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Osaka City University

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第 37 回 環論および表現論シンポジウム報告集

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信州大学理学部

編集:　浅芝秀人 (大阪市立大学)

２００５年１月
大阪市立大学

Organizing Committee of The Symposium on Ring Theory and Representation Theory

The Symposium on Ring Theory and Representation Theory has been held annually in Japan and the Proceedings have been published by the organizing committee. The first Symposium was organized in 1968 by H. Tominaga, H. Tachikawa, M. Harada and S. Endo. After their retirement, a new committee was organized in 1997 for managing the Symposium. The present members of the committee are Y. Hirano (Okayama Univ.), Y. Iwanaga (Shinshu Univ., 1997–2004), S. Koshitani (Chiba Univ.), K. Nishida (Shinshu Univ.) and M. Sato (Yamanashi Univ., 2004–).

The Proceedings of each Symposium is edited by program organizer. Anyone who wants these Proceedings should ask to the program organizer of each Symposium or one of the committee members.

The Symposium in 2005 will be held at Aichi University of Technology in Aichi Prefecture for Sep. 2–4, and the program will be arranged by T. Wakamatsu (Saitama Univ.).

Concerning several information on ring theory group in Japan containing schedules of meetings and symposiums as well as the addresses of members in the group, you should refer the following homepage, which is arranged by M. Sato (Yamanashi Univ.):

http://fuji.cec.yamanashi.ac.jp/ ring/ (in Japanese)

civil2.cec.yamanashi.ac.jp/ ring/japan/ (in English)

Yasuo Iwanaga
Nagano, Japan
December, 2004

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PREFACE

The 37th Symposium on Ring Theory and Representation Theory was held at Matsumoto City M-Wing from September 3 to 5 in 2004. The symposium and the proceedings are financially supported by Kenji Nishida (Shinshu University) Grant-in-Aid for Scientific Research (B)(1), No.14340007, JSPS.

This volume consists of eighteen articles presented at the symposium. We would like to thank all speakeres and coauthors for their contributions.

We would also like to express our thanks to all the members of the organizing committee for their helpful suggestions concerning the symposium, especially to Masahisa Sato for his administration of the web page and for his kind advice and help to publish the proceedings. Finally, we would like to express our gratitude to Kenji Nishida and all the staffs in Shinshu University for their great effort in the organization of the symposium.

Hideto Asashiba
Osaka, Japan
January, 2005

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REMARKS ON TRANSITIVITY OF EXCEPTIONAL SEQUENCES [1]

TOKUJI ARAYA

ABSTRACT. Let k be an algebraically closed field of characteristic 0. We denote by $\mathcal{C}$ the abelian k-category which has enough projectives (or enough injectives), and by $\mathcal{D}^b(\mathcal{C})$ the bounded derived category of $\mathcal{C}$.

A complex $E^\bullet \in \mathcal{D}^b$ is called *exceptional* if $\mathrm{RHom}(E^\bullet, E^\bullet) \cong k$, and a sequence $\epsilon = (\cdots, E_i^\bullet, E_{i+1}^\bullet, \cdots)$ of exceptional complexes is called an *exceptional sequence* if $\mathrm{RHom}(E_i^\bullet, E_j^\bullet) = 0$ for all $i > j$.

Let $\mathcal{C}$ be a category mod$A$ of finitely generated modules of a hereditary k-algebra A, or a category coh($\mathbf{X}$) of coherent sheaves of a weighted projective line $\mathbf{X}$ over k. In this case, for any exceptional sequence ϵ, the length of ϵ is smaller than or equal to the rank n of Grothendieck group of $\mathcal{C}$. An exceptional sequence ϵ is called *complete* if the length of ϵ is equal to n. It is shown by W. Crawley-Boevey (in the case of $\mathcal{C} = \mathrm{mod}A$) and by H. Meltzer (in the case of $\mathcal{C} = \mathrm{coh}(\mathbf{X})$) that the braid group B_n on n strings acts *transitively* on the set of complete exceptional sequences.

In this talk, we consider exceptional sequences on a translation quiver Γ.

1. Preliminaries

この講演を通じて $\Gamma = \mathbb{Z}\mathrm{A}_n$ を translation quiver とし、Γ_0 を Γ の頂点集合、τ を translation とする。

定義 1.1 $X, Y \in \Gamma_0$ とする。

1. X から Y へ arrow があるとき、$X \lessdot Y$ と表す。
2. X から Y へ path があるとき、$X < Y$ と表す。
3. X と Y の間に path がないとき、$X \nsim Y$ と表す。

定義 1.2 頂点集合 Γ_0 を以下のようにして、$\{\ (p,q) \mid 1 \leq p - q \leq n\}$ と同一視する。

[1] The detailed version of this paper will be submitted for publication elsewhere.

1. 一番下の τ-orbit 上の頂点を、$\cdots, (p, p-1), (p+1, p), (p+2, p+1), \cdots$ とする。

2. $X \in \Gamma_0$ に対し、$X \lessdot (p, q)$, $X \neq (p-1, q)$ のとき、$X = (p, q-1)$ と定義する。

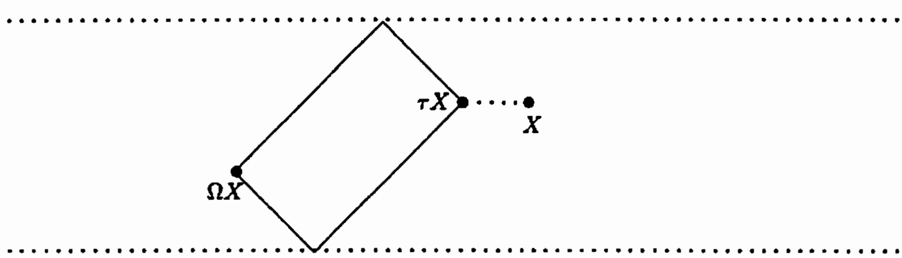
$X = (p, q), Y = (p', q') \in \Gamma_0$ に対し、X と Y が同じ τ-orbit にあるための必要十分条件は $p - q = p' - q'$ であることに注意する。

例 1.3 $\Gamma = \mathbb{Z}\mathbf{A}_4$ のとき、次のようになる。

$$
\begin{tikzcd}
 & (2,-2) \arrow[dr] \arrow[rr, dotted, no head] & & (3,-1) \arrow[dr] & \\
(1,-2) \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & (2,-1) \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & (3,0) \\
 & (1,-1) \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & (2,0) \arrow[ur] \arrow[dr] & \\
(0,-1) \arrow[ur] \arrow[rr, dotted, no head] & & (1,0) \arrow[ur] \arrow[rr, dotted, no head] & & (2,1)
\end{tikzcd}
$$

定義 1.4 syzygy functor $\Omega : \Gamma_0 \to \Gamma_0$ を、$\Omega(p, q) = (q, p - n - 1)$ と定義する。

例 1.5 各 $X \in \Gamma_0$ に対し、$\tau X, \Omega X$ は以下のような位置関係にあることに注意する。



k を標数 0 の代数的閉体とし、n を正の偶数とする。$R = k[x, y]/(y^2 - x^{n+1})$ を 1-次元次数付き環とし、mod$R$ を有限生成次数付き R-加群のなす圏で射は次数を保つものとする。さらに CMR を極大 CM 加群全体のなす充満部分圏とする。このとき、CMR の Auslander-Reiten quiver の射影加群でない極大 CM 加群全体から得られる full subquiver を Γ とおくと、$\Gamma = \mathbb{Z}\mathbf{A}_n$ である (c.f. [1])。定義 1.4 で定義している Ω は、この状況での syzygy 加群に対応している。

定義 1.6 各頂点 $X \in \Gamma_0$ に対し、$\mathrm{S}^+(X)$, $\mathrm{S}^-(X)$, $\mathrm{S}'^+(X)$, $\mathrm{S}'^-(X)$, $\Delta(X)$, $\Delta^+(X)$, $\Delta^-(X)$ を以下のように定義する。

1. $\mathrm{S}^+(X)$ は slice であり、$Y \in \mathrm{S}^+(X)$ ならば $X \leq Y$ である。

2. $S^-(X)$ は slice であり、$Y \in S^-(X)$ ならば $Y \le X$ である。

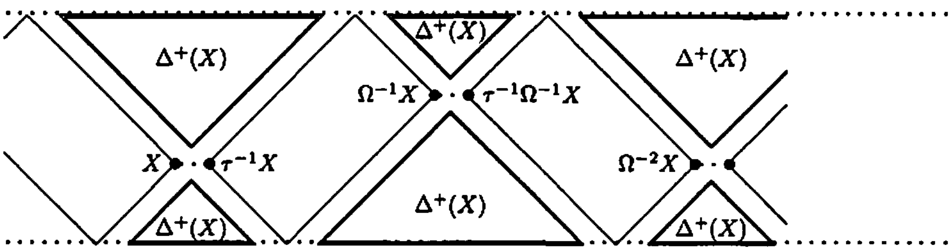
3. $S'^*(X) = S^* \setminus \{X\}$ とする。(ここで、$* = +, -$ である)

4. $\Delta(X) = \{ Y \in \Gamma_0 \mid X \nsim Y \}$ とする。

5. $\Delta^+(X) = \bigcup_{\ell \ge 0}(\Delta(\Omega^{-\ell}X) \cup S'^+(\Omega^{-\ell}X))$ とする。

6. $\Delta^-(X) = \bigcup_{\ell \le 0}(\Delta(\Omega^{-\ell}X) \cup S'^-(\Omega^{-\ell}X))$ とする。

各 $X \in \Gamma_0$ に対し、$\Delta^+(X)$ は以下のような位置関係にあることに注意する。



先程も述べたが、Γ は、1 次元次数付き環 $R = k[x,y]/(y^2 - x^{n+1})$ 上の極大 CM 加群のなす圏 CMR の Auslander-Reiten quiver の full subquiver を意識している。R-上での exceptional sequence の定義は以下の通りである。

定義 1.7 $R = \bigoplus_{i \ge 0} R_i$ を次数付き環で、$R_0 = k$ を標数 0 の代数的閉体とする。このとき、

1. 有限生成 R-加群 E が exceptional であるとは、$\begin{cases} \mathrm{Hom}(E,E) \cong k \\ \mathrm{Ext}^\ell(E,E) = 0 \ (\ell > 0) \end{cases}$ をみたすことである。

2. exceptional 加群の列 $\epsilon = (\cdots, E_i, E_{i+1}, \cdots)$ が exceptional sequence であるとは、$\mathrm{Ext}^\ell(E_i, E_j) = 0 \ (i > j, \ell \ge 0)$ をみたすことである。

$R = k[x,y]/(y^2 - x^{n+1})$ (n は正の偶数) のときには次のことがわかっている (c.f.[1],[2])。

補題 1.8 $R = k[x,y]/(y^2 - x^{n+1})$ (n は正の偶数) とし、Γ を CMR の Auslander-Reiten quiver の射影加群でない極大 CM 加群全体から得られる full subquiver とする。このとき、

1. 任意の直既約極大 CM 加群は exceptional である。

2. （射影加群でない）直既約極大 CM 加群 X, Y に対し、$\mathrm{Hom}(X,Y) \neq 0$ であるための必要十分条件は $X \leq Y$ (in Γ) である。

3. （射影加群でない）直既約極大 CM 加群 X, Y と正の整数 ℓ に対し、次は同値である。

   (a) $\mathrm{Ext}^{\ell}(X,Y) \neq 0$

   (b) $\Omega^{\ell} X \leq Y \leq \tau\Omega^{\ell-1}X$ (in Γ)

   (c) $\tau^{-1}\Omega^{-\ell+1}Y \leq X \leq \Omega^{-\ell}Y$ (in Γ)

注意 1.9 R, Γ を補題 1.8 の通りとし、X, Y を射影加群でない直既約極大 CM 加群とする。このとき、次は同値である。

1. すべての整数 ℓ に対し、$\mathrm{Ext}^{\ell}(X,Y) = 0$ である。

2. $X \in \Delta^{+}(Y)$ である。

3. $Y \in \Delta^{-}(X)$ である。

これらのことから、一般の translation quiver $\Gamma = \mathbb{Z}A_n$ に対し、exceptional sequence を以下のように定義する。

定義 1.10 頂点 $E_1, E_2, \cdots, E_r \in \Gamma_0$ に対し、列 $\epsilon = (E_1, E_2, \cdots, E_r)$ が exceptional sequence であるとは、次の条件をみたすことである。

$$E_i \in \bigcap_{j<i} \Delta^{+}(E_j) \ (1 < i \leq r)$$

この条件は次の条件と同値である。

$$E_i \in \bigcap_{j>i} \Delta^{-}(E_j) \ (1 \leq i < r)$$

2. Main results

主結果を述べるためにもう少し準備をする。

定義 2.1 $\epsilon = (E_1, E_2, \cdots, E_r)$ を exceptional sequence とする。E_i, E_j が次の二条件をみたすとき、$E_i <_{\epsilon} E_j$ と表す。

1. $E_i \in \mathrm{S}'^{-}(E_j)$ である。

2. もし $E_i \le E_m \le E_j$ ならば、 $m = i$ または $m = j$ である。

exceptional sequence の定義より次のことが成立していることが容易に確かめられる。

補題 2.2 $\epsilon = (E_1, E_2, \cdots, E_r)$ を exceptional sequence とする。このとき、次のどの列も exceptional sequence になる。

1. $E_{i-1} \not\sim E_i$ のとき、 $(E_1, E_2, \cdots, E_{i-2}, E_i, E_{i-1}, E_{i+1}, \cdots, E_r)$

2. $E_j \notin \mathrm{S}'^{-}(E_i)$ $(\forall j)$ のとき、$(E_1, E_2, \cdots, E_{i-1}, \Omega E_i, E_{i+1}, \cdots, E_r)$

3. $E_j \notin \mathrm{S}'^{+}(E_i)$ $(\forall j)$ のとき、$(E_1, E_2, \cdots, E_{i-1}, \Omega^{-1} E_i, E_{i+1}, \cdots, E_r)$

4. $E_i = (p, q)$, $E_j \lessdot_\epsilon E_i$ ならば $j = i - 1$ のとき、

$$(E_1, E_2, \cdots, E_{i-2}, E'_i, E_{i-1}, E_{i+1}, \cdots, E_r)$$

$$\text{但し、} E'_i = \begin{cases} (q, q') & (E_{i-1} = (p, q') \text{ のとき}) \\ (p', p - n - 1) & (E_{i-1} = (p', q) \text{ のとき}) \end{cases}$$

5. $E_i = (p, q)$, $E_i \lessdot_\epsilon E_j$ ならば $j = i + 1$ のとき、

$$(E_1, E_2, \cdots, E_{i-2}, E'_i, E_{i-1}, E_{i+1}, \cdots, E_r)$$

$$\text{但し、} E'_i = \begin{cases} (q + n + 1, q') & (E_{i+1} = (p, q') \text{ のとき}) \\ (p', p) & (E_{i+1} = (p', q) \text{ のとき}) \end{cases}$$

6. $E_{i-2} = (p, q')$, $E_{i-1} = (p', q)$, $E_i = (p, q)$ のとき、

$$(E_1, E_2, \cdots, E_{i-3}, E'_i, E_{i-2}, E_{i-1}, E_{i+1}, \cdots, E_r)$$

$$\text{但し、} E'_i = (p', q')$$

7. $E_i = (p, q)$, $E_{i+1} = (p', q)$, $E_{i+2} = (p, q')$ のとき、

$$(E_1, E_2, \cdots, E_{i-1}, E_{i+1}, E_{i+2}, E'_i, E_{i+3}, \cdots, E_r)$$

$$\text{但し、} E'_i = (p', q')$$

定義 2.3 ϵ, ϵ' を exceptional sequence とする。ϵ に補題 2.2 の変形を有限回行って ϵ' になるとき、$\epsilon \sim \epsilon'$ と表す。

定理 2.4 ϵ を exceptional sequence とする。このとき、exceptional sequence ϵ' と slice S で、$\epsilon \sim \epsilon'$, ϵ' は S に埋め込めるものが存在する。

証明 二段階に分けて証明する。

Step 1. $\epsilon = (E_1, E_2, \cdots, E_r)$ とおくとき、$\epsilon' = (E_1', E_2', \cdots, E_r')$ で、$\epsilon \sim \epsilon'$, $E_1' = E_1$, $E_i' \in \mathrm{S}^-(E_1') \cup \Delta(E_1') \cup \mathrm{S}^+(E_1')$ $(\forall i)$ をみたすものが存在する。

各 i に対し、ℓ_i を $E_i \in \Delta(\Omega^{-\ell_i}E_1) \cup \mathrm{S}^+(\Omega^{-\ell_i}E_1)$ をみたす数として定義する。
exceptional sequence の定義より、$\ell_i \geq 0$ であるので、$\ell = \sum_{i=1}^r \ell_i$ に関する帰納法で示す。$\ell = 0$ ならば $\epsilon' = \epsilon$ ととればいい。$\ell > 0$ のとき、$i = \min\{\, j \mid \ell_j > 0 \,\}$ とおく。
$E_j \notin \mathrm{S}'^-(E_i)$ $(\forall j)$ のとき、$\epsilon'' = (E_1, E_2, \cdots, E_{i-1}, \Omega E_i, E_{i+1}, \cdots, E_r)$ とおく。このとき、補題 2.2.2 より、$\epsilon \sim \epsilon''$ であり、$\Omega E_i \in \Delta(\Omega^{-(\ell_i-1)}E_1) \cup \mathrm{S}^+(\Omega^{-(\ell_i-1)}E_1)$ なので、帰納法の仮定より条件をみたす ϵ' をとることができる。
$E_j \in \mathrm{S}'^-(E_i)$ なる j が存在するとき、$E_j \lessdot_\epsilon E_i$ とする。このとき、E_j の取り方より、(必要ならば補題 2.2.1 を使うことで) $j = i-1$ としてよい。$E_j \lessdot_\epsilon E_i$ をみたす j が $i-1$ のときのみの場合には、ϵ'' を補題 2.2.4 のようにとる。$j \neq i-1$ なる j で $E_j \lessdot_\epsilon E_i$ をみたすものが存在するとき、(必要ならば補題 2.2.1 を使うことで) $j = i-2$ とできる。そして、ϵ'' を補題 2.2.6 のようにとる。いずれの場合でも $\epsilon \sim \epsilon''$ である。ここで、i の取り方から　$\ell_{i-1} = 0$ であり、$E_i' < E_{i-1}$ より $E_i' \in \Delta(E_1) \cup \mathrm{S}^+(E_1)$ である。よってこの場合も帰納法の仮定より条件をみたす ϵ' をとることができる。

Step 2. ϵ' を step 1. のようにとると、ϵ' はある slice S に埋め込むことができる。

$\epsilon' = (E_1', E_2', \cdots, E_r')$ とおく。exceptional sequence の定義より、任意の $i \neq j$ に対し $E_i' \in \mathrm{S}'^-(E_j') \cup \Delta(E_j') \cup \mathrm{S}'^+(E_j')$ をみたすことに注意する。さらに、$i \neq j$ ならば、E_i' と E_j' は異なる τ-orbit にあることに注意する。$E_i' = (p_i, q_i)$ とおき、$p_i - q_i = t_i$ とおく。このとき、S を次のようにとる。すべての E_i' は S に属しているとする。$1 \leq t \leq r$, $t \notin \{\, t_1, t_2, \cdots t_r \,\}$ に対し、$t_i = \max\{\, t_i \mid t_i < t \,\}$, $t_j = \min\{\, t_j \mid t_j > t \,\}$ をとる。さらに、$Y \in \mathrm{S}'^+(E_i') \cap \mathrm{S}'^+(E_j')$ をとる。そして、$X = (p, q)$ を $p - q = t$, $X \in \mathrm{S}^-(Y) \cap (\mathrm{S}'^+(E_i') \cup \mathrm{S}'^+(E_j'))$ ととり、この X が S に属するとすると、S は slice である。□

系 2.5 $\epsilon = (E_1, E_2, \cdots, E_r)$ を exceptional sequance とすると、$r \leq n$ である。

系 2.6 長さ n の任意の exceptional sequance ϵ, ϵ' に対し、$\epsilon \sim \epsilon'$ である。

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FROBENIUS FULL MATRIX ALGEBRAS AND GORENSTEIN TILED ORDERS

HISAAKI FUJITA AND YOSUKE SAKAI

Let D be a discrete valuation ring with a unique maximal ideal πD, and let Λ be a D-order. It is standard to reduce homological properties of Λ to those of the factor algebras $\Lambda/\pi\Lambda$ and such factor algebras are deserving of further study. (See [2].)

Let n be an integer with $n \geq 2$. In [1], we introduced an $n \times n$ $\mathbb{A}$-*full matrix algebra* over a field K, whose multiplication is determined by a *structure system* $\mathbb{A}$, that is, an n-tuple of $n \times n$ matrices with certain properties. $\mathbb{A}$-full matrix algebras are associative, basic, connected K-algebras. A prototype of $\mathbb{A}$-full matrix algebras is the class of factor algebras $\Lambda/\pi\Lambda$ of tiled D-orders Λ. Studying representation matrices of certain modules over $\mathbb{A}$-full matrix algebras, Frobenius $\mathbb{A}$-full matrix algebras are characterized by the shape of their structure systems $\mathbb{A}$. For a Gorenstein tiled D-order Λ, the factor algebra $\Lambda/\pi\Lambda$ is a Frobenius $\mathbb{A}$-full matrix algebra. In this paper we study the converse of this fact. Our main result is the following.

Theorem 1. (1) *For $2 \leq n \leq 7$, all Frobenius $n \times n$ $\mathbb{A}$-full matrix algebras are isomorphic to $\Lambda/\pi\Lambda$ for some Gorenstein tiled D-orders Λ. Moreover a list of them (up to isomorphism) is obtained.*

(2) *For each $n \geq 8$, there is a Frobenius $n \times n$ $\mathbb{A}$-full matrix algebra having no corresponding Gorenstein tiled D-orders.*

1. $\mathbb{A}$-FULL MATRIX ALGEBRAS

We begin by recalling $\mathbb{A}$-full matrix algebras. Let K be a field and $n \geq 2$ an integer. Let $\mathbb{A} = (A_1, \ldots, A_n)$ be an n-tuple of $n \times n$ matrices $A_k = (a_{ij}^{(k)}) \in \mathbb{M}_n(K)$ $(1 \leq k \leq n)$ satisfying the following three conditions.

(A1) $a_{ij}^{(k)} a_{il}^{(j)} = a_{il}^{(k)} a_{kl}^{(j)}$ for all $i, j, k, l \in \{1, \ldots, n\}$,

(A2) $a_{kj}^{(k)} = a_{ik}^{(k)} = 1$ for all $i, j, k \in \{1, \ldots, n\}$, and

(A3) $a_{ii}^{(k)} = 0$ for all $i, k \in \{1, \ldots, n\}$ such that $i \neq k$.

Let $A = \bigoplus_{1 \leq i,j \leq n} K u_{ij}$ be a K-vector space with basis $\{u_{ij} \mid 1 \leq i, j \leq n\}$. Then we define multiplication of A by using $\mathbb{A}$, that is,

$$u_{ik} u_{lj} := \begin{cases} a_{ij}^{(k)} u_{ij} & \text{if } k = l \\ 0 & \text{otherwise.} \end{cases}$$

Then $u_{11}, \ldots, u_{nn}$ are orthogonal primitive idempotents such that $u_{11} + \cdots + u_{nn} = 1_A$ an identity of A and $u_{ii} A u_{jj} \cong K$. Hence A is an associative, basic, connected K-algebra.

The detailed version of this paper will be submitted for publication elsewhere.

We call A an $n \times n$ $\mathbb{A}$-*full matrix algebra* with a *structure system* $\mathbb{A}$. We note that gl.dim$A = \infty$, because every entry of the Cartan matrix of A is 1.

In what follows, we assume that $a_{ij}^{(k)} = 0$ or 1 for all $1 \leq i, k, j \leq n$.

2. Tiled orders

Let D be a discrete valuation ring with a unique maximal ideal πD and $n \geq 2$ an integer. Let $\{\lambda_{ij} \mid 1 \leq i, j \leq n\}$ be a set of integers satisfying

$$\lambda_{ij} \geq 0, \quad \lambda_{ii} = 0, \quad \lambda_{ik} + \lambda_{kj} \geq \lambda_{ij}, \quad \text{and} \quad \lambda_{ij} + \lambda_{ji} > 0 \quad (\text{if } i \neq j)$$

for all $1 \leq i, j, k \leq n$. Then $\Lambda = (\pi^{\lambda_{ij}} D)$ is a subring of $\mathbb{M}_n(D)$, which we call an $n \times n$ *tiled D-order.*

Example 2. Let $\Lambda = (\pi^{\lambda_{ij}} D)$ be an $n \times n$ tiled D-order. Put $A := \Lambda/\pi\Lambda$, $K := D/\pi D$ and $u_{ij} := \pi^{\lambda_{ij}} e_{ij} + \pi\Lambda \in A$, where e_{ij}'s are the matrix units in $\mathbb{M}_n(D)$. Define $A_k = (a_{ij}^{(k)}) \in \mathbb{M}_n(K)$ $(1 \leq k \leq n)$ by

$$a_{ij}^{(k)} := \begin{cases} 1 & \text{if } \lambda_{ik} + \lambda_{kj} = \lambda_{ij} \\ 0 & \text{otherwise,} \end{cases}$$

and set $\mathbb{A} := (A_1, \ldots, A_n)$. Then note that

$$u_{ik} u_{lj} = \begin{cases} a_{ij}^{(k)} u_{ij} & \text{if } k = l \\ 0 & \text{otherwise.} \end{cases}$$

Hence A is an $\mathbb{A}$-full matrix algebra.

3. Representation matrices

Let $A = \bigoplus_{1 \leq i,j \leq n} u_{ij} K$ be an $n \times n$ $\mathbb{A}$-full matrix algebra, where $\mathbb{A} = (A_1, \ldots, A_n)$ and $A_k = (a_{ij}^{(k)}) \in \mathbb{M}_n(K)$ $(1 \leq k \leq n)$. Let M be a right A-module with dimension type $\underline{\dim} M = (1, \ldots, 1)$. Then M has a K-basis $\{v_i \mid 1 \leq i \leq n\}$ such that $v_i u_{ii} = v_i$ for all $1 \leq i \leq n$. Hence there exists a matrix $S = (s_{ij}) \in \mathbb{M}_n(K)$ such that $v_i u_{ij} = s_{ij} v_j$ for all $1 \leq i, j \leq n$. We call S a *representation matrix* of M.

Proposition 3. *For each indecomposable projective right A-module $u_{ii}A$, $\underline{\dim} u_{ii} A = (1, \ldots, 1)$ and it has a representation matrix $(a_{ij}^{(k)})_{k,j}$, that is, an $n \times n$ matrix whose (k, j)-entry is $a_{ij}^{(k)}$. Moreover $u_{ii}A$ is isomorphic to an injective* $\mathrm{Hom}_K(Au_{ll}, K)$ *if and only if $a_{il}^{(k)} = 1$ for all $1 \leq k \leq n$.*

Example 4. Let A be an $\mathbb{A}$-full matrix algebra where

$$\mathbb{A} = \begin{pmatrix} 1&1&1&1 & 0&1&1&0 & 0&0&1&0 & 0&0&1&1 \\ 1&0&0&1 & 1&1&1&1 & 1&0&1&1 & 0&0&0&1 \\ 1&1&0&1 & 0&1&0&0 & 1&1&1&1 & 0&0&0&1 \\ 1&1&0&0 & 0&1&0&0 & 1&1&1&0 & 1&1&1&1 \end{pmatrix}.$$

Then representation matrices of $u_{11}A, \dots, u_{44}A$ are given by

$$\begin{pmatrix} 1&1&1&1\\ 0&1&1&0\\ 0&0&1&0\\ 0&0&1&1 \end{pmatrix} \quad \begin{pmatrix} 1&0&0&1\\ 1&1&1&1\\ 1&0&1&1\\ 0&0&0&1 \end{pmatrix} \quad \begin{pmatrix} 1&1&0&1\\ 0&1&0&0\\ 1&1&1&1\\ 0&0&0&1 \end{pmatrix} \quad \begin{pmatrix} 1&1&0&0\\ 0&1&0&0\\ 1&1&1&0\\ 1&1&1&1 \end{pmatrix}.$$

Hence $u_{11}A, u_{22}A$ and $u_{44}A$ are injective but not $u_{33}A$.

4. Frobenius $\mathbb{A}$-full matrix algebras

By means of structure systems, we can characterize Frobenius $\mathbb{A}$-full matrix algebras. Let $A = \bigoplus_{1\le i,j\le n} u_{ij}K$ be an $n \times n$ $\mathbb{A}$-full matrix algebra, where $\mathbb{A} = (A_1, \dots, A_n)$ and $A_k = (a_{ij}^{(k)}) \in \mathbb{M}_n(K)$ $(1 \le k \le n)$.

Proposition 5. *The following are equivalent for an $\mathbb{A}$-full matrix algebra A.*

(1) *A is a Frobenius algebra.*

(2) *There exists a permutation σ of the set $\{1, \dots, n\}$ such that $\sigma(i) \ne i$ for all $1 \le i \le n$ and that $a_{i\sigma(i)}^{(k)} = 1$ for all $1 \le i, k \le n$.*

In this case σ is a Nakayama permutation of A, that is, $\mathrm{soc}(u_{ii}A) \cong \mathrm{top}(u_{\sigma(i)\sigma(i)}A)$ for all $1 \le i \le n$. Moreover, for all $1 \le i, k, j \le n$, $a_{ij}^{(k)} = a_{k\sigma(i)}^{(j)}$ holds.

Using Proposition 5, we can find structure systems $\mathbb{A}$ which define Frobenius $\mathbb{A}$-full matrix algebras. Let σ be a permutation of the set $\{1, \dots, n\}$ such that $\sigma(i) \ne i$ for all $1 \le i \le n$. Let T be the set of triples (i, k, j) of integers $1 \le i, k, j \le n$. Then we have a bijection

$$\varphi : T \to T,\ (i, k, j) \mapsto (k, j, \sigma(i)).$$

Decompose T into φ-orbits $\{T_\alpha\}_\alpha$, and put $I := \cup\{T_\alpha \mid (i, k, \sigma(i)) \in T_\alpha\}$, $Z := \cup\{T_\alpha \mid (i, k, i) \in T_\alpha, i \ne k\}$, and $X := \cup\{T_\alpha \mid T_\alpha \not\subset I \cup Z\}$. Then we have $T = I \cup Z \cup X$ (disjoint).

Proposition 6. (1) *Suppose that $\mathbb{A}$ is a Frobenius structure system with Nakayama permutation σ. Then there exists a φ-invariant subset Y of X such that*

$$a_{ij}^{(k)} = \begin{cases} 1 & \text{if } (i,k,j) \in I \cup Y \\ 0 & \text{otherwise.} \end{cases}$$

(2) *Let Y be a φ-invariant subset of X, and define $\mathbb{A}(Y) = (a_{ij}^{(k)})$ by*

$$a_{ij}^{(k)} := \begin{cases} 1 & \text{if } (i,k,j) \in I \cup Y \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathbb{A}(Y)$ is a Frobenius structure system whenever (A1) *holds for $\mathbb{A}(Y)$.*

(3) *For the empty subset $\emptyset$ of X, $\mathbb{A}(\emptyset)$ is a Frobenius structure system.*

5. φ-orbits for a cyclic permutation

In this section, we clarify the φ-orbits of T for a cyclic permutation $\sigma = (1\ 2\ \cdots\ n)$. First we count the number of φ-orbits of T.

Proposition 7. (1) *For a φ-orbit T_α of T, the number $|T_\alpha|$ of elements in T_α is $3n$ or n.*

(2) *T has a φ-orbit T_α with $|T_\alpha| = n$ if and only if n is not divisible by 3. In this case, T has a unique φ-orbit having n elements, which is contained in X.*

(3) *I has $n-1$ φ-orbits.*

(4) *Z has $n-2$ φ-orbits.*

(5) *If n is divisible by 3, then X has $(n-1)^2/3$ φ-orbits.*

(6) *If n is not divisible by 3, then X has $(n-2)(n-4)/3+1$ φ-orbits.*

Next we clarify the members of each φ-orbit of T.

Proposition 8. *Let T_α be a φ-orbit of T and put $T_\alpha^{(r)} := \{(i,k,j) \in T_\alpha \mid k = r\}$ for all $1 \le r \le n$.*

(1) *Suppose that $|T_\alpha| = 3n$. Then $|T_\alpha^{(r)}| = 3$ for each $r = 1,\dots,n$. If $(i,1,j) \in T_\alpha$, then*

$$T_\alpha^{(1)} = \{(i,1,j),\ (\sigma^{-j+1}(1),1,\sigma^{-j+2}(i)),\ (\sigma^{-i}(j),1,\sigma^{-i+1}(1))\}.$$

(2) *Suppose that $|T_\alpha| = n$. Then $|T_\alpha^{(r)}| = 1$ for each $r = 1,\dots,n$. If $n = 3t+1$ then $T_\alpha^{(1)} = \{(t+1,1,2t+2)\}$. If $n = 3t+2$ then $T_\alpha^{(1)} = \{(2t+2,1,t+2)\}$.*

6. Minimal Frobenius structure systems

Let A be a Frobenius $\mathbb{A}$-full matrix algebra with Nakayama permutation σ. Then it follows from Proposition 6 (1) that $\mathbb{A}$ is determined by a φ_σ-invariant subset Y of X. We call $\mathbb{A}$ a *minimal Forbenius structure system* if Y is minimal among φ_σ-invariant subset of X which define Frobenius full matrix algebras. For a cyclic permutation, minimal Frobenius structure systems are determined by the following theorem.

Theorem 9. *Let n be an integer with $n \ge 4$, and let $\sigma = (1\ 2\ \cdots\ n)$ be a cyclic permutation. Then the following statements hold.*

(1) *Let n be even. Then the φ-invariant subsets defining minimal Frobenius structure systems are just φ-orbits contained in X.*

(2) *Let n be odd and $n = 2s+1$ for some s. Then the φ-invariant subsets defining minimal Frobenius structure systems are just φ-orbits X_β contained in X such that X_β does not contain any element of the form $(s+1,1,k)$ for any k with $k \not\equiv s^2+1 \pmod{n}$.*

The following example illustrates Theorem 9.

Example 10. Let $n = 7$. Then X has 6 φ-orbits X_i $(1 \le i \le 6)$ such that

$$\begin{aligned}
X_1^{(1)} &= \{(4,1,3),(6,1,3),(6,1,5)\}\\
X_2^{(1)} &= \{(2,1,5),(3,1,7),(4,1,6)\}\\
X_3^{(1)} &= \{(2,1,6),(4,1,7),(3,1,5)\}\\
X_4^{(1)} &= \{(5,1,3),(5,1,4),(6,1,4)\}\\
X_5^{(1)} &= \{(2,1,4),(2,1,7),(5,1,7)\}\\
X_6^{(1)} &= \{(3,1,6)\}
\end{aligned}$$

Since $7 = 2\cdot 3+1$ and $3^2+1 \equiv 3 \pmod{7}$, there are minimal Frobenius structure systems corresponding to X_1, X_4, X_5, X_6, but not to X_2, X_3.

7. GORENSTEIN TILED ORDERS

A D-order Λ is *Gorenstein* if $\mathrm{Hom}_D(\Lambda, D)$ is projective as a right (or left) Λ-module. It is known that for an $n \times n$ tiled D-order $\Lambda = (\pi^{\lambda_{ij}} D)$, Λ is Gorenstein if and only if there exists a permutation σ such that $\lambda_{ik} + \lambda_{k\sigma(i)} = \lambda_{i\sigma(i)}$ for all $1 \leq i, k \leq n$. (See [4].) Since $\mathrm{id}_\Lambda \Lambda = \mathrm{id}_{\Lambda/\pi\Lambda} \Lambda/\pi\Lambda + 1$ and $\mathrm{id}_\Lambda \Lambda = \text{lattice-inj.dim}_\Lambda \Lambda + 1$, Λ is Gorenstein if and only if $\Lambda/\pi\Lambda$ is Frobenius. (We note that this fact follows from Proposition 5 in our context.) Hence some Frobenius $\mathbb{A}$-full matrix algebras are isomorphic to the factor algebras $\Lambda/\pi\Lambda$ for some Gorenstein tiled D-orders Λ. However the converse is not true.

Theorem 11. *For every integer $n \geq 8$, there exists a Frobenius $n \times n$ full matrix algebra which has no corresponding Gorenstein tiled orders.*

Proof. Let σ be a cyclic permutation $(1\ 2 \cdots\ n)$. Let X_β be a φ-orbit containing $(2, 1, 5)$. Then, since $n \geq 8$, it follows from Theorem 9 that X_β defines a minimal Frobenius structure system $\mathbb{A}$. Suppose that there exists a Gorenstein tiled D-order $\Lambda = (\pi^{\lambda_{ij}} D)$ such that $\Lambda/\pi\Lambda$ is a Frobenius $\mathbb{A}$-full matrix algebra. We may assume that $\lambda_{1j} = 0$ for all $1 \leq j \leq n$ by [3, Lemma 1.1]. Then we have $\lambda_{53} = 0$, which implies that $\lambda_{54} = 0$, so that $(3, 1, 5) \in X_\beta$, a contradiction. $\square$

In the ending part of the proof, we need a technical argument not included in this report, but the following example may be helpful to see the proof.

Example 12. When $n = 8$, the exponent matrix (λ_{ij}) of Gorenstein tiled D-orders $\Lambda = (\pi^{\lambda_{ij}} D)$ is given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & a & b & c & d & c & b \\ b & 0 & 0 & b & r & s & s & r \\ c & 0 & x & 0 & c & s & t & s \\ d & 0 & y & y & 0 & d & s & s \\ c & 0 & z & w & z & 0 & c & r \\ b & 0 & y & w & w & y & 0 & b \\ a & 0 & x & y & z & y & x & 0 \end{pmatrix}$$

where $x := a - b$, $y := a - c$, $z := a - d$, $w := a + b - c - d$, $r := b + c - a$, $s := c + d - a$, and $t := 2c + d - b - a$.

