Lambda) = n + 1$, we only have to show $n - 1 \leq \dim \underline{\mathrm{mod}}\Lambda$. The usual Koszul duality gives an equivalence $D^b(\mathrm{grmod}\,\Lambda) \to D^b(\mathrm{grmod}\,\Gamma)$ of triangulated categories, which induces an equivalence $\underline{\mathrm{grmod}}\Lambda \to D^b(\mathrm{coh}\,\mathbf{P}^{n-1})$ of triangulated categories [BBG]. But we need the following version of Koszul duality due to Keller.

4.6.1 Theorem [K;10.5] *There exists an equivalence* $D^b(\mathrm{mod}\,\Lambda) \to \langle\Gamma\rangle_{D\,\mathrm{diffgr}\,\Gamma,\infty}$ *of triangulated categories, which induces an equivalence* $\underline{\mathrm{mod}}\Lambda \to \langle\Gamma\rangle_{D\,\mathrm{diffgr}\,\Gamma,\infty}/\langle k\rangle_{D\,\mathrm{diffgr}\,\Gamma,\infty}$ *of triangulated categories.*

4.6.2 Proposition *For any* $(\mathcal{F}, d) \in D\,\mathrm{diffcoh}\,\mathbf{P}^{n-1}$, *there exists a point* $x \in \mathbf{P}^{n-1}$ *such that* $(\mathcal{F}_x, d_x) \in \langle\mathcal{O}_x\rangle_{D\,\mathrm{diff}\,\mathcal{O}_x}$.

PROOF Put $\mathcal{G} := \mathrm{Ker}\, d(1)/\mathrm{Im}\, d$ and take a point $x \in \mathbf{P}^{n-1}$ such that $\mathcal{G}_x$ is a projective $\mathcal{O}_x$-module. We have an exact sequence $(\mathcal{F}_x, d_x) \to \mathrm{Ker}\, d_x \xrightarrow{f} \mathcal{G}_x \to 0$ in $\mathrm{diff}\,\mathcal{O}_x$, and we can take $g \in \mathrm{Hom}_{\mathcal{O}_x}(\mathcal{G}_x, \mathrm{Ker}\, d_x)$ such that $gf = 1_{\mathcal{G}_x}$. Since the composition $\mathcal{G}_x \xrightarrow{g} \mathrm{Ker}\, d_x \subset (\mathcal{F}_x, d_x)$ is a quasi-isomorphism, we obtain the assertion.▮

4.6.3 Since Γ-modules of finite length vanishes by the functor $\mathrm{diffgr}\,\Gamma \to \mathrm{diffcoh}\,\mathbf{P}^{n-1}$, we have the following functors of triangulated categories for any $x \in \mathbf{P}^{n-1}$ by 4.4 and 4.6.1:

$$\underline{\mathrm{mod}}\Lambda \xrightarrow{\sim} \mathcal{T} := \langle \Gamma \rangle_{D\,\mathrm{diffgr}\,\Gamma,\infty} / \langle k \rangle_{D\,\mathrm{diffgr}\,\Gamma,\infty} \longrightarrow D\,\mathrm{diffcoh}\,\mathbf{P}^{n-1} \longrightarrow D\,\mathrm{diff}\,\mathcal{O}_x$$

Assume that $M \in \mathcal{T}$ satisfies $\mathcal{T} = \langle M \rangle_{\mathcal{T},n-1}$. Let $(\mathcal{F}, d) \in D\,\mathrm{diffcoh}\,\mathbf{P}^{n-1}$ be the image of M, $x \in \mathbf{P}^{n-1}$ the point obtained by 4.6.2, and $\mathfrak{p}$ the homogeneous prime ideal of Γ which defines x. Then $\Gamma/\mathfrak{p}$ is contained in $\mathcal{T}$ by the regularity of Γ. Since the image of $\Gamma/\mathfrak{p}$ in $D\,\mathrm{diff}\,\mathcal{O}_x$ is the simple $\mathcal{O}_x$-module k_x, we obtain $k_x \in \langle (\mathcal{F}_x, d_x) \rangle_{D\,\mathrm{diff}\,\mathcal{O}_x, n-1} \subseteq \langle \mathcal{O}_x \rangle_{D\,\mathrm{diff}\,\mathcal{O}_x, n-1}$ by the choice of x. This contradicts to 4.5 by $\mathrm{pd}_{\mathcal{O}_x} k_x = n-1$.▮

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A GENERALIZATION OF ZWARA'S THEOREM ON DEGENERATION OF MODULES

YUJI YOSHINO

This is a survey of my recent works on degenerations of modules and their G-dimension. For the datail of the contents, the reader should refer to the original papers [4], [5].

1. Degenerations of modules

In this section k always denotes a field and R is a k-algebra. Note that R may not be commutative nor Noetherian.

Definition 1.1. For finitely generated left R-modules M and N, we say that M degenerates to N along a discrete valuation ring, or N is a degeneration of M along a DVR, if there is a discrete valuation ring (V, tV, k) that is a k-algebra (where t is a prime element) and a finitely generated left $R \otimes_k V$-module Q which satisfies the following conditions:

(1) Q is flat as a V-module
(2) $Q/tQ \cong N$ as a left R-module.
(3) $Q[\frac{1}{t}] \cong M \otimes_k V[\frac{1}{t}]$ as a left $R \otimes_k V[\frac{1}{t}]$-module.

We have considered a different kind of degenerations in the previous paper [3], mainly for maximal Cohen-Macaulay modules over a commutative Cohen-Macaulay local ring. To distinguish it from the degenerations defined in Definition 1.1, we make the following definition.

Definition 1.2. In this definition we assume that k is an algebraically closed field to identify the affine line with k. And let R be a k-algebra as before. For finitely generated left R-modules M and N, we say that M degenerates to N along an affine line, or N is a degeneration of M along an affine line, if there is a finitely generated left module Q over $R \otimes_k k[t]$ which satisfies the following conditions:

(1) Q is flat as a $k[t]$-module.
(2) For any $c \in k$, let us denote $Q/(t-c)Q$ by Q_c, which is a finitely generated left R-module. Then, $Q_0 \cong N$ as a left R-module.
(3) There is a non-empty Zariski open subset U of $\mathbb{A}^1_k \cong k$ such that if $c \in U$, then $Q_c \cong M$ as a left R-module.

The following is our main theorem from [5].

Theorem 1.1. *The following conditions are equivalent for finitely generated left R-modules M and N.*

(1) *N is a degeneration of M along a DVR.*
(2) *There is a short exact sequence of finitely generated left R-modules*

$$0 \to Z \xrightarrow{\binom{\phi}{\psi}} M \oplus Z \to N \to 0,$$

such that the endomorphism ψ on Z is nilpotent, i.e. $\psi^n = 0$ for $n \gg 1$.

Remark 1.1. As G.Zwara has shown in [6], if R is an Artinian k-algebra, then the following conditions are equivalent for finitely generated left R-modules M and N:

(1) N is a degeneration of M along a DVR.
(2′) There is a short exact sequence of finitely generated left R-modules

$$0 \to Z \xrightarrow{\binom{\phi}{\psi}} M \oplus Z \to N \to 0.$$

Note here that we need not the nilpotency assumption for ψ. It is easy to see from Fitting Theorem that if R is an Artinian ring then the condition (2′) is equivalent to the condition (2). In this way, our theorem contains the theorem of Zwara.

By the proof of (2) $\Rightarrow$ (1) of the theorem we get the following result as a corollary.

Corollary 1.2. *Suppose that M degenerates to N along a DVR. Then as a discrete valuation ring V we can always take the ring $k[t]_{(t)}$.*

Remark 1.2. Assume that there is an exact sequence of finitely generated left R-modules

$$0 \to N' \xrightarrow{p} M \xrightarrow{q} N'' \to 0.$$

Then it is easy to see that M degenerates to $N' \oplus N''$ along a DVR. In fact, we have only to notice that there is an exact sequence

$$0 \to N' \xrightarrow{\binom{p}{0}} M \oplus N' \xrightarrow{\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}} N'' \oplus N' \to 0,$$

where the mapping $\psi : N' \to N'$ is the zero mapping, hence nilpotent.

We can prove an implication between degenerations as in the following theorem. But one should remark that the converse implication does not hold in general. See Remark 2.1.

Theorem 1.3. *Assume that k is an algebraically closed field and that R is a left Noetherian k-algebra. Let M and N be finitely generated left R-modules. If M degenerates to N along a DVR, then M degenerates to N along an affine line.*

2. Remarks for commutative Noetherian algebras

In the rest of the paper, we assume that R is a commutative Noetherian algebra over a field k. In this case we have the following result as a corollary of Theorem 1.1.

Corollary 2.1. *Suppose that M and N are R-modules of finite length. Then the following conditions are equivalent.*

(1) *N is a degeneration of M along a DVR.*
(2) *There is an exact sequence*

$$0 \to Z \xrightarrow{\binom{\phi}{\psi}} M \oplus Z \to N \to 0,$$

where Z is also a module of finite length.

In particular, if M degenerates to N along a DVR, then we must have an equality of the lengths; $\ell_R(M) = \ell_R(N)$.

Remark 2.1. There is an example where the opposite direction of the implication in Theorem 1.3 does not hold.

For example, let $R = k[[x]]$ be the formal power series ring over an algebraically closed field k and let $M = R/(x)$ and $N = R/(x^2)$. Since M and N have distinct lengths, N can never be a degeneration of M along a DVR by Corollary 2.1. On the other hand, consider the $R[t]$-module $Q = R[t]/(x^2 - tx)$. It is easy to see that $\mathrm{Ass}_{R[t]}Q = \{(x), (x-t)\}$, hence any nonzero element of $k[t]$ is a non-zero divisor on Q. This implies that Q is flat over $k[t]$. For any element $c \in k$, note that $Q_c \cong R/(x(x-c))$, hence that $Q_0 \cong R/(x^2)$ and $Q_c \cong R/(x)$ for $c \neq 0$, since $x - c$ is a unit in R. Therefore, from the definition, the module M degenerates to N along an affine line. Note that there is an exact sequence

$$0 \to R \xrightarrow{\binom{1}{x}} R/(x) \oplus R \xrightarrow{(x,-1)} R/(x^2) \to 0,$$

however the endomorphism $R \xrightarrow{x} R$ is not nilpotent.

3. Openness of the G-dimension zero property

Let R be a commutative Noetherian ring as before. Auslander and Bridger [1] give a definition of G-dimension, which we denote by $\text{G-dim}_R M$ for a finitely generated R-module M. In our subsequent work [4], we have shown the following fact:

Theorem 3.1. *If there is an exact sequence of finitely generated R-modules*

$$0 \to Z \to M \oplus Z \to N \to 0,$$

then we have an inequality $\text{G-dim}_R M \leqq \text{G-dim}_R N$.

Combining this with Theorem 1.1, we have the following result as a corollary

Corollary 3.2. *Assume that R is a Noetherian commutative algebra over a field k and let M and N be finitely generated R-modules. Suppose that N is a degeneration of M along a DVR. Then the inequality* $\text{G-dim}_R M \leqq \text{G-dim}_R N$ *holds.*

In particular, if N has G-dimension 0, then so does M in this case. We infer from this that if there is an algebraic set that parameterizes a family of finitely generated R-modules, then the set of points corresponding to modules with G-dimension 0 should form an open subset. By this property we may say that the property for a module having G-dimension 0 is an 'open' property. This generalizes a well-known fact that the maximal Cohen-Macaulay property for modules over a Gorenstein local ring is an open property.

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FINITE GENERATIONS OF RINGS OF DIFFERENTIAL OPERATORS OF SEMIGROUP ALGEBRAS

MUTSUMI SAITO AND WILLIAM N. TRAVES

1. Introduction

Let $A := \{ \mathbf{a}_1, \mathbf{a}_2, \dots , \mathbf{a}_n \}$ be a finite subset of $\mathbb{Z}^d$. We denote by $\mathbb{N}A$, $\mathbb{Z}A$, and $\mathbb{R}_{\geq 0}A$ the monoid, the abelian group, and the cone generated by A, respectively.

Let R_A denote the semigroup algebra $\mathbb{C}[\mathbb{N}A]$ of $\mathbb{N}A$. We consider two rings: the ring $D(R_A)$ of differential operators of R_A and its graded ring $\mathrm{Gr}(D(R_A))$ with respect to the order filtration. As a starting point for the study of $D(R_A)$, we have examined the finite generations of $D(R_A)$ and $\mathrm{Gr}(D(R_A))$.

While considering the finite generation of $\mathrm{Gr}(D(R_A))$, we encountered the notion of a scored semigroup; a semigroup $\mathbb{N}A$ is scored if the difference $(\mathbb{R}_{\geq 0}A \cap \mathbb{Z}A) \setminus \mathbb{N}A$ consists of a finite union of hyperlane sections of $\mathbb{R}_{\geq 0}A \cap \mathbb{Z}A$ parallel to facets of the cone $\mathbb{R}_{\geq 0}A$.

We have proved the following.

Theorem 1.1 (Theorem 3.2.12 in [1], Theorem 5.13, Theorem 6.3 in [2]).

1. $\mathrm{Gr}(D(R_A))$ *is finitely generated if and only if* R_A *is a scored semigroup algebra.*
2. $D(R_A)$ *is finitely generated for all semigroup algebras* R_A.

By the standard argument of filtered rings, we obtain the Noetherian properties of $D(R_A)$ from those of $\mathrm{Gr}(D(R_A))$;

Corollary 1.2 (Corollary 6.4 in [1]). *If the semigroup* $\mathbb{N}A$ *is scored, then* $D(R_A)$ *is left and right Noetherian.*

Here we exhibit the idea of the proof of the theorem, and explain the scored property. The scored property implies Serre's condition (S_2). However neither the scored property nor the Cohen-Macaulay property implies the other. The problem of the Noetherian properties of $D(R_A)$ is still open.

2. Motivation

In the theory of A-hypergeometric systems (or GKZ systems), the associative algebra composed of contiguity differential operators plays an important role. We [1] proved that this algebra (called the symmetry algebra) is anti-isomorphic to the algebra $D(R_A)$. Hence we expect that the study of $D(R_A)$ has some fruitful applications to the theory of A-hypergeometric systems.

We also expect that the study of $D(R_A)$ brings new insights into the ring R_A.

This report is based on [1] and [2].

3. Definitions

In this section, we briefly recall some fundamental facts about the rings of differential operators of semigroup algebras. Let $A := \{ \mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n \}$ be a finite set of column vectors in $\mathbb{Z}^d$. Sometimes we identify A with the matrix $(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n)$. **Throughout this paper, we assume that $\mathbb{Z}A = \mathbb{Z}^d$ for simplicity.**

The ring $\mathbb{C}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]\langle \partial_1, \dots, \partial_d \rangle$ of differential operators with Laurent polynomial coefficients is the ring of differential operators on the algebraic torus $(\mathbb{C}^\times)^d$, where $[\partial_i, t_j] = \delta_{ij}$, $[\partial_i, t_j^{-1}] = -\delta_{ij} t_j^{-2}$, and the other pairs of generators commute. Here $[\, ,\,]$ denotes the commutator and δ_{ij} is 1 if $i = j$ and 0 otherwise.

3.1. The Rings R_A and $D(R_A)$. The semigroup algebra $R_A := \mathbb{C}[\mathbb{N}A] = \bigoplus_{\mathbf{a} \in \mathbb{N}A} \mathbb{C} t^{\mathbf{a}}$ is the ring of regular functions on the affine toric variety defined by A, where $t^{\mathbf{a}} = t_1^{a_1} t_2^{a_2} \cdots t_d^{a_d}$ for $\mathbf{a} = {}^t(a_1, a_2, \dots, a_d)$. Its ring of differential operators $D(R_A)$ can be realized as a subring of the ring $\mathbb{C}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]\langle \partial_1, \dots, \partial_d \rangle$ of differential operators on the big torus as follows:

$$D(R_A) = \{ P \in \mathbb{C}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]\langle \partial_1, \dots, \partial_d \rangle \,:\, P(R_A) \subset R_A \}.$$

3.2. The Ring $\mathrm{Gr}(D(R_A))$. Next we explain the order filtration. A differential operator

$$P = \sum_{\mathbf{a} \in \mathbf{N}^d} c_{\mathbf{a}}(t) \partial^{\mathbf{a}} \in \mathbb{C}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]\langle \partial_1, \dots, \partial_d \rangle$$

is said to be of **order** k if $c_{\mathbf{a}} \neq 0$ for some $\mathbf{a}$ with $|\mathbf{a}| = k$ and $c_{\mathbf{a}} = 0$ for all $\mathbf{a}$ with $|\mathbf{a}| > k$, where $|\mathbf{a}| = a_1 + a_2 + \cdots + a_d$. Let $D_k(R_A)$ denote the set of differential operators in $D(R_A)$ of order at most k. Then $\{D_k(R_A)\}_{k \in \mathbf{N}}$ is called the **order filtration** of $D(R_A)$. We consider the graded ring $\mathrm{Gr}(D(R_A))$ of $D(R_A)$ with respect to the order filtration,

$$\mathrm{Gr}(D(R_A)) := \bigoplus_{k \in \mathbf{N}} D_k(R_A)/D_{k-1}(R_A),$$

where $D_{-1}(R_A) = 0$. The graded ring $\mathrm{Gr}(D(R_A))$ is a subring of the commutative ring

$$\mathrm{Gr}(\mathbb{C}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]\langle \partial_1, \dots, \partial_d \rangle) = \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_d^{\pm 1}, \xi_1, \xi_2, \dots, \xi_d],$$

where ξ_j is the element represented by ∂_j.

4. Scored Semigroups

We recall the definition of scored semigroups. To this end, let us define the primitive integral support function of a facet (maximal face) of the cone $\mathbb{R}_{\geq 0}A$. We denote by $\mathcal{F}$ the set of facets of the cone $\mathbb{R}_{\geq 0}A$. Given $\sigma \in \mathcal{F}$, we denote by F_σ the **primitive integral support function** of σ, i.e., F_σ is the uniquely determined linear form on $\mathbb{R}^d$ satisfying $F_\sigma(\mathbb{R}_{\geq 0}A) \geq 0$, $F_\sigma(\sigma) = 0$, and $F_\sigma(\mathbb{Z}^d) = \mathbb{Z}$.

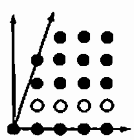
Definition 4.1. The semigroup $\mathbb{N}A$ is said to be **scored** if

$$\mathbb{N}A = \bigcap_{\sigma \in \mathcal{F}} \{ \mathbf{a} \in \mathbb{Z}^d \,:\, F_\sigma(\mathbf{a}) \in F_\sigma(\mathbb{N}A) \}. \tag{4.1}$$

Example 1. Let

$$A_1 = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \end{pmatrix}.$$

Then



$$\mathcal{F} = \{ \sigma_1 = \mathbb{R}_{\geq 0}\mathbf{a}_1,\ \sigma_3 = \mathbb{R}_{\geq 0}\mathbf{a}_3 \},$$

$F_{\sigma_1}(\theta_1, \theta_2) = \theta_2$, $F_{\sigma_3}(\theta_1, \theta_2) = 3\theta_1 - \theta_2$, and

$$\mathbb{N} \setminus F_{\sigma_1}(\mathbb{N}A_1) = \{ 1 \}, \quad \mathbb{N} \setminus F_{\sigma_3}(\mathbb{N}A_1) = \emptyset.$$

The semigroup $\mathbb{N}A_1$ is scored.

Remark 4.1. 1. By the definition of F_σ, the difference $\mathbb{N} \setminus F_\sigma(\mathbb{N}A)$ is finite for any $\sigma \in \mathcal{F}$.

2. The semigroup $\mathbb{N}A$ is scored if and only if

$$(\mathbb{R}_{\geq 0}A \cap \mathbb{Z}^d) \setminus \mathbb{N}A = \bigcup_{\sigma \in \mathcal{F}} \bigcup_{m \in \mathbb{N} \setminus F_\sigma(\mathbb{N}A)} F_\sigma^{-1}(m) \cap \mathbb{R}_{\geq 0}A \cap \mathbb{Z}^d.$$

Thus $\mathbb{N}A$ is scored if and only if the difference $(\mathbb{R}_{\geq 0}A \cap \mathbb{Z}A) \setminus \mathbb{N}A$ consists of a finite union of hyperlane sections of $\mathbb{R}_{\geq 0}A \cap \mathbb{Z}A$ parallel to facets of the cone $\mathbb{R}_{\geq 0}A$.

The semigroup ring $\mathbb{C}[\mathbb{N}A]$ is Cohen-Macaulay if and only if it satisfies Serre's condition (S_2) and the reduced homology modules of certain simplicial complexes vanish ([3, Theorem 4.1]). In our case, Serre's (S_2) condition can be stated as follows:

$$(S_2) \qquad \mathbb{N}A = \bigcap_{\sigma \in \mathcal{F}} (\mathbb{N}A + \mathbb{Z}(A \cap \sigma)).$$

Proposition 4.1 (Proposition 2.6 in [2]). *Any scored semigroup satisfies* (S_2).

Proof. Let $\mathbb{N}A$ be a scored semigroup. It is enough to show that for any facet $\sigma \in \mathcal{F}$ we have

$$(4.2) \qquad \mathbb{N}A + \mathbb{Z}(A \cap \sigma) = \{ \mathbf{a} \in \mathbb{Z}^d \,:\, F_\sigma(\mathbf{a}) \in F_\sigma(\mathbb{N}A) \}.$$

The inclusion '$\subset$' is clear from the definition of F_σ. To prove the other inclusion '$\supset$', let $\mathbf{a} \in \mathbb{Z}^d$ satisfy $F_\sigma(\mathbf{a}) \in F_\sigma(\mathbb{N}A)$. For every $\sigma' \in \mathcal{F}$ different from σ, there exists $\mathbf{a}_i \in A$ such that $\mathbf{a}_i \notin \sigma'$ and $\mathbf{a}_i \in \sigma$. Since $F_{\sigma'}(\mathbf{a}_i) > 0$ and $\mathbb{N} \setminus F_{\sigma'}(\mathbb{N}A)$ is finite, there exists $m_i \in \mathbb{N}$ such that $F_{\sigma'}(\mathbf{a} + m_i\mathbf{a}_i) \in F_{\sigma'}(\mathbb{N}A)$.

Doing this argument for every $\sigma' \in \mathcal{F}$ different from σ, we find $\mathbf{b} \in \mathbb{N}(A \cap \sigma)$ such that

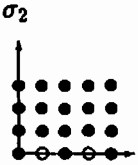
$$F_{\sigma'}(\mathbf{a} + \mathbf{b}) \in F_{\sigma'}(\mathbb{N}A) \qquad (\forall \sigma' \in \mathcal{F} \setminus \{\sigma\}).$$

Since

$$F_\sigma(\mathbf{a} + \mathbf{b}) = F_\sigma(\mathbf{a}) \in F_\sigma(\mathbb{N}A)$$

and $\mathbb{N}A$ is scored, we see $\mathbf{a} + \mathbf{b} \in \mathbb{N}A$. Hence $\mathbf{a} \in \mathbb{N}A + \mathbb{Z}(A \cap \sigma)$. □

Example 2.

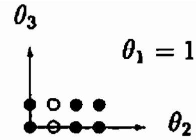


$$A_2 = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \end{pmatrix}.$$

The semigroup $\mathbb{N}A_2$ satisfies (S_2). Hence the semigroup ring $\mathbb{C}[\mathbb{N}A_2]$ is Cohen-Macaulay as well since the cone $\mathbb{R}_{\geq 0}A$ is simplicial. Thus $\mathbb{C}[\mathbb{N}A_2]$ is Cohen-Macaulay but not scored.

Example 3. Let

$$A_3 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$



Then the semigroup $\mathbb{N}A_3$ is clearly scored. However the semigroup ring $\mathbb{C}[\mathbb{N}A_3]$ is not Cohen-Macaulay ([3, Example 3.9]).

5. Graded Structure

Put $\theta_j := t_j\partial_j$ for $j = 1, 2, \dots, d$. Then it is easy to see that $\theta_j \in D(R_A)$ for all j. We introduce a $\mathbb{Z}^d$-grading on the ring $D(R_A)$ as follows: For $\mathbf{a} = {}^t(a_1, a_2, \dots, a_d) \in \mathbb{Z}^d$, set

$$D(R_A)_{\mathbf{a}} := \{P \in D(R_A) : [\theta_j, P] = a_j P \quad \text{for } j = 1, 2, \dots, d\}.$$

Then $D(R_A)$ is $\mathbb{Z}^d$-graded; $D(R_A) = \bigoplus_{\mathbf{a}\in\mathbb{Z}^d} D(R_A)_{\mathbf{a}} = \bigoplus_{\mathbf{a}\in\mathbb{Z}^d} t^{\mathbf{a}}\mathbb{I}(\Omega(\mathbf{a}))$, where $\mathbb{I}(\Omega(\mathbf{a}))$ is a certain ideal of $\mathbb{C}[\theta]$.

Since each $D_k(R_A)$ is $\mathbb{Z}^d$-graded – $D_k(R_A) = \bigoplus_{\mathbf{d}\in\mathbb{Z}^d} D_k(R_A) \cap D(R_A)_{\mathbf{d}}$ – the graded ring $\mathrm{Gr}(D(R_A))$ inherits the grading; $\mathrm{Gr}(D(R_A)) = \bigoplus_{\mathbf{d}\in\mathbb{Z}^d} \mathrm{Gr}(D(R_A))_{\mathbf{d}}$.

The following decomposition is a key to the finite generations of $D(R_A)$ and $\mathrm{Gr}(D(R_A))$.

Proposition 5.1 (Proposition 3.4 in [2]). *There exist $m \in \mathbb{N}$, $\mathbf{b}_i \in \mathbb{Z}^d$, and faces τ_i of $\mathbb{R}_{\geq 0}A$ with $i = 1, 2, \dots, m$ such that*

$$(5.1) \qquad (\mathbb{R}_{\geq 0}A \cap \mathbb{Z}^d) \setminus \mathbb{N}A = \coprod_{i=1}^{m} (\mathbf{b}_i + \mathbb{N}(A \cap \tau_i)).$$

The decomposition (5.1) is called a **Stanley decomposition** of $(\mathbb{R}_{\geq 0}A \cap \mathbb{Z}^d) \setminus \mathbb{N}A$. Note that it is not unique.

6. Finite Generation of $\mathrm{Gr}(D(R))$

The Serre's condition (S_2) means that every graded component $D(R_A)_{\mathbf{d}}$ is a singly generated $\mathbb{C}[\theta]$-module.

Proposition 6.1. *The $\mathbb{C}[\theta]$-modules $D(R_A)_{\mathbf{a}}$ are singly generated for all $\mathbf{a}$ if and only if the semigroup $\mathbb{N}A$ satisfies (S_2).*

Next we consider the scored property. Let $\mathbf{c} \in \mathbb{N}A$ be a vector such that $\mathbf{c} + \mathbb{R}_{\geq 0}A \cap \mathbb{Z}^d \subset \mathbb{N}A$. It is well known that such a vector always exists.

Proposition 6.2. *There exists $L \in \mathbb{N}$ independent of $\mathbf{d} \in \mathbf{c} + \mathbb{R}_{\geq 0}A \cap \mathbb{Z}^d$ such that the minimal degree of polynomials in $\mathbb{I}(\Omega(-\mathbf{d}))$ is equal to $\sum_{\sigma\in\mathcal{F}} F_\sigma(\mathbf{d}) + L$.*

Furthermore, $L = 0$ if and only if $\mathbb{N}A$ is scored.

Thus the scored property means that the degree of generators of graded components far from the hyperplanes grows linearly without a constant term. This fact intuitively explains Theorem 1.1 (1).

The proof of the finite generation of $\mathrm{Gr}(D(R_A))$ for a scored semigroup ring R_A goes as follows:

1. Fix a Stanley decomposition (5.1).

2. Let M be one plus the maximum of $F_\sigma(\mathrm{b}_i)$ ($\sigma \in \mathcal{F}$, $i = 1, \dots, m$).
3. Decompose $\mathbb{Z}^d$ into pieces;
$$\mathbb{Z}^d = \bigcup_\mu S_\mu,$$
where μ runs over all maps from $\mathcal{F}$ to $\{-\infty, \infty\} \cup \{k \in \mathbb{Z} : |k| < M\}$, and
$$S_\mu := \{\, \mathrm{d} \in \mathbb{Z}^d \,:\, F_\sigma(\mathrm{d}) = \mu(\sigma) \quad \text{for all } \sigma \in \mathcal{F}\,\}.$$
Here we agree that $F_\sigma(\mathrm{d}) = +\infty$ ($-\infty$, respectively) mean $F_\sigma(\mathrm{d}) \geq M$ ($\leq -M$, respectively).
4. Let $\mathrm{Ray}(A)$ denote the set of rays of the hyperplane arrangement determined by A, $\{(F_\sigma = 0) \,:\, \sigma \in \mathcal{F}\}$. For each ray $\rho \in \mathrm{Ray}(A)$, take d_ρ from $\rho \cap \mathbb{Z}^d$ so that $|F_\sigma(\mathrm{d}_\rho)| \geq M$ whenever it is not zero.
5. Put $F_\mu := \{\rho \in \mathrm{Ray}(A) \,:\, \rho \subset \sigma \text{ if } \mu(\sigma) \neq \pm\infty\}$.
6. Then there exists a finite subset $S_{\mu,\mathrm{fin}} \subset S_\mu$ such that $S_\mu = S_{\mu,\mathrm{fin}} + \sum_{\rho \in F_\mu} \mathbb{N}\mathrm{d}_\rho$.
7. Put $D(R_A)_{S_\mu} = \bigoplus_{\mathrm{a} \in S_\mu} D(R_A)_\mathrm{a}$. Then $\mathrm{Gr}(D(R_A))_{S_\mu}$ is generated by $\mathrm{Gr}(D(R_A))_\mathrm{a}$ ($\mathrm{a} \in S_{\mu,\mathrm{fin}}$) and $\mathrm{Gr}(D(R_A))_\rho$ ($\rho \in F_\mu$).
8. $\mathrm{Gr}(D(R_A))$ is finitely generated.

Example 4. (Continuation of Example 2)

Since $\mathbb{N}A_2$ satisfies (S_2), each $D(R_{A_2})_\mathrm{a}$ is singly generated. For $\mathrm{a} = {}^t(a_1, a_2)$, put
$$Q_\mathrm{a} := \begin{cases} t_1^{a_1} t_2^{a_2} & (a_1 \geq 0, a_2 \geq 1, \text{ or } a_1 \geq 0 \text{ even}, a_2 = 0) \\ t_1^{a_1} t_2 \partial_2^{|a_2|+1} & (a_1 \geq 0, a_2 < 0, \text{ or } a_1 \geq 0 \text{ odd}, a_2 = 0) \\ t_2^{a_2} \partial_1^{|a_1|} & (a_1 < 0, a_2 \geq 1, \text{ or } a_1 < 0 \text{ even}, a_2 = 0) \\ t_2 \partial_1^{|a_1|} \partial_2^{|a_2|+1} & (a_1, a_2 < 0, \text{ or } a_1 < 0 \text{ odd}, a_2 = 0). \end{cases}$$
By computing $\mathbb{I}(\Omega(\mathrm{a}))$, we see that $D(R_{A_2})_\mathrm{a}$ is generated by Q_a. Hence $L = 1$. Moreover we have
$$\mathrm{Gr}(D(R_{A_2}))/\langle t_1, \xi_1 \rangle = \mathbb{C}[t_2, t_2\xi_2, t_2\xi_2^2, t_2\xi_2^3, \cdots].$$
Since this is not a finitely generated algebra, neither is $\mathrm{Gr}(D(R_{A_2}))$.

7. Finite Generation of $D(R)$

The proof of the finite generation of $D(R_A)$ for a general affine semigroup ring R_A follows the same steps as that of $\mathrm{Gr}(D(R_A))$ for a scored semigroup ring made until Step 6. Steps 7 and 8 are the following:

7 Put $D(R_A)_{F_\mu,\mathbb{R}} = \bigoplus_{\mathrm{a} \in \sum_{\rho \in F_\mu} \mathbb{N}\mathrm{d}_\rho} D(R_A)_\mathrm{a}$. Then $D(R_A)_{S_\mu}$ is generated by $D(R_A)_\mathrm{a}$ ($\mathrm{a} \in S_{\mu,\mathrm{fin}} + \sum_{\rho \in F_\mu} \mathbb{N}_{<m+2}\mathrm{d}_\rho$) as a right $D(R_A)_{F_\mu,\mathbb{R}}$-module, where $\mathbb{N}_{<m+2}$ denotes the set of natural numbers less than $m+2$.

8 $D(R_A)$ is finitely generated.

Let us explain Step 7 in some more details. Let $\mathrm{d} = (\mathrm{d} - \mathrm{d}_\rho) + \mathrm{d}_\rho$ ($\mathrm{d} - \mathrm{d}_\rho \in S_\mu$, $\rho \in F_\mu$). Then there exists an ideal $I(\mathrm{d} - \mathrm{d}_\rho, \mathrm{d}_\rho)$ of $\mathbb{C}[\theta]$ such that
$$D(R_A)_{\mathrm{d}-\mathrm{d}_\rho} D(R_A)_{\mathrm{d}_\rho} = D(R_A)_\mathrm{d} I(\mathrm{d} - \mathrm{d}_\rho, \mathrm{d}_\rho).$$
When $\mathrm{d} - (m+1)\mathrm{d}_\rho \in S_\mu$, we can prove that $\sum_{k=1}^{m+1} I(\mathrm{d} - k\mathrm{d}_\rho, k\mathrm{d}_\rho) = \langle 1 \rangle$. Hence we see that $D(R_A)_{S_\mu}$ is generated by $D(R_A)_\mathrm{a}$ ($\mathrm{a} \in S_{\mu,\mathrm{fin}} + \sum_{\rho \in F_\mu} \mathbb{N}_{<m+2}\mathrm{d}_\rho$) as a right $D(R_A)_{F_\mu,\mathbb{R}}$-module.

Example 5. (Continuation of Example 4)

Recall that $Q_{\iota(0,-p)} = t_2\partial^{p+1}$ for $p \geq 1$. Then for $p, q \geq 1$

$$Q_{\iota(0,-p)}Q_{\iota(0,-q)} = Q_{\iota(0,-p-q)}(t_2\partial_2 - q).$$

Hence for $p \geq 3$

$$Q_{\iota(0,-p)} = Q_{\iota(0,-(p-1))}Q_{\iota(0,-1)} - Q_{\iota(0,-(p-2))}Q_{\iota(0,-2)}.$$

Thus $Q_{\iota(0,-p)}$ $(p \geq 3)$ are generated by $Q_{\iota(0,-1)}$ and $Q_{\iota(0,-2)}$.

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ON DEGENERATIONS OF MODULES OVER GENERAL RINGS

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In this paper, R is an associative ring with an identity and a module means a left R-module otherwise stated. We denote by $\mathbb{N}$ the set of positive integers.

Recall [6], [8] that for a finite dimensional algebra A over an algebraically closed field k and (finite) d-dimensional A-modules M, N, M degenerates to N (i.e., N lies in the $Gl_d(k)$-orbit closure of M) if and only if there is some exact tube, called (M, N)-tube, if and only if there is a short exact sequence $0 \to Z \to Z \oplus M \to N \to 0$. Our aims are to define the degeneration for infinite dimensional modules and to characterize the non-split exact sequence $0 \to N \to N \oplus N \to N \to 0$.

1. Degenerations

Our aim in this section is to generalize Zwara's results, that is, M degenerates to N if and only if there is an exact sequence $0 \to Z \to Z \oplus M \to N \to 0$.

Let N_i ($i \in \mathbb{N}$) be modules and

$$\mathbf{T}: N_1 \xleftarrow{\beta_1} N_2 \xleftarrow{\beta_2} N_3 \xleftarrow{\beta_3} \cdots$$

a sequence of homomorphisms. $\mathbf{T}[1]$ is a shift of $\mathbf{T}$, i.e.,

$$\mathbf{T}[1]: N_2 \xleftarrow{\beta_2} N_3 \xleftarrow{\beta_3} N_4 \xleftarrow{\beta_4} \cdots$$

A morphism $\alpha: \mathbf{T} \to \mathbf{T}[1]$ is the set of homomorphisms α_i $N_i \to N_{i+1}$, ($i \in \mathbb{N}$) which makes a commutative diagram;

$$\begin{array}{ccccccc} N_1 & \xleftarrow{\beta_1} & N_2 & \xleftarrow{\beta_2} & N_3 & \xleftarrow{\beta_3} & \cdots \\ {\scriptstyle\alpha_1}\downarrow & & {\scriptstyle\alpha_2}\downarrow & & {\scriptstyle\alpha_3}\downarrow & & \\ N_2 & \xleftarrow{\beta_2} & N_3 & \xleftarrow{\beta_3} & N_4 & \xleftarrow{\beta_4} & \cdots . \end{array}$$

A morphism $\alpha: \mathbf{T} \to \mathbf{T}[1]$ is said to be *"exact"* or *"exact tube"* if for each i, we have the short exact sequences

$$0 \longrightarrow N_{i+1} \xrightarrow{\binom{\beta_i}{\alpha_{i+1}}} N_i \oplus N_{i+2} \xrightarrow{(\alpha_i, -\beta_{i+1})} N_{i+1} \longrightarrow 0$$

with a monomorphism α_i and an epimorphism β_i.

For an exact tube $\alpha: \mathbf{T} \to \mathbf{T}[1]$ and $N := \mathrm{Cok}(\alpha_1)$, we have a short exact sequence of inverse systems;

$$0 \longrightarrow \mathbf{T} \xrightarrow{\alpha} \mathbf{T}[1] \longrightarrow (N, 1_N) \longrightarrow 0.$$

Here we consider that $(N, 1_N)$ is a sequence of identity maps of N,

$$(N, 1_N): N \xleftarrow{1_N} N \xleftarrow{1_N} N \xleftarrow{1_N} \cdots$$

The detailed version of this paper will be submitted for publication elsewhere.

Then, this induces the exact sequence of inverse limit;

$$0 \longrightarrow \varprojlim \mathbf{T} \xrightarrow{\varprojlim \alpha} \varprojlim \mathbf{T}[1] \longrightarrow N \longrightarrow 0,$$

because the inverse limit functor $\varprojlim$ is left exact and a right term N is the inverse limit of the inverse system $(N, 1_N)$. For an exact tube $\alpha: \mathbf{T} \to \mathbf{T}[1]$ and any module L, $\alpha \oplus 1_L: \mathbf{T} \oplus L \to \mathbf{T}[1] \oplus L$ is also an exact tube which makes a commutative diagram;

$$\begin{array}{ccccccc}
N_1 \oplus L & \xleftarrow{\beta_1 \oplus 1_L} & N_2 \oplus L & \xleftarrow{\beta_2 \oplus 1_L} & N_3 \oplus L & \xleftarrow{\beta_3 \oplus 1_L} & \cdots \\
\downarrow{\scriptstyle \alpha_1 \oplus 1_L} & & \downarrow{\scriptstyle \alpha_2 \oplus 1_L} & & \downarrow{\scriptstyle \alpha_3 \oplus 1_L} & & \\
N_2 \oplus L & \xleftarrow{\beta_2 \oplus 1_L} & N_3 \oplus L & \xleftarrow{\beta_3 \oplus 1_L} & N_4 \oplus L & \xleftarrow{\beta_4 \oplus 1_L} & \cdots .
\end{array}$$

Lemma 1.1. *Let $\alpha: \mathbf{T} \to \mathbf{T}[1]$ be an exact tube. If an exact sequence $0 \to \varprojlim \mathbf{T} \xrightarrow{\varprojlim \alpha} \varprojlim \mathbf{T}[1] \to N \to 0$ splits, then exact sequences $0 \to N_i \to N_{i+1} \to N \to 0$ split for all $i \geq 1$.*

Definition 1.2. Let M and N be non-zero modules. An exact tube $\alpha: \mathbf{T} \to \mathbf{T}[1]$ is called an (M, N)-tube if there is an isomorphism $\binom{f}{g}: \varprojlim \mathbf{T}[1] \to \varprojlim \mathbf{T} \oplus M$ with the property that $\varprojlim \mathbf{T}$ has no non-zero direct summand L such that $\varprojlim \alpha | L: L \to \varprojlim \alpha(L)$ is an isomorphism. In this case N is called a *"degeneration"* of M.

The following theorem generalizes the results in [1, 7, 8] on general rings and modules.

Theorem 1.3. *The following statements are equivalent;*

(1) *N is a degeneration of M.*

(2) *There is an exact sequence $0 \to Z \xrightarrow{\binom{f}{g}} Z \oplus M \to N \to 0$ for some module Z which has no non-zero direct summand L such that $f|L: L \to f(L)$ is an isomorphism.*

For a degeneration N of M, we constructed an exact sequence $0 \to Z \to Z \oplus M \to N \to 0$ that the module Z was realized by an inverse limit for a chain of epimorphisms. However, we can also construct it by using a direct limit. In fact, for an exact sequence $0 \to Z \xrightarrow{\binom{f}{g}} Z \oplus M \to N \to 0$, we can construct a commutative diagram consisting of direct systems ;

$$\begin{array}{lccccccc}
\bar{\mathbf{T}}: & Z & \xrightarrow{\iota_0} & Z \oplus M & \xrightarrow{\iota_1} & Z \oplus M^2 & \xrightarrow{\iota_2} & \cdots \\
\downarrow{\scriptstyle \alpha} & \downarrow{\scriptstyle h_0} & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} & & \\
\bar{\mathbf{T}}[1]: & Z \oplus M & \xrightarrow{\iota_1} & Z \oplus M^2 & \xrightarrow{\iota_2} & Z \oplus M^3 & \xrightarrow{\iota_3} & \cdots .
\end{array}$$

Here $h_i: Z \oplus M^i \to Z \oplus M^i \oplus M$ is given by $h_i\binom{z}{x} = \begin{pmatrix} f(z) \\ x \\ g(z) \end{pmatrix}$ and $\iota_i: Z \oplus M^i \to Z \oplus M \oplus M^i$ is given by $\iota_i\binom{z}{x} = \begin{pmatrix} z \\ 0 \\ x \end{pmatrix}$ for $z \in Z$, $x \in M^i$. Note that $h_0 = \binom{f}{g}$.

Then it induces the exact sequence of direct limits of direct systems;

$$0 \longrightarrow \varinjlim \bar{\mathbf{T}} \xrightarrow{\varinjlim \alpha} \varinjlim \bar{\mathbf{T}}[1] \longrightarrow N \longrightarrow 0.$$

Concretely, we have $\varinjlim \mathbf{T} = Z \oplus M^{(\mathbb{N})}$, where $M^{(\mathbb{N})}$ is the direct sum of $\mathbb{N}$-copies of M. On our definition of a degeneration, it is satisfied the following basic properties.

Corollary 1.4. (1) *If $0 \to A \to B \to C \to 0$ is an exact sequence, then $A \oplus C$ is a degeneration of B.*

(2) *If N is a degeneration of M, then $N^{(r)}$ is a degeneration of $M^{(r)}$, where $N^{(r)}$ is direct sum of r-copies of N and r is a cardinal number.*

(3) *If N is a degeneration of M, then N^{r} is a degeneration of M^{r}, where N^{r} is direct product of r-copies of N.*

Definition 1.5. Let $0 \to Z \xrightarrow{\binom{f}{g}} Z \oplus M \to N \to 0$ be an exact sequence which gives a degeneration N of M. A degeneration is called *"Fitting type"* if $f : Z \to Z$ is nilpotent.

On a degeneration for finitely generated modules $M \not\cong N$, the endomorphism f must be nilpotent for the induced exact sequence $0 \to Z \xrightarrow{\binom{f}{g}} Z \oplus M \to N \to 0$ [1], [7], [8].

Theorem 1.6. *Assume a module N is finitely generated and there is a Fitting type degeneration N of M. Then M is finitely generated and there is a finitely generated module Z such that $0 \to Z \xrightarrow{\binom{f}{g}} Z \oplus M \to N \to 0$ is exact and f is nilpotent.*

2. Exact sequences of the form $0 \to N \to N \oplus N \to N \to 0$

In this section, we give a non-split exact sequence of the form $0 \to N \to N \oplus N \to N \to 0$ and these examples make strange degenerations which do not happen for finitely generated modules over finite dimensional algebras.

We give the preliminary lemma.

Lemma 2.1. *The following statements are equivalent.*

(1) *Any exact sequence $0 \to M \to M \oplus N \to N \to 0$ splits for any modules M and N.*

(2) *Any exact sequence $0 \to N \to N \oplus N \to N \to 0$ splits for any modules N.*

An exact tube:

$$\begin{array}{ccccccc}
N_1 & \xleftarrow{\beta_1} & N_2 & \xleftarrow{\beta_2} & N_3 & \xleftarrow{\beta_3} & \cdots \\
\downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \\
N_2 & \xleftarrow{\beta_2} & N_3 & \xleftarrow{\beta_3} & N_4 & \xleftarrow{\beta_4} & \cdots
\end{array}$$

induces an inductive systems;

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N_i & \xrightarrow{\binom{\beta_{i-1}}{\alpha_i}} & N_{i-1} \oplus N_{i+1} & \xrightarrow{(\alpha_{i-1}, -\beta_i)} & N_i & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_i} & & \downarrow{\scriptstyle \begin{pmatrix} \alpha_{i-1} & 0 \\ 0 & \alpha_{i+1} \end{pmatrix}} & & \downarrow{\scriptstyle \alpha_i} & & \\
0 & \longrightarrow & N_{i+1} & \xrightarrow[\binom{\beta_i}{\alpha_{i+1}}]{} & N_i \oplus N_{i+2} & \xrightarrow[(\alpha_i, -\beta_{i+1})]{} & N_{i+1} & \longrightarrow & 0.
\end{array}$$

Proposition 2.2. *We set $N_{\aleph_0} = \varinjlim N_i$, $\binom{\beta_{\aleph_0}}{\alpha_{\aleph_0}} = \varinjlim \binom{\beta_i}{\alpha_{i+1}}$, $(\alpha_{\aleph_0}, -\beta_{\aleph_0}) = \varinjlim(\alpha_i, -\beta_{i+1})$. Then the short exact sequence*

$$0 \longrightarrow N_{\aleph_0} \xrightarrow{\binom{\beta_{\aleph_0}}{\alpha_{\aleph_0}}} N_{\aleph_0} \oplus N_{\aleph_0} \xrightarrow{(\alpha_{\aleph_0}, -\beta_{\aleph_0})} N_{\aleph_0} \longrightarrow 0$$

splits.

Aehle, Riedtmann and Zwara [1] defined a complexity for an (M, N)-tube where $M \not\cong N$ are finitely generated modules over a finite dimensional algebra, that is, a complexity for an (M, N)-tube is the minimal number such that exact sequences

$$0 \longrightarrow N_i \xrightarrow{\binom{\beta_{i-1}}{\alpha_i}} N_{i-1} \oplus N_{i+1} \xrightarrow{(\alpha_{i-1}, -\beta_i)} N_i \longrightarrow 0$$

split for all $i \geq m$.

Their results hold for an exact tube if there is such an m that the above property is satisfied. So we can define the complexity of an exact tube over general rings.

By Proposition 2.2, the following fact holds.

Proposition 2.3. *The complexity of an exact tube always exists and this is less than or equal to* $\aleph_0$.

The following example shows that there is a non-split exact sequence $0 \to N \to N \oplus N \to N \to 0$ which induces the strange degeneration.

Example 2.4. Let k be a field and $\mathbb{Q}$ the set of rational numbers. We set a semi-group algebra $R = \sum_{\substack{q \in \mathbb{Q}, \\ 0 \leq q \leq 1}} \oplus k z_q$. The multiplication is given by $z_p z_q = z_{pq}$. Then idempotents of R are only z_0, z_1 and z_1 is the identity of R.

Now we shall construct R-module N. It is defined as vector space with bases $\{ w_t | \, t \in \mathbb{Q}, -1 \leq t \leq 1 \}$ in the following;

$$N = \sum_{\substack{t \in Q, \\ -1 \leq q \leq 1}} \oplus K w_t.$$

The R-operation is defined by $z_q w_t = w_{qt}$. The generators of N are w_1, w_{-1} and the socle of N is $K w_0$.

We consider two R-homomorphisms f_1, f_2 : $N \to N$ by

$$f_1(w_t) = \begin{cases} w_t, & (t \leq 0), \\ w_{\frac{t}{2}}, & (t > 0), \end{cases} \qquad f_2(w_t) = \begin{cases} w_{\frac{t}{2}}, & (t \leq 0), \\ w_t, & (t > 0). \end{cases}$$

Then we have the non-split exact sequence;

$$0 \longrightarrow N \xrightarrow{\binom{f_1}{f_2}} N \oplus N \longrightarrow N \longrightarrow 0.$$

In particular, N has no non-zero direct summand L such that $f_1|L : L \to f_1(L)$ is not an isomorphism.

We set $M_q = \sum_{\substack{x \in \mathbb{Q} \\ 0 \leq x \leq q}} \oplus K w_x$ an ideal of R. Then we have the exact sequence ;

$$0 \longrightarrow M_{\frac{1}{2}}/M_{\frac{1}{4}} \longrightarrow M_1/M_{\frac{1}{4}} \longrightarrow M_1/M_{\frac{1}{2}} \longrightarrow 0.$$

For any rational numbers $0 \leq r < s \leq 1$, all of the modules of the form M_s/M_r are isomorphic, thus we have a non-split exact sequence;

$$0 \longrightarrow M \longrightarrow M \longrightarrow M \longrightarrow 0.$$

Hence M^2 is a degeneration of M by Corollary 1.4.

Crawley-Boevey and Krause studied a generic modules [2], [5]. This is determined to a homogeneous tube. In particular, for a tame concealed algebra, a generic module is uniquely constructed by a tubular family (it is a one-parameter family of tubes that almost all of them are homogeneous.)

Example 2.5 ([5]). Let Λ be an algebra over a field k. An exact tube $\mathbf{T}$ is called a homogeneous tube if it is a following exact diagrams

$$\begin{array}{ccccccccc} 0 & \longleftarrow & N_1 & \xleftarrow{\beta_1} & N_2 & \xleftarrow{\beta_2} & N_3 & \xleftarrow{\beta_3} & \cdots \\ \downarrow & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \\ N_1 & \xleftarrow{\beta_1} & N_2 & \xleftarrow{\beta_2} & N_3 & \xleftarrow{\beta_3} & N_4 & \xleftarrow{\beta_4} & \cdots . \end{array}$$

We have the exact sequence of inverse limits

$$0 \longrightarrow \varprojlim \mathbf{T} \xrightarrow{\psi} \varprojlim \mathbf{T}[1] \longrightarrow N \longrightarrow 0.$$

Then this also gives the following direct system of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \varprojlim \mathbf{T} & \xrightarrow{\psi^i} & \varprojlim \mathbf{T} & \longrightarrow & N_i & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \alpha_i} & & \\ 0 & \longrightarrow & \varprojlim \mathbf{T} & \xrightarrow{\psi^{i+1}} & \varprojlim \mathbf{T} & \longrightarrow & N_{i+1} & \longrightarrow & 0. \end{array}$$

Note that $\varprojlim \mathbf{T} \oplus N_i$ is a degeneration of $\varprojlim \mathbf{T}$ by Corollary 1.4. Otherwise, we also have the exact sequence of direct limits

$$0 \longrightarrow \varprojlim \mathbf{T} \longrightarrow M \longrightarrow \varinjlim \mathbf{T} \longrightarrow 0.$$

It is called *the universal exact sequence* corresponding to $\mathbf{T}$. In fact, M is uniquely determined to a tubular family, and it is called a *generic* module. By Corollary 1.4, we have that $\varinjlim \mathbf{T} \oplus \varprojlim \mathbf{T}$ is a degeneration of M.

Now, we shall introduce the behavior of the homogeneous tube explaining in [5]. Let $k[x]$ be a polynomial ring with one invariant and $k(x)$ be the quotient field of $k[x]$. Note that $k[x] \cong \varprojlim k[x]/x^i$ and $k(x)$ is the direct limit of the direct system $k[x] \xrightarrow{\cdot x} k[x] \xrightarrow{\cdot x} \cdots$. The short exact sequence $0 \to k[x] \to k(x) \to k(x)/k[x] \to 0$ is the universal exact sequence corresponding to the tube $(k[x]/x^i)_{i \in \mathbb{N}}$

Lemma 2.6 ([5]). *Let T be a homogeneous tube. Then T is isomorphic to* $\left(k[x]/x^i \otimes_{k[x]} (\varprojlim T)\right)_{i \in \mathbb{N}}$.

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LIFTING MODULES OVER RIGHT PERFECT RINGS

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We consider associative rings with identity and all modules considered are unitary right R-modules. An R-module M is said to be an *extending* module, if it satisfies the following property: For any submodule X of M, there exists a direct summand of M which contains X as an essential submodule, that is, for any submodule X of M, there exists a closure of X in M which is a direct summand of M. Dually, M is called a *lifting* module, if it satisfies the dual property: For any submodule X of M, there exists a direct summand of M which is a co-essential submodule of X, that is, for any submodule X of M, there exists a co-closure of X in M which is a direct summand of M (cf.[4]). Any submodule X of M has a closure in M, but not always has a co-closure in M. It seems that the difficulty of study of lifting modules is hidden here.

Okado [3] has studied the decomposition of extending modules over right noetherian rings and obtained the following:

Theorem R is a right noetherian if and only if every extending R-module is expressed as a direct sum of indecomposable (uniform) modules.

As a dual problem, we consider the following problem:

Problem Which ring R has the property that every lifting R-module has an indecomposable decomposition ?

Our purpose of this note is to study this problem and show the following:

Result 1 Every lifting module over right perfect rings is expressed as a direct sum of indecomposable (cyclic hollow) modules.

Result 2 Let R be a right perfect ring and let M be a lifting module.
(1) If M is generalized M-projective, then M has the exchange property.
(2) If M has the finite exchange property, then M has the exchange property.

Result 3 Any lifting module over right perfect rings has the exchange property.

1. Preliminaries

Let M be a module. A submodule S of M is said to be a *small* submodule of M (denoted by $S \ll M$) if $M \neq K + S$ for any proper submodule K of M. Let N and K be submodules of M with $K \subseteq N$. K is said to be a *co-essential* submodule of N in M if $N/K \ll M/K$ and we write $K \subseteq_c N$ in M in this case. Let X be a submodule of M. X is called *co-closed* submodule in M if X has no proper co-essential submodule in M. X' is called a *co-closure* of X in M if X' is a co-closed submodule of M with $X' \subseteq_c X$ in M. $K <_{\oplus} N$ means that K is a direct summand of N.

The detailed version of this paper has been submitted for publication elsewhere.

A module M is called a *lifting* module if, for any submodule X of M, there exists a direct summand X^* of M such that $X^* \subseteq_c X$.

Let $\{M_i \mid i \in I\}$ be a family of modules and let $M = \oplus_I M_i$. M is said to be a *lifting module for* the decomposition $M = \oplus_I M_i$ if, for any submodule X of M, there exist $X^* \subseteq M$ and $\overline{M_i} \subseteq M_i$ $(i \in I)$ such that $X^* \subseteq_c X$ and $M = X^* \oplus (\oplus_I \overline{M_i})$, that is, M is a lifting module and satisfies the internal exchange property in the direct sum $M = \oplus_I M_i$.

Let X be a submodule of M. A submodule Y of M is called a *supplement* of X in M if $M = X + Y$ and $X \cap Y \ll Y$. Note that any supplement submodule (hence any direct summand) of a module M is co-closed in M.

A module M is called *supplemented* if, X contains a supplement of Y in M whenever $M = X + Y$. Note that the module M is lifting if and only if M is supplemented and every supplement submodule of M is a direct summand. An epimorphism $P \xrightarrow{f} M \to 0$ with P projective, is called a *projective cover* of M if $\ker f \ll P$. The notion of a projective cover is dual to that of an injective hull. A ring R is *right perfect* if every right R-module has a projective cover. Now we consider the following condition:

(∗) Any submodule of M has a co-closure in M.

By [4, Theorem 1.3], we note that every module M over right perfect ring satisfies the condition (∗).

Proposition 1.1. *A ring R is right perfect if and only if every projective right R-module is lifting.*

Proof. Let P be a projective R-module. For any submodule A of P, consider a canonical epimorphism $\varphi : P \to P/A$. Since P/A has a projective cover, there exists a decomposition $P = P_1 \oplus P_2$ such that $P_2 \subseteq ker(\varphi) = A$ and $(\varphi|P_1) : P_1 \to P/A \to 0$ a projective cover. Thus $ker(\varphi|P_1) = P_1 \bigcap A \ll P$, hence P is lifting. Conversely, since any module M is an epimorphic image of a free module, there exist a projective module P and epimorphism $f : P \to M$. Since P is lifting, there exists a decomposition $P = A^* \oplus A^{**}$ such that $A^* \subseteq_c ker(f)$ in P. Thus $(f|_{A^{**}}) : A^{**} \to M \to 0$ is a projective cover. □

The following lemmas are due to Oshiro [4].