8. THE CASE OF $2 \leq n \leq 7$

For each n ($2 \leq n \leq 7$), we can verify that every Frobenius $n \times n$ $\mathbb{A}$-full matrix algebra A has a Gorenstein tiled D-order Λ such that $\Lambda/\pi\Lambda \cong A$. The following table shows that how many isomorphism classes of Frobenius $\mathbb{A}$-full matrix algebras are for each $n = 2, \ldots, 7$.

n	no. of iso. classes
2	1
3	1
4	3
5	4
6	21
7	17

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A construction of Auslander-Gorenstein rings*

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This is a summary of my joint work with T. Shiba.

Recall that a ring A is said to be an Auslander-Gorenstein ring if it is a left and right noetherian ring and for a minimal injective resolution $E^{\bullet}$ of A_A we have flat dim $E^n \leq n$ for all $n \geq 0$ (cf. [1]). It is well known that the group ring of a finite group over a commutative Gorenstein ring is an Auslander-Gorenstein ring. We refer to[1] for other examples of Auslander-Gorenstein rings. Our main aim is to construct another type of Auslander-Gorenstein rings.

Let R be a ring. In this talk, a ring A is said to be a Frobenius extension of R if it contains R as a subring and satisfies the conditions (F1) A_R and $_RA$ are finitely generated projective, (F2) $A_A \cong \mathrm{Hom}_R({}_AA_R, R_R)$ and $_AA \cong \mathrm{Hom}_R({}_RA_A, {}_RR)$, and (F3) the inclusion $R \to A$ is a split monomorphism of R-R-bimodules (cf. [2]). Again, the group ring of a finite group is a typical example of Frobenius extensions. If A is a Frobenius extension of R, then (1) inj dim A_A = inj dim R_R, (2) inj dim $_AA$ = inj dim $_RR$, and (3) A is an Auslander-Gorenstein ring if R is so. Therefore we will provide a way to construct Frobenius extensions of a given ring.

Let R be a ring, $n \geq 2$ an integer and ν a permutation of $I = \{1, 2, \cdots, n\}$. We will construct a family of Frobenius extensions A of R such that (i) $1_A = \Sigma_{i \in I}\, e_i$ with the e_i orthogonal idempotents, (ii) $e_iA_A \not\cong e_jA_A$ unless $i = j$, (iii) $xe_i = e_ix$ for all $i \in I$ and $x \in R$, (iv) $e_iA_A \cong \mathrm{Hom}_R({}_AAe_{\nu(i)R}, R_R)$ and $_AAe_{\nu(i)} \cong \mathrm{Hom}_R({}_Re_iA_A, {}_RR)$ for all $i \in I$, and (v) there exists $\eta \in \mathrm{Aut}(A)$ with $\eta(e_i) = e_{\nu(i)}$ for all $i \in I$. Furthermore, the rings e_iAe_i are local if R is so. In particular, if R is a quasi-Frobenius local ring, then A is a quasi-Frobenius ring with $\mathrm{soc}(e_iA_A) \cong e_{\nu(i)}A/e_{\nu(i)}J$ for all $i \in I$, where $J = \mathrm{rad}(A)$. In case ν has no fixed point, we can construct a desired Frobenius extension A of R as a skew matrix ring over R, the notion of which was first introduced in [3] (cf. also [4] and [5]). If ν has a fixed point, then we can not construct a desired Frobenius extension of R as a skew matrix ring over R, but we can construct a desired Frobenius extension B of R which contains an ideal V with B/V a skew matrix ring over R, where $V^2 \neq 0$ in general.

Throughout this note, rings are associative rings with identity. For a ring R, we denote by $R^{\times}$ the set of units and by rad(R) the Jacobson radical. We use the notation X_R (resp., $_RX$) to denote that the module X considered is a right(resp., left) R-module. Also, we use the notation $_SX_R$ to denote that X is an S-R-bimodule. We denote by Mod-R the category of right R-modules.

1. Frobenius extension of rings

In this section, we introduce a notion of Frobenius extension of rings (cf. [2]).

Definition 1.1. A ring A is said to be a Frobenius extension of a ring R if there exists an injective ring homomorphism $\varphi: R \to A$ satisfying the following conditions:

(F1) A_R and $_RA$ are finitely generated projective;

*The detailed version will be submitted for publication elsewhere.

(F2) $A_A \cong \mathrm{Hom}_R({}_AA_R, R_R)$ and ${}_AA \cong \mathrm{Hom}_R({}_RA_A, {}_RR)$; and

(F3) φ is a split monomorphism of R-R-bimodules.

Remark 1.2. Let A be a Frobenius extension of R. Then for an isomorphism $\phi : A_A \xrightarrow{\sim} \mathrm{Hom}_R({}_AA_R, R_R)$ we have a unique ring homomorphism $\theta : R \rightarrow A$ such that $x\phi(1) = \phi(1)\theta(x)$ for all $x \in R$. Similarly, for an isomorphism $\psi : {}_AA \xrightarrow{\sim} \mathrm{Hom}_R({}_RA_A, {}_RR)$ we have a unique ring homomorphism $\eta : R \rightarrow A$ such that $\psi(1)x = \eta(x)\psi(1)$ for all $x \in R$.

Definition 1.3 (cf. [1]). A ring R is said to be an Auslander-Gorenstein ring if it is left and right noetherian and for a minimal injective resolution $E^{\bullet}$ of R_R we have flat dim $E^n \leq n$ for all $n \geq 0$.

Definition 1.4. A ring R is said to be a quasi-Frobenius ring if it is left and right artinian and left and right selfinjective.

Proposition 1.5. *Let A be a Frobenius extension of R. Then the following hold.*

(1) inj dim A_A = inj dim R_R.

(2) inj dim ${}_AA$ = inj dim ${}_RR$.

(3) *A is an Auslander-Gorenstein ring if R is so.*

(4) *A is a quasi-Frobenius ring if R is so.*

Remark 1.6. The converse holds in (3) and (4) of Proposition 1.5. However, we do not need this fact in the present note.

In the following, taking Proposition 1.5 into account, we provide a way to construct Frobenius extensions of an arbitrary ring.

2. Skew matrix rings

In the following, we fix a ring R and a pair of $\sigma \in \mathrm{Aut}(R)$ and $c \in R$ such that

$$\sigma(c) = c \text{ and } xc = c\sigma(x) \text{ for all } x \in R.$$

In this section, we develop the construction of skew matrix rings given in [3], [4] and [5]. Let $n \geq 2$ be an integer and $I = \{1, 2, ..., n\}$. Let $\omega : I \times I \rightarrow \mathbb{Z}$ be a mapping and set

$$\lambda(i, j, k) = \omega(i, j) + \omega(j, k) - \omega(i, k)$$

for $i, j, k \in I$. We assume the following conditions are satisfied:

$$\omega(i, i) = 0 \text{ for all } i \in I \text{ and } \lambda(i, j, k) \geq 0 \text{ for all } i, j, k \in I.$$

Definition 2.1. Let A be a free right R-module with a basis $\{e_{ij}\}_{i,j \in I}$ and define the multiplication on A subject to the following axioms:

(A1) $xe_{ij} = e_{ij}\sigma^{\alpha(i,j)}(x)$ for all $i, j \in I$ and $x \in R$;
(A2) $e_{ij}e_{kl} = 0$ unless $j = k$; and
(A3) $e_{ij}e_{jk} = e_{ik}c^{\lambda(i,j,k)}$ for all $i, j, k \in I$.

Proposition 2.2. *The following hold.*
(1) *A is an associative ring with $1_A = \sum_{i \in I} e_{ii}$, where the e_{ii} are orthogonal idempotents.*
(2) *We have an injective ring homomorphism $\varphi: R \to A, x \mapsto \sum_{i \in I} e_{ii}x$ which is a split monomorphism of R-R-bimodules.*
(3) $e_{ii}A_A \cong e_{jj}A_A$ for all $i, j \in I$ *with $\lambda(i, j, i) = 0$. In case $c \notin R^{\times}$, the converse holds.*

In the following, we consider R as a subring of A via $\varphi: R \to A$. Note that A is a free left R-module with a basis $\{e_{ij}\}_{i,j \in I}$. Also, for any $i \in I$, since $xe_{ii} = e_{ii}x$ for all $x \in R$, Ae_{ii} is an A-R-bimodule and $e_{ii}A$ is an R-A-bimodule.

Remark 2.3. Denote by $\mathrm{M}_n(R)$ the $n \times n$ full matrix ring over R. Then for any $i \in I$ there exists a ring homomorphism of the form

$$\zeta: A \tilde{\to} \mathrm{M}_n(R), \textstyle\sum_{j,k \in I} e_{jk}x_{jk} \mapsto (\sigma^{-\alpha(i,j)}(c^{\lambda(i,j,k)}x_{jk}))_{j,k \in I}$$

which is an isomorphism if either $c \in R^{\times}$ or $\lambda = 0$. Also, if c is regular, then ζ is injective.

In the following, taking Remark 2.3 into account, we assume $c \notin R^{\times}$. However, for the sake of convenience, we do not exclude the case where $\lambda = 0$.

Definition 2.4. There exists a basis $\{\alpha_{ij}\}_{i,j \in I}$ for ${}_R\mathrm{Hom}_R(A_R, {}_RR_R)$ such that $a = \sum_{i,j \in I} e_{ij}\alpha_{ij}(a)$ for all $a \in A$. Similarly, there exists a basis $\{\beta_{ij}\}_{i,j \in I}$ for $\mathrm{Hom}_R({}_RA, {}_RR_R)_R$ such that $a = \sum_{i,j \in I} \beta_{ij}(a)e_{ij}$ for all $a \in A$.

Lemma 2.5. *For any $i, k \in I$ the following are equivalent.*
(1) *$\lambda(i, j, k) = 0$ for all $j \in I$.*
(2) *There exist isomorphisms of the form*

$$\phi_{ik}: e_{ii}A_A \tilde{\to} \mathrm{Hom}_R({}_AAe_{kkR}, R_R), a \mapsto \alpha_{ik}a,$$
$$\psi_{ki}: {}_AAe_{ii} \tilde{\to} \mathrm{Hom}_R({}_Re_{ii}A_A, {}_RR), a \mapsto a\beta_{ik}.$$

(3) *Either* $e_{ii}A_A \cong \mathrm{Hom}_R({}_AAe_{kkR}, R_R)$ *or* ${}_AAe_{kk} \cong \mathrm{Hom}_R({}_Re_{ii}A_A, {}_RR)$.

Proposition 2.6. *Assume R is a local ring. Then the following hold.*
(1) *Every e_{ii} is a local idempotent. In particular, A is a semiperfect ring.*
(2) *Assume either $A_A \cong \mathrm{Hom}_R({}_AA_R, R_R)$ or ${}_AA \cong \mathrm{Hom}_R({}_RA_A, {}_RR)$. Then there exists $\nu \in \mathrm{Aut}(I)$ such that $\lambda(i, j, \nu(i)) = 0$ for all $i, j \in I$.*

In the next Proposition, we refer to [6] for derived equivalence of rings.

Proposition 2.7. *Assume c is regular. Then for any $i \in I$ the following hold.*
(1) *A is derived equivalent to a generalized triangular matrix ring*

$$\begin{bmatrix} R & \mathrm{Ext}^1_A(Ae_{ii}A, e_{ii}A) \\ 0 & A/Ae_{ii}A \end{bmatrix}$$

(2) *Assume there exists $j \in I\setminus\{i\}$ such that $\lambda(j,k,i) = 0$ for all $k \in I$. Then $\mathrm{Ext}^1_A(A/Ae_{ii}A, e_{ii}A) \cong e_{jj}(A/Ae_{ii}A)$ as R-$A/Ae_{ii}A$-bimodules.*

In the following, we do not fix ω and use the notation A_ω to denote that the multiplication of A is defined by ω.

3. Classification of ω

In the following, with each $\omega : I\times I \to \mathbb{Z}$ we associate $\lambda : I\times I\times I \to \mathbb{Z}$ such that

$$\lambda(i,j,k) = \omega(i,j) + \omega(j,k) - \omega(i,k)$$

for all $i,j,k \in I$.

Lemma 3.1. *For any $\omega, \omega' : I\times I \to \mathbb{Z}$ the following are equivalent.*
(1) $\lambda = \lambda'$.
(2) *There exists $\chi : I \to \mathbb{Z}$ such that $\omega'(i,j) = \omega(i,j) - \chi(i) + \chi(j)$ for all $i,j \in I$.*

Definition 3.2. For $\omega, \omega' : I\times I \to \mathbb{Z}$, we set $\omega \equiv \omega'$ if the equivalent conditions of Lemma 3.1 are satisfied. Also, for $\omega : I\times I \to \mathbb{Z}$, we write $\omega \geq 0$ if $\omega(i,j) \geq 0$ for all $i,j \in I$.

Lemma 3.3. *Let $\omega, \omega' : I\times I \to \mathbb{Z}$ with $\omega \equiv \omega'$ and assume there exists $i_0 \in I$ such that $\omega(i_0,j) = \omega'(i_0,j)$ for all $j \in I$. Then $\omega = \omega'$*

Definition 3.4. We denote by Ω the set of $\omega : I\times I \to \mathbb{Z}$ such that $\omega(i,i) = 0$ for all $i \in I$ and $\lambda(i,j,k) \geq 0$ for all $i,j,k \in I$. Also, for each $\nu \in \mathrm{Aut}(I)$, we set $I(\nu) = \{i \in I \mid \nu(i) = i\}$ and denote by $\Omega(\nu)$ the set of $\omega \in \Omega$ such that
(1) $\lambda(i,j,\nu(i)) = 0$ for all $i \in I\setminus I(\nu)$ and $j \in I$, and
(2) there exists $t_\omega \geq 0$ such that $\lambda(i,j,i) = t_\omega$ for all $i \in I(\nu)$ and $j \in I\setminus\{i\}$.

Lemma 3.5. *For any $\omega \in \Omega$ and $i_0 \in I$ the following hold.*
(1) *If $\lambda(i_0,j,i_0) = 0$ for all $j \in I$, then $\omega \equiv 0$.*
(2) *If $\omega(i_0,j) = 0$ for all $j \in I$, then $\omega \geq 0$.*
(3) *There exists $\omega' \in \Omega$ such that $\omega \equiv \omega'$, $\omega' \geq 0$ and $\omega'(i_0,j) = 0$ for all $j \in I$.*

Definition 3.6. We denote by Ω_b the set of $\omega \in \Omega$ such that $\lambda(i,j,i) > 0$ for all $i,j \in I$

with $i \neq j$. Also, we set $\Omega_b(\nu) = \Omega_b \cap \Omega(\nu)$ for $\nu \in \mathrm{Aut}(I)$.

Proposition 3.7. *For any $\nu \in \mathrm{Aut}(I)$ we have $\Omega_b(\nu) \neq \emptyset$.*

4. Automorphisms of A_ω

In this section, we show that for any $\nu \in \mathrm{Aut}(I)$ and $\omega \in \Omega$ there exists $\eta \in \mathrm{Aut}(A_\omega)$ with $\eta(e_{ij}) = e_{\nu(i),\nu(j)}$ for all $i, j \in I$.

Lemma 4.1. *For any $\omega, \omega' \in \Omega$ the following hold.*

(1) *If there exists $\nu \in \mathrm{Aut}(I)$ such that $\omega' = \omega \circ (\nu \times \nu)$, then there exists a ring isomorphism of the form*

$$\xi_1 : A_\omega \xrightarrow{\sim} A_{\omega'}, \sum_{i,j \in I} e_{ij} x_{ij} \mapsto \sum_{i,j \in I} e_{ij} x_{\nu(i),\nu(j)}.$$

(2) *If there exists $\chi : I \to \mathbb{Z}$ such that $\omega'(i,j) = \omega(i,j) - \chi(i) + \chi(j)$ for all $i, j \in I$, then there exists a ring isomorphism of the form*

$$\xi_2 : A_\omega \xrightarrow{\sim} A_{\omega'}, \sum_{i,j \in I} e_{ij} x_{ij} \mapsto \sum_{i,j \in I} e_{ij} \sigma^{\chi(j)}(x_{ij}).$$

Proposition 4.2. *For any $\nu \in \mathrm{Aut}(I)$ and $\omega \in \Omega(\nu)$ the following hold.*

(1) *Define $\chi : I \to \mathbb{Z}$ as follows:*

$$\chi(i) = \begin{cases} \omega(i, \nu(i)) & \text{if } i \notin I(\nu), \\ t_\omega & \text{if } i \in I(\nu). \end{cases}$$

Then $\omega(\nu(i), \nu(j)) = \omega(i,j) - \chi(i) + \chi(j)$ for all $i, j \in I$.

(2) *There exist ring automorphisms of the form*

$$\theta : A_\omega \xrightarrow{\sim} A_\omega, \sum_{i,j \in I} e_{ij} x_{ij} \mapsto \sum_{i,j \in I} e_{ij} \sigma^{-\chi(j)}(x_{\nu(i),\nu(j)}),$$
$$\eta : A_\omega \xrightarrow{\sim} A_\omega, \sum_{i,j \in I} e_{ij} x_{ij} \mapsto \sum_{i,j \in I} e_{\nu(i),\nu(j)} \sigma^{\chi(j)}(x_{ij})$$

which are mutually inverse.

(3) *Let $e_0 = \sum_{i \in I \setminus I(\nu)} e_{i,\nu(i)} + \sum_{i \in I(\nu)} e_{ii} c^t$, where $t = t_\omega$ if $I(\nu) \neq \emptyset$. Then $\eta(e_0) = e_0$ and $ae_0 = e_0 \eta(a)$ for all $a \in A_\omega$.*

5. The case of ν with $I(\nu) = \emptyset$

In this section, we deal with the case of $\nu \in \mathrm{Aut}(I)$ with $I(\nu) = \emptyset$. Let $\omega \in \Omega_b(\nu)$ and $A = A_\omega$. We set $\alpha_0 = \sum_{i \in I} \alpha_{i,\nu(i)}$ and $\beta_0 = \sum_{i \in I} \beta_{i,\nu(i)}$ (see Definition 2.4). By Proposition 4.2, we have ring automorphisms

$$\theta : A_{\omega} \xrightarrow{\sim} A_{\omega}, \sum_{i,j \in I} e_{ij}x_{ij} \mapsto \sum_{i,j \in I} e_{ij}\sigma^{-\chi(j)}(x_{\nu(i),\nu(j)}),$$
$$\eta : A_{\omega} \xrightarrow{\sim} A_{\omega}, \sum_{i,j \in I} e_{ij}x_{ij} \mapsto \sum_{i,j \in I} e_{\nu(i),\nu(j)}\sigma^{\chi(j)}(x_{ij})$$

which are mutually inverse.

Lemma 5.1. *There exist isomorphisms of the form*

$$\phi : A_A \xrightarrow{\sim} \mathrm{Hom}_R({}_A Ae_{kkR}, R_R), a \mapsto \alpha_{ik}a,$$
$$\psi : {}_A A \xrightarrow{\sim} \mathrm{Hom}_R({}_R e_{ii}A_A, {}_R R), a \mapsto a\beta_{ik}.$$

Remark 5.2. The following hold.

(1) θ is the unique ring automorphism of A such that $\theta(e_{\nu(i),\nu(j)}) = e_{ij}$ for all $i, j \in I$ and $x\alpha_0 = \alpha_0\theta(x)$ for all $x \in R$.

(2) η is the unique ring automorphism of A such that $\eta(e_{ij}) = e_{\nu(i),\nu(j)}$ for all $i, j \in I$ and $\beta_0 x = \eta(x)\beta_0$ for all $x \in R$.

Theorem 5.3. *The ring A is a Frobenius extensin of R.*

Proposition 5.4. *Assume R is a quasi-Frobenius local ring. Then A is a quasi-Frobenius ring with* $\mathrm{soc}(e_{ii}A_A) \cong e_{\nu(i),\nu(i)}A/e_{\nu(i),\nu(i)}J$ *for all $i \in I$, where $J = \mathrm{rad}(A)$.*

6. Another base ring

In this section, we prepare another base ring S which we need in the next section. We fix an integer $t > 0$.

Definition 6.1. Let S be a free right R-module with a basis $\{e, v\}$ and define the multiplication on S subject to the following axioms:

(S1) $e^2 = e$, $v^2 = -vc^t$ and $ev = v = ve$; and

(S2) $xe = ex$ and $xv = v\sigma^t(x)$ for all $x \in R$.

Lemma 6.2. *The following hold.*

(1) *S is an associative ring with $1_S = e$.*

(2) *We have ring homomorphisms $\varphi : R \to S, x \mapsto ex$ and $\pi : S \to R, (ex + vy) \mapsto x$ with* $\pi\varphi = \mathrm{id}$.

(3) *S is a local ring if R is so.*

In the following, we consider R as a subring of S via $\varphi : R \to S$. Note that S is a free left R-module with a basis $\{e, v\}$.

Definition 6.3. There exists a basis $\{\alpha, \mu\}$ for ${}_R\mathrm{Hom}_R(S_R, {}_R R_R)$ such that $b = e\alpha(b) + v\mu(b)$ for all $b \in S$. Similarly, there exists a basis $\{\beta, \rho\}$ for $\mathrm{Hom}_R({}_R S, {}_R R_R)_R$ such that $b = \beta(b)e + \rho(b)v$ for all $b \in S$.

Lemma 6.4. *There exist isomorphisms of the form*

$$\phi: S_S \xrightarrow{\sim} \mathrm{Hom}_R({}_S S_R, R_R),\ b \mapsto \mu b,$$
$$\psi: {}_S S \xrightarrow{\sim} \mathrm{Hom}_R({}_R S_S, {}_R R),\ b \mapsto b\rho.$$

Theorem 6.5. *The ring S is a Frobenius extension of R.*

Proposition 6.6. *There exist ring automorphisms of the form*

$$\theta: S \xrightarrow{\sim} S,\ ex + vy \mapsto e\sigma^{-t}(x) + v\sigma^{-t}(y),$$
$$\eta: S \xrightarrow{\sim} S,\ ex + vy \mapsto e\sigma^{t}(x) + v\sigma^{t}(y)$$

which are mutually inverse.

Remark 6.7. The following hold.
(1) θ is the unique ring automorphism of S such that $\theta(v) = v$ and $x\mu = \mu\theta(x)$ for all $x \in R$.
(2) η is the unique ring automorphism of S such that $\eta(v) = v$ and $\rho x = \eta(x)\rho$ for all $x \in R$.

7. The case of v with $I(v) \neq \emptyset$

In this and the next sections, we deal with the case of v with $I(v) \neq \emptyset$. Let $\omega \in \Omega_b(v)$ and $t = t_\omega$. We construct a Frobenius extension B of R which contains an ideal V with $B/V \cong A_\omega$, where $V^2 \neq 0$ in general.

Remark 7.1. It may happen that $\omega \in \Omega_b(\tau)$ for some τ with $I(\tau) = \emptyset$. If this is the case, A_ω itself is a Frobenius extension of R.

Definition 7.2. Let B be a free right R-module with a basis $\{e_{ij}\}_{i,j \in I} \cup \{v_i\}_{i \in I(v)}$ and define the multiplication on B subject to the following axioms:
(B1) $xe_{ij} = e_{ij}\sigma^{\omega(i,j)}(x)$ for all $i, j \in I$ and $x \in R$;
(B2) $e_{ij}e_{kl} = 0$ unless $j = k$;
(B3) $e_{ij}e_{jk} = e_{ik}c^{\chi(i,j,k)}$ unless $i = k \in I(v)$ and $j \in I \setminus \{i\}$;
(B4) $e_{ij}e_{ji} = v_i + e_{ii}c^t$ for all $i \in I(v)$ and $j \in I \setminus \{i\}$;
(B5) $xv_i = v_i\sigma^t(x)$ for all $i \in I(v)$ and $x \in R$;
(B6) $v_ie_{jk} = 0 = e_{ij}v_k$ unless $i = j = k$;
(B7) $v_ie_{ii} = v_i = e_{ii}v_i$ for all $i \in I(v)$;
(B8) $v_iv_j = 0$ unless $i = j$; and
(B9) $v_iv_i = -v_ic^t$ for all $i \in I(v)$.

Proposition 7.3. *The following hold.*
(1) *B is an associative ring with $1_B = \sum_{i \in I} e_{ii}$, where the e_{ii} are orthogonal idempotents.*
(2) *We have an injective ring homomorphism $\varphi: R \to B,\ x \mapsto \sum_{i \in I} e_{ii}x$ which is a split*

monomorphism of R-R-bimodules.

(3) $e_{ii}B_B \cong e_{jj}B_B$ *only when* $i = j$.

(4) $V = \sum_{i \in I(v)} v_i R$ *is an ideal of* B *with* $B/V \cong A_\omega$.

In the following, we consider R as a subring of B via $\varphi : R \to B$. Note that B is a free left R-module with a basis $\{e_{ij}\}_{i,j \in I} \cup \{v_i\}_{i \in I(v)}$. Also, for any $i \in I$, since $xe_{ii} = e_{ii}x$ for all $x \in R$, Be_{ii} is a B-R-bimodule and $e_{ii}B$ is an R-B-bimodule.

Definition 7.4. There exists a basis $\{\alpha_{ij}\}_{i,j \in I} \cup \{\mu_i\}_{i \in I(v)}$ for ${}_R\mathrm{Hom}_R(B_R, {}_RR_R)$ such that $b = \sum_{i,j \in I} e_{ii}\alpha_{ij}(b) + \sum_{i \in I(v)} v_i\mu_i(b)$ for all $b \in B$. Similarly, we have a basis $\{\beta_{ij}\}_{i,j \in I} \cup \{\rho_i\}_{i \in I(v)}$ for $\mathrm{Hom}_R({}_RB, {}_RR_R)_R$ such that $b = \sum_{i,j \in I} \beta_{ij}(b)e_{ii} + \sum_{i \in I(v)} \rho_i(b)v_i$ for all $b \in B$. We set

$$\mu_0 = \sum_{i \in N(v)} \alpha_{i,\, v(i)} + \sum_{i \in I(v)} \mu_i \quad \text{and} \quad \rho_0 = \sum_{i \in N(v)} \beta_{i,\, v(i)} + \sum_{i \in I(v)} \rho_i.$$

Lemma 7.5. *The following hold.*

(1) *For any* $i \in I(v)$ *there exist isomorphisms of the form*

$$\phi_i : e_{ii}B_B \xrightarrow{\sim} \mathrm{Hom}_R({}_BBe_{iiR}, R_R),\ b \mapsto \mu_i b,$$
$$\psi_i : {}_BBe_{ii} \xrightarrow{\sim} \mathrm{Hom}_R({}_Re_{ii}B_B, {}_RR),\ b \mapsto b\rho_i.$$

(2) *For any* $i \in N(v)$ *there exist isomorphisms of the form*

$$\phi_i : e_{ii}B_B \xrightarrow{\sim} \mathrm{Hom}_R({}_BBe_{v(i),\, v(i)R}, R_R),\ b \mapsto \alpha_{i,\, v(i)}b,$$
$$\psi_i : {}_BBe_{v(i),\, v(i)} \xrightarrow{\sim} \mathrm{Hom}_R({}_Re_{ii}B_B, {}_RR),\ b \mapsto b\beta_{i,\, v(i)}.$$

(3) *There exist isomorphisms of the form*

$$\phi : B_B \xrightarrow{\sim} \mathrm{Hom}_R({}_BB_R, R_R),\ b \mapsto \mu_0 b,$$
$$\psi : {}_BB \xrightarrow{\sim} \mathrm{Hom}_R({}_RB_B, {}_RR),\ b \mapsto b\rho_0.$$

Theorem 7.8. *The ring* B *is a Frobenius extension of* R.

Proposition 7.9. *Assume* R *is a local ring. Then the following hold.*

(1) *Every* e_{ii} *is a local idempotent. In particular,* B *is a semiperfect ring.*

(2) *Assume further that* R *is a quasi-Frobenius local ring. Then* B *is a quasi-Frobenius ring with* $\mathrm{soc}(e_{ii}B_B) \cong e_{v(i),\, v(i)}B/e_{v(i),\, v(i)}J$ *for all* $i \in I$, *where* $J = \mathrm{rad}(B)$.

In the next section, we do not fix $\omega \in \Omega_b(v)$ and use the notation B_ω to denote that the multiplication of B is defined by ω.

8. Automorphisms of B_ω

In this section, we show that for any $\omega \in \Omega_b(\nu)$ there exists $\eta \in \mathrm{Aut}(B_\omega)$ such that $\eta(e_{ij}) = e_{\nu(i),\nu(j)}$ for all $i, j \in I$ and $\eta(v_i) = v_i$ for all $i \in I(\nu)$.

Lemma 8.1. *For any $\omega, \omega' \in \Omega_b(\nu)$ the following hold.*

(1) *If there exists $\tau \in \mathrm{Aut}(I)$ with $\tau\nu = \nu\tau$ such that $\omega' = \omega \circ (\tau \times \tau)$, then there exists a ring isomorphism of the form*

$$\xi_1 : B_\omega \xrightarrow{\sim} B_{\omega'}, b \mapsto \textstyle\sum_{i,j \in I} e_{ij}\alpha_{\tau(i),\tau(j)}(b) + \sum_{i \in I(\nu)} v_i\mu_i(b).$$

(2) *If there exists $\chi : I \to \mathbb{Z}$ such that $\omega'(i, j) = \omega(i, j) - \chi(i) + \chi(j)$ for all $i, j \in I$, then there exists a ring isomorphism of the form*

$$\xi_2 : B_\omega \xrightarrow{\sim} B_{\omega'}, b \mapsto \textstyle\sum_{i,j \in I} e_{ij}\sigma^{\chi(i)}(\alpha_{ij}(b)) + \sum_{i \in I(\nu)} v_i\sigma^{\chi(i)}(\mu_i(b)).$$

Proposition 8.2. *For any $\omega \in \Omega_b(\nu)$ the following hold.*

(1) *Define $\chi : I \to \mathbb{Z}$ as follows:*

$$\chi(i) = \begin{cases} \omega(i, \nu(i)) & \text{if } i \notin I(\nu), \\ t_\omega & \text{if } i \in I(\nu). \end{cases}$$

Then $\omega(\nu(i), \nu(j)) = \omega(i, j) - \chi(i) + \chi(j)$ for all $i, j \in I$.

(2) *There exist ring automorphisms of the form*

$$\theta : B_\omega \xrightarrow{\sim} B_\omega, b \mapsto \textstyle\sum_{i,j \in I} e_{ij}\sigma^{-\chi(i)}(\alpha_{\nu(i),\nu(j)}(b)) + \sum_{i \in I(\nu)} v_i\sigma^{-\chi(i)}(\mu_i(b)),$$
$$\eta : B_\omega \xrightarrow{\sim} B_\omega, b \mapsto \textstyle\sum_{i,j \in I} e_{\nu(i),\nu(j)}\sigma^{\chi(i)}(\alpha_{ij}(b)) + \sum_{i \in I(\nu)} v_i\sigma^{\chi(i)}(\mu_i(b))$$

which are mutually inverse.

(3) *Let $e_0 = \sum_{i \in N(\nu)} e_{i,\nu(i)} + \sum_{i \in I(\nu)} e_{ii}c^t$ and $v_0 = \sum_{i \in I(\nu)} v_i$, where $t = t_\omega$. Then $\eta(e_0) = e_0$ and $\eta(v_0) = v_0$.* Also, *$be_0 = e_0\eta(b)$ and $bv_0 = v_0\eta(b)$ for all $b \in B_\omega$.*

Remark 8.3. For any $\omega \in \Omega_b(\nu)$ the following hold.

(1) θ is the unique ring automorphism of B_ω such that $\theta(e_{\nu(i),\nu(j)}) = e_{ij}$ for all $i, j \in I$ and $x\mu_0 = \mu_0\theta(x)$ for all $x \in R$.

(2) η is the unique ring automorphism of B_ω such that $\eta(e_{ij}) = e_{\nu(i),\nu(j)}$ for all $i, j \in I$ and $\rho_0 x = \eta(x)\rho_0$ for all $x \in R$.

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HIGHER DIMENSIONAL AUSLANDER-REITEN THEORY ON MAXIMAL ORTHOGONAL SUBCATEGORIES[1]

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ABSTRACT. Auslander-Reiten theory, especially the concept of almost split sequences and their existence theorem, is fundamental to study categories which appear in representation theory, for example, modules over artin algebras [ARS][GR][R], their functorially finite subcategories [AS][S], their derived categories [H], Cohen-Macaulay modules over Cohen-Macaulay rings [Y], lattices over orders [A2,3][RS], and coherent sheaves on projective curves [AR][GL]. In these Auslander-Reiten theory, the number '2' is quite symbolic. For one thing, almost split sequences give minimal projective resolutions of simple objects of projective dimension '2' in functor categories. For another, Cohen-Macaulay rings and orders of Krull-dimension '2' have fundamental sequences and provide us one of the most beautiful situation in representation theory [A4][E][RV][Y], which is closely related to McKay's observation on simple singularities [M]. In this sense, usual Auslander-Reiten theory should be '2-dimensional' theory, and it would have natural importance to search a domain of higher Auslander-Reiten theory from the viewpoint of representation theory and non-commutative algebraic geometry (e.g. [V1,2][Ar][GL]). In this paper, we introduce $(n-1)$-orthogonal subcategories as a natural domain of '$(n+1)$-dimensional' Auslander-Reiten theory. We show that higher Auslander-Reiten translation and higher Auslander-Reiten duality can be defined quite naturally for such categories. Using them, we show that our categories have *n-almost split sequences*, which are completely new generalization of usual almost split sequences and give minimal projective resolutions of simple objects of projective dimension '$n+1$' in functor categories. We also show the existence of higher dimensional analogy of fundamental sequences for Cohen-Macaulay rings and orders of Krull-dimension '$n+1$'. We show that an invariant subring (of Krull-dimension '$n+1$') corresponding to a finite subgroup G of $\mathrm{GL}_{n+1}(k)$ has a natural maximal $(n-1)$-orthogonal subcategory.

1 From Auslander-Reiten theory

1.1 Let us recall M. Auslander's classical theorem [A1] below, which introduced a completely new insight to representation theory of algebras (see 2.3 for $\mathrm{dom.dim}\,\Gamma$).

Theorem A (Auslander correspondence) *There exists a bijection between the set of Morita-equivalence classes of representation-finite finite-dimensional algebras* Λ *and that of finite-dimensional algebras* Γ *with* $\mathrm{gl.dim}\,\Gamma \leq 2$ *and* $\mathrm{dom.dim}\,\Gamma \geq 2$. *It is given by* $\Lambda \mapsto \Gamma := \mathrm{End}_\Lambda(M)$ *for an additive generator* M *of* $\mathrm{mod}\,\Lambda$.

In this really surprising theorem, the representation theory of Λ is encoded in the structure of the homologically nice algebra Γ called an *Auslander algebra*. Since the category $\mathrm{mod}\,\Gamma$ is equivalent to the functor category on $\mathrm{mod}\,\Lambda$, Auslander correspondence

[1]The detailed version [I2,3] of this paper have been submitted for publication elsewhere.

gave us a prototype of the use of functor categories in representation theory. In this sense, Auslander correspondence was a starting point of later Auslander-Reiten theory [ARS] historically. Theoretically, Auslander correspondence gives a direct connection between two completely different concepts, i.e. a representation theoretic property 'representation-finiteness' and a homological property '$\mathrm{gl.dim}\,\Gamma \le 2$ and $\mathrm{dom.dim}\,\Gamma \ge 2$'. It is a quite interesting project to find correspondence between representation theoretic properties and homological properties (e.g. [I1]).

1.2 Let R be a complete regular local ring of dimension d and Λ a module-finite R-algebra. We call Λ an *isolated singularity* [A3] if $\mathrm{gl.dim}\,\Lambda \otimes_R R_{\mathfrak{p}} = \mathrm{ht}\,\mathfrak{p}$ holds for any non-maximal prime ideal $\mathfrak{p}$ of R. We call a left Λ-module M *Cohen-Macaulay* if it is a projective R-module. We denote by $\mathrm{CM}\,\Lambda$ the category of Cohen-Macaulay Λ-modules. Then $D_d := \mathrm{Hom}_R(\ ,R)$ gives a duality $\mathrm{CM}\,\Lambda \leftrightarrow \mathrm{CM}\,\Lambda^{op}$. We call Λ an *R-order* (or *Cohen-Macaulay R-algebra*) if $\Lambda \in \mathrm{CM}\,\Lambda$ [A2,3]. In this case, let $\underline{\mathrm{CM}}\Lambda := (\mathrm{CM}\,\Lambda)/[\Lambda]$ be the *stable category* and $\overline{\mathrm{CM}}\Lambda := (\mathrm{CM}\,\Lambda)/[D_d\Lambda]$ the *costable category*. A typical example of an order is a commutative complete local Cohen-Macaulay ring Λ containing a field since such Λ contains a complete regular local subring R [Ma]. Let $\mathbf{E} : 0 \to R \to E_0 \to \cdots \to E_d \to 0$ be a minimal injective resolution of the R-module R. We denote by $D := \mathrm{Hom}_R(\ ,E_d)$ the Matlis dual. Let us recall the fundamental theorems [A2,3][Y] below, where we will give the definition of (pseudo) almost split sequences in 2.5 (put $n := 1$ there).

Theorem B *(1)* (Auslander-Reiten translation) *There exists an equivalence* $\tau : \underline{\mathrm{CM}}\Lambda \to \overline{\mathrm{CM}}\Lambda$.

(2) (Auslander-Reiten duality) *There exist functorial isomorphisms* $\overline{\mathrm{Hom}}_\Lambda(Y,\tau X) \simeq D\,\mathrm{Ext}^1_\Lambda(X,Y) \simeq \underline{\mathrm{Hom}}_\Lambda(\tau^{-1}Y,X)$ *for any* $X,Y \in \mathrm{CM}\,\Lambda$.

Theorem C *(1)* $\mathrm{CM}\,\Lambda$ *has almost split sequences.*

(2) If $d = 2$*, then* $\mathrm{CM}\,\Lambda$ *has pseudo almost split sequeces.*

Consequently, almost all simple objects in the functor category $\mathrm{mod}(\mathrm{CM}\,\Lambda)$ have projective dimension 2. If $d = 2$, then all simple objects in the functor category $\mathrm{mod}(\mathrm{CM}\,\Lambda)$ have projective dimension 2. In this sense, we can say that *Auslander-Reiten theory for the case $d = 2$ is very nice.* Using (pseudo) almost split sequences, we can define the *Auslander-Reiten quiver* $\mathfrak{A}(\Lambda)$ of Λ (see 2.6 and put $n := 1$ there).