Lemma 1.2. *Any projective module satisfies the following condition:*

(D) If M_1 and M_2 are direct summands of M such that $M_1 \cap M_2 \ll M$ and $M = M_1 + M_2$, then $M = M_1 \oplus M_2$.

Lemma 1.3. *If M is a lifting module with the condition (D), then M satisfies the following condition:*

(D′) If $M_i <_\oplus M$ $(i \in I)$, $M = \sum_I M_i$ and $M_j \cap \sum_{i \neq j} M_i \ll M$, then $M = \sum_I M_i$ is direct.

2. Local Summands

$\sum \oplus_{\lambda \in \Lambda} X_\lambda \subseteq X$ is called a *local summand* of X, if $\sum \oplus_{\lambda \in F} X_\lambda <_\oplus X$ for every finite subset $F \subseteq \Lambda$.

The following lemma due to Oshiro is useful. For Okado's result above, this lemma was used.

Lemma 2.1 (cf. [5]). *If every local summand of M is a direct summand, then M has an indecomposable decomposition.*

By Lemma 1.2 and [4, Proposition 3.2], the following holds:

Lemma 2.2 (cf. [4]). *Every local summand of projective lifting modules is a direct summand.*

A module M is said to have the (finite) exchange property if, for any (finite) index set I, whenever $M \oplus N = \oplus_I A_i$ for modules N and A_i, then $M \oplus N = M \oplus (\oplus_I B_i)$ for submodules $B_i \subseteq A_i$.

A module M has the (finite) internal exchange property if, for any (finite) direct sum decomposition $M = \oplus_I M_i$ and any direct summand X of M, there exist submodules $\overline{M_i} \subseteq M_i$ such that $M = X \oplus (\oplus_I \overline{M_i})$.

The following is essentially due to Harada [1].

Theorem 2.3. *Let $M = \oplus_I M_\alpha$, where each M_α has a local endomorphism ring. Then the following conditions are equivalent:*

(1) *M has the internal exchange property (in the direct sum $M = \oplus_I M_\alpha$);*

(2) *M has the (finite) exchange property;*

(3) *Every local summand of M is a direct summand.*

3. Main Results

The following is a main result of our talk.

Theorem 3.1. *If R is a right perfect ring, then every local summand of lifting modules is direct summand.*

Sketch of Proof: Let M be a lifting module and let $\oplus_I X_i$ be a local summand of M. Since R is a right perfect ring, M has a projective cover, say $\ker f \ll P \xrightarrow{f} M \to 0$. By Lemma 1.1, P is projective lifting. So there exists a decomposition $P = P_i \oplus P_i^*$ such that $P_i \subseteq_c f^{-1}(X_i)$ in P. As X_i is co-closed in M, $f(P_i) = X_i$. By Lemma 3.1, $\sum_I P_i = \oplus_I P_i$. By Lemma 2.3, $\oplus_I P_i <_\oplus P$. Since $f(\oplus_I P_i)$ is co-closed in M, we see

$$\oplus_I X_i = f(\oplus_I P_i) <_\oplus M.$$

Thus any local summand of M is a direct summand.

□

The following result is a consequence of Lemma 2.1 and Theorem 3.1.

Theorem 3.2. *Any lifting module over right perfect rings has an indecomposable decomposition.*

A module H is said to be *hollow* if H is indecomposable lifting. By a proof of [9, Proposition 1], we see

Lemma 3.3. *Let X be a hollow module. If $X \oplus X$ has the internal exchange property, then X has a local endomorphism ring.*

By Theorem 3.2 and Lemma 3.3, the following holds:

Corollary 3.4. *Let R be a right perfect ring and M be a lifting module. If M has the finite exchange property, then it has exchange property.*

A module A is said to be *generalized B-projective (B-cojective)* if, for any homomorphism $f : A \to X$ and any epimorphism $g : B \to X$, there exist decompositions $A = A_1 \oplus A_2$, $B = B_1 \oplus B_2$, a homomorphism $h_1 : A_1 \to B_1$ and an epimorphism $h_2 : B_2 \to A_2$ such that $g \circ h_1 = f|_{A_1}$ and $f \circ h_2 = g|_{B_2}$ (cf. [8]). The concept of generalized projective is a dual one of generalized injective (cf. [6]). Note that every B-projective modules is generalized B-projective.

Lemma 3.5 (cf. [2]). *Let M_1 and M_2 be lifting modules and put $M = M_1 \oplus M_2$. Assume that M satisfies the condition $(*)$. Then M is lifting for $M = M_1 \oplus M_2$ if and only if M_i is generalized M_j-projective ($i \neq j$).*

By Lemma 3.5, 3.3, Theorem 3.1 and 2.3, we obtain the following:

Corollary 3.6. *Let R be a right perfect ring and M be a lifting module. If M is generalized M-projective, then M has the exchange property. In particular, any projective module over right perfect rings has the exchange property.*

Remark Recently, we obtained that any hollow module over right perfect rings has a local endomorphism ring. Thus, by Theorem 2.3, the following holds: Any lifting module over right perfect rings has the exchange property.

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NON-COMMUTATIVE VALUATION RINGS OF $K(X;\sigma,\delta)$ OVER A DIVISION RING K

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Abstract

Let K be a division ring with a σ-derivation δ, where σ is an endomorphism of K and $K(X;\sigma,\delta)$ be the quotient division ring of the Ore extension $K[X;\sigma,\delta]$ over K in an indeterminate X. Suppose that (σ,δ) is compatible with V, where V is a total valuation ring of K, then $R^{(1)} = V[X;\sigma,\delta]_{J(V)[X;\sigma,\delta]}$, the localization of $V[X;\sigma,\delta]$ at $J(V)[X;\sigma,\delta]$, is a total valuation ring of $K(X;\sigma,\delta)$. The aim of this paper is to describe non-commutative valuation rings B of $K(X;\sigma,\delta)$ such that $R^{(1)} \supseteq B, B\cap K = V$ and $X \in B$.

1. Introduction

Let K be a division ring, σ be an endomorphism of K and δ be a σ-derivation, i.e., an additive map $\delta : K \longrightarrow K$ such that $\delta(ab) = \sigma(a)\delta(b)+\delta(a)b$ for all $a,b \in K$. As usaual, $K[X;\sigma,\delta] = \{f(X) \mid f(X) = a_nX^n + \cdots + a_0, a_i \in K\}$ is the Ore extension over K with $Xa = \sigma(a)X + \delta(a)$ for any $a \in K$, where X is an indeterminate. Let V be a total valuation ring of K and (σ,δ) be compatible with V. Then, in [BT], they proved by Krull methed that $J(V)[X;\sigma,\delta]$ is a left Ore set of $V[X;\sigma,\delta]$ and $R^{(1)} = V[X;\sigma,\delta]_{J(V)[X;\sigma,\delta]}$, the localization of $V[X;\sigma,\delta]$ at $J(V)[X;\sigma,\delta]$, is a total valuation ring of $K(X;\sigma,\delta)$ such that $R^{(1)}\cap K = V, X \in R^{(1)}$. The aim of this paper is to describe non-commutative valuation rings B of $K(X;\sigma,\delta)$ such that $R^{(1)} \supseteq B, B\cap K = V$ and $X \in B$.

There are three types of non-commutative valuation rings as follows: Let Q be a simple Artinian ring and let R be an order in Q, i.e., R is a prime Goldie ring. We say that R is a *Dubrovin* valuation ring of Q if R is semi-hereditary and R is local, i.e., $R/J(R)$ is a simple Artinian ring, where $J(R)$

*The detailed version of this paper has been submitted for publication elsewhere is the Jacobson radical of R. Assume that Q is a division ring. A subring R of Q is said to be a *total* valuation ring of Q, if for any non-zero $q \in Q$, either $q \in R$ or $q^{-1} \in R$. In the case where K is a field, the definition of total valuation rings is the same as one of the equivalent definitions of [G, (16.3)]. However, in the case where K is not a field, the structure of total valuation rings is much different from one of commutative valuation rings (see, [D_2], [BMO]). Furthermore, a total valuation ring R of Q is said to be *invariant* if $q^{-1}Rq = R$ for all non-zero $q \in Q$. It is easy to see that a total valuation ring R is a Dubrovin valuation ring and the converse does not necessarily hold. We refer to [MMU] for more properties of non-commutative valuation rings.

2. Non-commutative valuation rings of $K(X;\sigma,\delta)$ contained in $R^{(1)}$

Let K be a division ring, σ be an endomorphism of K, δ be a σ-derivation and V be a total valuation ring of K. We assume that (σ,δ) is compatible with V, i.e., $\sigma(V) \subseteq V, \sigma(J(V)) \subseteq J(V), \delta(V) \subseteq V, \delta(J(V)) \subseteq J(V)$. In [BT], they proved that $J(V)[X;\sigma,\delta]$ is left localizable and $R^{(1)} = V[X;\sigma,\delta]_{J(V)[X;\sigma,\delta]}$, the localization of $V[X;\sigma,\delta]$ at $J(V)[X;\sigma,\delta]$, is a total valuation ring of $K(X;\sigma,\delta)$ with $R^{(1)} \cap K = V, X \in R^{(1)}$ and we studied some properties of $R^{(1)}$ (see [XKM]). In this paper we shall study non-commutative valuation rings B of $K(X;\sigma,\delta)$ such that $B \cap K = V, B \subseteq R^{(1)}$ and $X \in B$. For simplicity, we denote by $\mathcal{D}$ the set of all Dubrovin valuation rings B of $K(X;\sigma,\delta)$ such that $B \cap K = V$, $R^{(1)} \supseteq B$ and $X \in B$, by $\mathcal{T}$ the set of all total valuation rings B of $K(X;\sigma,\delta)$ such that $B \cap K = V$, $R^{(1)} \supseteq B$ and $X \in B$. Obviously, $\mathcal{T} \subseteq \mathcal{D}$. To get some information of $\mathcal{D}$ and $\mathcal{T}$, we use the following natural map.

Let $\varphi : R^{(1)} = V[X;\sigma,\delta]_{J(V[X;\sigma,\delta])} \longrightarrow \overline{R^{(1)}} = R^{(1)}/J(R^{(1)}) \cong \overline{V}(X;\overline{\sigma},\overline{\delta})$ be the natural homomorphism, where $\overline{\sigma}(\overline{v}) = [\sigma(v) + J(V)]$ and $\overline{\delta}(\overline{v}) = [\delta(v) + J(V)]$ for any $\overline{v} = [v + J(V)] \in \overline{V} = V/J(V)$. Set $R = \varphi^{-1}(\overline{V}[X;\overline{\sigma},\overline{\delta}]) = V[X;\sigma,\delta] + J(R^{(1)})$.

The following lemma is crucial to find out non-commutative valuation rings in $\mathcal{D}$.

Lemma 2.1 There is a one-to-one correspondence between $\mathcal{D}$ and the set of all Dubrovin valuation rings $\mathfrak{B}$ of $\overline{V}(X;\overline{\sigma},\overline{\delta})$ with $\mathfrak{B} \supseteq \overline{V}[X;\overline{\sigma},\overline{\delta}]$, which is given by $\varphi(B) = \mathfrak{B}$ and $\varphi^{-1}(\mathfrak{B}) = B$, where $B \in \mathcal{D}$.

Proof Let $B \in \mathcal{D}$ with $B \neq R^{(1)}$. Then $B \supsetneq J(R^{(1)})$ and $\varphi(B) = B/J(R^{(1)})$ is a Dubrovin valuation ring of $\overline{R^{(1)}}$ (see [MMU, (6.6)]).

Conversely, it is not difficult to prove that for any Dubrovin valuation ring $\mathfrak{B}$ of $\overline{V}(X;\overline{\sigma},\overline{\delta})$ with $\mathfrak{B} \supseteq \overline{V}[X;\overline{\sigma},\overline{\delta}]$, $B = \varphi^{-1}(\mathfrak{B})$ is in $\mathcal{D}$.

Remark: Since $R^{(1)}$ is a total valuation ring, we can replace "$\mathcal{D}$" and "Dubrovin valuation rings" by "$\mathcal{T}$" and "total valuation rings" respectively in Lemma 2.1.

We know from Lemma 2.1 that it suffices to study Dubrovin valuation rings $\mathfrak{B}$ of $\overline{V}(X;\overline{\sigma},\overline{\delta})$ with $\mathfrak{B} \supseteq \overline{V}[X;\overline{\sigma},\overline{\delta}]$ in order to describe Dubrovin valuation rings in $\mathcal{D}$. Let $\mathfrak{B}$ be any proper Dubrovin valuation ring of $\overline{V}(X;\overline{\sigma},\overline{\delta})$ with $\mathfrak{B} \supseteq \overline{V}[X;\overline{\sigma},\overline{\delta}]$. Then $J(\mathfrak{B}) \cap \overline{V}[X;\overline{\sigma},\overline{\delta}]$ is a non-zero ideal of $\overline{V}[X;\overline{\sigma},\overline{\delta}]$. Since $\overline{V}[X;\overline{\sigma},\overline{\delta}]$ is a left principal ideal domain (note that $\overline{V}$ is a division ring), some $\mathfrak{B}$ are obtained by the localization of $\overline{V}[X;\overline{\sigma},\overline{\delta}]$ at some maximal ideals. These show that there are close relatons between the ideal theory of $\overline{V}[X;\overline{\sigma},\overline{\delta}]$ and non-commutative valuation rings of $\overline{V}(X;\overline{\sigma},\overline{\delta})$. So we will firstly study the ideal theory of $K[X;\sigma,\delta]$, where K is a division ring, which is deeply depend on the properties of σ and δ. In order to study the ideal theory of $K[X;\sigma,\delta]$, we will introduce the following which was defined in [L] (also see [LL]). δ is said to be *a quasi-algebraic* σ-derivation if there exist $a_n = 1, a_{n-1}, \cdots, a_0 \in K, n > 0$ such that $\Sigma_{i=1}^{n} a_i \delta^i = D_{a_0,\sigma^n}$, where $D_{a_0,\sigma^n}(b) = a_0 b - \sigma^n(b) a_0$ for any $b \in K$. The *inner order* of σ, denoted by $o(\sigma)$, is defined by the smallest positive integer n such that $\sigma^n = I_a$, the inner automorphism induced by some $a \in K$; if no such natural number n exists, then $o(\sigma)$ is ∞. A monic polynomial $p(X)$ with deg $p(X) = n$ is said to be *invariant* if for any $a \in K, p(X)a = \sigma^n(a)p(X), p(X)X = (X+c)p(X)$ for some $c \in K$.

The following lemma was proved by Lam, Leung, Leroy and Matczuk (see [LLLM, 3.6]) after long history starting from Amitsur [A], in the case where $\sigma = 1$.

Lemma 2.2 *$K[X;\sigma,\delta]$ is simple if and only if δ is not a quasi-algebraic σ-derivation.*

In the case where δ is a quasi-algebraic σ-derivation, there exists a monic invariant polynomial $p(X)$ of minimal non-zero degree from Lemma 2.2 which plays a very important role to study the ideal theory of $K[X;\sigma,\delta]$ as follows.

Lemma 2.3 (Cauchon) *Let δ be a quasi-algebraic σ-derivation. Then there is a monic invariant polynomial $p(X)$ of minimal non-zero degree.*

(1) *If $o(\sigma) = \infty$, then $Z(K[X;\sigma,\delta]) = Z(K)_{\sigma,\delta}$, where $Z(S)$ stands for the center of S for any ring S and $Z(K)_{\sigma,\delta} = \{a \in Z(K) \mid \sigma(a) = a \text{ and } \delta(a) = 0\}$.*

(2) *If $o(\sigma) = n < \infty$, then $Z(K[X;\sigma,\delta]) = Z(K)_{\sigma,\delta}[\lambda p(X)^l]$ for some non-zero $\lambda \in K$ and some natural number l.*

Lemma 2.4 *Let δ be a quasi-algebraic σ-derivation and $p(X)$ be a monic* *invariant polynomial of minimal non-zero degree.*

(1) *If* $o(\sigma) = \infty$, *then* $P = K[X;\sigma,\delta]p(X)$ *is the only maximal ideal of* $K[X;\sigma,\delta]$.

(2) *If* $o(\sigma) = n < \infty$, *then any maximal ideal of* $K[X;\sigma,\delta]$ *is one of the following:* $P = K[X;\sigma,\delta]p(X)$, $M = K[X;\sigma,\delta]w(Y)$, *where* $Y = \lambda p(X)^l$ *for some* $\lambda \in K$ *and* $w(Y)$ *runs over all irreducible polynomials of* $Z(K)_{\sigma,\delta}[Y]$ *different from* Y.

We will study non-commutative valuation rings in $\mathcal{D}$ in the following five cases.

Theorem 2.5 (case 1) *If* $\overline{\delta}$ *is not a quasi-algebraic* $\overline{\sigma}$*-derivation, then* $\mathcal{D} = \{R^{(1)}\}$.

Proof Let $B \in \mathcal{D}$ with $B \neq R^{(1)}$. Then $\mathfrak{B} = \varphi(B)$ is a proper Dubrovin valuation ring of $\overline{V}(X;\overline{\sigma},\overline{\delta})$ containing $\overline{V}[X;\overline{\sigma},\overline{\delta}]$ and so $J(\mathfrak{B}) \cap \overline{V}[X;\overline{\sigma},\overline{\delta}]$ is a non-zero ideal of $\overline{V}[X;\overline{\sigma},\overline{\delta}]$ which is a contradiction to Lemma 2.2.

From now on we assume that $\overline{\delta}$ is a quasi-algebraic $\overline{\sigma}$-derivation. Then we can find a monic polynomial $p(X) \in V[X;\sigma,\delta]$ such that $\varphi(p(X)) = \overline{p(X)} \in \overline{V}[X;\overline{\sigma},\overline{\delta}]$ is a monic invariant polynomial of mininal non-zero degree. Let $P = Rp(X)$. At first we study the case $\overline{\sigma} \in \mathrm{Aut}(\overline{V})$. In this case $\overline{V}[X;\overline{\sigma},\overline{\delta}]$ is a Dedekind order in $\overline{V}(X;\overline{\sigma},\overline{\delta})$. Then we can use the following lemma which follows from [D_1, Theorems 3 and 4, §2].

Lemma 2.6 *Let* Q *be a simple Artinian ring and* S *be a Dedekind order in* Q. *Then there is a one-to-one correspondence between the set of all proper Dubrovin valuation rings of* Q *containing* S *and the set of all non-zero maximal ideals of* S, *which is given by* $\varphi : B \longrightarrow J(B) \cap S$ *and* $\varphi^{-1} : P \longrightarrow S_P$, *where* B *is a proper Dubrovin valuation ring of* Q *containing* S *and* P *is a non-zero maximal ideal of* S.

Theorem 2.7 *Suppose that* $\overline{\delta}$ *is a quasi-algebraic* $\overline{\sigma}$*-derivation and* $\overline{\sigma} \in$ *Aut*$(\overline{V})$.

(1) (case 2) *If* $o(\overline{\sigma}) = \infty$, *then* $\mathcal{D} = \{R^{(1)}, R_P\}$, *where* $P = Rp(X)$.

(2) (case 3) *If* $o(\overline{\sigma}) = m < \infty$, *then* $\mathcal{D} = \{R^{(1)}, R_P\} \cup \{R_M \mid M = Rw(X)$, *where* $w(X) \in V[X;\sigma,\delta]$ *such that* $\overline{w(X)}$ *is an irreducible polynomial of* $Z(\overline{V})_{\overline{\sigma},\overline{\delta}}[Y](Y = \overline{\lambda}\overline{p(X)}^l$ *for some* $\overline{\lambda} \in \overline{V}$ *and* $l \geq 1$ *as in Lemma 2.4* $\}$. *In particular,* $J(R_P) = R_Pp(X) = p(X)R_P$ *and* $J(R_M) = R_Mw(X) = w(X)R_M$.

Proof Since $p(X), w(X) \in U(R^{(1)})$, we easily have $P = \varphi^{-1}(\overline{V}[X;\overline{\sigma},\overline{\delta}]\overline{p(X)}) =$

$Rp(X) = p(X)R$ and $M = \varphi^{-1}(\overline{V}[X;\overline{\sigma},\overline{\delta}]\overline{w(X)}) = Rw(X) = w(X)R$, where $R = V[X;\sigma,\delta] + J(R^{(1)})$. Hence the theorem follows from Lemmas 2.3, 2.4 and 2.6.

Now we consider the case where $\overline{\delta}$ is a quasi-algebraic $\overline{\sigma}$-derivation and $\overline{\sigma} \notin \mathrm{Aut}(\overline{V})$. Let $\overline{p(X)}$ be a monic invariant polynomial of $\overline{V}[X;\overline{\sigma},\overline{\delta}]$ with minimal non-zero degree. Then $\overline{p(X)}\overline{V}[X;\overline{\sigma},\overline{\delta}] \subsetneq \overline{V}[X;\overline{\sigma},\overline{\delta}]\overline{p(X)}$, we can get two ascending chain:

$\overline{V}[X;\overline{\sigma},\overline{\delta}] \subsetneq \overline{p(X)}^{-1}\overline{V}[X;\overline{\sigma},\overline{\delta}]\overline{p(X)} \subsetneq \overline{p(X)}^{-2}\overline{V}[X;\overline{\sigma},\overline{\delta}]\overline{p(X)}^{2} \subsetneq \cdots$

$\overline{V} \subsetneq \overline{p(X)}^{-1}\overline{V}\overline{p(X)} \subsetneq \overline{p(X)}^{-2}\overline{V}\overline{p(X)}^{2} \subsetneq \cdots$

Let $\hat{\overline{V}} = \cup_{m=1}^{\infty}\overline{p(X)}^{-m}\overline{V}\overline{p(X)}^{m}$, a division ring containing $\overline{V}$, $\hat{\overline{R}} = \cup_{m=1}^{\infty}\overline{p(X)}^{-m}\overline{R}\overline{p(X)}^{m}$, where $\overline{R} = \overline{V}[X;\overline{\sigma},\overline{\delta}]$. For any $\alpha = \overline{p(X)}^{-l}\overline{a}\overline{p(X)}^{l} \in \hat{\overline{V}}$, define $\hat{\overline{\sigma}}(\alpha) = \overline{p(X)}^{-l}\overline{\sigma}(\overline{a})\overline{p(X)}^{l}$. We can check that $\hat{\overline{\sigma}}$ is well-defined and an automorphism of $\hat{\overline{V}}$ with $o(\hat{\overline{\sigma}}) = \infty$.

We define $\hat{\overline{\delta}}$ as follows; let $\alpha = \overline{p(X)}^{-l}\overline{a}\overline{p(X)}^{l}$ be any element of $\hat{\overline{V}}$, where $\overline{a} \in \overline{V}$. Since

$\overline{p(X)}^{l}X = (X + \overline{b_l})\overline{p(X)}^{l}$ for some $\overline{b_l} \in \overline{V}$, we have

$(X + \overline{b_l})\overline{a} = \overline{\sigma}(\overline{a})X + \overline{\delta}(\overline{a}) + \overline{b_l}\overline{a} = \overline{\sigma}(\overline{a})(X + \overline{b_l}) + \overline{a_l}$,

where $\overline{a_l} = \overline{\delta}(\overline{a}) + \overline{b_l}\overline{a} - \overline{\sigma}(\overline{a})\overline{b_l} \in \overline{V}$.

Define $\hat{\overline{\delta}}(\alpha) = \overline{p(X)}^{-l}\overline{a_l}\overline{p(X)}^{l} \in \hat{\overline{V}}$. Then

$$\begin{aligned}\overline{p(X)}^{l}X\alpha &= \overline{p(X)}^{l}X\overline{p(X)}^{-l}\overline{a}\overline{p(X)}^{l}\\ &= (X + \overline{b_l})\overline{a}\overline{p(X)}^{l}\\ &= (\overline{\sigma}(\overline{a})(X + \overline{b_l}) + \overline{a_l})\overline{p(X)}^{l}.\end{aligned}$$ So

$$\begin{aligned}X\alpha &= \overline{p(X)}^{-l}(\overline{\sigma}(a)(X + \overline{b_l}) + \overline{a_l})\overline{p(X)}^{l}\\ &= \overline{p(X)}^{-l}\overline{\sigma}(\overline{a})\overline{p(X)}^{l}X + \overline{p(X)}^{-l}\overline{a_l}\overline{p(X)}^{l} = \hat{\overline{\sigma}}(\overline{\alpha})X + \hat{\overline{\delta}}(\alpha).\end{aligned}$$

This means $\hat{\overline{\delta}}(\alpha) = X\alpha - \hat{\overline{\sigma}}(\overline{\alpha})X$. So $\hat{\overline{\delta}}$ is well-defined. Furthermore, we can easily check that $\hat{\overline{\delta}}$ is a $\hat{\overline{\sigma}}$-derivation and $\hat{\overline{R}} = \hat{\overline{V}}[X;\hat{\overline{\sigma}},\hat{\overline{\delta}}]$. Let $\hat{R} = \cup_{m=1}^{\infty}p(X)^{-m}Rp(X)^{m}$, an over ring of R. It is easy to see that $\hat{R} = \varphi^{-1}(\hat{\overline{R}})$, an Prüfer order in $K(X;\sigma,\delta)$. Let $\hat{\overline{P}} = \hat{\overline{R}}\overline{p(X)}$. Then $\hat{\overline{P}}$ is the unique maximal ideal of $\hat{\overline{R}}$. Let $\hat{P} = \varphi^{-1}(\hat{\overline{P}})$. We can get $\hat{R}_{\hat{P}} \in \mathcal{D}$.

We say that $\hat{\overline{V}}$ is *algebraic* over $\overline{V}$ if for any $\overline{\alpha} \in \hat{\overline{V}}$, there exist $\overline{c_s}, \cdots, \overline{c_0} \in \overline{V}$, not all zero, such that $\Sigma_{i=0}^{s}\bar{c}_i\bar{\alpha}^{i} = 0$. It is not difficult to prove the following lemma.

Lemma 2.8 *If $\hat{\overline{V}}$ is algebraic over $\overline{V}$, then any total valuation ring B of $\overline{V}(X;\overline{\sigma},\overline{\delta})$ with $\overline{B} \supseteq \overline{V}[X;\overline{\sigma},\overline{\delta}]$ containing $\hat{\overline{V}}[X;\hat{\overline{\sigma}},\hat{\overline{\delta}}]$.*

If $\overline{P}$ is a completely prime ideal of $\overline{V}[X;\overline{\sigma},\overline{\delta}]$, then it is easy to prove that $\hat{\overline{P}}$ is a completely prime ideal of $\hat{\overline{V}}[X;\hat{\overline{\sigma}},\hat{\overline{\delta}}]$. In this case $\hat{R}_{\hat{P}}$ is a total valuation ring of $K(X;\sigma,\delta)$. Now by Lemmas 2.1, 2.3,2.6, and 2.8, we can get that $\mathcal{T}=\{R^{(1)},\hat{R}_{\hat{P}}\}$. Thus we get the following theorem.

Theorem 2.9 (case 4) *Suppose that $\overline{\delta}$ is a quasi-algebraic $\overline{\sigma}$-derivation and $\overline{\sigma}\notin Aut(\overline{V})$. Then*

(1) $\hat{R}_{\hat{P}}\in\mathcal{D}$.

(2) *If $\hat{\overline{V}}$ is algebraic over $\overline{V}$ and $\hat{\overline{P}}$ is completely prime, then $\mathcal{T}=\{R^{(1)},\hat{R}_{\hat{P}}\}$.*

If $\hat{\overline{V}}$ is not algebraic over $\overline{V}$, then it becomes very sophisticated, even in the case where $\overline{V}$ is a field. In this case, $\hat{\overline{P}}$ is a completely prime ideal of $\overline{V}[X;\overline{\sigma},\overline{\delta}]$ and $\hat{R}_{\hat{P}}$ is a total valuation ring. By using some properties of commutative valuation rings in polynomial rings and localizations, we can construct total valuation rings $\{A_i\},\{B_i\}$ satisfying the following properties.

Theorem 2.10 (case 5) *Suppose that $\overline{\delta}$ is a quasi-algebraic $\overline{\sigma}$-derivation and $\overline{\sigma}\notin Aut(\overline{V})$. If $\hat{\overline{V}}$ is not algebraic over $\overline{V}$ and $\overline{V}$ is a field with $\alpha=tr.deg_{\overline{V}}\hat{\overline{V}}$, then α is infinite and there are $\{A_i\}$, $\{B_i\}\subseteq\mathcal{D}$ satisfying;*

(i) $\hat{R}_{\hat{P}}\supsetneq A_i$ *for any $i\in\Lambda$ and A_i are incomparable each other, where Λ is an index set with $|\Lambda|=\alpha$.*

(ii) $B_1\subsetneq B_2\subsetneq\cdots\subsetneq B_\alpha\subsetneq\hat{R}_{\hat{P}}$.

We have studied Dubrovin valuation rings in $\mathcal{D}$ classifying them into five cases. We can provide examples of total valuation rings satisfying the properties in each case. However, we will not mention the examples because of the limited pages. We refer the readers to the detailed version of this paper for examples.

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Non-semisimple Hopf algebras over a field of characteristic $p \neq 0$ using Personal Computer.

KENJI YOKOGAWA

§1 Abstract and Introduction

Let K be a field of characteristic $p \neq 0$ and H be a Hopf algebra of dimension p over K generated by a p-nilpotent element x, $x^p = 0$ (simply we say a *p-nilpotent* Hopf algebra. We doesnot assume that p-nilpotent Hopf algebra is cocommutative). A group ring $H = K[\sigma]$, $\sigma^p = 1$, is such a Hopf algebra with a p-nilpotent generator $\sigma - 1$ (we call Hopf algebra of this type as a Hopf algebra of *automorphism type*). A Hopf algebra generated by a derivation d, $d^p = 0$, is also such a Hopf algebra (we call as a Hopf algebra of *derivation type*). Another type of p-nilpotent Hopf algebra is not known. So the question whether these two types of Hopf algebra are the all of p-nilpotent Hopf algebras arises. p-nilpotent Hopf algebras are tiny, primitive, fundamental Hopf algebras and it seems easy to classify these algberas. But the recent method using semisimplicity does not work for these algebras and there is no useful methods for these Hopf algebras. Classifying these algberas needs too laborious calculations. Especially to check coassociativity is laborious. And this prevents to get good conjectures to these algebras.

Using a personal computer, the author tried to investigate these algebra in lower characteristic cases. And examining the great amount of calculations in details, the author found some induction rules and succeeded to determine the all p-nilpotent Hopf algebras in an arbitrary non-zero characteristic case. We get *for a suitable generator x of a p-nilpotent Hopf algebra, its diagonal map Δ has a following form;* $\Delta(x) = x \otimes 1 + 1 \otimes x + cx^{p-1} \otimes x - 2^{-1}cx^{p-2} \otimes x^2 + 3^{-1}cx^{p-3} \otimes x^3 + \cdots - (p-3)^{-1}cx^3 \otimes x^{p-3} + (p-2)^{-1}cx^2 \otimes x^{p-2} - (p-1)^{-1}cx \otimes x^{p-1}$, $c \in K$. As a Corollary, we get *if K is a prime field then there exists exactly p non-isomorphic p-nilpotent Hopf algebras and if K is an algebraically closed field then there exists exactly 2 non-isomorphic p-nilpotent Hopf algebras* .

† ここでは, Mathematica の使い方を中心に解説するが, 証明を付けた物はその内発表予定である.

In this paper, we discuss the utilization of a personal computer to mathematics.

Throughout of this paper, K denotes a field of characteristic $p \neq 0$, H denotes a p-nilpotent Hopf algebra over K unless otherwise stated. We shall use the terminology of [Sw]. Especially, the diagonal map of H is denoted by Δ, the augmentation map is denoted by ε, and the antipode is denoted by S. $\otimes$ means $\otimes_K$.

§2 Mathematical Soft Ware

数式処理 Software として、Mathematica, Maple などがある．これらの Software の主な機能として、数値計算, シンボル計算（文字式の計算）, 組み込み関数を用いた計算, グラフの表示, 行列計算, 連立一次方程式, 微積分, 微分方程式などがある. これらの機能の内, 特にシンボル計算（文字式の計算）に注目する.

Hopf algebra の coassociativity の計算が大変な訳であるが, 具体的にどれくらい大変かを示してみよう. この例では, 後ほど述べる Lemma 1 を用いて, $\Delta(x)$ を書き表している.
$p=5$ とし, $\Delta : H \to H\otimes H$, $\Delta(x) = x\otimes 1 + 1\otimes x + c_{11}x\otimes x + c_{21}x^2\otimes x + c_{12}x\otimes x^2 + c_{31}x^3\otimes x + c_{22}x^2\otimes x^2 + c_{13}x\otimes x^3 + c_{41}x^4\otimes x + c_{32}x^3\otimes x^2 + c_{23}x^2\otimes x^3 + c_{14}x\otimes x^4 + c_{42}x^4\otimes x^2 + c_{33}x^3\otimes x^3 + c_{24}x^2\otimes x^4 + c_{43}x^4\otimes x^3 + c_{34}x^3\otimes x^4 + c_{44}x^4\otimes x^4$ と書く, 但し x は p-nilpotent generator .

coassociativity を調べるには, $(\Delta\otimes 1)\Delta(x)$, $(1\otimes\Delta)\Delta(x)$ を計算する訳であるが, $(\Delta\otimes 1)\Delta(x)$ の部分は
$(\Delta\otimes 1)\Delta(x) = \Delta(x)\otimes 1 + 1\otimes x + c_{11}\Delta(x)\otimes x + c_{21}\Delta(x)^2\otimes x + c_{21}\Delta(x)\otimes x^2 + c_{31}\Delta(x)^3\otimes x + c_{22}\Delta(x)^2\otimes x^2 + c_{13}\Delta(x)\otimes x^3 + c_{41}\Delta(x)^4\otimes x + c_{32}\Delta(x)^3\otimes x^2 + c_{23}\Delta(x)^2\otimes x^3 + c_{14}\Delta(x)\otimes x^4 + c_{42}\Delta(x)^4\otimes x^2 + c_{33}\Delta(x)^3\otimes x^3 + c_{24}\Delta(x)^2\otimes x^4 + c_{43}\Delta(x)^4\otimes x^3 + c_{34}\Delta(x)^3\otimes x^4 + c_{44}\Delta(x)^4\otimes x^4$ となる.
最後の項 $c_{44}\Delta(x)^4\otimes x^4 = c_{44}\{x\otimes 1 + 1\otimes x + c_{11}x\otimes x + \cdots + c_{44}x^4\otimes x^4\}^4\otimes x^4$ である.

膨大な計算量であるのが判るであろう. しかも調べた所では, $p=7$ まで計算しないと一般形にたどり着かない. とても人間業では無い. この計算の部分を computer + 数式処理 Software に任せ, その結果から予想を出し、それを証明する訳である.

注 1: 私が使用した数式処理 SoftWare は「Mathematica」であるが, それ以外に「Maple」がある. 色々な点で「Maple」の方が優れているように思われる.

注 2: 最初 p-nilpotent Hopf algebra に cocommutative を仮定し, その後一般の場合を調べた. 数式処理 Software の利用方法を説明するのには, cocommutative の場合の方が適切なので, cocommutative を仮定して説明する.

§3 Preliminary

Lemma 1. *Let H be a Hopf algebra generated by a p-nilpotent element x (throughout H denotes such a Hopf algebra). Write $\Delta(x) = \sum c_{ij}x^i \otimes x^j$ $(0 \leq i, j < p)$, then $c_{00} = 0, c_{10} = 1, c_{i0} = 0 (1 \leq i < p), c_{01} = 1$ and $c_{0j} = 0$ $(1 \leq j < p)$.*

Proof. Use relations $(\varepsilon \otimes 1)\Delta(x) = (1 \otimes \varepsilon)\Delta(x) = x$.

$\Delta(x) = \sum c_{ij}x^i \otimes x^j$ として, multiplicative, linear に拡張すれば, Lemma 1 から常に $\Delta(x)^p = \Delta(x^p) = 0$ を満たすので, Δ は algebra-homo. になる. 即ち,

Lemma 2. *Let c_{ij} be coefficients satisfying the relation of Lemma 1. We write $\Delta(x) = \sum c_{ij}x^i \otimes x^j$ and extend to $\Delta : H \to H \otimes H$ multiplicatively and linearly, then Δ is an algebra homomorphism.*

Lemma 3. *$y \in H$ is also an p-nilpotent generator if and only if $y = a_1x + a_2x^2 + \ldots + a_{p-1}x^{p-1}$ with a_1 a unit of K.*

Lemma 3 から、良い generator を取る事が大切になる訳である. 実際に $p = 5$ の場合に mathematica を動かした結果を見せ, どのような generator を取れば良いか説明する.

Deltas $= \Delta(x), H = K[x], H \otimes H = K[X, Y], X = x \otimes 1, Y = 1 \otimes x$.
degree 4 以下の部分だけを考えるので, $c_{41} = c_{32} = c_{23} = c_{14} = c_{43} = c_{34} = 0$ とし,
$\Delta(x) = X + Y + c_{11}XY + c_{21}X^2Y + c_{12}XY^2 + c_{31}X^3Y + c_{22}X^2Y^2 + c_{13}XY^3$ と置く.
生成元変換 $s = a_1x + a_2x^2 + a_3x^3 + a_4x^4$ を行い, $U = s \otimes 1, V = 1 \otimes s$ と置

き, $\Delta(s)$ を U, V の綺麗な形で表わす訳である.
逆変換を, $x = b_1 s + b_2 s^2 + b_3 s^3 + b_4 s^4, X = b_1 U + b_2 U^2 + b_3 U^3 + b_4 U^4, Y = b_1 V + b_2 V^2 + b_3 V^3 + b_4 V^4$ として置く.(以下 X,Yの代わりにS,Tと書く)

以下は, Mathematica の結果であるが, Mathematica では c_{11} を $c11$, X^2 を X ^ 2 等と表わしている.

```
In[7]:=Deltax = X + Y + c11 X Y + c21 X ^ 2 Y + c12 X Y ^ 2 + c31 X ^ 3 Y + c22 X ^
2 Y ^ 2 + c13 X Y ^ 3;
In[8]:=S = b1 U + b2 U ^ 2 + b3 U ^ 3 + b4 U ^ 4;
In[9]:=T = b1 V + b2 V ^ 2 + b3 V ^ 3 + b4 V ^ 4;
In[10]:= deltas = a1 Deltax + a2 Deltax ^ 2 + a3 Deltax ^ 3 + a4 Deltax ^ 4
Out[10]= a1 ((X + Y + c11 X Y + c21 X ^ 2 Y + c31 X ^ 3 Y + c12 X Y ^ 2 + c22 X ^ 2
Y ^ 2 + c13 X Y ^ 3)) + a2 ((X + Y + c11 X Y + c21 X ^ 2 Y + c31 X ^ 3 Y + c12 X Y
^ 2 + c22 X ^ 2 Y ^ 2 + c13 X Y ^ 3)) ^ 2 + a3 ((X + Y + c11 X Y + c21 X ^ 2 Y + c31
X ^ 3 Y + c12 X Y ^ 2 + c22 X ^ 2 Y ^ 2 + c13 X Y ^ 3)) ^ 3 + a4 ((X + Y + c11 X Y +
c21 X ^ 2 Y + c31 X ^ 3 Y + c12 X Y ^ 2 + c22 X ^ 2 Y ^ 2 + c13 X Y ^ 3)) ^ 4)
In[11]:= deltas = deltas /. X -> S                注: ->は代入
Out[11]=a1 ((b1 U + b2 U ^ 2 + b3 U ^ 3 + b4 U ^ 4 + Y + c11 ((b1 U + b2 U ^ 2 + b3
U ^ 3 + b4 U ^ 4)) Y + c21 ((b1 U + b2 U ^ 2 + b3 U ^ 3 + b4 U ^ 4)) ^ 2 Y + c31 ((b1 U
+ b2 U ^ 2 + b3 U ^ 3 + b4 U ^ 4)) ^ 3 Y + c12 ((b1 U + b2 U ^ 2 + b3 U ^ 3 + b4 U ^
4)) Y ^ 2 + c22 ((b1 U + b2 U ^ 2 + b3 U ^ 3 + b4 U ^ 4)) ^ 2 Y ^ 2 + c13 ((b1 U + b2 U
^ 2 + b3 U ^ 3 + b4 U ^ 4)) Y ^ 3)) + a2(略) + a3(略) + a4(略)
In[12]:= deltas = deltas /. Y -> T
Out[12]= a1 ((b1 U + b2 U ^ 2 + b3 U ^ 3 + b4 U ^ 4 + b1 V + b2 V ^ 2 + b3 V ^ 3 +
b4 V ^ 4 + c11 ((b1 U + b2 U ^ 2 + b3 U ^ 3 + b4 U ^ 4)) ((b1 V + b2 V ^ 2 + b3 V ^ 3
+ b4 V ^ 4)) + c21 ((b1 U + b2 U ^ 2 + b3 U ^ 3 + b4 U ^ 4)) ^ 2 ((b1 V + b2 V ^ 2 + b3
V ^ 3 + b4 V ^ 4)) + c31 ((b1 U + b2 U ^ 2 + b3 U ^ 3 + b4 U ^ 4)) ^ 3 ((b1 V + b2 V ^ 2
+ b3 V ^ 3 + b4 V ^ 4)) + c12 ((b1 U + b2 U ^ 2 + b3 U ^ 3 + b4 U ^ 4)) ((b1 V + b2 V
^ 2 + b3 V ^ 3 + b4 V ^ 4)) ^ 2 + c22 ((b1 U + b2 U ^ 2 + b3 U ^ 3 + b4 U ^ 4)) ^ 2 ((b1
V + b2 V ^ 2 + b3 V ^ 3 + b4 V ^ 4)) ^ 2 + c13 ((b1 U + b2 U ^ 2 + b3 U ^ 3 + b4 U ^
4)) ((b1 V + b2 V ^ 2 + b3 V ^ 3 + b4 V ^ 4)) ^ 3)) + + a2(略) + a3(略) + a4(略) + 略
注: 以下で 5 乗で cut off
In[13]:= deltas = PolynomialRemainder[deltas, U ^ 5, U];
In[14]:=deltas = PolynomialRemainder[deltas, V ^ 5, V];
In[15]:=SS = Expand[deltas, U,V]
Out[15]= a1 b1 U + a2 b1 ^ 2 U ^ 2 + a1 b2 U ^ 2 + a3 b1 ^ 3 U ^ 3 + 2 a2 b1 b2 U ^ 3 +
a1 b3 U ^ 3 + a4 b1 ^ 4 U ^ 4 + 3 a3 b1 ^ 2 b2 U ^ 4 + a2 b2 ^ 2 U ^ 4 + 2 a2 b1 b3 U ^ 4
+ a1 b4 U ^ 4 + ((a1 b1 + 2 a2 b1 ^ 2 U + a1 b1 ^ 2 c11 U + 3 a3 b1 ^ 3 U ^ 2 + 2 a2 b1
b2 U ^ 2 + 2 a2 b1 ^ 3 c11 U ^ 2 +略)V + 長い省略
In[16]:= CoefficientList[SS, {U,V}]
Out[16]= {{0, a1 b1 = 1, a2 b1 ^ 2 + a1 b2, a3 b1 ^ 3 + 2 a2 b1 b2 + a1 b3, a4 b1 ^ 4 +
3 a3 b1 ^ 2 b2 + a2 b2 ^ 2 + 2 a2 b1 b3 + a1 b4}, {a1 b1, 2 a2 b1 ^ 2 + a1 b1 ^ 2 c11, 3
```

a3 b1 ^ 3 + 2 a2 b1 b2 + 2 a2 b1 ^ 3 c11 + a1 b1 b2 c11 + a1 b1 ^ 3 c12, 4 a4 b1 ^ 4 + 6 a3 b1 ^ 2 b2 + 2 a2 b1 b3 + 3 a3 b1 ^ 4 c11 + 4 a2 b1 ^ 2 b2 c11 + a1 b1 b3 c11 + 2 a2 b1 ^ 4 c12 + 2 a1 b1 ^ 2 b2 c12 + a1 b1 ^ 4 c13, 長い省略.
underline 部分に注目して
In[17]:= SSS = SS /. a2 −> - (1/2) a1 c11
Out[17]=a1 b1 U + a1 b2 U ^ 2 - 1/2 a1 b1 ^ 2 c11 U ^ 2 + a3 b1 ^ 3 U ^ 3 + a1 b3 U ^ 3 - a1 b1 b2 c11 U ^ 3 + a4 b1 ^ 4 U ^ 4 + 3 a3 b1 ^ 2 b2 U ^ 4 + a1 b4 U ^ 4 - 1/2 a1 b2 ^ 2 c11 U ^ 4 - a1 b1 b3 c11 U ^ 4 + ((a1 b1 + 3 a3 b1 ^ 3 U ^ 2 - a1 b1 ^ 3 c11 ^ 2 U ^ 2 + a1 b1 ^ 3 c21 U ^ 2 + 4 a4 b1 ^ 4 U ^ 3 + 6 a3 b1 ^ 2 b2 U ^ 3 + 3 a3 b1 ^ 4 c11 U ^ 3 - 2 a1 b1 ^ 2 b2 c11 ^ 2 U ^ 3 + 2 a1 b1 ^ 2 b2 c21 U ^ 3 - a1 b1 ^ 4 c11 c21 U ^ 3 + a1 b1 ^ 4 c31 U ^ 3 + 12 a4 b1 ^ 3 b2 U ^ 4 + 3 a3 b1 b2 ^ 2 U ^ 4 + 6 a3 b1 ^ 2 b3 U ^ 4 + 4 a4 b1 ^ 5 c11 U ^ 4 + 9 a3 b1 ^ 3 b2 c11 U ^ 4 - a1 b1 b2 ^ 2 c11 ^ 2 U ^ 4 - 2 a1 b1 ^ 2 b3 c11 ^ 2 U ^ 4 + 3 a3 b1 ^ 5 c21 U ^ 4 + a1 b1 b2 ^ 2 c21 U ^ 4 + 2 a1 b1 ^ 2 b3 c21 U ^ 4 - 3 a1 b1 ^ 3 b2 c11 c21 U ^ 4 + 3 a1 b1 ^ 3 b2 c31 U ^ 4 - a1 b1 ^ 5 c11 c31 U ^ 4)) V + 長い省略
これで UV の係数は 0 にする事が出来る. 次に, UV の係数が 0 と言う性質を変えない変換を考える. その為に最初から XY の係数を 0 とし、 UV の係数が 0 になる変換の条件を調べる.
In[18]:=SS = SS /. c11 −> 0
Out[18]=a1 b1 U + a2 b1 ^ 2 U ^ 2 + a1 b2 U ^ 2 + a3 b1 ^ 3 U ^ 3 + 2 a2 b1 b2 U ^ 3 + a1 b3 U ^ 3 + a4 b1 ^ 4 U ^ 4 + 3 a3 b1 ^ 2 b2 U ^ 4 + a2 b2 ^ 2 U ^ 4 + 2 a2 b1 b3 U ^ 4 + a1 b4 U ^ 4 + ((a1 b1 + 2 a2 b1 ^ 2 U + 3 a3 b1 ^ 3 U ^ 2 + 2 a2 b1 b2 U ^ 2 + a1 b1 ^ 3 c21 U ^ 2 + 4 a4 b1 ^ 4 U ^ 3 + 6 a3 b1 ^ 2 b2 U ^ 3 + 2 a2 b1 b3 U ^ 3 + (略))) V + 長い省略
UV の係数は 2 a2 b1 ^ 2 = 0 より, UV の係数が 0 ⇔ a2 = 0 を得る.
In[19]:=SS = SS /. a2 −> 0; In[20]:=SS = SS /. b2 −> 0
Out[20]=a1 b1 U + a3 b1 ^ 3 U ^ 3 + a1 b3 U ^ 3 + a4 b1 ^ 4 U ^ 4 + 略
In[21]:= CoefficientList[SS, {U,V}]
Out[21]= {{0, a1 b1, 0, a3 b1^3 + a1 b3, a4 b1^4 + a1 b4}, {a1 b1, 0, 3 a3 b1 ^ 3 + a1 b1 ^ 3 c12, 4 a4 b1 ^ 4 + a1 b1 ^ 4 c13, 6 a3 b1 ^ 2 b3 + 3 a3 b1 ^ 5 c12 + 2 a1 b1 ^ 2 b3 c12}, {0, 3 a3 b1 ^ 3 + a1 b1 ^ 3 c21, 6 a4 b1 ^ 4 + a1 b1 ^ 4 c22, 3 a3 b1 ^ 2 b3 + 6 a3 b1 ^ 5 c12 + 3 a3 b1 ^ 5 c21 + a1 b1 ^ 2 b3 c21, 12 a4 b1 ^ 3 b3 + 3 a3 b1 ^ 2 b4 + 12 a4 b1 ^ 6 c12 + 6 a3 b1 ^ 6 c13 + 4 a4 b1 ^ 6 c21 + a1 b1 ^ 2 b4 c21 + 3 a3 b1 ^ 6 c22 + 2 a1 b1 ^ 3 b3 c22}, 略
U ^ 2 V の係数 = 3 a3 b1 ^ 3 + a1 b1 ^ 3 c21, U ^ 2 V ^ 2 の係数 = 6 a4 b1 ^ 4 + a1 b1 ^ 4 c22 なので, a3, a4 を適当にとり, U ^ 2 V, U ^ 2 V ^ 2 の係数 = 0 と出来る.

同様に、U ^ 2 V, U ^ 2 V ^ 2 の係数 = 0 とする変換は, a3 = a4 = b3 = b4 = 0 である事が解る.

この計算結果を参考にして,

Theorem 4. *There exists a p-nilpotent generator x whose cocoefficients satisfy the relations;* $c_{11} = c_{21} = c_{22} = c_{32} = c_{33} = \cdots = c_{q\ q-1} = c_{qq} = 0$ $(p = 2q + 1)$.
Further if x and y are p-nilpotent generators which satisfy the above relations then $y = ax, a \neq 0,\ a \in K$.

Definition. Theorem 4 の条件を満たす p-nilpotent generator を *normal generator* と呼ぶ.

§4 Main Theorem

Theorem 5. *The cocoefficients of a normal generator x of a p-nilpotent Hopf algebra satisfy the relations;* $c_{10} = c_{01} = 1, c_{p-1\,1} = c$, $c_{p-k\,k} = (-1)^{k-1}k^{-1}c$, $(2 \leq k \leq p-1)$, $c \in K$, *the other terms equal to* 0 *and* $S(x) = -x$ *(S is an antipode).*

Conversely, an algebra generated by a p-nilpotent element whose cocoefficients and antipode satisfy the above relations becomes a p-nilpotent Hopf algebra.

Corollary 6. *If K is a prime field then there exists exactly p non-isomorphic p-nilpotent Hopf algebras, and if K is an algebraically closed field then there exists exactly 2 non-isomorphic p-nilpotent Hopf algebras.*

Theorem 5 の証明は
Step 1: $c_{ij} = 0$, $(0 \leq i, j \leq q)$ except $c_{10} = c_{01} = 0$, where $p = 2q + 1$
Step 2: $c_{ij} = 0$, $(q + 1 \leq i \leq 2q - 1,\ 0 \leq j \leq i - 2)$
Step 3: $c_{ij} = 0$, $(q + 1 \leq i \leq 2q,\ 2q + 2 - i \leq j \leq i)$
Step 4: $i + j = p$ の時, c_{ij} が Theorem 5 の条件を満たす事
Step 5: 逆に c_{ij} が Theorem 5 の条件を満たす時, coassociative になる事
の証明に分かれる.
$p = 7$ の場合, Step 1 で 下図の S1, Step 2 で 下図の S2, Step 3 で 下図の S3 が 0 を示し, Step 4 で 下図の S4 が条件を満たす事を示すわけです.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & S1 & S2 & S3 & S4 \\ 0 & 0 & 0 & 0 & S2 & S4 & S3 \\ 0 & S1 & 0 & 0 & S4 & S3 & S3 \\ 0 & S2 & S2 & S4 & S3 & S3 & S3 \\ 0 & S2 & S4 & S3 & S3 & S3 & S3 \\ 0 & S4 & S3 & S3 & S3 & S3 & S3 \end{bmatrix}$$

$p = 5$ の時は, Step 1 の部分は無いので, $p = 7$ の時, Mathematica の計算を見せよう.

$H = K[x], H\otimes H\otimes H = K[X,Y,Z], X = x\otimes 1\otimes 1, Y = 1\otimes x\otimes 1, Z = \otimes 1\otimes 1\otimes x$
と置き, $Dif = (\Delta\otimes 1)\Delta(x) - (1\otimes\Delta)\Delta(x)$ を計算する.
$Dif = 0 \Leftrightarrow$ coassociative に注意する.

```
In[1]:= De= X + Y + c31 X ^ 3 Y + c31 X Y ^ 3 + c41 X ^ 4 Y + c41 X Y ^ 4 + c51 X ^
5 Y + c42 X ^ 4 Y ^ 2 + c42 X ^ 2 Y ^ 4 + c51 X Y ^ 5 + c61 X ^ 6 Y + c52 X ^ 5 Y ^ 2 +
c43 X ^ 4 Y ^ 3 + c43 X ^ 3 Y ^ 4 + c52 X ^ 2 Y ^ 5 + c61 X Y ^ 6 + c62 X ^ 6 Y ^ 2 + c53
X ^ 5 Y ^ 3 + c44 X ^ 4 Y ^ 4 + c53 X ^ 3 Y ^ 5 + c62 X ^ 2 Y ^ 6 + c63 X ^ 6 Y ^ 3 + c54
X ^ 5 Y ^ 4 + c54 X ^ 4 Y ^ 5 + c63 X ^ 3 Y ^ 6 + c64 X ^ 6 Y ^ 4 + c55 X ^ 5 Y ^ 5 + c64
X ^ 4 Y ^ 6 + c65 X ^ 6 Y ^ 5 + c65 X ^ 5 Y ^ 6 + c66 X ^ 6 Y ^ 6;
In[2]:= De2 = Expand[De ^ 2, X];
In[3]:= DX = PolynomialRemainder[De2, X ^ 7, X];
In[4]:= DY = PolynomialRemainder[DX, Y ^ 7, Y];
In[5]:= De2 = PolynomialRemainder[DY, Z ^ 7, Z];
In[6]:= De3 = Expand[De ^ 3, X];
```

以下 De4, De5, De6 を計算しておく.

$(\Delta\otimes 1)\Delta(x)$ の計算

```
In[22]:= DeIdDe =De + Z + c31 De3 * Z + c31 De Z ^ 3 + c41 De4 * Z + c41 De Z ^ 4 +
c51 De5* Z + c42 De4* Z ^ 2 + c42 De2* Z ^ 4 +c51 De Z ^ 5 + c61 De6 * Z + c52 De5 *
Z ^ 2 + c43 De4* Z ^ 3 + c43 De3* Z ^ 4 + c52 De2 * Z ^ 5 + c61 De Z ^ 6 + c62 De6 * Z
^ 2 + c53 De5 * Z ^ 3 + c44 De4 * Z ^ 4 + c53 De3 * Z ^ 5 + c62 De2 * Z ^ 6 +c63 De6 *
Z ^ 3 + c54 De5 * Z ^ 4 + c54 De4 * Z ^ 5 +c63 De3 * Z ^ 6 + c64 De6 * Z ^ 4 + c55 De5
* Z ^ 5 + c64 De4 * Z ^ 6 + c65 De6 * Z ^ 5 + c65 De5 * Z ^ 6 +c66 De6 * Z ^ 6;
In[23]:= ExDeIdDe =Collect[DeIdDe, X];
In[24]:= DX = PolynomialRemainder[ExDeIdDe, X ^ 7, X];
In[25]:= DY = PolynomialRemainder[DX, Y ^ 7, Y];
In[26]:= ExDeIdDe = PolynomialRemainder[DY, Z ^ 7, Z];
In[27]:= ExDeIdDe = Expand[ExDeIdDe, X];
In[28]:= ExDeIdDe = PolynomialMod[ExDeIdDe, 7];
```

同様に $(1\otimes\Delta)\Delta(x)$ = ExIdDeDe を計算し、

```
In[56]:= Dif = ExDeIdDe - ExIdDeDe;
In[57]:= Dif = PolynomialMod[Dif, 7];
In[58]:= Expand[Dif, X];
In[59]:= CoefficientList[Dif, {X,Y,Z}]
Out[59]= {{{0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0},{0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0,
0}, {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}}, {{0, 0, 0, 0, 0, 0, 0}, {0,
0, 4c31, 3c41, 2 c42+2 c51, 4c31 ^ 2+2 c52+c61, 2c62}, {0, 0, c41, 4 c51, c31 ^ 2+3 c43+6
c61, 3 c31 c41+3 c53, 2 c41 ^ 2 + 4 c31 c42 + 2 c31 c51 + 3 c63}, 長い省略
In[60]:= Dif = Dif /. c31 -> 0;
```

In[61]:= Dif = Dif /. c41 -> 0;
In[62]:= Dif = Dif /. c51 -> 0;
In[63]:= Dif = Dif /. c42 -> 0;
p = 7 の場合は、単に c31 = 0 であるが, 一般には, ci1 = 0 から cij = 0 を示す帰納法になる. 又、c41 = 0, c42 = 0, c51 = 0 も一般化され、帰納法で証明できる. 但し, 帰納法のルールを見つけるには最低でも, p = 11 の時の計算が必要である.
In[64]:= Expand[Dif, X];
In[65]:= CoefficientList[Dif, {X,Y,Z}]
Out[65]={{{0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}}, {{0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 2 c52+c61, 2 c62}, {0, 0, 0, 0, 3 c43 + 6 c61, 3 c53, 3 c63}, {0, 0, 0, 4 c43 + c61, 4 c44, 4 c54, 4 c64}, {0, 0, 5 c52 + 6 c61, 5 c53, 5 c54, 5 c55, 5 c65}, {0, 0, 6 c62, 6 c63, 6 c64, 6 c65, 6 c66}, 長い省略
underline の部分に注目すれば, どこが 0 になるかよく判る.
In[66]:= Dif = Dif /. c62 -> 0; In[67]:= Dif = Dif /. c53 -> 0;
In[68]:= Dif = Dif /. c63 -> 0; In[69]:= Dif = Dif /. c44 -> 0;
In[70]:= Dif = Dif /. c54 -> 0; In[71]:= Dif = Dif /. c64 -> 0;c65, c66 も
In[78]:= Expand[Dif, X];
In[79]:= CoefficientList[Dif, {X,Y,Z}]
Out[79]= {{{0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}}, {{0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 2 c52 + c61, 0}, {0, 0, 0, 0, 3 c43 + 6 c61, 0, 0}, {0, 0, 0, 4 c43 + c61, 0, 0, 0}, {0, 0, 5 c52 + 6 c61, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 6 c43 c61 + 3 c52 c61 + 3 c61 ^ 2}}, {{0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 3 c43 + 2 c52, 0, 0}, {0, 0, 0, 6 c43 + 4 c52, 0, 0, 0}, 長い省略
この部分が、副対角線から 1 つずれた所である.