1.3 Let us recall Auslander's contribution [A4][Y] to McKay correspondence [M]. Let k be a field of characteristic zero and G a finite subgroup of $\mathrm{GL}_d(k)$ with $d \ge 2$. Recall that the *McKay quiver* $\mathfrak{M}(G)$ of G [M] is defined as follows: The set of vertices is the set $\mathrm{irr}\,G$ of isoclasses of irreducible representations of G. Let V be the representation of G acting on k^d through $\mathrm{GL}_d(k)$. For $X, Y \in \mathrm{irr}\,G$, we denote by d_{XY} the multiplicity of X in $V \otimes_k Y$, and draw d_{XY} arrows from X to Y.

Theorem D *Let G be a finite subgroup of* $\mathrm{GL}_2(\mathbb{C})$, $\Omega := \mathbb{C}[[x,y]]$ *and* $\Lambda := \Omega^G$ *the invariant subring. Assume that G does not contain pseudo-reflection except the identity. Then Λ is representation-finite with* $\mathrm{CM}\,\Lambda = \mathrm{add}_\Lambda\,\Omega$*, and the Auslander-Reiten quiver* $\mathfrak{A}(\Lambda)$ *of Λ coincides with the McKay quiver* $\mathfrak{M}(G)$ *of G.*

1.4 Aim We observed that Auslander-Reiten theory is 2-dimensional-like. Now we can state the aim of this paper. *For each $n \ge 1$, find a domain of $(n+1)$-dimensional*

Auslander-Reiten theory. Namely, *find natural categories $\mathcal{C}$ such that Theorems above replaced '2' and* $\mathrm{CM}\,\Lambda$ *by '$n+1$' and $\mathcal{C}$ respectively hold.*

2 Main results

2.1 Definition Let $\mathcal{A}$ be an abelian category, $\mathcal{B}$ a full subcategory of $\mathcal{A}$ and $n \geq 0$. For a functorially finite [AS] full subcategory $\mathcal{C}$ of $\mathcal{B}$, we put

$$\begin{aligned} \mathcal{C}^{\perp_n} &:= \{X \in \mathcal{B} \mid \mathrm{Ext}^i_{\mathcal{A}}(\mathcal{C}, X) = 0 \text{ for any } i\ (0 < i \leq n)\}, \\ {}^{\perp_n}\mathcal{C} &:= \{X \in \mathcal{B} \mid \mathrm{Ext}^i_{\mathcal{A}}(X, \mathcal{C}) = 0 \text{ for any } i\ (0 < i \leq n)\}. \end{aligned}$$

We call $\mathcal{C}$ a *maximal n-orthogonal subcategory* of $\mathcal{B}$ if

$$\mathcal{C} = \mathcal{C}^{\perp_n} = {}^{\perp_n}\mathcal{C}$$

holds. By definition, $\mathcal{B}$ is a unique maximal 0-orthogonal subcategory of $\mathcal{B}$.

2.2 Example Let Λ be a simple singularity of type Δ and dimension $d = 2$, $\mathcal{A} := \mathrm{mod}^{\mathbb{Z}}\,\Lambda$ the category of graded Λ-modules and $\mathcal{B} := \mathrm{CM}^{\mathbb{Z}}\,\Lambda$ the category of graded Cohen-Macaulay Λ-modules. Then the number of maximal 1-orthogonal subcategories of $\mathcal{B}$ is given as follows:

Δ	A_m	B_m, C_m	D_m	E_6	E_7	E_8	F_4	G_2
number	$\frac{1}{m+2}\binom{2m+2}{m+1}$	$\binom{2m}{m}$	$\frac{3m-2}{m}\binom{2m-2}{m-1}$	833	4160	25080	105	8

This is obtained by showing that maximal 1-orthogonal subcategories of $\mathcal{B}$ correspond bijectively to clusters of the cluster algebra of type Δ [I2,3]. See Fomin-Zelevinsky [FZ1,2] and Buan-Marsh-Reineke-Reiten-Todorov [BMRRT]. See also Geiss-Leclerc-Schröer [GLS].

2.3 For a finite-dimensional algebra Γ, we denote by $0 \to \Gamma \to I_0 \to I_1 \to \cdots$ a minimal injective resolution of the Γ-module Γ. Put $\mathrm{dom.dim}\,\Gamma := \inf\{i \geq 0 \mid I_i \text{ is not projective}\}$ [T]. The following theorem gives a higher dimensional version of Theorem A.

Theorem A′ ($(n+1)$-dimensional Auslander correspondence) *For any $n \geq 1$, there exists a bijection between the set of equivalence classes of maximal $(n-1)$-orthogonal subcategories $\mathcal{C}$ of* $\mathrm{mod}\,\Lambda$ *with additive generators M and finite-dimensional algebras Λ, and the set of Morita-equivalence classes of finite-dimensional algebras Γ with* $\mathrm{gl.dim}\,\Gamma \leq n+1$ *and* $\mathrm{dom.dim}\,\Gamma \geq n+1$. *It is given by* $\mathcal{C} \mapsto \Gamma := \mathrm{End}_\Lambda(M)$.

2.4 In the rest of this section, let R be a complete regular local ring of dimension d, Λ an R-order which is an isolated singularity, $\mathcal{A} := \mathrm{mod}\,\Lambda$ and $\mathcal{B} := \mathrm{CM}\,\Lambda$. For $n \geq 1$, we define functors τ_n and τ_n^- by

$$\tau_n := \tau \circ \Omega^{n-1} : \underline{\mathrm{CM}}\Lambda \to \overline{\mathrm{CM}}\Lambda \quad \text{and} \quad \tau_n^- := \tau^- \circ \Omega^{-(n-1)} : \overline{\mathrm{CM}}\Lambda \to \underline{\mathrm{CM}}\Lambda,$$

where $\Omega : \underline{\mathrm{CM}}\Lambda \to \underline{\mathrm{CM}}\Lambda$ is the syzygy functor and $\Omega^- : \overline{\mathrm{CM}}\Lambda \to \overline{\mathrm{CM}}\Lambda$ is the cosyzygy functor. For a subcategory $\mathcal{C}$ of $\mathrm{CM}\,\Lambda$, we denote by $\underline{\mathcal{C}}$ and $\overline{\mathcal{C}}$ the corresponding subcategories of $\underline{\mathrm{CM}}\Lambda$ and $\overline{\mathrm{CM}}\Lambda$ respectively.

Theorem B′ *Let $\mathcal{C}$ be a maximal $(n-1)$-orthogonal subcategory of* $\mathrm{CM}\,\Lambda$ *($n \geq 1$).*

(1) (n-Auslander-Reiten translation) *For any $X \in \mathcal{C}$, $\tau_n X \in \mathcal{C}$ and $\tau_n^- X \in \mathcal{C}$ hold. Thus $\tau_n : \underline{\mathcal{C}} \to \overline{\mathcal{C}}$ and $\tau_n^- : \overline{\mathcal{C}} \to \underline{\mathcal{C}}$ are mutually quasi-inverse equivalences.*

(2) (n-Auslander-Reiten duality) *There exist functorial isomorphisms $\overline{\mathcal{C}}(Y, \tau_n X) \simeq D\,\mathrm{Ext}^n_\Lambda(X, Y) \simeq \underline{\mathcal{C}}(\tau_n^- Y, X)$ for any $X, Y \in \mathcal{C}$.*

2.5 Definition Let $\mathcal{C}$ be a full subcategory of $\mathrm{CM}\,\Lambda$ and $J_\mathcal{C}$ the Jacobson radical of $\mathcal{C}$. We call an exact sequence

$$0 \to Y \xrightarrow{f_n} C_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} C_0 \xrightarrow{f_0} X \to 0$$
$$(\text{resp. } 0 \to Y \xrightarrow{f_n} C_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} C_0 \xrightarrow{f_0} X)$$

with terms in $\mathcal{C}$ an *n-almost split sequence* (resp. *pseudo n-almost split sequence*) if $f_i \in J_\mathcal{C}$ holds for any i and the following sequences are exact.

$$0 \to \mathcal{C}(\ ,Y) \xrightarrow{\cdot f_n} \mathcal{C}(\ ,C_{n-1}) \xrightarrow{\cdot f_{n-1}} \cdots \xrightarrow{\cdot f_1} \mathcal{C}(\ ,C_0) \xrightarrow{\cdot f_0} J_\mathcal{C}(\ ,X) \to 0$$
$$0 \to \mathcal{C}(X,\ ) \xrightarrow{f_0 \cdot} \mathcal{C}(C_0,\ ) \xrightarrow{f_1 \cdot} \cdots \xrightarrow{f_{n-1} \cdot} \mathcal{C}(C_{n-1},\ ) \xrightarrow{f_n \cdot} J_\mathcal{C}(Y,\ ) \to 0$$

We call $f_0 : C_0 \to X$ a *sink map* and $f_n : Y \to C_{n-1}$ a *source map*. We say that $\mathcal{C}$ *has n-almost split sequences* if, for any non-projective $X \in \mathrm{ind}\,\mathcal{C}$ (resp. non-injective $Y \in \mathrm{ind}\,\mathcal{C}$), there exists an n-almost split sequence $0 \to Y \to C_{n-1} \to \cdots \to C_0 \to X \to 0$. Similarly, we say that $\mathcal{C}$ *has pseudo n-almost split sequences* if, for any projective $X \in \mathcal{C}$ (resp. injective $Y \in \mathcal{C}$), there exists a pseudo n-almost split sequence $0 \to Y \to C_{n-1} \to \cdots \to C_0 \to X$.

Theorem C′ *Let $\mathcal{C}$ be a maximal $(n-1)$-orthogonal subcategory of $\mathrm{CM}\,\Lambda$ $(n \geq 1)$.*
(1) $\mathcal{C}$ has n-almost split sequences.
(2) If $d = n+1$, then $\mathcal{C}$ has pseudo n-almost split sequeces.

Consequently, almost all simple objects in the functor category $\mathrm{mod}\,\mathcal{C}$ have projective dimension $n+1$. If $d = n+1$, then all simple objects in the functor category $\mathrm{mod}\,\mathcal{C}$ have projective dimension $n+1$. In this sense, we can say that *$(n+1)$-dimensional Auslander-Reiten theory for the case $d = n+1$ is very nice.*

2.6 We will define the *Auslander-Reiten quiver* $\mathfrak{A}(\mathcal{C})$ of $\mathcal{C}$. For simplicity, we assume that the residue field k of R is algebraically closed. The set of vertices of $\mathfrak{A}(\mathcal{C})$ is $\mathrm{ind}\,\mathcal{C}$. For $X, Y \in \mathrm{ind}\,\mathcal{C}$, we denote by d_{XY} be the multiplicity of X in C for the sink map $C \to Y$, which equals to the multiplicity of Y in C' for the source map $X \to C'$. Draw d_{XY} arrows from X to Y.

Theorem D′ *Let G be a finite subgroup of $\mathrm{GL}_d(\mathbb{C})$, $\Omega := \mathbb{C}[[x_1, \cdots, x_d]]$ and $\Lambda := \Omega^G$ the invariant subring. Assume that G does not contain pseudo-reflection except the identity, and that Λ is an isolated singularity. Then $\mathcal{C} := \mathrm{add}_\Lambda \Omega$ is a maximal $(d-2)$-orthogonal subcategory of $\mathrm{CM}\,\Lambda$. Moreover, the Auslander-Reiten quiver $\mathfrak{A}(\mathcal{C})$ of $\mathcal{C}$ coincides with the McKay quiver $\mathfrak{M}(G)$ of G, i.e. there exists a bijection $\mathbb{H} : \mathrm{irr}\,G \to \mathrm{ind}\,\mathcal{C}$ such that $d_{XY} = d_{\mathbb{H}(X),\mathbb{H}(Y)}$ for any $X, Y \in \mathrm{irr}\,G$.*

3 Non-commutative crepant resolution and representation dimension

3.1 Let us generalize the concept of Van den Bergh's non-commutative crepant resolution [V1,2] of commutative normal Gorenstein domains to our situation.

Again let Λ be an R-order which is an isolated singularity. We call $M \in \mathrm{CM}\,\Lambda$ a *NCC resolution* of Λ if $\Lambda \oplus D_d\Lambda \in \mathrm{add}\,M$ and $\Gamma := \mathrm{End}_\Lambda(M)$ is an R-order with $\mathrm{gl.dim}\,\Gamma = d$. Our definition is slightly stronger than original non-commutative crepant resolutions in [V2] where M is assumed to be reflexive (not Cohen-Macaulay) and $\Lambda \oplus D_d\Lambda \in \mathrm{add}\,M$ is not assumed. But all examples of non-commutative crepant resolutions in [V1,2] satisfy our condition. For the case $d \geq 2$, we have the remarkable relationship below between NCC resolutions and maximal $(d-2)$-orthogonal subcategories.

Theorem *Let $d \geq 2$. Then $M \in \mathrm{CM}\,\Lambda$ is a NCC resolution of Λ if and only if* $\mathrm{add}\,M$ *is maximal $(d-2)$-orthogonal subcategory of* $\mathrm{CM}\,\Lambda$.

3.2 Conjecture It is interesting to study relationship among all maximal $(n-1)$-orthogonal subcategories of $\mathrm{CM}\,\Lambda$. Especially, we conjecture that *their endomorphism rings are derived equivalent.* It is suggestive to relate this conjecture to Van den Bergh's generalization [V2] of Bondal-Orlov conjecture [BO], which asserts that *all (commutative or non-commutative) crepant resolutions of a normal Gorenstein domain have the same derived category.* Since maximal $(n-1)$-orthogonal subcategories are analogy of non-commutative crepant resolutions from the viewpoint of 3.1, our conjecture is an analogy of Bondal-Orlov-Van den Bergh conjecture. We have the following partial solution.

Theorem *(1) Let $\mathcal{C}_i = \mathrm{add}\,M_i$ be a maximal 1-orthogonal subcategory of* $\mathrm{CM}\,\Lambda$ *and $\Gamma_i := \mathrm{End}_\Lambda(M_i)$ $(i = 1, 2)$. Then Γ_1 and Γ_2 are derived equivalent. In particular, $\#\,\mathrm{ind}\,\mathcal{C}_1 = \#\,\mathrm{ind}\,\mathcal{C}_2$ holds.*

(2) If $d \leq 3$, then all NCC resolutions of Λ have the same derived category.

3.3 Let us generalize the concept of Auslander's representation dimension [A1] to relate it to non-commutative crepant resolutios. For $n \geq 1$, define the *n-th representation dimension* $\mathrm{rep.dim}_n\,\Lambda$ of an R-order Λ which is an isolated singularity by

$$\mathrm{rep.dim}_n\,\Lambda := \inf\{\mathrm{gl.dim}\,\mathrm{End}_\Lambda(M) \mid M \in \mathrm{CM}\,\Lambda,\ \Lambda \oplus D_d\Lambda \in \mathrm{add}\,M,\ M \perp_{n-1} M\}.$$

Obviously $d \leq \mathrm{rep.dim}_n\,\Lambda \leq \mathrm{rep.dim}_{n'}\,\Lambda$ holds for any $n \leq n'$. For the case $d = 0$, $\mathrm{rep.dim}_1\,\Lambda$ coincides with the representation dimension defined in [A1]. We call Λ *representation-finite* if $\#\,\mathrm{ind}(\mathrm{CM}\,\Lambda) < \infty$. In the sense of (2) below, $\mathrm{rep.dim}_1\,\Lambda$ measures how far Λ is from being representation-finite.

Theorem *(1) Assume $d \leq n+1$. Then* $\mathrm{CM}\,\Lambda$ *has a maximal $(n-1)$-orthogonal subcategory $\mathcal{C}$ with $\#\,\mathrm{ind}\,\mathcal{C} < \infty$ if and only if* $\mathrm{rep.dim}_n\,\Lambda \leq n+1$.

(2) Assume $d \leq 2$. Then Λ is representation-finite if and only if $\mathrm{rep.dim}_1\,\Lambda \leq 2$.

(3) Λ has a NCC resolution if and only if $\mathrm{rep.dim}_{\max\{1,d-1\}}\,\Lambda = d$.

3.4 Conjecture It seems that no example of a maximal $(n-1)$-orthogonal subcategory $\mathcal{C}$ of $\mathrm{CM}\,\Lambda$ with $\#\,\mathrm{ind}\,\mathcal{C} = \infty$ is known. This suggests us to study

$$o(\mathrm{CM}\,\Lambda) := \sup_{\mathcal{C} \subseteq \mathrm{CM}\,\Lambda,\ \mathcal{C} \perp_1 \mathcal{C}} \#\,\mathrm{ind}\,\mathcal{C}\,.$$

We conjecture that $o(\mathrm{CM}\,\Lambda)$ *is always finite.* If Λ is a preprojective algebra of Dynkin type Δ, then Geiss-Schröer [GS] proved that $o(\mathrm{mod}\,\Lambda)$ equals to the number of positive roots of Δ. It would be interesting to find a geometric interpretation of $o(\mathrm{CM}\,\Lambda)$ for more general CM Λ. For some classes of CM Λ, one can calculate $o(\mathrm{CM}\,\Lambda)$ by using the theorem below. Especially, (1) seems to be interesting in the connection with known results for algebras with representation dimension at most 3 [IT][EHIS].

Theorem *(1)* $\mathrm{rep.dim}_1 \Lambda \leq 3$ *implies* $o(\mathrm{CM}\,\Lambda) < \infty$.

(2) If CM Λ *has a maximal* 1*-orthogonal subcategory* $\mathcal{C}$*, then* $o(\mathrm{CM}\,\Lambda) = \#\,\mathrm{ind}\,\mathcal{C}$.

3.5 Concerning our conjecture, let us recall the well-known proposition below which follows by a geometric argument due to Voigt's lemma ([P;4.2]). It is interesting to ask whether it is true without the restriction on R. If it is true, then any 1-orthogonal subcategory of CM Λ is 'discrete', and our conjecture asserts that it is finite. It is interesting to study the discrete structure of 1-orthogonal objects in CM Λ and the relationship to whole structure of CM Λ.

Proposition *Assume that R is an algebraically closed field. For any $n > 0$, there are only finitely many isoclasses of* 1*-orthogonal* Λ*-modules X with* $\dim_R X = n$.

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MORPHISMS REPRESENTED BY MONOMORPHISMS [1]

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ABSTRACT. We answer a question posed by Auslander and Bridger. Every homomorphism of modules is projective-stably equivalent to an epimorphism but is not always to a monomorphism. We prove that a map is projective-stably equivalent to a monomorphism if and only if its kernel is torsionless, that is, a first syzygy. If it occurs although, there can be various monomorphisms that are projective-stably equivalent to a given map. But in this case there uniquely exists a "perfect" monomorphism to which a given map is projective-stably equivalent.

1 Introduction

Let R be a commutative noetherian ring. Linear maps $f : A \to B$ and $f' : A' \to B'$ of finite R-modules are said to be projective-stably equivalent (pse for short) if the following diagram is commutative

$$\begin{array}{ccc} A \oplus P' & \xrightarrow{\begin{pmatrix} f & s \\ t & u \end{pmatrix}} & B \oplus Q' \\ \downarrow\cong & & \downarrow\cong \\ A' \oplus P & \xrightarrow{\begin{pmatrix} f' & s' \\ t' & u' \end{pmatrix}} & B' \oplus Q \end{array}$$

with some projective modules P, Q, P', Q' and R-linear maps s, t, u, s', t', u'. We say a morphism f is represented by monomorphisms ("rbm" for short) if there exists a monomorphism that is pse to f.

For any homomorphism $f : A \to B$ of R-modules, $(f \quad \rho_B) : A \oplus P_B \to B$ is surjective with a projective cover $\rho_B : P_B \to B$. Thus every morphism is represented by epimorphisms. The choice of epimorphism is unique; if an epimorphism f' is pse to f, then two sequences $0 \to \operatorname{Ker} f' \to A' \xrightarrow{f'} B' \to 0$ and $0 \to \operatorname{Ker}(f\ \rho_B) \to A \oplus P_B \xrightarrow{(f\ \rho_B)} B \to 0$ becomes isomorphic after splitting off common projective summands.

The formal analogy to the representations by monomorphisms fails both in existence and in uniqueness. Every morphism is not always represented by monomorphisms (Example 1). Even if a morphism f is rbm, the choice of monomorphism is not unique;

[1]The detailed version of this paper has been submitted for publication elsewhere.
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there may be two monomorphisms f' and f'' both pse to f and that $0 \to A' \xrightarrow{f'} B' \to \operatorname{Cok} f' \to 0$ and $0 \to A'' \xrightarrow{f''} B'' \to \operatorname{Cok} f'' \to 0$ are not isomorphic by splitting off common projective summands (Example 2).

The purpose of the paper is finding a condition of a given map to be rbm. Roughly speaking, our problem is to know when an exact sequence of modules

$$0 \to A \to B \xrightarrow{g} C \to 0$$

can be modified into an exact sequence

$$0 \to B \xrightarrow{f} C \to A' \to 0.$$

Of course the projective stabilization $\underline{\operatorname{mod} R}$ of $\operatorname{mod} R$ is not triangulated in general. So the obstruction for a given map to be rbm should be the obstruction for $\underline{\operatorname{mod} R}$ to be triangulated. Our first focus is an analogy to the homotopy category $\mathsf{K}(\operatorname{mod} R)$ of R-complexes. In [5, Theorem 2.6], the author showed a category equivalence between $\underline{\operatorname{mod} R}$

and a subcategory of $\mathsf{K}(\operatorname{mod} R)$. Due to this equivalence, we describe the obstruction of being rbm with a homology of a complex associated to the given map.

The problem was originally posed by Auslander and Bridger [1]. They proved that a map is rbm if and only if it is pse to a "perfect" monomorphism. An exact sequence of R-modules is called perfect if its R-dual is also exact. A perfect monomorphism refers to a monomorphism whose R-dual is an epimorphism. This is our next focal point. In the case that a map is rbm, the choice of a monomorphism is not unique, but then a perfect monomorphism pse to the given map is uniquely determined up to direct sum of projective modules. (Theorem 3.6.)

Looking at Theorem 3.6, we see that when a morphism is rbm, its pseudo-kernel is always the first syzygy of its pseudo-cokernel. So it is tempting to ask if the equivalent condition of rbm property is that the kernel is a submodule of a free module. This is our third point. Actually, we need to assume the total ring of fractions $Q(R)$ is Gorenstein: the condition is satisfied for instance if R is a domain.

> Theorem 4.8 : Suppose the total ring of fractions $Q(R)$ of a ring R is Gorenstein. A morphism f is rbm if and only if $\operatorname{Ker} f$ is a submodule of a free module.

Let us give easy examples:

Example 1 Set $R = k[[X,Y]]/(XY)$ with any field k, $g : R^2/\binom{X}{Y}R \to R/(X) \oplus R/(Y)$ with $g(\binom{a}{b} \operatorname{mod} \binom{X}{Y}R) = (a \operatorname{mod}(X), b \operatorname{mod}(Y))$. Since $\operatorname{Ker} g \cong R/(X,Y)$ is not a first syzygy, g is not rbm due to Theorem 4.8.

Example 2 Set $R = k[[X,Y]]/(XY)$ with any field k, $f : R^2/\binom{X}{Y}R \to R^2/\binom{X^2}{Y^2}R$ with $f(\binom{a}{b} \operatorname{mod} \binom{X}{Y}R) = \binom{Xa}{Yb} \operatorname{mod} \binom{X^2}{Y^2}R$. The map f is not a monomorphism; $\operatorname{Ker} f \cong R/(X) \oplus R/(Y)$ is a first syzygy. By Theorem 4.8, f is rbm. In fact, let $f' : R^2/\binom{X}{Y}R \to R^2/\binom{X^2}{Y^2}R \oplus R^2$ be defined as $f'(\binom{a}{b} \operatorname{mod} \binom{X}{Y}R) = (\binom{Xa}{Yb} \operatorname{mod} \binom{X^2}{Y^2}R, \binom{Ya}{Xb})$. Obviously f'

is a monomorphism that is pse to f. On the other hand, $f'' : R^2/\binom{X}{Y}R \to R^2/\binom{X^2}{Y^2}R \oplus R^2$ with $f''(\binom{a}{b} \bmod \binom{X}{Y}R) = (\binom{Xa}{Yb} \bmod \binom{X^2}{Y^2}R, \binom{Y^2a}{X^2b})$ is also a monomorphism and pse to f. We have two exact sequences

$$\theta_f : 0 \to R^2/\tbinom{X}{Y}R \xrightarrow{f'} R^2/\tbinom{X^2}{Y^2}R \oplus R^2 \to R^2/\tbinom{X}{Y}R \oplus R^2/\tbinom{Y}{X}R \to 0,$$

and

$$\sigma : 0 \to R^2/\tbinom{X}{Y}R \xrightarrow{f''} R^2/\tbinom{X^2}{Y^2}R \oplus R^2 \to R^2/\tbinom{X}{Y^2}R \oplus R^2/\tbinom{Y}{X^2}R \to 0,$$

that are not isomorphic. We see θ_f is perfect but σ is not.

2 Stable module category and homotopy category

Throughout the paper, R is a commutative noetherian ring, By an "R-module" we mean a finitely generated R-module. For an R-module M, $\rho_M : P_M \to M$ denotes a projective cover of M.

Definition 2.1 *The projective stabilization* $\underline{\mathrm{mod}}\,R$ *is defined as follows.*

- *Each object of* $\underline{\mathrm{mod}}\,R$ *is an object of mod* R.
- *For objects* A, B *of* $\underline{\mathrm{mod}}\,R$, *a set of morphisms from* A *to* B *is* $\underline{\mathrm{Hom}}_R(A,B) = \mathrm{Hom}_R(A,B)/\mathcal{P}(A,B)$ *where* $\mathcal{P}(A,B) := \{f \in \mathrm{Hom}_R(A,B) \mid$ *f factors through some projective module*$\}$. *Each element is denoted as* $\underline{f} = f \bmod \mathcal{P}(A,B)$.

A morphism mod R *is called* a stable isomorphism *if* $\underline{f}$ *is an isomorphism in* $\underline{\mathrm{mod}}\,R$. *If two R-modules A and A' are isomorphic in* $\underline{\mathrm{mod}}\,R$, *we say A and A' are stably isomorphic and write* $A \overset{st}{\cong} A'$.

Definition 2.2 *Morphisms* $f : A \to B$ *and* $f' : A' \to B'$ *in* mod R *are said to be projective-stably equivalent (pse for short) and denoted as* $f \overset{st}{\cong} f'$ *if if there exist stable isomorphisms* $\alpha : A \to A'$ *and* $\beta : B \to B'$ *such that* $\underline{\beta \circ f} = \underline{f' \circ \alpha}$.

Let $\mathcal{L}$ be a full subcategory of $\mathsf{K}(\mathrm{mod}\,R)$ defined as

$$\mathcal{L} = \{F^\bullet \in \mathsf{K}(\mathrm{proj}\,R) \mid \mathrm{H}^i(F^\bullet) = 0\ (i < 0),\quad \mathrm{H}_j(F^{\bullet}{}_{\bullet}) = 0\ (j \geq 0)\}.$$

Lemma 2.3 ([5] Proposition 2.3, Proposition 2.4)

1) *For* $A \in \underline{\mathrm{mod}}\,R$, *there exists* ${F_A}^\bullet \in \mathcal{L}$ *that satisfies*

$$\mathrm{H}^0(\tau_{\leq 0}{F_A}^\bullet) \overset{st}{\cong} A.$$

Such an ${F_A}^\bullet$ *is uniquely determined by A up to isomorphisms. We fix the notation* ${F_A}^\bullet$ *and call this a standard resolution of A.*

2) *For $\underline{f} \in \underline{\mathrm{Hom}}_R(A, B)$, there exists $f^\bullet \in \mathrm{Hom}_{\mathsf{K}(\mathrm{proj}\,R)}(F_A{}^\bullet, F_B{}^\bullet)$ that satisfies*

$$\underline{\mathrm{H}^0(\tau_{\leq 0} f^\bullet)} \cong \underline{f}.$$

Such an $f^\bullet$ is uniquely determined by $\underline{f}$ up to isomorphisms, so we use the notation $f^\bullet$ to describe a chain map with this property for given $\underline{f}$.

Theorem 2.4 ([5] Theorem 2.6) *The mapping $A \mapsto F_A{}^\bullet$ gives a functor from $\underline{\mathrm{mod}\,R}$ to $\mathsf{K}(\mathrm{mod}\,R)$, and this gives a category equivalence between $\underline{\mathrm{mod}\,R}$ and $\mathcal{L}$.*

For $f \in \mathrm{Hom}_R(A, B)$, there exists a triangle

$$C(f)^{\bullet-1} \xrightarrow{n_f{}^\bullet} F_A{}^\bullet \xrightarrow{f^\bullet} F_B{}^\bullet \xrightarrow{c_f{}^\bullet} C(f)^\bullet. \tag{2.1}$$

In general, $C(f)^\bullet$ does not belong to $\mathcal{L}$ but it satisfies the following:

$$\mathrm{H}^i(C(f)^\bullet) = 0 \ (i < -1), \quad \mathrm{H}_j(C(f)^\bullet{}_\bullet) = 0 \ (j > -1).$$

Definition and Lemma 2.5 ([5], Definition and Lemma 3.1) *As objects of $\underline{\mathrm{mod}\,R}$, $\underline{\mathrm{Ker}}\ \underline{f} := \mathrm{H}^{-1}(\tau_{\leq -1} C(f)^\bullet)$ and $\underline{\mathrm{Cok}}\ \underline{f} := \mathrm{H}^0(\tau_{\leq 0} C(f)^\bullet)$ are uniquely determined by $\underline{f}$, up to isomorphisms. We call these the pseudo-kernel and the pseudo-cokernel of $\underline{f}$.*

For a given map $f : A \to B$, from (2.1), we have an exact sequence

$$0 \to \underline{\mathrm{Ker}}\ \underline{f} \to A \oplus P \xrightarrow{(f\ \rho)} B \to 0 \tag{2.2}$$

with some projective module P. This characterizes the pseudo-kernel.

Lemma 2.6 *For a given $f \in \mathrm{Hom}_R(A, B)$, suppose $A \oplus P' \xrightarrow{(f\ p')} B$ is epimorphism with projective module P'. Then $\mathrm{Ker}\,(f\ p') \overset{st}{\cong} \underline{\mathrm{Ker}}\underline{f}$ and the sequence*

$$0 \ \to \ \mathrm{Ker}\,(f\ p') \ \to \ A \oplus P' \ \xrightarrow{(f\ p')} \ B \ \to \ 0$$

is isomorphic to 2.2 after splitting off some aplit exact sequence of projective modules.

Lemma 2.7 ([5] Lemma 3.6)

1) *There is an exact sequence*

$$0 \to \mathrm{Ker}\, f \to \underline{\mathrm{Ker}}\ f \to \Omega^1_R(\mathrm{Cok}\ \ f) \to 0.$$

2) *There is an exact sequence*

$$0 \to L \to \underline{\mathrm{Cok}}\ \underline{f} \to \mathrm{Cok}\ \ f \to 0$$

such that $\Omega^1_R(L)$ is the surjective image of $\mathrm{Ker}\, f$.

3 Representation by monomorphisms and perfect exact sequences

Definition 3.1 *A morphism $f : A \to B$ in* $\mathrm{mod}\, R$ *is said to be* represented by monomorphisms *(rbm for short) if some monomorphism $f' : A' \to B'$ in* $\mathrm{mod}\, R$ *is pse to f, that is, there exist stable isomorphisms $\alpha : A \to A'$ and $\beta : B \to B'$ such that $\underline{\beta \circ f} = \underline{f' \circ \alpha}$.*

Each morphism is not always rbm. It was Auslander and Bridger who first defined and studied "represented by monomorphisms" property.

Theorem 3.2 (Auslander-Bridger) *The following are equivalent for a morphism $f : A \to B$ in* $\mathrm{mod}\, R$.

1) *There exists a monomorphism $f' : A \to B \oplus P$ with a projective module P such that $f = s \circ f'$ via some split epimorphism $s : B \oplus P \to B$.*

2) *There exists a monomorphism $f' : A \to B \oplus P$ with a projective module P such that $f = s \circ f'$ via some split epimorphism $s : B \oplus P \to B$, and f'^{*} is an epimorphism.*

3) $\mathrm{Hom}_{\mathrm{R}}(B, I) \to \mathrm{Hom}_{\mathrm{R}}(A, I)$ *is surjective if I is an injective module.*

The condition 1) of Theorem 3.2 turns out to be equivalent to the rbm condition.

Lemma 3.3 *For a morphism $f : A \to B$ in* $\mathrm{mod}\, R$*, f is rbm if and only if there exists a monomorphism $f' : A \to B \oplus P$ with a projective module P such that $f = s \circ f'$ via some split epimorphism $s : B \oplus P \to B$.*

The most remarkable point in Auslander-Bridger's Theorem is that being rbm is equivalent to being represented by "perfect monomorphisms" whose R-dual is an epimorphism.

Definition 3.4 *An exact sequence $0 \to A \to B \to C \to 0$ of R-modules is called a perfect exact sequence or to be perfectly exact if its R-dual $0 \to \mathrm{Hom}_{\mathrm{R}}(C, R) \to \mathrm{Hom}_{\mathrm{R}}(B, R) \to \mathrm{Hom}_{\mathrm{R}}(A, R) \to 0$ is also exact. A monomorphism f is called a perfect monomorphism if* $\mathrm{Hom}_{\mathrm{R}}(f, R)$ *is an epimorphism.*

Proposition 3.5 ([5] Lemma 2.7) *The following are equivalent for an exact sequence*

$$\theta : 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0.$$

1) *θ is perfectly exact.*

2) $0 \to {F_A}^{*} \xrightarrow{f^{*}} {F_B}^{*} \xrightarrow{g^{*}} {F_C}^{*} \to 0$ *is exact.*

3) $F_C{}^{\bullet-1} \to F_A{}^\bullet \xrightarrow{f^\bullet} F_B{}^\bullet \xrightarrow{g^\bullet} F_C{}^\bullet$ *is a distinguished triangle in* $\mathsf{K}(\operatorname{mod} R)$.

For a morphism $f : A \to B$, $A \oplus P_B \xrightarrow{(f\ \rho_B)} B$ is an epimorphism with a projective cover $\rho_B : P_B \to B$. Thus each morphism is represented by epimorphisms. And the choice of the representing epimorphism is unique up to direct sum of projective modules, as we have seen in Lemma 2.6.

Unlikely, we already know an example of a morphism that is not rbm. And moreover, even if a given map is represented by a monomorphism, there would be another representing monomorphism. (Example 1 and Example 2.)

However, uniqueness theorem is obtained in this way. Due to Theorem 3.2, a morphism is rbm if and only if it is represented by a perfect monomorphism. And if this is the case, the representing perfect monomorphism is uniquely determined up to direct sum of projective modules.

Theorem 3.6 *Let $f : A \to B$ be a morphism in* $\operatorname{mod} R$. *Then f is rbm if and only if* $\mathrm{H}^{-1}(C(f)^\bullet)$ *vanishes. If this is the case, we have the following:*

1) *We have a perfect exact sequence*

$$\theta_f : 0 \to A \xrightarrow{\binom{f}{\epsilon}} B \oplus F_A{}^1 \xrightarrow{(c^f\ \pi)} \underline{\operatorname{Cok} f} \to 0.$$

2) *For any exact sequence of the form*

$$\sigma : 0 \to A \xrightarrow{\binom{f}{q}} B \oplus P' \xrightarrow{(g\ p)} C \to 0$$

with some projective module P', there is a commutative diagram

$$\begin{array}{ccccccccc}
\theta_f : & 0 & \to & A & \xrightarrow{\binom{f}{\epsilon}} & B \oplus F_A{}^1 & \xrightarrow{(c^f\ \pi)} & \underline{\operatorname{Cok} f} & \to & 0 \\
 & & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\
\sigma : & 0 & \to & A & \xrightarrow{\binom{f}{q}} & B \oplus P' & \xrightarrow{(g\ p)} & C & \to & 0
\end{array}$$

where α and β are stable isomorphisms.

3) *There is an exact sequence with some projective module Q and Q'*

$$0 \to Q' \to \underline{\operatorname{Cok} f} \oplus Q \xrightarrow{(\gamma\ \rho)} C \to 0.$$

In other words, $\underline{\operatorname{Ker}\gamma}$ *is projective.*

4) *If σ is also perfectly exact, then σ is isomorphic to θ_f up to direct sum of split exact sequences of projective modules.*

proof. We have a triangle

$$F_A{}^\bullet \xrightarrow{f^\bullet} F_B{}^\bullet \rightarrow C(f)^\bullet \xrightarrow{n_f{}^{\bullet+1}} F_A{}^{\bullet+1} \tag{3.3}$$

which induces a term-wise exact sequence of complexes in $\mathbb{C}(\operatorname{proj} R)$

$$0 \rightarrow F_A{}^\bullet \rightarrow C(n_f)^\bullet \rightarrow C(f)^\bullet \rightarrow 0 \tag{3.4}$$

Applying $\tau_{\leq 0}$ to the diagram above and taking homology, we get the following exact sequence of modules:

$$\theta_f:\ 0 \rightarrow \mathrm{H}^{-1}(C(f)^\bullet) \rightarrow A \xrightarrow{\binom{f}{\iota}} B \oplus F_A{}^1 \xrightarrow{(c^f\ \pi)} \underline{\operatorname{Cok} f} \rightarrow 0 \tag{3.5}$$

Suppose that $\mathrm{H}^{-1}(C(f)^\bullet) = 0$. Then $C(f)^\bullet \cong F_{\underline{\operatorname{Cok} f}}$, and the exact sequence 3.4 shows that θ_f is perfectly exact.