Step 5: は単純計算ではできない. automorphism type の Hopf algebra が Theorem 5 の条件を満たし, coassociative である事を利用して証明する.

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Valuations on coproducts of skew fields and free feilds

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Abstract. We consider valuations on coproducts of skew fields and free fields. We also introduce Cohn's skew field extensions $K \subset L$ such that the right dimension of L over K is finite , the left dimension of L over K is infinite, and K and L are isomorphic as rings.

1.序文及び Cohn の例

可換体上の free algebra の universal field of fractions を free field という. P.M.Cohn [4,5,6]は free field を用いて次のような斜体拡大を構成した. k を可換体, Λ を無限集合, n を自然数とする. k の標数 p が正で n を割り切り, $n = mp^r$, p□m である場合 ω を 1 の原始 m 乗根とし, それ以外の場合は ω を 1 の原始 n 乗根とする. $k < X >$ を集合 $X = \{x_{\lambda ij} | \lambda \in \Lambda, i, j = 1,2,3,\cdots\}$ 上の free k-algegra , E を free k-algebra $k < X >$ の universal field of fractions とする. $x_{\lambda ij}^{\alpha} = \omega^i x_{\lambda ij+1}$ で定義された E 上の準同型 α と $x_{\lambda ij}^{\delta} = x_{\lambda i+1 j}$ で定義された E 上の α -derivation δ を考える. $L = E(t;\alpha,\delta)$ を歪多項式環 $E[t;\alpha,\delta]$ の商体とし, K を E と t^n で生成された L の部分体とする. このとき L の K 上の右次元が n で L の K 上の左次元が無限であるが, さらにつぎのことがわかる.

命題 1(Cohn).上の斜体拡大 $K \subset L$ において k の標数 p が正で $n = p^r$ の場合において, K と L は k-多元環として同型である.

証明は[3]を参照されたい. Xue の定理[7,Theorem12.6]及びこの命題 1 より次の命題が導かれる.

<u>命題 2.森田 duality を持たない両側 artin かつ左 duo 環が存在する.</u>

The detailed version of this paper will be submitted for publication elsewhere.

上のCohnの例の様にartin環の研究においては中心上無限次元の斜体の研究が必要であり free field は重要な研究対象である．中心上無限次元の斜体は位相的に考えるとわかりやすいことが多い．Cohn や Lichtman による free field の可換付値の研究があるが，ここでは値群の可換性を仮定しない順序群を値群とする斜体の coproduct 及び free feild の付値について考察する．

定義:G を全順序群とし，G の単位元を 0 とかく．全ての $a \in G$ にたいして $\infty + a = a + \infty = \infty + \infty$ 及び $a < \infty$ とする様に，G に ∞ を加える．D を斜体とする．写像 $v : D \to G \cup \{\infty\}$ が D の元 x,y にたいして，次の条件をみたすとき写像 v を D の付値という．

(V_1) $v(xy) = v(x) + v(y)$,

(V_2) $v(x+y) \geq \min\{v(x), v(y)\}$,

(V_3) $v(1) = 0$ and $v(0) = \infty$.

D を斜体，X を集合とする．D と $k<X>$ の K 上の ring coproduct を $D_K<X> = D *_K K<X>$ と書く．その universal field of fractions を $D_K(X)$ と書く．k を可換体とする．free k-algebra $k<X>$ の universal field of fractions を $k(X)$ と書く．

2.結果及び証明

定理 1.D を付値 v をもつ斜体,K を D の部分斜体，D_i $(i \in I)$ を K を含む D の部分斜体，v_i $(i \in I)$ を v を D_i に制限した D_i $(i \in I)$ の付値とする．次のことが成り立つ．

(i)K 上の D_i $(i \in I)$ の ring coproduct $*_K D_i$ はすべての付値 v_i の拡大付値 w をもつ．

(ii)$*_K D_i$ から D の I 個の ring coproduct $*_K D$ への自然な写像が honest であれば，K 上の D_i $(i \in I)$ の field coproduct $\circ_K D_i$ は全ての v_i を拡大する付値 w をもつ．

(iii)D と I が可算ならば(i),(ii)の w の付値群 は $Z \times G$ とできる．ここで G は D の付値群 Z は整数からなる順序群である．

証明は以下の補題による．

補題 1([2, p.84, Theorem]).D を斜体 K を D の 2 重可換子環とし，X を集合とする．次のことが成り立つ．(i)任意の D の付値は $D_K(X)$ に拡大できる．(ii)D 上の付値が可換ならば $D_K(X)$ 上の拡大された付値もまた可換である．

補題 2．D を斜体，K を D の部分斜体，$\{x_i \mid i \in I\}$ を集合とすると $\{x_i^{-1}Dx_i \mid i \in I\}$ は $D_K(X)$ の中で field coproduct $\circ_K D_i$ を生成する．

補題 3([5, p.114, Corollary]).$K \subseteq E \subseteq F$ を斜体，X を集合とすると，$E_K(X) \subseteq F_K(X)$ となる．

定理 2.k を可換体，$X = \{x_i \mid i \in I\}$ を集合，$k<X>$ を k-free algebra，$k(X)$ を free field($k<X>$ の universal field of fractions)とする．v_i を $k(x_i)$ の k-付値とすると，全ての v_i の拡大となっている $k(X)$ の k-付値が存在する．

最後に[2,Theorem 3]を拡張した次の定理を報告する．

定理 3. k を可換体，$X = \{x_i \mid i \in I\}$ を集合，$k(X)$ を free field，G を順序群，$\{g_i \mid i \in I\}$ をその生成元とすると，$k(X)$ の k-付値 v で任意の i にたいして $v(x_i) = g_i$ となるものが存在する．

証明は次の補題を用いる．

補題 4([1, Theorem 3.4]).順序群 G が自由群 F の剰余群 $G \cong F/R$ としてあらわされたとする．このとき F に適当な順序をいれて上の準同系が順序群としての準同系にできる．

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BGG CORRESPONDENCE AND ARAMOVA-HERZOG'S THEORY ON EXTERIOR ALGEBRA

KOHJI YANAGAWA

This article is edited from my recent preprint [15] (some parts are abridged and some are extended). The formal version of [15] has been submitted to a journal.

ABSTRACT. Let $E = K\langle y_1, \dots, y_n \rangle$ be the exterior algebra. The *distinguished pairs* of a graded E-module N describe the growth of a minimal graded injective resolution of N. Römer gave a duality theorem between the distinguished pairs of N and those of its dual N^*. In this paper, we show that under Bernstein-Gel'fand-Gel'fand correspondence, his duality is translated into a natural corollary of Serre duality for (complexes of) graded $S = K[x_1, \dots, x_n]$-modules.

1. Background

While the results of this article are purely ring theoretic, they have a combinatorial background. More precisely, they are related to the theory of Stanley-Reisner rings. This section is devoted to an explanation of this background, and independent from the latter sections in some sense. *So one can skip this section.*

Let $S = K[x_1, \dots, x_n]$ be the polynomial ring over a field K. Set $[n] := \{1, \dots, n\}$. For an abstract simplicial complex $\Delta \subset 2^{[n]}$, set $I_\Delta := (\prod_{i \in F} x_i \mid F \subset [n],\ F \not\in \Delta)$ to be a monomial ideal of S. We call $K[\Delta] = S/I_\Delta$ the *Stanley-Reisner ring* of Δ. This is a central concept of combinatorial commutative algebra. See [5, 12]. It is easy to see that $\Delta^\vee := \{ F \subset [n] \mid [n] \setminus F \not\in \Delta \}$ is a simplicial complex again. We call $\Delta^\vee$ the *Alexander dual* of Δ. It is known that $\tilde{H}_i(|\Delta|; K) \cong \tilde{H}^{n-i-3}(|\Delta^\vee|; K)$ for all i, where $\tilde{H}_i(|\Delta|; K)$ is the ith reduced homology group of the geometric realization $|\Delta|$ of Δ with the coefficients in K. This isomorphism corresponds to the Alexander duality on the geometric realization $|\, 2^{[n]} \setminus [n] \,|$, which is homeomorphic to an $(i-2)$dimensional sphere.

Let M be a finitely generated graded S-module. The ijth Betti number $\beta_{i,j}(M) = \dim_K \operatorname{Tor}_i^S(K, M)_j$ of M is an important invariant. Following Bayer-Charalambous-Popescu [4], we say a Betti number $\beta_{k,m}(M) \neq 0$ is *extremal*, if $\beta_{i,j}(M) = 0$ for all $(i,j) \neq (k,m)$ with $i \geq k$ and $j - i \geq m - k$.

Theorem A (Bayer-Charalambous-Popescu, [4, Theorem 2.8]) *Let $\Delta \subset 2^{\{1,\dots,n\}}$ be a simplicial complex, and $K[\Delta] = S/I_\Delta$ the Stanley-Reisner ring. Then $\beta_{i,i+j}(K[\Delta])$ is extremal if and only if so is $\beta_{j,i+j}(I_{\Delta^\vee})$. Moreover, if this is the case, then $\beta_{i,i+j}(K[\Delta]) = \beta_{j,i+j}(I_{\Delta^\vee})$.*

We have $\beta_{i,n}(K[\Delta]) = \dim_K \tilde{H}_{n-i-1}(|\Delta|; K) =: \tilde{h}_{n-i-1}(|\Delta|)$ by Hochster's formula. If $\beta_{i,n}(K[\Delta]) \neq 0$ then it is always an extremal Betti numbers. The equality

$\tilde{h}_{n-i-1}(|\Delta|) = \beta_{i,n}(K[\Delta]) = \beta_{n-i,n}(I_{\Delta^\vee}) = \beta_{n-i+1,n}(K[\Delta^\vee]) = \tilde{h}_{i-2}(|\Delta^\vee|)$ induced by Theorem A corresponds to the usual Alexander duality. More generally, Theorem A gives an Alexander duality for (some) *iterated Betti numbers* (c.f. [8]).

Let $E = K\langle y_1, \ldots, y_n\rangle$ be the exterior algebra. To understand Theorem A, Aramova-Herzog [3] introduced *distinguished pairs* for a graded E-module N using its graded injective resolution. See Definition 2.7 below (*our convention to describe these pairs is different from the original one in* [3, 11]. *Please be careful, when you refer these papers*). Let $K\{\Delta\} = E/J_\Delta$, $J_\Delta := (\prod_{i\in F} y_i \mid F \subset [n],\ F \not\in \Delta)$, be the *exterior face ring* of Δ (c.f. [2, 8]). Then [3, Corollary 9.6] states that (d,i) is a distinguished pair for $K\{\Delta\}^* := \operatorname{Hom}_E(K\{\Delta\}, E)$ if and only if $\beta_{d+i-n,d}(K[\Delta])$ is extremal. But Römer [11] proved that (d,i) is distinguished for N if and only if so is $(d, 2n-d-i)$ for N^*. Since $k\{\Delta\}^* = J_{\Delta^\vee}$, his result implies Theorem A (their arguments can also treat the value of $\beta_{i,j}(K[\Delta])$).

Bernstein-Gel'fand-Gel'fand correspondence (BGG correspondence, for short) is a well known theorem which states that the derived category $D^b(\operatorname{gr} S)$ of finitely generated graded S-modules is equivalent to the similar category $D^b(\operatorname{gr} E)$ for E. In this article, we show that BGG correspondence translates Römer's duality into a natural consequence of Serre duality on $D^b(\operatorname{gr} S)$. A key point is that the duality functor $(-)^* = \operatorname{Hom}_E(-, E)$ on $D^b(\operatorname{gr} E)$ corresponds to the duality functor $R\operatorname{Hom}_S(-, \omega^\bullet)$ on $D^b(\operatorname{gr} S)$, where $\omega^\bullet$ is a dualizing complex of S.

The original paper [4] states Theorem A in the $\mathbb{Z}^n$-graded context, while the arguments in [3, 11] are hard to work in this context. But, since BGG correspondence also holds for $\mathbb{Z}^n$-graded modules, our method is powerful in this context too.

2. $\mathbb{Z}$-GRADED CASE

Let W be an n-dimensional vector space over a field K, and $S = \bigoplus_{i\geq 0} \operatorname{Sym}_i W$ the polynomial ring. We regard S as a graded ring with $S_i = \operatorname{Sym}_i W$. Let $\operatorname{Gr} S$ be the category of graded S-modules and their degree preserving S-homomorphisms, and $\operatorname{gr} S$ the full subcategory of $\operatorname{Gr} S$ consisting of finitely generated modules. Then there is an equivalence $D^b(\operatorname{gr} S) \cong D^b_{\operatorname{gr} S}(\operatorname{Gr} S)$. (For derived categories, consult [7].) So we will freely identify these categories. For $M = \bigoplus_{i\in\mathbb{Z}} M_i \in \operatorname{Gr} S$ and an integer j, $M(j)$ denotes the shifted module with $M(j)_i = M_{i+j}$. For $M^\bullet \in D^b(\operatorname{Gr} S)$, $M^\bullet[j]$ denotes the jth translation of $M^\bullet$, that is, $M^\bullet[j]$ is the complex with $M^\bullet[j]^i = M^{i+j}$. So, if $M \in \operatorname{Gr} S$, $M[j]$ is the cochain complex $\cdots \to 0 \to M \to 0 \to \cdots$, where M sits in the $(-j)$th position. If $M \in \operatorname{gr} S$ and $N \in \operatorname{Gr} S$, then $\operatorname{Hom}_S(M,N)$ has the structure of a graded S-module with $\operatorname{Hom}_S(M,N)_i = \operatorname{Hom}_{\operatorname{Gr} S}(M, N(i))$.

Let $\omega^\bullet \in D^b(\operatorname{gr} S)$ be a minimal graded injective resolution of $S(-n)[n]$. That is, $\omega^\bullet$ is a graded normalized dualizing complex of S. Then $\mathbf{D}_S(-) := \operatorname{Hom}^\bullet_S(-, \omega^\bullet)$ gives a duality functor from $D^b(\operatorname{gr} S)$ to itself. The ith cohomology of $\mathbf{D}_S(M^\bullet)$ is $\operatorname{Ext}^i_S(M^\bullet, \omega^\bullet)$. For $M^\bullet \in D^b(\operatorname{gr} S)$ and $i \in \mathbb{Z}$, set $d_i(M^\bullet) := \dim_S H^i(M^\bullet)$. Here the Krull dimension of the 0 module is $-\infty$.

Definition 2.1. We say $(d,i) \in \mathbb{N} \times \mathbb{Z}$ is a *distinguished pair* for a complex $M^\bullet \in D^b(\operatorname{gr} S)$, if $d = d_i(M^\bullet)$ and $d_j(M^\bullet) < d + i - j$ for all j with $j < i$.

Example 2.2. Let $M^\bullet \in D^b(\mathrm{gr}\, S)$ and $d = d_i(M^\bullet) \geq 0$. If $d = \max\{d_j(M^\bullet) \mid j \in \mathbb{Z}\}$, then (d, i) is distinguished for $M^\bullet$. Similarly, if $i = \min\{j \mid H^j(M^\bullet) \neq 0\}$, then (d, i) is also distinguished. Thus $M^\bullet$ has several distinguished pairs in general.

In the sequel, $\deg_S(M)$ denotes the multiplicity (c.f. [5, Definition 4.1.5]) of a module $M \in \mathrm{gr}\, S$, which is defined by the top term of the Hilbert polynomial of M.

Theorem 2.3. *For $M^\bullet \in D^b(\mathrm{gr}\, S)$, we have the following.*

(1) A pair (d, i) is distinguished for $M^\bullet$ if and only if $(d, -d-i)$ is distinguished for $\mathbf{D}_S(M^\bullet)$.

(2) If (d, i) is a distinguished pair for $M^\bullet$, then

$$\deg_S H^i(M^\bullet) = \deg_S \mathrm{Ext}_S^{-d-i}(M^\bullet, \omega^\bullet).$$

Proof. (1) Since the statement is "symmetric", it suffices to prove the direction $\Rightarrow$.

From the double complex $\mathrm{Hom}_S^\bullet(M^\bullet, \omega^\bullet)$, we have a spectral sequence $E_2^{p,q} = \mathrm{Ext}_S^p(H^{-q}(M^\bullet), \omega^\bullet) \Rightarrow \mathrm{Ext}_S^{p+q}(M^\bullet, \omega^\bullet)$. For simplicity, set $e_r^{p,q} := \dim_S E_r^{p,q}$. By [5, §8.1, Theorem 8.1.1], we have

$$e_2^{p,q} = \dim_S \mathrm{Ext}_S^p(H^{-q}(M^\bullet), \omega^\bullet) = \begin{cases} -p & \text{if } p = -d_{-q}(M^\bullet), \\ \leq -p & \text{if } -d_{-q}(M^\bullet) < p \leq 0, \\ -\infty & \text{otherwise.} \end{cases} \tag{2.1}$$

(I) By (2.1), we have $e_2^{-d,-i} = d$. On the other hand, we have $e_2^{p,q} < d$ for all $(p, q) \neq (-d, -i)$ with $p + q = -d - i$. In fact, the assertion follows from (2.1) if $p > -d$. So we may assume that $p < -d$ and $q = -d - i - p > -i$. Since (d, i) is distinguished, we have $d_{-q}(M^\bullet) < d + i + q = -p$. Thus $E_2^{p,q} = 0$ in this case. Anyway, we have $e_\infty^{p,q} < d$ for all $(p, q) \neq (-d, -i)$ with $p + q = -d - i$.

(II) Since $d_{i-j+1}(M^\bullet) < d+j-1 < d+j$ for all $j \geq 2$, we have that $E_2^{-d-j,-i+j-1} = 0$. So we have $E_r^{-d-j,-i+j-1} = 0$ for all $r \geq 2$. Next we will show that $d = e_2^{-d,-i} = e_3^{-d,-i} = \cdots = e_r^{-d,-i}$ by induction on r. Recall that $E_{r+1}^{-d,-i}$ is the cohomology of

$$E_r^{-d-r,-i+r-1} \to E_r^{-d,-i} \to E_r^{-d+r,-i-r+1}.$$

But we have seen that $E_r^{-d-r,-i+r-1} = 0$. Moreover, $e_r^{-d+r,-i-r+1} \leq e_2^{-d+r,-i-r+1} \leq d - r < d$ by (2.1), and $e_r^{-d,-i} = d$ by the induction hypothesis. Thus $e_{r+1}^{-d,-i} = d$. Hence $e_\infty^{-d,-i} = d$. From this fact and (I), we have that $\dim_S \mathrm{Ext}_S^{-d-i}(M^\bullet, \omega^\bullet) = d$.

(III) Finally, we will show that $\mathrm{Ext}_S^{-d-i-j}(M^\bullet, \omega^\bullet) < d + j$ for all $j > 0$. To see this, it suffices to show that $e_2^{p,q} < d + j$ for all $j > 0$ and all (p, q) with $p + q = -d - i - j$. If $p > -d - j$, the assertion is clear. If $p = -d - j$, then $q = -i$ and $d_{-q}(M^\bullet) = d < -p$. So $E_2^{p,q} = 0$ in this case. Hence we may assume that $p < -d - j$ and $-q = d + i + j + p < i$. Since (d, i) is distinguished, $d_{-q}(M^\bullet) < d + (i + q) = -j - p < -p$. So we have $E_2^{p,q} = 0$ in this case too.

(2) Since $\deg_S E_r^{-d,-i} = \deg_S E_{r+1}^{-d,-i}$ for all $r \geq 2$ by the argument in (II) of the proof of (1), we have $\deg_S E_2^{-d,-i} = \deg_S E_\infty^{-d,-i}$. Hence

$$\deg_S \mathrm{Ext}_S^{-d-i}(M^\bullet, \omega^\bullet) = \deg_S E_\infty^{-d,-i} = \deg_S E_2^{-d,-i} = \deg_S \mathrm{Ext}_S^{-d}(H^i(M^\bullet), \omega^\bullet),$$

where the first equality follows from (I) and (II). But, since $\dim(H^i(M^\bullet)) = d$, we have $\deg_S \operatorname{Ext}_S^{-d}(H^i(M), \omega^\bullet) = \deg_S H^i(M^\bullet)$. □

Remark 2.4. For the above theorem, only (2.1) and the fact that $\operatorname{inj.dim}_S \omega^\bullet < \infty$ are essential. So the theorem holds in much wider contexts.

(1) Theorem 2.3 (1) also holds for a noetherian local commutative ring R admitting a dualizing complex. The part (2) also holds for R, if we replace $\deg_S(-)$ by $l_{R_\mathfrak{p}}(- \otimes_R R_\mathfrak{p})$ for a prime ideal $\mathfrak{p} \subset R$ with $\dim R/\mathfrak{p} = d$.

(2) Let A be an associative ring with 1. For a left (or right) A-module M, set $j(M) := \min\{ i \mid \operatorname{Ext}_A^i(M, A) \neq 0 \}$. We say A is *Auslander Gorenstein* if A is left and right noetherian, $\operatorname{inj.dim}_A A = \operatorname{inj.dim} A_A < \infty$, and satisfies the following condition: For every finitely generated left (or right) A-module M and for all $i \geq 0$, we have $j(N) \geq i$ for all submodule $N \subset \operatorname{Ext}_A^i(M, A)$.

Familiar examples of Auslander Gorenstein rings include commutative Gorenstein local rings (in this case, $j(M) = \dim A - \dim M$), Weyl algebras, and universal enveloping algebras of finite dimensional Lie algebras. See [1] for further information.

If A is Auslander Gorenstein, then $-j(M)$ is an exact dimension function. If we use this "dimension" to define distinguished pairs for objects in $D^b(\mathrm{mod}_A)$ or $D^b(\mathrm{mod}_{A^{op}})$, Theorem 2.3 also holds for the duality functor $R\operatorname{Hom}_A(-, A)$ between $D^b(\mathrm{mod}_A)$ and $D^b(\mathrm{mod}_{A^{op}})$. More generally, the theorem holds for rings with *Auslander dualizing complexes* (see [17]).

Next, we assume that $A = \bigoplus_{i \geq 0} A_i$ is a graded $K(\cong A_0)$-algebra satisfying the following conditions.

(a) There is a polynomial $f(t) \in \mathbb{Q}[t]$ such that $f(i) = \dim_K A_i$ for $i \gg 0$.

(b) A is Auslander regular (i.e., A is Auslander Gorenstein and $\operatorname{gl.dim} A < \infty$).

(c) A is Cohen-Macaulay with respect to Gel'fand-Kirillov dimension (c.f. [1]).

Then a finitely generated graded A-module M has the Hilbert polynomial, and we can define the multiplicity $\deg_A(M)$. If we use Gel'fand-Kirillov dimension to define distinguished pairs, both (1) and (2) of Theorem 2.3 hold for A. So the under the additional assumption that A is Koszul, it might be interesting to generalize Corollary 2.8 below and related results to the quadratic dual ring $A^!$.

Let V be the dual vector space of W, and $E = \bigwedge V$ the exterior algebra. We regard E as a negatively graded ring with $E_{-i} = \bigwedge^i V$ (this is the opposite convention from [3, 11]). Let $\operatorname{gr} E$ be the category of finitely generated graded E-modules and their degree preserving E-homomorphisms. Here "E-module" means a left and right module N with $ea = (-1)^{(\deg e)(\deg a)} ae$ for all homogeneous $e \in E$ and $a \in N$.

Let $\{x_1, \ldots, x_n\}$ be a basis of W, and $\{y_1, \ldots, y_n\}$ its dual basis of V. Set $\mathrm{L}(N^\bullet) = \bigoplus_{i \in \mathbb{Z}} S \otimes_K N^i$ and $\mathrm{L}(N^\bullet)^m = \bigoplus_{i-j=m} S \otimes_K N^i_j$. The differential defined by

$$\mathrm{L}(N^\bullet)^m \supset S \otimes_K N^i_j \ni 1 \otimes z \mapsto \sum_{1 \leq l \leq n} x_l \otimes y_l z + (-1)^m (1 \otimes \delta^i(z)) \in \mathrm{L}(N^\bullet)^{m+1}$$

makes $\mathrm{L}(N^\bullet)$ a cochain complex of free S-modules. Here δ^i is the ith differential map of $N^\bullet$. Moreover, L gives a functor from $D^b(\operatorname{gr} E)$ to $D^b(\operatorname{gr} S)$. Similarly, we have a functor $\mathbf{R} : D^b(\operatorname{gr} S) \to D^b(\operatorname{gr} E)$. The following is a crucial result.

Theorem 2.5 (BGG correspondence, c.f.[6]). *The functors* L *and* R *give a category equivalence* $D^b(\operatorname{gr} S) \cong D^b(\operatorname{gr} E)$.

For $N \in \operatorname{gr} E$, then $N^\bullet := \operatorname{Hom}_E(N, E) \cong \operatorname{Hom}_K(N, K)(n)$ is a graded E-module again. $(-)^\bullet$ gives an exact duality functor on $\operatorname{gr} E$, and it can be extended to the duality functor $\mathbf{D}_E$ on $D^b(\operatorname{gr} E)$.

Proposition 2.6. *For* $N^\bullet \in D^b(\operatorname{gr} E)$, *we have*

$$\mathbf{D}_S \circ \mathbf{L}(N^\bullet) \cong \mathbf{L} \circ \mathbf{D}_E(N^\bullet)(-2n)[2n].$$

Proof. Since $\mathbf{L}(N^\bullet)$ consists of free S-modules, we have

$$\mathbf{D}_S \circ \mathbf{L}(N^\bullet) \cong \operatorname{Hom}_S^\bullet(\mathbf{L}(N^\bullet), S(-n)[n]).$$

It is easy to see that

$$\operatorname{Hom}_S^m(\mathbf{L}(N^\bullet), S(-n)[n]) \cong \bigoplus_{j-i=m+n} S(-n) \otimes_K (N_j^i)^\vee,$$

where $(-)^\vee$ means the graded K-dual. On the other hand,

$$\begin{aligned} \mathbf{L} \circ \mathbf{D}_E(N^\bullet)^m = \bigoplus_{i-j=m} S \otimes_K \mathbf{D}_E(N^\bullet)_j^i &= \bigoplus_{i-j=m} S(n) \otimes_K (N_{-n-j}^{-i})^\vee \\ &= \bigoplus_{j-i=m-n} S(n) \otimes_K (N_j^i)^\vee. \end{aligned}$$

So we can easily construct a quasi-isomorphism $\mathbf{D}_S \circ \mathbf{L}(N^\bullet) \to \mathbf{L} \circ \mathbf{D}_E(N^\bullet)(-2n)[2n]$. □

For $N^\bullet \in D^b(\operatorname{gr} E)$, we have $H^i(\mathbf{L}(N^\bullet))_j \cong \operatorname{Ext}_E^{j+i}(K, N^\bullet)_j$ by [6, Theorem 3.7]. So the Laurent series $P_i(t) = \sum_{j \in \mathbb{Z}} (\dim_K \operatorname{Ext}_E^{j+i}(K, N^\bullet)_j) \cdot t^j$ is the Hilbert series of the finitely generated graded S-module $H^i(\mathbf{L}(N^\bullet))$. If $H^i(\mathbf{L}(N^\bullet)) \neq 0$, there exists a Laurent polynomial $Q_i(t) \in \mathbb{Z}[t, t^{-1}]$ such that

$$P_i(t) = \frac{Q_i(t)}{(1-t)^d},$$

where $d = d_i(\mathbf{L}(N^\bullet)) = \dim_S H^i(\mathbf{L}(N^\bullet))$. Set $e_i(N^\bullet) := Q_i(1) = \deg_S H^i(\mathbf{L}(N^\bullet))$.

Definition 2.7. Let $N^\bullet \in D^b(\operatorname{gr} E)$. We say $(d, i) \in \mathbb{N} \times \mathbb{Z}$ is a *distinguished pair* for $N^\bullet$ if and only if it is distinguished for $\mathbf{L}(N^\bullet)$ (in the sense of Definition 2.1).

Note that a distinguished pair (d, i) for $N^\bullet$ and $e_i(N^\bullet)$ concern the growth of the "$(-i)$-linear strand" of a minimal graded injective resolution of $N^\bullet$. The above definition is (essentially) same as that of [3, 11]. But our convention to describe these pairs is different from the original one.

Corollary 2.8 (c.f. [11, Theorem 3.8]). *Let* $N^\bullet \in D^b(\operatorname{gr} E)$. *A pair* (d, i) *is distinguished for* $N^\bullet$ *if and only if* $(d, 2n-d-i)$ *is distinguished for* $\mathbf{D}_E(N^\bullet)$. *If this is the case, we have* $e_i(N^\bullet) = e_{2n-d-i}(\mathbf{D}_E(N^\bullet))$.

Proof. For the first statement, it suffices to prove the direction $\Rightarrow$. By Theorem 2.3, $(d, -d-i)$ is a distinguished pair for $\mathbf{D}_S \circ \mathbf{L}(N^\bullet) \cong \mathbf{L} \circ \mathbf{D}_E(N^\bullet)(-2n)[2n]$. For a complex $M^\bullet \in D^b(\operatorname{gr} S)$, we have $H^j(M^\bullet(-2n)[2n]) = H^{2n+j}(M^\bullet)(-2n)$ and $d_j(M^\bullet(-2n)[2n]) = d_{2n+j}(M^\bullet)$. Thus $(d, 2n-d-i)$ is distinguished for $\mathbf{L} \circ \mathbf{D}_E(N^\bullet)$. The last equality follows from Theorem 2.3 (2). □

For a module $N \in \operatorname{gr} E$, $d_i(\mathbf{L}(N))$ can be 0 quite often. But we have the following.

Proposition 2.9. *Assume that a module $N \in \operatorname{gr} E$ does not have a free summand. If (d, i) is a distinguished pair for N, then we have $d > 0$.*

Proof. Let $0 \to N \to I^0 \to I^1 \to \cdots$ (resp. $\cdots \to I^{-1} \to I^0 \to N \to 0$) be a minimal injective (resp. projective) resolution of N. For $j \geq 0$, set $\Omega_j(N) := (\ker(I^j \to I^{j+1}))[-j]$. Obviously, $0 \to \Omega_j(N) \to I^j \to I^{j+1} \to \cdots$ is a minimal injective resolution. On the other hand, since N does not have a free summand, $\cdots \to I^{-1} \to I^0 \to \cdots \to I^{j-1} \to \Omega_j(N) \to 0$ is a minimal projective resolution. If $d_i(\mathbf{L}(N)) > 0$, then $d_i(\mathbf{L}(\Omega_j(N))) = d_i(\mathbf{L}(N))$ for all $j \geq 0$. But, if $d_i(\mathbf{L}(N)) = 0$, then $d_i(\mathbf{L}(\Omega_j(N))) = -\infty$ for $j \gg 0$. On the other hand, since a minimal injective resolution of $N^\bullet$ is the dual of a minimal projective resolution of N, we have $d_i(\mathbf{L}(N^\bullet)) = d_i(\mathbf{L}(\Omega_j(N)^\bullet))$ for all i and all $j \geq 0$. So $N^\bullet$ and $\Omega_j(N)^\bullet$ have the same distinguished pairs. For a contradiction, we assume that $(0, i)$ is a distinguished pair for N. Then $(0, 2n-i)$ is distinguished for $N^\bullet$ and $\Omega_j(N)^\bullet$. So $(0, i)$ is distinguished for $\Omega_j(N)$ for all $j \geq 0$. This contradicts the above observation. □

We say a distinguished pair (d, i) is *positive*, if $d > 0$. Since [3, 11] study a distinguished pair for a module, they only treat a positive one.

Remark 2.10. When $N^\bullet$ is a module, Corollary 2.8 was proved in [11, Theorem 3.8]. On the other hand, for positive distinguished pairs, we can prove the corollary from [11, Theorem 3.8] directly: Let $I^\bullet$ be an injective resolution of $N^\bullet$ and $P^\bullet$ a projective resolution of $I^\bullet$. From the quasi-isomorphism $f : P^\bullet \to I^\bullet$, we have the exact complex $(T^\bullet, \partial^\bullet) := \operatorname{cone}(f)$. Then $N := \ker \partial_0$ (resp. $N^\bullet$) has the same positive distinguished pairs as $N^\bullet$ (resp. $\mathbf{D}_E(N^\bullet)$).

A variant of BGG correspondence gives an equivalence $\underline{\operatorname{gr}} E \cong D^b(\operatorname{Coh}(\mathbb{P}^{n-1}))$ of triangulated categories, where $\underline{\operatorname{gr}} E$ is the stable category, and $\operatorname{Coh}(\mathbb{P}^{n-1})$ is the category of coherent sheaves on $\mathbb{P}^{n-1} = \operatorname{Proj} S$. More precisely, the composition of the functor $\mathbf{L} : \operatorname{gr} E \to D^b(\operatorname{gr} S)$ and the natural functor $D^b(\operatorname{gr} S) \to D^b(\operatorname{Coh}(\mathbb{P}^{n-1}))$ induces this equivalence. Note that the functor $\operatorname{gr} S \ni M \to \tilde{M} \in \operatorname{Coh}(\mathbb{P}^{n-1})$ ignores modules of finite length. Hence if $d_i(M^\bullet) = 0$ then $H^i(\tilde{M}^\bullet) = 0$.

In the rest of this section, we assume that K is algebraically closed. Let $N \in \operatorname{gr} E$. Following [2], we say $v \in E_{-1} = V$ is *N-regular* if $\operatorname{Ann}_N(v) = vN$. It is easy to see that v is N-regular if and only if it is $N^\bullet$-regular. We say $V_E(N) = \{v \in V \mid v \text{ is } \textit{not}\ N\text{-regular}\}$ is the *rank variety* of N (see [2]). [2, Theorem 3.1] states that $V_E(N)$ is an algebraic subset of $V = \operatorname{Spec} S$, and $\dim V_E(V) = \max\{ d_i(\mathbf{L}(N)) \mid i \in \mathbb{Z}\}$. By the above remark, $V_E(N) = V_E(N^\bullet)$. We can refine this observation using the grading of N.

Recall that S can be seen as the Yoneda algebra $\operatorname{Ext}_E^\bullet(K,K)$, and $\operatorname{Ext}_E^\bullet(K,N)$ has the S-module structure by the Yoneda product. By the same argument as [2, Theorem 3.9], we have that

$$V_E(N) = \{\, v \in V \mid \xi(v) = 0 \text{ for all } \xi \in \operatorname{Ann}_S(\operatorname{Ext}_E^\bullet(K,N))\,\}.$$

But $[\operatorname{Ext}_E^{\bullet+i}(K,N)]_\bullet := \bigoplus_{j\in\mathbb{Z}} \operatorname{Ext}_E^{j+i}(K,N)_j$ is an S-module which is isomorphic to $H^i(\mathbf{L}(N))$ (see the proof of [6, Proposition 2.3]. Note that $\operatorname{Ext}_E^i(K,K) = [\operatorname{Ext}_E^i(K,K)]_i$ for all i), and we have $\operatorname{Ext}_E^\bullet(K,N) \cong \bigoplus_{j\in\mathbb{Z}}[\operatorname{Ext}_E^{\bullet+i}(K,N)]_\bullet$. Set

$$V_E^i(N) = \{\, v \in V \mid \xi(v) = 0 \text{ for all } \xi \in \operatorname{Ann}_S([\operatorname{Ext}_E^{\bullet+i}(K,N)]_\bullet)\,\}.$$

We have $V_E(N) = \bigcup_i V_E^i(N)$ and $d_i(\mathbf{L}(N)) = \dim V_E^i(N)$. For an algebraic set $X \subset \operatorname{Spec} S$ of dimension d, set $\operatorname{Top}(X)$ to be the union of the all irreducible components of X of dimensions d.

Proposition 2.11. *If (d,i) is a distinguished pair for $N \in \operatorname{gr} E$, then we have* $\operatorname{Top}(V_E^i(N)) = \operatorname{Top}(V_E^{2n-d-i}(N^\bullet))$.

Proof. By the proof of Theorem 2.3, $\operatorname{Ann}_S(H^i(\mathbf{L}(N)))$ has the same top dimensional components as $\operatorname{Ann}_S(H^{-d-i}(\mathbf{D}_S \circ \mathbf{L}(N)))$. □

In the above situation, we have $V_E^i(N) \neq V_E^{2n-d-i}(N^\bullet)$ in general.

3. Squarefree case

In this section, we regard $S = K[x_1,\dots,x_n]$ as an $\mathbb{N}^n$-graded ring with $\deg x_i = (0,\dots,0,1,0,\dots,0)$ where 1 is in the ith position. Similarly, $E = K\langle y_1,\dots,y_n\rangle$ is a $-\mathbb{N}^n$-graded ring with $\deg y_i = -\deg x_i$. Let ${}^*\operatorname{gr} S$ (resp. ${}^*\operatorname{gr} E$) be the category of finitely generated $\mathbb{Z}^n$-graded S-modules (resp. E-modules). The functors $\mathbf{L}$ and $\mathbf{R}$ defining the BGG correspondence $D^b(\operatorname{gr} S) \cong D^b(\operatorname{gr} E)$ also work in the $\mathbb{Z}^n$-graded context. That is, the functors $\mathbf{L} : D^b({}^*\operatorname{gr} E) \to D^b({}^*\operatorname{gr} S)$ and $\mathbf{R} : D^b({}^*\operatorname{gr} S) \to D^b({}^*\operatorname{gr} E)$ are defined by the same way as the $\mathbb{Z}$-graded case, and they give an equivalence $D^b({}^*\operatorname{gr} S) \cong D^b({}^*\operatorname{gr} E)$, see [14, Theorem 4.1]. Note that the dualizing complex $\omega^\bullet$ of S is $\mathbb{Z}^n$-graded, and $\mathbf{D}_S(-) = \operatorname{Hom}_S^\bullet(-,\omega^\bullet)$ is also a duality functor on $D^b({}^*\operatorname{gr} S)$. Similarly, $\mathbf{D}_E(-) = \operatorname{Hom}_E(-,E)$ is a duality functor on $D^b({}^*\operatorname{gr} E)$. As Proposition 2.6, for $N^\bullet \in D^b({}^*\operatorname{gr} E)$, we have $\mathbf{D}_S \circ \mathbf{L}(N^\bullet) \cong \mathbf{L} \circ \mathbf{D}_E(N^\bullet)(-\mathbf{2})[2n]$ in $D^b({}^*\operatorname{gr} S)$. Here we set $\mathbf{j} := (j,j,\dots,j) \in \mathbb{N}^n$ for $j \in \mathbb{Z}$.

For $\mathbf{a} = (a_1,\dots,a_n) \in \mathbb{Z}^n$, set $\operatorname{supp}(\mathbf{a}) := \{i \mid a_i > 0\} \subset [n] := \{1,\dots,n\}$ and $|\mathbf{a}| = \sum_{i=1}^n a_i$. We say $\mathbf{a} \in \mathbb{Z}^n$ is *squarefree* if $a_i = 0, 1$ for all $i \in [n]$. When $\mathbf{a} \in \mathbb{Z}^n$ is squarefree, we sometimes identify $\mathbf{a}$ with $\operatorname{supp}(\mathbf{a})$.

Definition 3.1 ([13]). We say a $\mathbb{Z}^n$-graded S-module M is *squarefree*, if the following conditions are satisfied.

(a) M is $\mathbb{N}^n$-graded (i.e., $M_\mathbf{a} = 0$ if $\mathbf{a} \notin \mathbb{N}^n$) and finitely generated.
(b) The multiplication map $M_\mathbf{a} \ni y \mapsto (\prod x_i^{b_i}) \cdot y \in M_{\mathbf{a}+\mathbf{b}}$ is bijective for all $\mathbf{a}, \mathbf{b} \in \mathbb{N}^n$ with $\operatorname{supp}(\mathbf{a}+\mathbf{b}) = \operatorname{supp}(\mathbf{a})$.

For a simplicial complex $\Delta \subset 2^{[n]}$, the Stanley-Reisner ideal $I_\Delta := (\prod_{i\in F} x_i \mid F \notin \Delta)$ and the Stanley-Reisner ring $K[\Delta] := S/I_\Delta$ are squarefree modules. Note that if M is squarefree then $M_{\mathbf{a}} \cong M_F$ as K-vector spaces for all $\mathbf{a} \in \mathbb{N}^n$ with $\mathrm{supp}(\mathbf{a}) = F$. Let Sq_S be the full subcategory of ${}^*\mathrm{gr}\, S$ consisting of squarefree modules. In ${}^*\mathrm{gr}\, S$, Sq_S is closed under kernels, cokernels and extensions ([13, Lemma 2.3]), and we have that $D^b(\mathrm{Sq}_S) \cong D^b_{\mathrm{Sq}_S}({}^*\mathrm{gr}\, S)$. If $M^\bullet \in D^b(\mathrm{Sq}_S)$, then $\mathbf{D}_S(M^\bullet) \in D^b_{\mathrm{Sq}_S}({}^*\mathrm{gr}\, S)$ (see [14]). So $\mathbf{D}_S$ gives a duality functor on $D^b(\mathrm{Sq}_S)$.

Remark 3.2. This remark is an advertisement for my recent paper [16], and quite independent from other parts of this article.

Let B be an $(n-1)$-simplex which is the geometric realization of $2^{[n]}$. From a squarefree S-module M, we can construct a k-sheaf M^+ on B (we are thinking classical topology on B). For example, $k[\Delta]^+ \cong j_* \underline{k}_{|\Delta|}$, where $\underline{k}_{|\Delta|}$ is the constant sheaf on $|\Delta|$ and $j : |\Delta| \to B$ is the embedding map. We have an isomorphism $H^i(B, M^+) \cong [H^{i+1}_{\mathfrak{m}}(M)]_0$ for all $i \geq 1$, and an exact sequence $0 \to [H^0_{\mathfrak{m}}(M)]_0 \to M_0 \to H^0(B, M^+) \to [H^1_{\mathfrak{m}}(M)]_0 \to 0$, where $H^i_{\mathfrak{m}}(-)$ is the local cohomology module with support in the maximal ideal $\mathfrak{m} := (x_1, \dots, x_n)$. We have that $\omega^\bullet_{K[\Delta]} := \mathrm{Hom}_S(K[\Delta], \omega^\bullet) \in D^b(\mathrm{Sq}_S)$, and $j^\bullet(\omega^\bullet_{K[\Delta]})^+$ is Verdier's dualizing complex of $|\Delta|$ (c.f. [9]). Moreover, Serre duality for $R\mathrm{Hom}_{K[\Delta]}(-, \omega^\bullet_{K[\Delta]})$ corresponds to the Poincaré-Verdier duality on $|\Delta|$ in our context. See [16] for detail.

Definition 3.3 (Römer [11]). A $\mathbb{Z}^n$-graded E-module $N = \bigoplus_{\mathbf{a}\in\mathbb{Z}^n} N_{\mathbf{a}}$ is *squarefree* if N is finitely generated and $N = \bigoplus_{F \subset [n]} N_{-F}$.

For example, any monomial ideal of E is a squarefree module. Let Sq_E be the full subcategory of ${}^*\mathrm{gr}\, E$ consisting of squarefree E-modules. Then we have the functors $\mathcal{S} : \mathrm{Sq}_E \to \mathrm{Sq}_S$ and $\mathcal{E} : \mathrm{Sq}_S \to \mathrm{Sq}_E$ giving an equivalence $\mathrm{Sq}_S \cong \mathrm{Sq}_E$. Here $\mathcal{S}(N)_F = N_{-F}$ for $N \in \mathrm{Sq}_E$, and the multiplication map $\mathcal{S}(N)_F \ni z \mapsto x_i z \in \mathcal{S}(N)_{F\cup\{i\}}$ for $i \notin F$ is given by $\mathcal{S}(N)_F = N_{-F} \ni z \mapsto (-1)^{\alpha(i,F)} y_i z \in N_{-(F\cup\{i\})} = \mathcal{S}(N)_{F\cup\{i\}}$, where $\alpha(i, F) = \#\{ j \in F \mid j < i \}$. See [11] for further information. Of course, $\mathcal{S}$ and $\mathcal{E}$ can be extended to the functors between $D^b(\mathrm{Sq}_S)$ and $D^b(\mathrm{Sq}_E)$.

If $N \in \mathrm{Sq}_E$, then $N^* = \mathrm{Hom}_E(N, E)$ is squarefree again. So $(-)^*$ gives the duality functor $\mathbf{D}_E$ on $D^b(\mathrm{Sq}_E)$. We have the *Alexander duality functor* $\mathbf{A} := \mathcal{S} \circ \mathbf{D}_E \circ \mathcal{E}$ on Sq_S (or $D^b(\mathrm{Sq}_S)$). For example, $\mathbf{A}(K[\Delta]) = I_{\Delta^\vee}$, where $\Delta^\vee$ is the Alexander dual complex of Δ (see §1). In general, we have $\mathbf{A}(H^i(M^\bullet))_F = (H^{-i}(M^\bullet)_{[n]\setminus F})^\vee$.

Remark 3.4. Let $\Lambda := \langle e_{F,G} \mid F, G \subset [n], F \subset G \rangle$ be the incidence algebra of the Boolean lattice $2^{[n]}$ with the coefficients in K, and mod_Λ the category of finitely generated right Λ-modules. Then we have a functor $\Psi : \mathrm{mod}_\Lambda \to \mathrm{Sq}_S$ giving an equivalence $\mathrm{mod}_\Lambda \cong \mathrm{Sq}_S$. See [14, Proposition 2.2]. Since $\mathrm{inj.dim}\, \Lambda < \infty$, $R\mathrm{Hom}_\Lambda(-, \Lambda[n])$ gives a duality functor from $D^b(\mathrm{mod}_\Lambda)$ to $D^b(\mathrm{mod}_{\Lambda^{\mathrm{op}}})$. But the map defined by $e_{F,G} \mapsto e_{G^c, F^c}$ gives a ring isomorphism $\Lambda^{\mathrm{op}} \cong \Lambda$, where $F^c := [n] \setminus F$. Under this isomorphism $R\mathrm{Hom}_\Lambda(-, \Lambda[n])$ gives the duality functor $\mathbf{D}_\Lambda$ from $D^b(\mathrm{mod}_\Lambda)$ to itself. Then we have $\mathbf{D}_S \cong \Psi \circ \mathbf{D}_\Lambda \circ \Psi^{-1}$. Similarly, the duality $\mathbf{A}$ on Sq_S corresponds to the duality $\mathrm{Hom}_k(-, k)$ on mod_Λ. See [14, Remark 3.3].

An associated prime ideal of $M \in {}^*\mathrm{gr}\, S$ is of the form $P_F = (x_i \mid i \notin F)$ for some $F \subset [n]$. Let M be a squarefree module. A monomial prime ideal P_F is a minimal prime of M if and only if F is a maximal element of the set $\{G \subset [n] \mid M_G \neq 0\}$. The following is a squarefree version of Definition 2.1.

Definition 3.5. We say $(F, i) \in 2^{[n]} \times \mathbb{Z}$ is a *distinguished pair* for a complex $M^\bullet \in D^b(\mathrm{Sq}_S)$, if P_F is a minimal prime of $H^i(M^\bullet)$ and $H^j(M^\bullet)_G = 0$ for all j with $j < i$ and $G \supset F$ with $|G| < |F| + i - j$.

Theorem 3.6. *Let $M^\bullet \in D^b(\mathrm{Sq}_S)$. A pair (F, i) is distinguished for $M^\bullet$ if and only if $(F, -|F| - i)$ is distinguished for $\mathrm{D}_S(M^\bullet)$. If this is the case, $\dim_K H^i(M^\bullet)_F = \dim_K H^{-|F|-i}(\mathrm{D}_S(M^\bullet))_F$.*

Proof. Note that the spectral sequence $E_2^{p,q} = \mathrm{Ext}_S^p(H^{-q}(M^\bullet), \omega^\bullet) \Rightarrow \mathrm{Ext}_S^{p+q}(M^\bullet, \omega^\bullet)$ is $\mathbb{Z}^n$-graded, and $E_r^{p,q}$ is squarefree for all p, q and $r \geq 2$. So we can prove the theorem in a similar way to Theorem 2.3. See [15] for detail. □

If $N^\bullet \in D^b(\mathrm{Sq}_E)$, then it is easy to see that $\mathbf{L}(N^\bullet)(-1) \in D^b(\mathrm{Sq}_S)$. So $\mathcal{L}(-) := \mathbf{L}(-)(-1)$ gives a functor from $D^b(\mathrm{Sq}_E)$ to $D^b(\mathrm{Sq}_S)$. Moreover, we have $\mathcal{L} \cong \mathbf{A} \circ \mathrm{D}_S \circ \mathcal{S}$ by [14, Proposition 4.3].

Definition 3.7. Let $N^\bullet \in D^b(\mathrm{Sq}_E)$. We say (F, i) is a *distinguished pair* for $N^\bullet$ if it is a distinguished pair for $\mathcal{L}(N^\bullet) \in D^b(\mathrm{Sq}_S)$ in the sense of Definition 3.5.

The next result can be proved by the same way as Corollary 2.8 using Theorem 3.6.

Proposition 3.8. *Let $N^\bullet \in D^b(\mathrm{Sq}_E)$. A pair (F, i) is distinguished for $N^\bullet$ if and only if $(F, 2n - |F| - i)$ is distinguished for $\mathrm{D}_E(N^\bullet)$. If this is the case, we have $\dim_K H^i(\mathcal{L}(N^\bullet))_F = \dim_K H^{2n-|F|-i}(\mathcal{L} \circ \mathrm{D}_E(N^\bullet))_F$.*

If $M^\bullet \in D^b({}^*\mathrm{gr}\, S)$, then $\mathrm{Tor}_i^S(K, M^\bullet) := H^{-i}(K \otimes_E P^\bullet)$ is a $\mathbb{Z}^n$-graded module, where $P^\bullet$ is a graded free resolution of $M^\bullet$. Set $\beta_{i,\mathbf{a}}(M^\bullet) := \dim_K \mathrm{Tor}_i^S(K, M^\bullet)_\mathbf{a}$ for $\mathbf{a} \in \mathbb{Z}^n$. We say $\beta_{i,\mathbf{a}}(M^\bullet)$ is the $(i, \mathbf{a})$th Betti number of $M^\bullet$. If $M^\bullet \in D^b(\mathrm{Sq}_S)$ and $\beta_{i,\mathbf{a}}(M^\bullet) \neq 0$, then $\mathbf{a}$ is squarefree (see [14]). Now we back to extremal Betti numbers mentioned in §1.

Definition 3.9 ([4]). A Betti number $\beta_{i,F}(M^\bullet) \neq 0$ is *extremal* if $\beta_{j,G}(M^\bullet) = 0$ for all $(j, G) \neq (i, F)$ with $j \geq i$, $G \supset F$, and $|G| - j > |F| - i$.

Proposition 3.10 (c.f. [3]). *Let $M^\bullet \in D^b(\mathrm{Sq}_S)$ and $N^\bullet := \mathcal{E}(M^\bullet) \in D^b(\mathrm{Sq}_E)$. A pair (F, i) is distinguished for $\mathrm{D}_E(N^\bullet)$ if and only if $\beta_{i+|F|-n,F}(M^\bullet)$ is an extremal Betti number. If this is the case, then we have $\beta_{i+|F|-n,F}(M^\bullet) = H^i(\mathcal{L} \circ \mathrm{D}_E(N^\bullet))_F$.*

Proof. For $j \in \mathbb{Z}$ and $G \subset [n]$, we have the following.

$$\begin{aligned}
\beta_{j,G}(M^\bullet) &= \dim_K[H^{|G|-j-n}(\mathrm{D}_S \circ \mathbf{A}(M^\bullet))]_{[n]\setminus G} \quad \text{(by [14, Corollary 3.6])} \\
&= \dim_K[H^{n+j-|G|}(\mathbf{A} \circ \mathrm{D}_S \circ \mathbf{A}(M^\bullet))]_G \\
&= \dim_K[H^{n+j-|G|}(\mathcal{L} \circ \mathcal{E} \circ \mathbf{A}(M^\bullet))]_G \\
&= \dim_K[H^{n+j-|G|}(\mathcal{L} \circ \mathrm{D}_E(N^\bullet))]_G.
\end{aligned}$$

The assertion easily follows from this equality. □

Corollary 3.11 (c.f. [4, 11, 10]). *Let* $M^\bullet \in D^b(\mathrm{Sq}_S)$. *A Betti number* $\beta_{i,F}(M^\bullet)$ *is extremal if and only if so is* $\beta_{|F|-i,F}(\mathrm{A}(M^\bullet))$. *If this is the case,* $\beta_{i,F}(M^\bullet) = \beta_{|F|-i,F}(\mathrm{A}(M^\bullet))$.

Proof. If $\beta_{i,F}(M^\bullet)$ is extremal, then $(F, n+i-|F|)$ is a distinguished pair for $\mathbf{D}_E \circ \mathcal{E}(M^\bullet)$ by Proposition 3.10. By Proposition 3.8, $(F, n-i)$ is a distinguished pair for $\mathcal{E}(M^\bullet) \cong \mathbf{D}_E \circ \mathcal{E} \circ \mathrm{A}(M^\bullet)$. So $\beta_{|F|-i,F}(\mathrm{A}(M^\bullet))$ is extremal. The converse implication can be proved by the same way. The last equality follows from Proposition 3.8. □

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FACTORIZATIONS OF ENVELOPING ALGEBRAS OF THREE DIMENSIONAL LIE ALGEBRAS AND THEIR APPLICATIONS

JUN MORITA

1. Gauss Eliminations

To solve a linear equation generically, for example,

$$\begin{cases} a\ x + b\ y = r \\ c\ x + d\ y = s \end{cases}$$

we usually use the so-called Gauss elimination, which can be expressed as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & \frac{ad-bc}{a} \end{pmatrix} \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix}.$$

That is, we see

$$\begin{pmatrix} 1 & 0 \\ -\frac{c}{a} & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & \frac{ad-bc}{a} \end{pmatrix},$$

$$\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{a}{ad-bc} \end{pmatrix} \begin{pmatrix} a & b \\ 0 & \frac{ad-bc}{a} \end{pmatrix} = \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix}.$$

This idea leads to certain decompositions of groups of Lie type.

Let G be a split semisimple algebraic group or a Kac-Moody group (over a field). We take subgroups U, T, V of G as a standard maximal uppertriangular unipotent subgroup, a standard maximal diagonal subgroup, a standard maximal lowertriangular unipotent subgroup, respectively. Roughly speaking, $\Omega = VTU$ is an open dense (or very large) subset of G. Then, more precisely we obtain the Gauss decomposition:

$$G = UVTU.$$

There is also a strong version. Namely,

$$G = Z(G) \cup \bigcup_{g \in G} g(VhU)g^{-1}$$

for all $h \in T$, where $Z(G)$ is the center of G. This is called a strong Gauss decomposition of G. Using such a decomposition, we can immediately find that every noncentral element can be expressed as a product of two unipotent elements.

Philosophically we can think of an additive version of these decompositions. Here we introduce some idea of additive Gauss decompositions.

2. Additive Gauss Decompositions

Let F be a field of characteristic 0, and L be a Lie algebra over F. The universal enveloping algebra of L is denoted by $U(L)$. Let V be an L-module.

Theorem (cf. [1]). Ler L be a three dimensional Lie algebra over F. If L is generated by two elements x and y, Then, $U(L) = \sum_{i,j,k} x^i y^j x^k$. Furthermore, if both x and y act on V as locally nilpotent operators, then V is locally finite as an L-module.