Conversely, suppose that f is rbm; there is an exact sequence

$$\sigma: 0 \rightarrow A \xrightarrow{\binom{f}{q}} B \oplus P' \xrightarrow{(g\ p)} C \rightarrow 0.$$

The maps $\tilde{f} = \binom{f}{q}$ and $\tilde{g} = (g\ p)$ produce the similar diagram as (3.3) :

$$\begin{tikzcd}
F_A{}^\bullet \arrow[r, "\tilde{f}^\bullet"] \arrow[d, "\tilde{\alpha}^\bullet"] & F_{B\oplus P'}{}^\bullet \arrow[r] \arrow[d, equal] & C(\tilde{f})^\bullet \arrow[r] \arrow[d, "\tilde{\gamma}^\bullet"] & F_A{}^{\bullet+1} \arrow[d, "\tilde{\alpha}^{\bullet+1}"] \\
C(\tilde{g})^{\bullet-1} \arrow[r] & F_{B\oplus P'}{}^\bullet \arrow[r, "\tilde{g}^\bullet"] & F_C{}^\bullet \arrow[r] & C(\tilde{g})^\bullet
\end{tikzcd} \tag{3.6}$$

Since $A \overset{st}{\cong} \underline{\operatorname{Ker}\tilde{g}}$, $\tau_{\leq 0}\tilde{\alpha}^\bullet = 0$ is an isomorphism, equivalently $\tau_{\leq -1}C(\tilde{\alpha})^\bullet = 0$ hence $\tau_{\leq -2}C(\tilde{\gamma})^\bullet = 0$. From the long exact sequence of homology groups $\mathrm{H}^{-2}(C(\tilde{\gamma})^\bullet) \rightarrow \mathrm{H}^{-1}(C(\tilde{f})^\bullet) \rightarrow \mathrm{H}^{-1}(F_C{}^\bullet)$, we get $\mathrm{H}^{-1}(C(\tilde{f})^\bullet) = 0$. Obviously, $\mathrm{H}^{-1}(C(\tilde{f})^\bullet) \cong \mathrm{H}^{-1}(C(f)^\bullet)$ hence $\mathrm{H}^{-1}(C(f)^\bullet) = 0$. Now it remains to prove 2) - 4) in the case $\mathrm{H}^{-1}(C(f)^\bullet) = 0$.

2) Applying $\tau_{\leq 0}$ to the diagram (3.6) and taking homology, we get the following diagram with exact rows:

$$\begin{tikzcd}
\theta_{\tilde{f}}:\ 0 \arrow[r] & A \arrow[r, "\left(\begin{smallmatrix} f \\ q \\ \iota \end{smallmatrix}\right)"] \arrow[d, "\tilde{\alpha}'"] & B \oplus P' \oplus F_A{}^1 \arrow[r] \arrow[d, "\tilde{\beta}'"] & \underline{\operatorname{Cok}\binom{f}{q}} \arrow[r] \arrow[d, "\tilde{\gamma}'"] & 0 \\
\sigma_{\tilde{g}}:\ 0 \arrow[r] & \underline{\operatorname{Ker}(g\ p)} \arrow[r] & B \oplus P' \oplus P_C \arrow[r, "(g\ p\ \rho c)"] & C \arrow[r] & 0.
\end{tikzcd}$$

Notice that $\tilde{\alpha}'$ and $\tilde{\beta}'$ are stable isomorphisms. The upper row is a direct sum of θ_f and a trivial complex, and the lower row is that of σ and a trivial complex. Splitting off trivial complexes we get a desired diagram:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\binom{f}{\iota}"] \arrow[d, "\alpha'"] & B \oplus F_A{}^1 \arrow[r, "(c^f\ \pi)"] \arrow[d, "\beta'"] & \underline{\operatorname{Cok} f} \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A \arrow[r, "\binom{f}{q}"] & B \oplus P' \arrow[r, "(g\ p)"] & C \arrow[r] & 0
\end{tikzcd}$$

3) As we see above, $\tilde{\gamma}^\bullet : C(\tilde{f})^\bullet = F_{\underline{\mathrm{Cok}}\tilde{f}}{}^\bullet \to F_C{}^\bullet$ has $\tau_{\leq -2}C(\tilde{\gamma})^\bullet = 0$. We may consider $\tilde{\gamma}^\bullet$ as $\tilde{\gamma}_i = \mathrm{id} \quad (i \leq -1)$ hence $Q' = \underline{\mathrm{Ker}}\tilde{\gamma}$ is projective;

$$0 \to Q' \to \underline{\mathrm{Cok}}\tilde{f} \oplus P_C \xrightarrow{(\tilde{\gamma}'\ \rho_C)} C \to 0.$$

Since $\underline{\mathrm{Cok}}\tilde{f} \overset{st}{\cong} \underline{\mathrm{Cok}}f$ and $\tilde{\gamma}' \overset{st}{\cong} \gamma$, the above sequence is the desired sequence $0 \to Q' \to \underline{\mathrm{Cok}}f \oplus Q \to C \to 0$ with some projective module Q.

4) Suppose σ is perfect. From Proposition 3.5, $F_C{}^{\bullet -1} \to F_A{}^\bullet \xrightarrow{f^\bullet} F_B{}^\bullet \xrightarrow{g^\bullet} F_C{}^\bullet$ is a distinguished triangle, and $F_C{}^\bullet \cong C(f)^\bullet$, hence the induced sequence σ is isomorphic to θ_f. (q.e.d.)

4 Representation by monomorphisms and torsionless modules.

In the previous section, we see that a given map f is represented by monomorphisms if and only if $\mathrm{H}^{-1}(C(f)^\bullet) = 0$. If this is the case, $\underline{\mathrm{Ker}}f = \mathrm{Cok}\, d_{C(f)}{}^{-2}$ is the first syzygy of $\underline{\mathrm{Cok}}f = \mathrm{Cok}\, d_{C(f)}{}^{-1}$. So it is natural to ask the converse: Is a given map f represented by monomorphisms if $\underline{\mathrm{Ker}}f$ is a first sygyzy? This section deals with the problem. As a conclusion, the answer is yes if the total ring of fractions $Q(R)$ of R is Gorenstein. What is more, if $Q(R)$ is Gorenstein, instead of a pseudo-kernel, we can use a (usual) kernel to describe rbm condition.

The next is well known. See [1] and [4] for the proof.

Definition and Lemma 4.1 *The following are equivalent for an R-module M.*

1) *The natural map $\phi : M \to M^{**}$ is a monomorphism.*

2) $\mathrm{Ext}^1_R(\mathrm{Tr}\, M, R) = 0$

3) *M is a first syzygy; there exists a monomorphism from M to a projective module.*

If M satisfies these conditions, M is said to be torsionless. [1]

To solve our problem, the special kind of maps is a key. For $M \in \mathrm{mod}\ R$, consider a module $J^2M = \mathrm{Tr}\,\Omega^1_R\,\mathrm{Tr}\,\Omega^1_R M$. Since $\mathrm{Tr}\, J^2M$ is a first sygyzy, we have $\mathrm{Ext}^1_R(J^2M, R) = 0$, which means $\mathrm{H}_{-1}(F_{J^2M}{}^*{}_\bullet) = 0$ and $\tau_{\geq -2}F_{J^2M}{}^*{}_\bullet$ is a projective resolution of $\mathrm{Tr}\,\Omega^1_R M = \mathrm{Cok}\,(d_{F_{J^2M}}{}^{-2})^* = \mathrm{Cok}\,(d_{F_M}{}^{-2})^*$. The identity map on $\mathrm{Tr}\,\Omega^1_R M$ induces a chain map $(F_M)^*_\bullet \to (F_{J^2M})^*_\bullet$ and its R-dual $\psi_M{}^\bullet : F_{J^2M}{}^\bullet \to F_M{}^\bullet$ subsequently.

Lemma 4.2 *The map $\psi_M : J^2M \to M$ is rbm if and only if an R-module M has* $(\mathrm{Ext}^1_R(M, R))^* = 0$.

[1] In [1], Auslander and Bridger use the term "1-torsion free" for "torsionless". Usually a module M is called torsion-free if the natural map $M \to M \otimes Q(R)$ is injective.

proof. From Theorem 3.6, ψ_M is rbm if and only if $\mathrm{H}^{-1}(C(\psi_M)^*) = 0$. By definition, ψ_M^{-1} and ψ_M^{-2} are identity maps hence ψ_M^i are identity maps for $i \leq -1$. We may assume $\tau_{\leq -2}C(\psi_M)^\bullet = 0$, which implies $\mathrm{H}^{-1}(C(\psi_M)^\bullet) = \operatorname{Ker} d_{C(\psi_M)}{}^{-1}$. As $\tau_{\geq -1}C(\psi_M)_\bullet^*$ is a projective resolution of $\operatorname{Cok}(d_{\psi_M}{}^{-1})^* \cong \mathrm{H}_{-1}(C(\psi_M)_\bullet^*)$, we get $\mathrm{H}^{-1}(C(\psi_M)^\bullet) \cong (\mathrm{H}_{-1}(C(\psi_M)_\bullet^*))^*$. A triangle $F_{J^2M}{}^\bullet \xrightarrow{\psi_M{}^\bullet} F_M{}^\bullet \to C(\psi_M)^\bullet \to F_{J^2M}{}^{\bullet+1}$ induces an R-dual triangle $F_{J^2M}{}^*_{\bullet+1} \to C(\psi_M)_\bullet^* \to F_M{}^*_\bullet \to F_{J^2M}{}^*_\bullet$ which produces an exact sequence of modules

$$0 \to \mathrm{H}_{-1}(C(\psi_M)_\bullet^*) \to \mathrm{H}_{-1}(F_M{}^*_\bullet) \to \mathrm{H}_{-1}(F_{J^2M}{}^*_\bullet) \to 0.$$

As we see in the discussion above, $\mathrm{H}_{-1}(F_{J^2M}{}^*_\bullet) = 0$. Hence $\mathrm{H}_{-1}(C(\psi_M)_\bullet^*) \cong \mathrm{H}_{-1}(F_M{}^*_\bullet) = \mathrm{Ext}_R^1(M,R)$, and we get $\mathrm{H}^{-1}(C(\psi_M)^\bullet) \cong (\mathrm{Ext}_R^1(M,R))^*$. (q.e.d.)

The above result is generalized as follows:

Lemma 4.3 *Let $f : A \to B$ be a morphism in* $\operatorname{mod} R$. *Suppose* $(\mathrm{Ext}_R^1(B,R))^* = 0$. *If* $\underline{\operatorname{Ker} f}$ *is projective, then f is rbm.*

proof. We may assume $\tau_{\leq -2}C(f)^\bullet = 0$. Similarly as in the proof of Lemma 4.2, we have $\mathrm{H}^{-1}(C(f)^\bullet) \cong (\mathrm{H}_{-1}(C(f)_\bullet^*))^*$. Since $\underline{\operatorname{Ker} f}$ is projective, f induces a stable isomorphism $J^2A \overset{st}{\cong} J^2B$, and via this stable isomorphism, ψ_B is projective stably equivalent to $f \circ \psi_A$, equivalently $\psi_B{}^\bullet \cong f^\bullet \circ \psi_A{}^\bullet$ in $\mathsf{K}(\operatorname{mod} R)$. We have a triangle

$$C(\psi_A)^\bullet \to C(\psi_B)^\bullet \to C(f)^\bullet \to C(\psi_A)^{\bullet+1}$$

and its R-dual

$$C(\psi_A)^*{}_{\bullet+1} \to C(f)^*{}_\bullet \to C(\psi_B)^*{}_\bullet \to C(\psi_A)^*{}_\bullet$$

which induce an exact sequence of modules

$$0 \to \mathrm{H}_{-1}(C(f)_\bullet^*) \to \mathrm{H}_{-1}(C(\psi_B)_\bullet^*)$$

Note that $\mathrm{H}_{-1}(C(\psi_B)^*)_\bullet) = \mathrm{Ext}_R^1(B,R)$. The assumption $(\mathrm{Ext}_R^1(B,R))^* = 0$ equivalently says $\mathrm{Ext}_R^1(B,R)_\mathfrak{p} = 0$ for any associated prime ideal $\mathfrak{p}$ of R. A submodule has the same property; $\mathrm{H}_{-1}(C(f)_\bullet^*)_\mathfrak{p} = 0$ for any associated prime ideal $\mathfrak{p}$ of R therefore $(\mathrm{H}_{-1}(C(f)_\bullet^*))^* = 0$. (q.e.d.)

Proposition 4.4 *Let $f : A \to B$ a morphism of* $\operatorname{mod} R$. *Suppose* $(\mathrm{Ext}_R^1(B,R))^* = 0$. *Then f is rbm if and only if* $\underline{\operatorname{Ker} f}$ *is torsionless.*

proof. We already get the "only if " part and have only to show the "if" part. Adding a projective cover of B to f, we get an exact sequence

$$\sigma_f : 0 \to \underline{\operatorname{Ker} f} \xrightarrow{\binom{n^f}{q}} A \oplus P_B \xrightarrow{(f\ \rho_B)} B \to 0.$$

Due to Theorem 3.6, we have a perfect exact sequence θ_{n^f}, because n^f is rbm:

$$\begin{array}{ccccccccc} \theta_{n^f}: & 0 & \to & \underline{\mathrm{Ker} f} & \xrightarrow{\binom{n^f}{\epsilon}} & A \oplus F^1_{\underline{\mathrm{Ker} f}} & \xrightarrow{(c^{n^f}\ \pi)} & \underline{\mathrm{Cok}\, n^f} & \to & 0 \\ & & & \downarrow{\scriptstyle st}\, \cong & & \downarrow{\scriptstyle st}\, \cong & & \downarrow{\scriptstyle \omega_f} & & \\ \sigma_f & 0 & \to & \underline{\mathrm{Ker} f} & \xrightarrow{\binom{n^f}{q}} & A \oplus P_B & \xrightarrow{(f\ \rho_B)} & B & \to & 0. \end{array}$$

From Theorem 3.6 3), we know $\underline{\mathrm{Ker}\omega_f}$ is projective. With the assumption $(\mathrm{Ext}^1_R(B,R))^* = 0$, we can apply Lemma 4.3 and get that ω_f is rbm. From the equation $\underline{f} = \underline{\omega_f} \circ \underline{c^{n^f}}$, f is rbm if c^{n^f} is rbm. Since

$$F_{\underline{\mathrm{Ker} f}}{}^\bullet \xrightarrow{n^{f\bullet}} F_A{}^\bullet \xrightarrow{c^{n^f\bullet}} C(n_f)^\bullet \to F_{\underline{\mathrm{Ker} f}}{}^{\bullet+1}$$

is a triangle, $C(c^{n^f})^\bullet \cong F^{\bullet+1}_{\underline{\mathrm{Ker} f}}$; $\mathrm{H}^{-1}(C(c^{n^f})^{*}) \cong \mathrm{H}^0(F_{\underline{\mathrm{Ker} f}}{}^*) \cong \mathrm{Ext}^1_R(\mathrm{Tr}\, \underline{\mathrm{Ker} f}, R)$. Hence c^{n^f} is rbm if and only if $\underline{\mathrm{Ker} f}$ is torsionless. (q.e.d.)

Lemma 4.5 *Let the sequence of R-modules $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be exact. Suppose $(\mathrm{Ext}^1_R(C,R))^* = 0$. If A and C are torsionless, then so is B.*

proof. From the assumption, $A \cong \underline{\mathrm{Ker} g}$ is torsionless. Due to Proposition 4.4, g is rbm; there exists an exact sequence

$$\theta_g : 0 \to B \xrightarrow{\binom{g}{\epsilon}} C \oplus Q \to \underline{\mathrm{Cok} g} \to 0$$

with a projective module Q and a map $\epsilon : B \to Q$. Since C is a submodule of some projective module, so is B. (q.e.d.)

Proposition 4.6 *The following are equivalent for a noetherian ring R.*

1) *$Q(R)$ is Gorenstein.*

2) *$Q(R)$ is Gorenstein of dimension zero.*

3) *$(\mathrm{Ext}^1_R(M,R))^* = 0$ for each $M \in \mathrm{mod}\, R$.*

4) *Ψ_M is rbm for each $M \in \mathrm{mod}\, R$.*

If R is a local ring with the maximal ideal m, *the above conditions are also equivalent to the following.*

5) *$\Psi_{R/\mathrm{m}}$ is rbm.*

proof. As $Q(R)$ is always of dimension zero, we get 1) $\Leftrightarrow$ 2).

3) $\Leftrightarrow$ 4) is already shown in Lemma 4.2.

4) $\Rightarrow$ 5) is obvious.

5) $\Rightarrow$ 1). The condition 5) is equivalent to $\mathrm{Ext}^1_{\mathrm{R}}(R/\mathbf{m}, R) \otimes Q(R) \cong \mathrm{Ext}^1_{\mathrm{Q(R)}}(R/\mathbf{m} \otimes Q(R), Q(R)) = 0$, which means $Q(R)$ is Gorenstein. (q.e.d.)

In the case $Q(R)$ is Gorenstein, every morphism in $\mathrm{mod}\, R$ satisfies the hypotheses of Proposition 4.4 and Lemma 4.5. Thus with the condition $Q(R)$ is Gorenstein, when discussing rbm property, we can deal with normal kernel as well as pseudo-kernel.

Proposition 4.7 *Suppose $Q(R)$ is Gorenstein. For a given morphism f,* $\mathrm{Ker}\, f$ *is torsionless if and only if* $\underline{\mathrm{Ker} f}$ *is torsionless.*

proof. From Lemma 2.7, there is an exact sequence $0 \to \mathrm{Ker}\, f \to \underline{\mathrm{Ker}}\, f \to \Omega^1_R(\mathrm{Cok}\, f) \to 0$. So the "if" part is obvious, and the "only if" part comes from Lemma 4.5. (q.e.d.)

Theorem 4.8 *Suppose $Q(R)$ is Gorenstein. The following are equivalent for a morphism $f : A \to B$ in* $\mathrm{mod}\, R$.

1) *f is rbm.*

2) $\mathrm{Ker}\, f$ *is torsionless.*

3) $\underline{\mathrm{Ker} f}$ *is torsionless.*

4) $\mathrm{H}^{-1}(C(f)^{\bullet}) = 0$.

5) $\Omega^1_R(\underline{\mathrm{Cok} f}) \overset{st}{\cong} \underline{\mathrm{Ker} f}$.

6) *There exists f' such that $f' \overset{st}{\cong} f$ and* $\mathrm{Ker}\, f'$ *is torsionless.*

7) *For any f' with $f' \overset{st}{\cong} f$,* $\mathrm{Ker}\, f'$ *is torsionless.*

proof. Implications 5) $\Rightarrow$ 3), 7) $\Rightarrow$ 2) and 7) $\Rightarrow$ 6) are obvious. We already showed 1) $\Leftrightarrow$ 4) in Theorem 3.6, 1) $\Leftrightarrow$ 3) in Proposition ??, and 3)$\Leftrightarrow$ 2) in Corollary 4.7. Implications 3) $\Rightarrow$ 7) and 6) $\Rightarrow$ 3) are obtained from "if" and "only if " part of Corollary 4.7 respectively.

4) $\Rightarrow$ 5). It comes directly from $\mathrm{Cok}\, {d_{C(f)}}^0 = \underline{\mathrm{Ker} f}$ and $\mathrm{Cok}\, {d_{C(f)}}^{-1} = \underline{\mathrm{Cok} f}$. (q.e.d.)

Remark 4.9 *Takashima gives an easy proof for Theorem 4.8 using the torsion theory [7].*

Corollary 4.10 *The following are equivalent for a noetherian ring R.*

1) *$Q(R)$ is Gorenstein.*

2) *Every morphism with torsionless kernel is rbm.*

proof.

1) $\Rightarrow$ 2). It comes directly from Theorem 4.8.

2) $\Rightarrow$ 1). For every $M \in \operatorname{mod} R$, $\operatorname{Ker}\psi_M$ is torsionless. Because $\underline{\operatorname{Ker}\psi_M}$ is projective and $\operatorname{Ker}\psi_M$ is a submodule of $\underline{\operatorname{Ker}\psi_M}$ from Lemma 2.7 1). So if 2) holds, ψ_M is rbm for any $M \in \operatorname{mod} R$, which implies 1) from Proposition 4.6. (q.e.d.)

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MORITA DUALITY AND RING EXTENSIONS *

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ABSTRACT. Let A be a ring with Morita duality induced by a bimodule ${}_BQ_A$ and let R be a ring extension of A. Müller proved the fundamental result that if R_A and $\mathrm{Hom}_A(R,Q)_A$ are linearly compact, then R has a Morita duality induced by ${}_S\mathrm{Hom}_A(R,Q)_R$, where $S = \mathrm{End}_R(\mathrm{Hom}_A(R,Q))$. We improve this result by showing the existence of a category equivalence between certain categories of A-rings and B-rings whenever A and B are two Morita dual rings. We also generalize and unify a result of Fuller-Haack about Morita duality of semigroup rings and a result of Mano about self-duality of finite centralizing extensions.

1 研究の背景

環の Morita duality や self-duality が様々な拡大環にどのように遺伝するかについては、多くの研究者によって調べられてきた．基礎となるのは次の Müller の結果である．

定理 A ([1, Proposition 7.3] 参照). A を両側加群 ${}_BQ_A$ によって定められる Morita duality をもつ環，R を A の拡大環とする．R_A と $\mathrm{Hom}_A(R,Q)_A$ が linearly compact であれば，R は両側加群 ${}_S\mathrm{Hom}_A(R,Q)_R$ によって定められる Morita duality をもつ．ただし $S = \mathrm{End}_R(\mathrm{Hom}_A(R,Q))$ である．

この定理の特別な場合として，Fuller-Haack は次の定理を示した．

定理 B ([1, Corollary 9.4] 参照). G を有限半群とする．環 A が環 B に右 Morita dual であれば，半群環 AG は半群環 BG に右 Morita dual である．特に環 A が self-duality をもてば，半群環 AG も self-duality をもつ．

この半群環の self-duality を一般化する形で，真野は次の定理を証明した．

定理 C ([1, Theorem 9.2] 参照). 環 A は両側加群 ${}_AQ_A$ によって定められる self-duality をもつとする．A の拡大環 R が条件

(1) R は $r_1,\dots,r_n$ を基底とする自由右 A 加群である．

*The detailed version of this note will be submitted for publication elsewhere.

(2) 各 r_i は A のすべての元と可換である.

(3) $r_i r_j = \sum_{k=1}^{n} r_k a_{ijk}$ $(a_{ijk} \in A)$ と表すとき，各 a_{ijk} は Q のすべての元と可換である.

を満たすならば，R も self-duality をもつ.

今回の研究において，2つの環 A と B が Morita dual のとき，ある種の A-環の圏と B-環の圏の間に圏同値が存在することを示すことによって，定理 A を改良した (定理 2). またその応用として，自由であるような有限中心的拡大 (定理 C の条件 (1), (2) を満たす拡大環) の Morita duality を決定し，定理 B と定理 C を統合・一般化することができた (定理 4).

以下この報告集では，すべての環は単位元をもち，すべての加群は単位的であるとする.

2 Morita duality の定義と両側加群の圏

A と B を環とする. Mod-A, B-Mod によって，それぞれ右 A 加群全体，左 B 加群全体の圏を表す. **Morita duality** とは，次の条件を満たす Mod-A の充満部分圏 $\mathcal{A}$ と B-Mod の充満部分圏 $\mathcal{B}$ の間の duality (すなわち反変同値) $F : \mathcal{A} \rightleftarrows \mathcal{B} : G$ である:

(1) $A_A \in \mathcal{A}$, ${}_BB \in \mathcal{B}$.

(2) $\mathcal{A}$ と $\mathcal{B}$ は部分加群と剰余加群で閉じている.

実際には，F, G は適当な両側加群 ${}_BQ_A$ を用いて $F \cong \mathrm{Hom}_A(-, Q)$, $G \cong \mathrm{Hom}_B(-, Q)$ として表現可能である. ここで，${}_BQ_A$ は忠実平衡的両側加群で，Q_A と ${}_BQ$ は移入的余生成加群となる. 逆に，このような両側加群 ${}_BQ_A$ に対して，双対圏手 $\mathrm{Hom}_A(-, Q)$, $\mathrm{Hom}_B(-, Q)$ は Q-反射的な加群 (定義は後述) からなる充満部分圏の間の Morita duality となるため，両側加群 ${}_BQ_A$ は **Morita duality を定義する**という. このように，Morita duality は片側加群の圏に対して定義されるが，今回の研究ではある種の両側加群の圏に注目した.

以下，この論文を通して，両側加群 ${}_BQ_A$ は Morita duality を定めるとし，双対圏手を $(-)^* = \mathrm{Hom}_A(-, Q)$, $(-)^\# = \mathrm{Hom}_B(-, Q)$ とおく. 右 A 加群 X (左 B 加群 Y) は，標準的な評価写像 $X \to X^{*\#}$ $(Y \to Y^{\#*})$ が同型写像であるとき，**Q-反射的** (Q-reflexive)

であるという．両側加群の圏を次のように定める．

$$\mathcal{M}_{A-A} = \{{}_AL_A \mid L_A, L^*{}_A \text{は } Q\text{-反射的}\},$$
$$ {}_B\mathcal{M}_A = \{{}_BM_A \mid M_A, {}_BM \text{は } Q\text{-反射的}\},$$
$$ {}_{B-B}\mathcal{M} = \{{}_BN_B \mid {}_BN, {}_BN^{\#} \text{は } Q\text{-反射的}\}.$$

ただし射は両側準同型写像である．よく知られているように，${}_BQ_A$ が Morita duality を定めるとき，右 A 加群 X (左 B 加群 Y) が Q-反射的であることと linearly compact であることは同値である．したがって，定理 A の拡大環 R に対する仮定は ${}_AR_A \in \mathcal{M}_{A-A}$ であることを意味する．Morita duality を制限することによって，これらの両側加群の圏について，次の補題が成り立つ．

補題 1. (1) ${}_AA_A \in \mathcal{M}_{A-A}$，${}_BQ_A \in {}_B\mathcal{M}_A$，${}_BB_B \in {}_{B-B}\mathcal{M}$.

(2) $\mathcal{M}_{A-A}$，${}_B\mathcal{M}_A$，${}_{B-B}\mathcal{M}$ は，部分両側加群，剰余両側加群，両側加群の拡大で閉じている．

(3) 双対圏手の対 $(-)^* : \mathcal{M}_{A-A} \rightleftarrows {}_B\mathcal{M}_A : (-)^{\#}$ と $(-)^* : {}_B\mathcal{M}_A \rightleftarrows {}_{B-B}\mathcal{M} : (-)^{\#}$ は duality である．したがって，これらの合成 $(-)^{**} : \mathcal{M}_{A-A} \rightleftarrows {}_{B-B}\mathcal{M} : (-)^{\#\#}$ は圏同値である．

3 環の圏とその同値

環 R と環準同型写像 $A \xrightarrow{f} R$ の対 (R, f) を A-環 (A-ring) という．A の拡大環や剰余環は A-環である．2 つの A-環 (R, f) と (R', f') の間の射 $\phi : (R, f) \to (R', f')$ を $\phi \circ f = f'$ を満たす環準同型写像 $\phi : R \to R'$ として定める．以後 A-環 (R, f) を単に R で表す．A-環 R は (A, A) 両側加群と見なすことができ，A-環の射 $R \to R'$ は (A, A) 両側準同型写像となる．

A-環の圏と B-環の圏の充満部分圏を，それぞれ

$$\mathcal{R}_A = \{R \mid {}_AR_A \in \mathcal{M}_{A-A}\},\ {}_B\mathcal{R} = \{S \mid {}_BS_B \in {}_{B-B}\mathcal{M}\}$$

によって定めれば，定理 A の精密化の次の定理を述べることができる．

定理 2. (1) $(-)^{**} : \mathcal{R}_A \rightleftarrows {}_B\mathcal{R} : (-)^{\#\#}$ は圏同値を定める．

(2) 各 $R \in \mathcal{R}_A$ に対して，R^* は Morita duality を定める (R^{**}, R) 両側加群となる．

$R \in \mathcal{R}_A$ のとき，${}_AR_A \in \mathcal{M}_{A-A}$ であるから，$R^{**} \in {}_{B-B}\mathcal{M}$ であるが，実際には R^{**} は B-環の構造をもつ．さらに，$(-)^{**}$ は A-環の射を B-環の射に写すことを確かめて，定理2の(1)を得る．また，環として $R^{**} \cong \mathrm{End}_R(R^*)$ であるから，定理2の(2)は定理Aそのものである．この種の結果においては，通常(定理A，B，Cでも) $\mathrm{End}_R(R^*)$ が用いられる．こちらの方が環構造は明らかであるが，圏手として見なす場合は R^{**} の方が扱いやすい．また，後述の定理4でも R^{**} の方が計算は容易である．

定理2より，環 A と B が Morita dual のとき，A と B のある種の拡大環との間には，Morita dual であるという関係の元で，同型の意味で1対1対応が存在する．次の系は，対応する A-環と B-環のそれぞれ A-，B-部分環全体は一種の annihilator によって1対1に対応することを示している．

系 3. $R \in \mathcal{R}_A$ に対して，$U = R^*$，$S = U^* = R^{**}$ とおく．R の A-部分環 R' に対して，

$$U' = \{u \in U \mid u(R') = 0\},\ S' = \{s \in S \mid s(U') = 0\}$$

とおけば，U/U' は Morita duality を定める (S', R') 両側加群となる．また，対応 $R' \mapsto S'$ は R の A-部分環全体と S の B-部分環全体との間の1対1対応を与える．

この系より，両側加群 ${}_BQ_A$ が Morita duality を定めるとき，例えば n 次行列環 $R = M_n(A)$ の A-部分環も Morita duality をもつことや，Morita dual な環や Morita duality を定める両側加群が annihilator によって具体的に記述できることが分かる．

4 有限中心的拡大

R を A の拡大環とする．A の任意の元と可換な R の元 $r_1, \ldots, r_n$ で，$R = \sum_{i=1}^n r_iA$ となるようなものが存在するとき，R は A の**有限中心的拡大** (**finite centralizing extension**) と言われる．両側加群 ${}_AM_A$ が $M = mA = Am$ と書けるとき，$M \in \mathcal{M}_{A-A}$ であることが分かる．A の有限中心的拡大環 R は，(A, A) 両側加群として，この ${}_AM_A$ のような両側加群の有限個の直和の剰余両側加群であるから，補題1より R は $\mathcal{R}_A$ に属する．したがって定理2より有限中心的拡大環に Morita duality は遺伝する．この事実はもちろんよく知られているが，次の定理において，有限中心的拡大 $R = \sum_{i=1}^n r_iA$ において，$r_1, \ldots, r_n$ が自由 A-基底になっているとき，R と Morita dual な環を完全に決定した．

なお，有限中心的拡大の一般化として，有限正規拡大と有限三角拡大がある ([1, Sections 8–9] 参照)．これらの拡大環に Morita duality が遺伝することもすでに示されているが，上の有限中心的拡大の場合と同じ論法で示すことができる．有限正規拡大や有限三角拡大については，たとえ生成元が自由基底になっていても，Morita dual な環を具体的に計算することは難しいものと思われる．

Morita duality を定める両側加群 ${}_BQ_A$ は忠実平衡であるから，A，B の中心は同型，すなわち，環同型写像 $\alpha : \mathrm{Cen}(A) \xrightarrow{\cong} \mathrm{Cen}(B)$ が存在する．ここで α は，任意の $q \in Q$ に対して $qa = \alpha(a)q$，によって定められる．また次の定理4の条件 (3) における a_{ijk} が $\mathrm{Cen}(A)$ に属することが直ちに分かる．条件 (3) は，条件 (3)** と対比するために書いただけで，実際には何の制約も与えておらず，単に R は自由であるような A の有限中心的拡大であるという意味である．以上の注意の下，定理4を次のように記述できる．

定理 4. 両側加群 ${}_BQ_A$ は Morita duality を定めるとする．R は A の拡大環で，条件

(1) $r_1, \ldots, r_n$ は R の A-自由基底である．

(2) 各 r_i は A の任意の元と可換である．

(3) $r_i r_j = \sum_{k=1}^{n} r_k a_{ijk}$ $(a_{ijk} \in A)$.

を満たすとする．$U = R^*$，$S = U^*$ とおく．このとき，U は Morita duality を定める (S, R) 両側加群で，S は条件

(1)** $r_1^{**}, \ldots, r_n^{**}$ は S の B-自由基底である．

(2)** 各 r_i^{**} は B の任意の元と可換である．

(3)** $r_i^{**} r_j^{**} = \sum_{k=1}^{n} r_k^{**} \alpha(a_{ijk})$ $(\alpha(a_{ijk}) \in B)$.

を満たす．ただし r_i^{**} は $r_i^{**}(u) = u(r_i)$ $(u \in U)$ によって定義される S の元である．

この定理は，R の積の構造定数 a_{ijk} と S の積の構造定数 $\alpha(a_{ijk})$ はほとんど「同じ」であることを示している．半群環の場合，構造定数には 0 か 1 しか現れないが，同型写像 α はそれらを保つから，特別な場合として Fuller-Haack の定理 B を得る．また $A = B$ で $\alpha(a_{ijk}) = a_{ijk}$ の場合として，真野の定理 C を得る．

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ON INDECOMPOSABILITY OF A MODULE GIVEN BY BRAUER CONSTRUCTION

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ABSTRACT. In representation theory of finite groups one of the most important and well-known conjectures is Broué's abelian defect group conjecture. In this note we introduce a sort of technical result which is useful to prove Broué's abelian defect group conjecture for a few examples which are new.

1. Introduction

In representation theory of finite groups, there has been a very important problem, namely,

How are representations of a finite group G over a field k of prime characteristic p similar to those of its subgroup H containing a Sylow p-subgroup P of G?

$$\begin{array}{c}\text{representations of } G \text{ over } k \\ \updownarrow \\ \text{representations of } H \text{ over } k\end{array}$$

An origin of this problem is, of course, due to Richard Brauer (1901–77). From this point of view, one of the most important and interesting (and also well-known) conjectures in representation theory of finite groups is *Broué's abelian defect group conjecture.* Actually, Michel Broué conjectures the following.

(1.1) Broué's abelian defect group conjecture, (see [2], [3] and [5]) . For any prime p, if a block algebra A of $\mathcal{O}G$ has an abelian defect group P then A and its Brauer corresponding block algebra B in $\mathcal{O}N_G(P)$ should be **derived equivalent**, that is,

$$\mathrm{D}^b(\text{mod-}A) \underset{\textit{equivalent}}{\simeq} \mathrm{D}^b(\text{mod-}B)$$

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The detailed final vertion of this paper will be published, see [7].

as triangulated categories, where $\mathcal{O}$ is a complete discrete valuation ring whose residue field is an algebraically closed field k of characteristic p, and mod-A is the category of finitely generated right A-modules, and $\mathrm{D}^b(\mathfrak{A})$ is the bouded derived category of an abelian category $\mathfrak{A}$.

As well-known, there is a beautiful result of Jeremy Rickard ([9], [10]), which characterizes such a derived equivalence completely and which is a generalization of a Morita equivalence from modules to complexes. As a matter of fact, in **(1.1)** above, a stronger conclusion is expected. Namely, **derived equivalent** should be replaced by **splendidly Rickard equivalent**, which is due to Jeremy Rickard [11].

In this note we present a theorem which is a sort of technical one, but also useful to prove Broué's abelian defect group conjecture, which is used in a work of the author, Kunugi and Waki [6].

Actually, we have a result which is a joint work with **Markus Linckelmann**. We present it in the next section.

2. The main result

First we introduce notation and terminology we need to state our main result.

(2.1) Notation and assumption. Let G be a finite group, and let $\mathcal{O}$ and k be as in **(1.1)**. We call an $\mathcal{O}G$-lattice of finite rank *a finitely generated $\mathcal{O}G$-module.* For any subgroup L of G let $\Delta L = \{(\ell,\ell) \in L \times L \mid \ell \in L\}$, and we consider $\mathcal{O}G$ as a right $\mathcal{O}[\Delta L]$-module via $a(\ell,\ell) = \ell^{-1}a\ell$ for $a \in \mathcal{O}G$ and $\ell \in L$. Then, we next define $(\mathcal{O}G)^{\Delta L}$ as $(\mathcal{O}G)^{\Delta L} = \{a \in \mathcal{O}G \mid a(\ell,\ell) = a for \forall \ell \in L\} = \{a \in \mathcal{O}G \mid \ell^{-1}a\ell = a for \forall \ell \in\}$. For a p-subgroup R of G let $\mathrm{Br}_{\Delta R}$ be the Brauer map (homomorphism) with respect to R. Namely, $\mathrm{Br}_{\Delta R}$ is defines as

$$\mathrm{Br}_{\Delta R}: \ (\mathcal{O}G)^{\Delta R} \longrightarrow kC_G(R), \quad \sum_{g\in G} \alpha_g g \ \mapsto \sum_{g\in C_G(R)} \bar{\alpha}_g g,$$

where $\alpha_g \in \mathcal{O}$ and $\bar{\alpha}_g$ is its image in k

Note the Brauer map $\mathrm{Br}_{\Delta R}$ is a surjective algebra-homomorphism. For a p-subgroup R which is contained in both of G and L and an $(\mathcal{O}G,\mathcal{O}L)$-bimodule Y we define the Brauer construction (quotient) $Y(\Delta R)$ by

$$Y(\Delta R) = \Big(Y^{\Delta R} / \sum_{R' \subsetneqq R} \mathrm{Tr}\uparrow_{\Delta R'}^{\Delta R}(Y^{\Delta R'})\Big) \otimes_{\mathcal{O}} k$$

where Tr is the transfer (trace) map. Clearly, the Brauer construction $?(\Delta R)$ induces an additive functor from the category $\mathcal{O}G$-mod-$\mathcal{O}L$ of the finitely generated $(\mathcal{O}G,\mathcal{O}L)$-bimodules to the category $kC_G(R)$-mod-$kC_L(G)$. In particular if $Y = \mathcal{O}G$ we get $Y(\Delta R) = (\mathcal{O}G)(\Delta R) \cong kC_G(R)$, and we identify these since we have a commutative diagram

$$\begin{array}{ccc} (\mathcal{O}G)^{\Delta R} & \xrightarrow{\mathrm{Br}_{\Delta R}} & kC_G(R) \\ {\scriptstyle \text{canonical epimorphism}}\downarrow & & \| \\ (\mathcal{O}G)(\Delta R) & \xrightarrow[\simeq]{} & kC_G(R) \end{array}$$

First, let (P, e) be a Brauer pair in G, that is, P is a p-subgroup of G and e is a block of $kC_G(P)$. Then, by results of Alperin-Broué [1, Theorem 3.4] and Broué-Puig [4, Theorem 1.8], it follows that, for any subgroup Q of P there uniquely exists a block e_Q of $kC_G(Q)$ such that $(Q, e_Q) \leqslant (P, e)$. For other notation and terminology, see books of Nagao-Tsushima [8] and Thévenaz [12].

Now, we can state the main result of this note; that is,

(1.2) Theorem(Koshitani-Linckelmann, see [7]). *Let b be a block (block idempotent) of a block algebra A of $\mathcal{O}G$ (so that $A = b\mathcal{O}G$), and let (P, e) be a maximal b-Brauer pair in G, and hence P is a defect group of b and e is a block of $kC_G(P)$ such that $Br_{\Delta P}(b)e = e$. Next, let $H = N_G(P, e) = \{g \in N_G(P) \,|\, g^{-1}eg = e\}$ be the inertial group of e in $N_G(P)$. For each subgroup Q let e_Q be the same as above, and let f_Q be a block of $kC_H(Q)$ such that $(Q, f_Q) \leqslant (P, e)$ since $C_G(P) = C_H(P)$. Moreover, let f be a primitive idempotent of $(b\mathcal{O}G)^{\Delta H}$ such that $Br_{\Delta P}(f) \cdot e = e$, and let $X = \mathcal{O}Gf$. Then we have the following.*

(i) *X is an indecomposable $\mathcal{O}[G \times H]$-module with vertex ΔP.*

(ii) *If Q is a subgroup of $Z(P)$, the center of P, then $e_Q \cdot X(\Delta Q) \cdot f_Q$ is a unique indecomposable direct summand of $e_Q kC_G(Q) f_Q$ with vertex ΔP as a $(kC_G(Q), kC_H(Q))$-bimodule.*

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SOME TOPICS ON DERIVED EQUIVALENT BLOCKS OF FINITE GROUPS

NAOKO KUNUGI

1. INTRODUCTION

Let G be a finite group. Let k be an algebraically closed field of characteristic $\ell > 0$. We denote the principal block of kG by $B_0(G)$.

We say that two finite groups G and H have the same ℓ-local structure if G and H have a common Sylow ℓ-subgroup P such that whenever Q_1 and Q_2 are subgroups of P and $f : Q_1 \to Q_2$ is an isomorphism, then there is an element $g \in G$ such that $f(x) = x^g$ for all $x \in Q_1$ if and only if there is an element $h \in H$ such that $f(x) = x^h$ for all $x \in Q_1$.

There is a well known conjecture due to Broué.

Conjecture 1.1(Broué [1, 2]). Let G and H be finite groups having the same ℓ-local structure with common Sylow ℓ-subgroup P. If P is abelian then the principal blocks of G and H would be derived equivalent.

If P is not abelian, then there is a counterexample to this conjecture. However , there are some examples that P is not abelian and there is a derived equivalence between the principal blocks of G and H. We will give such examples in §3.

2. GENERAL THEORY

In this section, let G and H be finite groups having the same ℓ-local structure with common Sylow ℓ-subgroup P. We say that a complex of $(B_0(G), B_0(H))$-bimodules is *splendid* if each indecomposable summand of each term of the complex is a direct summand of a module of the form $kG \otimes_{kQ} kH$ for a subgroup Q of P.

Definition 2.1. Let $X^\bullet$ be a splendid complex of $(B_0(G), B_0(H))$-bimodules. We say that $X^\bullet$ induces a *splendid stable equivalence* if we have isomorphisms

$$X^\bullet \otimes_{B_0(H)} X^{\bullet *} \cong B_0(G) \oplus Z_1, \quad X^{\bullet *} \otimes_{B_0(G)} X^\bullet \cong B_0(H) \oplus Z_2$$

where Z_1 and Z_2 are homotopy equivalent to complexes of projective bimodules.

Definition 2.2. Let $X^\bullet$ be a splendid complex of $(B_0(G), B_0(H))$-bimodules. We say that $X^\bullet$ induces a *splendid equivalence* if we have isomorphisms

$$X^\bullet \otimes_{B_0(H)} X^{\bullet *} \cong B_0(G) \oplus Z_1, \quad X^{\bullet *} \otimes_{B_0(G)} X^\bullet \cong B_0(H) \oplus Z_2 \quad .$$

where Z_1 and Z_2 are homotopy equivalent to 0. The complex $X^\bullet$ is called a *splendid tilting complex.*

The detailed version of this paper will be submitted for publication elsewhere.

By the definition, splendid equivalences induce derived equivalences and homotopy equivalences.

Theorem 2.1 (Rouquier [12]). *Let $X^\bullet$ be a splendid complex of $(B_0(G), B_0(H))$-bimodules. Then the following are equivalent.*

(1) *The complex $X^\bullet$ induces a splendid stable equivalence between $B_0(G)$ and $B_0(H)$.*

(2) *For every non-trivial subgroup Q of P, the complex $X^\bullet(\Delta(Q))$ induces a splendid equivalence between $B_0(C_G(Q))$ and $B_0(C_H(Q))$, where $\Delta(Q)$ is a diagnal subgroup and*

$$X(Q) = X^{\Delta(Q)} / \sum_{R<Q} \mathrm{Tr}_R^Q X^{\Delta(R)}.$$

In our example in §3 we will use the following method when we prove splendid equivalences.

(Step 1) Construct a splendid tilting complex between $B_0(C_G(Q))$ and $B_0(C_H(Q))$ for every non-trivial subgroup Q of P.

(Step 2) Construct a splendid stable equivalence F from $B_0(G)$ to $B_0(H)$ by gluing the splendid tilting complexes obtained in Step 1 (by using the above theorem).

(Step 3) Calculate $F(S)$ for the simple $B_0(G)$-modules.

(Step 4) Lift the stable equivalence in Step 2 to a splendid equivalence by looking at the modules calcutated in Step 3.

3. General linear groups and unitary groups

Let q be a power of a prime. Assume that ℓ is odd and ℓ^e divides $q+1$ but ℓ^{e+1} does not divide $q+1$ for some $e > 0$. Under this condition, we consider representations of the general linear group $GL(n, q^2)$ and the unitary group $GU(n, q^2)$ for small n. Note that if $\ell > n$ then the principal ℓ-block of $GL(n, q^2)$ is Morita equivalent to its Brauer correspondent by Puig's result(see [8]).

3.1. $GL(2, q^2)$ **and** $GU(2, q^2)$. We have isomorphisms

$$B_0(GL(2, q^2)) \cong kZ_{\ell^e} \otimes B_0(SL(2, q^2)), \quad B_0(GU(2, q^2)) \cong kZ_{\ell^e} \otimes B_0(SU(2, q^2)).$$

The blocks $B_0(SL(2, q^2))$ and $B_0(SU(2, q^2))$ have cyclic defect groups, and they are splendid equivalent by Rouquier's result in [11]. Therefor the principal blocks $B_0(GL(2, q^2))$ and $B_0(GU(2, q^2))$ are splendid equivalent.

3.2. $GL(3, q^2)$ **and** $GU(3, q^2)$ **in characteristic** $\ell > 3$. In this case, Sylow ℓ-subgroups of $GL(3, q^2)$ and $GU(3, q^2)$ are abelian. As in case $n = 2$, we have isomorphisms

$$B_0(GL(3, q^2)) \cong kZ_{\ell^e} \otimes B_0(SL(3, q^2)), \quad B_0(GU(3, q^2)) \cong kZ_{\ell^e} \otimes B_0(SU(3, q^2)).$$

In [5], Waki and the author showed that $B_0(SU(3, q^2))$ and its Brauer correspondent, which is isomorphic to the Brauer correspondent of $B_0(SL(3, q^2))$, are splendid equivalent. Therefore $B_0(SL(3, q^2))$ and $B_0(SU(3, q^2))$ are splendid equivalent since as we mentioned above $B_0(SL(3, q^2))$ and its Brauer correspondent are Morita(Puig) equivalent by Puig's result. Hence we also have $B_0(GL(3, q^2))$ and $B_0(GU(3, q^2))$ are splendid equivalent.

3.3. $GL(3, q^2)$ **and** $GU(3, q^2)$ **in characteristic 3.** In this case Sylow 3-subgroups of $GL(3, q^2)$ and $GU(3, q^2)$ are not abelian. Our main result in this paper is the following theorem.

Theorem 3.1 (with T. Okuyama). *Assume that 3^e divides $q+1$ but 3^{e+1} does not divide $q+1$ for $e>0$. Then*
(1) *The principal 3-blocks $B_0(PSL(3,q^2))$ and $B_0(PSU(3,q^2))$ are splendid equivalent.*
(2) *The principal 3-blocks $B_0(SL(3,q^2))$ and $B_0(SU(3,q^2))$ are splendid equivalent.*
(3) *The principal 3-blocks $B_0(PGL(3,q^2))$ and $B_0(PGU(3,q^2))$ are splendid equivalent.*
(4) *The principal 3-blocks $B_0(GL(3,q^2))$ and $B_0(GU(3,q^2))$ are splendid equivalent.*

Remark 3.1. If $e=1$, then the result for (1) has been obtained by [6, 4, 3] and the result for (3) has been obtained by Usami and the author.

4. Outline of proof of theorem

In this section, we give an outline of a proof of Theorem 3.1 (1) and (2). Let $G = SL(3,q^2)$, $H = SU(3,q^2)$, $\overline{G} = PSL(3,q^2)$ and $\overline{H} = PSU(3,q^2)$. Let P be a common Sylow 3-subgroup of G and H. We denote the image of a subgroup L of G (or H) in $\overline{G}$ (or $\overline{H}$) by $\overline{L}$. For each subgroup $\overline{R}$ of $\overline{P}$, let $\overline{G}_R := C_{\overline{G}}(\overline{R})$, $\overline{H}_R := C_{\overline{H}}(\overline{R})$, and let $\overline{M}_R$ be the Scott module of $\overline{G}_R \times \overline{H}_R$ with vertex $\Delta(\overline{R'})$, where $\overline{R'}$ is a Sylow 3-subgroup of $\overline{G}_R$ and $\overline{H}_R$.

(Step 1). There is essentially one subgroup of P(up to conjugate), which we denote by Q, containing $Z(P)$ such that $B_0(C_G(Q))$ and $B_0(C_H(Q))$ are not Morita equivalent. Then $C_G(Q) \cong GL(2,q^2)$ and $C_H(Q) \cong GU(2,q^2)$. Let $\overline{M}_Q \to k_{\overline{G}_Q\times\overline{H}_Q}$ be a $\Delta(\overline{Q'})$-projective cover of $k_{\overline{G}_Q\times\overline{H}_Q}$ and $\overline{N}_Q \to \Omega_{\Delta(\overline{Q'})}(k_{\overline{G}_Q\times\overline{H}_Q})$ be a $\Delta(\overline{Q})$-projective cover of $\Omega_{\Delta(\overline{Q'})}(k_{\overline{G}_Q\times\overline{H}_Q})$. Then we have a splendid tilting complex for $B_0(\overline{G}_Q)$ and $B_0(\overline{H}_Q)$ of the form

$$0 \longrightarrow \overline{N}_Q \longrightarrow \overline{M}_Q \longrightarrow 0.$$

For a subgroup $\overline{R}$ of $\overline{P}$ not contained in $\overline{Q}$, the blocks $B_0(\overline{G}_R)$ and $B_0(\overline{H}_R)$ are Morita equivalent and the Scott module $\overline{M}_R$ gives a splendid tilting complex for these two blocks.

(Step 2). Let M be the Scott module of $G\times H$ with vertex $\Delta(P)$. Let $M \to k_{G\times H}$ be a $\Delta(P)$-projective cover of $k_{G\times H}$ and $N \to \Omega_{\Delta(P)}(k_{G\times H})$ be a $\Delta(Q)$-projective cover of $\Omega_{\Delta(P)}(k_{G\times H})$. Consider the following complex

$$M^\bullet: \quad 0 \longrightarrow N \longrightarrow M \longrightarrow 0.$$

and set $\overline{M}^\bullet = \mathrm{Inv}_{Z(P)\times 1}(M^\bullet)$. Then the complex $\overline{M}^\bullet$ is a splendid complex, and for each non-trivial subgroup $\overline{R}$ of $\overline{P}$, the complex $\overline{M}^\bullet(\Delta(\overline{R}))$ coincides with the complex in (Step 1). Therefore by Rouquier's theorem (Theorem 2.1) we can see that the complex $\overline{M}^\bullet$ induces a splendid stable equivalence between $B_0(\overline{G})$ and $B_0(\overline{H})$.

(Step 3). Let $F = - \otimes_{B_0(G)} \overline{M}^\bullet$. The principal block of $B_0(\overline{G})$ has 5 simple modules k, S, T_1, T_2 and T_3 and the principal block of $B_0(\overline{H})$ has 5 simple modules $k, \varphi, \theta_1, \theta_2$ and θ_3. Then we have the following lemma.

Lemma 4.1. *There exist exact sequences*

$$0 \longrightarrow \Omega^{-1}(U(k,\varphi)) \longrightarrow \Omega(F(S)) \longrightarrow k\oplus k \longrightarrow 0$$

and

$$0 \longrightarrow \Omega^{-1}(U(k,\varphi,\theta_i)) \longrightarrow \Omega^2(F(T_i)) \longrightarrow k \longrightarrow 0$$

for $i = 1,2$ and 3, where $U(k,\varphi)$ is a uniserial module of length 2 with top k, and $U(k,\varphi,\theta_i)$ is a uniserial module of length 3 with top k and socle θ_i.

(Step 4). It follows from Lemma 4.1 that the tilting complex defined by a sequence $\{\theta_1,\theta_2,\theta_3\}$, $\{\varphi,\theta_1,\theta_2,\theta_3\}$ and $\{\varphi,\theta_1,\theta_2,\theta_3\}$ of subsets of the set of simple modules (see [6]) gives a derived equivalence between $B_0(\overline{G})$ and $B_0(\overline{H})$. The equivalence is a lift of the stable equivalence given by F (see [7]), and therefore $B_0(G)$ and $B_0(H)$ are splendid equivalent.

Now we have the splendid tilting complex for $B_0(\overline{G})$ and $B_0(\overline{H})$ of the form

$$\overline{X}^{\bullet}: \quad 0 \longrightarrow \overline{Q}_3 \longrightarrow \overline{Q}_2 \longrightarrow \overline{Q}_1 \oplus \overline{N} \longrightarrow \overline{M} \longrightarrow 0$$

where $\overline{M} = \mathrm{Inv}_{Z(P)\times 1}(M)$ and $\overline{N} = \mathrm{Inv}_{Z(P)\times 1}(N)$ and $\overline{Q}_1$, $\overline{Q}_2$, and $\overline{Q}_3$ are projective bimodules. Since $\mathrm{Inv}_{Z(P)\times 1}(-)$ induces a one to one correspondence between the set of trivial source $k[G\times H]$-modules with vertex $\Delta(Z(P))$ and the set of projective $k[\overline{G}\times\overline{H}]$-modules, we have a tilting complex of the form

$$X^{\bullet}: \quad 0 \longrightarrow Q_3 \longrightarrow Q_2 \longrightarrow Q_1 \oplus N \longrightarrow M \longrightarrow 0$$

for $B_0(G)$ and $B_0(H)$, where Q_1, Q_2 and Q_3 are direct sums of trivial source $k[G\times H]$-modules with vertex $\Delta(Z(P))$ and $\mathrm{Inv}_{Z(P)\times 1}(X^{\bullet}) = \overline{X}^{\bullet}$ (see [12, A.4]).

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GAUSS SUMS ARE JUST CHARACTERS OF MULTIPLICATIVE GROUPS OF FINITE FIELDS [1]

KAORU MOTOSE

This paper is a summary of some papers [4,5,6] such that using special commutative group algebras, we could prove alternatively some reciprocity theorems, prime decompositions of Gauss sums and Lenstra's primality test.

1. Group Algebra Map(F, K)

Let $A = \mathrm{Map}(F, K)$ be the set of all mappings from a finite field $F = F_q$ of order q to a field K where q is a power of a prime p. Then we define the convolution product in A by the following

$$(f * g)(c) = \sum_{\substack{a,b \in F \\ a+b=c}} f(a)g(b)$$

for $f, g \in A$ and $c \in F$. This product together with the usual sum and the scalar product gives the structure of a commutative algebra over K. If there is no chance of confusion we shall denote the product $f * g$ by the usual notation fg.

Let u_a be the characteristic function of $a \in F$, namely, u_a is defined by the following

$$u_a(b) := \begin{cases} 1 & \text{if } b = a \\ 0 & \text{if } b \neq a. \end{cases}$$

Then we have the following equations.

$$u_a u_b = u_{a+b} \text{ and } f = \sum_{a \in F} f(a) u_a \text{ for } f \in A.$$

Thus $\{u_a \mid a \in F\}$ forms a basis of the group algebra A of the additive group of F over K. We denote by $\hat{F}$ the set of all characters of the multiplicative group $F^* = F \setminus \{0\}$, by $\chi^{(k)}$ k-th power of $\chi \in \hat{F}$ with respect to the convolution product and by ϵ the trivial character. We set $\epsilon(0) = 1$ and $\chi(0) = 0$ for $\chi \neq \epsilon \in \hat{F}$. Thus we have $\hat{F} \subset A$. We set $J(f_1, f_2, \ldots, f_n) = (f_1 f_2 \cdots f_n)(1)$ for $f_1, f_2, \ldots, f_n \in A$ which is

[1] This paper is a summary of some papers [4,5,6] that was already published. This paper was financially supported by Fund for the Promotion of International Scientific Research B-2, 2004, Aomori, Japan.

usually called the Jacobi sum.

2. Gauss sums and Jacobi sums

It is easy to see that $\epsilon * \epsilon = q\epsilon$ and $\lambda * \epsilon = 0$ for nontrivial $\lambda \in \widehat{F}$. We have the following another relations which are important to our object.

Lemma 1. *Assume that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are nontrivial elements in $\widehat{F}$ and $q-1 \neq 0$ in K. Then we have the next equations in each case.*

(1) *In case $\lambda_1\lambda_2\ldots\lambda_n \neq \epsilon$, we have*

$$\lambda_1 * \lambda_2 * \cdots * \lambda_n = J(\lambda_1, \lambda_2, \ldots, \lambda_n)\lambda_1\lambda_2\cdots\lambda_n.$$

(2) *In case $\lambda_1\lambda_2\cdots\lambda_n = \epsilon$, we have*

$$\lambda_1 * \lambda_2 * \cdots * \lambda_n = \lambda_n(-1)J(\lambda_1, \lambda_2, \ldots, \lambda_{n-1})(qu_0 - \epsilon)$$

where $J(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}) = 1$ if $n = 2$.

For $\chi \in \widehat{F}$, we can write $\chi = \Sigma_{a\in F_p}\chi(a)u_a$. On the other hand Gauss sums is defined by

$$g(\chi) = \sum_{a\in F} \chi(a)\zeta_p^{\mathrm{tr}(a)}$$

where $\zeta_p := e^{\frac{2\pi i}{p}}$, $q = p^r$ and $\mathrm{tr}(a) = a + a^p + \cdots + a^{p^{r-1}}$ for $a \in F$. Hence, in case $K = C$ the complex number field, a map $\chi \mapsto g(\chi)$ ($u_a \mapsto \zeta_p^{\mathrm{tr}(a)}$) is the natural homomorphism from A to C. Therefore, it is natural to think of χ as Gauss sum $g(\chi)$. It is easy to see $\widehat{F}$ forms a basis of A because $u_a = \frac{1}{q-1}\Sigma_{\chi\in\widehat{F}}\chi(a^{-1})\chi$ if $q-1 \neq 0$ in K.

3. Quadratic characters for odd primes

In this section, we shall have evaluation of the quadratic character $\eta \in A$ for an odd prime q. Using the character table and a permutation $b \mapsto b^{-1}$ on F^*, we can see easily the next proposition.

Proposition 2.

(1) $\det[u_{ab^{-1}}]_{a,b} = (\epsilon - u_0) * \prod^*_{\chi\neq\epsilon} \chi$ *where $\prod^*$ means the product of all nontrivial multiplicative characters with respect to the convolution product.*

(2) $\det[u_{ab}]_{a,b} = (-1)^{\frac{q^2-1}{8}} q^{\frac{q-3}{2}}\eta$ *where q is odd.*

The next needs for evaluation of η. This follows from Proposition 2.

Lemma 3.

(1) $\prod_{k=1}^{q-1}(u_0 - u_1^k) = qu_0 - \epsilon.$

(2) $\eta = u_1^{\frac{(q^2-1)(q-1)}{16}} \prod_{k=1}^{\frac{q-1}{2}}(u_0 - u_1^k).$

(3) $\eta = (-1)^{\frac{q-1}{2}} v^{\frac{q(q^2-1)}{8}} \prod_{k=1}^{\frac{q-1}{2}}(v^k - v^{-k})$ *where* $v = u_{\frac{q+1}{2}}.$

We can see the evaluation of ordinary Gauss sum

$$g(\eta) = i^{\frac{(q-1)^2}{4}}\sqrt{q}$$

from Lemma 3 and the equation $\prod_{k=1}^{\frac{n-1}{2}} 2\sin(\frac{k\pi}{n}) = \sqrt{n}$ for an odd n.

4. Prime decompositions of Gauss sums

In this section, using commutative group algebras, we shall give an alternative proof of theorem about the prime decomposition of the Gauss sum which was essentially used in the proof of Stickelberger relation (see [1]).

Let m be a natural number, let p be a prime which does not divide m, let f be the order of p mod m, and $q = p^f$. Moreover let O be the ring of algebraic integers in $Q(\zeta_{q-1})$ and let P be a prime ideal containing p, where ζ_{q-1} is a primitive $(q-1)$-th root of 1. Then it is well known that q is the order of a finite field $F = O/P$.

We consider the Gauss sum $g_a = g(\chi^a) = \sum_{\alpha \in F} \chi^a(\alpha)\zeta_p^{\mathrm{tr}(\alpha)}$ where χ is a generator of $\widehat{F}$ and $\mathrm{tr}(\alpha)$ is the trace of α. Let $\mathcal{P}$ be the ideal generated by P and $\{1-\zeta_p^k \mid 0 < k < p\}$ in the ring of algebraic integers $\mathcal{O}$ of $Q(\zeta_{(q-1)p})$. It is easy to see $\mathcal{P}$ is the prime ideal generated by P and $1 - \zeta_p$. We set $a^* = b_0 + b_1 + \cdots + b_{f-1}$ for a positive integer $a = b_0 + b_1 p + \cdots + b_{f-1}p^{f-1}$ where $0 < a < q$ and $0 \le b_k < p$.

The next follows essentially from [3, Proposition 3.2] and this was used essentially for the Stickelberger relation (see [1]).

Theorem 4. $\mathrm{ord}_{\mathcal{P}}(g_a) = a^*$ *for* $0 < a < q$, *namely,* $\mathcal{P}^{a^*}$ *divides exactly* g_a.

Proof. Let ν be a natural homomorphism from $\mathrm{Map}(F, O)$ to $\mathrm{Map}(F, O/P)$ and let $\mathcal{J}$ be the ideal generated by P and $\{u_0 - u_\alpha | \alpha \in F\}$. Since $\nu(\chi^c)^{[p]} = 0$ for $\chi^c \neq 1$, we obtain that $\nu(\chi^c)$ is contained in $\nu(\mathcal{J})$, the radical of the group algebra $\mathrm{Map}(F, O/P)$, and so $\chi^c \in \mathcal{J}$. [3, Proposition 3.2] together with this implies that $\gamma\chi^a \in \mathcal{J}^{a^*}$ for the Jacobi sum $\gamma \in O \setminus P$. The character $u_\beta \mapsto \zeta_p^{\mathrm{tr}(\beta)}$ induces the epimorphism $\phi : \mathrm{Map}(F, O) \to \mathcal{O}$ with $\phi(\mathcal{J}) = \mathcal{P}$ and $\phi(\gamma\chi^a) = \gamma g_a$. Thus we have $\mathrm{ord}_{\mathcal{P}}(g_a) \ge a^*$.

On the other hand, $\mathrm{ord}_{\mathcal{P}}(g_a) + \mathrm{ord}_{\mathcal{P}}(g_{q-1-a}) = f(p-1) = a^* + (q-1-a)^*$ follows from $g_a g_{q-1-a} = \chi^a(-1)q$ and $\mathrm{ord}_{\mathcal{P}}(p) = p-1$. This completes our proof.

Remark 5. [3, Proposition 3.3] shows that $\{\chi^a | a^* = k\}$ forms a basis of $\nu(\mathcal{J})^k/\nu(\mathcal{J})^{k+1}$ and so $\mathrm{ord}_{\mathcal{J}}(\chi^a) = a^*$, namely, a^* is the maximum integer s such that $\chi^a \in \mathcal{J}^s$.

Loewy series of $\mathrm{Map}(F, O/P)$ are computed from this. (a^* is the maximum integer s with $\chi^a \in \mathcal{J}^s$)

5. Reciprocity theorems and Lenstra's primality test

The next lemma is essential in proving quadratic, cubic and biquadratic reciprocity theorems, and Lenstra's primality test.

Lemma 6. *Let ℓ be the order of $\chi \in \widehat{F}$, let n be a prime number with $(n,q)=1$ and let e and s be natural numbers with $n^e \equiv s \bmod \ell$. Then*

$$\chi^{-es}(n) \equiv (jq)^{\frac{n^e - s}{\ell}} \chi^{[s]}(1) \mod n \quad \textit{where } j = \chi(-1)\chi^{[\ell-1]}(1).$$

Theorem 7 (Lenstra). *Let n be an odd integer and let r be a prime divisor of n. Let T be a finite set consisting of 2 and odd primes p satisfying $(n,p)=1$ and $n^{p-1} \not\equiv 1 \bmod p^2$. We set $t = \prod_{p \in T} p$. Let S be the set of primes q satisfying $(n,q)=1$ and $(q-1) \mid t$. We set $s = \prod_{q \in S} q$.*

We assume there exists an integer c such that $c^{\frac{n-1}{2}} \equiv -1 \bmod n$, and $(jq)^{\frac{n^{p-1}-1}{p}} \equiv \chi_q(n) \bmod n$ for every $p \in T$, $q \in S$ and $\chi_q \in \widehat{F}$ with order p. Then we have $r \equiv n^i \bmod s$ for some $i < t$.

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RELATIVE INVARIANTS OF GROUPS ACTING ON KRULL DOMAINS AND APPLICATIONS

HARUHISA NAKAJIMA

Abstract. In this paper, we will study on relative invariants of a group G consisting of automorphims of a Krull domain R and give a criterion for R_χ to be a free R^G-module of rank one for a 1-cocycle χ of G in the unit group $\mathrm{U}(R)$, in terms of local 1-cocycles which is similar to one in [N1, S1, S2]. On the other hand, suppose that G is equal to the centralizer of an algebraic torus G^0. Consider an affine factorial G-variety X with trivial units, over an algebraically closed field K of characteristic zero. Let V be the K-dual of a finite dimensional generating G-submodule $V^\vee$ of $\mathcal{O}(X)$ having a K-basis Ω consisting of weight vectors of G^0 such that Ω does not degenerate under $V\backslash\{0\} \to \mathbb{P}(V)$. Suppose that the action (X, G) is stable (cf. [K1, P]) and consider the G-submodule W of V such that G is diagonal on V/W and $W \ni x \mapsto x+w \in V$ induces $W//G_w \cong V//G$, , for some $w \in V$. As an application of the result on modules of relative invariants mentioned above, we show that, for a minimal W, $\mathrm{Cl}(X//G) \hookrightarrow \mathfrak{y}(G_w/s_W(G_w))$ whose cokernel is finite. Here $\mathfrak{y}(G_w/s_W(G_w))$ is a group of rational characters of G_w modulo its largest pseudo-reflection subgroup on W. Some related results are discussed.

1. Introduction.

Let $Q(R)$, $\mathrm{U}(R)$ and $\mathrm{Ht}_1(R)$ denote the total quotient ring of a commutative ring R, the unit group of R and the set consisting of all prime ideals $\mathfrak{p}$ of R of height one, respectively. For a ring extension $S \hookrightarrow R$, put

$$\mathrm{Ht}_1(R,S) := \{\mathfrak{p} \in \mathrm{Ht}_1(R) \mid \mathfrak{p} \cap S \in \mathrm{Ht}_1(S)\}$$

and, moreover for any $\mathfrak{Q} \in \mathrm{Ht}_1(S)$, put

$$X_{\mathfrak{Q}}(R) := \{\mathfrak{p} \in \mathrm{Ht}_1(R) \mid \mathfrak{p} \cap S = \mathfrak{Q}\}.$$

In the case where R is a Krull domain, let $\mathrm{v}_{R,\mathfrak{p}}$ be the discrete valuation of R defined by $\mathfrak{p} \in \mathrm{Ht}_1(R)$ and, for a subset Ω of $Q(R)$, let $\mathrm{div}_R(\Omega)$ be the divisor on R associated to the divisorialization $\widetilde{\Omega \cdot R}$ of $\Omega \cdot R$, if it is a fractional ideal of R. Moreover, $\mathrm{supp}_R(\Omega)$ stands for the set

$$\{\mathfrak{p} \in \mathrm{Ht}_1(R) \mid \mathrm{v}_{R,\mathfrak{p}}(\Omega \cdot R) \neq 0\}.$$

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This is an expository article on some results of the author which shall be published in the forthcoming papers with detailed proofs.

If R is a Krull domain, then a subring S of R satisfying the condition that $Q(S)\cap R$ is so and we can define the reduced ramification index $e(\mathfrak{p},\mathfrak{p}\cap S)$ of $\mathfrak{p}\in \mathrm{Ht}_1(R,S)$ over $\mathfrak{p}\cap S$ (cf. [N4]). In this case, the ring R is said to be unramified over S at a subset $\Gamma \subseteq \mathrm{Ht}_1(R,S)$, if R is unramified over S at each $\mathfrak{p}\in\Gamma$ (cf. [N4]). Let us consider an action of a group G on R as ring automorphisms. For a 1-cocycle χ of G in the unit group $\mathrm{U}(R)$ of R, we denote by R_χ the R^G-module of χ-invariants or invariants relative to χ, where R^G denotes the subring of R of invariants of G.

(1.1) Suppose that X is an affine normal variety over an algebraically closed field K of characteristic $p\geq 0$, whose structure sheaf is denoted to $\mathcal{O}$, with a regular action of an affine algebraic K-group G. Suppose that $Q(\mathcal{O}(X)^G)=Q(\mathcal{O}(X))^G$ and that $\mathcal{O}(X)^{\mathcal{R}_u(G)}$ is noetherian, where $\mathcal{R}_u(G)$ is the unipotent radical of G. Regard $Z^1(G,\mathrm{U}(\mathcal{O}(X)))$, the group of 1-cocyles of G, as an additive group and also $\mathfrak{X}(G)$, the rational character group of G, as an additive subgroup. We can choose a finite subset Γ of $\mathrm{Ht}_1(\mathcal{O}(X),\mathcal{O}(X)^G)$ in such a way that

$$\begin{aligned}\Gamma \supseteq &\mathrm{supp}_{\mathcal{O}(X)}(\mathcal{O}(X)_\chi)\cap \mathrm{Ht}_1(\mathcal{O}(X),\mathcal{O}(X)^G)\\ &(\forall\chi\in Z^1(G,\mathrm{U}(\mathcal{O}(X)))\ \text{ with }\ \mathcal{O}(X)_\chi\neq\{0\})\end{aligned}$$

and $X_{\mathfrak{p}\cap\mathcal{O}(X)^G}(\mathcal{O}(X))\subseteq\Gamma$ for all $\mathfrak{p}\in\Gamma$. For any $\mathfrak{p}\in\Gamma$, let $I_G(\mathfrak{p}\cap\mathcal{O}(X)^{G^0})$ denote the inertia group of G at $\mathfrak{p}\cap\mathcal{O}(X)^{G^0}$ (cf. [N4]). Set $U=\cap_{\mathfrak{p}\in\Gamma}(\mathcal{O}(X)\backslash\mathfrak{p})$ and let $\pi_\mathfrak{p}\in\mathfrak{p}$ ($\mathfrak{p}\in\Gamma$) be a relative invariant of G^0 whose associated 1-cocycle is denoted to $\delta_{\pi_\mathfrak{p}}\in Z^1(G^0,\mathrm{U}(U^{-1}\mathcal{O}(X)))$ such that $\pi_\mathfrak{p}$ generates $\mathfrak{p}U^{-1}\mathcal{O}(X)$. Let $\Delta_\mathfrak{p}$ be the 1-cocycle in $Z^1(I_G(\mathfrak{p}\cap\mathcal{O}(X)^{G^0}),\mathrm{U}(U^{-1}R))$ defined by $\prod_{\mathfrak{q}\in I_G(\mathfrak{p}\cap\mathcal{O}(X)^{G^0})_\mathfrak{p}}\pi_\mathfrak{q}$. For any $\chi\in Z^1(G,\mathrm{U}(\mathcal{O}(X)))$ with $\mathcal{O}(X)_\chi\neq\{0\}$, $s_\mathfrak{p}(\chi)$ stands for the smallest $a\in\mathbb{Z}_0$ satisfying

$$\chi|_{I_G(\mathfrak{p}\cap\mathcal{O}(X)^{G^0})}\equiv a\cdot\Delta_\mathfrak{p}\mod B^1(I_G(\mathfrak{p}\cap\mathcal{O}(X)^{G^0}),\mathrm{U}(U^{-1}\mathcal{O}(X))).$$

We can choose a nonnegative integer $b_\mathfrak{p}(\chi)$ ($\mathfrak{p}\in\Gamma$) such that

$$\chi|_{G^0}-\sum_{\mathfrak{p}\in\Gamma}s_\mathfrak{p}(\chi)\cdot\delta_{\pi_\mathfrak{p}}\equiv\sum_{\mathfrak{p}\in\Gamma}b_\mathfrak{p}(\chi)\cdot\delta_{\pi_\mathfrak{p}}\mod B^1(G^0,\mathrm{U}(U^{-1}\mathcal{O}(X)))$$

and, for any minimal subset Δ of Γ satisfying

$$\left\{\sum_{\mathfrak{p}\in\Delta}\mathrm{N}\cdot\delta_{\pi_\mathfrak{p}}\right\}\cap B^1(G^0,\mathrm{U}(U^{-1}R))\neq\emptyset,$$

$\{b_\mathfrak{p}(\chi)\mid\mathfrak{p}\in\Delta\}$ contains 0.

As an application of our main result in Sect. 2, we obtain

Theorem 1.2. *Under the circumstances as in (1.1), for $\chi\in Z^1(G,\mathrm{U}(\mathcal{O}(X)))$ such that $\mathcal{O}(X)_\chi\neq\{0\}$, the following conditions are equivalent:*

(1) $\mathcal{O}(X)_\chi\cong\mathcal{O}(X)^G$ *as* $\mathcal{O}(X)^G$*-modules.*

(2) $\dim_{Q(\mathcal{O}(X)^G)}(\mathcal{O}(X)_\chi\otimes_{\mathcal{O}(X)^G}Q(\mathcal{O}(X)^G))=1$, *and the intersection of*

$$\sum_{\mathfrak{p}\in\Gamma}(b_\mathfrak{p}(\chi)+s_\mathfrak{p}(\chi))\cdot\mathrm{div}_R(\mathfrak{p})+\sum_{\mathfrak{p}\in\mathrm{Ht}_1(\mathcal{O}(X))\backslash\mathrm{Ht}_1(\mathcal{O}(X),\mathcal{O}(X)^G)}\mathbb{Z}\cdot\mathrm{div}_{\mathcal{O}(X)}(\mathfrak{p})$$

and $\{\mathrm{div}_{\mathcal{O}(X)}(f)\mid f\in\mathcal{O}(X)_\chi\backslash\{0\}\}$ *is nonempty.*

If these conditions are satisfied, then $\mathcal{O}(X)_\chi$ is generated by the element f as an $\mathcal{O}(X)^G$-module such that $\{\mathrm{div}_{\mathcal{O}(X)}(f)\}$ is equal to the non-empty set of divisors stated in (2).

This can be applied to establishing a reduction of invariants of certain reductive algebraic groups to ones of their stabilizers, which is similar to slice method.

(1.3) Furthermore suppose that G^0 is an algebraic torus. Let $\mathcal{O}(X)_{\mathrm{st}}$ be the largest G-invariant affine K-subalgebra of $\mathcal{O}(X)$ such that the action of G on $\mathrm{Spec}(\mathcal{O}(X)_{\mathrm{st}})$ is stable (e.g. [K1, P]) and X_{st} the affine variety defined by $\mathcal{O}(X_{\mathrm{st}}) = \mathcal{O}(X)_{\mathrm{st}}$ (cf. [W1, N3]). Consider a finite dimensional rational G-submodule $V^\vee$ of $\mathcal{O}(X)$ which generates $\mathcal{O}(X)$ as a K-algebra. Let V denote the dual space of $V^\vee$ on which G acts naturally. We have a canonical G-equivariant closed embedding $X \hookrightarrow V$ and identify the dual of V with $V^\vee$. A pair (W, w) is defined to be a paralleled linear hull of (X, G) through $V^\vee$ or, simply, of (V, G), if W is a G-submodule of V_{st} such that G is diagonalizable on V_{st}/W, w is a nonzero vector of V_{st} such that $W \cap < G\cdot w >_K = \{0\}$ and the morphism

$$(\bullet + w) : W \ni x \mapsto x + w \in V_{\mathrm{st}}$$

induces the isomorphism

$$\pi_{V_{\mathrm{st}}//G_w, V//G} \circ (\bullet + w)//G_w : W//G_w \xrightarrow{\sim} V_{\mathrm{st}}//G.$$

Here $(\bullet + w)//G_w : W//G_w \longrightarrow V_{\mathrm{st}}//G_w$ is the algebraic quotient of $(\bullet + w)$ and

$$\pi_{V_{\mathrm{st}}//G_w, V//G} : V_{\mathrm{st}}//G_w \longrightarrow V_{\mathrm{st}}//G$$

is induced by $\mathcal{O}(V_{\mathrm{st}})^G \hookrightarrow \mathcal{O}(V_{\mathrm{st}})^{G_w}$. Under the assumption that $G = G^0$, the pair (W, w) seems to be initially defined and used by H. P. Kraft and D. H. Wehlau (cf. §1 of [K2] and [W1, W2]). A paralleled linear hull (W_0, w_o) of (V, G) is said to be minimal, if W_0 is minimal in the subspaces W which admit paralleled linear hulls (W, w) of (V, G) for some w 's. A element of $\sigma \in G$ is said to be a pseudo-reflection on V (resp. a generalized-reflection on $\mathcal{O}(X)$), if $\dim(\sigma-1)(V) = \mathrm{ht}((\sigma-1)(V)\cdot\mathcal{O}(V)\cap\mathcal{O}(V)^G) = 1$ (resp. $\mathrm{ht}((\sigma-1)(\mathcal{O}(X))\cdot\mathcal{O}(X)) = \mathrm{ht}((\sigma-1)(\mathcal{O}(X))\cdot\mathcal{O}(X)\cap\mathcal{O}(X)^G) = 1$). We denote by $\mathcal{R}_V(G)$ (resp. by $\mathcal{R}_{\mathcal{O}}(X)(G)$) the subgroup of G generated by all pseudo-reflections (resp. generalized-reflections) of G on V (resp on $\mathcal{O}(X)$).