The key formulas to establish the above theorem are as follows:

$$(\mathrm{A}_k) \quad yxy^k \equiv \tfrac{k}{k+1} xy^{k+1} + \tfrac{1}{k+1} y^{k+1}x \mod U_k,$$

$$(\mathrm{B}_k) \quad y^kxy \equiv \tfrac{1}{k+1} xy^{k+1} + \tfrac{k}{k+1} y^{k+1}x \mod U_k,$$

$$(\mathrm{C}_k) \quad yU_k \subset U_{k+1}, \quad U_k \subset U_{k+1},$$

where $U_k = \sum_{0\le m\le k}(Fxy^m + Fy^mx + Fy^m)$.

Proof of Theorem. Suppose that L is generated by two elements x, y. Let $z = [x,y]$. Now we want to show $U(L) = \sum_{i,j,k\ge 0} Fx^iy^jx^k$. Put $\mathfrak{X} = \sum_{i,j,k\ge 0} Fx^iy^jx^k \subseteq U(L)$ and let U_k be defined as above. Clearly $x\mathfrak{X} \subseteq \mathfrak{X}, \mathfrak{X}x \subseteq \mathfrak{X}$ and $U_k \subseteq \mathfrak{X}$ for all $k \ge 0$. We claim

$$y(x^\ell y^m x^n) \in \mathfrak{X},$$
$$z(x^\ell y^m x^n) \in \mathfrak{X}.$$

and show this by induction on ℓ. If $\ell = 0$, then we see $y(y^mx^n) \in \mathfrak{X}$ and using (A_m) we get

$$\begin{aligned} z(y^mx^n) &= (xy - yx)(y^mx^n) \\ &= xy^{m+1}x^n - yxy^mx^n \\ &\in Fxy^{m+1}x^n + (Fxy^{m+1} + Fy^{m+1}x + U_m)x^n \subseteq \mathfrak{X}. \end{aligned}$$

Let $\ell > 0$. Then, we obtain, using our inductive assumption, that

$$\begin{aligned} y(x^\ell y^m x^n) &= (xy - z)(x^{\ell-1}y^mx^n) \\ &\in x\mathfrak{X} + \mathfrak{X} \subseteq \mathfrak{X} \end{aligned}$$

and, letting $[z,x] = ax + by + cz$ for $a, b, c \in F$, we also get using our inductive assumption that

$$\begin{aligned} z(x^\ell y^m x^n) &= (xz + ax + by + cz)(x^{\ell-1}y^mx^n) \\ &\in x\mathfrak{X} + \mathfrak{X} + \mathfrak{X} + \mathfrak{X} \subseteq \mathfrak{X}. \end{aligned}$$

Hence, our induction method shows $y\mathfrak{X} \subseteq \mathfrak{X}$. Since $\mathfrak{X}$ is a left ideal of $U(L)$ containing 1, we obtain $\mathfrak{X} = U(L)$. Therefore,

$$U(L) = \mathfrak{X} = \sum_{i,j,k\ge 0} Fx^iy^jx^k = U(Fx)\,U(Fy)\,U(Fx).$$

Q.E.D.

The decomposition described in the above theorem may be rewritten as

$$U(L) = U(Fx)\,U(Fy)\,U(Fx) = U(Fy)\,U(Fx)\,U(Fy),$$

which we call an additive Gauss decomposition. If we take

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

as generators of $s\ell_2 = s\ell_2(F)$, then we have

$$U(s\ell_2) = U(Fe)\, U(Ff)\, U(Fe) = U(Ff)\, U(Fe)\, U(Ff).$$

If $\mathfrak{H} = Fx \oplus Fy \oplus Fz$ is a Heisenberg Lie algebra with $[x,y] = z$ and $[x,z] = [y,z] = 0$, then we also have

$$U(\mathfrak{H}) = U(Fx)\, U(Fy)\, U(Fx) = U(Fy)\, U(Fx)\, U(Fy).$$

We note that a three dimensional Lie algebra L is not two generated if and only if L is abelian or isomorphic to the Lie algebra $M = Fx \oplus Fy \oplus Fz$ satisfying $[x,y] = 0$, $[x,z] = x$, $[y,z] = y$. Hence, almost all three dimensional Lie algebras have additive Gauss decompositions.

3. Applications

Let $\mathfrak{g}$ be a Kac-Moody Lie algebra or a Borcherds Lie algebra over F with the so-called standard Cartan subalgebra $\mathfrak{h}$, and $\mathfrak{g}_+$ (resp. $\mathfrak{g}_-$) the standard maximal positive (resp. negative) nilpotent subalgebra of $\mathfrak{g}$ corresponding to positive (resp. negative) roots with respect to $\mathfrak{h}$. Using additive Gauss decompositions of $s\ell_2$ and $\mathfrak{H}$, we obtain

$$U(\mathfrak{g}) = U(\mathfrak{g}_\pm)\, U(\mathfrak{g}_\mp)\, U(\mathfrak{g}_\pm).$$

Hence, all Kac-Moody Lie algebras and all Borcherds Lie algebras have additive Gauss decompositions.

If e and f are locally nilpotent on an $s\ell_2$-module V, then h is diagonalizable on V, since V is a locally finite $s\ell_2$-module. Furthermore, we take standard generators

$$e_1, \dots, e_n, h_1, \dots, h_n, f_1, \dots, f_n$$

for the derived subalgebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ of a Kac-Moody Lie algebra $\mathfrak{g}$. If the e_i and the f_i are all locally nilpotent on a $\mathfrak{g}$-module V, then the h_i are simultaneously diagonalizable on V. Hence, this gives a sufficient condition for a $\mathfrak{g}$-module V to be integrable.

Let $L = L_{-1} \oplus L_0 \oplus L_1$ be a three graded Lie algebra of dimension three with $\dim L_\varepsilon = 1$ for $\varepsilon = 0, \pm 1$. If $U(L) = U(L_1)\, U(L_{-1})\, U(L_1)$, then $L \simeq s\ell_2$ or $L \simeq \mathfrak{H}$. This gives a new caracterization of $s\ell_2$ and $\mathfrak{H}$.

We can also apply our method to quantum groups. Let $\mathbf{C}$ be the field of complex numbers, and we fix a nonzero element $q \in \mathbf{C}$ with the property that $q^m \neq 1$ for all $m = 1, 2, 3 \dots$. We denote by $U_q(s\ell_2)$ the associative $\mathbf{C}$-algebra generated by $t^{\pm 1}, e, f$ with the following defining relations as usual:

$$t^{\pm 1} t^{\mp 1} = 1, \quad t e t^{-1} = q^2 e, \quad t f t^{-1} = q^{-2} f, \quad [e,f] = \frac{t - t^{-1}}{q - q^{-1}}.$$

The algebra $U_q(s\ell_2)$ is called the quantum group associated with $s\ell_2$. For each natural number k we define a q-integer, $[k] = [k]_q = (q^k - q^{-k})/(q - q^{-1})$, as a nonzero complex number, and here we put $[0] = 0$ for convenience. Then, for $n \geq 0$, we obtain:

$$(A_n^+) : fef^n = \frac{[n]}{q[n+1]} ef^{n+1} + \frac{q^n}{[n+1]} f^{n+1}e + \frac{[n]}{q^n} f^n t,$$

$$(B_n^+) : f^n ef = \frac{1}{q^n[n+1]} ef^{n+1} + \frac{q[n]}{[n+1]} f^{n+1}e + \frac{[n]}{q^n} f^n t,$$

$$(A_n^-) : fef^n = \frac{q[n]}{[n+1]} ef^{n+1} + \frac{1}{q^n[n+1]} f^{n+1}e + q^n[n] f^n t^{-1},$$

$$(B_n^-) : f^n ef = \frac{q^n}{[n+1]} ef^{n+1} + \frac{[n]}{q[n+1]} f^{n+1}e + q^n[n] f^n t^{-1}.$$

Using these relations, we can establish the following result.

$$\begin{aligned} U_q(s\ell_2) &= (\textstyle\sum_{i,j,k\geq 0} \mathbb{C}e^i f^j e^k) + (\sum_{i,j,k\geq 0} \mathbb{C}e^i f^j e^k)t \\ &= \mathbb{C}[e]\,\mathbb{C}[f]\,\mathbb{C}[e]\,(\mathbb{C} \oplus \mathbb{C}t). \end{aligned}$$

As a direct consequence of this, we obtain the following. Let V be an infinite dimensional $U_q(s\ell_2)$-module. Suppose that both operators e and f are locally nilpotent on V. Then:
(1) For each $v \in V$, the submodule, $U_q(s\ell_2)v$, generated by v is finite dimensional.
(2) The operator t is diagonalizable on V.
(3) V is a direct sum of finite dimensional irreducible $U_q(s\ell_2)$-modules. In particular, V is completely reducible.

Using the results (1),(2),(3), we can produce a sufficient condition for a representation of a quantum group to be integrable.

Remark. We note that this article is a survay of the successive papers [1], [5], [6].

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SOME APPLICATIONS OF KOSZUL DUALITY

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ABSTRACT. In this paper, we will discuss some applications of Koszul duality for a connected graded Koszul algebra A. Since A is an Artin-Schelter regular Koszul algebra if and only if its Koszul dual $A^!$ is a Frobenius Koszul algebra, there is potential interaction between two research areas, Noncommutative Algebraic Geometry and the study of Frobenius algebras.

1. INTRODUCTION

Throughout, let $A = \oplus_{i=0}^{\infty} A_i$ be a finitely generated connected graded algebra over a field k. The augmentation ideal of A is denoted by $\mathfrak{m} = \oplus_{i=1}^{\infty} A_i$. We often view k as a graded left A-module by the identification $k = A/\mathfrak{m}$. For a graded left A-module $M = \oplus_{i=-\infty}^{\infty} M_i$ and an integer $n \in \mathbb{Z}$, we define a graded left A-module $M(n)$ by $M(n) = M$ as an ungraded left A-module but $M(n)_i = M_{n+i}$ for all $i \in \mathbb{Z}$. A linear resolution of M is a resolution of the form

$$\cdots \to \bigoplus A(-2) \to \bigoplus A(-1) \to \bigoplus A \to M \to 0,$$

that is, a free resolution in which each differential is given by right multiplication of a matrix whose entries are all degree 1 elements (linear elements) of A. We say that A is Koszul if k has a linear resolution as a graded left A-module.

If A is a Koszul algebra, then its dual $A^!$ is again a Koszul algebra. An important classical result is that there is a duality, known as the Koszul duality, between the categories of graded left modules having linear resolutions over A and $A^!$, respectively. Recently, it has been shown that the Koszul duality can be extended to a duality between the derived categories of finitely generated graded left modules over A and $A^!$, respectively. Through this extended Koszul duality, we are more able to translate results on A to those on $A^!$.

It is known that A is an Artin-Schelter regular Koszul algebra, which is one of the most important algebras in Noncommutative Algebraic Geometry, if and only if $A^!$ is a Frobenius Koszul algebra. Further, the extended Koszul duality induces an equivalence between the derived category of a quantum projective space, which is a projective scheme associated to a noetherian Artin-Schelter regular Koszul algebra, and the graded stable category of the corresponding Frobenius Koszul algebra. So there is potential interaction between two research areas, Noncommutative Algebraic Geometry and the study of Frobenius algebras. In this paper, we will give some examples of such interaction.

This is an expository paper based on the results of the author and others. The detailed version of some of the results in this paper has been submitted for publication elsewhere.

2. Koszul Duality

Let L, M and N be graded left A-modules. The Yoneda product is a map

$$\begin{array}{ccc} \mathrm{Ext}^i_A(N,L) \otimes \mathrm{Ext}^j_A(M,N) & \to & \mathrm{Ext}^{i+j}_A(M,L) \\ [\beta] \otimes [\alpha] & \mapsto & [\beta \circ \alpha_i] \end{array}$$

defined by the following diagram:

$$\begin{array}{ccccccccc} \cdots \to & F^{i+j} & \to \cdots \to & F^j & \to \cdots \to & M & \to 0 \\ & \downarrow{\scriptstyle \alpha_i} & & \downarrow{\scriptstyle \alpha} & & & \\ \cdots \to & G^i & \to \cdots \to & N & \to 0 & & \\ & \downarrow{\scriptstyle \beta} & & & & & \\ & L & & & & & \end{array}$$

where

$$\cdots \to F^2 \to F^1 \to F^0 \to M \to 0, \text{ and}$$
$$\cdots \to G^2 \to G^1 \to G^0 \to N \to 0$$

are free resolutions of M and N, respectively, and α_i is a lift of α.

We define a graded vector space $E(M)$ over k by $E(M)_i = \mathrm{Ext}^i_A(M,k)$. Then $E(k)$ has a structure of a connected graded algebra over k, and $E(M)$ has a structure of a graded left $E(k)$-module by the Yoneda product.

Let A be a quadratic algebra over k, that is, $A = T(V)/(R)$ where V is a finite dimensional vector space over k, $T(V)$ is a tensor algebra over V, and $R \subset V \otimes V$ is a subvector space. We define the dual of A by $A^! = T(V^*)/(R^\perp)$ where V^* is the dual vector space of V and

$$R^\perp = \{\lambda \in V^* \otimes V^* \mid \lambda(r) = 0 \text{ for all } r \in R\}.$$

If A is a Koszul algebra, then it is known that A is quadratic and $A^! \cong E(k)$ as graded algebras. In this case, $E(E(M))$ has a graded left module structure over $(A^!)^! \cong A$. The following is an important classical result.

Theorem 2.1. [17] *(Koszul duality) If A is a Koszul algebra, then $A^!$ is also a Koszul algebra. In this case, if M is a graded left A-module having a linear resolution, then $E(M)$ is a graded left $A^!$-module having a linear resolution, and*

$$E(E(M)) \cong M$$

as graded left A-modules. That is, E defines a duality

$$E : \mathrm{Lin}\, A \to \mathrm{Lin}\, A^!,$$

where $\mathrm{Lin}\, A$ *is the category of graded left A-modules having linear resolutions.*

If A is a Koszul algebra, then we call $A^!$ the Koszul dual of A.

Example 1. A free algebra

$$T = k\langle x_1, \cdots, x_d\rangle$$

is a Koszul algebra where

$$T^! \cong k\langle x_1, \cdots, x_d\rangle/(x_i x_j)_{i,j=1,\cdots,d}.$$

Example 2. A polynomial algebra

$$S = k[x_1, \cdots, x_d] = k\langle x_1, \cdots, x_d\rangle/(x_ix_j - x_jx_i)_{i,j=1,\cdots,d}$$

is a Koszul algebra where

$$S^! \cong k\langle x_1, \cdots, x_d\rangle/(x_ix_j + x_jx_i, x_i^2)_{i,j=1,\cdots,d}$$

is an exterior algebra.

Example 3. An algebra

$$A = k\langle x, y\rangle/(y^2)$$

is a Koszul algebra where

$$A^! \cong k\langle x, y\rangle/(x^2, xy, yx).$$

By this duality, we can translate some of the properties of A to those of $A^!$. For example, the following result is immediate.

Lemma 2.2. *Let A be a Koszul algebra. Then A is regular, that is, A has finite global dimension, if and only if $A^!$ is finite dimensional over k.*

3. Extended Koszul Duality

The functor E above does not define a duality for the categories of arbitrary modules. So the above Koszul duality has limited applications. In order to overcome this limitation, we will extend the above Koszul duality to derived categories. If we assume the existence of a balanced dualizing complex defined below, then the duality behaves particularly well.

Definition 3.1. [20] A balanced dualizing complex of A is a complex D_A of graded A-A bimodules satisfying the following conditions, viewing D_A as a complex of graded left and right A-modules:

- D_A has finite injective dimension,
- $h^i(D_A)$ are finitely generated for all i,
- $\operatorname{Ext}^i_A(D_A, D_A) \cong \begin{cases} A & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}$ as graded A-A bimodules, and
- $\operatorname{H}^i_{\mathfrak{m}}(D_A) \cong \begin{cases} A^* & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}$ as graded A-A bimodules,

where

$$\operatorname{H}^i_{\mathfrak{m}}(-) = \lim_{n\to\infty} \operatorname{Ext}^i_A(A/\mathfrak{m}^n, -)$$

is the i-th local cohomology functor.

A balanced dualizing complex plays an essential role in Noncommutative Algebraic Geometry. Many noetherian algebras of importance have balanced dualizing complexes. For example, finite dimensional algebras, noetherian commutative algebras, and FBN algebras (including noetherian PI algebras) have balanced dualizing complexes (see [19]).

We will now extend the Koszul duality to derived categories as follows. Let $\operatorname{grmod} A$ be the category of finitely generated graded left A-modules and $\mathcal{D}^b(\operatorname{grmod} A)$ the derived category of bounded complexes in $\operatorname{grmod} A$.

Theorem 3.1. *(Koszul duality)* [4], [12] *Let A be a Koszul algebra. If both A and $A^!$ are noetherian and having balanced dualizing complexes, then there is a duality*

$$\bar{E} : \mathcal{D}^b(\text{grmod } A) \to \mathcal{D}^b(\text{grmod } A^!).$$

We refer to [4] and [12] for the definition of $\bar{E}(X)$ for a complex X of graded left A-modules. If M is a finitely generated graded left A-module, then $\bar{E}(M)$ is a complex of left $A^!$-modules defined by

$$\bar{E}(M)^i = A^! \otimes M_i^*(-i).$$

The following lemma says that the duality in the above theorem is in fact an extension of the classical Koszul duality.

Lemma 3.2. [12] *Let A be a Koszul algebra and M a finitely generated graded left A-module. Then M has a linear resolution if and only if $\bar{E}(M) \cong E(M)$ in $\mathcal{D}^b(\text{grmod } A^!)$.*

By this extended Koszul duality, we are more able to translate results on A to those on $A^!$. We will see such examples below.

4. Gorenstein Condition

One of the starting points of Noncommutative Algebraic Geometry was to classify Artin-Schelter regular algebras of global dimension 3, defined below.

Definition 4.1. [2] We say that A satisfies Gorenstein condition if, for some integer d, we have

$$\text{Ext}_A^i(k, A) \cong \begin{cases} k & \text{if } i = d, \\ 0 & \text{if } i \neq d. \end{cases}$$

We say that A is Artin-Schelter regular (AS-regular, for short) if A is regular and satisfying Gorenstein condition.

If A is noetherian and having a balanced dualizing complex (e.g. commutative), then A satisfies Gorenstein condition if and only if A has finite injective dimension as a left module over itself, so the above definition of Gorenstein condition agrees with the commutative one. Note that a noetherian regular algebra is AS-regular if and only if A has a balanced dualizing complex.

The first application of the extended Koszul duality is below.

Theorem 4.1. [7] [12] *Let A be a Koszul algebra. Then A satisfies Gorenstein condition if and only if $A^!$ satisfies Gorenstein condition.*

Since a finite dimensional algebra satisfies Gorenstein condition if and only if it is Frobenius, we can recapture the following result as a corollary.

Corollary 4.2. [17] *Let A be a Koszul algebra. Then A is AS-regular if and only if $A^!$ is Frobenius.*

5. Hilbert Series

In this section and the next section, we will discuss rationality of the Hilbert series and the Poincaré series, respectively, of a finitely generated graded left A-module M.

If $\dim_k M_i < \infty$ for all i, then we define the Hilbert series of M by

$$H_M(t) = \sum_{i=-\infty}^{\infty} (\dim_k M_i)t^i \in \mathbb{Z}[[t, t^{-1}]].$$

Let us calculate the Hilbert series of a few examples of graded algebras.

Example 4. If $T = k\langle x_1, \cdots, x_d\rangle$ is a free algebra, then

$$H_T(t) = \sum_{i=0}^{\infty} d^i t^i = \frac{1}{1-dt} \in \mathbb{C}(t).$$

Example 5. If $S = k[x_1, \cdots, x_d]$ is a polynomial algebra, then

$$H_S(t) = \sum_{i=0}^{\infty} \binom{d-1+i}{d-1} t^i = \frac{1}{(1-t)^d} \in \mathbb{C}(t).$$

Example 6. If $A = k\langle x, y\rangle/(y^2)$, then

$$\begin{aligned}
A_0 &= k.\\
A_1 &= kx + ky.\\
A_2 &= kx^2 + kxy + kyx.\\
A_3 &= kx^3 + kx^2y + kxyx + kyx^2 + kyxy.\\
A_4 &= kx^4 + kx^3y + kx^2yx + kxyx^2 + kyx^3 + kxyxy + kyx^2y + kyxyx.\\
&\cdots
\end{aligned}$$

If $a_i = \dim_k A_i$, then, by induction,

$$a_0 = 1, a_1 = 2, a_i = a_{i-2} + a_{i-1} \text{ for } i \geq 2$$

(Fibonacci sequence). We can prove that

$$H_A(t) = \sum_{i=0}^{\infty} a_i t^i = \frac{1+t}{1-t-t^2} \in \mathbb{C}(t).$$

There was a natural conjecture by V. E. Govorov (1972) that "if A is finitely presented, then $H_A(t) \in \mathbb{C}(t)$". Although counterexamples were constructed by J. B. Shearer [16] (1980) and D. Anick [1] (1982), we will still expect that reasonably nice algebras, such as left noetherian algebras, have the following stronger property.

Definition 5.1. We say that A has the property (H) if $H_M(t) \in \mathbb{C}(t)$ for all finitely generated graded left A-modules M.

In fact, many left noetherian algebras have the property (H).

Example 7.

1. Every finite dimensional algebra has the property (H).
2. Every quotient algebra of a left noetherian regular algebra has the property (H). In particular, every noetherian commutative algebra has the property (H).
3. (Stafford-Zhang [18], 1994) Every FBN algebra has the property (H). In particular, every noetherian PI algebra has the property (H).

6. Poincaré Series

If $\dim_k \operatorname{Ext}^i_A(M,k) < \infty$ for all i, then we define the Poincaré series of M by

$$P^M_A(t) = \sum_{i=0}^{\infty} \dim_k \operatorname{Ext}^i_A(M,k)t^i \in \mathbb{Z}[[t]].$$

There was a similar but more restrictive conjecture by Serre-Kaplansky (1965) that "if A is finite dimensional and commutative, then $P^k_A(t) \in \mathbb{C}(t)$". Although a counterexample was constructed by D. Anick [1] (1982), finding a class of algebras having the following stronger property remains interesting.

Definition 6.1. We say that A has the property (P) if $P^M_A(t) \in \mathbb{C}(t)$ for all finitely generated graded left A-modules M.

We know that the following classes of algebras have the property (P).

Example 8. 1. Every left noetherian regular algebra has the property (P).
2. (T. H. Gulliksen [6], 1974) Every commutative complete intersection algebra has the property (P).

Recently, Martínez and Zacharia proved the following result.

Theorem 6.1. [10] *If A is a finite dimensional Koszul algebra such that $A^!$ is left noetherian, then A has the property (P).*

The second application of the extended Koszul duality is to generalize the above theorem.

Lemma 6.2. [12] *Let A be a Koszul algebra and M a finitely generated graded left A-module. If A is noetherian and having a balanced dualizing complex , then*

$$P^M_A(t) = \sum_{i:\ finite} H_{h^i(E(M))}(t)t^i.$$

The following theorem is an immediate consequence of the above lemma.

Theorem 6.3. [12] *Let A be a noetherian Koszul algebra having a balanced dualizing complex. If $A^!$ has the property (H), then A has the property (P).*

As we have seen, every finite dimensional algebra is noetherian and having a balanced dualizing complex, and every left noetherian regular algebra has the property (H), so the above theorem contains that of Martínez and Zacharia

7. Grothendieck Group

In the following two sections, we will apply the Koszul duality to the study of a quantum projective space and the graded stable category of a Frobenius Koszul algebra.

Let $\operatorname{fdim} A$ be the full subcategory of $\operatorname{grmod} A$ consisting of finite dimensional modules. Artin and Zhang [3] defined the noncommutative projective scheme associated to A to be the quotient category $\operatorname{tails} A = \operatorname{grmod} A/\operatorname{fdim} A$. The image of $M \in \operatorname{grmod} A$ in $\operatorname{tails} A$ is denoted by $\widetilde{M}$. To understand this category, if A is left noetherian, then $\widetilde{M} \cong \widetilde{N}$ in $\operatorname{tails} A$ if and only if $\oplus_{i=n}^{\infty} M_i \cong \oplus_{i=n}^{\infty} N$ in $\operatorname{grmod} A$ for $n \gg 0$. By Serre, if A is commutative and generated by degree 1 elements, then $\operatorname{tails} A$ is equivalent to the category of coherent modules over Proj A. If A is a noetherian AS-regular Koszul algebra, then we call tails A a quantum projective space.

Since $\sim$: grmod $A \to$ tails A is an exact functor, it induces a functor $\sim$: $\mathcal{D}^b(\text{grmod } A) \to \mathcal{D}^b(\text{tails } A)$.

Theorem 7.1. [13] *Let A be a Koszul algebra such that both A and $A^!$ are noetherian and having balanced dualizing complexes. Then, for $X \in \mathcal{D}^b(\text{grmod } A)$, $\widetilde{X} \cong 0$ in $\mathcal{D}^b(\text{tails } A)$ if and only if $\bar{E}(X)$ has finite projective dimension. That is, $\bar{E}$ induces a duality*

$$\bar{E} : \mathcal{D}^b(\text{tails } A) \to \mathcal{D}^b(\text{grmod } A^!)/\mathcal{P}(A^!),$$

where $\mathcal{P}(A^!)$ is the full subcategory of $\mathcal{D}^b(\text{grmod } A^!)$ consisting of complexes having finite projective dimension.

Now, suppose that A is a noetherian AS-regular Koszul algebra. Since $A^!$ is Frobenius, $\mathcal{D}^b(\text{grmod } A^!)/\mathcal{P}(A^!) \cong \underline{\text{grmod}}A^!$, where $\underline{\text{grmod}}A^!$ is the stable category of grmod $A^!$. Moreover, since the duality $\text{Hom}_{A^!}(-, A^!)$: grmod $A^! \to \text{grmod}(A^!)^{op}$ induces a duality $\underline{\text{grmod}}A^! \to \underline{\text{grmod}}(A^!)^{op}$, where $(A^!)^{op}$ is the opposite graded algebra of $A^!$, we recapture the following classical result.

Theorem 7.2. [5] [9] *If A is a noetherian AS-regular Koszul algebra, then there is an equivalence of categories*

$$\mathcal{D}^b(\text{tails } A) \cong \underline{\text{grmod}}(A^!)^{op}.$$

By this equivalence, we can translate results on a quantum projective space to those on the graded stable category of the corresponding Frobenius Koszul algebra, and vice versa. For example, let A and B be noetherian AS-regular Koszul algebras. Then the above theorem says that tails A and tails B are derived equivalent if and only if $(A^!)^{op}$ and $(B^!)^{op}$ are graded stable equivalent. Moreover, we can calculate the Grothendieck group.

Theorem 7.3. [14] [15] *If A is a left noetherian regular Koszul algebra, then*

$$K_0(\text{tails } A) \cong \mathbb{Z}[t]/(P_A^k(-t)).$$

The above theorem can be translated as follows.

Theorem 7.4. *If A is a Frobenius Koszul algebra such that $A^!$ is noetherian, then*

$$K_0(\underline{\text{grmod}}A) \cong \mathbb{Z}[t]/(H_A(-t)).$$

Example 9. If $A = k\langle x_1, \cdots, x_d\rangle/(x_ix_j + x_jx_i, x_i^2)_{i,j=1,\cdots,d}$ is an exterior algebra, then $H_A(t) = (1+t)^d$, so

$$K_0(\underline{\text{grmod}}A) \cong \mathbb{Z}[t]/((1-t)^d).$$

8. Serre Duality

Let M and N be graded left A-modules. The set of morphisms $M \to N$ in $\underline{\text{grmod}}A$ is denoted by $\underline{\text{Hom}}_A(M, N)$. The i-th stable cohomology is defined by

$$\underline{\text{Ext}}_A^i(M, N) = \lim_{n \geq \max\{-i,0\}} \underline{\text{Hom}}_A(\Omega^{n+i}M, \Omega^n N),$$

where $\Omega^n M$ is the n-th syzygy of M. If $\sigma : A \to A$ is a graded algebra automorphism, then we define a graded left A-module ${}^\sigma M$ by ${}^\sigma M = M$ as a graded vector space over k but with the new action $a \cdot m = \sigma(a)m$.

First, let A be a Frobenius algebra of Loewy length n. Then there exists a graded algebra automorphism $\nu : A \to A$, called the Nakayama automorphism, such that $A^* \cong {}^{\nu}A(n)$ as graded A-A bimodules.

Theorem 8.1. [8] *Let A be a Frobenius algebra of Loewy length n, ν the Nakayama automorphism, and M, N finitely generated graded left A-modules. Then, for any integer i, there is a natural isomorphism*

$$\underline{\mathrm{Ext}}^i_A(M,N) \cong \underline{\mathrm{Ext}}^{-1-i}_A(N, {}^{\nu}M(n))^*.$$

Next, let A be a noetherian AS-regular Koszul algebra. Since $A^!$ is Frobenius, we have the Nakayama automorphism $\nu : A^! \to A^!$, which canonically induces a graded algebra automorphism $\sigma : A \to A$. The above theorem can be translated as follows.

Theorem 8.2. [8] *(Serre duality) Let A be a noetherian AS-regular Koszul algebra of global dimension n, and M, N finitely generated graded left A-modules. Then, for any integer i, there is a natural isomorphism*

$$\mathrm{Ext}^i_{\mathrm{tails}\, A}(\widetilde{M}, \widetilde{N}) \cong \mathrm{Ext}^{n-1-i}_{\mathrm{tails}\, A}(\widetilde{N}, \widetilde{{}^{\sigma}M(-n)})^*.$$

Now, we will rewrite the formulas in the above two theorems using a balanced dualizing complex D_A of A. First, let A be a Frobenius algebra of Loewy length n and ν the Nakayama automorphism. The functor

$$\mathcal{N}(-) = \mathrm{Hom}_A(-, A)^* : \mathrm{grmod}\, A \to \mathrm{grmod}\, A$$

is called the Nakayama equivalence. Since A is Frobenius,

$$D_A \cong \mathrm{H}^0_{\mathfrak{m}}(A)^* \cong A^* \cong {}^{\nu}A(n)$$

as graded A-A-bimodules, so $\mathcal{N}$ can be written as

$$\mathcal{N}(-) \cong D_A \otimes_A - : \mathrm{grmod}\, A \to \mathrm{grmod}\, A.$$

Hence we can rewrite the formula in Theorem 8.1 as

$$\underline{\mathrm{Ext}}^i_A(M,N) \cong \underline{\mathrm{Ext}}^{-1-i}_A(N, \mathcal{N}(M))^* \cong \underline{\mathrm{Ext}}^{-1-i}_A(N, D_A \otimes_A M)^*.$$

On the other hand, if A is a noetherian AS-regular algebra, then we can rewrite the formula in Theorem 8.2 as

$$\mathrm{Ext}^i_{\mathrm{tails}\, A}(\widetilde{M}, \widetilde{N}) \cong \mathrm{Ext}^{-1-i}_{\mathrm{tails}\, A}(\widetilde{N}, \widetilde{D_A \otimes_A M})^*$$

by [21]. So these two formulas look strikingly alike.

9. A GENERALIZATION OF NAKAYAMA EQUIVALENCE

Let D_A be a balanced dualizing complex of A. By the previous section, it is interesting to view the functor $D_A \otimes_A -$ as a generalization of the Nakayama equivalence. Unless A is Frobenius, it no longer induces an equivalence of categories of graded left A-modules. However, it induces an equivalence of the following derived categories.

Theorem 9.1. [11] *(Foxby equivalence) If A is a noetherian algebra having a balanced dualizing complex D_A, then there is an equivalence of categories*

$$D_A \otimes^L_A - : \mathcal{P}(A) \to \mathcal{I}(A),$$

where $\mathcal{I}(A)$ is the full subcategory of $\mathcal{D}^b(\mathrm{grmod}\, A)$ consisting of complexes having finite injective dimension.

Although we do not know how to generalize the Nakayama automorphism to an arbitrary algebra, the following may be an interesting idea. Let A be a noetherian algebra having a balanced dualizing complex D_A. Then it is known that A satisfies Gorenstein condition if and only if there exists a graded algebra automorphism $\sigma : A \to A$ such that D_A is isomorphic to a "shift" of ${}^{\sigma}A$ in the derived categories of graded A-A bimodules. We may consider σ as a generalization of the Nakayama automorphism because if A is a Frobenius algebra of Loewy length n and $\nu : A \to A$ is the Nakayama automorphism, then we have seen that $D_A \cong {}^{\nu}A(n)$ as graded A-A bimodules.

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ON MODULES OF G-DIMENSION ZERO OVER NON-GORENSTEIN LOCAL RINGS

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1. Introduction

Throughout this note, we assume that all rings are commutative and noetherian, and that all modules are finitely generated.

G-dimension was defined by Auslander [1] and was deeply studied by Auslander and Bridger [2]. This is a homological invariant for modules analogous to projective dimension.

A Cohen-Macaulay local ring is called to be of finite Cohen-Macaulay representation type if there are only finitely many isomorphism classes of indecomposable maximal Cohen-Macaulay modules. Such a ring has been well researched for a long time. In several cases, all the isomorphism classes of indecomposable maximal Cohen-Macaulay modules over such a ring have already been classified completely. (See [14] for the details.)

Over a Gorenstein local ring, a module has G-dimension zero if and only if it is a maximal Cohen-Macaulay module. (Recall that a Gorenstein ring is always Cohen-Macaulay.) Hence a Gorenstein local ring has only finitely many isomorphism classes of indecomposable modules of G-dimension zero if and only if it is of finite Cohen-Macaulay representation type. Thus we are interested in a non-Gorenstein local ring which has only finitely many isomorphism classes of indecomposable modules of G-dimension zero, especially interested in determining all the isomorphism classes of indecomposable modules of G-dimension zero over such a ring.

Let R be such a ring. Our guess is that the only isomorphism class of indecomposable module of G-dimension zero is the isomorphism class of R. In other words, we guess that the following holds:

Conjecture 1.1. Let R be a non-Gorenstein local ring. Suppose that there exists a non-free R-module of G-dimension zero. Then there exist infinitely many isomorphism classes of indecomposable R-modules of G-dimension zero.

Indeed, over a certain artinian local ring having a non-free module of G-dimension zero, Yoshino [16] actually constructed a family of modules of G-dimension zero with continuous parameters.

For a ring R, let us denote by $\mathrm{mod}R$ the category of all finitely generated R-modules, and by $\mathcal{G}(R)$ the full subcategory of $\mathrm{mod}R$ consisting of all R-modules of G-dimension zero. The main result of this note is the following theorem, which extends [16, Theorem 6.1].

Theorem 1.2. *Let R be a henselian non-Gorenstein local ring of depth zero. Suppose that there exists a non-free R-module in $\mathcal{G}(R)$. Then the residue class field of R does not admit a $\mathcal{G}(R)$-precover as an R-module. In particular, the category $\mathcal{G}(R)$ is not contravariantly finite in* $\mathrm{mod}R$.

The detailed version [8] of this note has been submitted for publication elsewhere.

It is easily seen that this theorem gives a positive answer for Conjecture 1.1 if the depth of the local ring R is zero:

Corollary 1.3. *Let R be a henselian non-Gorenstein local ring of depth zero. Suppose that there exists a non-free R-module of G-dimension zero. Then there exist infinitely many isomorphism classes of indecomposable R-modules of G-dimension zero.*

We should remark that the above corollary especially asserts that Conjecture 1.1 holds if R is artinian.

2. Outline of the proof of Theorem 1.2

Throughout this section, R is always a commutative noetherian local ring with residue class field k. All R-modules considered in this section are finitely generated.

We define $(-)^*$ to be the dual functor $\mathrm{Hom}_R(-, R)$ from $\mathrm{mod}R$ to itself, and denote by $\Omega_R^n M$ the nth syzygy module of an R-module M. To begin with, we recall the definition of G-dimension.

Definition 2.1. Let M be an R-module.

(1) If the following conditions hold, then we say that M has *G-dimension zero*, and write $\text{G-dim}_R M = 0$.
   i) The natural homomorphism $M \to M^{**}$ is isomorphic.
   ii) $\mathrm{Ext}_R^i(M, R) = 0$ for every $i > 0$.
   iii) $\mathrm{Ext}_R^i(M^*, R) = 0$ for every $i > 0$.

(2) If $\text{G-dim}_R(\Omega_R^n M) = 0$ for a non-negative integer n, then we say that M has *G-dimension at most n*, and write $\text{G-dim}_R M \leq n$. If such an integer n does not exist, then we say that M has *infinite G-dimension*, and write $\text{G-dim}_R M = \infty$.

As follows, G-dimension possesses many properties similar to those of projective dimension. For the proofs, we refer to [2], [6], [7], and [13].

Proposition 2.2. (1) *The following conditions are equivalent.*
   i) *R is Gorenstein.*
   ii) $\text{G-dim}_R M < \infty$ *for any R-module M.*
   iii) $\text{G-dim}_R k < \infty$.

(2) *Let M be a non-zero R-module with* $\text{G-dim}_R M < \infty$. *Then* $\text{G-dim}_R M = \mathrm{depth} R - \mathrm{depth}_R M$.

(3) *Let $0 \to L \to M \to N \to 0$ be a short exact sequence of R-modules. If two of L, M, N have finite G-dimension, then so does the third.*

(4) *Let M be an R-module, and n a non-negative integer. Then* $\text{G-dim}_R(\Omega_R^n M) = \sup\{\text{G-dim}_R M - n, 0\}$.

(5) *Let M, N be R-modules. Then* $\text{G-dim}_R(M \oplus N) = \sup\{\text{G-dim}_R M, \text{G-dim}_R N\}$.

Here we introduce the notion of a cover of a module.

Definition 2.3. Let $\mathcal{X}$ be a full subcategory of $\mathrm{mod}R$, and let $\phi : X \to M$ be a homomorphism from $X \in \mathcal{X}$ to $M \in \mathrm{mod}R$.

(1) We call ϕ an *$\mathcal{X}$-precover* (or a *right $\mathcal{X}$-approximation*) of M if for any homomorphism $\phi' : X' \to M$ with $X' \in \mathcal{X}$ there exists a homomorphism $f : X' \to X$ such that $\phi' = \phi f$.

(2) Assume that ϕ is an $\mathcal{X}$-precover of M. We call ϕ an $\mathcal{X}$*-cover* (or a *right minimal* $\mathcal{X}$*-approximation*) of M if any endomorphism f of X with $\phi = \phi f$ is an automorphism.

A full subcategory $\mathcal{X}$ of mod$R$ is said to be *contravariantly finite* in mod$R$ if every $M \in$ mod$R$ has an $\mathcal{X}$-precover.

The following proposition helps us to see whether a given precover is a cover or not.

Proposition 2.4. [15, Lemma (2.2)] *Let* $\mathcal{X}$ *be a full subcategory of* mod$R$, *and let*

$$0 \to N \xrightarrow{\psi} X \xrightarrow{\phi} M$$

be an exact sequence in mod$R$ *where* ϕ *is an* $\mathcal{X}$*-precover. Suppose that* R *is henselian. Then the following conditions are equivalent.*

i) ϕ *is not an* $\mathcal{X}$*-cover.*

ii) *There exists a non-zero submodule* L *of* N *such that* $\psi(L)$ *is a direct summand of* X.

Let $\mathcal{X}$ be a full subcategory of mod$R$. We say that $\mathcal{X}$ is closed under direct summands provided that for any object M of $\mathcal{X}$ and any direct summand N of M we have N is also an object of $\mathcal{X}$. Note by Proposition 2.2.5 that the category $\mathcal{G}(R)$ is closed under direct summands. Using the above proposition, we can easily prove that the existence of an $\mathcal{X}$-precover in fact implies the existence of an $\mathcal{X}$-cover if $\mathcal{X}$ is a full subcategory of mod$R$ which is closed under direct summands. Hence we have the following.

Corollary 2.5. *Let* $\mathcal{X}$ *be a full subcategory of* mod$R$ *which is closed under direct summands. Suppose that* R *is henselian. Then an* R*-module admits an* $\mathcal{X}$*-cover if and only if it admits an* $\mathcal{X}$*-precover.*

We say that a full subcategory $\mathcal{X}$ of mod$R$ is closed under extensions if for any short exact sequence

$$0 \to L \to M \to N \to 0$$

in mod$R$ with $L, N \in \mathcal{X}$, we have M is also an object of $\mathcal{X}$. Note by 2.2.2 and 2.2.3 that the category $\mathcal{G}(R)$ is closed under extensions. The lemma below is so-called Wakamatsu's Lemma, which plays an important role in the notion of a cover. For the proof, see [11] or [12, Lemma 2.1.1].

Lemma 2.6 (Wakamatsu). *Let* $\mathcal{X}$ *be a full subcategory of* mod$R$ *which is closed under extensions, and let*

$$0 \to N \to X \xrightarrow{\phi} M$$

be an exact sequence in mod$R$ *where* ϕ *is an* $\mathcal{X}$*-cover of* M. *Then* $\mathrm{Ext}_R^1(X', N) = 0$ *for any* $X' \in \mathcal{X}$.

Now, we shall give the outline of the proof of our theorem.

Outline of the proof of Theorem 1.2. Let R be a henselian non-Gorenstein local ring of depth zero such that there exists a non-free module in $\mathcal{G}(R)$. Suppose that the residue field k of R has a $\mathcal{G}(R)$-precover as an R-module. We want to derive contradiction. By Corollary 2.5, there exists a short exact sequence

$$0 \to L \xrightarrow{\theta} Z \xrightarrow{\pi} k \to 0$$

of R-modules such that π is a $\mathcal{G}(R)$-cover. Dualizing this sequence, we obtain an exact sequence

$$0 \to k^* \xrightarrow{\pi^*} Z^* \xrightarrow{\theta^*} L^*.$$

Set $C = \mathrm{Im}(\theta^*)$, and let $\alpha : Z^* \to C$ be the surjection induced by θ^* and $\beta : C \to L^*$ be the natural embedding.

We shall show that the homomorphism $\alpha : Z^* \to C$ is a $\mathcal{G}(R)$-cover of C. Fix $X \in \mathcal{G}(R)$. To prove that any homomorphism $X \to C$ is factored as $X \to Z^* \xrightarrow{\alpha} C$, we may assume that X is non-free and indecomposable. Applying the functor $\mathrm{Hom}_R(X, -)$ to the above exact sequence, we get an exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(X, k^*) \xrightarrow{\mathrm{Hom}_R(X,\pi^*)} \mathrm{Hom}_R(X, Z^*) \xrightarrow{\mathrm{Hom}_R(X,\theta^*)} \mathrm{Hom}_R(X, L^*).$$

We can prove that the homomorphism $\mathrm{Hom}_R(X, \theta^*)$ is a split epimorphism. Since $\mathrm{Hom}_R(X, \theta^*) = \mathrm{Hom}_R(X, \beta) \cdot \mathrm{Hom}_R(X, \alpha)$ and $\mathrm{Hom}_R(X, \beta)$ is an injection, the homomorphism $\mathrm{Hom}_R(X, \beta)$ is an isomorphism. Therefore $\mathrm{Hom}_R(X, \alpha)$ is a split epimorphism, and hence it is especially a surjection. This means that the homomorphism $\alpha : Z^* \to C$ is a $\mathcal{G}(R)$-precover of C. Assume that α is not a $\mathcal{G}(R)$-cover. Then Proposition 2.4 shows that k^* and Z^* have some common non-zero summand. Since k^* is a k-vector space, the R-module Z^* has a summand isomorphic to the R-module k, and hence $k \in \mathcal{G}(R)$ by Proposition 2.2.5. It follows from Proposition 2.2.1 that R is Gorenstein, which contradicts the assumption of the theorem. Therefore α must be a $\mathcal{G}(R)$-cover of C.

Thus we can apply Lemma 2.6, and get $\mathrm{Ext}^1_R(Y, k^*) = 0$ for every $Y \in \mathcal{G}(R)$. Since the local ring R has depth zero, that is to say, k^* is a non-zero k-vector space, every module in $\mathcal{G}(R)$ is free, which is contrary to the assumption of our theorem. This contradiction proves our theorem. □

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TILTING COMPLEXES ASSOCIATED WITH A SEQUENCE OF IDEMPOTENTS

MITSUO HOSHINO AND YOSHIAKI KATO

This note is a summary of our paper ([HK3]).

Rickard [Ri2] showed that the Brauer tree algebras with the same numerical invariants are derived equivalent to each other. Let A be a Brauer tree algebra corresponding to a Brauer tree whose edges are labelled $1, 2, \dots, n$. Note that there exists a partition of the edges $\{1, \dots, n\} = E_0 \cup \cdots \cup E_l$, where E_s consists of the edges i for which there exists a sequence of edges $i_0, i_1, \dots, i_s = i$ such that i_0 is adjacent to the exceptional vertex and for any $0 \leq r < l$, $i_r \neq i_{r+1}$ and i_r, i_{r+1} have a vertex in common. He constructed a tilting complex $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ such that $P^j = 0$ for $j > 0$ and $j < -l$, $P^{-j} \in \mathrm{add}(\oplus_{i \in E_{l-j}} e_i A)$, where $e_i \in A$ is a local idempotent corresponding to the edge i, for $0 \leq j \leq l$ and $\mathrm{End}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet)$ is a Brauer "star" algebra with the same numerical invariants as A. On the other hand, Okuyama [Ok] pointed out recently that for Brauer tree algebras A, B with the same numerical invariants there exists a sequence of Brauer tree algebras $B_0 = A, B_1, \dots, B_l = B$ such that B_{r+1} is the endomorphism algebra of a tilting complex for B_r of term length two defined by an idempotent. See König and Zimmermann [KZ] for another example of derived equivalences which are iterations of derived equivalences induced by tilting complexes of term length two. We will formulate these results.

Let A be a noetherian ring and $e_0, e_1, \dots, e_l \in A$ a sequence of idempotents such that $\mathrm{add}(e_0 A_A) = \mathcal{P}_A$, $e_{i+1} \in e_i A e_i$ for $0 \leq i < l$ and $\mathrm{Ext}^j_A(A/Ae_iA, e_iA) = 0$ for $0 \leq j < i \leq l$. First, we will show that there exists a tilting complex $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ such that $P^i = 0$ for $i > 0$ and $i < -l$, $P^{-i} \in \mathrm{add}(e_iA)$ for $0 \leq i \leq l$ and $\mathrm{H}^{-j}(P^\bullet) \in \mathrm{Mod}\text{-}(A/Ae_iA)$ for $0 \leq j < i \leq l$ (Proposition 1.1), and that such a tilting complex $P^\bullet$ is essentially unique. Next, we will show that there exists a sequence of rings $B_0 = A, B_1, \dots, B_l = \mathrm{End}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet)$ such that for any $0 \leq i < l$, B_{i+1} is the endomorphism ring of a tilting complex for B_i of term length two defined by an idempotent (Theorem 1.2). Furthermore, in case A is a selfinjective artin algebra over a commutative artin ring R and $\mathrm{add}(e_iA_A) = \mathrm{add}(D({}_AAe_i))$ for $1 \leq i \leq l$, where $D = \mathrm{Hom}_R(-, E(R/\mathrm{rad}\, R))$, we will show that $\mathrm{End}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet)$ is a selfinjective artin R-algebra whose Nakayama permutation coincides with that of A (Proposition 2.3). Finally, we deal with the case where A is a finite dimensional algebra over a field k and $\mathrm{add}(e_iA_A) = \mathrm{add}(D({}_AAe_i))$ for $1 \leq i \leq l$, where $D = \mathrm{Hom}_k(-, k)$. We will construct a two-sided tilting complex which corresponds to $P^\bullet$ (Section 3). Simultaneously, we will provide a sufficient condition for an algebra B containing A as a subalgebra to be derived equivalent to A (Theorem 4.1).

Throughout this note, rings are associative rings with identity and modules are unitary modules. Unless otherwise stated, modules are right modules. For a ring A, we denote by A^{op} the opposite ring of A and consider left A-modules as A^{op}-modules. In case A is a finite dimensional algebra over a field k, we denote by A^{e} the enveloping algebra

The detailed version has been published in another place.

$A^{\mathrm{op}} \otimes_k A$. Sometimes, we use the notation X_A (resp., ${}_AX$) to signify that the module X considered is a right (resp., left) A-module. We denote by Mod-A the category of A-modules and by $\mathcal{P}_A$ the full additive subcategory of Mod-A consisting of finitely generated projective modules. For an object X in an additive category $\mathcal{A}$, we denote by $\mathrm{add}(X)$ the full additive subcategory of $\mathcal{A}$ consisting of objects isomorphic to direct summands of finite direct sums of copies of X. For an additive category $\mathcal{A}$, we denote by $\mathsf{K}(\mathcal{A})$ the homotopy category of cochain complexes over $\mathcal{A}$ and by $\mathsf{K}^{\mathrm{b}}(\mathcal{A})$ the full subcategories of $\mathsf{K}(\mathcal{A})$ consisting of bounded complexes. In case $\mathcal{A}$ is an abelian category, we denote by $\mathsf{D}(\mathcal{A})$ the derived category of cochain complexes over $\mathcal{A}$. Also, we denote by $\mathrm{Z}^i(X^\bullet)$, $\mathrm{Z}'^i(X^\bullet)$ and $\mathrm{H}^i(X^\bullet)$ the i-th cycle, the i-th cocycle and the i-th cohomology of a complex $X^\bullet$, respectively. Finally, we use the notation $\mathrm{Hom}^\bullet(-,-)$ (resp., $-\otimes^\bullet -$) to denote the single complex associated with the double hom (resp., tensor) complex. We refer to [RD], [Ve] and [BN] for basic results in the theory of derived categories. Also, we refer to [Ri1, Ri3] for definitions and basic properties of tilting complexes and two-sided tilting complexes, and to e.g. [Br], [Ri3], [Ri4], [Ro], [RZ] and so on for recent progress.

1. General case

In this section, we will show that a certain sequence of idempotents $e_0, e_1, \dots, e_l$ in a ring A defines a tilting complex $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ of term length $l+1$ and that there exists a sequence of rings $B_0 = A, B_1, \dots, B_l = \mathrm{End}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet)$ such that for any $0 \le i < l$, B_{i+1} is the endomorphism ring of a tilting complex for B_i of term length two defined by an idempotent.

Let $e_0, e_1, \dots$ be idempotents in A such that $\mathrm{add}(e_0A) = \mathcal{P}_A$ and $e_{i+1} \in e_iAe_i$ for all $i \ge 0$.

Proposition 1.1. *Assume A is right noetherian. Let $l \ge 0$ and assume $\mathrm{Ext}^j_A(A/Ae_iA, e_iA) = 0$ for $0 \le j < i \le l$. Then there exists a tilting complex $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ such that $P^i = 0$ for $i > 0$ and $i < -l$, $P^{-i} \in \mathrm{add}(e_iA)$ for $0 \le i \le l$ and $\mathrm{H}^{-j}(P^\bullet) \in \mathrm{Mod}\text{-}(A/Ae_iA)$ for $0 \le j < i \le l$.*

Theorem 1.2. *Let $l \ge 0$ and assume $\mathrm{Ext}^j_A(A/Ae_iA, e_iA) = 0$ for $0 \le j < i \le l$. Let $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ be a tilting complex such that $P^i = 0$ for $i > 0$ and $i < -l-1$, $P^{-i} \in \mathrm{add}(e_iA)$ for $0 \le i \le l+1$ and $\mathrm{H}^{-j}(P^\bullet) \in \mathrm{Mod}\text{-}(A/Ae_iA)$ for $0 \le j < i \le l+1$. Then the following hold.*

(1) *There exists a tilting complex $\bar{P}^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ such that $\bar{P}^i = 0$ for $i > 0$ and $i < -l$, $\bar{P}^{-i} \in \mathrm{add}(e_iA)$ for $0 \le i \le l$, $\mathrm{H}^{-j}(\bar{P}^\bullet) \in \mathrm{Mod}\text{-}(A/Ae_iA)$ for $0 \le j < i \le l$ and $e_{l+1}A[l]$ is a direct summand of $\bar{P}^\bullet$. Furthermore, we have a distinguished triangle in $\mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ of the form*

$$P[l] \to \bar{P}^\bullet \to P^\bullet \to$$

with $P = P^{-l-1} \oplus e_{l+1}A$.

(2) *Let $B = \mathrm{End}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(\bar{P}^\bullet)$ and $f \in B$ the composite of canonical homomorphisms $\bar{P}^\bullet \to e_{l+1}A[l] \to \bar{P}^\bullet$. Then $\mathrm{Hom}_B(B/BfB, fB) = 0$ and there exists a tilting complex $Q^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_B)$ such that $Q^i = 0$ for $i \ne 0, -1$, $Q^{-1} \in \mathrm{add}(fB)$, $\mathrm{H}^0(Q^\bullet) \in \mathrm{Mod}\text{-}(B/BfB)$ and $\mathrm{End}_{\mathsf{K}(\mathrm{Mod}\text{-}B)}(Q^\bullet) \cong \mathrm{End}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet)$.*

2. The case of artin algebras

In this section, we will apply the results of the preceding section to the case where A is an artin algebra over a commutative artin ring R. We set $D = \mathrm{Hom}_R(-, E(R/\mathrm{rad}\, R))$.

According to Proposition 1.1, we have the following.

Proposition 2.1. *Let $e_0, e_1, e_2, \ldots$ be idempotents in A such that $\mathrm{add}(e_0 A) = \mathcal{P}_A$ and $e_{i+1} \in e_i A e_i$ for $i \geq 0$. Let $l \geq 0$ and assume $\mathrm{add}(e_i A_A) = \mathrm{add}(D({}_A A e_i))$ for $1 \leq i \leq l$. Then there exists a tilting complex $P^\bullet \in \mathsf{K}^\mathrm{b}(\mathcal{P}_A)$ such that $P^i = 0$ for $i > 0$ and $i < -l$, $P^{-i} \in \mathrm{add}(e_i A)$ for $0 \leq i \leq l$ and $\mathrm{H}^{-j}(P^\bullet) \in \mathrm{Mod}\text{-}(A/Ae_iA)$ for $0 \leq j < i \leq l$.*

Proposition 2.2. *Let $e_0, e_1, e_2, \ldots$ be idempotents in A such that $\mathrm{add}(e_0 A) = \mathcal{P}_A$ and $e_{i+1} \in e_i A e_i$ for $i \geq 0$. Let $l \geq 0$ and assume $\mathrm{add}(e_i A_A) = \mathrm{add}(D({}_A A e_i))$ for $1 \leq i \leq l+1$. Let $P^\bullet \in \mathsf{K}^\mathrm{b}(\mathcal{P}_A)$ be a tilting complex such that $P^i = 0$ for $i > 0$ and $i < -l-1$, $P^{-i} \in \mathrm{add}(e_i A)$ for $0 \leq i \leq l+1$ and $\mathrm{H}^{-j}(P^\bullet) \in \mathrm{Mod}\text{-}(A/Ae_iA)$ for $0 \leq j < i \leq l+1$. Then the following hold.*

1. *There exists a tilting complex $\bar{P}^\bullet \in \mathsf{K}^\mathrm{b}(\mathcal{P}_A)$ such that $\bar{P}^i = 0$ for $i > 0$ and $i < -l$, $\bar{P}^{-i} \in \mathrm{add}(e_i A)$ for $0 \leq i \leq l$, $\mathrm{H}^{-j}(\bar{P}^\bullet) \in \mathrm{Mod}\text{-}(A/Ae_iA)$ for $0 \leq j < i \leq l$ and $e_{l+1}A[l]$ is a direct summand of $\bar{P}^\bullet$. Furthermore, we have a distinguished triangle in $\mathsf{K}^\mathrm{b}(\mathcal{P}_A)$ of the form*

$$P[l] \to \bar{P}^\bullet \to P^\bullet \to$$

   with $P = P^{-l-1} \oplus e_{l+1}A$.

2. *Let $B = \mathrm{End}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(\bar{P}^\bullet)$ and $f \in B$ the composite of canonical homomorphisms $\bar{P}^\bullet \to e_{l+1}A[l] \to \bar{P}^\bullet$. Then $\mathrm{add}(fB_B) = \mathrm{add}(D({}_B Bf))$ and there exists a tilting complex $Q^\bullet \in \mathsf{K}^\mathrm{b}(\mathcal{P}_B)$ such that $Q^i = 0$ for $i \neq 0, -1$, $Q^{-1} \in \mathrm{add}(fB)$, $\mathrm{H}^0(Q^\bullet) \in \mathrm{Mod}\text{-}(B/BfB)$ and $\mathrm{End}_{\mathsf{K}(\mathrm{Mod}\text{-}B)}(Q^\bullet) \cong \mathrm{End}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet)$.*

Consider next the case of A being selfinjective. Let $\{e_1, \ldots, e_n\}$ be a basic set of orthogonal local idempotents in A and $I_0 = \{1, \ldots, n\}$. Set $\nu = D \circ \mathrm{Hom}_A(-, A)$. Then there exists a permutation σ of I_0, called the Nakayama permutation, such that $\nu(e_i A) \cong e_{\sigma(i)} A$ for all $i \in I_0$.