Theorem 1.4. *Under the circumstances as in (1.3), moreover suppose that $p = 0$ and X is an affine conical factorial variety with a regular conical action of G. Let $V^\vee$ be the rational homogeneous G-submodule of $\mathcal{O}(X)$ minimally generating $\mathcal{O}(X)$ as a K-algebra. Suppose that $Z_G(G^0) = G$. Let (W, w) be a minimal paralleled linear hull of (V, G) through $V^\vee$, where V denotes the K-dual $(V^\vee)^\vee$ of $V^\vee$. Then*

(1) *The quotient morphism π_{W,G_w} (resp. $\pi_{X_{\mathrm{st}},G_w}$) is no-blowing-up of codimension one, and for any $\mathfrak{p} \in \mathrm{Ht}_1(\mathcal{O}(X_{\mathrm{st}})^{G_w})$ and $\mathfrak{q} \in \mathrm{Ht}_1(\mathcal{O}(W)^{G_w})$, the actions of G_w on $X_{\mathfrak{p}}(\mathcal{O}(X_{\mathrm{st}}))$ and $X_{\mathfrak{q}}(\mathcal{O}(W))$ are transitive.*
(2) $\mathrm{Cl}(\mathcal{O}(X)^G) \cong \mathrm{Cl}(\mathcal{O}(X_{\mathrm{st}})^{G_w}) \cong \mathrm{Cl}(\mathcal{O}(V)^G) \cong \mathfrak{X}(G_w/\mathfrak{R}_{\mathcal{O}(X_{\mathrm{st}})}(G_w))$.
(3) *If $X_{\mathrm{st}}//G_w$ is an affine space, then so is $V//G$.*
(4) *Suppose that G_w is solvable. Then $V//G$ is an affine space if and only if $\mathcal{O}(X)^G$ is factorial.*

The last assertion can be regarded as a generalization of [W2] and Proposition 3.6 of [N1]. Similarly by the use of paralleled linear hulls, we can determine representations of diagonalizable groups with the algebra of invariants which are hypersurfaces. Further applications to invariants of non-semisimple reductive groups shall be given by the author elsewhere.

The notations $\mathbb{N}$ and $\mathbb{Z}$ are standard and $\mathbb{Z}_0$ stands for the set of all nonnegative integers.

2. Modules of relative invariants.

From now on to the end of Proposition 2.10, suppose that R is a Krull domain with an action of a group G through the automorphism group $\mathrm{Aut}(R)$ of R. The group of cocycles of degree 1 including character groups are represented as additive groups.

Proposition 2.1. *For $\chi \in Z^1(G, U(R))$, $R_\chi \cong R^G$ as R^G-modules if and only if $\dim_{\mathcal{Q}(R^G)}(\mathcal{Q}(R^G) \otimes_{R^G} R_\chi) = 1$ and $\frac{1}{f}R \cap \mathcal{Q}(R^G) = R^G$ for some $f \in R_\chi$.*

We derive the next theorem from Proposition 2.1.

Theorem 2.2. *For $\chi \in Z^1(G, U(R))$, the R^G-module R_χ is free and of rank one if and only if the following two conditions are satisfied:*

(1) $\dim_{\mathcal{Q}(R^G)}(\mathcal{Q}(R^G) \otimes_{R^G} R_\chi) = 1$

(2) *There exists a nonzero element f satisfying*

$$\begin{aligned} &\forall \mathfrak{q} \in \mathrm{Ht}_1(R^G) \quad \Rightarrow \\ &\exists \mathfrak{P} \in X_{\mathfrak{q}}(R) \ \textit{such that} \ \mathrm{v}_{R,\mathfrak{P}}(f) < \mathrm{e}(\mathfrak{P}, \mathfrak{q}).. \end{aligned} \tag{2.2.1}$$

If these conditions are satisfied, then $R_\chi = R^G \cdot f$ for any nonzero $f \in R_\chi$ such that (3.3.1) holds for f. □

The following two propositions play an important role in defining the local datum which shall be introduced in (2.5).

Proposition 2.3. *Let $\chi \in Z^1(G, U(R))$ and let H be a normal subgroup of G of a finite index. Suppose that R^H is unramified over R^G at $\mathrm{supp}_R(R_\chi) \cap \mathrm{Ht}_1(R, R^G)$. Then the following conditions are equivalent:*

(1) *R_χ is an R^G-free module of rank one and*

$$\dim_{\mathcal{Q}(R^H)}(\mathcal{Q}(R^H) \otimes_{R^H} R_{\chi|_H}) = 1.$$

(2) *There are nonzero $g \in R_{\chi|_H}$ and $u \in \mathbf{U}(R^H)$ such that $R_{\chi|_H} = R^H \cdot g$ and $R_\chi \ni g \cdot u$.*

If these conditions are satisfied, then $R_\chi = R^G \cdot gu$ for the elements g and u in (2).

Proposition 2.4. *Suppose that $\mathcal{Q}(R^G) = \mathcal{Q}(R)^G$ and let $\chi \in Z^1(G, \mathbf{U}(R))$ such that $R_\chi \neq \{0\}$. Let U be a multiplicative system of R invariant under the action of G such that*

$$\mathfrak{p} \cap U = \emptyset \quad (\forall \mathfrak{p} \in \bigcup_{\mathfrak{p} \in \mathrm{supp}(R_\chi) \cap \mathrm{Ht}_1(R, R^G)} X_{\mathfrak{p} \cap R^G}(R)),$$

$$\mathfrak{p} \cap U \neq \emptyset \quad (\forall \mathfrak{p} \in \mathrm{supp}(R_\chi) \text{ satisfying } \mathrm{ht}(\mathfrak{p} \cap R^G) \geq 2).$$

Then we have

(1) *For any* $\mathfrak{q} \in \mathrm{supp}(R_\chi) \cap \mathrm{Ht}_1(R, R^G)$ *and* $\mathfrak{p} \in X_{\mathfrak{q} \cap R^G}(R)$, *the following congruence holds;*

$$\mathrm{v}_{R,\mathfrak{p}}(R_\chi) \equiv \mathrm{v}_{U^{-1}R,\mathfrak{p}U^{-1}R}((U^{-1}R)_\chi) \mod \mathrm{e}(\mathfrak{p}, \mathfrak{q} \cap R^G).$$

(2) *If* $R_\chi \cong R^G$ *as* R^G*-modules,* $(U^{-1}R)_\chi \cong (U^{-1}R)^G$ *as* $(U^{-1}R)^G$*-modules and* $(U^{-1}R)_\chi = R_\chi \cdot (U^{-1}R)^G$.

(2.5) We explain our notations and circumstances which shall be considered as follows. Suppose that $\mathcal{Q}(R^G) = \mathcal{Q}(R)^G$. Let $\chi \in Z^1(G, U(R))$ be a 1-cocycle such that $R_\chi \neq \{0\}$ and Γ a finite subset of $\mathrm{Ht}_1(R, R^G)$ such that

$$\Gamma \supseteq \mathrm{supp}_R(R_\chi) \cap \mathrm{Ht}_1(R, R^G)$$

and $\Gamma \supseteq X_{\mathfrak{p} \cap R^G}(R)$ for any $\mathfrak{p} \in \Gamma$. Let H be a normal subgroup of G of a finite index stabilizing each $\mathfrak{p}$ in Γ. Put $U = \cap_{\mathfrak{p} \in \Gamma}(R \backslash \mathfrak{p})$. Let $\pi_\mathfrak{p}$ be an element which generates $\mathfrak{p}U^{-1}R$ and, for any subgroup N of G, set

$$\pi_{N\mathfrak{p}} = \prod_{\mathfrak{q} \in N\mathfrak{p}} \pi_\mathfrak{q}.$$

Let $\Delta_{N\pi_\mathfrak{p}} \in Z^1(N, \mathrm{U}(U^{-1}R))$ be the cocycle defined by

$$N \ni \tau \longmapsto \Delta_{N\pi_\mathfrak{p}}(\tau) = \frac{\tau(\pi_{N\mathfrak{p}})}{\pi_{N\mathfrak{p}}} \in \mathrm{U}(U^{-1}R)$$

and $\delta_{\pi_\mathfrak{p}} \in Z^1(H, \mathrm{U}(U^{-1}R))$ the cocycle defined by

$$H \ni \sigma \longmapsto \delta_{\pi_\mathfrak{p}}(\sigma) = \frac{\sigma(\pi_\mathfrak{p})}{\pi_\mathfrak{p}} \in \mathrm{U}(U^{-1}R).$$

Moreover we donote by $s_{I_G(\mathfrak{p} \cap R^H)\mathfrak{p}}(\chi)$ the infimum of

$$\{a \in \mathbb{Z}_0 \mid \chi|_{I_G(\mathfrak{p} \cap R^H)} \equiv a \cdot \Delta_{I_G(\mathfrak{p} \cap R^H)\mathfrak{p}} \mod B^1(I_G(\mathfrak{p} \cap R^H), \mathrm{U}(U^{-1}R))\}$$

and denote by χ^H the cocycle

$$\chi|_H - \sum_{\mathfrak{p} \in \Gamma} s_{I_G(\mathfrak{p} \cap R^H)\mathfrak{p}}(\chi) \cdot \delta_\mathfrak{p} \in Z^1(H, \mathrm{U}(U^{-1}R)).$$

The independence of the number $s_{I_G(\mathfrak{p} \cap R^H)\mathfrak{p}}(\chi)$ on the choice of the subgroup H is guaranteed in the next lemma.

Lemma 2.6. *Under the same circumstances as in (2.5), the following properties hold for each* $\mathfrak{p} \in \Gamma$*:*

(1) $\mathrm{e}(\mathfrak{q}, \mathfrak{q} \cap R^G) = \mathrm{e}(\mathfrak{p}, \mathfrak{p} \cap R^G)$ *for all* $\mathfrak{q} \in I_G(\mathfrak{p} \cap R^H)$.
(2) $\mathrm{ord}(\Delta_{I_G(\mathfrak{p}\cap R^H)_\mathfrak{p}} \mod B^1(I_G(\mathfrak{p} \cap R^H), \mathrm{U}(U^{-1}R))) = \mathrm{e}(\mathfrak{p}, \mathfrak{p} \cap R^G)$.
(3) $0 \leq s_{I_G(\mathfrak{p}\cap R^H)_\mathfrak{p}}(\chi) < \mathrm{e}(\mathfrak{p}, \mathfrak{p} \cap R^G)$ *and*

$$s_{I_G(\mathfrak{p}\cap R^H)_\mathfrak{p}}(\chi) \equiv \mathrm{v}_{R,\mathfrak{p}}(R_\chi) \mod \mathrm{e}(\mathfrak{p}, \mathfrak{p} \cap R^G).$$

In order to define the characteristic divisor D_χ, we need

Lemma 2.7. *Under the same circumstances as in (2.5), there exist non-negative integers* $a_\mathfrak{p}(\chi)$ $(\forall \mathfrak{p} \in \Gamma)$ *such that* $\{a_\mathfrak{p}(\chi) \mid \mathfrak{p} \in X_\Omega(R)\}$ *contains* 0*, for any restriction* $\Omega \in \{\mathfrak{p} \cap R^H \mid \mathfrak{p} \in \Gamma\}$ *and the following congruence holds;*

$$\chi^H \equiv \sum_{\mathfrak{p}\in\Gamma} a_\mathfrak{p}(\chi)\mathrm{e}(\mathfrak{p}, \mathfrak{p} \cap R^G) \cdot \delta_{\pi_\mathfrak{p}} \mod B^1(H, \mathrm{U}(U^{-1}R)).$$

Furthermore:

(1) $(a_\mathfrak{p}(\chi) \mid \mathfrak{p} \in \Gamma)$ *is uniquely determined by* χ.
(2) *Extending* $a_\mathfrak{p}(\chi) = 0$ *for any* $\mathfrak{p} \in \mathrm{Ht}_1(R, R^G)\backslash\Gamma$*, we see that* $(a_\mathfrak{p}(\chi) \mid \mathfrak{p} \in \mathrm{Ht}_1(R, R^G))$ *is independent on the choice of* Γ.
(3) *There is an element* $u \in \mathrm{U}(U^{-1}R)$ *such that the* $(U^{-1}R)^G$*-module* $(U^{-1}R)_\chi$ *is generated by the element*

$$u \cdot \prod_{\mathfrak{p}\in\Gamma} \pi_\mathfrak{p}^{a_\mathfrak{p}(\chi)\mathrm{e}(\mathfrak{p},\mathfrak{p}\cap R^G)+s_{I_G(\mathfrak{p}\cap R^H)_\mathfrak{p}}(\chi)}.$$

(2.8) Under the same circumstances as in (2.5) and Lemma 2.7, let D_χ denote the divisor

$$\sum_{\mathfrak{p}\in\mathrm{supp}_R(R_\chi)\cap\mathrm{Ht}_1(R,R^G)} \{a_\mathfrak{p}(\chi)\mathrm{e}(\mathfrak{p}, \mathfrak{p} \cap R^G) + s_{I_G(\mathfrak{p}\cap R^H)_\mathfrak{p}}(\chi)\} \cdot \mathrm{div}_R(\mathfrak{p})$$

in $\mathrm{Div}(R)$ and we call D_χ the *characteristic divisor of* χ. Let $\mathcal{D}_G$ denote the subgroup

$$\sum_{\mathfrak{p}\in\mathrm{Ht}_1(R)\backslash\mathrm{Ht}_1(R,R^G)} \mathbb{Z} \cdot \mathrm{div}_R(\mathfrak{p})$$

of $\mathrm{Div}(R)$.

Using the residue class of the characteristic divisor, we obtain the main result of this section which can be regarded as a generalization of the Stanley criterion (cf. [S1, S2]) for R_χ to be a free R^G-module of rank one:

Theorem 2.9. *Under the circumstances as in (2.5) and Lemma 2.7, $R_\chi \cong R^G$ as R^G-modules if and only if* $\dim_{\mathcal{Q}(R^G)}(R_\chi \otimes_{R^G} \mathcal{Q}(R^G)) = 1$ *and*

$$(D_\chi + \mathcal{D}_G) \cap \{\mathrm{div}_R(f) \mid f \in R_\chi \backslash \{0\}\} \neq \emptyset.$$

In the case where these equivalent conditions are satisfied,

$$(D_\chi + \mathcal{D}_G) \cap \{\mathrm{div}_R(f) \mid f \in R_\chi \backslash \{0\}\} = \{\mathrm{div}_R(f_\chi)\}$$

for some $f_\chi \in R_\chi$ and $R_\chi = R^G \cdot f_\chi$.

The criterion in the case of algebraic group actions is reduced to Theorem 2.9, by the aid of the following two results.

Proposition 2.10. *Suppose that $\mathcal{Q}(R^G) = \mathcal{Q}(R)^G$. Let H be a normal subgroup of G of a finite index and $\chi \in Z^1(G, \mathrm{U}(R))$. If $R_\chi \cong R^G$ as R^G-modules, then*

$$\mathrm{supp}_R(R_\chi) \cap \mathrm{Ht}_1(R, R^G) \subseteq \left\{ \bigcap_{f \in R_{\chi|_H} \backslash \{0\}} \mathrm{supp}_R(f) \right\}$$
$$\cup \{\mathfrak{p} \in \mathrm{Ht}_1(R, R^G) \mid I_G(\mathfrak{p} \cap R^H)|_{R^H} \neq \{1\}\}.$$

Corollary 2.11. *Suppose that G is an affine algebraic group over an algebraically closed field K. Suppose that R is an affine normal domain over K on which G acts K-rationally as K-algebra automorphisms. Suppose that $\mathcal{Q}(R^G) = \mathcal{Q}(R)^G$ and $R^{R_u(G^0)}$ is noetherian. Then*

$$\left\{ \bigcup_{\chi \in \{\chi \in Z^1(G, \mathrm{U}(R)) \mid R_\chi \cong R^G\}} \mathrm{supp}_R(R_\chi) \right\} \cap \mathrm{Ht}_1(R, R^G)$$

is a finite set.

(2.12) Proof of Theorem 1.2. By Corollary 2.11, we see that the finite set Γ in (1.1) exists. If Δ is a minimal subset of Γ such that there exist $a_\mathfrak{p} \in \mathbb{N}$ ($\mathfrak{p} \in \Delta$) and a unit u of $U^{-1}\mathcal{O}(X)$, satisfying

$$u \cdot \prod_{\mathfrak{p} \in \Delta} \pi_\mathfrak{p}^{a_\mathfrak{p}} \in (U^{-1}\mathcal{O}(X))^{G^0},$$

then, for $\mathfrak{p} \in \Delta$,

$$\{\mathfrak{p}U^{-1}\mathcal{O}(X) \mid \mathfrak{p} \in \Delta\} = X_{\mathfrak{p}U^{-1}\mathcal{O}(X) \cap (U^{-1}\mathcal{O}(X))^{G^0}}(U^{-1}\mathcal{O}(X)) \tag{2.12.1}$$

For any subset Δ of Γ, the equality (2.12.1) holds for a prime ideal $\mathfrak{p} \in \Delta$ if and only if Δ is minimal in the subsets Δ's satisfying

$$\left\{ \sum_{\mathfrak{p} \in \Delta} \mathbb{N} \cdot \delta_{\pi_\mathfrak{p}} \right\} \cap B^1(G^0, \mathrm{U}(U^{-1}R)) \neq \emptyset.$$

Hence, by (2.5), such a subset Δ as in (2.11.1) is identical with $X_{\mathfrak{p} \cap \mathcal{O}(X)^{G^0}}(\mathcal{O}(X))$ for a prime ideal $\mathfrak{p} \in \Delta$. Consequently the assertion of Theorem 1.2 follows from Theorem 2.9. $\square$

3. Applications.

In this section, suppose that R is an affine factorial domain with trivial units and G^0 is an algebraic torus over an algebraically closed field K of characteristic zero. A subset $\{f_1, \ldots, f_n\}$ of n elements in R is said to be (R, G)-*basic*, if each f_i is a relative invariant of G which is prime in R and $R \cdot f_i \neq R \cdot f_j$ for any $1 \leq i < j \leq n$. Moreover, a subset $\{f_1, \ldots, f_n\}$ of n elements is said to be (R, G)-*free*, if it is (R, G)-basic and, for any $a_i \in \mathbb{Z}_0$, there exists a character $\chi \in \mathfrak{X}(G)$ such that

$$R_\chi = R^G \cdot \prod_{i=1}^{n} f_i^{a_i}.$$

We further say that $\{f_1, \ldots, f_n\}$ of n elements is (R, G)-*afforded*, if it is (R, G^0)-basic, the K-subspace $< f_1, \ldots, f_n >_K$ of R is G-invariant and $\{f_1, \ldots, f_n\}$ forms a K-basis of this space. There exists a G-afforded generating system $\{g_1, \ldots, g_n\}$, which is denoted to Ω for a convenience sake, of R of n elements as a K-algebra. We denote by $(K^\times \circ \Omega)^G_{R,\mathrm{st}}$ the set consisting of all g_i's such that $R \cdot g_i \cap R^G \neq \{0\}$ and g_i are relative invariants of G. The subring R_{st} is an affine factorial K-domain on which G acts naturally (cf. Sect.1 and [N3]) and $(K^\times \circ \Omega)^G_{R,\mathrm{st}} \subseteq R_{\mathrm{st}}$. Let $\Omega_{R,G}$ denote the set

$$\{g_i \in (K^\times \circ \Omega)^G_{R,\mathrm{st}} \mid \mathrm{ht}(R \cdot g_i \cap R^G) = 1 \text{ and } \mathfrak{X}_{R \cdot g_i \cap R^G}(R)^G = \mathfrak{X}_{R \cdot g_i \cap R^G}(R)\}.$$

For any pair g_{i_1}, g_{i_2} of elements in $\Omega_{R,G}$, define the relation $g_{i_1} \equiv_G g_{i_2}$, if $R \cdot g_{i_1} \cap R^G = R \cdot g_{i_2} \cap R^G$. Then we denote by $\Omega^{\equiv_G}_{R,G}$ the set consisting of all complete systems of representatives of $\Omega_{R,G}$ modulo $\equiv_G$.

As an application of Theorem 2.9, we have

Theorem 3.1. *Let $\{f_1, \ldots, f_m\}$ be a subset of R_{st} of m elements. Then $\{f_1, \ldots, f_m\}$ is (R, G)-free if and only if there exists a system $\Upsilon \in \Omega^{\equiv_G}_{R,G}$ such that*

$$(K^\times \circ \Omega)^G_{R,\mathrm{st}} \backslash \Upsilon \cap K^\times \circ f_i \neq \emptyset \ \ (1 \leq i \leq m).$$

Let V be the dual space of $< \Omega >_K$ and denote by ρ the canonical K-morphism $\mathcal{O}(V) \to R$ defined by the inclusion $\Omega \subseteq R$. We easily see that $(K^\times \circ \Omega)^G_{R,\mathrm{st}} = (K^\times \circ \Omega)^G_{\mathcal{O}(V),\mathrm{st}}$.

Corollary 3.2. *The restriction of the mapping*

$$2^{\mathcal{O}(V_{\mathrm{st}})} \ni \Gamma \longmapsto \rho(\Gamma) \in 2^{R_{\mathrm{st}}}$$

is a one-to-one correspondence between the set consisting of all finite $(\mathcal{O}(V), G)$-free subsets of $\mathcal{O}(V_{\mathrm{st}})$ and the set consisting of all finite (R, G)-free subsets of R_{st}. Moreover the equality $\Omega^{\equiv_G}_{R,G} = \Omega^{\equiv_G}_{\mathcal{O}(V),G}$ holds.

Proof. Thanks to Theorem 3.1, the proof of the first assertion is reduced to show that, for a subset Γ of Ω, the condition that Γ is $(\mathcal{O}(V), G^0)$-free is equivalent to the one that Γ is (R, G^0)-free. The last equality follows easily from Theorem 3.1 and the first assertion. □

Proposition 3.3. *Suppose that $\{f_1, \dots, f_m\}$ is a maximal (R,G)-free subset of R of m elements. Then there is not a non-empty $(R, \cap_{i=1}^m G_{f_i})$-free subset consisting relative invariants of G. Especially in the case where G is connected, we have* $\mathrm{ht}(R \cdot f \cap R^{\cap_{i=1}^m G_{f_i}}) = 1$ *and* $\sharp(X_{R\cdot f \cap R^{(\cap_{i=1}^m G_{f_i})^0}}(R)) = 1$, *for any prime element $f \in R$ such that $R \cdot f \cap R^{\cap_{i=1}^m G_{f_i}} \neq \{0\}$.*

Proposition 3.4. *For a subspace W of V, the following conditions are equivalent:*

(1) *(W, w) is a paralleled linear hull of (V,G) for a nonzero $w \in V_{\mathrm{st}}$.*
(2) *There exists a system $\Upsilon \in \Omega^{\equiv G}_{\mathcal{O}(V),G}$ and a subset $\mathcal{F}_W$ of $(K^\times \circ \Omega)^G_{\mathcal{O}(V),\mathrm{st}} \backslash \Upsilon$ such that*

$$W = \{x \in V_{\mathrm{st}} \mid F(x) = 0 \ \ (\forall F \in \mathcal{F}_W)\},$$

where Ω is regarded as a subset of $\mathcal{O}(V)$.

Proof. (2) $\Rightarrow$ (1) : By Theorem 3.1, we see that the set $\mathcal{F}_W$ is $(\mathcal{O}(V_{\mathrm{st}}), G)$-free. For each $z \in \mathcal{F}_W$, let $\chi_z \in \mathfrak{X}(G)$ to satisfy $\mathcal{O}(V_{\mathrm{st}})_{\chi_z} \ni z$. Since $(V_{\mathrm{st}}^\vee)_{\chi_z} = K \cdot z$, for each $z \in \mathcal{F}_W$, we can choose a unique vector $z^\vee$ from $(V_{\mathrm{st}})_{-\chi_z}$ in such a way that $z^\vee(z) = 1$. Putting $w = \sum_{z\in\mathcal{F}_W} z^\vee$, we infer that (W, w) is a paralleled linear hull of (V, G).

(1) $\Rightarrow$ (2) : There are weight vectors v_i $(1 \le i \le m)$ of G in V_{st} whose reduction modulo W form a K-basis of V_{st}/W such that $w = \sum_{i=1}^m v_i$. Let $\{v_1^\vee, \dots v_m^\vee\}$ be a basis of $W^\perp = (V_{\mathrm{st}}/W)^\vee$ dual to $\{v_1 \mod W, \dots, v_m \mod W\}$. Then, since

$$\pi_{V_{\mathrm{st}}//G_w, V//G} \circ (\bullet + w)//G_w : W//G_w \xrightarrow{\sim} V_{\mathrm{st}}//G,$$

the set $\{v_1^\vee, \dots v_m^\vee\}$ is $(\mathcal{O}(X), G)$-free. Thus the assertion follows from Theorem 3.1. $\square$

Consequently (W, w) is a minimal paralleled linear hull of (V, G) for some $w \in V$ if and only if

$$W = \{x \in V_{\mathrm{st}} \mid F(x) = 0 \ \ (\forall F \in (K^\times \circ \Omega)^G_{\mathcal{O}(V),\mathrm{st}} \backslash \Upsilon)\},$$

for some system $\Upsilon \in \Omega^{\equiv G}_{\mathcal{O}(V),G}$.

(3.5) Suppose that $g \in R_\chi$ a fixed relative invariant of G such that $\{g\}$ is (R,G)-free and, for a convenience sake, put $H = \mathrm{Ker}\chi$ and $S = R^H$. Furthermore, suppose that the action $(\mathrm{Spec}(R), G)$ is stable and $Z_G(G^0) = G$. As in [M] (e.g, [N2]), we similarly introduce the following subgroups of $\mathrm{Div}(S)$;

$$E^*(G,S) = \bigoplus_{\mathfrak{p}\in \mathrm{Ht}_1(S^G)} \mathbb{Z} \cdot \left(\sum_{\mathfrak{q}\in X_{\mathfrak{p}}(S)} \mathrm{e}(\mathfrak{q},\mathfrak{p}) \cdot \mathrm{div}_S(\mathfrak{q}) \right) \oplus \bigoplus_{\mathfrak{q}\in \mathrm{Ht}_1(S), \mathrm{ht}(\mathfrak{q}\cap S^G)\ge 2} \mathbb{Z} \cdot \mathrm{div}_S(\mathfrak{q}),$$

$$\Delta_{S\cdot g}(G,S) = \bigoplus_{\mathfrak{q}\in \mathrm{Ht}_1(S)\backslash Y_{S\cdot g}(S), \mathfrak{q}\cap S^G \neq (0)} \mathbb{Z} \cdot \mathrm{div}_S(\mathfrak{q}),$$

where $Y_{S\cdot g}(S) = X_{S\cdot g\cap S^G}(S)$, if $\mathrm{ht}(S\cdot g\cap S^G) = 1$, and $Y_{S\cdot g}(S) = \{S\cdot g\}$, otherwise. Let $\Phi^* : E^*(G,S) \to \mathrm{Div}(S^G)$ be a homomorphism defined by

$$\sum_{\mathfrak{p}\in \mathrm{Ht}_1(S^G)} a_{\mathfrak{p}}\cdot \mathcal{D}_{\mathfrak{p}} + \sum_{\mathfrak{q}\in \mathrm{Ht}_1(S),\mathrm{ht}(\mathfrak{q}\cap S^G)\geq 2} b_{\mathfrak{q}}\cdot \mathrm{div}_S(\mathfrak{q}) \longmapsto \sum_{\mathfrak{p}\in \mathrm{Ht}_1(S^G)} a_{\mathfrak{p}}\cdot \mathrm{div}_{S^G}(\mathfrak{p}) \quad (a_{\mathfrak{p}}, b_{\mathfrak{q}} \in \mathbb{Z}),$$

where $\mathcal{D}_{\mathfrak{p}} = \sum_{\mathfrak{q}\in X_{\mathfrak{p}}(S)} \mathrm{e}(\mathfrak{q},\mathfrak{p})\cdot \mathrm{div}_S(\mathfrak{q})$.

Under the same circumstances as in (3.5), we have Lemma 3.6, Proposition 3.6 and Proposition 3.7 as follows:

Lemma 3.6. *The canonical homomorphism* $\mathrm{Div}(S)^G \to \mathrm{Cl}(S)$ *is an epimorphism.*

Proof. For any $\mathfrak{p} \in \mathrm{Ht}_1(S)$, choose a prime f from R in such a way that $R\cdot f\cap R^H = \mathfrak{p}$. Then, expressing $H\cdot Rf = \{R\cdot f_1,\dots,R\cdot f_{\sharp(H\cdot Rf)}\}$, we see

$$a := \prod_{1\leq i\leq \sharp(H\cdot Rf)} f_i$$

is an invariant relative to some $\psi \in \mathfrak{X}(H)$ such that $R\cdot a\cap R^H = R\cdot a\cdot R_{-\psi}$. As $Z_G(G^0) = G$, there is a character $\delta \in \mathfrak{X}(G^0\cdot H)$ such that $\delta|_H = \delta$ and $\delta(\mathrm{Ker}((G^0\cdot H) \to \mathrm{Aut}(R)))$ is trivial. Since δ is regarded as a character of the group $G^0\cdot H|_R$, the module R_δ contains a nonzero element b. So $R^H\cdot \frac{b}{a}\cdot (R\cdot a)^H = (R\cdot b)^H$. By the stability of the action of H on R (cf. [P, N3]), we see $\frac{b}{a} \in Q(R)^H = Q(R^H)$, which implies that

$$\mathrm{div}_{R^H}((R\cdot b)\cap R^H) \equiv \mathrm{div}_{R^H}(\mathfrak{p}) \mod \mathrm{Prin}(R^H).$$

From this, we infer that the homomorphism

$$\mathrm{Div}(R^H)^{G^0\cdot H} \hookrightarrow \mathrm{Div}(R^H) \longrightarrow \mathrm{Cl}(R^H)$$

is surjective. □

The next two propositions are based on Lemma 3.6, in which the assumption that G is a central extension of G^0 is essential.

Proposition 3.7. *Suppose that* $\mathrm{ht}((S\cdot g\cap S^G) \geq 2$. *Then the sequence*

$$0 \to \mathrm{Prin}(S)\cap \Delta_{S\cdot g}(G,S) \to \Delta_{S\cdot g}(G,S) \to \mathrm{Cl}(S) \to 0$$

is exact and

$$\Phi^*(\mathrm{Prin}(S)\cap \Delta_{S\cdot g}(G,S)) = \mathrm{Prin}(S^G).$$

Moreover, Φ^* *induces the isomorphism* $\mathrm{Cl}(S) \cong \mathrm{Cl}(S^G)$.

Proposition 3.8. *Suppose that* $\mathrm{ht}(S \cdot g \cap S^G) = 1$. *Let* $\mathfrak{q}_0 \in \mathrm{Ht}_1(S)$ *satisfy the equality* $X_{S \cdot g \cap S^G}(S) \backslash \{S \cdot g\} = \{\mathfrak{q}_0\}$. *Set*

$$\nabla(G, S) := \Delta_{S \cdot g}(G, S) \oplus \mathbb{Z} \cdot \mathrm{div}_S(\mathfrak{q}_0) \subseteq \mathrm{Div}(S)$$

and let $\Theta : E^*(G, S) \to \nabla(G, S)$ *be the composite*

$$E^*(G, S) \hookrightarrow \mathrm{Div}(S) \xrightarrow{\mathrm{pr}} \nabla(G, S).$$

Then the square

$$\begin{array}{ccc} \mathrm{Cl}(S^G) & \xrightarrow{\sim} & \mathrm{Cl}(S) \\ {\scriptstyle can.}\uparrow & & \uparrow{\scriptstyle can.} \\ \mathrm{Div}(S^G) & \xrightarrow[\Theta \circ \Psi_{S^G, S}]{\sim} & \nabla(G, S) \end{array}$$

is a commutative diagram with horizontal isomorphisms.

Applying Proposition 3.7 and Proposition 3.8 inductively to a maximal (R, G)-free set, by the aid of Theorem 3.1, we now establish

Theorem 3.9. *Suppose that* $Z_G(G^0) = G$. *Let* (W, w) *be a minimal paralleled linear hull of* (V, G) *through* $< \Omega >_K$. *Then*

(1) *The inclusions* $R_{\mathrm{st}}^{G_w} \hookrightarrow R_{\mathrm{st}}$ *and* $\mathcal{O}(W)^{G_w} \hookrightarrow \mathcal{O}(W)$ *are no-blowing-up of codimension one and, for each* $\mathfrak{p} \in \mathrm{Ht}_1(R_{\mathrm{st}}^{G_w})$ *and* $\mathfrak{q} \in \mathrm{Ht}_1(\mathcal{O}(W)^{G_w})$, *the actions of* G_w *on* $X_{\mathfrak{p}}(R_{\mathrm{st}})$ *and* $X_{\mathfrak{q}}(\mathcal{O}(W))$ *are, respectively, transitive.*

(2) *We have the monomorphism*

$$\begin{aligned} \mathrm{Cl}(R^G) \cong \mathrm{Cl}(R_{\mathrm{st}}^{G_w}) &\cong \mathfrak{X}(G_w / \mathcal{R}_{R_{\mathrm{st}}}(G_w)) \\ &\xrightarrow{\varphi} \mathfrak{X}(G_w / \mathcal{R}_W(G_w)) \cong \mathrm{Cl}(\mathcal{O}(V)^G) \end{aligned}$$

with its cokernel, isomorphic to a subgroup of the abelialization of the finite group $\mathcal{R}_{R_{\mathrm{st}}}(G_w) / \mathcal{R}_W(G_w)|_{R_{\mathrm{st}}}$.

Proof of Theorem 1.4. The first and second assertions follow from Theorem 3.9. Suppose that $\mathrm{Cl}(R^G)$ is finite. Then, the assertion (2) of Theorem 3.9 implies that G_w is also finite. Hence the third assertion follows from [C, R1] and, in the case where G_w is solvable, applying Proposition 3.6 of [N1] to the G_w-module W, we see that $\mathcal{O}(V)^G$ is a polynomial ring over K if and only if it is factorial. Thus the last assertion follows from this observation and Theorem 3.9. □

As in [D] and [N1], it shall be expected that Theorem 2.9 and Theorem 3.9 imply the detailed description of $\mathrm{Cl}(\mathcal{O}(X)^G)$ in terms of group theory. Finally, we point out that these theorems seem somewhat useful in studying invariants of non-semisimple reductive algebraic groups which are treated in [P] for semisimple groups.

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NAKAYAMA ISOMORPHISMS FOR THE MAXIMAL QUOTIENT RING OF A LEFT HARADA RING

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ABSTRACT. From several results of Kado and Oshiro, we see that if the maximal quotient ring of a given left Harada ring R of type $(*)$ has a Nakayama automorphism, then R has a Nakayama isomorphism. This result poses a question whether if the maximal quotient ring of a given left Harada ring R has a Nakayama isomorphism, then R has a Nakayama isomorphism. In this paper, we shall show that a basic ring of the maximal quotient ring of a given Harada ring has a Nakayama isomorphism if and only if its Harada ring has a Nakayama isomorphism.

INTRODUCTION

Let R be a basic left Harada ring. Then we have a complete set

$$\{e_{11}, \ldots, e_{1n(1)}, \ldots, e_{m1}, \ldots, e_{mn(m)}\}$$

of primitive idempotents for R such that for each $i = 1, \ldots, m$

(a) $e_{i1}R$ is injective as a right R-module;
(b) $J(e_{i,k-1}R) \cong e_{ik}R$ for each $k = 2, \ldots, n(i)$.

We call R a ring of type $(*)$ if there exists an unique g_i in $\{e_{in(i)}\}_{i=1}^{m}$ for each $i = 1, \ldots, m$ such that the socle of $e_{i1}R$ is isomorphic to $g_iR/J(g_iR)$ and the socle of Rg_i is isomorphic to $Re_{i1}/J(Re_{i1})$.

Oshiro [10] showed the following;

Result A ([10, Theorem 2]). *Suppose that R is a left Harada ring which is not of type $(*)$. Then there exists a series of left Harada rings and surjective ring homomorphisms:*

$$T_1 \overset{\phi_1}{\twoheadrightarrow} T_2 \overset{\phi_2}{\twoheadrightarrow} \cdots \overset{\phi_{n-1}}{\twoheadrightarrow} T_n \overset{\phi_n}{\twoheadrightarrow} R$$

such that

(1) *T_1 is of type $(*)$, and*
(2) *$\mathrm{Ker}\,\phi_i$ is a simple ideal of T_i for any $i \in \{1, \ldots, n\}$.*

Kado and Oshiro [7] showed the following results;

Result B ([7, Proposition 5.3]). *If every basic QF rings has a Nakayama automorphism, then every basic left Harada ring of type $(*)$ has a Nakayama isomorphism.*

The detailed version of this paper will be submitted for publication elsewhere.

Result C ([7, Proposition 5.4]). *Let S be a two-sided ideal of R that is simple as a left ideal and as a right ideal. If R has a Nakayama isomorphism, then R/S has a Nakayama isomorphism.*

Moreover Kado showed the following;

Result D ([6, Corollary]). *The maximal quotient ring of a left Harada ring of type $(*)$ is a QF ring.*

Using these four results, we see that if the maximal quotient ring of a given left Harada ring R of type $(*)$ has a Nakayama automorphism, then R has a Nakayama isomorphism. So this result poses a question whether if the maximal quotient ring of a given left Harada ring R has a Nakayama isomorphism, then R has a Nakayama isomorphism. In this paper, we shall show that the maximal quotient ring of a given left Harada ring R has a Nakayama isomorphism iff R has a Nakayama isomorphism.

Throughout this paper, we assume that all rings are associative rings with identity and all modules are unitary. By M_R (resp. ${}_RM$), we means that M is a right (resp. left) R-module, respectively. We denote the set of primitive idempotents of R by $\mathrm{Pi}(R)$, and denote a complete set of primitive idempotents of R by $\mathrm{pi}(R)$.