Proposition 2.3. *Let $I_0 \supset I_1 \supset I_2 \supset \cdots$ be a descending sequence of nonempty σ-stable subsets of I_0 and $e^{(i)} = \sum_{j \in I_i} e_j$ for $i \geq 0$. Then for any $l \geq 0$ there exists a tilting complex $P^\bullet \in \mathsf{K}^\mathrm{b}(\mathcal{P}_A)$ such that $P^i = 0$ for $i > 0$ and $i < -l$, $P^{-i} \in \mathrm{add}(e^{(i)} A)$ for $0 \leq i \leq l$ and $\mathrm{H}^{-j}(P^\bullet) \in \mathrm{Mod}\text{-}(A/Ae^{(i)}A)$ for $0 \leq j < i \leq l$. Furthermore, $\mathrm{End}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet)$ is a selfinjective artin algebra whose Nakayama permutation coincides with σ.*

3. Two-sided tilting complexes

Let A be a finite dimensional algebra over a field k and $D = \mathrm{Hom}_k(-, k)$. Our aim is to construct two-sided tilting complexes which correspond to tilting complexes constructed in Proposition 2.1. According to Proposition 2.2, we have only to deal with tilting complexes of term length two.

We will first construct a two-sided tilting complex $T^\bullet$ corresponding to the following tilting complex $S^\bullet$. Recall that an idempotent $e \in A$ is called local if eAe is a local ring. Let $\{e_1, \ldots, e_n\}$ be a basic set of orthogonal local idempotents in A and J the Jacobson radical of A. We fix a nonempty subset I_0 of $I = \{1, \ldots, n\}$ and define $S^\bullet$ as the mapping cone of the multiplication map

$$\rho : \bigoplus_{i \in I_0} Ae_i \otimes_k e_i A \to A.$$

We set $e = \sum_{i \in I_0} e_i$, $B = \mathrm{End}_{\mathsf{K}(\mathsf{Mod}\text{-}A)}(S^\bullet)$ and $d_{ij} = \dim_k e_i A e_j$ for $i, j \in I_0$. We assume the following conditions are satisfied:

(a_1) there exists a permutation σ of I_0 such that $e_i A_A \cong D({}_A A e_{\sigma(i)})$ for all $i \in I_0$;
(a_2) $e_i J e_{\sigma(i)} \neq 0$ for all $i \in I_0$; and
(a_3) $e_i A e_i / e_i J e_i \cong k$ for all $i \in I_0$.

Proposition 3.1. *The following hold.*
(1) $S^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ *is a tilting complex with* $\mathrm{H}^0(S^\bullet) \in \mathrm{Mod}\text{-}(A/AeA)$.
(2) *The left multiplication of A on each homogeneous component of $S^\bullet$ gives rise to an injective k-algebra homomorphism* $\varphi : A \to B$.
(3) ${}_A(B/A)_A \cong \oplus_{i,j \in I_0} ({}_A A e_i \otimes_k e_j A_A)^{(\alpha_{ij})}$, *where*

$$\alpha_{ij} = \begin{cases} d_{ji} - 2 & \text{if } i = j = \sigma(j), \\ d_{ji} - 1 & \text{if } j \neq \sigma(j) \text{ and } i \in \{j, \sigma(j)\}, \\ d_{ji} & \text{otherwise.} \end{cases}$$

(4) *For any* $i \in I_0$, $e_i B_B \cong \oplus_{j \in I_0} \mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\text{-}A)}(S^\bullet, e_{\sigma(j)} A[1])^{(\mu_{ij})}$, *where*

$$\mu_{ij} = \begin{cases} d_{ji} - 1 & \text{if } i = \sigma(j), \\ d_{ji} & \text{otherwise.} \end{cases}$$

Proposition 3.2. *For any $i \in I_0$ there exists a local idempotent $f_i \in e_i B e_i$ such that* $f_i B_B \cong \mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\text{-}A)}(S^\bullet, e_{\sigma(i)} A[1])$. *Furthermore, the following hold.*
(1) $f_i B_B \not\cong f_j B_B$ *unless* $i = j$.
(2) $f_i B_B \cong D({}_B B f_{\sigma(i)})$ *for all* $i \in I_0$.
(3) $f_i B f_j \cong e_i A e_j$ *for all* $i, j \in I_0$.
(4) $e_i B_B \cong \oplus_{j \in I_0} f_j B_B{}^{(\mu_{ij})}$ *for all* $i \in I_0$.
(5) $f_i B_A \cong \oplus_{j \in I_0} e_j A_A{}^{(\mu_{ji})}$ *for all* $i \in I_0$.

4. Derived equivalent extension algebras

Let A be the same as in Section 3. We will show that an algebra B containing A as a subalgebra satisfying (3) of Proposition 3.1 and (1) $\sim$ (5) of Proposition 3.2 is derived equivalent to A.

More precisely, let B be a finite dimensional k-algebra containing A as a subalgebra and for each $i \in I_0$ take a local idempotent $f_i \in e_i B e_i$. We assume the following conditions are satisfied:

(b_1) ${}_A(B/A)_A \cong \oplus_{i,j \in I_0} ({}_A A e_i \otimes_k e_j A_A)^{(\alpha_{ij})}$;
(b_2) $f_i B_B \not\cong f_j B_B$ unless $i = j$ and $f_i B_B \cong D({}_B B f_{\sigma(i)})$ for all $i \in I_0$;

(b_3) $f_iBf_j \cong e_iAe_j$ for all $i, j \in I_0$;
(b_4) $e_iB_B \cong \oplus_{j\in I_0} f_jB_B{}^{(\mu_{ij})}$ for all $i \in I_0$; and
(b_5) $f_iB_A \cong \oplus_{j\in I_0} e_jA_A{}^{(\nu_{ij})}$ for all $i \in I_0$.

Theorem 4.1. *Denote by $T^\bullet$ the mapping cone of the multiplication map*

$$\delta : \bigoplus_{i\in I_0} {}_BBf_i \otimes_k e_iA_A \to {}_BB_A.$$

Then $T^\bullet$ is a two-sided tilting complex with $T^\bullet \cong S^\bullet$ in $\mathsf{K}(\text{Mod-}A)$ if

$$\alpha_{ij} = \begin{cases} d_{ji} - 2 & \text{if } i = j = \sigma(j), \\ d_{ji} - 1 & \text{if } j \neq \sigma(j) \text{ and } i \in \{j, \sigma(j)\}, \\ d_{ji} & \text{otherwise,} \end{cases} \quad \mu_{ij} = \nu_{ji} = \begin{cases} d_{ji} - 1 & \text{if } i = \sigma(j), \\ d_{ji} & \text{otherwise.} \end{cases}$$

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Let's use cyclotomic polynomials in your lecture for your students [1]

Kaoru MOTOSE

In this paper, using cyclotomic polynomial, we will look again proofs of some fundamental theorems on finite fields and rational integers. We will suggest some applications to a code and a cipher (cryptography). Moreover, we will present some relations between cyclotomic polynomials of prime order and quadratic Gauss sums. If you can find some materials in this paper for your students, please use those in your lecture.

Cyclotomic polynomials of order n is defined by

$$\Phi_n(x) = \prod_{(k,n)=1} (x - \zeta_n^k), \quad \text{where } \zeta_n = e^{\frac{2\pi i}{n}} \text{ and } 1 \le k < n.$$

Classifying roots of $x^n - 1$ by orders, we obtain $x^n - 1 = \prod_{d|n} \Phi_d(x)$. We can see also $\Phi_n(x) \in Z[x]$ from the induction on n, the above equation and the division algorithm for monic polynomials in $Z[x]$.

1. Orders of elements.

Let q be a prime divisor of a Mersenne number $\Phi_p(2) = 2^p - 1$ where p is prime. Then p is the order $|2|_q$ of 2 mod q and so $p < q$ because $2^{q-1} \equiv 1 \bmod q$ and so p is a divisor of $q - 1$. This shows that there exists infinitely many prime numbers. In this argument, $p = |2|_q$ is most important. We can generalize this to the next theorem. This proof is very easy but this theorem is fundamental for cyclotomic polynomials.

Theorem 1. *Let R be a commutative ring containing $Z/\ell Z$. Assume $\Phi_n(\alpha) = 0$ for $\alpha \in R$. Then $n = \ell^e|\alpha|_\ell$ where $|\alpha|_\ell$ means the order of α and $e \ge 0$.*

Proof. Since $\Phi_n(x)$ divides $x^n - 1$, we have $\alpha^n = 1$. Hence $|\alpha|_\ell$ is a divisor of n and so we can write $n = \ell^e|\alpha|_\ell \cdot t$ where ℓ does not divide t. We set $s = \ell^e|\alpha|_\ell$ and assume $t > 1$. Then $\alpha^s = 1$ and noting $\Phi_n(x)g(x) = \frac{x^{st}-1}{x^s-1} = (x^s)^{t-1} + \cdots + (x^s)^2 + x^s + 1$ for some $g(x) \in Z[x]$, we have a contradiction that ℓ divides t from the next equation

$$0 = \Phi_n(\alpha)g(\alpha) = (\alpha^s)^{t-1} + (\alpha^s)^{t-2} + \cdots + (\alpha^s)^2 + \alpha^s + 1 = t.$$

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From this result, we can prove a special case of Dirichlet theorem with respect to arithmetic progressions, namely, the set $\Delta = \{ns + 1 \mid s = 1, 2, \cdots\}$ contains infinite primes. Setting $p_0 = 1$, let p_k be a prime divisor of $\Phi_{p_{k-1}n}(p_{k-1}n)$ for $k = 1, 2, \cdots$ and set $R_k = Z/p_kZ$. Then it follows from the above theorem that $p_k \in \Delta$ for $k = 1, 2, \cdots$.

The next proposition is an easy consequence of the above theorem.

Proposition 1. *Let G be a finite subgroup of the multiplicative group of a field K. Then G is cyclic.*

Proof. We set $m = |G|$. Then G is contained in the set of roots of $x^m - 1$ in K which has at most m elements. Thus, we obtain $x^m - 1 = \prod_{\alpha \in G}(x - \alpha)$. Hence, $\Phi_m(x)$ has a root $\beta \in G$ since $\Phi_m(x)$ divides $x^m - 1$. If K is of characteristic $p > 0$, then p is not a divisor of m because $x^m - 1$ has no multiple roots, and so $m = |\beta|_p$ by Theorem 1. If K is of characteristic zero, then our assertion is trivial.

In many primality tests, it is essential to find the orders of elements of commutative rings. Thus Theorem 1 is important. A cipher is considered like RSA from the next proposition.

Proposition 2. *If d is a divisor of $\Phi_n(a)$ and d is not divided by the maximal prime divisor of n, then $a^{d-1} \equiv 1 \bmod d$.*

Proof. Let p be a prime divisor of d and so of $\Phi_n(a)$. Then $n = |a|_p$ is a divisor of $p - 1$, equivalently, $p \equiv 1 \bmod n$. Hence $d \equiv 1 \bmod n$ Since $a^n \equiv 1 \bmod d$, we have our result.

The next theorem follows from Theorem 1 and an inequality (see [2])

$$(a+1)^{\varphi(n)} \geq \Phi_n(a) \geq (a-1)^{\varphi(n)}.$$

Theorem 2 (Bang). *If $n \geq 3, a \geq 2$ and $(n, a) \neq (6, 2)$, then there exists a prime p with $n = |a|_p$.*

2. Factorizations of cyclotomic polynomials over fields

It is fundamental that cyclotomic polynomial is irreducible in $Q[x]$. We can see from this that $\Phi_n(x)$ is irreducible over $Q(\zeta_m)$ for $(n, m) = 1$ where $\zeta_m = e^{\frac{2\pi i}{m}}$. The next theorem state about irreducible factors of cyclotomic polynomials over arbitrary fields.

Theorem 3. *Let K be an arbitrary field. Then every irreducible factor $f(x)$ of $\Phi_n(x)$ in $K[x]$ has the same degree $[L:K]$ where L is the minimal splitting field of $\Phi_n(x)$ over K.*

Proof. We assume the characteristic p of K is positive. Let $f(x)$ be an arbitrary irreducible factor of $\Phi_n(x)$ in $K[x]$, let $\alpha \in L$ be a root of $f(x)$ and let m be the order of α in L. Then $n = p^e m$ with $(m,p) = 1$ from [2, Theorem 1]. Thus, we can see from the equation $x^m - 1 = \prod_{d|m} \Phi_d(x)$ and [2, Theorem 1] that

$$\Phi_m(x) = \prod_k (x - \alpha^k), \text{ where } (k,m) = 1 \text{ and } 1 \leq k < m.$$

Let $\beta \in L$ be an arbitrary root of $\Phi_n(x)$. Then the minimal polynomial of β over K is an irreducible factor of $\Phi_n(x)$. Thus $\beta = \alpha^t$ for some t from the same argument in the above. Hence we have $L = K(\alpha)$ and so $\deg f(x) = [L:K]$. In case $p = 0$, our result is trivial.

The above theorem is important for finite fields. The next corollary can be proved from the above theorem (see also [2]).

Corollary. *Let q be a power of a prime p. If p is not divisor of n, then $\Phi_n(x) \in F_q[x]$ is a product of irreducible polynomials of the same degree $|q|_n$.*

Examples. 1. We have $\Phi_7(x) = (x^3+x+1)(x^3+x^2+1)$ over F_2 by $3 = |2|_7$. This factor is used often as an exercise in the code theory.

2. We obtain

$$\Phi_{23}(x) = (x^{11}+x^9+x^7+x^6+x^5+x+1)(x^{11}+x^{10}+x^6+x^5+x^4+x^2+1)$$

over F_2 from $11 = |2|_{23}$ by $\Phi_{11}(2) = 2^{11} - 1 = 23 \cdot 89$. These factors are generator polynomials of Golay code. This code was used in planetary probe Voyagers. On the other hand, this code is closely related to Mathieu Group.

We can find an irreducible polynomial of degree n over an arbitrary F_q and for every n from the proof of the next proposition.

Proposition 3. *Let p be a prime and let q be a power of p. For an arbitrary n, There exists an irreducible polynomial of degree n in $F_q[x]$.*

Proof 1. It follows from $n = |q|_{q^n-1}$ that $\Phi_{q^n-1}(x) \in F_q[x]$ has an irreducible factor of degree n.

Proof 2. In case $n \geq 3$ and $(n, q) \neq (6, 2)$, then we can find a (prime) divisor r of $\Phi_n(q)$ with $n = |q|_r$. Hence $\Phi_r(x) \in F_q[x]$ has an irreducible factor of degree n. In case $n = 2$, $\Phi_{q+1}(x) \in F_q[x]$ has an irreducible factor of degree 2 because $2 = |q|_{q+1}$. In case $n = 6$ and $q = 2$, we obtain $\Phi_9(x) = \Phi_3(x^3) = x^6 + x^3 + 1$ over F_2 is irreducible from $6 = |2|_9$.

Examples. 1. We have $\Phi_{2^4-1}(x) = \Phi_{15}(x) = (x^4+x^3+1)(x^4+x+1)$ over F_2. These factors are primitive polynomials of order $2^4 - 1 = 15$. The class of x is a generator of $F^*_{2^4}$.

2. It follows from $4 = |2|_5$ that $\Phi_5(x)$ is irreducible over F_2.

3. A method of a factorization of a number

Let n be a number, let m be the product of distinct prime divisors of n, let p be a fixed prime divisor of m and let $m' = \frac{m}{p}$. We can see easily the next equations

$$\Phi_n(x) = \Phi_m(x^{\frac{n}{m}}) \text{ and } \Phi_m(x) = \prod_{d|m'} \Phi_p(x^d)^{\mu(\frac{m'}{d})}$$

where μ is Möbius function. The above equation and the next proposition show us that factorizations of $\Phi_n(a)$, especially, cyclotomic polynomials of prime orders are essential in factorizations of numbers (see [2]).

Proposition 4. *For a natural number n, let a and m be natural numbers such that $(am, n) = 1$ and $a^m \equiv 1 \bmod n$. Then $n = \prod_{d|m}(n, \Phi_d(a))$.*

4. Discriminants of cyclotomic polynomials of prime orders and quadratic Gauss sums

We set $\zeta = e^{\frac{2\pi i}{p}}$ for an odd prime p. Let χ be a linear character of the multiplicative group F_p^* of a prime field F_p. We consider Gauss sums $g(\chi) = \sum_{t \in F_p^*} \zeta^t \chi(t)$, the following matrices A and character vectors χ defined by

$$A = \begin{pmatrix} \zeta & \zeta^2 & \cdots & \zeta^{p-1} \\ \zeta^2 & \zeta^4 & \cdots & \zeta^{2(p-1)} \\ \vdots & \vdots & \vdots & \vdots \\ \zeta^{p-1} & \zeta^{2(p-1)} & \cdots & \zeta^{(p-1)^2} \end{pmatrix}, \ \chi = \begin{pmatrix} \chi(1) \\ \chi(2) \\ \vdots \\ \chi(p-1) \end{pmatrix}$$

Trace of A is $g(\eta) - 1$ (see [3]) where η is the quadratic character. The next is easy but important.

Lemma 1. We have $|A^2| = (-1)^{\frac{p-1}{2}} p^{p-2}$ and $A\chi = g(\chi)\bar{\chi}$.

Gauss' result $g(\eta) = i^{\frac{(p-1)^2}{4}}\sqrt{p}$ follows from computing value $|A|$ by two ways, namely, by Vandermonde determinant $|A|$ and by using the canonical form of A with respect to linear characters from $A\chi = g(\chi)\bar{\chi}$ (see [3]). The next theorem follows from the discriminant $|A|^2$ of $\Phi_p(x)$.

Proposition 5 (Quadratic reciprocity). *Let p and q be distinct odd primes and let $\left(\frac{q}{p}\right)$ be a Legendre symbol. Then $\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}} \left(\frac{p}{q}\right)$.*

Proof. We have $|A|^2 = |A^2| = (-1)^{\frac{p-1}{2}} p^{p-2}$ by Lemma 1. Hence we have the next equation since $p-2$ is odd.

$$|A|^{q-1} = (|A|^2)^{\frac{q-1}{2}} = (-1)^{\frac{p-1}{2}\frac{q-1}{2}} (p^{\frac{q-1}{2}})^{p-2} \equiv (-1)^{\frac{p-1}{2}\frac{q-1}{2}} \left(\frac{p}{q}\right) \mod q.$$

Let $A^{(k)} = (\zeta^{stk})$ be the matrix of k-th powers of all entries in A, let r be a primitive root of p, and let σ_r be a cyclic odd permutation $(1, c_1, \ldots, c_{p-2})$ where $c_k \equiv r^k \bmod p$. Then we have $|A^{(r)}| = \mathrm{sgn}(\sigma_r)|A| = -|A|$. Thus, setting $r^s \equiv q \bmod p$, we can see

$$|A|^q \equiv |A^{(q)}| = |A^{(r^s)}| = (-1)^s |A| = \left(\frac{q}{p}\right) |A| \mod qZ[\zeta].$$

We product $|A|$ on both sides of the above equation and divide by the integer $|A|^2 \not\equiv 0 \bmod q$. Then we have

$$\left(\frac{q}{p}\right) \equiv |A|^{q-1} \equiv (-1)^{\frac{p-1}{2}\frac{q-1}{2}} \left(\frac{p}{q}\right) \mod q.$$

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ON CALCULATIONS OF MODULAR IRREDUCIBLE CHARACTERS WITH THE HELP OF COMPUTERS

KATSUSHI WAKI

ABSTRACT. Let p be a prime. We denote p-modular system by (K, R, F) where F is an algebraically closed field of characteristic p. Let G be a finite group. It is not so easy to clculate of irreducible modular characters correspondent to simple FG-modules from ordinary characters correspondent to simple KG-modules. I will introduce a standard way written in [2] for the calculation of irreducible modular characters without a concrete construction of simple FG-modules. In the appendix, I also show a log of short demonstrations of the calculations of the decomposition matrix for the block of defect 2 in J_4 by GAP[3].

1. NOTATION

For an element $x \in G$, $o(x)$ denotes the order of x. Then we call

$$\begin{array}{lcl} x : p\text{-singular} & \leftrightarrow & p \mid o(x) \\ x : p\text{-regular} & \leftrightarrow & p \nmid o(x) \end{array}$$

Let $\mathrm{Irr}(G)$ ($\mathrm{IBr}(G)$) be a set of irreducible ordinary (Brauer) characters of G. Let $C(G)$ be a set of all class functions of G. For simplification of description, for any $\varphi \in \mathrm{IBr}(G)$, $\varphi(x)$ is defined 0 on p-singular elements x. So we can see that $\mathrm{IBr}(G)$ is a subset of $C(G)$.

Let $\mathbb{Z}$ be a set of integers and $\mathbb{N}$ be a set of natural numbers. Let X be a subset of $C(G)$ then we define $\langle X\rangle_{\mathbb{Z}}$ be a set of all $\mathbb{Z}$-linear combinations of X. We can also put $\mathbb{N}$ instead of $\mathbb{Z}$. For example, $\langle \mathrm{Irr}(G)\rangle_{\mathbb{N}}$ is a set of all ordinary characters of G and $\langle \mathrm{IBr}(G)\rangle_{\mathbb{N}}$ is a set of all Brauer characters of G.

It is well-known that $\mathrm{Irr}(G)$ ($\mathrm{IBr}(G)$) is $\mathbb{Z}$-basis of $\langle \mathrm{Irr}(G)\rangle_{\mathbb{Z}}$ ($\langle \mathrm{IBr}(G)\rangle_{\mathbb{Z}}$).

Moreover for an ordinary character χ, Let

$$\widehat{\chi}(x) := \begin{cases} \chi(x) & \text{if } x \text{ is } p\text{-regular} \\ 0 & \text{else} \end{cases}$$

Then $\widehat{\chi}$ is in $\langle \mathrm{IBr}(G)\rangle_{\mathbb{N}}$. In particular, we can find positive integers $d_{\chi\varphi}$ for $\chi \in \mathrm{Irr}(G)$ such that $\widehat{\chi} = \sum_{\varphi\in \mathrm{IBr}(G)} d_{\chi\varphi}\cdot\varphi$. We call the matrix $\{d_{\chi\varphi}\}_{\substack{\chi\in\mathrm{Irr}(G)\\ \varphi\in\mathrm{IBr}(G)}}$ the decomposition matrix of G. Thus if we get the decomposition matrix, we can get all irreducible Brauer characters. Let Φ_φ be the projective indecomposable character with respect to an irreducible Brauer character φ. Then $\mathrm{IPr}(G) := \{\Phi_\varphi \mid \varphi \in \mathrm{IBr}(G)\}$ is a set of all projective indecomposable characters of G. For λ, $\mu \in C(G)$, Scalar product $\langle\lambda, \mu\rangle$ is $\frac{1}{|G|}\sum_{x\in G}\lambda(x)\overline{\mu(x)}$. For any $\varphi \in \mathrm{IBr}$, $\Phi_\varphi = \sum_{\chi\in\mathrm{Irr}(G)} d_{\chi\varphi}\cdot\chi$. Thus we can get the decomposition matrix if we get all indecomposable projective characters in B.

2. Basic Set and System of Atom

In this section, we introduce two $\mathbb{Z}$-basis of a set of $\mathbb{Z}$-linear combinations of characters. We can get all irreducible Brauer characters from these $\mathbb{Z}$-basis. We start to show the elementary theorem.

Theorem 2.1. *For $\Phi_\varphi \in IPr(G)$ and $\psi \in IBr(G)$,*

$$\langle \Phi_\varphi, \psi \rangle = \delta_{\varphi\psi}$$

Next proposition is used to find characters which is NOT projective characters.

Proposition 2.2. *For $\Phi \in \langle Irr(G) \rangle_{\mathbb{N}}$, if $\Phi(x) = 0$ ($\forall x$: p-singuler), and $\langle \Phi, \varphi \rangle \geqq 0$ ($\forall \varphi \in IBr(G)$) then Φ is a projective character.*

Let B be a p-block of G. We denote $\mathrm{Irr}(B)$, $\mathrm{IBr}(B)$ and $\mathrm{IPr}(B)$ sets of characters of $\mathrm{Irr}(G)$, $\mathrm{IBr}(G)$ and $\mathrm{Irr}(G)$ in B, respectively. The matrix $D_B = \{d_{\chi\varphi}\}_{\substack{\chi \in \mathrm{Irr}(B) \\ \varphi \in \mathrm{IBr}(B)}}$ is called the decomposition matrix of B.

Let s be the number of irreducible ordinary characters in B. Let t be the number of irreducible Brauer characters in B.

Definition 2.3. Let BS and PS be $\mathbb{Z}$-basis of $\langle \mathrm{IBr}(B) \rangle_{\mathbb{Z}}$ and $\langle \mathrm{IPr}(B) \rangle_{\mathbb{Z}}$, respectively. We call BS is a **basic set** of $\mathrm{IBr}(B)$ if and only if $\mathrm{BS} \subset \langle \mathrm{IBr}(B) \rangle_{\mathbb{N}}$ and PS is a **basic set** of $\mathrm{IPr}(B)$ if and only if $\mathrm{PS} \subset \langle \mathrm{IPr}(B) \rangle_{\mathbb{N}}$.

We denote $M_t(\mathbb{Z})$ a set of $t \times t$-matrix over $\mathbb{Z}$.

Lemma 2.4. *Let BS and PS be $\mathbb{Z}$-basis of $\langle IBr(B) \rangle_{\mathbb{Z}}$ and $\langle IPr(B) \rangle_{\mathbb{Z}}$. Let $t \times t$-matrix $U := \langle BS, PS \rangle = \{\langle \varphi, \Phi \rangle \mid \varphi \in BS,\ \Phi \in PS\}$ Then both BS and PS are basic sets if and only if $U^{-1} \in M_t(\mathbb{Z})$. In paticular, if U is the identity matrix, BS equals $IBr(B)$ and PS equals $IPr(B)$.*

Definition 2.5. Let BA and PA be $\mathbb{Z}$-basis of $\langle \mathrm{IBr}(B) \rangle_{\mathbb{Z}}$ and $\langle \mathrm{IPr}\langle B) \rangle_{\mathbb{Z}}$, respectively. We call BA is a **system of atom** of $\mathrm{IBr}(B)$ if and only if $\forall \varphi \in \mathrm{IBr}(B)$; $\varphi \in \langle \mathrm{BA} \rangle_{\mathbb{N}}$ and PA is a **system of atom** of $\mathrm{IPr}(B)$ if and only if $\forall \Phi \in \mathrm{IPr}(B)$; $\Phi \in \langle \mathrm{PA} \rangle_{\mathbb{N}}$.

Next lemma is used to check indecomposability of projective characters.

Lemma 2.6. *For any $\Phi \in \langle IPr(B) \rangle_{\mathbb{N}}$, if $\Phi \in PA$ then $\Phi \in IPr(B)$.*

Definition 2.7. Let BS be a basic set of $\mathrm{IBr}(B)$. We call BS is **special** if any characters in $\mathrm{IBr}(B)$ are liftable s.t. $\forall \varphi \in \mathrm{BS}$; $\exists \chi \in \mathrm{Irr}(U)$ $\varphi = \chi$ on p-regular elements.

Last lemma help us to get outline of decomposition matrix.

Lemma 2.8. *If B has a special basic set of $IBr(B)$ then D_B can be like*

$$\left(\begin{array}{cccc} 1 & & 0 & \\ * & 1 & & \\ * & * & \ddots & \\ * & * & * & 1 \\ \hline * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{array}\right).$$

3. Construction of BS, PS, BA and PA

3.1. How to make BS.

Lemma 3.1. *Let $S \subset \langle IBr(B)\rangle_{\mathbb{N}}$. Then S is a basic set of $IBr(B)$ if and only if $\forall \chi \in Irr(B)$, $\widehat{\chi} \in \langle S\rangle_{\mathbb{Z}}$ and S is linearly independent over $\mathbb{Z}$.*

The algorithm to get a basic set of $\mathrm{IBr}(B)$ is the following. Let $\mathrm{Irr}(B) = \{\chi_1, \chi_2, \dots, \chi_s\}$. Assume that we have a set of k Brauer characters $\mathcal{B} = \{\beta_1, \dots, \beta_k\}$ such that $\mathcal{B}$ is linearly independent over $\mathbb{Z}$ and there is a number $r \leqq s$ such that $\{\widehat{\chi_1}, \dots, \widehat{\chi_r}\} \subset \langle \mathcal{B}\rangle_{\mathbb{Z}}$

Step 1 if $r = s$ then $\mathcal{B}$ is a basic set and finish else goto Step 2.
Step 2 if $r < s$ and $\widehat{\chi_{r+1}} \in \langle \mathcal{B}\rangle_{\mathbb{Z}}$ then $r := r+1$ and goto Step 1 else goto Step 3.
Step 3 if $\widehat{\chi_{r+1}} \notin \langle \mathcal{B}\rangle_{\mathbb{Q}}$ then add $\beta_{k+1} := \widehat{\chi_{r+1}}$ to $\mathcal{B}$, $r := r+1$, $k := k+1$ and goto Step 1 else goto Step 4.
Step 4 if $\widehat{\chi_{r+1}} \in \langle \mathcal{B}\rangle_{\mathbb{Q}} - \langle \mathcal{B}\rangle_{\mathbb{Z}}$ then

(i) Find the smallest integer $m_1 > 0$ such that $m_1\widehat{\chi_{r+1}} = \sum_{i=1}^{k} z_i\beta_i \in \langle \mathcal{B}\rangle_{\mathbb{Z}}$.
(ii) Let m be the minimum in $\{gcd(m_1, z_j) \mid 1 \leqq j \leqq k\}$.
(iii) Find $a, b \in \mathbb{Z}$ such that $m = am_1 + bz_j$ for a number j of $m = gcd(m_1, z_j)$.
(iv) Let c_i be the minimum of $\{x \in \mathbb{N} \mid x + bz_im_1^{-1} \geqq 0\}$ for $i \neq j$.
(v) Let $\widetilde{\beta_j} := b\widehat{\chi_{r+1}} + a\beta_j + \sum_{i\neq j} c_i\beta_i$

Then $m_1\widetilde{\beta_j} := m\beta_j + m_1\sum_{i\neq j}(c_i + bz_im_1^{-1})\beta_i$ is in $\langle \mathcal{B}\rangle_{\mathbb{N}}$ and is a Brauer character. Thus $\widetilde{\beta_j}$ is a Brauer character, too. So replace $\widetilde{\beta_j}$ as β_j in $\mathcal{B}$ then $\{\widehat{\chi_1}, \dots, \widehat{\chi_{r+1}}\} \subset \langle \mathcal{B}\rangle_{\mathbb{Z}}$. Let $r := r+1$ and goto Step 1.

3.2. How to make PS.

Theorem 3.2. *Let Mx be a set of all maximal subgroups of G and $P_{Mx} := \{\psi^G \mid \psi \in IPr(H), H \in Mx\}$. Then $\langle IPr(G)\rangle_{\mathbb{Z}} = \langle P_{Mx}\rangle_{\mathbb{Z}}$.*

From this theorem, we can make PS from $\{\Psi_B \mid \Psi \in P_{Mx}\}$ with the same way in 3.1. $\{\widehat{\chi} \mid \chi \in \mathrm{Irr}(B)\}$.

3.3. How to make BA and PA. Assume that we have succeed to make BS $= \{\varphi_1, \dots \varphi_t\}$ and PS $= \{\Phi_1, \dots \Phi_t\}$. Find subset $BA := \{\Phi_1^*, \dots, \Phi_t^*\} \subset \langle \mathrm{IBr}(B)\rangle_{\mathbb{Z}}$ such that $\langle \Phi_i^*, \Phi_j\rangle = \delta_{ij}$. Find subset $PA := \{\varphi_1^*, \dots, \varphi_t^*\} \subset \langle \mathrm{IPr}(B)\rangle_{\mathbb{Z}}$ such that $\langle \varphi_i^*, \varphi_j\rangle = \delta_{ij}$. We can get the next lemma from Theorem 2.1.

Lemma 3.3. *Above BA and PA are systems of atoms of $IBr(B)$ and $IPr(B)$.*

Thus good BS and PS make better BA and PA. In the next section, we show that BA and PA make better BS ans PS, repectively. And the best BS and PS are $\mathrm{IBr}(B)$ and $\mathrm{IPr}(B)$ from lemma 2.4.

4. Indecmposablility check of projective characters

Definition 4.1. Let Φ be a projective character and $\Phi = \sum_{i=1}^{t} n_i\varphi_i^* \ (n_i \geqq 0)$. We call Φ' is a part of Φ if $\Phi' = \sum_{i=1}^{t} n_i'\varphi_i^* \ (0 \leqq n_i' \leqq n_i)$.

Theorem 4.2. $\forall\Phi'$*: a part of* Φ*;* $\exists\varphi$ *: a Brauer character such that* $\langle\Phi', \varphi\rangle < 0$ *or* $\langle\Phi - \Phi', \varphi\rangle < 0$ *then* Φ *is indecomposable.*

Let Br (Ps) be all Brauer (projective) characters which we got.

4.1. Subtract indecomposable projective characters from decomposable one. Let I be a suset of $\{1, \ldots, t\}$ such that $\forall i \in I$; Φ_i: indecomposable projective character.

Definition 4.3. For $i \in I$ and $j \notin I$, Let m_{ij} be the maximum of $\{n \in \mathbb{N} \mid \forall\varphi \in Br, \langle\varphi, \Phi_j - n\Phi_i\rangle \geqq 0\}$. We call this m_{ij} a possible multiplicity of Φ_i in Φ_j.

Definition 4.4. Let $i \in I$ and $\varphi \in$ BS such that $\langle\varphi, \Phi_i\rangle > 0$ and $\varphi = \sum_{i=1}^{t} n_i\Phi_i^* \ (n_i \geqq 0)$. We call φ' a bit of φ with respect to Φ_i if φ' satifies the following conditions.

(a) $\varphi' = \sum_{i=1}^{t} n_i'\Phi_i^* \ (0 \leqq n_i' \leqq n_i)$.
(b) if $j \in I$ then $n_j' = \delta_{ij}$ else $n_j' \leqq m_{ij}$.
(c) $\forall\Psi \in Pr$, $\langle\varphi', \Psi\rangle \geqq 0$ and $\langle\varphi - \varphi', \Psi\rangle \geqq 0$.

Lemma 4.5. *For* $i \in I$*, let* β_i *be the irreducible Brauer character corresponds to* Φ_i*. If* $\varphi \in BS$ *and* $\langle\varphi, \Phi_i\rangle > 0$ *then* β_i *is a bit of* φ *with respect to* Φ_i*. In paticular, if* φ *has only one bit with respcet to* Φ_i *then this bit is the irreducible Brauer character corresponds to* Φ_i*.*

For $i \in I$ and $j \notin I$,

$$m(i, j, \varphi) := \min\{\langle\varphi', \Phi_j\rangle \mid \varphi' \text{ is a bit of } \varphi \text{ with respect to } \Phi_i\}$$

Theorem 4.6. *For* $i \in I$ *and* $j \notin I$*,* $\Phi_j - Max\{m(i, j, \varphi) \mid \varphi \in BS, \ \langle\varphi, \Phi_i\rangle > 0\}\,\Phi_i$ *is a projective character.*

Example

(Calculation of the decomposotion matrix of the defect 2 block in J_4)

```
GAP4, Version: 4.3fix4 of December 20, 2002, i686-pc-linux-gnu-gcc
gap> Read("BrauerCT2.g");
gap> ct:=CharacterTable("J4");;
gap> sub_ct:=List(Maxes(ct){[1,2]},CharacterTable);;
gap> BB:=FindProjectiveCharacters(ct,sub_ct,3,20);;
This is 1 th - try.
This is 2 th - try.
gap> DisplayBlocksInfo(BB[1]);
There are 20 blocks.
```

```
+++ Block [J4]B1 of defect 3 has 14 irr. char. and 9 irr. Brauer char. +++
+++ Block [J4]B2 of defect 3 has 14 irr. char. and 9 irr. Brauer char. +++
+++ Block [J4]B3 of defect 2 has 9 irr. char. and 5 irr. Brauer char. +++
+++ Block [J4]B4 of defect 1 has 3 irr. char. and 2 irr. Brauer char. +++
+++ Block [J4]B5 of defect 1 has 3 irr. char. and 2 irr. Brauer char. +++
+++ Block [J4]B6 of defect 1 has 3 irr. char. and 2 irr. Brauer char. +++
+++ Block [J4]B7 of defect 1 has 3 irr. char. and 1 irr. Brauer char. +++
+++ Block [J4]B8 of defect 0 has 1 irr. char. and 1 irr. Brauer char. +++
+++ Block [J4]B9 of defect 0 has 1 irr. char. and 1 irr. Brauer char. +++
+++ Block [J4]B10 of defect 0 has 1 irr. char. and 1 irr. Brauer char. +++
+++ Block [J4]B11 of defect 0 has 1 irr. char. and 1 irr. Brauer char. +++
+++ Block [J4]B12 of defect 0 has 1 irr. char. and 1 irr. Brauer char. +++
+++ Block [J4]B13 of defect 0 has 1 irr. char. and 1 irr. Brauer char. +++
+++ Block [J4]B14 of defect 0 has 1 irr. char. and 1 irr. Brauer char. +++
+++ Block [J4]B15 of defect 0 has 1 irr. char. and 1 irr. Brauer char. +++
+++ Block [J4]B16 of defect 0 has 1 irr. char. and 1 irr. Brauer char. +++
+++ Block [J4]B17 of defect 0 has 1 irr. char. and 1 irr. Brauer char. +++
+++ Block [J4]B18 of defect 0 has 1 irr. char. and 1 irr. Brauer char. +++
+++ Block [J4]B19 of defect 0 has 1 irr. char. and 1 irr. Brauer char. +++
+++ Block [J4]B20 of defect 0 has 1 irr. char. and 1 irr. Brauer char. +++
gap> B2A:=ShallowCopy(BB[1][3]);;
gap> SetBS(B2A);; SetPS(B2A);; SetBA(B2A);; SetPA(B2A);; SetPIM(B2A,10);;
gap> DisplayProjectiveCharacters(B2A);
+++ Block [J4]B3 of defect 2 has 16 proj. characters. +++
 Irr:    14 21 25 27 28 30 31 35 41   : Indec. Flag
----+-----+--+--+--+--+--+--+--+--+-:-------------
   1: [  1  .  .  .  1  1  .  .  1] :  (*)
   2: [  .  1  1  .  1  1  2  2  2] :
   3: [  .  .  1  3  3  3  4  4  7] :
   4: [  .  .  1  2  2  2  3  3  5] :
   5: [  .  .  .  1  .  1  .  1  1] :  (*)
   6: [  .  .  .  1  3  1  3  1  4] :
   7: [  .  .  1  2  3  2  4  3  6] :
   8: [  .  .  2  1  4  1  6  3  7] :
   9: [  .  .  3  2  3  2  6  5  8] :
  10: [  1  6  5  3  2 10  6 14 10] :
  11: [  .  2  5  6  5  8 10 13 16] :
  12: [  .  3  6  5  4  8 10 14 15] :
  13: [  1  3  4  5  5  9  8 12 14] :
  14: [  .  6  7  3  3  9 10 16 13] :
  15: [  .  2  8  5  4  7 12 15 17] :
  16: [  2  6  5  4  7 12 10 15 16] :
gap> B2A.rank;
5
gap> TryToSubstractIndecProjectiveCharacters(B2A,5,6,10);
1
gap> AddProjRec(B2A,LinearCombOfProj([1,-1],B2A.proj{[6,5]}),false);;
```

```
gap> SetBS(B2A);; SetPS(B2A);; SetBA(B2A);; SetPA(B2A);; SetPIM(B2A,10);;
gap> DisplayProjectiveCharacters(B2A);
+++ Block [J4]B3 of defect 2 has 16 proj. characters. +++
 Irr:    14 21 25 27 28 30 31 35 41   : Indec. Flag
----+-----+--+--+--+--+--+--+--+--+-:-------------
   1: [  1  .  .  .  1  1  .  .  1] :  (*)
   2: [  .  1  1  .  1  1  2  2  2] :
   3: [  .  .  1  2  3  2  4  3  6] :
   4: [  .  .  1  2  2  2  3  3  5] :
   5: [  .  .  .  1  .  1  .  1  1] :  (*)
   6: [  .  .  .  .  3  .  3  .  3] :
   7: [  .  .  .  1  3  1  3  1  4] :
   8: [  .  .  2  1  4  1  6  3  7] :
   9: [  .  .  3  2  3  2  6  5  8] :
  10: [  1  6  5  3  2 10  6 14 10] :
  11: [  .  2  5  6  5  8 10 13 16] :
  12: [  .  3  6  5  4  8 10 14 15] :
  13: [  1  3  4  5  5  9  8 12 14] :
  14: [  .  6  7  3  3  9 10 16 13] :
  15: [  .  2  8  5  4  7 12 15 17] :
  16: [  2  6  5  4  7 12 10 15 16] :
gap> AddProjRec(B2A,LinearCombOfProj([1/3],B2A.proj{[6]}),false);;
gap> SetBS(B2A);; SetPS(B2A);; SetBA(B2A);; SetPA(B2A);; SetPIM(B2A,10);;
gap> DisplayProjectiveCharacters(B2A);
+++ Block [J4]B3 of defect 2 has 14 proj. characters. +++
 Irr:    14 21 25 27 28 30 31 35 41   : Indec. Flag
----+-----+--+--+--+--+--+--+--+--+-:-------------
   1: [  1  .  .  .  1  1  .  .  1] :  (*)
   2: [  .  1  1  .  1  1  2  2  2] :
   3: [  .  .  1  2  2  2  3  3  5] :
   4: [  .  .  .  1  .  1  .  1  1] :  (*)
   5: [  .  .  .  .  1  .  1  .  1] :  (*)
   6: [  .  .  2  1  4  1  6  3  7] :
   7: [  .  .  3  2  3  2  6  5  8] :
   8: [  1  6  5  3  2 10  6 14 10] :
   9: [  .  2  5  6  5  8 10 13 16] :
  10: [  .  3  6  5  4  8 10 14 15] :
  11: [  1  3  4  5  5  9  8 12 14] :
  12: [  .  6  7  3  3  9 10 16 13] :
  13: [  .  2  8  5  4  7 12 15 17] :
  14: [  2  6  5  4  7 12 10 15 16] :
gap> TryToSubstractIndecProjectiveCharacters(B2A,5,3,10);
2
gap> AddProjRec(B2A,LinearCombOfProj([1,-2],B2A.proj{[3,5]}),false);;
gap> SetBS(B2A);; SetPS(B2A);; SetBA(B2A);; SetPA(B2A);; SetPIM(B2A,10);;
4 th proj. char. in PS is indec. because it is in PA.
5 th proj. char. in PS is indec. because it is in PA.
```

```
gap> DisplayProjectiveCharacters(B2A);
+++ Block [J4]B3 of defect 2 has 12 proj. characters. +++
 Irr:    14 21 25 27 28 30 31 35 41  : Indec. Flag
----+-----+--+--+--+--+--+--+--+--+-:-------------
   1: [  1  .  .  .  1  1  .  .  1] :  (*)
   2: [  .  1  1  .  1  1  2  2  2] :
   3: [  .  .  1  2  .  2  1  3  3] :
   4: [  .  .  .  1  .  1  .  1  1] :  (*)
   5: [  .  .  .  .  1  .  1  .  1] :  (*)
   6: [  .  .  2  1  4  1  6  3  7] :
   7: [  .  .  3  2  3  2  6  5  8] :
   8: [  1  6  5  3  2 10  6 14 10] :
   9: [  .  3  6  5  4  8 10 14 15] :
  10: [  .  6  7  3  3  9 10 16 13] :
  11: [  .  2  8  5  4  7 12 15 17] :
  12: [  2  6  5  4  7 12 10 15 16] :
gap> TryToSubstractIndecProjectiveCharacters(B2A,4,3,10);
2
gap> AddProjRec(B2A,LinearCombOfProj([1,-2],B2A.proj{[3,4]}),false);;
gap> SetBS(B2A);; SetPS(B2A);; SetBA(B2A);; SetPA(B2A);; SetPIM(B2A,10);;
gap> DisplayProjectiveCharacters(B2A);
+++ Block [J4]B3 of defect 2 has 8 proj. characters. +++
 Irr:    14 21 25 27 28 30 31 35 41  : Indec. Flag
----+-----+--+--+--+--+--+--+--+--+-:-------------
   1: [  1  .  .  .  1  1  .  .  1] :  (*)
   2: [  .  1  1  .  1  1  2  2  2] :
   3: [  .  .  1  .  .  .  1  1  1] :  (*)
   4: [  .  .  .  1  .  1  .  1  1] :  (*)
   5: [  .  .  .  .  1  .  1  .  1] :  (*)
   6: [  1  6  5  3  2 10  6 14 10] :
   7: [  .  6  7  3  3  9 10 16 13] :
   8: [  2  6  5  4  7 12 10 15 16] :
gap> TryToSubstractIndecProjectiveCharacters(B2A,3,2,10);
1
gap> AddProjRec(B2A,LinearCombOfProj([1,-1],B2A.proj{[2,3]}),false);;
gap> SetBS(B2A);; SetPS(B2A);; SetBA(B2A);; SetPA(B2A);; SetPIM(B2A,10);;
gap> DisplayProjectiveCharacters(B2A);
+++ Block [J4]B3 of defect 2 has 8 proj. characters. +++
 Irr:    14 21 25 27 28 30 31 35 41  : Indec. Flag
----+-----+--+--+--+--+--+--+--+--+-:-------------
   1: [  1  .  .  .  1  1  .  .  1] :  (*)
   2: [  .  1  .  .  1  1  1  1  1] :
   3: [  .  .  1  .  .  .  1  1  1] :  (*)
   4: [  .  .  .  1  .  1  .  1  1] :  (*)
   5: [  .  .  .  .  1  .  1  .  1] :  (*)
   6: [  1  6  5  3  2 10  6 14 10] :
   7: [  .  6  7  3  3  9 10 16 13] :
```

```
   8: [  2  6  5  4  7 12 10 15 16] :
gap> TryToSubstractIndecProjectiveCharacters(B2A,5,2,10);
1
gap> AddProjRec(B2A,LinearCombOfProj([1,-1],B2A.proj{[2,5]}),false);;
gap> SetBS(B2A);; SetPS(B2A);; SetBA(B2A);; SetPA(B2A);; SetPIM(B2A,10);;
gap> DisplayProjectiveCharacters(B2A);
+++ Block [J4]B3 of defect 2 has 5 proj. characters. +++
 Irr:   14 21 25 27 28 30 31 35 41  : Indec. Flag
----+-----+--+--+--+--+--+--+--+--+-:------------
   1: [  1  .  .  .  1  1  .  .  1] :  (*)
   2: [  .  1  .  .  .  1  .  1  .] :  (*)
   3: [  .  .  1  .  .  .  1  1  1] :  (*)
   4: [  .  .  .  1  .  1  .  1  1] :  (*)
   5: [  .  .  .  .  1  .  1  .  1] :  (*)
```

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Self-Duality of Quasi-Harada Rings and Locally Distributive Rings *

Kazutoshi Koike

Abstract

Recently Y. Baba [1] proved that a QH ring (quasi-Harada ring) R has a self-duality if gRg is a local serial ring with some condition, where g is an idempotent of R with gR a minimal faithful right R-module. Motivated by this result, we also investigate self-duality of QH rings and obtain several results including an improvement of the result of Y. Baba and applications to self-duality of locally distributive rings.

最近馬場 [1] は両側 QH 環 (quasi-Harada 環) の self-duality について研究し，特に A がある条件 (**) を満たす local serial 環であれば，A の Auslander 環 (有限表現型の環に対して有限生成直既約加群全体の直和の自己準同型環として定義される環) は self-duality をもつことを示した．この結果について，筆者は「条件 (**) を落とせないか?」という疑問を抱き，片側 QH 環の self-duality の研究を開始したが，もっと一般的な形でいろいろな結果を得ることができ，その系として (**) だけでなく local 性も落とすことができた．

以下この報告集では，すべての環は単位元をもち，すべての加群は単位的であるとする．加群 M に対して，その radical, socle, 移入包絡を，それぞれ $J(M)$, $S(M)$, $E(M)$ で表す．

1 いろいろなクラスの環の self-duality

まず最初にさまざまな self-duality の定義を思い出しておこう．環 R, S に対して，Morita duality を定める両側加群 ${}_SU_R$ が存在するとき，R は S に右 **Morita dual** であるといい，この報告集では，記号 $S \overset{\text{dual}}{\approx} R$ によって表すことにする．特に $R \overset{\text{dual}}{\approx} R$ のとき，R は **self-duality** をもつという．self-duality の一般化として，環の列 $R_1 = R, R_2, \ldots, R_{n+1} = R$ で，$R_1 \overset{\text{dual}}{\approx} R_2 \overset{\text{dual}}{\approx} \ldots \overset{\text{dual}}{\approx} R_{n+1}$ となるものが存在するとき，R は **almost self-duality** をもつという．self-duality を定める両側加群 ${}_RU_R$ について，R の任意の原始巾等元 e に対して，$S(eU) \cong eR/J(eR)$ が成り立つとき，${}_RU_R$ は **weakly symmetric self-duality** を定めるという．また，R の任意のイデアル I に対して，$l_R l_U(I) = I$ が成り立つとき，${}_RU_R$

*The detailed version of this note will be submitted for publication elsewhere.

は **good self-duality** を定めるとをいう．ここで l_R や l_U は左 annihilator を表す．${}_RU_R$ が good self-duality を定めれば，weakly symmetric self-duality を定める．almost self-duality, weakly self-duality, good self-duality は任意の corner 環に遺伝し，good self-duality は剰余環にも遺伝する．ここで，環 R の **corner** 環とは eRe (e は R の巾等元) の形の環である．

次に今回の研究の主題となる locally distributive 環と QH 環の定義を与えておく．両側アルチン環 R は，任意の原始巾等元 e に対して eR_R, ${}_RRe$ が分配的 (すなわち部分加群全体の束が分配的) なとき，**locally distributive** であるという．明らかに serial 環は locally distributive 環である．また左アルチン環 R は，任意の直既約射影的右 R 加群が擬移入的 (quasi-injective) であるとき左 **QH** 環 **(quasi-Harada** 環**)** であるという．(左 QH 環の概念は左 H 環の一般化として，馬場・岩瀬 [2] によって定義された．) 片側 H 環や両側 QH 環は QF-3 である．定義より直ちに分かるように，左 QH 環は右 QF-2 である．ここで，semiprimary 環 R が右 **QF-2** であるとは，任意の直既約射影的右 R 加群は単純な socle をもつことをいう．

すでに知られている self-duality の例も交えて，主結果の一部を表形式にまとめておく．•印を付けたものは今回筆者が示したものである．

強 ⟶ 弱

	(G) good self-duality	(W) weakly symmetric self-duality	(S) self-duality	(A) almost self-duality
↓ (1) QF 環		×	○	
(2) 左 H 環			×	○
(3) 左 QH 環			×	?
(3′) (+ イデアルは有限個)			×	○•
↓ (4) serial 環	○	|	|	
(5) locally distributive 環	?	⇔ ?	⇔• ?	?
(5′) (+ 右 serial)	○•	|	|	
(5″) (+ 右 QF-2)	?	⇔ ?	⇔• ?	○•
(6) 完全アルチン環	?	?	⇔• ?	?

この表において，例えば，1 行 S 列は「すべての QF 環は self-duality をもつ」ということを示している．(G) good self-duality，(W) weakly symmetric self-duality，(S) self-duality，(A) almost self-duality は，この順に弱くなっているので，1 つに○が付けば，その右側の○は省略する．また，1 行 W 列の×は，○の否定，すなわち「weakly symmetric self-duality をもたない QF 環が存在する」ということを表している．(このような QF 環の例や 2 行 S 列，2 行 W 列との関係，また 2 行 A 列の左 H 環の almost self-duality の存

在については，それぞれ Kraemer [6]，加戸・大城 [3]，筆者 [4, 5] を参照のこと．) 左 H 環は almost self-duality(2 行 A 列) をもつから，左 H 環の一般化である左 QH 環の almost self-duality (3 行 A 列) が問題になる．これに対する解答は得られなかったが，左 QH 環にイデアルは有限個のみであるという条件を付加すると，almost self-duality をもつ (3′ 行 A 列) ことを証明できた．(なお，2 行 S 列の例は 3′ 行 S 列の例にもなっている．)

一方，QF 環と並んで self-duality がよく知られている serial 環は，実際には good self-duality をもつ (4 行 G 列)．locally distributive 環は serial 環の一般化であるから，その self-duality が問題となるが，これは東屋の予想「すべての完全アルチン環は self-duality をもつ」(すなわち「上の表の 6 行 S 列は○である」) と密接に関係している．完全アルチン環とは locally distributive 環を含む環のクラスなので，東屋の予想が正しければ，すべての locally distributive 環は self-duality をもつ (すなわち，5 行 S 列は○である) ことになるが，この問題は未解決である．今回筆者は，locally distributive 環に，右 QF-2 を仮定すれば almost self-duality をもつ (5″ 行 A 列) ことと，さらに右 serial を仮定すれば good self-duality をもつ (5′ 行 G 列) ことを示した．(locally distributive 環，完全環や東屋の予想については，Xue [7] の 4 章を参照のこと．)

なお，この表で "⇔" は，その左右の duality の存在の有無が同値であることを示している．ここでも * 印が付いたものは，今回の研究において加戸・大城 [3] の手法を用いて筆者が示してものであるが，本報告集ではこれ以上は触れない．

2 diagonally complete な部分環の Morita duality

以下，本論に入ろう．今回の研究において，筆者が diagonally complete と名付けたある種の部分環が大変重要な役割を果たした．R' を環 R の部分環とする．R の直交巾等元の完全系 $\{e_1,\ldots,e_n\}$ で，$\bigoplus_{i=1}^n e_iRe_i \subset R'$ となるものが存在するとき，R' は R の **diagonally complete** な部分環であるという．この $\{e_1,\ldots,e_n\}$ に関する条件は，R' は R を行列表示したときの対角成分を含んでいることと同値である．diagonally complete な部分環の Morita duality について，次が成り立つ．

定理 1. 両側加群 ${}_SU_R$ は Morita duality を定めるとし，$\{e_1,\ldots,e_n\}$, $\{f_1,\ldots,f_n\}$ をそれぞれ R, S の直交巾等元の完全系で，各 i に対して $S(f_iU)$ と $e_iR/J(e_iR)$ に現れる単純部分加群は同じであるようなものとする．$\bigoplus_{i=1}^n e_iRe_i$ を含む R の diagonally complete な部分環 R' に対して，$S' = \bigoplus_{i,j} l_{f_iSf_j} l_{f_jUe_i}(e_iR'e_j)$ とおく．このとき，$S' \overset{\text{dual}}{\approx} R'$ で，$R' \mapsto S'$ は $\bigoplus_{i=1}^n e_iRe_i$ を含む R の diagonally complete な部分環全体と $\bigoplus_{i=1}^n f_iSf_i$ を含む S の diagonally complete な部分環全体との間の 1 対 1 対応を定める．

系 2. 左右の Morita duality や good self-duality は diagonally complete な部分環に遺伝する．

3　左 QH 環の構造

左 QH 環 R の構造を記述するために重要な役割を果たす 2 つの環 $\Lambda(R)$ と $\Gamma(R)$ がある．1 つ目は $\Lambda(R) = \mathrm{End}_R(E(R_R))$ と定める．これについては，次の簡単だが重要な補題が成り立つ．

補題 3. 左 QH 環 R に対して，$S = \{\alpha \in \Lambda(R) \mid \alpha(R) \leq R\}$, $I = \{\alpha \in \Lambda(R) \mid \alpha(R) = 0\}$ とおけば，S は $\Lambda(R)$ の diagonally complete な部分環，I は S のイデアルであり，$R \cong S/I$ が成り立つ．

今回の研究で $\Lambda(R)$ も再び左 QH 環になることが分かった．R が左 H 環のときには，$\Lambda(R)$ だけでなく S も左 H 環になり，S や I の形も完全に決定される．この場合の表示 $R \cong S/I$ は，いわゆる「左 H 環の行列表現」に他ならない．

E を左 QH 環 R の直交原始巾等元の完全系とし，$f = \sum\{e \in E \mid S(R_R)e \neq 0\}$ とおき，もう 1 つの環 $\Gamma(R)$ を $\Gamma(R) = fRf$ と定める．$\Gamma(R)$ も再び左 QH 環になる (馬場・岩瀬 [2, Proposition 8(1)])．f と対称的に，$g = \sum\{e \in E \mid eS({}_RR) \neq 0\}$ とおけば，これらの環について，次の関係が成り立つ．

$$
\begin{array}{rcccl}
 & S & \overset{\text{factor}}{\underset{\text{ring}}{\twoheadrightarrow}} & R & \\
 & {\scriptstyle \text{d.c. subring}}\ \cap & & \cup\ {\scriptstyle \text{corner ring}} & \\
\mathrm{End}_R(E(R_R)) = & \Lambda(R) & \overset{\text{dual}}{\approx} & \Gamma(R) & \\
 & \overset{\text{eq.}}{\approx}\updownarrow & & \| & \\
(R\text{ が QF-3 のとき})\ & gRg & \overset{\text{dual}}{\approx} & fRf &
\end{array}
$$

ここで “$\overset{\text{eq.}}{\approx}$” は Morita equivalent，“factor ring” は剰余環，“d.c.subring” は diagonally complete な部分環，“corner ring” は corner 環であることを表す．この図式は，corner 環 $\Gamma(R)$ は R より小さいが $\Lambda(R)$ と Morita dual であり，$\Lambda(R)$ の diagonally complete な部分環の剰余環として R は復元できること，R が QF-3 のときには gRg を $\Lambda(R)$ の代わりに考えても良いことを示している．

上の図式において，R が QF-3 であれば，gR は極小忠実右加群で，gRg は $\Lambda(R)$ と Morita equivalent である．一方 serial 環は good self-duality をもつ．したがって，系 2 と上の図式より得られる次の定理は，冒頭で触れた馬場氏の定理 [1, Theorem 5] を導く．ただし，$Q^r_{\max}(R)$ は R の極大右商環を表す．

定理 4. 左 QH 環 R が good self-duality をもつことと，環 $Q^r_{\max}(R)$，$\Lambda(R)$，$\Gamma(R)$ (R が QF-3 のとき gRg も) のどれか 1 つが good self-duality をもつこととは同値である．

左 QH 環は左アルチン的で左 Morita duality をもつが，右側についても次を得た．

定理 5. 左 QH 環は右 Morita duality をもち，特に右アルチン的である．

左 QH 環 R に対して，Γ を次々に取り続けて $R=\Gamma^0(R),\Gamma^1(R),\Gamma^2(R),\ldots$ を考えれば，Γ の定義より，いずれこの操作は終了し，右 Kasch 環 (任意の単純右加群が右 socle に現れるような環) に到達するが，それは QF 環であることが分かる．同様に，Λ を次々取り続けて，$R=\Lambda^0(R),\Lambda^1(R),\Lambda^2(R),\ldots$ を考えれば，次の定理を得る．

定理 6. 左 QH 環 R に対して，次が成り立つ．

(1) 各 $i\geq 0$ に対して，$\Lambda^i(R)\overset{\text{dual}}{\approx}\Gamma\Lambda^{i-1}(R)\overset{\text{dual}}{\approx}\cdots\overset{\text{dual}}{\approx}\Gamma^{i-1}\Lambda(R)\overset{\text{dual}}{\approx}\Gamma^i(R)$ で，これらも左 QH 環である．

(2) 各 $i\geq 1$ に対して，$\Lambda^i(R)$ の diagonally complete な部分環 S_i と，S_i のイデアル I_i が存在して，$\Lambda^{i-1}(R)\cong S_i/I_i$ となる．

(3) ある $m\geq 0$ が存在して，$\Lambda^m(R)\overset{\text{eq.}}{\approx}\Gamma^m(R)$ となり，これらは QF 環である．

この定理は，左 H 環の場合と同様に，すべての左 QH 環は QF 環から構成可能であることを示している．この状況を図示すると次のようになる．

$$
\begin{array}{ccccccccc}
 & & & & & & S_1 & \twoheadrightarrow & R \\
 & & & & & & \cap & & \cup \\
 & & & & S_2 & \twoheadrightarrow & \Lambda(R) & \overset{\text{dual}}{\approx} & \Gamma(R) \\
 & & & & \cap & & \cup & & \cup \\
 & & S_{m-1}(R) & \twoheadrightarrow & & & \vdots & & \vdots \\
 & & \cap & & & & \cup & & \cup \\
S_m & \twoheadrightarrow & \Lambda^{m-1}(R) & \overset{\text{dual}}{\approx} & \cdots & \overset{\text{dual}}{\approx} & \Gamma^{m-2}\Lambda(R) & \overset{\text{dual}}{\approx} & \Gamma^{m-1}(R) \\
\cap & & \cup & & & & \cup & & \cup \\
\Lambda^m(R) & \overset{\text{dual}}{\approx} & \Gamma\Lambda^{m-1}(R) & \overset{\text{dual}}{\approx} & \cdots & \overset{\text{dual}}{\approx} & \Gamma^{m-1}\Lambda(R) & \overset{\text{dual}}{\approx} & \Gamma^m(R)
\end{array}
$$

ここで "$\cup$" は corner 環，"$\cap$" は diagonally complete な部分環，"$\twoheadrightarrow$" は剰余環であることを意味する．

したがって左 QH 環の性質で，この図式の右側の縦の列の梯子を下がる (corner 環に遺伝する) ものや，左下から右上の階段を上る (diagonally complete な部分環と剰余環に遺伝する) ものは，QF 環に帰着される．

4 self-duality への応用

最後に self-duality への応用 (2 節の表の○*) について述べておく．

左 QH 環は右 QF-2 であるが，locally distributive 環については逆も成り立つ．

補題 7. locally distributive 右 QF-2 環は左 QH 環である．

片側 serial 環の corner 環も片側 serial である．また，片側 serial な QF 環は両側 serial となり，good self-duality をもつ．したがって，補題 7 と上の図式，系 2 より次が成り立つ．

定理 8. すべての locally distrbutive 右 serial 環は good self-duality をもつ.