We call a one-sided artinian ring R right (resp. left) QF-3 ring if $E(R_R)$ (resp. $E({}_RR)$) is projective, respectively.

We denote the maximal left (resp. right) quotient ring of R by $Q_\ell(R)$ (resp. $Q_r(R)$), respectively, and denote the maximal left and maximal right quotient ring of R by $Q(R)$. If a ring is QF-3, its maximal left quotient ring and its right quotient ring coincide by [16, Theorem 1.4].

1. Maximal quotient ring

We list some basic results, which several authors showed, for our main result in this paper. Recall that for $e, f \in \mathrm{Pi}(R)$, we say that the pair $(eR : Rf)$ is an i-pair if $\mathrm{S}(eR) \cong fR/J(fR)$ and $\mathrm{S}(Rf) \cong Re/J(Re)$.

Lemma 1 ([5]). *Let R be a one-sided artinian ring, and let $e \in \mathrm{Pi}(R)$. Then the following conditions are equivalent:*

(1) *eR is injective.*

(2) *There exists some $f \in \mathrm{Pi}(R)$ such that $(eR : Rf)$ is an i-pair.*

In this case, Rf is also injective.

Let R be a left perfect ring. Then R has a primitive idempotent e with $S(R_R)e \neq 0$. If R is QF-3, then the primitive idempotent e with $S(R_R)e \neq 0$ are characterized as follows;

Lemma 2 ([4, Theorem 2.1]). *Let R be a one-sided artinian* QF-3 *ring, and let $e \in \mathrm{Pi}(R)$. Then ${}_RRe$ is injective if and only if $\mathrm{S}(R_R)e \neq 0$.*

We call $e \in \mathrm{Pi}(R)$ right (resp. left) S-primitive if $\mathrm{S}(R_R)e \neq 0$ (resp. $e\,\mathrm{S}({}_RR) \neq 0$), respectively.

The following statement, which Storrer [15, Proposition 4.8] showed, is helpful in this paper.

Lemma 3 ([15, Proposition 4.8]). *Let R and $Q = Q(R)$ be left perfect. Then*

(1) *If e is a right S-primitive idempotent for R, then so is it for Q.*

(2) *if* e_1, e_2 *are right* S*-primitive idempotents for* R*, then* $e_1R \cong e_2R$ *if and only if* $e_1Q \cong e_2Q$.

(3) *If* e *is a right* S*-primitive idempotent for* Q*, then there exists a right* S*-primitive idempotent* $e' \in R$ *such that* $eQ \cong e'Q$.

A ring R is called a left Harada ring if it is left artinian and its complete set $\mathrm{pi}(R)$ of orthogonal primitive idempotents is arranged as follows:

$$\mathrm{pi}(R) = \bigcup_{i=1}^{m} \{e_{ij}\}_{j=1}^{n(i)},$$

where

(a) each $e_{i1}R_R$ is an injective module for each $i = 1, 2, \dots, m$.

(b) $e_{i,k-1}R_R \cong e_{ik}R$, or $J(e_{i,k-1}R_R) \cong e_{ik}R$ for each i and each $k = 2, 3, \dots, n(i)$.

(c) $e_{ik}R \not\cong e_{jt}R$ for $i \neq j$.

Remark 1. Let R be a left Harada ring. Then $Q(R)$ is also a left Harada ring (See [6, Theorem 4]) and a complete set $\mathrm{pi}(Q)$ of orthogonal primitive idempotents for Q coincides with $\mathrm{pi}(R)$ (See [6, p.248]).

Using Remark 1, Kado showed the following;

Proposition 4 ([6, Proposition 2]). *Let* R *be a left Harada ring, and let* $(eR : Rf)$ *be an i-pair for* $e, f \in \mathrm{pi}(R)$. *Then* $(eQ(R) : Q(R)f)$ *is an i-pair* .

Recall the following notation [6, p.249]. Let $\theta : fR \to eR$ be an R-monomorphism such that $\mathrm{Im}\,\theta = J(eR)$, where $e, f \in \mathrm{Pi}(R)$. Then by [15, Proposition 4.3], θ can be uniquely extended to a $Q_r(R)$-homomorphism $\theta^* : fQ_r(R) \to eQ_r(R)$.

We shall need the following results.

Lemma 5 ([6, Proposition 3]). *Let* R *be a basic and left Harada ring, and* $Q = Q(R)$ *and* θ *as above. Then the following hold.*

(1) *If* e *is not right* S*-primitive, then the extension* $\theta^* : fQ \to eQ$ *is an isomorphism.*

(2) *If* e *is right* S*-primitive, then the extension* $\theta^* : fQ \to eQ$ *is a monomorphism such that* $\mathrm{Im}\,\theta^* = J(eQ)$.

Remark 2 (cf. [15, Lemma 4.2]). Let $\{g_i\} \cup \{f_j\}$ be a complete set of orthogonal primitive idempotents for R, where the g_i are right S-primitive and the f_j are not right S-primitive. We denote g_0 by $g_0 = \sum g_i$. Then $Q(R)g_0 = Rg_0$ and $Q(R)g = Rg$ for every right S-primitive idempotent g of R.

Let R be a basic left artinian ring, and let $\{e_1, e_2, \dots , e_n\}$ be a complete set of orthogonal primitive idempotents for R and let

$$S = \mathrm{End}_R(\oplus_{i=1}^{n} E(Re_i/J(Re_i)))$$

be the endomorphism ring of a minimal injective cogenerator for R-mod. Let f_i be the primitive idempotent for S corresponding to the projection

$$\oplus_{i=1}^{n} E(Re_i/J(Re_i)) \to E(Re_i/J(Re_i)).$$

Then we call a ring isomorphism $\tau : R \to S$ a Nakayama isomorphism if $\tau(e_i) = f_i$ for each $i = 1, 2, \dots, n$. By [3, p.42], the existence of a Nakayama isomorphism does not depend on the choice of the complete set $\{e_1, e_2, \dots, e_n\}$ of orthogonal primitive idempotents. (See [7, Remark on p.387].)

It is important whether the maximal quotient ring of a basic artinian ring is basic since a Nakayama isomorphism is defined on a basic ring. Here we shall study the case that the maximal quotient ring of a given left Harada ring is basic.

Theorem 6 (cf. [2, Corollary 22]). *Let R be a basic and left Harada ring and $Q = Q(R)$. Then Q is a basic ring if and only if R either is* QF *or satisfies the following; $n(i) = 1$ or 2 and Re_{i1} is injective for any i. In this case $R = Q$.*

Proof. Note that both R and Q are artinian QF-3. Let $\mathrm{pi}(R) = \bigcup_{i=1}^{m}\{e_{ij}\}_{j=1}^{n(i)}$ be a complete set of orthogonal primitive idempotent for R satisfying the following conditions:

(a) $e_{i1}R_R$ is injective for each $i = 1, 2, \dots, m$,
(b) $e_{i,j+1}R_R \cong J(e_iR_R)$ for $j = 1, 2, \dots, n(i) - 1$.

We have a complete set $\{Rg_1, \dots, Rg_m\}$ of pairwise non-isomorphic indecomposable injective projective left R-modules, such that the $(e_{i1}R : Rg_i)$ are i-pair for each $i = 1, \dots, m$ since R is basic and artinian QF-3.

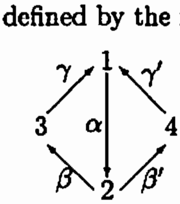
Assume that Q is basic. Let $e_{i,k+1}, e_{ik} \in \{e_{ij}\}_{j=2}^{n(i)}$. Then we have an R-monomorphism $\theta_{ik} : e_{i,k+1}R \to e_{ik}R$ such that $\mathrm{Im}\,\theta = J(e_{ik}R)$. If e_{ik} is not right S-primitive, then $e_{ik+1}Q \cong e_{ik}Q$ by Lemma 5. This contradicts that Q is basic. Hence e_{ik} is right S-primitive for $k = 1, 2, \dots, n(i) - 1$. Since the Re_{ik} are injective for each $k = 1, 2, \dots, n(i) - 1$ by Lemma 2, there exists some Rg in $\{Rg_1, \dots, Rg_m\}$ such that $Re_{ik} \cong Rg$. However R is basic, so we see that $n(i) \le 2$ and e_{i1} is right S-primitive.

In case $n(i) = 1$ for every $i = 1, \dots, m$, then R is QF.

In case $n(i) = 2$ for some $i \in \{1, \dots, m\}$. If $e_{in(i)}$ is right S-primitive, then ${}_RRe_{in(i)}$ is injective by Lemma 2. Hence $e_{in(i)}$ is not right S-primitive since ${}_RRe_{i1}$ is injective and so $\{Rg_1, \dots, Rg_m\} = \{Re_{11}, \dots, Re_{m1}\}$.

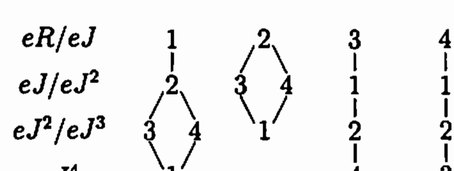
Conversely, first, assume that R is QF. Since ${}_RRe$ is injective for any $e \in \mathrm{pi}(R)$, e is right S-primitive by Lemma 2. Thus, $eQ \not\cong fQ$ for any $e, f \in \mathrm{pi}(R) = \mathrm{pi}(Q)$ by Lemma 3. Therefore Q is basic. Next, assume that R satisfies $n(i) = 1$ or 2 and Re_{i1} is injective for any i. Then e_{i1} is left S-primitive and so $eQ = eR$ by Remark 2. Hence $J(eQ) = J(eR)$. Therefore it is also clear to see that $R = Q$. □

Example 1. We shall give a basic left Harada ring R with $J(R)^5 = 0$, which is not QF. Let R be an algebra over a field K defined by the following quiver;



with the relations $\gamma\beta = \gamma'\beta'$, $\alpha\gamma\beta = 0$, and $\beta'\alpha\gamma = 0$.

The composition diagrams of the Loewy factors of the indecomposable projective modules of R_R is the following.



Then R is a left Harada ring which is not QF since e_1R_R, e_3R_R and e_4R_R are injective and $e_2R_R \cong J(e_1R)$. Moreover e_1, e_3, e_4 are right S-primitive. Hence $e_1Q(R) = e_1R$, $e_3Q(R) = e_3R$ and $e_4Q(R) = e_4R$ are injective and $e_2Q(R) \cong J(e_1Q(R))$. Therefore $R = Q(R)$.

Example 2. We shall give a basic Harada ring R with $J(R)^6 = 0$, but $Q(R)$ is not basic. Let R be an algebra over a field K defined by the following quiver;

$$\begin{tikzcd}
& 1 \arrow[dd, "\alpha"] & \\
3 \arrow[ur, "\gamma"] & & 4 \arrow[ul, "\gamma'"'] \\
& 2 \arrow[ul, "\beta"] \arrow[ur, "\beta'"'] &
\end{tikzcd}$$

with the relations $0 = \beta\alpha\gamma\beta = \beta'\alpha\gamma'\beta' = \beta\alpha\gamma = \beta'\alpha\gamma'$, and $\gamma\beta = \gamma'\beta'$. Then the composition diagrams of the Loewy factors of the indecomposable projective modules of R_R is the following.

$$\begin{tikzcd}[row sep=small, column sep=tiny]
e_iR/e_iJ & & 1 \arrow[d, dash] & & & 2 \arrow[dl, dash] \arrow[dr, dash] & & 3 \arrow[d, dash] & 4 \arrow[d, dash] \\
e_iJ/e_iJ^2 & & 2 \arrow[dl, dash] \arrow[dr, dash] & & 3 \arrow[dr, dash] & & 4 \arrow[dl, dash] & 1 \arrow[d, dash] & 1 \arrow[d, dash] \\
e_iJ^2/e_iJ^3 & 3 \arrow[dr, dash] & & 4 \arrow[dl, dash] & & 1 \arrow[d, dash] & & 2 \arrow[d, dash] & 2 \arrow[d, dash] \\
e_iJ^4/e_iJ^5 & & 1 \arrow[d, dash] & & & 2 & & 4 & 3 \\
e_iJ^5 & & 2 & & & & & &
\end{tikzcd}$$

Then since e_1R_R, e_3R_R and e_4R_R are injective and $e_2R_R \cong J(e_1R)$, R is a left Harada ring which is not QF. Hence $e_2Q(R) \cong e_1Q(R)$ since e_1 is not right S-primitive. Therefore $Q(R)$ is not basic.

2. Nakayama isomorphism

In this section, we study the Nakayama isomorphisms for the representative matrix ring of a basic left Harada ring and its maximal quotient ring. Let R be a basic left Harada ring, and let $\mathrm{pi}(R) = \bigcup_{i=1}^{m} \{e_{ij}\}_{j=1}^{n(i)}$ be a complete set of orthogonal primitive idempotents as in Theorem 6. Furthermore, let $R^\bullet$ be the representative matrix ring of R. $R^\bullet$ is represented as block matrices as follows:

$$R^\bullet = \begin{pmatrix} R^\bullet_{11} & \cdots & R^\bullet_{1m} \\ & \cdots & \\ R^\bullet_{m1} & \cdots & R^\bullet_{mm} \end{pmatrix},$$

where $R^\bullet_{ij} = P_{ij}$ for $j \neq \sigma(i)$ and $R^\bullet_{i\sigma(i)} = P^\bullet_{i\sigma(i)}$ (See [7, Section 4]).

Here, adding one row and one column to $R^\bullet$, we make an extended matrix ring $W_i(R)$ of R as follows:

$$\begin{pmatrix}
R^\bullet_{11} & \cdots & \cdots & R^\bullet_{1i} & Y_1 & R^\bullet_{1,i+1} & \cdots & R^\bullet_{1m} \\
\vdots & & & \vdots & \vdots & \vdots & & \vdots \\
R^\bullet_{i1} & \cdots & \cdots & R^\bullet_{ii} & Y_i & R^\bullet_{i,i+1} & \cdots & R^\bullet_{im} \\
X_1 & \cdots & X_{i-1} & X_i & Q & X_{i+1} & \cdots & X_m \\
R^\bullet_{i+1,1} & \cdots & \cdots & R^\bullet_{i+1,i} & Y_{i+1} & R^\bullet_{i+1,i+1} & \cdots & R^\bullet_{i+1,m} \\
\vdots & & & \vdots & \vdots & \vdots & & \vdots \\
R^\bullet_{m1} & \cdots & \cdots & R^\bullet_{mi} & Y_m & R^\bullet_{m,i+1} & \cdots & R^\bullet_{mm}
\end{pmatrix},$$

where X_k is the last row of R^*_{ik} ($k = 1,\dots,m,\ k \neq i$), Y_k is the last column of R^*_{ki} ($k = 1,\dots,m$), $X_i = (P^*_{in(i),i1} \cdots P^*_{in(i),in(i)-1} J(P^*_{in(i),in(i)}))$, and $Q = P^*_{in(i),in(i)}$.

Then $W_i(R)$ naturally becomes a ring by operations of R^*. We call this the *i-th extended ring* of R.

Proposition 7 ([7, Proposition 5.11]). *If $W_i(R)$ has a Nakayama isomorphism, then R also has a Nakayama isomorphism.*

Let R be a basic and left Harada ring, and let

$$\mathrm{pi}(R) = \bigcup_{i=1}^{m} \{e_{ij}\}_{j=1}^{n(i)}$$

be a complete set of orthogonal primitive idempotents of R satisfying the following;

(1) $e_{i1}R_R$ is injective for each $i = 1, 2, \dots, m$.

(2) $e_{ij}R \cong J(e_{i,j-1}R)$ for each $j = 2, \dots, n(i)$.

Then (See [7, p.388]), for any e_{ij} in $\mathrm{pi}(R)$, there exists some g_i in $\mathrm{pi}(R)$ with Rg_i injective such that $E(Re_{ij}/J(Re_{ij})) \cong Rg_i/S_{j-1}(Rg_i)$, where $S_j(Rg_i)$ is the j-th socle of Rg_i. We denote the generator $g_i + S_{j-1}(Rg_i)$ of $Rg_i/S_{j-1}(Rg_i)$ by g_{ij} for each $i = 1, 2, \dots, m, j = 1, 2, \dots, n(i)$. Then by [7, Proposition 3.2], a minimal injective cogenerator $G = \oplus_{i,j} Rg_{ij}$ is finitely generated. Therefore we note that R is left Morita dual to $\mathrm{End}_R(G)$ by [1, Theorem 30.4]. We call this $\mathrm{End}({}_RG)$ the *dual ring* of R. We denote the dual ring of R by $T(R)$.

For the proof of proposition 8 below, we denote

$$\begin{pmatrix} 0 & & & \cdots & & & 0 \\ 0 & \cdots & 0 & R^*_{ij} & 0 & \cdots & 0 \\ 0 & & & \cdots & & & 0 \end{pmatrix} \subseteq R^*$$

by $[R^*_{ij}]$ and

$$\begin{pmatrix} 0 & & & \cdots & & & 0 \\ 0 & \cdots & 0 & R^*_{ij} & 0 & \cdots & 0 \\ 0 & & & \cdots & & & 0 \end{pmatrix} \subseteq W_i(R)$$

by $[R^*_{ij}]^w$.

By using the result that Kado and Oshiro [7, Proposition 5,11] showed, we shall show the following proposition. The proposition is essential in this paper.

Proposition 8. *$W_i(R)$ has a Nakayama isomorphism if and only if so does R.*

Proof. ($\Rightarrow$). By Proposition 7 ([7, Proposition 5,11]). ($\Leftarrow$). As [7, Proposition 5.11], let e_{ij} be the matrix of R^* such that the (ij, ij)-component is the unity and other components are zero, and let w_{ij} be the matrix of $W_i(R)$ such that the (ij, ij)-component is the unity and other components are zero. Note that the size of the columns in $W_i(R)$ is $n(i) + 1$. Let Ψ be the natural embedding homomorphism;

$$\begin{pmatrix} R^*_{11} & \cdots & R^*_{1m} \\ & \cdots & \\ R^*_{m1} & \cdots & R^*_{mm} \end{pmatrix}$$

$$\downarrow \Psi$$

$$
\begin{array}{c}
\begin{pmatrix}
R^*_{11} & \cdots & \cdots & R^*_{1i} & 0 & R^*_{1,i+1} & \cdots & R^*_{1m} \\
\vdots & & & \vdots & \vdots & \vdots & & \vdots \\
R^*_{i1} & \cdots & \cdots & R^*_{ii} & 0 & R^*_{i,i+1} & \cdots & R^*_{im} \\
0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
R^*_{i+1,1} & \cdots & \cdots & R^*_{i+1,i} & 0 & R^*_{i+1,i+1} & \cdots & R^*_{i+1,m} \\
\vdots & & & \vdots & \vdots & \vdots & & \vdots \\
R^*_{m1} & \cdots & \cdots & R^*_{mi} & 0 & R^*_{m,i+1} & \cdots & R^*_{mm}
\end{pmatrix}, \\
\overbrace{i+1}
\end{array}
$$

where $R^*_{ij} \to R^*_{ij}$ are identity maps for all i, j. Moreover let h_{ij} be the matrix of $T(R)$ such that the (ij, ij)-component is the unity and other components are zero, and let v_{ij} be the matrix of $W_i(T(R))$ such that the (ij, ij)-component is the unity and other components are zero. Note that the size of the columns in $W_i(T(R))$ is $n(i)+1$. Let

$$
\begin{pmatrix}
T(R)_{11} & \cdots & T(R)_{1m} \\
 & \cdots & \\
T(R)_{m1} & \cdots & T(R)_{mm}
\end{pmatrix}
$$

be the representative matrix ring $T(R)^*$ of $T(R)$, and let $T(W_i(R))$ be the dual ring of $W_i(R)$ as follows;

$$
\begin{pmatrix}
T(R)_{11} & \cdots & T(R)_{1i} & {}^tY_1 & T(R)_{1,i+1} & \cdots & T(R)_{1m} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
T(R)_{i1} & \cdots & T(R)_{ii} & {}^tY_i & T(R)_{i,i+1} & \cdots & T(R)_{im} \\
{}^tX_1 & \cdots & {}^tX_i & {}^tQ & {}^tX_{i+1} & \cdots & {}^tX_m \\
T(R)_{i+1,1} & \cdots & T(R)_{i+1,i} & {}^tY_{i+1} & T(R)_{i+1,i+1} & \cdots & T(R)_{i+1,m} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
T(R)_{m1} & \cdots & T(R)_{mi} & {}^tY_m & T(R)_{m,i+1} & \cdots & T(R)_{mm}
\end{pmatrix}.
$$

Letting $\Psi_{T(R)}$ be the natural embedding homomorphism;

$$
\begin{pmatrix}
T(R)_{11} & \cdots & T(R)_{1m} \\
 & \cdots & \\
T(R)_{m1} & \cdots & T(R)_{mm}
\end{pmatrix}
$$

$$\downarrow \Psi_{T(R)}$$

$$
\begin{array}{c}
\begin{pmatrix}
T(R)_{11} & \cdots & \cdots & T(R)_{1i} & 0 & T(R)_{1,i+1} & \cdots & T(R)_{1m} \\
\vdots & & & \vdots & \vdots & \vdots & & \vdots \\
T(R)_{i1} & \cdots & \cdots & T(R)_{ii} & 0 & T(R)_{i,i+1} & \cdots & T(R)_{im} \\
0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
T(R)_{i+1,1} & \cdots & \cdots & T(R)_{i+1,i} & 0 & T(R)_{i+1,i+1} & \cdots & T(R)_{i+1,m} \\
\vdots & & & \vdots & \vdots & \vdots & & \vdots \\
T(R)_{m1} & \cdots & \cdots & T(R)_{mi} & 0 & T(R)_{m,i+1} & \cdots & T(R)_{mm}
\end{pmatrix}, \\
\overbrace{i+1}
\end{array}
$$

where $T(R)_{ij} \to T(R)_{ij}$ are identity maps for all i, j. We note that $T(W_i(R)) = W_i(T(R))$ (See [7, Proposition 5.11]).

Assume that $\varphi : R^* \to T(R)^*$ is a Nakayama isomorphism with $\varphi(e_{ij}) = h_{ij}$. (i.e., $\varphi([r_{kl}]) \in [T(R)_{kl}]$ for any $[r_{kl}] \in [R^*_{ij}]$, where (k,l)-componentwise of R^*_{ij} corresponds to (k,l)-componentwise of $T(R)_{ij}$.) We consider the following diagram;

$$\begin{array}{ccc} W_i(R) & & W_i(T(R)) \\ {\scriptstyle\Psi}\uparrow & & \uparrow{\scriptstyle\Psi_{T(R)}} \\ R^* & \xrightarrow{\varphi} & T(R). \end{array}$$

Here we define a map $\bar{\varphi} : W_i(R) \to W_i(T(R))$ as follows;

(a) $\bar{\varphi}([r_{kl}]^w) = [\varphi([r_{kl}])]^w \in [T(R)_{kl}]^w$
for any $[r_{kl}]^w \in [R^*_{kl}]^w; 1 \le k \le m, 1 \le l \le m$;
(b) $\bar{\varphi}([x]^w) \in [{}^tX_k]^w$ for any $[x]^w \in [X_k]; k = 1, \dots, m$;
(c) $\bar{\varphi}([y]^w) \in [{}^tY_l]^w$ for any $[y]^w \in [Y_l]^w; l = 1, \dots, m$;
(d) $\bar{\varphi}([q]^w) \in [{}^tQ]^w$ for any $[q]^w \in [Q]^w$.

Since $\varphi(e_{ij}) = h_{ij}$, $\bar{\varphi}$ is well-defined. Moreover it is satisfied $\bar{\varphi}(w_{i,n(i)+1}) = f_{i,n(i)+1}$. Then we can easily check that $\bar{\varphi}$ is a Nakayama isomorphism. $\square$

Remark 3. We shall define a special case of an extended ring for a given ring R. Let $\{e_1, e_2, \dots, e_n\}$ be a complete set of orthogonal primitive idempotents for R. Then for primitive idempotent e_i in R, we define R_{e_i} as follows;

$$\begin{pmatrix} e_1Re_1 & \cdots & e_1Re_i & Y_1 & e_1Re_{i+1} & \cdots & e_1Re_n \\ \vdots & \ddots & \vdots & \vdots & \vdots & & \\ e_iRe_1 & \cdots & e_iRe_i & Y_i & e_iRe_{i+1} & \cdots & e_iRe_n \\ X_1 & \cdots & X_i & U & X_{i+1} & \cdots & X_n \\ e_{i+1}Re_1 & \cdots & e_{i+1}Re_i & Y_{i+1} & e_{i+1}Re_{i+1} & \cdots & e_{i+1}Re_n \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \\ e_nRe_1 & \cdots & e_nRe_i & Y_n & e_nRe_{i+1} & \cdots & e_nRe_n \end{pmatrix},$$

where the X_j are e_iRe_j for $j = 1, \dots, i-1, i+1, \dots, n$, X_i is $J(e_iRe_i)$, the Y_k are e_kRe_i for $k = 1, \dots, n$ and U is e_iRe_i. Then R_{e_i} is a ring by usual matrix operations.

Remark 4. Proposition 8 says that a basic left Harada ring R has a Nakayama isomorphism if and only if so does R_e for $e \in \mathrm{pi}(R) = \bigcup_{i=1}^m \{e_{ij}\}_{j=1}^{n(i)}$.

We denote a basic ring of $Q(R)$ by $\mathrm{Q}^b(R)$.

Remark 5. If R is a one-sided artinian QF-3 ring, the number of right S-primitive idempotents for R coincides with that of left S-primitive idempotents for R.

Theorem 9. *Let R be a basic and left Harada ring and let $Q = Q(R)$. Then Q has a Nakayama isomorphism if and only if so does R.*

Proof. If Q is basic, then $R = Q$ by Theorem 6. Hence we may assume that Q is not basic. Let $\mathrm{pi}(R) = \bigcup_{i=1}^m \{e_{ij}\}_{j=1}^{n(i)}$ be a complete set of primitive idempotents for R as given in the proof of Theorem 6. Then if $\{e_{ij}\}_{j=1}^{n(i)}$ has no right S-primitive idempotents, then $e_{i1}Q \cong e_{ij}Q$ for $j = 2, \dots, n(i)$ by Lemma 5. If $\{e_{ij}\}_{j=1}^{n(i)}$ has only one right S-primitive idempotent, say e_{ik}, then

$$\begin{cases} e_{i1}Q \cong e_{ij}Q & \text{for } j = 2, \dots, k; \\ e_{i,k+1}Q \cong J(e_{ik}Q) & \text{and} \\ e_{i,k+1}Q \cong e_{ij}Q & \text{for } j = k+2, \dots, n(i). \end{cases}$$

Moreover, if $\{e_{ij}\}_{j=1}^{n(i)}$ has two right S-primitive idempotents, say, e_{ik}, e_{it} $(k < t)$, then

$$\begin{cases} e_{i1}Q \cong e_{ij}Q & \text{for } j = 2, \dots, k; \\ e_{i,k+1}Q \cong J(e_{ik}Q) & \text{and} \\ e_{i,k+1}Q \cong e_{ij}Q & \text{for } j = k+2, \dots, t; \\ e_{i,t+1}Q \cong J(e_{it}Q) & \text{and} \\ e_{i,t+1}Q \cong e_{ij}Q & \text{for } j = t+2, \dots, n(i). \end{cases}$$

Repeating the same argument and Remark 5, we have the following sequences for $i = 1, \dots, m$;

$$\begin{array}{ccccccc} e_{i1}Q & > & e_{i1}J(Q) & & & & \\ & & \wr\uparrow & & & & \\ & & e_{i,k_1+1}Q & > & J(e_{i,k_1+1}Q) & & \\ & & & & \wr\uparrow & & \\ & & & & e_{i,k_2+1}Q & & \cdots, \end{array}$$

where e_{ik_t} is right S-primitive. Hence the complete set of the primitive idempotents $\mathrm{pi}(Q^b)$ for Q^b is $\bigcup_{i=1}^m \{e_{i1}, e_{i,k_t+1}\}_{t\geq 1} \subseteq \mathrm{pi}(R) = \mathrm{pi}(Q)$ and $e_{i1}\, Q^b$ is injective. Since e_{i1} is left S-primitive, $e_{i1}R = e_{i1}Q$ by Remark 2 and so $e_{i1}Re_{i1} = e_{i1}Qe_{i1}$. Hence we have a ring isomorphism from Q^b to a subring of R.

(i) We choose $\{e_{h1}\}_{h=1}^{n(h)} \subset \mathrm{pi}(R)$ with $e_{hn(h)}$ right S-primitive. We put $e_h = e_{h1} + \cdots + e_{hn(h)}$. Then by Lemma 3 and Lemma 5, we $e_hR = e_hQ$. (ii) We choose $\{e_{h1}\}_{h=1}^{n(h)} \subset \mathrm{pi}(R)$ without right S-primitive. By Remark 3, $Q^b_{e_{h1}}$ is isomorphism to a ring with the complete set $\bigcup_{i\neq h}\{e_{i1}, e_{i,k_t+1}\}_{t\geq 1} \cup \{e_{h1}, e_{h2}\}$ of primitive idempotents. Similarly repeating $n(h) - 2$ times, we can make an extended ring with the complete set $\cup_{i\neq h}\{e_{i1}, e_{i,k_t+1}\}_{t\geq 1} \cup \{e_{hj}\}_{j=1}^{n(h)}$ of primitive idempotents.

Letting,

$$
Q^b = \begin{pmatrix}
* & & e_{11}Re_{h1} & & * & \\
 & & \vdots & & & \\
e_{h1}Re_{11} & \cdots & e_{h1}Re_{h1} & \cdots & e_{h1}Re_{m1} & \cdots \\
 & & \vdots & & & \\
* & & e_{m1}Re_{h1} & & * & \\
 & & \vdots & & &
\end{pmatrix},
$$

$$
Q^b_{e_{h1}} = \begin{pmatrix}
* & & e_{11}Re_{h1} & e_{11}Re_{h1} & & * & \\
 & & \vdots & \vdots & & & \\
e_{h1}Re_{11} & \cdots & e_{h1}Re_{h1} & e_{h1}Re_{h1} & \cdots & e_{h1}Re_{m1} & \cdots \\
e_{h1}Re_{11} & \cdots & J(e_{h1}Re_{h1}) & e_{h1}Re_{h1} & \cdots & e_{h1}Re_{m1} & \cdots \\
 & & \vdots & \vdots & & & \\
* & & e_{m1}Re_{h1} & e_{m1}Re_{h1} & & * & \\
 & & \vdots & \vdots & & &
\end{pmatrix}.
$$

For two submodules

$$
A = \begin{array}{c}
\\ h1 > \\ \\
\end{array}
\begin{pmatrix}
0 & & \cdots & \cdots & & & 0 \\
e_{h1}Re_{11} & \cdots & e_{h1}Re_{h1} & e_{h1}Re_{h1} & \cdots & e_{h1}Re_{m1} & \cdots \\
0 & & \cdots & \cdots & & & 0
\end{pmatrix},
$$

$$
B = \begin{array}{c}
\\ h1 > \\ \\ \\
\end{array}
\begin{pmatrix}
0 & & \cdots & \cdots & & & 0 \\
0 & & \cdots & \cdots & & & 0 \\
e_{h1}Re_{11} & \cdots & J(e_{h1}Re_{h1}) & e_{h1}Re_{h1} & \cdots & e_{h1}Re_{m1} & \cdots \\
0 & & \cdots & \cdots & & & 0
\end{pmatrix}
$$

of $Q^b_{e_{h1}}$, $J(A) \cong B$ by [13, Theorem 1].

Hence as a ring isomorphism,

$$\begin{pmatrix} * & & e_{11}Re_{h1} & e_{11}Re_{h2} & & * & \\ & & \vdots & \vdots & & & \\ e_{h1}Re_{11} & \dots & e_{h1}Re_{h1} & e_{h1}Re_{h2} & \dots & e_{h1}Re_{m1} & \dots \\ e_{h2}Re_{11} & \dots & e_{h2}Re_{h1} & e_{h2}Re_{h2} & \dots & e_{h2}Re_{m1} & \dots \\ & & \vdots & \vdots & & & \\ * & & e_{m1}Re_{h1} & e_{m1}Re_{h2} & & * & \\ & & \vdots & \vdots & & & \end{pmatrix}$$

$$\cong \begin{pmatrix} * & & e_{11}Re_{h1} & e_{11}Re_{h1} & & * & \\ & & \vdots & \vdots & & & \\ e_{h1}Re_{11} & \dots & e_{h1}Re_{h1} & e_{h1}Re_{h1} & \dots & e_{h1}Re_{m1} & \dots \\ e_{h1}Re_{11} & \dots & J(e_{h1}Re_{h1}) & e_{h1}Re_{h1} & \dots & e_{h1}Re_{m1} & \dots \\ & & \vdots & \vdots & & & \\ * & & e_{m1}Re_{h1} & e_{m1}Re_{h1} & & * & \\ & & \vdots & \vdots & & & \end{pmatrix}$$

by [13, Theorem 1] again.

(iii) We choose $\{e_{h1}\}_{h=1}^{n(h)} \subset \mathrm{pi}(R)$ with some right S-primitive idempotents. Then we denote a right S-primitive idempotent of $\{e_{h1}\}_{h=1}^{n(h)}$ by e_{hk_t}. We reset

$$\{e_{h1}\}_{h=1}^{n(h)} = \{e_{h1}, \dots, e_{hk_1}, \dots, e_{hk_2}, \dots\}.$$

Then the complete set $\mathrm{pi}(Q^b)$ of Q^b is $\bigcup_{i=1}^{m}\{e_{i1}, e_{i,k_t+1}\}_{t\geq 1}$. First by the same argument above for e_{i1}, e_{i,k_1+1}, we have a ring isomorphic to a ring with the complete set $\{e_{i1}, \dots, e_{i,k_1+1}\} \subset \mathrm{pi}(R)$. Next, by [13, Theorem 1], repeating the same argument like (ii), for e_{i,k_1+1}, e_{i,k_2+1}, we have a ring isomorphism to a ring with the complete set $\{e_{i1}, \dots, e_{ik_1}, e_{ik_1+1}, \dots, e_{ik_2}, e_{i,k_2+1}\}$. Hence the suitable extended ring of Q^b is isomorphic to R. Therefore, by Proposition 8, Q^b has a Nakayama isomorphism if and only if so does R. □

3. Another question

Oshiro's result(Result A) in the introduction also poses another question whether there exist surjective ring homomorphisms:

$$\begin{array}{ccccccccc} Q(T_1) & \overset{\bar{\phi}_1}{\twoheadrightarrow} & Q(T_2) & \overset{\bar{\phi}_2}{\twoheadrightarrow} & \dots & \overset{\bar{\phi}_{n-1}}{\twoheadrightarrow} & Q(T_n) & \overset{\bar{\phi}_n}{\twoheadrightarrow} & Q(R) \\ \vee & & \vee & & & & \vee & & \vee \\ T_1 & \overset{\phi_1}{\twoheadrightarrow} & T_2 & \overset{\phi_2}{\twoheadrightarrow} & \dots & \overset{\phi_{n-1}}{\twoheadrightarrow} & T_n & \overset{\phi_n}{\twoheadrightarrow} & R. \end{array}$$

However K. Koike informed the author the following examples;

Example 3. Let Q be a local serial ring, and $J(Q) \neq 0, J(Q)^2 = 0$. Then $J(Q) = S(Q)$. We put

$$R = \begin{pmatrix} Q & Q \\ J & Q \end{pmatrix} / \begin{pmatrix} 0 & J \\ 0 & J \end{pmatrix},$$

where $J = J(Q)$. Then R is a serial ring of an admissible sequence (3,2) and so we see that $R = Q(R)$. Also

$$T_1 = \begin{pmatrix} Q & Q \\ J & Q \end{pmatrix}, \quad T_2 = \begin{pmatrix} Q & Q \\ J & Q \end{pmatrix} / \begin{pmatrix} 0 & J \\ 0 & 0 \end{pmatrix},$$

$$Q(T_1) = \begin{pmatrix} Q & Q \\ Q & Q \end{pmatrix}, \qquad Q(T_2) = T_2.$$

$\begin{pmatrix} J & J \\ J & J \end{pmatrix}$ is a unique non-trivial ideal of $Q(T_1)$. Hence there does not exist a surjective ring homomorphism $Q(T_1)$ to $Q(T_2)$.

Example 4. We put

$$T = \begin{pmatrix} \mathbf{K} & \mathbf{K} & \mathbf{K} \\ 0 & \mathbf{K} & \mathbf{K} \\ 0 & 0 & \mathbf{K} \end{pmatrix}, I = \begin{pmatrix} 0 & 0 & \mathbf{K} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $\mathbf{K}$ is a field, and $R = T/I$. Then R is a serial ring of an admissible sequence (2,2,1) and we have a natural map

$$T = T_1 \to R.$$

However the maximal quotient ring $Q(T)$ of T is the full matrix algebra with degree 3 over a field $\mathbf{K}$ and $Q(R) = R$. Since $Q(T)$ is semisimple, there does not exist a surjective ring homomorphism $Q(T)$ to $Q(R)$.

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THE HOLONOMIC RANK FORMULA FOR A-HYPERGEOMETRIC SYSTEM [1]

Go Okuyama

1. Introduction

Given a finite set A of d-dimensional integral vectors which belong to one hyperplane off the origin in $\mathbb{Q}A$ and a parameter vector $\beta \in \mathbb{C}^d$, Gel'fand, Kapranov and Zelevinskii [5] defined a system of differential equations, called an A-hypergeometric system $M_A(\beta)$. They proved that the holonomic rank of $M_A(\beta)$ equals the normalized volume of the convex hull of A and the origin (denote by vol(A)) for any β when the semigroup ring $\mathbb{C}[\mathbb{N}A]$ determined by A is Cohen-Macaulay. In general, the rank is not less than the volume (see [1], [13], Theorem 3.5.1). Meanwhile Adolphson [1] showed that even when $\mathbb{C}[\mathbb{N}A]$ is not Cohen-Macaulay, the holonomic rank equals vol(A), as long as β is generic in a certain sense. After Strumfels and Takayama showed that the holonomic rank can actually be greater than vol(A) for non-generic parametes β, Cattani, D'Andrea and Dickenstein showed that if the covex hull of A is a segment, then there exists a rank-jumping parameter whenever $\mathbb{C}[\mathbb{N}A]$ is not a Cohen-Macaulay ring. Saito, who generalized this result by using different methods, showed that there exist rank-jumping parameters for any non-Cohen-Macaulay simplicial semigroup ring $\mathbb{C}[\mathbb{N}A]$. Matusevich [6] showed that, if the toric ideal defined by A is generic in a certain sense and non-Cohen-Macaulay, then there exists a rank-jumping parameter. However, when we fix a parameter β, it is not well-known how the holonomic rank is described explicitly except when the covex hull of A is simplicial (see [10], Theorem 6.3). In this paper, using combinatorial notion, we provide a rank formula in the case where the rank of A is three.