イデアルは有限個に限るという性質は，上の図式の梯子を下り，階段を上る性質で，イデアルが有限個のとき，almost self-duality の存在は階段を上る性質である．また有限表現型の環や locally distrbutive 環のイデアルは有限個しかない．したがって次の定理を得る．

定理 9. イデアルが有限個しかないような左 QH 環は almost self-duality をもつ．特に，有限表現型の左 QH 環や locally distrbutive 右 QF-2 環は almost self-duality をもつ.

これらの結果は，東屋の予想の特別な場合「すべての locally distributive 環は self-duality をもつ」の部分的な解答と見なせる．やはり QF 環の議論に帰着して次の結果も得た．

定理 10. R を locally distributive 右 QF-2 環，$\{e_1, \ldots, e_n\}$ を R の直交原始巾等元の基本系とし，$S(R_R)e_i \neq 0$ であるような e_i の個数を m とする．

(1) $m \leq 2$ ならば R は good self-duality をもつ．特に $S(R_R)$ が homogeneous ならば R は good self-duality をもつ.

(2) $n \leq 3$ ならば R は self-dualiy をもつ.

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On S_R(H)-blocks II[1]

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1. INTRODUCTION

Let G be a finite group, p a prime divisor of the order of G and (K, R, k) a p-modular system, i.e., R is a complete discrete valuation ring with maximal ideal (π), K is the quotient field of R of characteristic 0 and $k(:= R/(\pi))$ is the residue class field of R of characteristic p. Moreover, we assume that K contains the $|G|$th roots of unity.

For a subset X of G, $\widehat{X}$ denotes the sum of all elements of X in the group algebra oG, where o is R, K or k.

We consider the *Hecke algebra* $S_o(H) := \mathrm{End}_{oG}(\widehat{H}oG)$ for a subgroup H of G.

As $e_H := \widehat{H}/|H|$ is the idempotent of KG (which is the central primitive idempotent of KH correponding to the trivial character 1_H of H), $S_K(H) = e_H KGe_H$, where we identify $f \in S_K(H)$ with $f(e_H) \in e_H KGe_H$ as usual.

For $\chi \in \mathrm{Irr}(G)$, let e_χ be the central primitive idempotent of KG corresponding to χ and put $\Phi_H^G := \{\chi \in \mathrm{Irr}(G); (\chi_{|H}, 1_H)_H \neq 0\}$. Then we have that $\{e_\chi e_H; \chi \in \Phi_H^G\}$ is the set of all central primitive idempotents of $S_K(H)$ in KG (see [C-R, (11.26) Corollary]).

As $S_R(H) \subset K \otimes_R S_R(H) = S_K(H)$, for a central idempotent ε of $S_R(H)$, there exists a non-empty subset β of Φ_H^G such that $\varepsilon = \sum_{\chi\in\beta} e_\chi e_H$. Here the element of this form is denoted by ε_β and if ε_β is a centrally primitive, β (or $\varepsilon_\beta S_R(H)$) is called an $S_R(H)$*-block* (cf. Definition 2.1).

Also, we have $S_R(H)/\pi S_R(H) \simeq S_k(H)$ as $\widehat{H}RG$ is a permutation module. Hence the set of $S_R(H)$-blocks corresponds bijectively to the set of $S_k(H)$-blocks.

On the other hand, the multiplication induces the K-algebra homomorphism ϕ from $Z(KG)$ to $Z(S_K(H))$. Using the map ϕ, G.R.Robinson [Ro] has proved that $Z(S_R(H)) \simeq A_R(H)$ as R-algebras, where $A_R(H)$ denotes the endomorpism ring $\mathrm{End}_{R[G\times G]}(RG\widehat{H}RG)$. Then so each $S_R(H)$-block corresponds to a (central) primitive idempotent of $A_R(H)$ and to a unique indecomposable direct summand M_β of $RG\widehat{H}RG$ as $R[G \times G]$-modulue. Therefore we can define a *defect group* for each $S_R(H)$-block in $G \times G$ (see Definition 2.2 and Remark 2.2).

Now we recall that for any $S_R(H)$-block β there exists a unique p-block B such that $\beta \subset \mathrm{Irr}(B)$ (see Proposition 2.3). Also if e_B is a central primitive idempotent, i.e., a block idempotent, of RG with the condition $\phi(e_B) \neq 0$, then $\phi(e_B) = \sum_{\beta\in\mathfrak{B}} \varepsilon_\beta$ as $\phi(e_B) \in Z(S_R(H))$, where $\mathfrak{B}$ is the suitable non-empty subset of $S_R(H)$-blocks. Hence $\mathrm{Irr}(B) \cap \Phi_H^G = \sum_{\beta\in\mathfrak{B}} \beta$ is a (disjoint) union of $S_R(H)$-blocks.

In this note, we shall exhibit some results on $S_R(H)$-blocks (which are mainly proved in [Ro] and [H-T]) and show the following example :

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[1] The final and detailed version of this note will be submitted for publication elsewhere.

Example 8 Let p be an odd prime, $G := \mathfrak{A}_p$ and $H := \mathfrak{A}_t$ for $\frac{p+1}{2} < t \le p$, where $\mathfrak{A}_n$ denotes the alternating group of degree n. Then

(1) $\mathrm{Irr}(B_0) \cap \Phi_H^G = \{[p-i, 1^i]_{|G}; 0 \le i \le p-t\}$, where $[p-i, 1^i]$ is an irreducible character of $\mathfrak{S}_p$.
(2) $\beta_0 = \mathrm{Irr}(B_0) \cap \Phi_H^G$ and $\delta_H(\beta_0) \in Syl_p(G \times G)$.

The notation is almost standard. Concerning some basic facts and terminologies used here, we refer to [C-R] and [N-T] for example.

2. Definitions and G.R.Robinson's results on $S_R(H)$-blocks

At first we recall the definition of $S_R(H)$-blocks by G.R.Robinson [Ro] (he call them $A_R(H)$-blocks), which is equivalent to our definition (see section 1).

Definition 2.1. ([Ro]) For any central primitive idempotent ε of $S_R(H)$ (or $A_R(H)$), there exists a minimal non-empty subset β of Φ_H^G which satisfies the following two conditions :

(a) $\varepsilon = \phi(e_\beta)$, where $e_\beta := \sum_{\chi\in\beta} e_\chi$.
(b) $\frac{1}{|G|}\sum_{\chi\in\beta}\chi(\widehat{Hg})\chi(1) \in R$ for any $g \in G$.

We call such sets $S_R(H)$-blocks ($A_R(H)$-blocks) of irreducible characters of G.

Remark 2.1. (1) The above condition (b) is equivalent to the next condition :

$(b)'$ $\frac{1}{|G|}\sum_{\chi\in\beta}\chi(\widehat{Hx})\chi(1) \in R$ for any $x \in H\backslash G/H$, where $H\backslash G/H$ is a complete set of representatives for (H,H)-double cosets of G.

(2) There exists a unique $S_R(H)$-block for any $\chi \in \Phi_H^G$. In particular, the trivial character 1_G of G is always in Φ_H^G for any subgroup H of G. Then there exists the $S_R(H)$-block which has 1_G. So such $S_R(H)$-block is called *the principal* $S_R(H)$-*block* and is denoted by β_0.

According to [Ro], we define *defect groups* for $S_R(H)$-blocks.

Definition 2.2. ([Ro]) For any $S_R(H)$-block β, there exists a minimal subgroup D of $G\times G$ and $\lambda \in \mathrm{End}_{RD}(RG\widehat{H}RG)$ with the condition : $\mathrm{Tr}_D^{G\times G}(\lambda) = \varepsilon_\beta$.
Then D is called *a defect group* for β and denoted by $\delta_H(\beta)$.

Remark 2.2. ([Ro, the above remark of Lemma 2.1]) By the theory of G-algebras, we have

(1) $\delta_H(\beta)$ is a p-subgroup of $G\times G$ and uniquely determined up to $G\times G$-conjugacy.
(2) $\delta_H(\beta)$ is a vertex of M_β.

The next proposition tells us some relations between p-blocks and $S_R(H)$-blocks.

Proposition 2.1. ([Ro, Remark of Lemma 2.1]) *If $H = \{1\}$, then* $\mathrm{Irr}(B)$ *is an $S_R(\{1\})$-block for any p-block B of G. Moreover, a defect group of an $S_R(\{1\})$-block* $\mathrm{Irr}(B)$ *is the diagonal subgroup $\delta(B)^{\Delta} := \{(x,x) \in G \times G; x \in \delta(B)\}$, where $\delta(B)$ is a (usual) defect group of B.*

The following property is one of the characterization of p-blocks.

Corollary 2.2. *For any p-block B of G,* $\mathrm{Irr}(B)$ *is a minimal subset of* $\mathrm{Irr}(G)$ *which satisfies the following condition :* $\dfrac{1}{|G|}\displaystyle\sum_{\chi \in \mathrm{Irr}(B)} \chi(x)\chi(1) \in R$ *for any $x \in G$.*

Proof. We take $H = \{1\}$. Then the statement is clear from Definition 2.1 and the above proposition. □

Now we exhibit some important and useful results by G.R.Robinson in [Ro].

Proposition 2.3. ([Ro, Lemma 2.1,Lemma 2.3(i),(iii),Corollary 2.4])

(1) *For any $S_R(H)$-block β and $x, y \in G$,* $\dfrac{|\delta_H(\beta)|}{|C_G(x)||C_G(y)|}\displaystyle\sum_{\chi\in\beta}\chi(x)\chi(y) \in R$.

In particular, $\dfrac{|\delta_H(\beta)|}{|G \times G|}\displaystyle\sum_{\chi\in\beta}\chi(1)^2 \in R$.

(2) *β is contained in a single p-block B of G in the usual sence, and if B has a defect group D, then $\delta_H(\beta)$ is contained (up to conjugacy) in $D \times D$.*

(3) *Let $C := core_G(H)$. Then there is a bijection between the set of $S_R(H)$-blocks of G and the set of $S_R(H/C)$-block of G/C. In particular, if $C = H$, i.e., H is normal in G, then the $S_R(H)$-blocks of G are precisely the p-blocks of $R[G/H]$.*

(4) *For the principal $S_R(H)$-block β_0,$\beta_0 = \{1_G\}$ if and only if H contains a Sylow p-subgroup of G.*

Corollary 2.4. (see [H1, Corollary 3]) *The followings hold.*

(1) *If $\sum_{\chi\in\beta}\chi(1)^2$ is prime to p for an $S_R(H)$-block β, then a defect group of β is a Sylow p-subgroup of $G \times G$. In particular, if H contains a Sylow p-subgroup of G, then a defect group of β_0 is a Sylow p-subgroup of $G \times G$.*

(2) *If $\chi \in \Phi_H^G$ is in p-block B of defect 0, then $\{\chi\} =$* $\mathrm{Irr}(B)$ *is an $S_R(H)$-block and whose defect group is unit element $\{(1,1)\}$ of $G \times G$.*

3. Othogonality Relation

In this section we assume H is a p'-subgroup of G and consider only those blocks such that $\phi(e_B) \neq 0$.

In this case $e_H \in RG$, i.e., $\widehat{H}RG = e_H RG$ is a projective RG-module and kH is a semisimple k-algebra.

For any $\varphi \in \mathrm{IBr}(G)$, let S_φ (resp. P_φ) be an irreducible kG-module (resp. an indecomposable projective RG-module) corresponding to φ and $\Psi_H^G := \{\varphi \in \mathrm{IBr}(G); k_H | S_{\varphi\downarrow H}\}$. Here we mention that $\Psi_H^G = \{\varphi \in \mathrm{IBr}(G); P_\varphi | e_H RG\}$ by Robinson's reciprocity. We let furthermore $\beta_i{}^\bullet := \{\varphi \in \mathrm{IBr}(B); P_\varphi | \varepsilon_{\beta_i}(e_H RG)\}$ for $0 \le i \le t$, where $\mathfrak{B} = \{\beta_i\}_{i=0}^t$ (see section 1).

Proposition 3.1. (see [H-T, Proposition 3]) *The followings hold.*

(1) $\mathrm{IBr}(B) \cap \Psi_H^G = \bigcup_{i=0}^t \beta_i{}^\bullet$.

(2) *The decomposition matrix D_B of B has the following form :*

$$(3.1) \qquad D_B = \left(\begin{array}{cccc|c} D_{\beta_0} & 0 & \cdots & 0 & * \\ 0 & D_{\beta_1} & \cdots & 0 & * \\ \vdots & \vdots & \ddots & \vdots & * \\ 0 & 0 & \cdots & D_{\beta_t} & * \\ \hline 0 & 0 & \cdots & 0 & * \\ \vdots & \vdots & \cdots & \vdots & * \\ 0 & 0 & \cdots & 0 & * \end{array}\right) = (D_B{}'|D_B{}''),$$

where $D_B{}'$ denotes the set of the first $|\mathrm{IBr}(B) \cap \Psi_H^G|$ columns of D_B and $D_B{}''$ the rest.

Corollary 3.2. (see [H-T, Lemma 7 and Corollary 8]) *If* $\Phi_H^G = \mathrm{Irr}(G)$, *then* $\mathrm{Irr}(B)$ *is an $S_R(H)$-block for any p-block B. Moreover, if* $\mathrm{Irr}(B) \subset \Phi_H^G$, *then* $\mathrm{Irr}(B)$ *is an $S_R(H)$-block.*

Remark 3.1. Let $G := \mathfrak{A}_5$:the alternating group of degree 5, $C_3 := \langle(1\,2\,3)\rangle \in Syl_3(G)$ and $p = 3$. (So C_3 is not a $3'$-subgroup.) Then $\Phi_{C_3}^G = \mathrm{Irr}(G)$. But $\mathrm{Irr}(B_0) = \beta_0 \cup \beta_1$,where $\beta_0 = \{1_G\}$ and $\beta_1 = \{\chi_2, \chi_3\}$ with $\chi_2(1) = 4$ and $\chi_3(1) = 5$. Then the above statement is not true in general.

In the rest of this note $\mathfrak{S}_n$ (resp. $\mathfrak{A}_n$) denotes the symmetric (resp. the alternating) group of degree n. For a *partition* of n the *Young diagram* associated with λ is denoted by $[\lambda]$ and λ' denotes the conjugate of λ. (So $[\lambda']$ is the transposed diagram of $[\lambda]$.) Also, the same notation $[\lambda]$ means an irreducible character of $\mathfrak{S}_n$ corresponding to the Young diagram $[\lambda]$. (For example $[n] := [(n)]$ means the trivial character $1_{\mathfrak{S}_n}$, too.) Also B_0 denotes the principal p-block of G.

Using the above notations, we have the following examples.

Example 1. Let $G = \mathfrak{S}_4$, $H := \langle(1\,2)\rangle$ and $p = 3$.

Then $\Phi_H^G = \mathrm{Irr}(G) \setminus \{[1^4]\} = \beta_0 \cup \beta_1 \cup \beta_2$, where $\beta_0 = \{[4], [2^2]\} = \mathrm{Irr}(B_0) \cap \Phi_H^G, \beta_1 = \{[3,1]\} = \mathrm{Irr}(B_1)$ and $\beta_2 = \{[2,1^2]\} = \mathrm{Irr}(B_2)$. Moreover, $\delta_H(\beta_0) \in Syl_3(G \times G)$ and $\delta_H(\beta_1) = \{(1,1)\}$ for $i = 1, 2$ by Corollary 2.4. Also the decomposition matrices D_{B_0} and D_{β_0} are followings :

$$D_{B_0} = \begin{array}{c} [4] \\ [2^2] \\ [1^4] \end{array} \left(\begin{array}{c|c} 1 & 0 \\ 1 & 1 \\ \hline 0 & 1 \end{array}\right), D_{\beta_0} = \begin{array}{c} [4] \\ [2^2] \end{array} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Example 2. (see [H-T, Remark 12] and [H1, Example 10])

Let $G := \mathfrak{S}_5$, $H := \langle (1\,2)(3\,4), (1\,3)(2\,4) \rangle$ and $p = 5$. Then $\mathrm{Irr}(B_0) \cap \Phi_H^G = \beta_0 \cup \beta_1$ with $\beta_0 = \{[5], [4,1]\}$ and $\beta_1 = \{[2,1^3], [1^5]\}$. Moreover, $\delta_H(\beta_i) \in Syl_5(G \times G)$ $(i = 0, 1)$ by Corollary 2.4. Also the decomposition matrix of B_0 is the following form :

$$D_{B_0} = \begin{matrix} [5] \\ [4,1] \\ [3,1^2] \\ [2,1^3] \\ [1^5] \end{matrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{matrix} [5] \\ [4,1] \\ [2,1^3] \\ [1^5] \\ [3,1^2] \end{matrix} \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \end{array}\right).$$

So the decomposition matrices of $S_R(H)$-blocks in B_0 are followings:

$$D_{\beta_0} = \begin{matrix} [5] \\ [4,1] \end{matrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, D_{\beta_1} = \begin{matrix} [2,1^3] \\ [1^5] \end{matrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The next theorem is the orthogonality relation for the $S_R(H)$-block.

Theorem 3.3. ([H-T, Theorem 5]) *Let β be an $S_R(H)$-block. Then we have* $\sum_{\chi \in \beta} \chi(xe_H)\chi(y) = 0$ *for any $y \in G - G_{p'}$ and $x \in G_{p'}$ such that $\langle x, H \rangle$ is a p'-subgroup.*

Remark 3.2. (see [H-T, Remark 12] and [H1, Remark 6-1])

(1) If $H = \{1\}$ in the above theorem, we get the second orthogonality relation for the (usual) p-block.

(2) We recall Example 2 and take $x := (4\,5) \in G_{5'}$ and $y := (1\,2\,3\,4\,5) \in G - G_{5'}$ in the above theorem. Then $Hx \subset HxH \subset G_{5'}$ but $\langle H, x \rangle = G \not\subset G_{5'}$. This means that the assumption, $\langle x, H \rangle$ is a p'-subgroup, is not satisfied. Moreover, $\sum_{\chi \in \beta_0} \chi(xe_H)\chi(y) \neq 0$. So the above theorem is not true for any p-regular element $x \in G$.

(3) <u>Question</u> Does the following statement hold?

$$\sum_{\chi \in \beta} \chi(xe_H)\chi(y) \equiv 0 \pmod{(\pi)} \text{ for any } x \in G_{p'} \text{ and } y \in G - G_{p'}.$$

4. Some examples of the principal $S_R(H)$-blocks

We use the same notations in section 3.

Example 3. ([H2, Example 1]) Let $H := G$ and $\mathrm{char}\, k = p$.

(1) $\Phi_G^G = \{1_G\}$.

(2) $\beta_0 = \Phi_G^G$ and $\delta_G(\beta_0) \in Syl_p(G \times G)$.

Example 4. ([H1, Example 12]) Let $G := \mathfrak{S}_n$ and $H := \mathfrak{S}_{n-1}$, $(n \geq 2)$.

(1) $\Phi_H^G = \{[n], [n-1, 1]\}$.
(2) (a) If p does not devide n, then $\beta_0 = \{[n]\}$ and $\delta_H(\beta_0) \in Syl_p(G \times G)$.
(b) If p devides n, then $\beta_0 = \{[n], [n-1, 1]\} (= \Phi_H^G)$.
In particular, if p is odd prime, then $\delta_H(\beta_0) \in Syl_p(G \times G)$.

For the alternating groups the same statements hold. (We notice that we correct the range of n in [H2, Example 5].)

Example 5. (cf. [H2, Example 5]) Let $G := \mathfrak{A}_n$ and $H := \mathfrak{A}_{n-1}$, $(n \geq 4)$.

(1) $\Phi_H^G = \{1_G, \chi\}$, where $\chi(1) = n - 1$.
(2) (a) If p does not divide n, then $\beta_0 = \{1_G\}$ and $\delta_H(\beta_0) \in Syl_p(G \times G)$.
(b) If p divides n, then $\beta_0 = \{1_G, \chi\} (= \Phi_H^G)$.
In particular, if p is odd prime, then $\delta_H(\beta_0) \in Syl_p(G \times G)$.

If G is the symmetric group of degree p, then we have the following two examples.

Example 6. ([H1, Example 11]) Let $G := \mathfrak{S}_p, H := \mathfrak{S}_t$ $(1 \leq t \leq p)$ and char$k = p$. Then

(1) $\beta_0 = \mathrm{Irr}(B_0) \cap \Phi_H^G = \{[p-i, 1^i]; 0 \leq i \leq p-t\}$.
(2) $\delta_H(\beta_0) =_{G \times G} \begin{cases} P^\Delta & t = 1 \\ P \times P & 2 \leq t \leq p \end{cases}$, where P is a Sylow p-subgroup of G.

Example 7. ([H2, Example 7]) Let p be an odd prime, $G := \mathfrak{S}_p$ and $H := \mathfrak{A}_t$ for $\dfrac{p+1}{2} < t \leq p$.

(1) $\mathrm{Irr}(B_0) \cap \Phi_H^G = \{[p-i, 1^i]; 0 \leq i \leq p-t\} \cup \{[j+1, 1^{p-j-1}]; 0 \leq j \leq p-t\}$.
(2) $\beta_0 = \{[p-i, 1^i]; 0 \leq i \leq p-t\}$ and $\delta_H(\beta_0) \in Syl_p(G \times G)$.

One of the purposes of this note is to show the next example.

Example 8. Let p be an odd prime, $G := \mathfrak{A}_p$ and $H := \mathfrak{A}_t$ for $\dfrac{p+1}{2} < t \leq p$. Then

(1) $\mathrm{Irr}(B_0) \cap \Phi_H^G = \{[p-i, 1^i]_{|G}; 0 \leq i \leq p-t\}$, where $[p-i, 1^i]$ is an irreducible character of $\mathfrak{S}_p$.
(2) $\beta_0 = \mathrm{Irr}(B_0) \cap \Phi_H^G$ and $\delta_H(\beta_0) \in Syl_p(G \times G)$.

Proof. We may assume that $t \neq p$ by Example 3. Put $\widetilde{G} := \mathfrak{S}_p$ and $\widetilde{B}_0$ (resp. $\widetilde{\beta}_0$) denotes the principal p-block (resp. the principal $S_R(H)$-block) of $\widetilde{G}$. At first we mention that $[p-i, 1^i]_{|G} = [i+1, 1^{p-i-1}]_{|G}$ is irreducible for any $0 \leq i \leq p-t$ as $\dfrac{p+1}{2} < t \leq p$ and $[i+1, 1^{p-i-1}]$ is the conjugate of $[p-i, 1^i]$.

(1) As $\mathrm{Irr}(B_0) \cap \Phi_H^G = \{\widetilde{\chi}_{|G}; \widetilde{\chi} \in \mathrm{Irr}(\widetilde{B}_0) \cap \Phi_H^{\widetilde{G}}\}$ and Example 7(1), the assertion holds.

(2) We may only prove the first half because the later half follows from Example 7(2). Now put $I := \{[p-i, 1^i]; [p-i, 1^i]_{|G} \in \beta_0\} \subset \widetilde{\beta}_0$. So we may show that $I = \widetilde{\beta}_0$ by (1). As $p > 2$, for any $x \in H \setminus G/H$,

$$\frac{1}{|\tilde{G}|}\sum_{\tilde{\chi}\in I}\tilde{\chi}(\widehat{Hx})\tilde{\chi}(1)=\frac{1}{2|G|}\sum_{\tilde{\chi}\in I}\tilde{\chi}_{|G}(\widehat{Hx})\tilde{\chi}_{|G}(1)=\frac{1}{2|G|}\sum_{\chi\in\beta_0}\chi(\widehat{Hx})\chi(1)\in R,$$

where $\chi := \tilde{\chi}_{|G}$. Also, $\frac{1}{|\tilde{G}|}\sum_{\tilde{\chi}\in I'}\tilde{\chi}(\widehat{Hx})\tilde{\chi}(1)\in R$ for $I' := \{[i+1, 1^{p-i-1}]; [p-i, 1^i] \in I\}$.

Therefore $\frac{1}{|\tilde{G}|}\sum_{\tilde{\chi}\in I\cup I'}\tilde{\chi}(\widehat{Hx})\tilde{\chi}(1)\in R$ as $I\cap I' = \emptyset$.

On the other hand, for $\tau := (1\ 2) \in \tilde{G}/G$, $\frac{1}{|\tilde{G}|}\sum_{\tilde{\chi}\in I}\tilde{\chi}(\widehat{Hx\tau})\tilde{\chi}(1) = -\frac{1}{|\tilde{G}|}\sum_{\tilde{\chi}'\in I'}\tilde{\chi}'(\widehat{Hx\tau})\tilde{\chi}'(1)$.

Then $\frac{1}{|\tilde{G}|}\sum_{\tilde{\chi}\in I\cup I'}\tilde{\chi}(\widehat{Hx\tau})\tilde{\chi}(1) = 0$. Hence $I\cup I'$ is a union of $S_R(H)$-blocks of $\tilde{G}$ as $H\setminus\tilde{G}/H = \{x, x\tau; x\in H\setminus G/H\}$. Thus $\tilde{\beta}_0 \subset I\cup I'$. Moreover, $\tilde{\beta}_0\subset I$ since $\tilde{\beta}_0\cap I' = \emptyset$. This means $I = \tilde{\beta}_0$ and the assertion holds. □

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ON RINGS WITH THE SAME SET OF PROPER IDEALS

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Abstract: We investigate pairs of rings with a set of common ideals.

In 1980's, a series of papers appeared in Canadian Journal of Mathematics ([1],[2]) that investigated pairs of commutative rings with the same set of prime ideals. We consider some generalizations of the study in the noncommutative setting. Throughout, all rings are assumed to be associative (but not necessarily commutative) with an identity element. The term "*subring*" will be used for a *unital subring*. Thus, not only a subring inherits its binary operations from its overring, but also they have the same identity element.

Consider $H = Hom_{\mathbb{R}}(V, V)$, where V is a vector space over $\mathbb{R}$ with $\dim_{\mathbb{R}}(V) = \aleph_{\omega_0}$ (ω_0 is the first limit ordinal). The center of H is isomorphic to $\mathbb{R}$ and hence, it has subfields K and F such that $K \not\subset F$ and $F \not\subset K$. Let $M = \{f \in Hom_{\mathbb{R}}(V, V) \mid \dim f(V) < \aleph_{\omega_0}\}$. Then $S = M + K$ and $R = M + F$ are an example of a pair of rings with the same set of prime ideals. Further more, S and R have infinitely many ideals and all of their proper ideals are prime ideals. A curiosity therefore arises for a pair of rings with the same set of proper ideals. By our first theorem, the only possible pairs of subrings of a commutative ring with the same set of proper ideals are fields.

Theorem 1. *Two distinct subrings R and S of a ring are division ring if and only if they have the same set of proper right ideals.*

Proof. Since $R \neq S$, they cannot have two distinct maximal right ideals in common. Let M be the unique maximal ideal of R and S, and suppose that $0 \neq a \in M$. Then, since R and S have the same set of proper right ideals, we have $aR = aS$. Further, since $1-m$ is invertible for any $m \in M$, we must have $aR = aS \neq aM$. Thus aS/aM is a one-dimensional vector space over the division ring S/M and $aS/aM = aR/aM$ is also a one-dimensional vector space over R/M. This is a contradiction since $R/M \neq S/M$. Thus, $M = 0$ and hence R and S are division rings. □

We now state two propositions on a pair of rings with an ideal in common.

Proposition 1. *Let R and S be subrings of a ring and suppose that they have a common ideal I. If P is a prime ideal of R, then $\bar{P} = \{a \in S \mid IaI \subseteq P\}$ is either S or a prime ideal of S.*

[1] The detailed version of this paper has been submitted elsewhere.

Proposition 2 Let R and S be subrings of a ring having a common ideal I. If P is a primitive ideal of R, then $\bar{P} = \{a \in S \mid IaI \subseteq P\}$ is either S or a primitive ideal of S.

Our second theorem yields that a pair of rings has the same set of prime ideals if and only if they have the same set of maximal ideals. We denote the set of prime ideals of a ring R by *Spec(R);* the set of maximal ideals of a ring R by *Max(R);* and the set of primitive ideals of a ring R by $Prim(R)$.

Theorem 2. *Let $R \neq S$ be subrings of an arbitrary ring. Then the following statements are equivalent:*

(a) $Max(S) \supseteq Max(R)$
(b) $Max(S) \subseteq Max(R)$
(c) $Spec(S) = Spec(R)$
(d) $Prim(S) = Prim(R)$

Proof. If $Max(S) \supseteq Max(R)$, then R has a unique maximal ideal M. Let N be another maximal ideal of S. Then since $S = M + N$, there exist $m \in M$ and $n \in N$ such that $1 = m + n$. But then $n = 1 - m \in R \setminus M$ and hence $RnR = R$. Hence, $M^2 = MRnRM \subseteq N$. Since N is a prime ideal of S, this is a contradiction. Therefore, $Max(S) = Max(R) = \{M\}$. This shows the equivalence of the statement (a) and (b).
Suppose now that $Max(S) = Max(R) = \{M\}$ and let $P \neq M$ be a prime ideal of R. Then, by Proposition 1, $\bar{P} = \{a \in S \mid MaM \subset P\}$ is a prime ideal of S. Since M is the unique maximal ideal of S, we have $\bar{P} \subseteq M$, and so $\bar{P}$ is an ideal of R. Since $M\bar{P}M \subset P$, we obtain $\bar{P} \subset P$, and therefore $P = \bar{P}$ is a prime ideal of S. Since a primitive ideal is prime, the equivalence of the statement (a), (b), and (d) can be shown similarly by using Proposition 2. □

For a ring T, let $\mathbb{S}(T)$ be the set of all subrings S of T with $Spec(S) = Spec(T)$. We note that if *T is a ring with unique maximal ideal M, then*
$\mathbb{S}(T) = \{p^{-1}(S) \mid S$ *is a simple subring of* $T/M\}$ *where* $p: T \to T/M$ *is the canonical epimorphism.*

A ring is called *fully idempotent* if every ideal of R is idempotent. A commutative fully idempotent ring is Von Neumann regular. However, the class of fully idempotent rings strictly contains the class of regular rings.

Proposition 3. *let R and S be fully idempotent subrings of a ring. Then R and S have the same set of proper ideals if and only if R and S have the same set of prime ideals.*

We are in a position to give a few examples.

Example 1. An example of a pair of rings having the same set of maximal (therefore prime) ideals but the set of proper ideals are not identical.

Let $R=\mathbb{Q}(\sqrt{3})\oplus\mathbb{R}$ and $S=\mathbb{Q}(\sqrt{2})\oplus\mathbb{R}$ be additive abelian groups with multiplication defined by $(a, b)(c, d)=(ac, ad+bc)$. Then R and S have a unique maximal ideal $M=0\oplus\mathbb{R}$. Let $I=0\oplus\mathbb{Q}(\sqrt{2})$. Then I is an ideal of S but not of R. □

Example 2. An example of a pair of rings that have a nonzero ideal in common but the set of prime ideals are not identical.

Let K be a field, and $K[x]$ and $K[y]$ be two polynomial rings over K. Consider the ring $S=K[x]\oplus K[y]$ and its subring $R=\{(a+xf(x),a)\in S \mid a\in K, f(x)\in F[x]\}$. Then R and S have common ideal $I=\{(xf(x), 0)\mid f(x)\in K[x]\}$. Clearly $P=\{(0,0)\}$ is a prime ideal of R, but it is not a prime ideal of S. □

Example 3. An example of a pair of rings that are not fully idempotent but have the same set of prime ideals.

Let $\bar{R}$ be the ring consisting of countable matrices over $\mathbb{R}$ of the form

$$\begin{pmatrix} A_m & & & \\ & a & & \\ & & a & \\ & & & \ddots \end{pmatrix}$$

where $a\in\mathbb{R}$ and A_m is an arbitrary $m\times m$ matrix over $\mathbb{R}$ and m is allowed to be any integer.

Let $\bar{S}=\bar{M}+F$ where F is a subfield of the center of $\bar{R}$ and $\bar{M}=\begin{pmatrix} A_m & & & \\ & 0 & & \\ & & 0 & \\ & & & \ddots \end{pmatrix}$

Let $S=\bar{S}\oplus\bar{M}$ and $R=\bar{R}\oplus\bar{M}$ be additive abelian groups with multiplication defined by $(a, b)(c, d)=(ac, ad+bc)$. Then, S and R have the unique common maximal ideal $M=\{(m_1, m_2)\mid m_1, m_2\in\bar{M}\}$ and hence, they have the same set of prime ideals. However, the ideal $I=\{(0, m)\mid m\in\bar{M}\}$ is not idempotent. □

Next, we investigate properties that pass through a pair of rings with common ideals. By Theorem 2, if two subrings R and S of a ring have the common maximal ideal, then they have the same set of prime ideals. Thus, in particular, if S is prime, then so is R. For a ring R, let $B(R)$ denote its prime radical, and $J(R)$ denote its Jacobson radical. Using Propositions 1 and 2, one can prove Lemma 1 below and hence Proposition 4 holds.

Lemma 1. Let R and S be subrings of a ring having a common ideal I.

(a) If $B(R) \subset I$, then $IB(S)I \cap I \subset B(R)$.
(b) If $J(R) \subset I$, then $IB(S)I \cap I \subset J(R)$.

Proposition 4. Let R and S be subrings of a ring having a common ideal I.

(a) If R is a semiprime ring and if $r_S(I) = \ell_S(I) = 0$, then S is a semiprime ring.
(b) If R is a semiprimitive ring and if $r_S(I) = \ell_S(I) = 0$, then S is a semiprimitive ring.

Let $R \subseteq S$ be rings with a common ideal I, and let P be a prime ideal of R with $I \not\subset P$. Then "lying over" holds, i.e., there exists a prime ideal Q in S such that $Q \cap R = P$. (See for example Rowen [4]).

Proposition 5. Let $R \subseteq S$ be rings with a common ideal I. If P is a prime ideal of S with $I \not\subset P$, then $P \cap R$ is a prime ideal of R.

Using Propositions 2, one can prove Lemma 2 below and hence Proposition 6 holds.

Lemma 2. Let $R \subseteq S$ be rings with a common ideal I. If $B(S) \subset I$, then $B(R) \cap I \subset B(S)$.

Proposition 6. Let $R \subseteq S$ be rings with a common ideal I. Then if S is a semiprime ring and if I is an essential ideal of S, then R is a semiprime ring.

A ring all of whose (two sided) ideal is idempotent is called a *fully idempotent ring.* A fully idempotent ring is in particular, a semiprime ring.

Proposition 7. Let R and S be subrings of a ring having the common maximal ideal M. Then if R is fully idempotent, then so is S and in this case they have the same set of proper ideals.

Every right ideal of a von Neumann regular is idempotent. A ring all of whose right ideal is idempotent is called a *fully right idempotent ring* and has received some attention in the literature.

Proposition 8. Let R and S be subrings of a ring having the common maximal ideal M. Then if R is fully right idempotent, then so is S and in this case they have the same set of proper ideals.

The next natural question is whether or not the "regularity" passes through two rings having the common maximal ideals.

Example 4. Let W denote the n-th Weyl algebra over a field of characteristic zero. It is well known that W is a simple Noetherian domain, and hence W is an Ore domain. Let D denote the filed of fraction of W. Let R be the set of countable matrices over D of the form

$$\begin{pmatrix} A_m & & & \\ & a & & \\ & & a & \\ & & & \ddots \end{pmatrix}$$

where $a \in D$ and A_m is an arbitrary $m \times m$ matrix over D and m is allowed to be any integer. Let S be the same set of matrices except $a \in W$. Then R and S have the unique maximal ideal M that consists of countable matrices of the form

$$\begin{pmatrix} A_m & & & \\ & 0 & & \\ & & 0 & \\ & & & \ddots \end{pmatrix}.$$

While it is easy to see that R is a von Neumann regular ring, S is not von Neumann regular since $S/M \simeq W$: a simple Northerian but not an Artinian ring. □

Neither the descending nor the ascending chain condition passes through a pair of rings with the same set of ideals in general.

Proposition 9. Let $R \subset S$ be rings with the common maximal ideal. If S satisfies a polynomial identity, then R is right Noetherian if and only if S is right Noetherian and S/M is finitely generated right R/M-module.

For a right R-module M and an ideal I of a ring R, consider $P_I(M) = \{m \in M \mid mI = 0\}$. M is said to be *split* in P_I if $P_I(M)$ is a direct summand of M, and P_I is said to be *splitting* if every R-module M *splits* in P_I. For a non-zero ideal I of a prime fully right idempotent ring R, Theorem 1.3 of Hirano-Tsutusi [3] yields that P_I is splitting if and only if R/I is semisimple Artinian. Therefore, Example 4 shows that "P_I splitting property" does not in general pass through a pair of rings with the same set of proper ideals.

Proposition 10. Let R and S be non-prime subrings of a ring having the common maximal ideal M. Then P_I is splitting for every ideal I of R if and only if P_I is splitting for every ideal I of S.

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Finitely Cogenerated Distributive Modules*

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概要

In this paper we consider a generalization of Vámos's proposition which asserts that finitely generated artinian and distributive modules are cyclic. Also, we show its duality as an answer to the problem which is shown by him (See [Vam78]).

1 導入

この文書中では、全ての "A" は特に断わりのない限り、non-zero identity を持ち、全ての module は unitary な left A-module とする。

初めにいくつかの定義について言及する。module を *distributive* と呼ぶのは、その submodule 全体による lattice が distributive law を満たすときである。2 つの module M_1 と M_2 が *unrelated pair* を成すと言うのは、以下の条件を満たすときである。

> $L_1/N_1 \simeq L_2/N_2$ となるような submodule の列 $N_i \leq L_i \leq M_i$ $(i = 1, 2)$ は、常に $L_1 = N_1$ かつ $L_2 = N_2$ となる。

以下の 2 つの remark (Remark 2 は [Ste74, Proposition 1.3] である) と [Erd87] の結果から、distributive module 周辺の研究において unrelated pair という概念が重要な役割を演じていることが分かる。

Remark 1. module M に対して以下の条件は互いに同値である。

(1) M は distributive である。
(2) 任意の M の submodule N に対し、N と M/N は distributive となる。
(3) N と M/N が distributive であり、かつ unrelated pair をなすような M の submodule N が存在する。

*This is not in a final form. This note is a summary of the paper [ShiOX]

Remark 2. M_i $(i \in I)$ を distributive module とする。また、M をその direct sum としたとき、以下の条件は互いに同値となる。

(1) M は distributive である。
(2) 任意の異なる subscript $i, j \in I$ に対して、M_i と M_j は unrelated pair をなす。

適当な module X に対して module M が以下の条件を満たしているとき、M を *X-cyclic* と呼ぶ。

> $\sum_{\alpha \in \mathrm{Hom}_A(X,M)} \mathrm{Im}\alpha = M$ であるとき、epimorphism $\alpha \in \mathrm{Hom}_A(X, M)$ が存在する。

また、同様に以下の条件を満たしているとき、M を *X-cocyclic* と呼ぶ。

> $\bigcap_{\beta \in \mathrm{Hom}_A(M,X)} \mathrm{Ker}\beta = 0$ であるとき、monomorphism $\beta \in \mathrm{Hom}_A(M, X)$ が存在する。

明らかに "A-cyclic" と "cyclic" は互いに同値な条件である。Section 2 において、X-cyclic module と X-cocyclic module の基本的な性質について考える。

module M を *semilocal* と呼ぶのは、factor module $M/\mathrm{Rad}(M)$ が semisimple のときである ($\mathrm{Rad}(M)$ は M の Jacobson radical)。Section 3 において、これらの概念を使って以下の theorem を得る。これは [Vam78] の主結果の一般化となるもの、また、双対となるものを含んでいる。後者は [Vam78] で提出された問題の現代的な解と考えることが出来る。

Theorem 1. *distributive module M* に対して、以下が成り立つ。

(1) M は、*finitely generated semilocal* であれば、任意の *projective module P* に対して *P-cyclic* である。
(2) M は、*finitely cogenerated* であれば、任意の *injective module Q* に対して *Q-cocyclic* である。

この報告書の目的は、上記 theorem を証明して実際に結果を得ることである。

2 X-cyclic module と X-cocyclic module

まず初めに以下の 2 つの lemma を示す。これらの結果は比較的容易に導くことが出来るだろう。

Lemma 2. *2 つの non-zero module M* と *X* に対して、以下の条件は互いに同値である。

(1) $\sum_{\alpha \in \mathrm{Hom}_A(X,M)} \mathrm{Im}\alpha \neq M$
(2) 全ての $\alpha \in \mathrm{Hom}_A(X,M)$ に対して $\alpha\kappa = 0$ を満たすような、*non-zero module* V と *non-zero homomorphism* $\kappa \in \mathrm{Hom}_A(M,V)$ が存在する。
(3) $\mathrm{Hom}_A(1_X,\varphi) : \mathrm{Hom}_A(X,V) \to \mathrm{Hom}_A(X,M)$; $\gamma \mapsto \gamma\varphi$ が *surjective* となるような、*module* V と *surjective* でない *homomorphism* $\varphi \in \mathrm{Hom}_A(V,M)$ が存在する。

Lemma 3. *2* つの *non-zero module* M と X に対して、以下の条件は互いに同値である。

(1) $\bigcap_{\beta \in \mathrm{Hom}_A(M,X)} \mathrm{Ker}\beta \neq 0$
(2) 全ての $\beta \in \mathrm{Hom}_A(M,X)$ に対して $\lambda\beta = 0$ を満たすような、*non-zero module* W と *non-zero homomorphism* $\lambda \in \mathrm{Hom}_A(W,M)$ が存在する。
(3) $\mathrm{Hom}_A(\psi,1_X) : \mathrm{Hom}_A(W,X) \to \mathrm{Hom}_A(M,X)$; $\delta \mapsto \psi\delta$ が *surjective* となるような、*module* W と *injective* でない *homomorphism* $\psi \in \mathrm{Hom}_A(M,W)$ が存在する。

また、これらの lemma と関連して、以下の 2 つ proposition が分かる。

Proposition 4. *2* つの *non-zero module* M と X に対して、以下の条件は互いに同値である。

(1) M は X*-cyclic* である。
(2) M と X は以下のどちらか一方の条件を満たす。

- *epimorphism* $\alpha_0 \in \mathrm{Hom}_A(X,M)$ が存在する。
- M と X は *Lemma 2* の条件を満たす。

Proposition 5. *2* つの *non-zero module* M と X に対して、以下の条件は互いに同値である。

(1) M は X*-cocyclic* である。
(2) M と X は以下のどちらか一方の条件を満たす。

- *monomorphism* $\beta_0 \in \mathrm{Hom}_A(M,X)$ が存在する。
- M と X は *Lemma 3* の条件を満たす。

以上の結果から、以下に示す (抽象的な) 例が得られる。これは "cyclic" と "X-cyclic" の違いの一部を示すものと考えることが出来る。

Example 1. *3* つ *simple module* M_1, M_2, M_3 について考える。このとき、M_1 と M_2 は *isomorphic* であり、M_3 はそれらと *isomorphic* でない

ならば、以下が成り立つ。ただし、M を M_1, M_2, M_3 の *direct sum* と、N を M_1, M_2 の *direct sum* とする。

(1) M は M_1-*cyclic* であり、N は M の *epimorphic image* であるが、N は M_1-*cyclic* ではない。
(2) M は M_1-*cocyclic* であり、N は M の *submodule* であるが、N は M_1-*cocyclic* ではない。

この章の最後に、以下の remark も示しておく。

Remark 3. 一般的な条件として以下が成り立つ。

(1) 全ての module は 0-cyclic であり 0-cocyclic である。
(2) simple module と 0 は任意の module X に対して X-cyclic であり X-cocyclic である。
(3) 全ての local module は任意の module X に対して X-cyclic である。
(4) 全ての colocal module は任意の module X に対して X-cocyclic である。

3 Theorem 1 の証明について

章の初めに以下の remark ([Ste74, Proposition 1.2] と同等のもの) について触れておこう。

Remark 4. module M を module M_i $(i \in I)$ の direct sum とする。このとき、以下の条件は互いに同値である。

(1) 全ての異なる subscript $i, j \in I$ に対して、M_i と M_j は unrelated pair をなす。
(2) $H \cap K = \emptyset$ を満たす I の任意の finite subset H, K に対して、$\bigoplus_{h \in H} M_h$ と $\bigoplus_{k \in K} M_k$ は unrelated pair をなす。
(3) M の任意の submodule N に対して、$N = \bigoplus_{i \in I}(N \cap M_i)$ が成立する。

この remark から、以下の proposition を得ることが出来る。

Proposition 6. *2 つの module* X と $M = \bigoplus_{i=1}^{h} M_i$ について考える。任意の異なる *subscript* $i, j \in \{1, \ldots, h\}$ に対して M_i と M_j が *unrelated pair* をなすならば、以下が成り立つ。

(1) 全ての *subscript* $i \in \{1, \ldots, h\}$ に対して *epimorphism* $\alpha_i \in \mathrm{Hom}_A(X, M_i)$ が存在するならば、*epimorphism* $\alpha \in \mathrm{Hom}_A(X, M)$ が存在する。
(2) 全ての *subscript* $i \in \{1, \ldots, h\}$ に対して *monomorphism* $\beta_i \in \mathrm{Hom}_A(M_i, X)$ が存在するならば、*monomorphism* $\beta \in \mathrm{Hom}_A(M, X)$ が存在する。

この結果から、以下の corollary を得ることが出来る。

Corollary 7. *2 つの module X と $M = \bigoplus_{i=1}^{h} M_i$ について考える。任意の異なる subscript $i,j \in \{1,\dots,h\}$ に対して M_i と M_j が unrelated pair をなすならば、以下が成り立つ。*

(1) 任意の *subscript* $i \in \{1,\dots,h\}$ に対して M_i が X-*cyclic* であるならば、M も X-*cyclic* である。
(2) 任意の *subscript* $i \in \{1,\dots,h\}$ に対して M_i が X-*cocyclic* であるならば、M も X-*cocyclic* である。

更に以下の remark を示す。

Remark 5. P を projective module、M を module、N を M の small submodule とする。このとき、epimorphism $\alpha \in \mathrm{Hom}_A(P, M/N)$ が存在するならば、$\alpha = \alpha'\pi$ (ただし π は M から M/N への natural epimorphism) となる epimorphism $\alpha' \in \mathrm{Hom}_A(P, M)$ が存在する。

Remark 6. Q を injection module、M を module、N を M の essential submodule とする。このとき、monomorphism $\beta \in \mathrm{Hom}_A(N, Q)$ が存在するならば、$\beta = \iota\beta'$ (ただし ι は N から M への canonical injection) となる monomorphism $\beta' \in \mathrm{Hom}_A(M, Q)$ が存在する。

この remark から、以下の lemma を得ることが出来る。

Lemma 8. *P を projective module、M を module、N を M の small submodule とする。このとき、M が P-cyclic であることと M/N が P-cyclic であることは互いに同値である。*

Lemma 9. *Q を injective module、M を module、N を M の essential submodule とする。このとき、M が Q-cocyclic であることと N が Q-cocyclic であることは互いに同値である。*

以上の結果から、以下の P-cyclic module と Q-cocyclic module に関する重要な proposition を得ることが出来る。

Proposition 10. *P を projective module、M を module、N を M の small submodule とする。このとき、以下の条件は互いに同値である。*

(1) M は P-*cyclic* である。
(2) 以下の 2 つの条件を満たすような *decomposition* $M/N = \bigoplus_{i=1}^{h} M_i/N$ が存在する。
    - 異なる任意の *subscript* $i,j \in \{1,\dots,h\}$ に対して、M_i/N と M_j/N は *unrelated pair* をなす。

- 全ての $M_1/N, \ldots, M_h/N$ は *P-cyclic* である。

Proposition 11. Q を *injective module*、M を *module*、N を M の *essential submodule* とする。このとき、以下の条件は互いに同値である。

(1) M は *Q-cocyclic* である。
(2) 以下の 2 つの条件を満たすような *decomposition* $N = \bigoplus_{i=1}^{h} N_i$ が存在する。
   - 異なる任意の *subscript* $i, j \in \{1, \ldots, h\}$ に対して、N_i と N_j は *unrelated pair* をなす。
   - 全ての $N_1, \ldots, N_h$ は *Q-cocyclic* である。

ここまでの結果から、Theorem 1 を得ることが出来る。

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GLOBAL DIMENSION IN LEFT SERIAL ALGEBRAS

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Let A be a finite dimentional basic conected algebra over an algebraically closed field k, and n is the number of the non isomorphic simple right modules of A. If A is serial and $gl.dim.A$ is finite, then $gl.dim.A \leq 2n-2$ and $l(A) \leq 2n-1$ [3].

If A is quasi-hereditary, then the global dimension $gl.dim.A$ is less than or equal to $2n-2$, and the Loewy length $l(A)$ of A is less than or equal to $2^n - 1$[4]. If A is serial and $gl.dim.A$ is finite, then $gl.dim.A \leq 2n-2$ and $l(A) \leq 2n-1$. On the other hand, Yamagata constructed the algebras of large global dimension with few simple modules [5], and Deng constructed the algebras of arbitrary finite global dimension with $n \geq 2$ arbitrary [1]. So, if we consider the relationship between $gl.dim.A$ and n, we need some conditions on algebras. In this note we show that the algebra whose quiver contains unique oriented cycle has global dimension less than or equal to $2n-2$ if it has finite global dimension. Moreover, in case of quasi-hereditary algebras, we compute the global dimension of an algebra whose quiver contains some essentital oriented cycles.

1. The algebra whose quiver contains at most one oriented cycle

First we fix some notations. Let Q_A be the quiver of A, $\{1, 2, \cdots, n\}$ be its vertices, and $\{e_1, \cdots, e_n\}$ be the set of corresponding primitive idempotents of A.

A vertex i of Q_A is called a sink(resp., source) vertex if all arrows which contain i have i as ending(resp., starting) point. If vertex i is a sink vertex, then Ae_i is a simple projective module. Dually, if vertex i is a source vertex, then e_iA is a simple injective module.

Lemma 1.1. *Suppose that Q_A has a sink or source vertex, that is there is a primitive idempotent e of A with simple projective module Ae or simple injective module eA. Let $B = A/AeA$. Then*
$gl.dim.A \leq gl.dim.B + 1$,
$l(A) \leq l(B) + 1$,
and these bounds are sharp.

Proof. The second inequality is trivial. For the first, assume that Ae is simple projective. Since AeA is direct sum of Ae, ${}_AAeA$ is projective as left A-module. Let ${}_AX$ be an arbitary left A-module. If ${}_AAeAX \neq 0$, $AeAX$ is direct sum of Ae and projective. Hence, $proj.dim._AX \leq proj.dim._A(X/AeAX)$. If $AeAX = 0$, ${}_AX$ is a B-module. It is enough to show that if X is a B-module, then

$$proj.dim._AX \leq 1 + proj.dim._BX.$$

This is the same as the Statement 1 of [4].

In case of eA is simple injective, we can show by duality. □

Remark 1.1. In the above Lemma, the word "simple" can be replaced by "semi simple".

Lemma 1.2. *If A is an algebra whose quiver has no oriented cycles then $gl.dim.A \leq n-1$.*

Proof. If there are no sink vertices, each vertex is the starting point of some arrow. Then there exists an infinite chain of arrows and this forms an oriented cycle.

So we may assume that n is a sink vertex. Set $B = A/Ae_nA$. Since B has no oriented cycles, using Lemma 1.1 we complete the proof. □

Example 1. In the above lemma, the bound of global dimension is sharp. Let A be an algebra defined by the following quiver with relations $\{\alpha_{i+1}\alpha_i = 0 (1 \le i \le n-2)\}$.

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} n$$

Then the global dimension of A is $n - 1$.

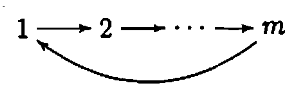
Next theorem is due to Gustafson[3].

Theorem 1.3 (Gustafson). *Let A be a serial algebra. If the global dimension of A is finite, then*
$gl.dim.A \le 2n - 2$,
$l(A) \le 2n - 1$,
and these bounds are sharp.

Serial algebras have at most one oriented cycle. So, next theorem is generalization of Theorem 1.3.

Theorem 1.4. *Let A be an algebra whose quiver contains at most one oriented cycle. If the global dimension of A is finite, then*
$gl.dim.A \le 2n - 2$,
$l(A) \le 2n - 1$,
and these bounds are sharp.

Proof. Using Lemma 1.2, we may assume Q_A has unique oriented cycle. If Q_A has a sink or source vertex, take the corresponding idempotent e and make new algebra $B = A/AeA$. Then Q_B contains one oriented cycle and by the Lemma 1.1 $gl.dim.A \le gl.dim.B + 1$. We continue this process until we get the algebra U that has no simple projective nor simple injective module. Then Q_U is the following shape.



U must be a serial algebra. So the Theorem 1.3, $gl.dim.U \le 2m - 2$ and $l(U) \le 2m - 1$ where m is the number of the non isomorphic simple left modules of U. Using Lemma 1 repeatedly, we can prove the theorem.

For the sharpness of bounds, these are the same as the Gustafson's Example[3]. □

If A is a left or right (not necessarily both) serial algebra and has finite global dimension, the condition of the theorem holds. So the following corollary is the direct consequence of the Theorem 1.4.

Corollary 1.5. *Let A be a left or right serial algebra. If the global dimension of A is finite, then*
$gl.dim.A \le 2n - 2$,
$l(A) \le 2n - 1$,
and these bounds are sharp.

More over if one of the equality holds, A is a serial algebra.

2. Quasi-hereditary algebras with some oriented cycles

Let N be the Jacobson radical of A. An idempotent e is said to be a heredity idempotent of A if $eNe = 0$ and AeA is projective as a right A-module.

Let $\{e_1, e_2, \cdots, e_n\}$ be fixed ordering of the complete set of primitive orthogonal idempotents of A. An algebra A is said to be quasi-hereditary algebra with respect to this ordering if for any $1 \leq t \leq n$, $\overline{\varepsilon_t}$ is a heredity idempotent of $A/A\varepsilon_{t+1}A$, where $\varepsilon_j = e_j + e_{j+1} + \cdots + e_n$ for $1 \leq j \leq n$ and $\varepsilon_{n+1} = 0$. Such a sequence called a heredity sequence.

The following lemma is due to Dlab and Ringel[4].

Lemma 2.1. *Suppose that A has a heredity idempotent e. Let $B = A/AeA$. Then*
$gl.dim.A \leq gl.dim.B + 2$
$l(A) \leq 2l(B) + 1$.

We call an oriented cycle is essential when it is made by the distinct vertices.

Theorem 2.2. *Let A be a quasi-hereditary algebra with m essential oriented cycles. Then*
$gl.dim.A \leq n + m - 1$,
$l(A) \leq 2^m(n - m + 1)$,
and these bouds are sharp.

Proof. If Ae_n is simple projective or e_nA is simple injective, we use Lemma 1.1 and consider the algebra $B = A/Ae_nA$. Then B is again quasi-hereditary algebra with respect to $\{e_1, e_2, \cdots, e_{n-1}\}$ which has same number of essentital oriented cycles. Otherwise, if n is not sink nor source vertex, then n is belong to some essential oriented cycle. In this case, we use Lemma 2.1 and consider the algebra $B = A/Ae_nA$. B is again quasi-hereditary algebra with respect to $\{e_1, e_2, \cdots, e_{n-1}\}$ that has $m - 1$ or less essential oriented cycles.

We continue these processes. Second case does occur at most m times. So we conclude that

$$gl.dim.A \leq 2m + (n - m) = n + m - 1,$$
$$l(A) \leq 2^m(n - m + 1).$$

□

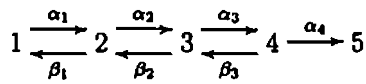
Remark 2.1. In the above theorem, we may assume $m \leq n - 1$. Since otherwise these bounds exceed $2n - 2$ and $2^n - 1$ respectively.

Corollary 2.3. *Let A be a left serial quasi-hereditary algebra. Then $gl.dim.A \leq n$ and $l(A) \leq 2n - 1$, and these bounds are sharp.*

Example 2. Let A be a serial algebra of second type with relation $\{\alpha_{i+1}\alpha_i = 0 (1 \leq i \leq n-2), \alpha_1\alpha_n = 0\}$. Where $n \geq 3$ and α_i is an arrow from i to $i + 1$ $(1 \leq i \leq n - 1)$, and α_n is an arrow from n to 1. Then S_n (the simple module corresponding to the vertex n) is of projective dimension n and the other simple modules are of projective dimension less than n. So $gl.dim.A = n$. A is quasi-hereditary with heredity idempotent e_n.

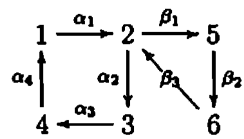
Example 3 (Gustafson). Let A be a serial algebra of second type with relation $\{\alpha_1\alpha_2 \cdots \alpha_n = 0\}$ Then S_i is of projective dimension one, for $2 \leq i \leq n$, while S_1 is of projective dimension 2. Hence, $gl.dim.A = 2$, and A is quasi-hereditary. The length of Ae_2 is $2n - 1$, and this is the maximal length among the indecomposable projective modules of A. We have $l(A) = max\{l(e_iA)\} = 2n - 1$.

Example 4. Let A be the algebra of



with relation $\{\alpha_4\alpha_3 = \alpha_3\alpha_2 = \alpha_2\alpha_1 = \beta_3\beta_2 = \beta_2\beta_1 = 0, \alpha_1\beta_1 = \beta_2\alpha_2, \alpha_2\beta_2 = \beta_3\alpha_3\}$. Then A is a quasi-hereditary algebra with respect to $\{e_1, e_2, e_3, e_4, e_5\}$ and has 3 essential oriented cycles. Ae_5 is simple projective module. This is the case of $n = 5$ and $m = 3$ in Theorem 2.2, and $gl.dim.A = 7$. This shows that the bound of global dimension is sharp.

Example 5. Let A be the algebra of



with relation $\{\alpha_4\alpha_3 = \alpha_3\alpha_2 = \alpha_1\alpha_4 = 0, \beta_2\beta_1 = \beta_1\beta_3 = 0, \beta_1\alpha_1 = \alpha_2\beta_3 = 0\}$. Then A is a quasi-hereditary algebra with respect to $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ and has 2 essential oriented cycles. Ae_6 is not simple. Projective dimension of S_6 is 7. Indeed,

$$P_6 \longrightarrow P_5 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_4 \oplus P_6 \longrightarrow P_3 \oplus P_5 \longrightarrow P_2 \longrightarrow P_6 \longrightarrow S_6 \longrightarrow 0$$

is the minimal projective resolution of S_6. Where P_i and S_i are coressponding indecomposable projective module Ae_i and simple left module Ae_i/Ne_i respectively. This is maxmal among S_i, so $gl.dim.A = 7$. This is the case of $n = 6$ and $m = 2$ in Theorem 2.2, and $gl.dim.A = 7 = 6 + 2 - 1$ is maximal.

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Derivations of polynomial rings over a field of characteristic zero

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Let k be a commutative ring containing $\mathbb{Q}$ and let A be a commutative k-algebra containing k. A k-linear mapping $d : A \to A$ is called a *k-derivation* of A if it satisfies the Leibniz rule: $d(ab) = ad(b) + bd(a)$.

Let d be a k-derivation of A. W denote by A^d the kernel of d, that is,

$$A^d = \text{Ker d} = \{a \in A;\ d(a) = 0\}.$$

This set is a k-subalgebra of A which we call the *ring of constants* of d. If A is a domain and k is a field, then we denote by A_0 the field of quotients of A and we denote also by d the unique extension of d to A_0. In this case A_0^d is a subfield of A_0 containing k.

Let D be a family of k-derivations of A. Then we have the ring of constants $A^D = \bigcap_{d \in D} A^d = \{a \in A;\ d(a) = 0 \ \text{ for all } \ d \in D\}$. We will mostly consider derivations of a polynomial ring in a finite set of variables. In such a case the rings of the form A^D are not interesting for us. It is known ([23], [22]) that in this case every ring A^D is of the form A^d for some k-derivation d of A.

If d is a derivation of A, then we denote by $\text{Nil}(d)$ the following subset of A:

$$\text{Nil}(d) := \{a \in A;\ \exists_{n \geqslant 0}\ d^n(a) = 0\}.$$

This subset is also a k-subalgebra of A and we have: $k \subseteq A^d \subseteq \text{Nil}(d) \subseteq A$. We say that a derivation d is *locally nilpotent* if $A = \text{Nil}(d)$.

Assume now that $A = k[X] := k[x_1, \ldots, x_n]$ is the polynomial ring over k. In this case we know a description of all k-derivations of A. If d is a k-derivation of $k[X]$, then we have the polynomials $f_1 := d(x_1), \ldots, f_n := d(x_n)$, belonging to $k[X]$, and then

$$d = f_1 \frac{\partial}{\partial x_1} + \cdots + f_n \frac{\partial}{\partial x_n}.$$

Every k-derivation d of $k[X]$ is uniquely determined by a sequence $(f_1, \ldots, f_n)$ of polynomials from $k[X]$. If d is a k-derivation of $k[X]$, then d is locally nilpotent if and only if $x_1, \ldots, x_n \in \mathrm{Nil}(d)$.

Locally nilpotent derivations of polynomial rings play an important role in algebra and algebraic geometry. It is well known that many open famous problems may be formulated using derivations or locally nilpotent derivations and their rings of constants. On the list of such problems are: the cancellation problem, the embedding problem, the linearization problem, the Jacobian conjecture, tame generator conjecture, the fourteenth problem of Hilbert and others (see, for example, [17], [12], [10]). On this lecture we present some old and new results concerning the fourteenth problem of Hilbert.

Let k be a field of characteristic zero, $n \geqslant 1$, $k[X] := k[x_1, \ldots, x_n]$ the polynomial ring over k, and $k(X) := k(x_1, \ldots, x_n)$ the field of rational functions over k. Assume that L is a subfield of $k(X)$ containing k. The fourteenth problem of Hilbert is the following question ([21]).

Is the ring $L \cap k[X]$ finitely generated over k?

In 1954 Zariski ([29]) proved that the answer is affirmative if $\mathrm{tr.deg}_k L \leqslant 2$. It is known, by a famous counterexample of Nagata ([21]), that if $\mathrm{tr.deg}_k L \geqslant 4$, then it is possible to obtain a negative answer. If $\mathrm{tr.deg}_k L = 3$, then the problem is still open.

Assume that d is a k-derivation of $k[X]$. Then we have the field $L := \left(k[X]^d\right)_0$, the field of quotients of the constant ring $k[X]^d$, which is a subfield of $k(X)$ containing k. The intersection $L \cap k[X]$ is equal to $k[X]^d$. So, in this case, the fourteenth problem of Hilbert is the following question.

Is the ring $k[X]^d$ finitely generated over k?

In this question k is of characteristic zero. If $\mathrm{char}(k) > 0$, then it is easy to show ([25]) that the answer is affirmative. So, let again $\mathrm{char}(k) = 0$. Using the above mentioned result of Zariski it is not difficult to show (see [25]) that if $n \leqslant 3$, then the answer is affirmative. What does happen for $n \geqslant 4$?

In 1993 Derksen ([7]) showed that the ring from the Nagata counterexample is of the form $k[X]^d$ for some derivation d of $k[X]$ with $n = 32$. Thus, he proved:

Theorem 1 (Derksen). *There exists a k-derivation d of $k[X] := k[x_1, \ldots, x_{32}]$ such that $k[X]^d$ is not finitely generated over k.*

If G is a subgroup of $\mathrm{Aut}_k(k[X])$, the group of all k-automorphisms of $k[X]$, then we denote by $k[X]^G$ the subalgebra of invariants of G, that is,

$$k[X]^G = \{f \in k[X];\ \sigma(f) = f \text{ for all } \sigma \in G\}.$$

In 1994 the author, inspired by the above Derksen theorem, proved:

Theorem 2. *If $G \subseteq \mathrm{GL}_n(k)$ is a connected algebraic group, then there exists a k derivation d of $k[X]$ such that $k[X]^G = k[X]^d$.*

In 1990 Roberts ([27]) gave a new counterexample to the fourteenth problem of Hilbert with $n = 7$. In 1994 Deveney and Finston ([9]) realized the Roberts counterexample in the following form.

Theorem 3. *Let d be the k-derivation of $k[X] := k[x_1, x_2, x_3,\ y_1, y_2, y_3, y_4]$ defined by $d(x_1) = d(x_2) = d(x_3) = 0$ and*

$$d(y_1) = x_1^3, \quad d(y_2) = x_2^3, \quad d(y_3) = x_3^3, \quad d(y_4) = (x_1x_2x_3)^2.$$

Then the ring $k[X]^d$ is not finitely generated over k.

Using this theorem one can easily deduce that if $n \geqslant 7$, then there always exists a k-derivation d of $k[X]$ such that the ring of constants $k[X]^d$ is not finitely generated over k.

Observe that the derivation d from the above theorem is locally nilpotent. This derivation has no slice. We say that a locally nilpotent k-derivation d of a k-algebra A has a *slice* if there exists an element $s \in A$ such that $d(s) = 1$. It is well known (see for example [12] or [22]) that if A is a finitely generated k-algebra and d is a locally nilpotent k-derivation of A having a slice, then the ring of constants A^d is finitely generated over k.

Similar examples of derivations for $n \geqslant 7$ one can find, for instance, in [16] and [13]. Later, in 1998, Freudenburg ([14]) constructed a locally nilpotent derivation with the same property for $n = 6$.

Theorem 4 (Freudenburg). *Let d be the k-derivation of $k[X] := k[x_1, x_2, y_1, y_2, y_3,\ y_4]$ defined by $d(x_1) = d(x_2) = 0$ and*

$$d(y_1) = x_1^3, \quad d(y_2) = x_2^3y_1, \quad d(y_3) = x_2^3y_2, \quad d(y_4) = x_1^2x_2^2.$$

Then the ring $k[X]^d$ is not finitely generated over k.

In 1999, Daigle and Freudenburg ([4]) gave a similar example with $n = 5$.

Theorem 5 (Daigle and Freudenburg). *Let d be the k-derivation of the polynomial ring $k[X] := k[a, b, x, y, z]$ defined by $d(a) = d(b) = 0$ and*

$$d(x) = a^2, \quad d(y) = ax + b, \quad d(z) = y.$$

Then the ring $k[X]^d$ is not finitely generated over k.

Observe that also in this case the derivation d is locally nilpotent. So, if $n \leqslant 3$, then $k[X]^d$ is always finitely generated over k, and if $n \geqslant 5$, then there exists a k-derivation (even locally nilpotent) of $k[X]$ with non-finitely generated ring of constants.

For $n = 4$ the problem is open. In this case there is no counterexample for arbitrary derivations and we do not know if $k[X]^d$ is finitely generated for locally nilpotent derivations. In the last case we know, by the result of Maubach ([19]), that $k[X]^d$ is finitely generated if d is locally nilpotent and the polynomials $d(x_1), d(x_2), d(x_3)$ and $d(x_4)$ are monomials. Recently Daigle and Freudenburg ([6]) proved that the ring of constants of any triangular derivation of $k[x_1, x_2, x_3, x_4]$ is finitely generated over k. We say that a derivation d of $k[X]$ is *triangular* if $d(x_i) \in k[x_1, \ldots, x_{i-1}]$ for all $i = 1, \ldots, n$. Every triangular derivation of $k[X]$ is of course locally nilpotent.

Let d be a k-derivation of $k[X] = k[x_1, \ldots, x_n]$, where k is a ring containing $\mathbb{Q}$. If $k[X]^d \neq k$ and $k[X]^d$ is finitely generated over k then we denote by $\gamma(d)$ the minimal number of polynomials in $k[X] \setminus k$ which generate $k[X]^d$ over k. Moreover, we assume that $\gamma(d) = 0$ iff $k[X]^d = k$, and $\gamma(d) = \infty$ iff $k[X]^d$ is not finitely generated over k. We already know from the previous section that there exist a natural number n and a k-derivation d of $k[X]$ such that $\gamma(d) = \infty$.

If k is not a domain, then there always exist k-derivations of $k[X]$ (even for $n = 1$) with non-finitely generated ring of constants. It follows from the following proposition which we may find in [12].

Proposition 6. *Assume that k contains two nonzero elements a and b such that $ab = 0$. Let d be the k-derivation of $k[X]$ defined by: $d(x_1) = ax_1$, $d(x_2) = 0, \ldots, d(x_n) = 0$. Then $\gamma(d) = \infty$.*
More exactly, $k[X]^d = k[x_2, x_3, \ldots, x_n, bx_1, bx_1^2, bx_1^3, \ldots]$.

If $d = 0$, then it is clear that $\gamma(d) = n$. If $n = 1$, k is a domain and $d \neq 0$, then of course $\gamma(d) = 0$. Now we assume that k is a domain, $d \neq 0$ and $n \geqslant 2$.

It is known ([28], [11]) that if k is a field then every Dedekind k-subalgebra of $k[X]$ is a polynomial ring in one variable over k. As a consequence of this fact we obtain

Theorem 7. *Let d be a nonzero k-derivation of $k[X]$, where k is a field of characteristic zero. If* $\text{tr.deg}_k(k[X]^d) \leqslant 1$, *then* $\gamma(d) \leqslant 1$.

It follows from the above theorem that if k is a field of characteristic zero and d is a nonzero k-derivation of $k[x,y]$, then $\gamma(d) \leqslant 1$. The same is true when k is a UFD.

Theorem 8 ([1]). *Let k be a* UFD *containing* $\mathbb{Z}$. *If d is a nonzero k-derivation of $k[x,y]$, then* $\gamma(d) \leqslant 1$.

If k is not a UFD (even if k is noetherian), then the following proposition shows that Theorem 8 does not always hold.

Proposition 9 (Berson). *Let* $k := \mathbb{C}[t^2, t^3]$. *Consider the k-derivation d of $k[x,y]$ defined by*

$$d(x) = t^3, \quad d(y) = -t^2.$$

Then k is not a UFD, and $k[x,y]^d$ is not finitely generated over k.

Observe that the derivation d from the above example is locally nilpotent. Similar examples of locally nilpotent derivations of $k[x,y]$ with non-finitely generated rings of constants one can find in [2].

Now let $n = 3$. We already know from if k is a field, then $\gamma(d) < \infty$. However in this case the number $\gamma(d)$ is unbounded. Strelcyn and the author ([26]) proved that if $n \geqslant 3$ and $r \geqslant 0$, then there exists a k-derivation d of $k[X]$ such that $\gamma(d) = r$.

In some cases the number $\gamma(d)$ is bounded. As a consequence of Theorem 8 we obtain that if k is a *UFD* containing $\mathbb{Z}$ and d is a nonzero k-derivation of $k[x,y,z]$ such that $d(x) = 0$, then $\gamma(d) \leqslant 2$. For locally nilpotent derivations over a field the number $\gamma(d)$ is also bounded.

Theorem 10 (Miyanishi [20]). *If d is a nonzero locally nilpotent derivation of $k[x, y, z]$, where k is a field of characteristic zero, then* $k[x,y,z]^d = k[f,g]$ *for some algebraically independent polynomials* $f, g \in k[x,y,z]$.

There exists also the following homogeneous version of this theorem.

Theorem 11 ([30], [3]). *Let d be a nonzero homogeneous locally nilpotent derivation of $k[x, y, z]$. Assume that the degrees of x, y, z are positive. Then* $k[x,y,z]^d = k[f,g]$, *for some homogeneous polynomials* $f, g \in k[x,y,z]$.

Note also the following recent two results for $n = 4$.

Theorem 12 ([8]). *If k is a field of characteristic zero and d is a nonzero triangular derivation of $k[x_1, x_2, x_3, x_4]$ with a slice, then $\gamma(d) = 3$.*

Theorem 13 ([5]). *For any integer $n \geqslant 3$ there exists a triangular derivation d of $k[x_1, x_2, x_3, x_4]$ such that $n \leqslant \gamma(d) \leqslant n+1$.*

Various facts and results concerning this subject we may find also in [12], [18], [15], [24], [22].

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STABLE EQUIVALENCES INDUCED FROM GENERALIZED TILTING MODULES

TAKAYOSHI WAKAMATSU

1. Introduction

In the paper [5], for a genuine tilting module ${}_BT_A$, H. Tachikawa and the author constructed a stably equivalent functor $\underline{\text{mod}}\text{-}T(A) \xrightarrow{\approx} \underline{\text{mod}}\text{-}T(B)$, where $T(A) = A \ltimes \mathrm{D}A$ and $T(B) = B \ltimes \mathrm{D}B$ denote the trivial extension algebras. Y. Miyashita [2] and the author [7] independently defined generalized tilting modules and, in the papers [8, 9], the author constructed a stably equivalent functor $\underline{\text{mod}}\text{-}T(A) \xrightarrow{\approx} \underline{\text{mod}}\text{-}T(B)$ for a generalized tilting module ${}_BT_A$ under some conditions, as a generalization of the above result with Tachikawa. M. Auslander and R. Buchbeitz [1] developed the theory of approximations of modules. To construct stably equivalent functors, the author also considered approximations of modules independently by different name. We call a faithfully balance bimodule ${}_BT_A$ a generalized tilting module if the condition $\mathrm{Ext}^n({}_BT, {}_BT) = 0 = \mathrm{Ext}^n(T_A, T_A)$ is satisfied for any $n > 0$. Miyashita's generalized tilting module is a generalized tilting module ${}_BT_A$ in our sense with a restriction $pd({}_BT), pd(T_A) < \infty$. J. Rickard [3] defined tilting complexes and developed the theory of triangulated equivalences of derived categories of module categories. He observed that the existence of stable equivalence $\underline{\text{mod}}\text{-}T(A) \xrightarrow{\approx} \underline{\text{mod}}\text{-}T(B)$ for a Miyashita' tilting module ${}_BT_A$ follows from his theory. However, there are many generalized tilting modules with infinite projective and injective dimension, hence, our construction of stable equivalences does not follow from the Rickard's theory in general. In the paper [10], the author proved that any symmetric algebra Λ with separable factor algebra $\Lambda/\operatorname{rad}\Lambda$ can be constructed as $\Lambda(\varphi, \psi)$ from an admissible system $({}_AM_A, \varphi, \psi)$ (see the next section for definitions of these) if it has no semisimple direct factors. The description of symmetric algebra as the form $\Lambda(\varphi, \psi)$ is a generalization of trivial extension algebra, because the algebra $\Lambda(\varphi, \psi)$ is identical with $A \ltimes \mathrm{D}A$ if $M = 0$. In that paper, we gave a way of transforming the algebra $\Lambda(\varphi, \psi)$ into $\Lambda(\varphi^T, \psi^T)$ by using generalized tilting modules ${}_BT_A$ and constructed stably equivalent functor $\underline{\text{mod}}\text{-}\Lambda(\varphi, \psi) \xrightarrow{\approx} \underline{\text{mod}}\text{-}\Lambda(\varphi^T, \psi^T)$, under quite restricted conditions. The purpose of this paper is to construct such equivalences under reasonable conditions.

Troughout this paper, all algebras and modules are finite dimensional over the ground field K. The duality functor $\mathrm{Hom}_K(?, K)$ is always denoted by D.

2. Symmetric Algebras

2.1. Let K be a field and A a finite dimensional K-algebra. We call a pair (φ, ψ) of A-bimodule homomorphisms

$$\varphi : {}_AM \otimes_A M_A \to {}_AM_A$$

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and

$$\psi : {}_A M \otimes_A M_A \to {}_A S_A$$

an admissible system for a quasi-Frobenius algebra if the following three conditions are satisfied:

(S-1) The homomorphism φ is associative and nilpotent.

(S-2) The homomorphism ψ is φ-associative in the sense that the equality

$$\psi(\varphi(m_1 \otimes m_2) \otimes m_3) = \psi(m_1 \otimes \varphi(m_2 \otimes m_3))$$

holds for any elements m_1, m_2, $m_3 \in M$.

(S-3) The module ${}_A S_A$ is an injective cogenerator and the homomorphism ψ is non-degenerate in the sense that one of the conditions $\psi(m \otimes M) = 0$ and $\psi(M \otimes m) = 0$ implies $m = 0$ for $m \in M$.

It is easy to check that the algebra $\Lambda(\varphi, \psi) = A \oplus M \oplus S$ becomes a quasi-Frobenius algebra with the multiplication

$$(a, m, s) \cdot (a', m', s') = (aa', am' + ma' + \varphi(m \otimes m'), as' + sa' + \psi(m \otimes m')).$$

2.2. An admissible system (φ, ψ) for a quasi-Frobenius algebra is called an admissible system for a symmetric algebra if the injective cogenerator ${}_A S_A$ is isomorphic to the module ${}_A \mathrm{D} A_A$ and the condition

(S-4) $\psi(m \otimes m')(1_A) = \psi(m' \otimes m)(1_A)$

is satisfied for any elements m, $m' \in M$.

Proposition 1. [10] *For an admissible system (φ, ψ) for a symmetric algebra, the corresponding algebra $\Lambda(\varphi, \psi) = A \oplus M \oplus \mathrm{D} A$ is always symmetric. Conversely, any symmetric algebra Λ with separable factor algebra $\Lambda / \operatorname{rad} \Lambda$ is of the form $\Lambda(\varphi, \psi)$ for a suitable admissible system (φ, ψ) whenever it has no semisimple direct factors.*

It should be noted that for a given symmetric algebra Λ there is a wide choice of subalgebras A and admissible systems (φ, ψ) over A to express as $\Lambda = \Lambda(\varphi, \psi)$ generally.

2.3. We next consider to transform the symmetric algebras $\Lambda(\varphi, \psi)$ by using bimodules. Let ${}_B T_A$ be a bimodule and $({}_A M_A; \varphi, \psi)$ an admissible system for a symmetric algebra. We put

$${}_B(M^T)_B = {}_B T \otimes_A Hom_A(T, M)_B$$

and define two homomorphisms

$$\varphi^T : {}_B(M^T \otimes_B M^T)_B \to {}_B(M^T)_B$$

and

$$\psi^T : {}_B(M^T \otimes_B M^T)_B \to {}_B D B_B$$

by

$$\varphi^T((t \otimes f) \otimes (t' \otimes f')) = t \otimes \varphi(f(t') \otimes f'(?))$$

and

$$\psi^T((t \otimes f) \otimes (t' \otimes f')) = \psi(f(t') \otimes f'(?t))(1_A),$$

respectively.

Proposition 2. *The following assertions are equivalent:*

(1) $(M^T; \varphi^T, \psi^T)$ *is an admissible system for a symmetric algebra over* B.

(2) *The homomorphism* ψ^T *is non-degenerate.*

(3) *The homomorphism* $\theta_{T,M} : M^T = T \otimes_A \mathrm{Hom}_A(T, M) \to \mathrm{Hom}_A(T, T \otimes_A M)$ *defined by* $\theta_{T,M}(t \otimes f) = (t \otimes f(?)) = (t' \mapsto t \otimes f(t'))$ *is bijective.*

The third condition in the proposition does not depend on the choice of the algebra homomorphism $B \to \mathrm{End}(T_A)$. Therefore, from an admissible system $(M; \varphi, \psi)$ satisfying the conditions, we can construct many symmetric algebras $\Lambda(\varphi^T, \psi^T)$ by using any algebra homomorphisms $B \to \mathrm{End}(T_A)$. Note that the conditions are always satisfied if the bimodule ${}_AM_A$ is of the form $\bigoplus_{L,R} {}_AL \otimes_K R_A$.

2.4. In order to define a stable equivalence between $\Lambda(\varphi, \psi)$ and $\Lambda(\varphi^T, \psi^T)$ for a tilting module ${}_BT_A$ satisfying some conditions, it will be necessary to describe modules over such symmetric algebras by using modules over smaller algebras. Let X be a right module over the symmetric algebra $\Lambda = \Lambda(\varphi, \psi)$. Then, since A is a subalgebra of Λ, X can be seen as a right module over A and we have two A-homomorphisms

$$\alpha : X \otimes_A M_A \to X_A$$

and

$$\beta : X \otimes_A \mathrm{D} A_A \to X_A$$

by $\alpha(x \otimes m) = x \cdot m$ and $\beta(x \otimes s) = x \cdot s$ for elements $x \in X, m \in M$ and $s \in \mathrm{D} A$. It is easy to check that the followiong condisions are satisfied:

(M-1) $\alpha \cdot (\alpha \otimes M) = \alpha \cdot (X \otimes \varphi) + \beta \cdot (X \otimes \psi) : X \otimes_A M \otimes_A M \to X$.

(M-2) $\beta \cdot (\beta \otimes \mathrm{D} A) = 0 : X \otimes_A \mathrm{D} A \otimes_A \mathrm{D} A \to X$.

(M-3) $\alpha \cdot (\beta \otimes M) = 0 : X \otimes_A \mathrm{D} A \otimes_A M \to X$.

(M-4) $\beta \cdot (\alpha \otimes \mathrm{D} A) = 0 : X \otimes_A M \otimes_A \mathrm{D} A \to X$.

Conversely, for any module X_A and two homomorphisms $\alpha : X \otimes_A M_A \to X_A$ and $\beta : X \otimes_A \mathrm{D} A_A \to X_A$ satisfying the above conditions, we can define a Λ-module structure on X by

$$x \cdot (a, m, s) = xa + \alpha(x \otimes m) + \beta(x \otimes s)$$

for elements $x \in X$ and $(a, m, s) \in \Lambda$. Therefore, we may identify a module $X_{\Lambda(\varphi,\psi)}$ with a module X_A which possesses two homomorphisms α and β satisfying the four conditions above. We call X_A the underlying module and two homomorphisms α and β the structure maps.

3. Generalized Tilting Modules

3.1. We call a bimodule ${}_BT_A$ a generalized tilting module if it satisfies the following two conditions:

(GT-1) ${}_BT_A$ is faithfully balanced, i.e. $B = \mathrm{End}(T_A)$ and $\mathrm{End}({}_BT) = A$.

(GT-2) $\mathrm{Ext}^n({}_BT, {}_BT) = 0 = \mathrm{Ext}^n(T_A, T_A)$ hold for all $n > 0$.

We call an exact sequence

$$0 \to X_A \to T_0 \to T_1 \to \cdots$$

a left dominant T-resolution of X_A provided

(L-1) $T_k \in \mathrm{add}(T_A)$ for any $k = 0, 1, \cdots$, and
(L-2) The functor $\mathrm{Hom}_A(?, T)$ preserves the exactness.

Dually, an exact sequence

$$\cdots \to T_1 \to T_0 \to X_A \to 0$$

is called a right dominant T-resolution of X_A if the conditions below are satisfied:
(R-1) $T_k \in \mathrm{add}(T_A)$ for any $k = 0, 1, \cdots$.
(R-2) The functor $\mathrm{Hom}_A(T, ?)$ preserves the exactness.

We denote by $\mathcal{C}og(T_A)$ the class of all A-modules with left dominant T-resolutions and, similarly, by $\mathcal{G}en(T_A)$ the class of all mudules with right dominant T-resolutions. Define the module classes $\mathcal{C}(T_A)$ and $\mathcal{D}(T_A)$ by

$$\mathcal{C}(T_A) = \bigcap_{n>0} \mathrm{Ker}\,\mathrm{Ext}^n(T_A, ?) \cap \mathcal{G}en(T_A), \quad \mathcal{D}(T_A) = \bigcap_{n>0} \mathrm{Ker}\,\mathrm{Ext}^n(?, T_A) \cap \mathcal{C}og(T_A).$$

Proposition 3. [7, 8, 9] *The functors* $\mathrm{Hom}({}_BT_A, ?)$ *and* $(? \otimes_B T_A)$ *induce an equivalence* $\mathcal{C}(T_A) \approx \mathcal{D}(\mathrm{D}\,T_B)$. *Dually, the functors* $\mathrm{Hom}({}_A\mathrm{D}\,T_B, ?)$ *and* $(? \otimes_A \mathrm{D}\,T_B)$ *induce an equivalence* $\mathcal{D}(T_A) \approx \mathcal{C}(\mathrm{D}\,T_B)$. *Moreover, defining the classes* $\mathcal{PC}$ *and* $\mathcal{IC}$ *for a module class* $\mathcal{C}$ *by* $\mathcal{PC} = \{W \mid \forall C \in \mathcal{C}, \mathrm{Ext}^{n>0}(W, C) = 0\}$ *and* $\mathcal{IC} = \{V \mid \forall C \in \mathcal{C}, \mathrm{Ext}^{n>0}(C, V) = 0\}$, *we have* $\mathcal{PC}(T_A) \subseteq \mathcal{D}(T_A)$, $\mathcal{ID}(T_A) \subseteq \mathcal{C}(T_A)$ *and* $\mathcal{IPC}(T_A) = \mathcal{C}(T_A)$, $\mathcal{PID}(T_A) = \mathcal{D}(T_A)$. *Further, we have the equivalences* $\mathcal{ID}(T_A) \approx \mathcal{PC}(\mathrm{D}\,T_B)$ *and* $\mathcal{PC}(T_A) \approx \mathcal{ID}(\mathrm{D}\,T_B)$ *by restricting the above equivalences.*

3.2. For an admissible system $(M; \varphi, \psi)$ over the algebra A and a generalized tilting module ${}_BT_A$, we want to know when the induced system (M^T, φ^T, ψ^T) becomes an admissible system over the algebra B.

Proposition 4. *Let* ${}_BT_A$ *be a generalized tilting module and* (φ, ψ) *an admissible system for a symmetric algebra defined over* ${}_AM_A$. *Assume that the modules* M_A *and* $T \otimes_A M_A$ *are members of the class* $\mathcal{C}(T_A)$. *Then, the following assertions are equivalent:*

(1) *The homomorphism* $\theta_{T,M} : T \otimes_A \mathrm{Hom}_A(T, M) \to \mathrm{Hom}_A(T, T \otimes_A M)$ *is bijective.*
(2) *The sequence*

$$\cdots \to \mathrm{Hom}_A(T, P_1 \otimes_A M) \to \mathrm{Hom}_A(T, P_0 \otimes_A M) \to \mathrm{Hom}_A(T, T \otimes_A M) \to 0$$

is exact, where $\cdots \to P_1 \to P_0 \to T \to 0$ *is a projective resolution of* T_A.
(3) $\Omega^n(T) \otimes_A M_A \in \mathcal{C}(T_A)$ *for all* $n > 0$.
(4) *The functor* $(? \otimes_A M_A)$ *sends the modules in the class* $\mathcal{PC}(T_A)$ *into* $\mathcal{C}(T_A)$.

In the case $pd(T_A) < \infty$, we can say more.

Proposition 5. *Let* ${}_BT_A$ *be a generalized tilting module with* $pd(T_A) < \infty$ *and* (φ, ψ) *an admissible system for a symmetric algebra over* ${}_AM_A$. *Then,* $M_A \in \mathcal{C}(T_A)$ *implies* $\Omega^n(T) \otimes_A M_A \in \mathcal{C}(T_A)$ *for all* $n \geq 0$. *Therefore, the condition* $T \otimes_A M_A \in \mathcal{C}(T_A)$ *and the bijectivity of the homomorphism* $\theta_{T,M}$ *follow automatically.*

3.3. **Approximations.** In order to define stable functors in the next section, we will need to use approximations of modules with respect to the tilting modules ${}_BT_A$ and ${}_A\mathrm{D}\,T_B$. To define the kernel functor Ker: $\underline{\text{mod}}$-$\Lambda \to \underline{\text{mod}}$-$\Gamma$, we will use $\mathcal{PC}(T_A)$-approximations of A-modules. Dually, to define the cokernel functor Cok : $\underline{\text{mod}}$-$\Gamma \to \underline{\text{mod}}$-$\Lambda$, we will use $\mathcal{ID}(\mathrm{D}\,T_B)$-approximations of B-modules.

For a module X_A, a homomorphism $W \xrightarrow{\gamma} X$ is called a right $\mathcal{PC}(T_A)$-approximation of X if (1) $W \in \mathcal{PC}(T_A)$ and (2) $\mathrm{Hom}(W', \gamma)$ is surjective for any module W' in the class $\mathcal{PC}(T_A)$. Since the class $\mathcal{PC}(T_A)$ contains a generator A_A, if X has a right $\mathcal{PC}(T_A)$-approximation homomorphism γ it must be surjective. Further, since $\mathcal{PC}(T_A)$ is closed under extensions, we have $\mathrm{Ker}\,\gamma \in \mathcal{C}(T_A) = \mathcal{IPC}(T_A)$. Conversely, from an exact sequence

$$0 \to V \to W \xrightarrow{\gamma} X \to 0$$

with $V \in \mathcal{C}(T_A)$ and $W \in \mathcal{PC}(T_A)$, we have a right $\mathcal{PC}(T_A)$-approximation γ of X as easily seen. We denote by

$$\mathrm{Cok}(\mathcal{C}(T_A), \mathcal{PC}(T_A))$$

the class of all modules X_A with right $\mathcal{PC}(T_A)$-approximations. The module class

$$\mathrm{Ker}(\mathcal{ID}(\mathrm{D}\,T_B), \mathcal{D}(\mathrm{D}\,T_B))$$

is defined in the dual manner. Hence, the class $\mathrm{Ker}(\mathcal{ID}(\mathrm{D}\,T_B), \mathcal{D}(\mathrm{D}\,T_B))$ consists of all modules Y_B for which there are exact sequences

$$0 \to Y \xrightarrow{\delta} S \to Q \to 0$$

with $S \in \mathcal{ID}(\mathrm{D}\,T_B)$ and $Q \in \mathcal{D}(\mathrm{D}\,T_B)$.

Proposition 6. [9] *The conditions*

$$\mathrm{Cok}(\mathcal{C}(T_A), \mathcal{PC}(T_A)) = \textit{mod-}A \quad \textit{and} \quad \mathrm{Ker}(\mathcal{ID}(\mathrm{D}\,T_B), \mathcal{D}(\mathrm{D}\,T_B)) = \textit{mod-}B$$

hold under the one of the assumptions below:

(1) *Both* $\mathrm{pd}({}_BT)$ *and* $\mathrm{pd}(T_A)$ *are finite.*
(2) A *or* B *is representation-finite.*

4. Construction of Stable Equivalences

4.1. Let $({}_AM_A, \varphi, \psi)$ be an admissible system for a symmetric algebra over A. Throughout this section, we assume that ${}_BT_A$ is a tilting module satisfying the following conditions:

(1) The equality $\mathrm{Cok}(\mathcal{C}(T_A), \mathcal{PC}(T_A)) = \text{mod-}A$ and $\mathrm{Ker}(\mathcal{ID}(\mathrm{D}\,T_B), \mathcal{D}(\mathrm{D}\,T_B)) = \text{mod-}B$ hold.
(2) The modules M_A and $T \otimes_A M_A$ are in the class $\mathcal{C}(T_A)$.
(3) The homomorphism $\theta_{T,M}$ is bijective.

Then, by Proposition 2, (M^T, φ^T, ψ^T) is also an admissible system for a symmetric algebra over B. We put $\Lambda = \Lambda(\varphi, \psi)$ and $\Gamma = \Lambda(\varphi^T, \psi^T)$. The main purpose of this paper is to show that the stable categories $\underline{\text{mod}}\text{-}\Lambda$ and $\underline{\text{mod}}\text{-}\Gamma$ are equivalent. Put ${}_BN_B = M^T$ and ${}_AU_B = \mathrm{Hom}_A(T, M) \cong \mathrm{Hom}_B(T, N)$.

Proposition 7. *The following are true:*

(1) ${}_BN$ *and* ${}_BN \otimes_B T$ *are in* $\mathcal{C}({}_BT)$.
(2) *The homomorphism*

$$\theta_{T,N} : \mathrm{Hom}_B(T, N) \otimes_B T \to \mathrm{Hom}_B(T, N \otimes_B T), \;\; g \otimes t \mapsto (t' \mapsto g(t') \otimes t)$$

is bijective.
(3) *The system* (M, φ, ψ) *is identical with* $((M^T)^T, (\varphi^T)^T, (\psi^T)^T)$.
(4) *The isomorphisms* ${}_AM_A \cong {}_AU \otimes_B T_A$ *and* ${}_BN_B \cong {}_BT \otimes_A U_B$ *hold.*

Therefore, we have

$$\Lambda = A \oplus (U \otimes_B T) \oplus (\mathrm{D}T \otimes_B T)$$

and

$$\Gamma = B \oplus (T \otimes_A U) \oplus (T \otimes_A \mathrm{D}T).$$

Then, from $\Gamma \otimes_B T = T \oplus (T \otimes_A U \otimes_B T) \oplus (T \otimes_A \mathrm{D}T \otimes_B T) = T \otimes_A \Lambda$, we get a bimodule

$$_\Gamma\Theta_\Lambda = {_\Gamma\Gamma} \otimes_B T = T \otimes_A \Lambda_\Lambda.$$

4.2. **Cokernel Functor.** Let $(Y_B; \sigma, \tau)$ be a Γ-module and

$$0 \to Y_B \xrightarrow{\delta} S(Y) \to Q(Y) \to 0$$

an exact sequence with $S(Y) \in \mathcal{ID}(\mathrm{D}T_B)$ and $Q(Y) \in \mathcal{D}(\mathrm{D}T_B)$. We define a Λ-homomorphism

$$\ell_Y : Y \otimes_B T = Y \otimes_\Gamma \Theta \to \mathrm{Hom}_B(\mathrm{D}T, S(Y)) \otimes_A \Lambda$$

by using the structure maps σ, τ and the $\mathcal{ID}(\mathrm{D}T_B)$-approximation homomorphism δ. We observe that the module $\mathrm{Hom}_B(\mathrm{D}T, S(Y)) \otimes_A \Lambda$ as an A-module is isomorphic to the direct sum of three modules $\mathrm{Hom}_B(\mathrm{D}T, S(Y))$, $\mathrm{Hom}_B(\mathrm{D}T, S(Y)) \otimes_A (U \otimes_B T)$ and $\mathrm{Hom}_B(\mathrm{D}T, S(Y)) \otimes_A (\mathrm{D}T \otimes_B T)$. Three components of the map ℓ_Y are defined as follows:

$$Y \otimes_B T \xrightarrow{\tau^*} \mathrm{Hom}_B(\mathrm{D}T, Y) \xrightarrow{\mathrm{Hom}(\mathrm{D}T,\delta)} \mathrm{Hom}_B(\mathrm{D}T, S(Y)),$$

$$Y \otimes_B T \xrightarrow{\delta \otimes T} S(Y) \otimes_B T \xrightarrow{\epsilon^{-1} \otimes T} \mathrm{Hom}_B(\mathrm{D}T, S(Y)) \otimes_A \mathrm{D}T \otimes_B T$$

and

$$Y \otimes_B T \xrightarrow{\sigma^*} \mathrm{Hom}_B(U, Y) \xrightarrow{\mathrm{Hom}(U,\delta)} \mathrm{Hom}_B(U, S(Y)) \xrightarrow{\approx} \mathrm{Hom}_B(\mathrm{D}T, S(Y)) \otimes_A U \otimes_B T,$$

where τ^* and σ^* are the adjoint maps of τ and σ and the isomorphism $\mathrm{Hom}_B(U, S(Y)) \xrightarrow{\approx} \mathrm{Hom}_B(\mathrm{D}T, S(Y)) \otimes_A U \otimes_B T$ is given by composing the isomorphisms

$$\mathrm{Hom}_B(\mathrm{D}T, S(Y)) \otimes_A \mathrm{Hom}_B(U, \mathrm{D}T) \to \mathrm{Hom}_B(U, S(Y)), \quad f \otimes g \mapsto f \cdot g$$

and $\mathrm{Hom}_B(\mathrm{D}T, S(Y)) \otimes_A \mathrm{Hom}_B(U, \mathrm{D}T) \cong \mathrm{Hom}_B(\mathrm{D}T, S(Y)) \otimes_A U \otimes_B T$.

Proposition 8. *The map $\ell_Y : Y \otimes_\Gamma \Theta \to \mathrm{Hom}_B(\mathrm{D}T, S(Y)) \otimes_A \Lambda$ is an injective homomorphism of Λ-modules.*

By the above result, we have a Λ-module $\mathrm{Cok}(Y) = \mathrm{Cok}(\ell_Y)$ from any Γ-module Y, by using its structure maps σ, τ and left $\mathcal{ID}(\mathrm{D}T_B)$-approximation homomorphism δ. It is checked that the correspondence $Y \mapsto \mathrm{Cok}(Y)$ defines a functor $\mathrm{Cok} : \text{mod-}\Gamma \to \underline{\text{mod}}\text{-}\Lambda$, which we call the cokernel functor. Further, we have

Proposition 9. *The functor* $\mathrm{Cok} : \textit{mod-}\Gamma \to \underline{\textit{mod}}\textit{-}\Lambda$ *induces a functor* $\underline{\textit{mod}}\textit{-}\Gamma \to \underline{\textit{mod}}\textit{-}\Lambda$.

4.3. **Kernel Functor.** The kernel functor Ker : $\underline{\mathrm{mod}}$-$\Lambda \to \underline{\mathrm{mod}}$-$\Gamma$ is defined in the dual manner. For a Λ-module (X_A, α, β), using an exact sequence

$$0 \to V(X) \to W(X) \xrightarrow{\gamma} X \to 0$$

with $V(X) \in \mathcal{C}(T_A)$ and $W(X) \in \mathcal{PC}(T_A)$, we first define a surjective homomorphism of Γ-modules

$$p_X : \mathrm{Hom}_B(\Gamma, W(X) \otimes_A \mathrm{D}T) \to \mathrm{Hom}_\Lambda(\Theta, X) = \mathrm{Hom}_A(T, X).$$

Three components of the map p_X is given by

$$\mathrm{Hom}_A(T, \mathrm{Hom}_B(\mathrm{D}T, W(X) \otimes_A \mathrm{D}T)) \xrightarrow{\mathrm{Hom}(T,\eta^{-1})} \mathrm{Hom}_A(T, W(X)) \xrightarrow{\mathrm{Hom}(T,\gamma)} \mathrm{Hom}_A(T, X),$$

$$W(X) \otimes_A \mathrm{D}T \xrightarrow{\gamma \otimes \mathrm{D}T} X \otimes_A \mathrm{D}T \xrightarrow{\beta^*} \mathrm{Hom}_A(T, X) \quad \cdot$$

and

$$\mathrm{D}\,\mathrm{Hom}_A(W(X), \mathrm{D}U) \cong W(X) \otimes_A U \xrightarrow{\gamma \otimes U} X \otimes_A U \xrightarrow{\alpha^*} \mathrm{Hom}_A(T, X),$$

where we used the isomorphisms

$$\mathrm{Hom}_B(T \otimes_A \mathrm{D}T, W(X) \otimes_A \mathrm{D}T) \cong \mathrm{Hom}_A(T, \mathrm{Hom}_B(\mathrm{D}T, W(X) \otimes_A \mathrm{D}T))$$

and

$$\omega : \mathrm{Hom}_B(T \otimes_A U, W(X) \otimes_A \mathrm{D}T) \cong \mathrm{D}\,\mathrm{Hom}_A(W(X), \mathrm{D}U).$$

The isomorphism ω is the composition map of the canonical isomorphism

$$\mathrm{Hom}_B(T \otimes_A U, W(X) \otimes_A \mathrm{D}T) \cong \mathrm{D}(\mathrm{Hom}_A(T, \mathrm{D}U) \otimes_B \mathrm{Hom}_A(W(X), T))$$

with the inverse of the dual of

$$\mathrm{Hom}_A(T, \mathrm{D}U) \otimes_B \mathrm{Hom}_A(W(X), T) \xrightarrow{\approx} \mathrm{Hom}_A(W(X), \mathrm{D}U) \quad (f \otimes g \mapsto f \cdot g).$$

Then, the correspondence $X \mapsto \mathrm{Ker}(X) = \mathrm{Ker}(p_X)$ defines a functor mod-$\Lambda \to \underline{\mathrm{mod}}$-$\Gamma$ and it induces the kernel functor $\underline{\mathrm{mod}}$-$\Lambda \to \underline{\mathrm{mod}}$-$\Gamma$ of stable categories.

4.4. **Stable Equivalence.** We have now two functors Cok : $\underline{\mathrm{mod}}$-$\Gamma \to \underline{\mathrm{mod}}$-$\Lambda$ and Ker : $\underline{\mathrm{mod}}$-$\Lambda \to \underline{\mathrm{mod}}$-$\Gamma$. In order to prove the equivalence $\underline{\mathrm{mod}}$-$\Lambda \approx \underline{\mathrm{mod}}$-$\Gamma$ by using those functors, we have to study $\mathcal{ID}(\mathrm{D}T_B)$-approximations of the B-modules of the form $\mathrm{Ker}(X)_B$ and also $\mathcal{PC}(T_A)$-approximations of the A-modules of the form $\mathrm{Cok}(Y)_A$. We will describe only $\mathcal{ID}(\mathrm{D}T_B)$-approximations of the modules $\mathrm{Ker}(X)$ since the arguments are symmetric. First, we have an exact sequence

$$0 \to \mathrm{Ker}(X) \to \mathrm{Hom}_A(T, W(X)) \oplus W(X) \otimes_A U \oplus W(X) \otimes_A \mathrm{D}T \to \mathrm{Hom}_A(T, X) \to 0.$$

Since $W(X) \in \mathcal{PC}(T_A) \subseteq \mathcal{C}og(T_A)$, we have a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & V(X) & \to & W(X) & \to & X & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & V(X) & \to & T(X) & \to & V_1(X) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & W_1(X) & = & W_1(X) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

with $W_1(X) \in \mathcal{PC}(T_A)$ and $V_1(X) \in \mathcal{C}(T_A)$. Therefore, we have an exact sequence

$$0 \to \mathrm{Hom}_A(T, W(X)) \to \mathrm{Hom}_A(T, X) \oplus \mathrm{Hom}_A(T, T(X)) \to \mathrm{Hom}_A(T, V_1(X)) \to 0$$

and get the

$$0 \to \mathrm{Ker}(X) \to \mathrm{Hom}_A(T, T(X)) \oplus W(X) \otimes_A U \oplus W(X) \otimes_A \mathrm{D}\,T \to \mathrm{Hom}_A(T, V(X)) \to 0.$$

From the $\mathcal{ID}(\mathrm{D}\,T_B)$-approximations

$$0 \to \mathrm{Hom}_A(T, T(X)) \to S(\mathrm{Hom}_A(T, T(X))) \to Q(\mathrm{Hom}_A(T, T(X))) \to 0$$

and

$$0 \to W(X) \otimes_A U \to S(W(X) \otimes_A U) \to Q(W(X) \otimes_A U) \to 0,$$

we obtain an exact sequence

$$0 \to \mathrm{Ker}(X) \to S(\mathrm{Hom}_A(T, T(X))) \oplus S(W(X) \otimes_A U) \oplus W(X) \otimes_A \mathrm{D}\,T \to Q \to 0$$

for which the sequence

$$0 \to \mathrm{Hom}_A(T, V(X)) \to Q \to Q(\mathrm{Hom}_A(T, T(X))) \oplus Q(W(X) \otimes_A U) \to 0$$

is exact. The left and right terms in the sequence are members of the class $\mathcal{D}(\mathrm{D}\,T_B)$ which is closed under extensions. Hence, the middle term Q_B is also in $\mathcal{D}(\mathrm{D}\,T_B)$ and, therefore, the momomorphism

$$\mathrm{Ker}(X) \to \mathrm{Hom}_A(T, W(X)) \oplus W(X) \otimes_A U \oplus W(X) \otimes_A \mathrm{D}\,T$$

is a left $\mathcal{ID}(\mathrm{D}\,T_B)$-approximation of the module $\mathrm{Ker}(X)_B$. Then, using the above approximation of $\mathrm{Ker}(X)$, we can calculate the module $\mathrm{Cok}(\mathrm{Ker}(X))$.

Proposition 10. *The module* $\mathrm{Cok}(\mathrm{Ker}(X))_\Lambda$ *is isomorphic to the direct sum of* X_Λ *with projective modules*

$$\mathrm{Hom}_B(\mathrm{D}\,T, S(\mathrm{Hom}_A(T, T(X)))) \otimes_A \Lambda_\Lambda \ \textit{and}\ \mathrm{Hom}_B(\mathrm{D}T, S(W(X) \otimes_A U)) \otimes_A \Lambda_\Lambda.$$

Conbining the above result with its dual, we have

Theorem 11. *Let* $({}_AM_A, \varphi, \psi)$ *be an admissible system for a symmetric algebra and* ${}_BT_A$ *a generalized tilting module with the properties*

(1) $\mathrm{Ker}(\mathcal{ID}(\mathrm{D}\,T_B), \mathcal{D}(\mathrm{D}\,T_B)) = \textit{mod-}B$, $\mathrm{Cok}(\mathcal{C}(T_A), \mathcal{PC}(T_A)) = \textit{mod-}A$,
(2) M_A, $T \otimes_A M_A \in \mathcal{C}(T_A)$, *and*
(3) $\theta_{T,M} : T \otimes_A \mathrm{Hom}(T, M) \to \mathrm{Hom}_A(T, T \otimes_A M)$ *is bijective.*

Then, for the symmetric algebras $\Lambda(\varphi, \psi)$ *and* $\Lambda(\varphi^T, \psi^T)$, *we have an equivalence of stable module categories* $\underline{\mathrm{mod}}\text{-}\Lambda(\varphi, \psi) \approx \underline{\mathrm{mod}}\text{-}\Lambda(\varphi^T, \psi^T)$.

5. Tilting Complexes

5.1. For a (generalized) tilting module ${}_BT_A$ with $pd({}_BT)$, $pd(T_A) < \infty$, J. Rickard [3] showed the eistence of derived equivalences $\mathcal{D}^b(A) \approx \mathcal{D}^b(B)$ and $\mathcal{D}^b(T(A)) \approx \mathcal{D}^b(T(B))$, where $T(A) = A \ltimes \mathrm{D}\,A$ and $T(B) = B \ltimes \mathrm{D}\,B$ stand for trivial extension algebras. He also proved that the existence of a derived equivalence $\mathcal{D}^b(\Lambda_1) \approx \mathcal{D}^b(\Lambda_2)$ implies the existence of a derived equivalence $\underline{\mathrm{mod}}\text{-}\Lambda_1 \approx \underline{\mathrm{mod}}\text{-}\Lambda_2$ for selfinjective algebras Λ_1 and Λ_2. Therefore, for a (generalized) tilting module ${}_BT_A$ with $pd({}_BT)$, $pd(T_A) < \infty$, the existence of a stable equivalence $\underline{\mathrm{mod}}\text{-}T(A) \approx \underline{\mathrm{mod}}\text{-}T(B)$ follows from his results stated above. In that paper, he actually proved that the projective resolution $P^\bullet$ of T_A becomes a tilting complex with

$B \cong \mathrm{End}_{\mathcal{D}^b(A)}(P^\bullet)$ and, similarly, the induced complex $P^\bullet \otimes_A T(A)$ becomes also a tilting complex with $T(B) \cong \mathrm{End}_{\mathcal{D}^b(T(A))}(P^\bullet \otimes_A T(A))$. In this section, we remark that those results are still true in the case of symetric algebras $\Lambda(\varphi, \psi)$ of admissible systems (φ, ψ).

5.2. For an algebra A, by A-modules we always mean finitely generated A-modules as before. Denote by $\mathcal{K}(A)$ the homotopy category of all complexes consisting of A-modules. For bounded complexes $P^\bullet$ and $Q^\bullet$ consisting of projective modules, we have

$$\mathrm{Hom}_{\mathcal{D}^b(A)}(P^\bullet, Q^\bullet) = \mathrm{Hom}_{\mathcal{K}(A)}(P^\bullet, Q^\bullet).$$

We start with a simple observation.

Proposition 12. *Let $P^\bullet$ and $Q^\bullet$ be bounded complexes consisting of projective A-modules and ${}_AV_A$ an A-bimodule. Then, we have an isomorphism*

$$\mathrm{Hom}_{\mathcal{K}(A)}(P^\bullet, Q^\bullet \otimes_A V) \cong \mathrm{D}\,\mathrm{Hom}_{\mathcal{K}(A)}(Q^\bullet, P^\bullet \otimes_A \mathrm{D}\, V)$$

which is natural on $P^\bullet$ and $Q^\bullet$.

Corollary 13. *Let A be a Frobenius algebra with Nakayama automorphism $\nu \in \mathrm{Aut}_K(A)$, i.e. ${}_AA_A \cong {}_\nu \mathrm{D}\, A_A$, and $P^\bullet$, $Q^\bullet$ bounded complexes consisting of projective modules. Then, we have $\mathrm{End}_{\mathcal{K}(A)}(P^\bullet, Q^\bullet) \cong \mathrm{Hom}_{\mathcal{K}(A)}(Q^\bullet, P^\bullet_\nu)$. Terefore, if A is symmetric then so is $\mathrm{End}_{\mathcal{K}(A)}(P^\bullet)$.*

By the above corollary, as endomorphism rings of bounded complexes consisting of projective modules, we can construct many symmetric algebras from a given symmetric algebra. Now, we back to the consideration of the induced complexes. Let $P^\bullet$ be a bounded complex consisting of projective A-modules. For an admissible system $({}_AM_A; \varphi, \psi)$, we have the induced system

$$({}_{\mathrm{End}_{\mathcal{K}(A)}(P^\bullet)}\mathrm{Hom}_{\mathcal{K}(A)}(P^\bullet, P^\bullet \otimes_A M)_{\mathrm{End}_{\mathcal{K}(A)}(P^\bullet)}; \varphi^{P^\bullet}, \psi^{P^\bullet}),$$

where the maps $\varphi^{P^\bullet}$ and $\psi^{P^\bullet}$ are defined as follows: First we define a natural map c from

$$\mathrm{Hom}_{\mathcal{K}(A)}(P^\bullet, P^\bullet \otimes_A M) \otimes_{\mathrm{End}_{\mathcal{K}(A)}(P^\bullet)} \mathrm{Hom}_{\mathcal{K}(A)}(P^\bullet, P^\bullet \otimes_A M)$$

to $\mathrm{Hom}_{\mathcal{K}(A)}(P^\bullet, P^\bullet \otimes_A M \otimes_A M)$ by the correspondence $f^\bullet \otimes g^\bullet \mapsto (f^\bullet \otimes M) \cdot g^\bullet$. Next, we have two morphisms $\varphi' = \mathrm{Hom}(P^\bullet P^\bullet \otimes \varphi)$ from $\mathrm{Hom}_{\mathcal{K}(A)}(P^\bullet, P^\bullet \otimes_A M \otimes_A M)$ to $\mathrm{Hom}_{\mathcal{K}(A)}(P^\bullet, P^\bullet \otimes_A M)$ and $\psi' = \mathrm{Hom}(P^\bullet, P^\bullet \otimes \psi)$ from $\mathrm{Hom}_{\mathcal{K}(A)}(P^\bullet, P^\bullet \otimes_A M \otimes_A M)$ to $\mathrm{Hom}_{\mathcal{K}(A)}(P^\bullet, P^\bullet \otimes_A \mathrm{D}\, A)$. We also have an isomorphism

$$\mu : \mathrm{Hom}_{\mathcal{K}(A)}(P^\bullet, P^\bullet \otimes_A \mathrm{D}\, A) \overset{\approx}{\to} \mathrm{D}\,\mathrm{End}_{\mathcal{K}(A)}(P^\bullet)$$

by Proposition 12. Finally, we put $\varphi^{P^\bullet} = \varphi' \cdot c$ and $\psi^{P^\bullet} = \mu \cdot \psi' \cdot c$.

Proposition 14. *The induced system $(\varphi^\bullet, \psi^\bullet)$ over $\mathrm{End}_{\mathcal{K}(A)}(P^\bullet)$ is an admissible system for a symmetric algebra and the isomorphism*

$$\Lambda(\varphi^{P^\bullet}, \psi^{P^\bullet}) \cong \mathrm{End}_{\mathcal{K}(\Lambda(\varphi,\psi))}(P^\bullet \otimes_A \Lambda(\varphi, \psi))$$

holds.

Let ${}_BT_A$ be a (generalized) tilting module with finite projective dimension on both side and $P^\bullet$ a projective resolution of T_A. Then $P^\bullet$ is a tilting comples over A. By modifying the proof of Rickard [3], we have

Proposition 15. *Assume that M_A is in the class $C(T_A)$. Then, the induced complex $P^\bullet \otimes_A \Lambda(\varphi, \psi)$ is a tilting complex again over $\Lambda(\varphi, \psi)$.*

Under the assumption of Proposition 15, it is easily checked that the isomorphism $\mathrm{Hom}_{\mathcal{K}(A)}(P^\bullet, P^\bullet \otimes_A M) \cong \mathrm{Hom}_A(T, T \otimes_A M)$ holds. In this case, we also have an isomorphism $\theta_{T,M} : T \otimes_A \mathrm{Hom}_A(T, M) \cong \mathrm{Hom}_A(T, T \otimes_A M)$. Then, easy verification shows that the system $(\varphi^{P^\bullet}, \psi^{P^\bullet})$ is identical with (φ^T, ψ^T). Therefore, we have an isomorphism

$$\Lambda(\varphi^T, \psi^T) \cong \mathrm{End}_{\mathcal{K}(\Lambda(\varphi,\psi))}(P^\bullet \otimes_A \Lambda(\varphi, \psi)).$$

In the paper [3], J. Rickrd also proved that any Brauer tree algebra can be transformed into a star algebra which is uniserial, by using tilting complexes. We observe here that such transformations are realized by applying our construction successively. In fact, any Brauer tree algebra can be represented as an algebra $\Lambda(\varphi, \psi)$ of an admissible system (φ, ψ) over an iterated tilted algebra A of type A_n in such a way that there is an APR-tilting module ${}_BT_A$ and the corresponding algebra $\Lambda(\varphi^T, \psi^T)$ is a Brauer tree algebra whose number of radical-generators outside of the exceptional cycle in the Brauer quiver is smaller than that of the original algebra $\Lambda(\varphi, \psi)$. Similar process is explained in the previous paper [6].

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Foundation of the Representation Theory of Artin Algebras, Using the Gabriel-Roiter Measure.

Claus Michael Ringel

1. The Setting.

Let Λ be an artin algebra (this means that Λ is an associative ring with 1, its center is a commutative artinian ring and Λ is finitely generated as a module over its center), we always may (and will) assume that Λ is connected (this means that the center is a local ring). Let $\operatorname{Mod}\Lambda$ denote the category of all (left) Λ-modules and $\operatorname{mod}\Lambda$ the full subcategory of all finitely generated modules. Usually, we will deal with finitely generated modules and call them just *modules*, given such a module M, we denote by $|M|$ its length (this is the length of any composition series, recall that this is an invariant of the module according to the Jordan-Hölder theorem).

Our interest concerns indecomposable modules: given an arbitrary, not necessarily finitely generated module M and submodules M_1, M_2 of M, then we write $M = M_1 \oplus M_2$ provided $M_1 \cap M_2 = 0$ and $M_1 + M_2 = M$ and call this a *direct decomposition* of M; we say that M is *indecomposable*, provided M is nonzero and the only direct decompositions $M = M_1 \oplus M_2$ are those with $M_1 = 0$ or $M_2 = 0$. Of course, any finitely generated Λ-module can be written as a finite direct sum of indecomposable modules, and such a decomposition is unique up to isomorphism (according to the Theorem of Krull-Remak-Schmidt); the reason for this uniqueness is the fact that any indecomposable module of finite length has a local endomorphism ring.

The main problem of representation theory is to find invariants for modules and to describe the isomorphism classes of all the indecomposable modules for which such an invariant takes a fixed value. A typical such invariant is the length of a module: the simple modules are those of length 1 (and there is just a finite number of such modules), the information concerning the indecomposable modules of length 2 is stored in the quiver (in case we deal with a finite dimensional algebra over some algebraically closed field) or the "species" of Λ. Given any invariant γ, as a first question one may look for values *of finite type:* these are those values v such that there are only finitely many isomorphisms classes of indecomposable modules M with $\gamma(M) = v$. The invariant to be discussed here is the Gabriel-Roiter measure.

The Gabriel-Roiter measure was introduced (under the name "Roiter measure") by Gabriel in [G] in order to clarify the intricate induction scheme used by Roiter [Ro] in his proof of the first Brauer-Thrall conjecture. Gabriel's analysis of Roiter's proof is a quite non-trivial achievement and it merits to add his name to the concept. Indeed, the definition of what we call the Gabriel-Roiter measure seems to be strange on first sight, but as we are going to show it embodies a complete theory. Recall that the first Brauer-Thrall conjecture [Ri3] asserted that an artin algebra of bounded representation type is of finite representation type (here, *bounded representation type* means that there is a bound on the length of the indecomposable representations, and *finite representation type* means that there are only finitely many isomorphism classes of indecomposables). Roiter's proof of this conjecture marks the beginning of the new representation theory of finite dimensional algebras. Despite the fame of the result, the actual paper of Roiter (and also Gabriel's interpretation) was apparently forgotten in the meantime. There was a later proof of the first Brauer-Thrall conjecture by Auslander and it is this proof, or its modification due to Yamagata, which usually is presented. Auslander's proof has the advantage that it works for artinian rings, not only for artin algebras, but the usual references do not even exploit this, but use it as a striking application of the Auslander-Reiten theory for artin algebras (which it is). It is worthwhile to recall the old proof of Roiter and the methods involved. These methods can be used and should be used as a kind of foundation for the representation theory of artin algebras: the Gabriel-Roiter measure seems to be an important first invariant to be studied when dealing with the representations of an artin algebra. One of the reasons that this has not been done may stem from the fact that both Roiter as well as Gabriel work from the beginning only with algebras of bounded representation type (thus with algebras which are shown to be of finite representation type). However, and this will be our main objective, the Gabriel-Roiter measure can be introduced and used for arbitrary artin algebras, and it unfolds its real strength when dealing with algebras of infinite representation type! (Actually, there is a footnote in Gabriel's paper asserting that one may waive the restriction of dealing with bounded representation type, but apparently this was overlooked.)

The main topic to be discussed here will be cogeneration of modules: Recall that given two modules X, Y, one says that X is *cogenerated by* Y provided the intersection of the kernels of all maps $X \to Y$ is zero. In case X is of finite length, it is immediate to see that X is cogenerated by Y if and only if X can be embedded into a finite direct sum of copies of Y. Cogeneration yields a kind of partial ordering of the isomorphism classes of Λ-modules. Namely, there is the following observation:

Assume that X, Y are non-zero modules of finite length such that X is cogenerated by Y and Y is cogenerated by X, then there is an indecomposable module Z which is a direct summand of X as well as of Y.

Proof: By assumption, there exist embeddings $f: X \to Y^n$ and $g: Y \to X^m$ for some natural numbers n, m. Obviously, this yields an inclusion map $h: X \to X^{nm}$ which factors through Y^n. Since for any module X the radical of the endomorphism ring of X annihilates some non-zero element of X, we conclude that there is an indecomposable direct summand X' with inclusion $m: X' \to X'$ and a projection $p: X^{mn} \to X'$ such that the composition $phm: X' \to X'$ is invertible. Since this invertible map phm factors through Y^n, the module X' occurs as a direct summand of Y^n and therefore of Y.

The Gabriel-Roiter measure μ provides a tool for a better understanding of the cogeneration of modules. It allows to index the isomorphism classes of the Λ-modules by a totally ordered set (say a set of real numbers with their usual ordering) so that cogenerations are possible only in the given order: *Assume that X, Y are non-zero modules of finite length and without any common indecomposable direct summand. If and X is cogenerated by Y, then $\mu(X) < \mu(Y)$.*

For the proofs of the Main Proposition and Theorems 1, 2 and 3, see [R5].

2. The Basic Definitions.

Let $\mathbb{N}_1 = \{1, 2, \dots\}$ be the set of natural numbers. Note that we use the symbol $\subset$ to denote proper inclusions. Let $\mathcal{P}(\mathbb{N}_1)$ be the set of all subsets $I \subseteq \mathbb{N}_1$. We consider this set as a **totally ordered** set as follows: If I, J are different subsets of $\mathbb{N}_1$, write $I < J$ provided the smallest element in $(I \setminus J) \cup (J \setminus I)$ belongs to J. It is easy to see that $\mathcal{P}(\mathbb{N}_1)$ with this ordering is complete. Also note that $I \subseteq J \subseteq \mathbb{N}_1$ implies that $I \leq J$.

The Gabriel-Roiter measure of a module of finite length will be a finite set of natural numbers. We want to provide a more intuitive understanding of the Gabriel-Roiter measure, in particular of the total ordering as described above. In order to do so, we are going to embed the set $\mathcal{P}_f(\mathbb{N}_1)$ of all finite subsets of $\mathbb{N}_1$ into the ordered set $\mathbb{Q}$ of all rational numbers (in section 5 we will extend this embedding to an embedding of all the possible Gabriel-Roiter measures for arbitrary, not necessarily finitely generated modules over an artin algebra into the ordered set of real numbers).

Lemma 1. *The map $r\colon \mathcal{P}_f(\mathbb{N}_1) \to \mathbb{Q}$ given by $r(I) = \sum_{i \in I} \frac{1}{2^i}$ for $I \in \mathcal{P}_f(\mathbb{N}_1)$ is injective, its image is contained in the interval $[0, 1]$ and it preserves and reflects the ordering.*

Proof: The essential consideration is the following: Let I, J belong to $\mathcal{P}_f(\mathbb{N}_1)$ with $I < J$. Then $r(I) = r(I \cap J) + r(I \setminus J)$ and $r(J) = r(I \cap J) + r(J \setminus I)$. Let a be the smallest element in $J \setminus I$. Then $r(J \setminus I) \geq \frac{1}{2^a} = \sum_{i > a} \frac{1}{2^i} > r(I \setminus J)$, since $I \setminus J$ is a proper subset of $\{i \in \mathbb{N}_1 \mid i > a\}$.

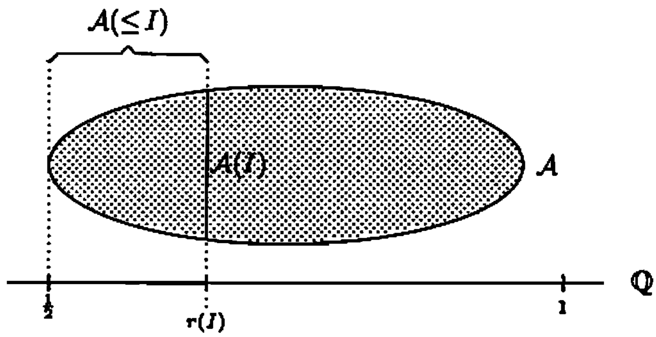
For a (not necessarily finitely generated) Λ-module M, let $\mu(M)$ be the supremum of the sets $\{|M_1|, \dots, |M_t|\}$ in the complete totally ordered set $(\mathcal{P}(\mathbb{N}_1), \leq)$, where $M_1 \subset M_2 \subset \cdots \subset M_t$ is a chain of indecomposable submodules of M. We call $\mu(M)$ the *Gabriel-Roiter measure* of M. Note that the Gabriel-Roiter measure of a module M only depends on its submodule lattice: if M and N are modules with isomorphic submodule lattices, then $\mu(M) = \mu(N)$.

Examples. Let M be an indecomposable module of length t.

- $\mu(M) = \{1\}$ iff M is simple (thus $t = 1$).
- $\mu(M) = \{1, 2\}$ iff M is indecomposable and $t = 2$.
- $\mu(M) = \{1, 2, \dots, t\}$ iff M is uniform (i.e. its socle is simple).
- $\mu(M) = \{1, t\}$ iff M is local and has Loewy length at most 2.

We will use the Gabriel-Roiter measure μ (or the composition $r\mu$) in order to visualize the category $\operatorname{mod}\Lambda$. As abbreviation, let us write $\mathcal{A} = \operatorname{mod}\Lambda$. For any finite subset $I \subset \mathbb{N}_1$, we denote by $\mathcal{A}(I)$ the class of indecomposable Λ-modules M

with $\mu(M) = I$, and we say that I is a *Gabriel-Roiter measure for* Λ provided $\mathcal{A}(I)$ is non-empty. Similarly, let $\mathcal{A}(\leq I)$ be the class of indecomposable Λ-modules M with $\mu(M) \leq I$.



If M is an indecomposable Λ-module of finite length, we call any filtration

$$M_1 \subset M_2 \subset \cdots \subset M_{t-1} \subset M_t = M$$

with $\mu(M) = \{|M_1|, |M_2|, \ldots, |M_{t-1}|, |M_t|\}$ a *Gabriel-Roiter filtration* of M; if M is of length at least 2 (thus $t \geq 2$) the module M_{t-1} will be said to be a *Gabriel-Roiter submodule* of M. Thus a Gabriel-Roiter filtration exhibits an iterated sequence of Gabriel-Roiter submodules (in section 5, we will consider also Gabriel-Roiter filtrations of infinitely generated modules, again using iterated sequences of Gabriel-Roiter submodules). Given a proper inclusion $X \subset Y$ of indecomposable finite length modules, then X is a Gabriel-Roiter submodule of Y iff $\mu(Y) = \mu(X) \cup \{|Y|\}$. In particular, if X is a Gabriel-Roiter submodule of Y, then for every monomorphism $f\colon X \to Y$, also $f(X)$ is a Gabriel-Roiter submodule of Y.

Gabriel-Roiter submodules of a given indecomposable module are usually not unique, not even unique up to isomorphism (all have however the same length). For example, for the Kronecker quiver, all the indecomposables of length 2 are Gabriel-Roiter submodules of the indecomposable injective module of length 3.

3. The Cogeneration Property.

Main Property (Gabriel). *Let $X, Y_1, \ldots, Y_t$ be indecomposable Λ-modules of finite length and assume that there is a monomorphism $f\colon X \to \bigoplus_{i=1}^t Y_i$.*
(a) *Then $\mu(X) \leq \max \mu(Y_i)$.*
(b) *If $\max \mu(Y_i)$ starts with $\mu(X)$, then there is some j such that $\pi_j f$ is injective, where $\pi_j\colon \bigoplus_i Y_i \to Y_j$ is the canonical projection.*

Note that (b) immediately implies:
(b′) *If $\mu(X) = \max \mu(Y_i)$, then f splits.*

The assertions (a) and (b′) have been formulated and proven by Gabriel in [G] using the additional assumption that Λ is of bounded representation type.

We conclude: add $\mathcal{A}(\leq I)$ *is closed under submodules and any monomorphism* $f: X \to Y$ *with* X *in* $\mathcal{A}(I)$ *and* Y *in* add $\mathcal{A}(\leq I)$ *splits* (if $\mathcal{X}$ is a class of indecomposable Λ-modules, we denote by add $\mathcal{X}$ the class of all finite direct sums of modules in $\mathcal{X}$). The latter assertion may be reformulated as follows: the modules in add $\mathcal{A}(I)$ are "relative injective" inside add $\mathcal{A}(\leq I)$.

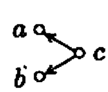
Corollary 1. *If $M_1, \ldots, M_t$ are (not necessarily finitely generated) indecomposable Λ-modules, then $\mu(\bigoplus M_i) = \max \mu(M_i)$.*

Proof: Since M_i is a submodule of $M = \bigoplus M_i$, we have $\max \mu(M_i) \leq \mu(\bigoplus M_i)$. Conversely, $\mu(M)$ is the supremum of $\mu(M')$, where M' is a finitely generated indecomposable submodule of M, thus we have to show $\mu(M') \leq \max \mu(M_i)$. Now $M' \subseteq \bigoplus M_i'$, where M_i' is a finitely generated submodule of M_i. We can write $M_i' = \bigoplus_j M_{ij}$ with indecomposable modules M_{ij}. Note that M_{ij} is a submodule of M_i, thus $\mu(M_{ij}) \leq \mu(M_i)$. According to part (a) of Main Property, we get $\mu(M') \leq \max_{ij} \mu(M_{ij}) \leq \max_i M_i$, this concludes the proof.

Corollary 2. *Let M be an indecomposable module and N a Gabriel-Roiter submodule of M. Then, for any proper submodule N' of M containing N, the embedding $N \subseteq N'$ splits.*

Proof: First consider the case where N' is indecomposable. Assume f' is not an isomorphism. Then $\mu(N) \cup \{|N'|, |M|\} \leq \mu(M)$. However, by assumption $\mu(M) = \mu(N) \cup \{|M|\}$ and $\mu(N) \cup \{|M|\} < \mu(N) \cup \{|N'|, |M|\}$, a contradiction. Now, consider the general case: Write $N' = \bigoplus_i N_i$ with indecomposable modules N_i. The Main Property (a) asserts that $\mu(N) \leq \max \mu(N_i)$ and trivially $\max \mu(N_i) \leq \mu(M)$. Since $\mu(M)$ starts with $\mu(N)$, the same is true for $\max \mu(N_i)$, thus by (b') there is some j such that the map $\pi_j f'$ is injective, where $\pi_j: N' \to N_j$ is the canonical projection. Besides the monomorphisms $\pi_j f': N \to N_j$, there also exists a monomorphism $N_j \to N' \to M$. Since the latter is a proper monomorphism, and N_j is indecomposable, we are in the first case, thus we know that $\pi_j f'$ is an isomorphism, thus f' is a split monomorphism.

The property of the inclusion map $f: N \to M$ in Corollary 2 may be called *mono-irreducibility*, in parallelity to the Auslander-Reiten notion of irreducibility: f is a non-invertible monomorphism and any factorization $f = f''f'$ of f using monomorphisms f' and f'' implies that f' is a split monomorphism of f'' is a split epimorphism (thus isomorphism). Irreducible monomorphism are mono-irreducible; however there are obvious mono-irreducible maps which are not irreducible: for example consider the path algebra of the quiver



The inclusion map of the simple module $S(a)$ into its injective envelope is mono-irreducible, however it factors through the projective cover of $S(c)$, thus it is not irreducible. Also, there is the following phenomenon: Given indecomposable modules

X, Y, there may be irreducible monomorphisms $f\colon X \to Y$ and also a monomorphism $g\colon X \to Y$ which is not even mono-irreducible. For example, take the hereditary algebra $\tilde{A}_{21}$, let S be simple projective and P the indecomposable projective of length 4. Then $\operatorname{Hom}(S, P)$ is 2-dimensional and the non-zero maps are monomorphisms. Thus the monomorphisms (up tp scalar multiplication) $S \to P$ form a projective line; one of these equivalence classes is not mono-irreducible (it factors through an indecomposable length 2 submodule), the remaining ones are irreducible, thus mono-irreducible.

Corollary 3. *Let N be a Gabriel-Roiter submodule of the indecomposable module M. Then M/N is indecomposable.*

Proof of Corollary 3: Assume $M/N = Q_1 \oplus Q_2$ with non-zero modules Q_1, Q_2. For $i = 1, 2$, write $Q_i = N_i/N$, where $N \subset N_i \subset M$. According to Corollary 2, we find submodules N_i' of N_i such that $N_i = N \oplus N_i'$. Then $M = N \oplus N_1 \oplus N_2$, in contrast to the fact that M is indecomposable.

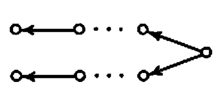
This corollary asserts, in particular, that *any indecomposable module M of length at least 2 occurs as the middle term of an exact sequence*

$$0 \to N \to M \to M/N \to 0,$$

where all three terms $N, M, M/N$ are indecomposable. (This exact sequence has the following additional property: its equivalence class in $\operatorname{Ext}^1(M/N, N)$ is annihilated by the radical of $\operatorname{End}(M/N)$, where we view $\operatorname{Ext}^1(M/N, N)$ as usual as a right $\operatorname{End}(M/N)$-module.)

Also we see: *If M and N are indecomposable modules with $|N| < |M|$ and $\mu(M) = \mu(N) \cup \{|M|\}$, then the cokernel of* **any** *monomorphism $f\colon N \to M$ is indecomposable.* One should be aware that there are plenty of pairs of modules N, M such that there do exist monomorphisms $f_1, f_2\colon N \to M$ such that the kernel of f_1 is indecomposable whereas the kernel of f_2 is not (for example, let Λ be the path algebra of the Kronecker quiver and let N, M be preprojective Λ-modules of length 1 and 5, respectively).

One may wonder about the possible modules which occur as factor modules M/N, where M is indecomposable and N is a Gabriel-Roiter submodule. For the path algebra of a quiver of type A_n, all these factors are serial and of length at most $\frac{n+1}{2}$, a factor of length $\frac{n+1}{2}$ occurs for the sincere representation of the quiver of type A_n (n odd) with a unique source



and with arms of equal length.

4. Main Results.

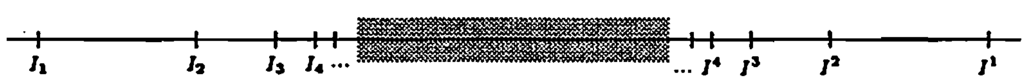
The indecomposable Λ-modules of length at most n belong to the classes $\mathcal{A}(I)$ with $I \subseteq \{1, 2, \ldots, n\}$, and there are just finitely many such classes. Thus as soon as we exhibit (as we will do now) an infinite list of Gabriel-Roiter measures for Λ, this implies that Λ cannot be of bounded representation type. Thus, the following theorem strengthens the assertion of the first Brauer-Thrall conjecture. In contrast to the assertion of the first Brauer-Thrall conjecture, the statement is meaningfull even in case Λ is a finite ring (i.e. a ring with finitely many elements). Recall that a Gabriel-Roiter measure I is said to be *of finite type* provided there are only finitely many isomorphism classes in $\mathcal{A}(I)$.

Theorem 1. *Let Λ be of infinite representation type. Then there are Gabriel-Roiter measures I_t, I^t for Λ with*

$$I_1 < I_2 < I_3 < \quad \cdots \quad < I^3 < I^2 < I^1$$

such that any other Gabriel-Roiter measure I for Λ satisfies $I_t < I < I^t$ for all $t \in \mathbb{N}_1$, and all these Gabriel-Roiter measures I_t and I^t are of finite type.

We call the modules in $\bigcup_t \mathcal{A}(I_t)$ (or the additive category with these indecomposable modules) the *take-off part* of the category $\mathcal{A}$, and $\bigcup_t \mathcal{A}(I^t)$ (or the additive category with these indecomposable modules) the *landing part* of $\mathcal{A}$. The remaining indecomposables (those which do not belong to the take-off part or the landing part) are said to form the *central part*. It is the central part which should be of particular interest in future:



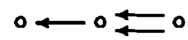
Note that *for any n, there are only finitely many isomorphism classes of indecomposable modules of length n which belong to the take-off part* (since they belong to only finitely many classes $\mathcal{A}(I_t)$ and any class $\mathcal{A}(I_t)$ is of finite type). Similarly, *there are only finitely many isomorphism classes of indecomposable modules of length n which belong to the landing part.*

It is obvious that the modules in $\mathcal{A}(I_1)$ are just the simple modules, those in $\mathcal{A}(I_2)$ are the local modules of Loewy length 2 of largest possible length. On the other hand, the modules in $\mathcal{A}(I^1)$ are the indecomposable injective modules of largest possible length. For general t, it seems to be difficult to characterize the modules in $\mathcal{A}(I_t)$ or $\mathcal{A}(I^t)$ in a direct way.

Recall that Auslander-Smalø have introduced in [AS] the notion of preprojective and preinjective modules (actually with reference to the work of Roiter and Gabriel).

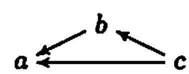
Theorem 2. *The modules in the landing part are preinjective.*

Since modules which have infinitely many different Gabriel-Roiter measures cannot have bounded length, we obtain in this way a new proof for the assertion that the indecomposable preinjective modules are of unbounded length ([AS],5.11). But note that usually there will exist preinjective indecomposables which do not belong to the landing part. For example, any simple module belongs to $\mathcal{A}(I_1)$, thus a simple injective module is preinjective and in the take-off part, thus not in the landing part. Also, there may exist preinjective modules Q such that $\mathcal{A}(\mu(Q))$ is infinite, as the example of the radical-square-zero algebra with quiver



shows: take for Q the indecomposable injective module of length 2. But there may be even infinitely many isomorphism classes of preinjective indecomposables which do not belong to the landing part:

Example. Consider the tame hereditary algebra of tye $\widetilde{A}_{21}$



For a tame hereditary algebra, the Auslander-Smalø preinjectives are just those modules which belong to the preinjective component.

We denote by $S(x)$ the simple module corresponding to the vertex x, thus $S(a)$ is projective and $S(c)$ is injective. The top composition factors of the preinjective indecomposable modules are injective, all but at most one socle composition factors are projective, the exceptional one will be of the form $S(b)$. Now, in case the socle is projective, then the GR-measures are as follows:

$$\cdots > 1235689, 10 > 123567 > 1234,$$

the general form is

$$123|56|89|\cdots|3i-1, 3i|\cdots|3n-1, 3n|3n+1,$$

with $n \geq 0$. For $n = 4$, it looks as follows

and for $n \geq 1$, the GR-filtration starts with $M_1 \subset M_2 \subset M_3$, where M_3 is the indecomposable length 3 module seen left: it is uniform, but not serial.

On the other hand, those preinjectives with $S(b)$ in the socle have GR-measure

$$123|6|9|\cdots|3i|\cdots|3n|3n+2,$$

with $n \geq 0$. For small $n \geq 1$, we obtain the values

$$\cdots > 12369, 11 > 12368 > 1235.$$

Here is the picture for $n = 4$

now, for $n \geq 1$, the GR-filtration starts with $M_1 \subset M_2 \subset M_3$, where M_3 is the serial length 3 module seen right.

It follows that all the preinjective modules with $S(b)$ in the socle belong to the central part.

In contrast to Theorem 3, the modules in the take-off part are usually not preprojective. Here is an example: Let $\Lambda = k[X,Y]/\langle XY, X^3, Y^3\rangle$ and A the ideal generated by X^2 and Y^2 (these elements actually form a basis of A). The take-off part for Λ is the same as the take-off part for Λ/A and these modules are the preprojective Λ/A-modules, but none of them is preprojective as a Λ-module.

Note that there is no dualization principle concerning the take-off and the landing part (whereas the notions of preprojectivity and the preinjectivity are dual ones)! If we want to invoke dual considerations, then we have to work with a corresponding Gabriel-Roiter comeasure which is based on looking at indecomposable factor modules in contrast to the Gabriel-Roiter measure which is based on indecomposable submodules. This will be done in section 7.

It is usually difficult to specify the position of the possible Gabriel-Roiter measures. But here is such an assertions, dealing with uniform modules:

Proposition. *Let $I^1 = (1, 2, ..., t)$ and $1 \leq s < t$. Assume the following: for any simple Λ-module with injective envelope $Q(S)$ of length greater than s, there are only finitely many indecomposable Λ-modules with a submodule of the form S. Then $(1, 2, \ldots, s)$ is a landing measure.*

Proof: We show that any indecomposable module M with $\mu(M) > (1, 2, \ldots, s)$ has a composition factor of the form S, such that $|Q(s)| > s$. Thus assume that $\mu(M) > (1, 2, \ldots, s)$ and take a Gabriel-Roiter-filtration of M. The first s submodules in the filtration are uniform of length i with $1 \leq i \leq s$. In particular, M contains a uniform module U of length s. Let S be its socle, thus U embeds into $Q(S)$, and this is a proper embedding, since otherwise $U = Q(S)$ would be a direct summand of M. However M is indecomposable and of length greater than s. This shows that $|Q(S)| > s$ and $S \subseteq U \subseteq M$ is a submodule of M. By assumption, there are only finitely many such isomorphism classes. This shows that there are only finitely many isomorphism classes of indecomposable modules M with $\mu(M) > (1, 2, \ldots, s)$, thus $(1, 2, \ldots, s)$ belongs to the landing part.

5. Infinitely generated modules.

Up to now, we have concentrated on Λ-modules of finite length, however the Gabriel-Roiter measure was introduced above for all Λ-modules M, not just those of finite length. Note that by definition $\mu(M)$ is the supremum of $\mu(M')$, where M' are the finitely generated submodules of M (or just the indecomposable ones).

We extend the notion of a Gabriel-Roiter filtration as follows: In case there exists a (countable) chain of submodules

$$M_1 \subset M_2 \subset \cdots \subseteq \bigcup_i M_i = M \quad \text{such that} \quad \mu(M) = \{|M_i| \mid i\},$$

then we call this chain a *Gabriel-Roiter filtration* of M. Of course, a finitely generated Λ-module M has a Gabriel-Roiter filtration if and only if M is indecomposable. As a consequence of Gabriel's Main Property we show now that also any infinitely generated module with a Gabriel-Roiter filtration is indecomposable:

Corollary 4. *Any module M with a Gabriel-Roiter filtration is indecomposable.*

Proof: We can assume that there is given an infinite chain

$$M_1 \subset M_2 \subset \cdots \subseteq \bigcup_i M_i = M$$

such that M_i is a Gabriel-Roiter submodule of M_{i+1}, for all $i \geq 1$. Assume that there is given a direct decomposition $M = U \oplus V$ with U, V both nonzero. Note that if $U \cap M_i = 0$ for all i, then $U = U \cap M = U \cap (\bigcup M_i) = \bigcup (U \cap M_i) = 0$. This shows that there is some index s such that $U \cap M_s \neq 0$ and also $V \cap M_s \neq 0$. Choose finitely generated submodules $U' \subseteq U$ and $V' \subseteq V$ such that $M_s \subseteq M' = U' \oplus V'$, and decompose $U' = \bigoplus U_i$, $V' = \bigoplus V_j$ with indecomposable modules U_i and V_j. Finally, choose t such that $M' \subseteq M_t$.

Now we consider the Gabriel-Roiter measures: We get

$$\mu(M_s) \leq \max\{\mu(U_i), \mu(V_j)\} \leq \mu(M_t)$$

(the first inequality is Main Property (a), the second is trivial). Since M_s and M_t belong to a Gabriel-Roiter filtration, it follows that $\mu(M_t)$ starts with $\mu(M_s)$, thus also $\max\{\mu(U_i), \mu(V_j)\}$ starts with $\mu(M_s)$ and we can apply Main Property (b). Without loss of generality, we can assume that the composition of the inclusion $M_s \to \bigoplus_i U_i \oplus \bigoplus_j V_j = M'$ and the projection $\pi_1^U \colon M' \to U_1$ is injective (where $i = 1$ is one of the indices). Recall that there is a non-zero element $v \in V \cap M_s$. Since $M_s \subseteq M' = U' \oplus V'$, we can write $v = u' + v'$ with $u' \in U'$ and $v' \in V'$. However $u' = v - v' \in U' \cap V = 0$ shows that $v = v'$ belongs to V'. Since v belongs to $V' = \bigoplus V_j$, it is mapped under π_1^U to zero. This contradicts the fact that π_1^U is injective.

Theorem 3. *Let Λ be of infinite representation type. There do exist modules which have an infinite Gabriel-Roiter filtration*

$$M_1 \subset M_2 \subset \cdots \subseteq \bigcup_i M_i = M$$

such that all the modules M_i belong to the take-off part.

Note that according to Corollary 4, such a module M is indecomposable. Also, any finitely generated submodule M' of M is contained in some M_t, thus belongs to the take-off part. In particular, for any natural number n, M has only finitely many isomorphism classes of submodules of length n. In general, Theorem 3 will provide a large number of indecomposable Λ-modules M, however all these modules have the same Gabriel-Roiter measure! For example, if K is the Kronecker quiver and k is a countable and algebraically closed field, then all the "torsionfree kK-modules of rank 1" (see [Ri2]) occur in this way, and $\mu(M) = \{1, 2, 4, 6, 8, \dots\}$. On the other hand, for the tame algebra of type $\widetilde{A}_{21}$, there is only one such module M, namey the string module corresponding to

its Gabriel-Roiter measure is $\{1, 2, 4, 5, 7, 8, \dots\}$.

The existence of infinitely generated indecomposables for any artin algebra of infinite representation type was first shown by Auslander [A]. For a discussion of the question whether a union of a chain of indecomposable modules of finite length is indecomposable or not, we refer to [Ri1].

Let us note that there are indecomposable modules without a Gabriel-Roiter filtration. Of course, any module with a Gabriel-Roiter filtration is countably generated, here is an example of a countable generated indecomposable module without a Gabriel-Roiter filtration: We consider again the tame hereditary algebra of type $\widetilde{A}_{21}$ and take the Prüfer module for the simple module $S(b)$ which is neither projective nor injective:

its Gabriel-Roiter measure is {1,2,4,5,7,8,...}, but there is no corresponding sequence of submodules which exhaust all of M.

We have introduced above an embedding of $\mathcal{P}_f(\mathbb{N}_1)$ into $\mathbb{Q}$. In order to deal also with modules with are not finitely generated, we consider the set $\mathcal{P}_l(\mathbb{N}_1)$ of all subsets I of $\mathbb{N}_1$ such that for any $n \in \mathbb{N}_1$, there is $n' \geq n$ with $n' \notin I$.

Lemma 2. *The Gabriel-Roiter measure $\mu(M)$ of any module M belongs to $\mathcal{P}_l(\mathbb{N}_1)$.*

Proof. There is $m \in \mathbb{N}_1$ such that any indecomposable injective Λ-module has length at most m. Let $\mu(M) = \{a_1 < a_2 < \cdots < a_i < \cdots\}$ and assume that for some n we have $a_{n+t} = a_n + t$ for all $t \in \mathbb{N}_1$. Let $s = m \cdot a_n$

There is a chain of indecomposable submodules $M_1 \subset M_2 \subset \cdots \subset M_{n+s}$ with $|M_i| = a_i$ for $1 \leq i \leq n+s$. Since $|M_{n+t}| = a_{n+t} = a_{n+t-1} + 1 = |M_{n+t-1}| + 1$, we see that M_{n+t-1} is a maximal submodule of M_{n+t}. Since M_{n+t} is indecomposable, the socle of M_{n+t} has to be contained in M_{n+t-1}. Inductively, we see that the socle of M_{n+t} is contained in M_n, for any $t \geq 1$, in particular, the socle of M_{n+s} is contained in M_n, thus M_{n+s} can be embedded into the injective envelope of M_n. Since any indecomposable injective module is of length at most m, the injective envelope of M_n has length at most $m \cdot a_n$, thus $|M_{n+s}| \leq m \cdot a_n$. But $|M_{n+s}| = |M_n| + s = (m+1)a_n > m \cdot a_n$, a contradiction.

The embedding of $\mathcal{P}_f(\mathbb{N}_1)$ into $\mathbb{Q}$ (thus into $\mathbb{R}$) extends to an embedding of $\mathcal{P}_l(\mathbb{N}_1)$ into the real interval $[0,1]$:

Lemma 1′. *The map $r\colon \mathcal{P}_l(\mathbb{N}_1) \to \mathbb{R}$ given by $r(I) = \sum_{i \in I} \frac{1}{2^i}$ for $I \in \mathcal{P}_l(\mathbb{N}_1)$ is injective, its image is contained in the interval $[0,1]$ and it preserves and reflects the ordering.*

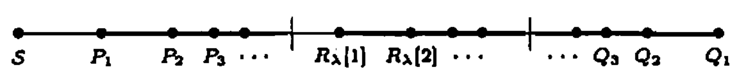
Remark: The map r can be defined not just on $\mathcal{P}_l(\mathbb{N}_1)$, but on all of $\mathcal{P}(\mathbb{N}_1)$, however it will no longer be injective (indeed, for any element I in $\mathcal{P}(\mathbb{N}_1) \setminus \mathcal{P}_l(\mathbb{N}_1)$, there is a unique finite set I' with $r(I) = r(I')$). Of course, one easily may change the definition of r in order to be able to embed all of $\mathcal{P}(\mathbb{N}_1)$ into $\mathbb{R}$: just use say 3 instead of 2 in the denominator. However, our interest lies in the Gabriel-Roiter measures which occur for finite dimensional algebras and Lemma 2 assures us that the definition of r as proposed is sufficient for these considerations.

5. Examples.

Example 1. The Kronecker quiver $\widetilde{A}_{11}$. We have referred to this quiver already several times, it has vertices a, b and two arrows $b \to a$; its representations are called *Kronecker modules.* There are two simple Kronecker modules, the projective simple module $S(a)$ and the injective simple module $S(b)$. If M is a Kronecker module, its *dimension vector* is of the from $\dim M = (d_a, d_b)$, where d_a is the Jordan-Hölder multiplicity of $S(a)$, and d_b that of $S(b)$. The dimension vectors of the indecomposable modules are of the form (x,y) with $|x-y| \leq 1$. Here is the complete list of the indecomposable representations in case k is algebraically closed:

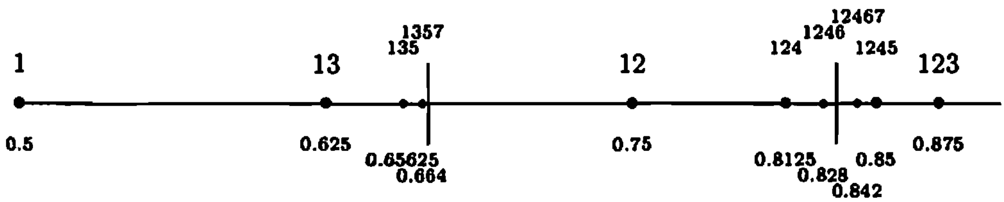
- The preprojectives P_n for $n \in \mathbb{N}_0$, with $\dim P_n = (n+1, n)$ and $\mu(P_n) = \{1, 3, 5, \ldots, 2n+1\}$.
- The preinjectives Q_n for $n \in \mathbb{N}_0$, with $\dim Q_n = (n, n+1)$ and $\mu(Q_n) = \{1, 2, 4, 6, \ldots, 2n, 2n+1\}$.
- The regular modules $R_\lambda[n]$ for $\lambda \in \mathbb{P}^1(k)$ and $n \in \mathbb{N}_1$, with $\dim R_\lambda[n] = (n,n)$ and $\mu(R_\lambda[n]) = \{1, 2, 4, 6, \ldots, 2n\}$.

The totally ordered set of all the Gabriel-Roiter measures for the Kronecker quiver looks as follows:



Here $\mathcal{S} = \mathcal{A}(\{1\}) = \{S(a), S(b)\}$. Note that there are precisely two accumulation points, indicated by the dotted vertical lines, they correspond to the only two Gabriel-Roiter measures for infinitely generated modules: to the left, there is $\{1, 3, 5, 7, \ldots\}$, this is the Gabriel-Roiter measure for all indecomposable torsionfree modules; to the right, there is $\{1, 2, 4, 6, 8, \ldots\}$, this is the Gabriel-Roiter measure for the so-called Prüfer modules (an account of the structure theory for infinitely generated Kronecker modules can be found for example in [Ri2]).

A more precise picture of the Gabriel-Roiter measures for the Kronecker algebra is the following; here the upper sequences are the measures I, the lower numbers the corresponding values $r(I)$:



In case k is not algebraically closed, we have to take into account field extensions of k, or better indecomposable $k[T]$-module of finite length N, where $k[T]$ is the polynomial ring over k in one variable T. Any indecomposable $k[T]$-module N of length n and with a simple submodule of dimension d gives rise to a regular Kronecker module with dimension vector (nd, nd) and Gabriel-Roiter measure $\{1, 3, 5, \dots, 2d{-}1, 2d; 4d, 6d, \dots, 2nd\}$. Thus we see that *the Gabriel-Roiter measure for the path algebra $k\Delta$ of a quiver Δ may depend on k* (and usually will).

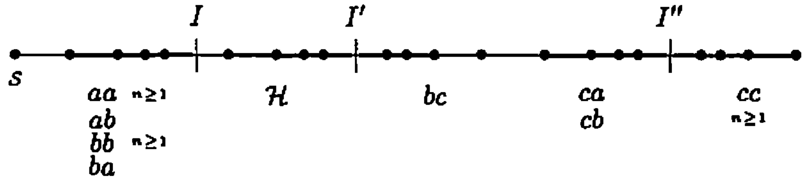
Example 2. The tame hereditary algebra of type $\widetilde{A}_{21}$. Also this algebra has been referred to before, we want to stress here some features which one should be aware of. In order to list all the indecomposable Λ-modules, we use that Λ is a string algebra. Thus the indecomposable modules are the string and the band modules. Again, we restrict to the case of k being algebraically closed.

There is a unique one-parameter family of band modules; they are of the form $R_\lambda[n]$, where $\lambda \in k \setminus \{0\}$ and $n \in \mathbb{N}_1$, with Gabriel-Roiter measure $\mu(R_\lambda) = \{1, 2, 3; 6, 9, \cdots, 3n\}$.

In order to write down the string modules, we use words in $\alpha, \beta, \gamma^{-1}$; the relevant distinction is given by fixing the vertices x, y such that the word starts in x and ends in y (always $n \in \mathbb{N}_0$):

xy	property	dimension	GR-measure
aa	preprojective	$3n{+}1$	$1,2,4,5,7,8,\dots,3n-2,3n-1,3n+1$
ab	preprojective	$3n{+}2$	$1,2,4,5,7,8,\dots,3n-2,3n-1,3n+1,3n+2$
ac	homogeneous	$3n{+}3$	$1,2,3;6,9,\dots,3n$
ba	regular, non-homog.	$3n{+}3$	$1,2,4,5,7,8,\dots,3n-2,3n-1,3n+1,3n+3$
bb	regular, non-homog.	$3n{+}1$	$1,2,4,5,7,8,\dots,3n-2,3n-1,3n+1$
bc	preinjective	$3n{+}2$	$1,2,3;6,9,\dots,3n;3n+2$
ca	regular, non-homog.	$3n{+}2$	$1,2,3;5,6,8,9,\dots,3n-1,3n,3n+2$
cb	regular, non-homog.	$3n{+}3$	$1,2,3;5,6,8,9,\dots,3n+2,3n+3$
cc	preinjective	$3n{+}1$	$1,2,3;5,6,8,9,\dots,3n-1,3n;3n+1$

The set of Gabriel-Roiter measures for Λ has the following structure:



Here, $\mathcal{H}$ denotes the class of all homogeneous modules (the bands as well as the strings of type ac), whereas S are the simple modules.

Some observations:

(1) There are many "maximal" GR-measures I (maximality should mean that no other GR-measure starts with I), in particular see ba, but also bc and cc.

(2) The take-off part contains all the preprojective modules, but in addition also half of the non-homogeneous tube (namely all the regular modules which have the simple module $S(b)$ as submodule.

(3) The landing part contains only half of the preinjective modules (also the modules bc are preinjective)

(4) The GR-measure apparently does not distinguish modules which have quite different behaviour, see aa and bb (however, aa and bb will be distinguished in case we invoke the dual concepts, see the next appendix)

(5) There are three accumulation points I, I', I'':

$$I = \{1,2,4,5,7,8,10,11,\dots\}$$
$$I' = \{1,2,3,6,9,12,15,\dots\}$$
$$I'' = \{1,2,3,5,6,8,9,11,12,\dots\}$$

The first one I is the Gabriel-Roiter measure of the torsionfree modules; I' is the Gabriel-Roiter measure for all the Prüfer modules arising from homogeneous tubes; I'' is that of the Prüfer module containing the 2-dimensional indecomposable regular module as a submodule.

(6) There is one additional Prüfer module, it contains the simple module $S(b)$ as a submodule: this module does not have a Gabriel-Roiter filtration!

6. Dualization

Dualization. Almost all the considerations presented above can be dualized and then they yield corresponding dual results. This means that instead of looking at filtrations

$$0 = M_0 \subset M_1 \subset \cdots \subset M_t = M$$

with M_i indecomposable for $1 \le i \le t$, we now look at such filtrations with M/M_{i-1} indecomposable for $1 \le i \le t$. We prefer to use now the opposite order on $\mathcal{P}(\mathbb{N}_1)$, we denote it by $\le^*$ (and $<^*$), thus $I \le^* J$ iff $J \le I$. For a (not necessarily finitely generated) Λ-module M, let $\mu^*(M)$ be the infimum of the sets $\{|M_1|, \ldots, |M_t|\}$ in $(\mathcal{P}(\mathbb{N}_1), \leq^*)$, where $M_1 \subset M_2 \subset \cdots \subset M_t$ is a chain of submodules of M with M/M_{i-1} indecomposable for $1 \leq i \leq t$, we call $\mu^*(M)$ the *Gabriel-Roiter comeasure* of M. We say that J is a Gabriel-Roiter comeasure for Λ provided there exists an indecomposable module M with $\mu^*(M) = J$.

In order to visualize $(\mathcal{P}_l(\mathbb{N}_1), \leq^*)$, we use the embedding $r^* : (\mathcal{P}_l(\mathbb{N}_1), \leq^*) \to \mathbb{R}$ given by $r^*(I) = -r(I)$. Note that for any non-zero module M, we have $-1 \leq r^*(\mu(M)) \leq 0$. (Actually, it may be advisable to rescale r and r^* so that $r(\{1\}) = r^*(\{1\}) = 0$ and $r(\mathbb{N}_1) = 1, r^*(\mathbb{N}_1) = -1$.)

The dual version of Main Property reads as follows:

Main Property*. *Let $Y_1, \ldots, Y_t, Z$ be indecomposable Λ-modules of finite length and assume that there is an epimorphism $g \colon \bigoplus_{i=1}^t Y_i \to Z$.*
(a) *Then* $\max \mu^*(Y_i) \leq^* \mu^*(Z)$.
(b) *If* $\max \mu^*(Y_i)$ *starts with* $\mu^*(Z)$, *then there is some j such that gu_j is surjective, where $u_j \colon Y_j \to \bigoplus_i Y_i$ is the canonical inclusion.*
(b′) *If* $\mu^*(Z) = \max \mu^*(Y_i)$, *then g splits.*

As a consequence, we see that the class of modules which are direct sums of modules M with $I \leq^* \mu^*(M)$ for some set $I \subseteq \mathbb{N}_1$ is closed under factor modules. In this way, one obtains a second interesting filtration of the category of all Λ-modules by subcategories, now these subcategories are closed under factor modules.

Let us formulate the dual versions of Theorem 1 and Theorem 2:

Theorem 1*. *Let Λ be of infinite representation type. Then there are Gabriel-Roiter comeasures J_t, J^t for Λ with*

$$J_1 < J_2 < J_3 < \quad \cdots \quad < J^3 < J^2 < J^1$$

such that any other Gabriel-Roiter comeasure J for Λ satisfies $J_t < J < J^t$ for all $t \in \mathbb{N}_1$, and all these Gabriel-Roiter comeasures J_t and J^t are of finite type.

We do not have a suggestion how to call the modules in $\bigcup_t \mathcal{A}(J_t)$ or in $\bigcup_t \mathcal{A}(J^t)$. The indecomposable modules which belong neither to $\bigcup_t \mathcal{A}(J_t)$ nor to $\bigcup_t \mathcal{A}(J^t)$ may be said to be form the *$*$-central part.*

Note that for any n, there are only finitely many isomorphism classes of indecomposable modules of length n which belong to $\bigcup_t \mathcal{A}(J_t)$ or to $\bigcup_t \mathcal{A}(J^t)$.

The modules in $\mathcal{A}(J^1)$ are just the simple modules, those in $\mathcal{A}(J^2)$ are the uniform modules of Loewy length 2 of largest possible length. On the other hand, the modules in $\mathcal{A}(J_1)$ are the indecomposable projective modules of largest possible length.

Theorem 2*. *The modules in $\bigcup_t \mathcal{A}(J_t)$ are preprojective.*

There does not seem to exist a dual version of Theorem 3, since Theorem 3 deals with infinitely generated modules. It is the assertion of Corollary 4 which breaks down. For example, consider again the Kronecker quiver and let Q_n be the preinjective module of length $2n+1$. Then Q_{n-1} is a Gabriel-Roiter factor module of Q_n, for $n \geq 1$, and the sequences of epimorphisms

$$\cdots \longrightarrow Q_2 \longrightarrow Q_1 \longrightarrow Q_0$$

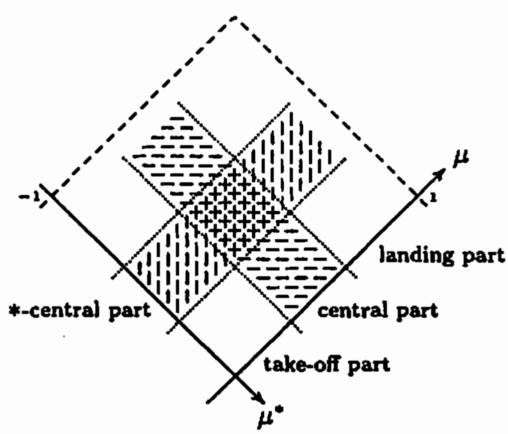
may be called Gabriel-Roiter cofiltrations. If we form the inverse limits, we obtain infinite direct sums of Prüfer modules; in particular, such an inverse limit module is not indecomposable.

7. The Rhombic Picture.

We are going to use now both the measure and the comeasure at the same time. Given a pair (J,I) of finite subsets I, J of $\mathbb{N}_1$, we may consider the module class

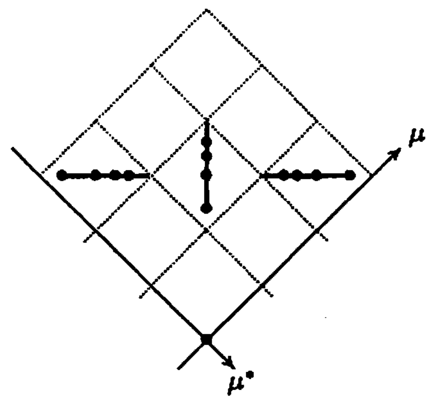
$$\mathcal{A}(J,I) = \{M \mid M \quad \text{indecomposable}, \quad \mu^*(M) = J, \ \mu(M) = I\},$$

thus we attach to a module M the pair $(\mu^*(M), \mu(M))$. The possible pairs (J,I) can be considered (via r^* and r) as elements in the rational plane $\mathbb{Q}^2$:



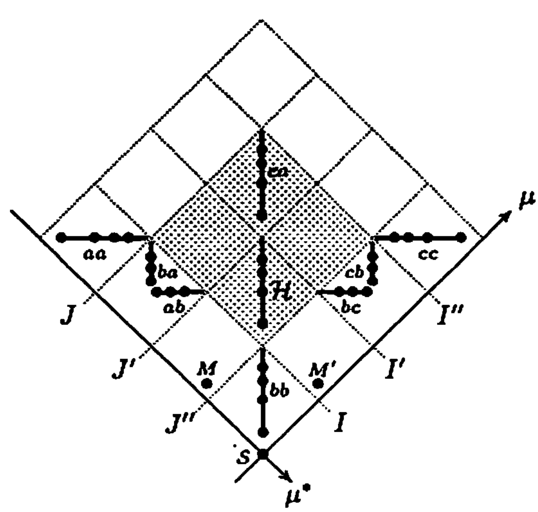
The horizontally dashed region is the central part (in between the take-off part and the landing part); the vertically dashed region is the *-central part. The main information one should keep in mind: *The only possible pairs (J,I) of finite subsets of $\mathbb{N}_1$ such that $\mathcal{A}(J,I)$ contains infinitely many isomorphism classes, are those which belong both to the central and the *-central part.*

Example 1: The Kronecker quiver, with k algebraically closed. The picture which we obtain is nearly the same as the commonly accepted visualization, the only exception being the position of the simple modules. One should be aware that the commonly accepted visualization with the preprojectives and the preinjectives being drawn horizontally and the tubes being drawn vertically in the middle was based mainly on the feeling that this arrangement reflects much of the structure of the category, but for the actual position of the individual modules there was no further mathematical justification. The rhombic picture should be seen as a definite reassurance in this case (but it suggests deviations in other cases).



Even for the Kronecker quiver, one should be aware that there does exist a deviation, namely the position of the simple modules. Of course, they are usually drawn far apart, one at the left end, the other at the right end, now they are located at the same position: in the middle lower corner. But note that the rhombic picture for the Kronecker quiver and the algebra $k[X,Y]/(X,Y)^2$ do not differ, and the usual Auslander-Reiten picture for the latter algebra puts its unique simple module precisely at this position (and bends down the preprojective modules on the left as well as the preinjective modules on the right to form half circles).

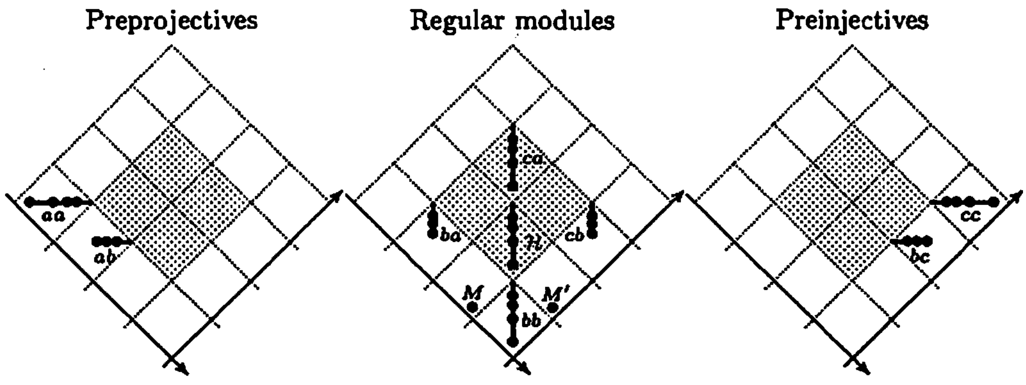
Example 2. The tame hereditary algebra of type $\widetilde{A}_{21}$. Here is the rhombic picture, for k algebraically closed:



Two modules have to be specified separately, the indecomposable modules M, M' of length 3 and Loewy length 2: M is local, M' uniform; note that M has type ba,

M' type cb. The accumulation points I, I', I'' for the Gabriel-Roiter measure are marked on the μ-axis; similarly, the accumulation points J, J', J'' for the Gabriel-Roiter comeasure are marked on the μ^*-axis (note that $J = I'', J' = I', J'' = I$ in $\mathcal{P}(\mathbb{N}_1)$). The intersection of the central and the $*$-central part has been dotted, this region contains for every $n \in \mathbb{N}_1$ a $\mathbb{P}^1(k)$-family of indecomposable representations of length $3n$.

One immediately realizes that the rhombic picture again corresponds quite well to the commonly used visualization, at least after deleting the simple modules. The preprojectives and the preinjectives are arranged horizontally, the regular modules vertically (there is one exceptional tube of rank 2, it has four types of indecomposable modules, namely the types ca, ba (including M), bb and cd (inclusing M'). Let us take apart these three parts of the category:



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