1.1. **Definition of A-hypergeometric system.** Let $A = (a_1, \ldots, a_n) = (a_{ij})$ be a $d \times n$-matrix of rank d with coefficients in $\mathbb{Z}$. Let k be a field of characteristic zero and $\mathbb{N}$ the set of nonnegative integers. We denote the set $\{a_1, \ldots, a_n\}$ by A as well. Let $\mathcal{F}_A$ denote the face lattice of the cone

$$\mathbb{Q}_{\geq 0}A := \left\{ \sum_{j=1}^{n} c_j a_j \mid c_j \in \mathbb{Q}_{\geq 0} \right\}.$$

Let $\mathbb{N}A$ denote the semigroup generated by A and by $k[\mathbb{N}A]$ its semigroup ring contained in the Laurant polynomial ring $k[t_1^{\pm}, \ldots, t_d^{\pm}]$. For a face σ in $\mathcal{F}_A$, we denote by $\mathbb{N}(A \cap \sigma)$ the semigroup generated by $A \cap \sigma$, and by $\mathbb{Z}(A \cap \sigma)$ the group generated by

[1] The detailed version of this paper has been submitted for publication elsewhere.

$A \cap \sigma$. When $A \cap \sigma = \emptyset$, we agree that $\mathbb{N}(A \cap \sigma) = \mathbb{Z}(A \cap \sigma) = 0$. We consider the $\boldsymbol{k}$-algebra homomorphism $\phi_A : \boldsymbol{k}[\partial_1, \ldots, \partial_n] \to \boldsymbol{k}[\mathbb{N}A]$ defined by

$$\phi_A\left(\sum_{u \in \mathbb{N}^n} c_u \partial^u\right) := \sum_{u \in \mathbb{N}^n} c_u t^{Au},$$

where $c_u \in \boldsymbol{k}$, $\partial^u := \partial_1^{u_1} \cdots \partial_n^{u_n}$, and $t^{Au} := t_1^{(Au)_1} \cdots t_d^{(Au)_d}$. We denote by $I_A(\partial)$ the kernel of ϕ_A and call it the toric ideal of A. Since ϕ_A is an epimorphism, we have

$$\boldsymbol{k}[\partial]/I_A(\partial) \overset{\bar{\phi}_A}{\cong} \boldsymbol{k}[\mathbb{N}A].$$

Given a column parameter vector $\beta = {}^t(\beta_1, \ldots, \beta_d) \in \boldsymbol{k}^d$, let $H_A(\beta)$ denote the left ideal of the n-th Weyl algebra

$$D = \boldsymbol{k}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle$$

generated by $I_A(\partial)$ and $\sum_{j=1}^n a_{ij}\theta_j - \beta_i$ $(i = 1, \ldots, d)$, where $\theta_j := x_j\partial_j$. We call the quotient D-module $M_A(\beta) := D/H_A(\beta)$ the A-hypergeometric system with parameter β. This system was introduced in the late eighties by Gel'fand, Graev, and Zelevinski (see [4]); its systematic study was started by Gel'fand, Zelevinski, and Kapranov (see, e.g. [5]).

1.2. **Known results on the holonomic rank of $M_A(\beta)$.** In this note, we define the holonomik rank of the A-hypergeometric system $\operatorname{rank}(M_A(\beta))$ as follows:

$$\operatorname{rank}(M_A(\beta)) := \dim_{\boldsymbol{k}(x)}(\boldsymbol{k}(x) \otimes_{\boldsymbol{k}[x]} M_A(\beta)).$$

Here $\boldsymbol{k}(x) = \boldsymbol{k}(x_1, \ldots, x_n)$ is the field of rational functions. One of the results shown in [5] about the holonomic rank of A-hypergeometric system is that $\operatorname{rank}(M_A(\beta)) = \operatorname{vol}(A)$ for any $\beta \in \boldsymbol{k}^d$ when the semigroup ring $\boldsymbol{k}[\mathbb{N}A]$ is a *Cohen-Macaulay* ring. Here $\operatorname{vol}(A)$ means the normalized volume of the convex hull in $\mathbb{Q}^d$ of A and the origin. This equality can fail if $\boldsymbol{k}[\mathbb{N}A]$ is not a Cohen-Macaulay ring. However, even if we drop the assumption that $\boldsymbol{k}[\mathbb{N}A]$ is a Cohen-Macaulay ring, we have

$$\operatorname{rank}(M_A(\beta)) \geq \operatorname{vol}(A)$$

for any $\beta \in \boldsymbol{k}^d$, and the equality holds for *generic* β. So we write $j_A(\beta)$ for the gap between the holonomic rank and the volume in this talk.

Moreover, in fact, Matusevich, Miller and Walther [7] completely showed that the rank of $M_A(\beta)$ is indepent of β, that is, $j_A(\beta) = 0$ for any β if and only if $\mathbb{C}[\mathbb{N}A]$ is Cohen-Macaulay. However, given a parameter β, we do not know the formula of the rank of $M_A(\beta)$ except when the convex hull of A is *simplicial.*

2. Main Result

2.1. **Combinatorial term $F_A(\beta)$.** As in the previous section, in order to compute the gap $j_A(\beta)$, we introduce a combinatorial term as follows. First, for $\lambda \in \mathbb{Z}A$ and β, we define the subset $\mathcal{J}(\lambda; \beta)$ of $\mathcal{F}_A$ by

$$\mathcal{J}(\lambda; \beta) := \{\sigma \in \mathcal{F}_A |\ \lambda \notin \mathbb{N}A + \mathbb{Z}(A \cap \sigma), \beta - \lambda \in \boldsymbol{k}(A \cap \sigma)\}.$$

Second, we define a preorder on $\mathbb{Z}A$ as follows:

$$\lambda < \mu \underset{\text{def}}{\iff} \text{for any } \sigma \in \mathcal{J}(\lambda; \beta), \lambda + \mathbb{Z}(A \cap \sigma) = \mu + \mathbb{Z}(A \cap \sigma).$$

Then we have the following proposition on this set.

Proposition 2.1. (1) *The set $\mathcal{J}(\lambda;\beta)$ does not contain $\mathbb{Q}_{\geq 0}A$.*
(2) *If $\lambda \in \mathbb{N}A$, then we have $\mathcal{J}(\lambda;\beta) = \emptyset$.*
(3) *If $\lambda < \mu$, then we have $\mathcal{J}(\lambda;\beta) \subset \mathcal{J}(\mu;\beta)$.*

Now, we consider the subset of $\mathbb{Z}A \setminus \mathbb{N}A$:

$$E_A(\beta) = \{\lambda \in \mathbb{Z}A \setminus \mathbb{N}A \mid \mathcal{J}(\lambda;\beta) \neq \varnothing\}.$$

We denote by $F_A(\beta)$ the inductive limit of the set $(E_A(\beta), <)$ which we regard as an inductive system. In other words, $F_A(\beta)$ coincides with the set of maximal elements in $((\mathbb{Z}A \setminus \mathbb{N}A)/\sim, <)$, where $\sim$ means the equivalence relation defined by

$$\lambda \sim \mu \underset{\text{def}}{\iff} \lambda < \mu \text{ and } \lambda > \mu.$$

Since $\lambda \in \beta + \bigcup_{\tau \in \mathcal{F}_A} k(A \cap \tau)$ for any $\lambda \in F_A(\beta)$ and $[\mathbb{Z}A \cap \mathbb{Q}\tau : \mathbb{Z}(A \cap \tau)] < \infty$ for any face τ, we see that $F_A(\beta)$ is a finite set.

2.2. **Main result.** Let $d = 3$ to the end of this note. First assume that the cardinality of $F_A(\beta)$ is one. Let $F_A(\beta) = \{\lambda\}$ and $\tilde{\mathcal{J}}(\lambda;\beta)$ denote the set of maximal elements in $\mathcal{J}(\lambda;\beta)$. Then the sets $\tilde{\mathcal{J}}(\lambda;\beta)$ can be classified into four cases:

(1): $\tilde{\mathcal{J}}(\lambda;\beta) = \emptyset$,
(2): $\tilde{\mathcal{J}}(\lambda;\beta)$ consists of one proper face σ,
(3): $\tilde{\mathcal{J}}(\lambda;\beta)$ consists of all facets,
(4): none of the above.

For each case, we have the following theorem:

Theorem 2.2. *Let $d = 3$. Assume that the cardinality of $F_A(\beta)$ is one. Then we have the following.*

(1) *$\tilde{\mathcal{J}}(\lambda;\beta)$ satisfies the case (1) $\Rightarrow j_A(\beta) = 0$,*
(2) *$\tilde{\mathcal{J}}(\lambda;\beta)$ satisfies the case (2) $\Rightarrow$*

$$j_A(\beta) = \begin{cases} 0 & \textit{if } \sigma \textit{ is a facet,} \\ \mathrm{vol}(A \cap \sigma) & \textit{if } \sigma \textit{ is an edge,} \\ 2 & \textit{if } \sigma = \{0\}, \end{cases}$$

(3) *$\tilde{\mathcal{J}}(\lambda;\beta)$ satisfies the case (3) $\Rightarrow j_A(\beta) = 0$,*
(4) *$\tilde{\mathcal{J}}(\lambda;\beta)$ satisfies the case (4) $\Rightarrow j_A(\beta) = \sum_{\sigma \in \tilde{\mathcal{J}}(\lambda;\beta):edges}(\mathrm{vol}(A \cap \sigma)) + m - 1$. Here m means the number of connected components of the finite graph $G_\lambda = \{\sigma \in \mathcal{F}_A \mid \{0\} \neq \sigma \subset \tau$ for some $\tau \in \tilde{\mathcal{J}}(\lambda;\beta)\}$ with respect to the inclusion relation.*

Second not assume that the cardinality of $F_A(\beta)$ is one. In this case, it suffices that for each $\lambda \in F_A(\beta)$ we compute the number determind by the previous theorem, that is, we compute the right hand side of the equality in the theorem, regarding $F_A(\beta)$ as the singleton set $\{\lambda\}$. For each $\lambda \in F_A(\beta)$, let l_λ denote the number in this meaning. Then we have the rank formula as desired:

Theorem 2.3. *Let $d = 3$. Then we have $j_A(\beta) = \sum_{\lambda \in F_A(\beta)} l_\lambda$.*

3. Examples

Example 1. Let $A_1 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 5 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$. Then we have $\mathrm{vol}(A_1) = 7$.

First we consider the case where $\beta = {}^t(1,2,0)$. Then we have $F_{A_1}(\beta) = \{\beta\}$ and $\tilde{\mathcal{J}}(\beta;\beta) = \{\mathbb{Q}_{\geq 0}a_1, \mathbb{Q}_{\geq 0}a_4\}$. Hence we have $j_{A_1}(\beta) = 1$.

Second we consider the case where $\beta = {}^t(2/5,1,0)$. Then we have $F_{A_1}(\beta) = \{{}^t(1,1,0), {}^t(1,4,0)\}$ and $\tilde{\mathcal{J}}({}^t(1,1,0);\beta) = \{\mathbb{Q}_{\geq 0}a_1\}$ and $\tilde{\mathcal{J}}({}^t(1,4,0);\beta) = \{\mathbb{Q}_{\geq 0}a_4\}$. Hence $F_{A_1}(\beta)$ is semisimple. Since $\mathbb{Q}_{\geq 0}a_1$ and $\mathbb{Q}_{\geq 0}a_4$ are both edges, we have $j_{A_1}(\beta) = 1 + 1 = 2$.

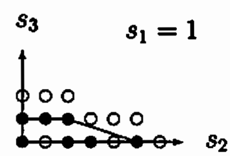

FIGURE 1. The set A_1

Example 2. Let $A_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 & 2 \end{pmatrix}$. Then we have $\mathrm{vol}(A_2) = 6$.

Let $\beta = {}^t(1,1,1)$. Then we have $F_{A_2}(\beta) = \{\beta\}$ and $\tilde{\mathcal{J}}(\beta;\beta) = \{\{0\}\}$. Hence we have $j_{A_2}(\beta) = 2$.

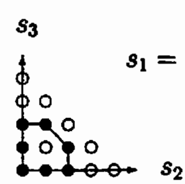

FIGURE 2. The set A_2

Example 3. Let $A_3 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$. Then we have $\mathrm{vol}(A_3) = 6$.

Let $\beta = {}^t(0,1,0)$. Then we have $F_{A_3}(\beta) = \{\beta\}$ and $\tilde{\mathcal{J}}(\beta;\beta) = \{\mathbb{Q}_{\geq 0}a_1 + \mathbb{Q}_{\geq 0}a_4, \mathbb{Q}_{\geq 0}a_3 + \mathbb{Q}_{\geq 0}a_6\}$. Hence we have $j_{A_3}(\beta) = 1$.

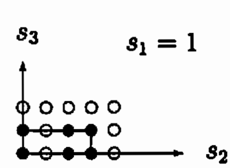

FIGURE 3. The set A_3

Example 4. Let $A_4 = \begin{pmatrix} 3 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$; not homogeneous. Then we have $\mathrm{vol}(A_4) = 12$.

First we consider the case where $\beta = {}^t(1,1,0)$. Then we have $F_{A_4}(\beta) = \{\beta\}$ and $\tilde{\mathcal{J}}(\beta;\beta) = \{\mathbb{Q}_{\geqq 0}a_1\}$. Hence we have $j_{A_4}(\beta) = \mathrm{vol}(A_4 \cap \mathbb{Q}_{\geqq 0}a_1) = 3$.

Second we consider the case where $\beta = {}^t(2,2,0)$. Then $F_{A_4}(\beta) = \{\beta\}$ and $\tilde{\mathcal{J}}(\beta;\beta) = \{\{0\}\}$. Hence we have $j_{A_4}(\beta) = 2$.

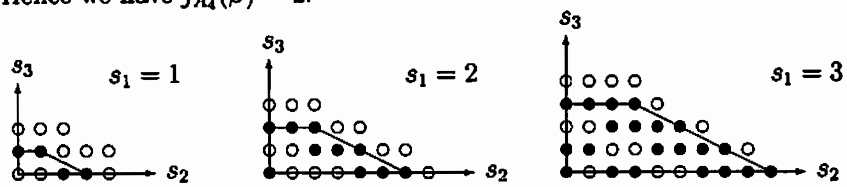

FIGURE 4. The set A_4

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ON THE DECOMPOSABILITY OF A SYZYGY OF THE RESIDUE FIELD

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1. Introduction

Throughout the present paper, we assume that all rings are commutative noetherian local rings and all modules are finitely generated modules.

Dutta [10] proved the following theorem in his research into the homological conjectures:

Theorem 1.1 (Dutta). *Let $(R, \mathfrak{m}, k)$ be a local ring. Suppose that the nth syzygy module of k has a non-zero direct summand of finite projective dimension for some $n \geq 0$. Then R is regular.*

Since G-dimension is similar to projective dimension, this theorem naturally leads us to the following question:

Question 1.2. Let $(R, \mathfrak{m}, k)$ be a local ring. Suppose that the nth syzygy module of k has a non-zero direct summand of finite G-dimension for some $n \geq 0$. Then is R Gorenstein?

It is obviously seen from the indecomposability of k that this question is true if $n = 0$. Hence this question is worth considering just in the case where $n \geq 1$.

We are able to answer in this paper that the above question is true if $n \leq 2$. Furthermore, we can even determine the structure of a ring satisfying the assumption of the above question for $n = 1, 2$.

2. Main results

For a local ring R, we denote by $\operatorname{mod} R$ the category of finitely generated R-modules. First of all, we recall the definition of G-dimension.

Definition 2.1. (1) We denote by $\mathcal{G}(R)$ the full subcategory of $\operatorname{mod} R$ consisting of all R-modules M satisfying the following three conditions:

(i) M is reflexive,

(ii) $\operatorname{Ext}^i_R(M, R) = 0$ for every $i > 0$,

(iii) $\operatorname{Ext}^i_R(M^*, R) = 0$ for every $i > 0$.

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(2) Let M be an R-module. If n is a non-negative integer such that there is an exact sequence

$$0 \to G_n \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$$

of R-modules with $G_i \in \mathcal{G}(R)$ for every i, then we say that M has *G-dimension at most* n, and write $\text{G-dim}_R M \leq n$. If such an integer n does not exist, then we say that M has *infinite G-dimension*, and write $\text{G-dim}_R M = \infty$.

For properties of G-dimension, we refer to [3] or [9].

Proposition 2.2. *Let $(R, \mathfrak{m}, k)$ be a local ring. Suppose that there is a direct sum decomposition $\mathfrak{m} = I \oplus J$ where I, J are non-zero ideals of R. Let M be a non-free indecomposable module in $\mathcal{G}(R)$. Then there exist elements $x, y \in \mathfrak{m}$ such that*

(1) *$I = (x)$ and $J = (y)$,*
(2) *$(0 : x) = (y)$ and $(0 : y) = (x)$,*
(3) *M is isomorphic to either (x) or (y).*

Hence the minimal free resolution of k is as follows:

$$\cdots \xrightarrow{\left(\begin{smallmatrix} y & 0 \\ 0 & x \end{smallmatrix}\right)} R^2 \xrightarrow{\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)} R^2 \xrightarrow{\left(\begin{smallmatrix} y & 0 \\ 0 & x \end{smallmatrix}\right)} R^2 \xrightarrow{(x\ y)} R \longrightarrow k \longrightarrow 0.$$

PROOF. The modules M^* and ΩM are also non-free indecomposable modules in $\mathcal{G}(R)$. There are isomorphisms

$$\begin{aligned} M^* &\cong \operatorname{Hom}_R(M, \mathfrak{m}) \\ &= \operatorname{Hom}_R(M, I \oplus J) \\ &\cong \operatorname{Hom}_R(M, I) \oplus \operatorname{Hom}_R(M, J). \end{aligned}$$

The indecomposability of M^* implies that either $\operatorname{Hom}_R(M, I) = 0$ or $\operatorname{Hom}_R(M, J) = 0$. We may assume that

$$\operatorname{Hom}_R(M, J) = 0. \tag{2.2.1}$$

There is an exact sequence

$$0 \to \Omega M \to R^n \to M \to 0. \tag{2.2.2}$$

Dualizing this by J, we obtain another exact sequence

$$\operatorname{Hom}_R(M, J) \to J^n \to \operatorname{Hom}_R(\Omega M, J).$$

We have $\operatorname{Hom}_R(\Omega M, J) \neq 0$ by (2.2.1). Applying the above argument to the module ΩM yields

$$\operatorname{Hom}_R(\Omega M, I) = 0. \tag{2.2.3}$$

Also, dualizing (2.2.2) by I, we get an exact sequence

$$0 \to \operatorname{Hom}_R(M, I) \to I^n \to \operatorname{Hom}_R(\Omega M, I),$$

and hence $M^* \cong \operatorname{Hom}_R(M, I) \cong I^n$. The indecomposability of M^* implies that $n = 1$ (i.e. M is cyclic), and $M^* \cong I$.

We also have

$$\begin{aligned} M &\cong M^{**} \\ &\cong \operatorname{Hom}_R(M^*, \mathfrak{m}) \\ &\cong \operatorname{Hom}_R(M^*, I) \oplus \operatorname{Hom}_R(M^*, J). \end{aligned}$$

Note that $\mathrm{Hom}_R(M^*, I)$ is isomorphic to $\mathrm{Hom}_R(I, I)$, which contains the identity map of I. Hence $\mathrm{Hom}_R(M^*, I) \neq 0$ and therefore

$$\mathrm{Hom}_R(M^*, J) = 0.$$

Applying the above argument to the module M^*, we see that M^* is also cyclic and $M \cong M^{**} \cong I$. Thus, we have shown that $M \cong M^* \cong I$ and these modules are cyclic. Noting (2.2.3) and applying the above argument to the module ΩM, we see that $\Omega M \cong (\Omega M)^* \cong J$ and these modules are cyclic.

Now, writing $I = (x)$ and $J = (y)$, we can prove $(x) = (0 : y)$ and $(0 : x) = (y)$. Thus we obtain the minimal free resolutions of (x) and (y):

$$\begin{cases} \cdots \xrightarrow{y} R \xrightarrow{x} R \xrightarrow{y} R \to (x) \to 0, \\ \cdots \xrightarrow{x} R \xrightarrow{y} R \xrightarrow{x} R \to (y) \to 0. \end{cases}$$

Taking the direct sum of these exact sequence, we get

$$\cdots \xrightarrow{\left(\begin{smallmatrix} y & 0 \\ 0 & x \end{smallmatrix}\right)} R^2 \xrightarrow{\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)} R^2 \xrightarrow{\left(\begin{smallmatrix} y & 0 \\ 0 & x \end{smallmatrix}\right)} R^2 \longrightarrow \mathfrak{m} \longrightarrow 0.$$

Joining this to the natural exact sequence $0 \to \mathfrak{m} \to R \to k \to 0$ constructs the minimal free resolution of k in the assertion. $\square$

We denote by $\mathrm{edim}\, R$ the embedding dimension of a local ring R. When a homomorphic image of a regular local ring is given, we can choose a minimal presentation of the ring in the following sense:

Proposition 2.3. *Let R be a homomorphic image of a regular local ring. Then there exist a regular local ring $(S, \mathfrak{n})$ and an ideal I of S contained in $\mathfrak{n}^2$ such that $R \cong S/I$.*

Here we introduce a famous result due to Tate [17, Theorem 6]. See also [5, Remarks 8.1.1(3)].

Lemma 2.4 (Tate). *Let $(S, \mathfrak{n}, k)$ be a regular local ring, I an ideal of S contained in $\mathfrak{n}^2$, and $R = S/I$ a residue class ring. Suppose that the complexity of k over R is at most one. (In other words, the set of all the Betti numbers of the R-module k is bounded.) Then I is a principal ideal.*

We denote by $\beta_i^R(M)$ the ith Betti number of a module M over a local ring R. Handling the above results, we can determine the structure of a local ring with decomposable maximal ideal having a non-free module of G-dimension zero, as follows:

Theorem 2.5. *Let $(S, \mathfrak{n}, k)$ be a regular local ring, I an ideal of S contained in $\mathfrak{n}^2$, and $R = S/I$ a residue class ring. Suppose that there exists a non-free R-module in $\mathcal{G}(R)$. Then the following conditions are equivalent:*

(1) *The maximal ideal of R is decomposable;*
(2) *$\dim S = 2$ and $I = (xy)$ for some regular system of parameter x, y of S.*

PROOF. Let $\mathfrak{m} = \mathfrak{n}/I$ be the maximal ideal of R.

(2) $\Rightarrow$ (1): It is easy to see that $\mathfrak{m} = xR \oplus yR$ and that xR, yR are non-zero.

(1) $\Rightarrow$ (2): First of all, note from the condition (1) that R is not an integral domain, hence is not a regular local ring.

Proposition 2.2 says that $\mathfrak{m} = xR \oplus yR$ for some $x, y \in \mathfrak{n}$, and that $\beta_i^R(k) = 2$ for every $i \geq 2$. It follows from Lemma 2.4 that I is a principal ideal. Hence R is a hypersurface.

We write $I = (f)$ for some $f \in \mathfrak{n}^2$. Since $\mathfrak{m}$ is decomposable, the local ring R is not artinian. (Over an artinian Gorenstein local ring, the intersection of non-zero ideals is also non-zero; cf. [8, Exercise 3.2.15].) Hence we have $0 < \dim R < \operatorname{edim} R = 2$, which says that $\dim R = 1$ and $\dim S = 2$.

Note that $\mathfrak{n} = (x, y, f)$. Because $\operatorname{edim} S = \dim S = 2$, one of the elements x, y, f belongs to the ideal generated by the other two elements. Noting that the images of elements x, y in $\mathfrak{m}$ form a minimal system of generators of $\mathfrak{m}$, we see that $f \in (x, y)$, and hence x, y is a regular system of parameters of S.

On the other hand, noting $xR \cap yR = 0$, we get $xy \in I = (f)$. Write $xy = cf$ for some $c \in S$. Since the associated graded ring $\operatorname{gr}_{\mathfrak{n}}(S)$ is a polynomial ring over k in two variables $\overline{x}, \overline{y} \in \mathfrak{n}/\mathfrak{n}^2$, we especially have $\overline{xy} \neq 0$ in $\mathfrak{n}^2/\mathfrak{n}^3$, namely, $xy \notin \mathfrak{n}^3$. It follows that $c \notin \mathfrak{n}$ because $f \in \mathfrak{n}^2$. Therefore the element c is a unit of S, and thus $I = (xy)$. $\square$

Using Theorem 2.5 and Cohen's structure theorem, we obtain the following corollary.

Corollary 2.6. *Let $(R, \mathfrak{m})$ be a complete local ring. The following conditions are equivalent:*

(1) *There is a non-free module in $\mathcal{G}(R)$, and $\mathfrak{m}$ is decomposable;*
(2) *R is Gorenstein, and $\mathfrak{m}$ is decomposable;*
(3) *There are a complete regular local ring S of dimension two and a regular system of parameters x, y of S such that $R \cong S/(xy)$.*

The finiteness of G-dimension is independent of completion. Thus, Corollary 2.6 not only gives birth to a generalization of [15, Proposition 2.3] but also guarantees that Question 1.2 is true if $n = 1$.

As far as here, we have observed a local ring whose maximal ideal is decomposable. From here to the end of this paper, we will observe a local ring such that the second syzygy module of the residue class field is decomposable. We begin with the following theorem, which implies that Question 1.2 is true if $n = 2$.

Theorem 2.7. *Let $(R, \mathfrak{m}, k)$ be a local ring. Suppose that $\mathfrak{m}$ is indecomposable and that $\Omega_R^2 k$ has a non-zero proper direct summand of finite G-dimension. Then R is a Gorenstein ring of dimension two.*

PROOF. Replacing R with its $\mathfrak{m}$-adic completion, we may assume that R is a complete local ring. In particular, note that R is Henselian.

We have $\Omega_R^2 k = M \oplus N$ for some non-zero R-modules M and N with $\operatorname{G-dim}_R M < \infty$. There is an exact sequence

$$0 \longrightarrow M \oplus N \xrightarrow{(f,g)} R^e \longrightarrow \mathfrak{m} \longrightarrow 0$$

of R-modules, where $e = \operatorname{edim} R$. Setting $A = \operatorname{Coker} f$ and $B = \operatorname{Coker} g$, we get exact sequences

$$\text{(2.7.1)} \qquad \begin{cases} 0 \to M \xrightarrow{f} R^e \xrightarrow{\alpha} A \to 0, \\ 0 \to N \xrightarrow{g} R^e \xrightarrow{\beta} B \to 0. \end{cases}$$

It is easily observed that there are exact sequences

$$\text{(2.7.2)} \qquad 0 \longrightarrow R^e \xrightarrow{\binom{\alpha}{\beta}} A \oplus B \longrightarrow \mathfrak{m} \longrightarrow 0$$

and

$$(2.7.3) \qquad \begin{cases} 0 \longrightarrow M \xrightarrow{\beta f} B \longrightarrow \mathfrak{m} \longrightarrow 0, \\ 0 \longrightarrow N \xrightarrow{\alpha g} A \longrightarrow \mathfrak{m} \longrightarrow 0. \end{cases}$$

Using (2.7.1), (2.7.2) and (2.7.3), we can prove that $\mathrm{Ext}_R^2(k, R) \neq 0$. (Hence depth $R \leq 2$.)

Fix a non-free indecomposable module $X \in \mathcal{G}(R)$. Applying the functor $\mathrm{Hom}_R(X, -)$ to (2.7.2) gives an exact sequence

$$0 \to (X^*)^e \to \mathrm{Hom}_R(X, A) \oplus \mathrm{Hom}_R(X, B) \to \mathrm{Hom}_R(X, \mathfrak{m}) \to 0$$

and an isomorphism

$$(2.7.4) \qquad \mathrm{Ext}_R^1(X, A) \oplus \mathrm{Ext}_R^1(X, B) \cong \mathrm{Ext}_R^1(X, \mathfrak{m}).$$

We have $(X^*)^e \in \mathcal{G}(R)$ and $\mathrm{Hom}_R(X, \mathfrak{m}) \in \mathcal{G}(R)$, hence

$$\mathrm{Hom}_R(X, A) \in \mathcal{G}(R).$$

Take the first syzygy module of X; we have an exact sequence

$$0 \to \Omega X \to R^n \to X \to 0.$$

Dualizing this sequence by A, we obtain an exact sequence

$$0 \to \mathrm{Hom}_R(X, A) \to A^n \to \mathrm{Hom}_R(\Omega X, A) \to \mathrm{Ext}_R^1(X, A) \to 0.$$

Divide this into two short exact sequences

$$(2.7.5) \qquad \begin{cases} 0 \to \mathrm{Hom}_R(X, A) \to A^n \to C \to 0, \\ 0 \to C \to \mathrm{Hom}_R(\Omega X, A) \to \mathrm{Ext}_R^1(X, A) \to 0 \end{cases}$$

of R-modules. Since ΩX is also a non-free indecomposable module in $\mathcal{G}(R)$, applying the above argument to ΩX instead of X shows that the module $\mathrm{Hom}_R(\Omega X, A)$ also belongs to $\mathcal{G}(R)$. We have $\text{G-dim}_R(A^n) < \infty$ by the first sequence in (2.7.1). Hence it follows from (2.7.5) that $\text{G-dim}_R C < \infty$, and

$$(2.7.6) \qquad \text{G-dim}_R(\mathrm{Ext}_R^1(X, A)) < \infty.$$

On the other hand, applying the functor $\mathrm{Hom}_R(X, -)$ to the natural exact sequence $0 \to \mathfrak{m} \to R \to k \to 0$, we get an exact sequence

$$0 \to \mathrm{Hom}_R(X, \mathfrak{m}) \to X^* \to \mathrm{Hom}_R(X, k) \to \mathrm{Ext}_R^1(X, \mathfrak{m}) \to 0.$$

There is an isomorphism $\mathrm{Hom}_R(X, k) \cong \mathrm{Ext}_R^1(X, \mathfrak{m})$, hence $\mathrm{Ext}_R^1(X, \mathfrak{m})$ is a k-vector space. Since $\mathrm{Ext}_R^1(X, A)$ is contained in $\mathrm{Ext}_R^1(X, \mathfrak{m})$ by (2.7.4),

$$(2.7.7) \qquad \mathrm{Ext}_R^1(X, A) \text{ is a } k\text{-vector space.}$$

Using (2.7.6) and (2.7.7), we can prove that the local ring R is Gorenstein.

Since the only number i such that $\mathrm{Ext}_R^i(k, R) \neq 0$ is the Krull dimension of R if R is Gorenstein, it follows from the above two claims that R is a Gorenstein local ring of dimension two, which completes the proof of the theorem. □

The above theorem interests us in the observation of a Gorenstein local ring of dimension two such that the second syzygy module of the residue class field is decomposable. We introduce here a related result due to Yoshino and Kawamoto.

A homomorphic image of a convergent power series ring over a field k is called an *analytic* ring over k. Any complete local ring containing a field is an analytic ring over its coefficient field, and it is known that any analytic local ring is Henselian; see [14, Chapter VII]. Yoshino and Kawamoto observed the decomposability of the fundamental module of an analytic normal domain.

Theorem 2.8 (Yoshino-Kawamoto). *Let R be an analytic normal local domain of dimension two. Suppose that the residue class field of R is algebraically closed and has characteristic zero. Then the following conditions are equivalent:*

(1) *The fundamental module of R is decomposable;*

(2) *R is an invariant subring of a regular local ring by a cyclic group. (In other words, R is a cyclic quotient singularity.)*

For the details of this theorem, see [21, Theorem (2.1)] or [19, Theorem (11.12)].

With the notation of the above theorem, suppose in addition that R is a complete Gorenstein ring such that $\Omega_R^2 k$ is decomposable. Then R satisfies the condition (1) in the above theorem. Hence the proof of the above theorem shows that R is of finite Cohen-Macaulay representation type (i.e. there exist only finitely many non-isomorphic maximal Cohen-Macaulay R-modules); see [21] or [19]. Therefore it follows from a theorem of Herzog [12] that R is a hypersurface. Thus the local ring R is a rational double point of type (A_n) for some $n \geq 1$ by [21, Proposition (4.1)], namely, R is isomorphic to

$$k[[X, Y, Z]]/(XY - Z^{n+1}).$$

From a more general viewpoint, we can give a characterization as follows:

Theorem 2.9. *Let $(S, \mathfrak{n}, k)$ be a regular local ring, I an ideal of S contained in $\mathfrak{n}^2$, and $R = S/I$ a residue class ring. Suppose that R is a Henselian Gorenstein ring of dimension two. Then the following conditions are equivalent:*

(1) *$\Omega_R^2 k$ is decomposable;*

(2) *$\dim S = 3$ and $I = (xy - zf)$ for some regular system of parameters x, y, z of S and $f \in \mathfrak{n}$.*

It is necessary to prepare three elementary lemmas to prove this theorem. The first one is both well-known and easy to check, and we omit the proof.

Lemma 2.10. *Let $(S, \mathfrak{n}, k)$ be a regular local ring of dimension three and $R = S/(f)$ a hypersurface with $f \in \mathfrak{n}^2$. Then $f = xf_x + yf_y + zf_z$ for some $f_x, f_y, f_z \in \mathfrak{n}$, and the minimal free resolution of k over R is as follows:*

$$\cdots \xrightarrow{C} R^4 \xrightarrow{D} R^4 \xrightarrow{C} R^4 \xrightarrow{D} R^4 \xrightarrow{C} R^4 \xrightarrow{B} R^3 \xrightarrow{A} R \longrightarrow k \longrightarrow 0,$$

where

$$A = \begin{pmatrix} x & y & z \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & -z & y & f_x \\ z & 0 & -x & f_y \\ -y & x & 0 & f_z \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & -f_z & f_y & x \\ f_z & 0 & -f_x & y \\ -f_y & f_x & 0 & z \\ -x & -y & -z & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & -z & y & f_x \\ z & 0 & -x & f_y \\ -y & x & 0 & f_z \\ -f_x & -f_y & -f_z & 0 \end{pmatrix}.$$

Lemma 2.11. *Let $(R, \mathfrak{m}, k)$ be a local ring and $x \in \mathfrak{m} - \mathfrak{m}^2$ an R-regular element. Then we have a split exact sequence*

$$0 \to k \xrightarrow{\theta} \mathfrak{m}/x\mathfrak{m} \xrightarrow{\pi} \mathfrak{m}/xR \to 0,$$

where θ is defined by $\theta(\overline{a}) = \overline{xa}$ for $\overline{a} \in R/\mathfrak{m} = k$ and π is the natural surjection.

PROOF. Let $x_1, x_2, \dots, x_n$ be a minimal system of generators of $\mathfrak{m}$ with $x_1 = x$. Define a homomorphism $\varepsilon : \mathfrak{m}/x\mathfrak{m} \to k$ by $\varepsilon(\sum_{i=1}^n \overline{x_i a_i}) = \overline{a_1}$. We easily see that the composite map $\varepsilon\theta$ is the identity map of k, which means that θ is a split-monomorphism. □

Lemma 2.12. *Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring of dimension one. Then the following conditions are equivalent:*

(1) *R is a discrete valuation ring;*
(2) *$\mathfrak{m}^*$ is a cyclic R-module.*

PROOF. (1) ⇒ (2): This implication is obvious since the maximal ideal $\mathfrak{m}$ is a free R-module of rank one.

(2) ⇒ (1): We have $\mathfrak{m}^* \cong R/I$ for some ideal I of R. Dualizing the natural exact sequence $0 \to \mathfrak{m} \to R \to k \to 0$, we obtain an exact sequence

$$\mathrm{Hom}_R(k, R) \to R \to \mathfrak{m}^*.$$

Since $\mathrm{Hom}_R(k, R) = 0$ by the assumption that R is Cohen-Macaulay, there is an injective homomorphism $R \to R/I$. We easily observe that $I = 0$, equivalently, $\mathfrak{m}^* \cong R$. This implies the condition (1). □

Let R be a local ring and I an ideal of R. We recall that the *grade* of I is defined to be the infimum of the integers n such that $\mathrm{Ext}_R^n(R/I, R) \neq 0$, and is denoted by $\operatorname{grade} I$. As is well-known, it coincides with the length of any maximal R-sequence in I. Now let us prove Theorem 2.9.

PROOF OF THEOREM 2.9. (2) ⇒ (1): We have $xy - zf = x \cdot 0 + y \cdot x + z \cdot (-f)$. Lemma 2.10 gives a finite free presentation

$$R^4 \xrightarrow{C} R^4 \longrightarrow \Omega_R^2 k \longrightarrow 0$$

of the R-module $\Omega_R^2 k$, where $C = \begin{pmatrix} 0 & f & x & x \\ -f & 0 & 0 & y \\ -x & 0 & 0 & z \\ -x & -y & -z & 0 \end{pmatrix}$. Putting $P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, we obtain

$$PCQ = \begin{pmatrix} U & 0 \\ 0 & {}^tU \end{pmatrix}$$

where $U = \left(\begin{smallmatrix} x & f \\ z & y \end{smallmatrix}\right)$. It is easily seen that the matrices P, Q are invertible. Denoting by M (resp. N) the cokernel of the homomorphism defined by the matrix U (resp. ${}^tU$), we get an isomorphism $\Omega_R^2 k \cong M \oplus N$.

(1) ⇒ (2): First of all, note that the local ring R is not regular. We denote by $\mathfrak{m}$ the maximal ideal $\mathfrak{n}/I$ of R.

We can choose an element $z \in \mathfrak{n} - \mathfrak{n}^2$ whose image in $\mathfrak{m}$ is an R-regular element and that the module $\mathfrak{m}/zR$ is decomposable. Put $\overline{(-)} = (-) \otimes_S S/(z)$. Note that $\overline{S}$ is also a regular local ring because z is a minimal generator of the maximal ideal $\mathfrak{n}$ of S (see the proof of Proposition 2.3). Since the maximal ideal $\mathfrak{m}\overline{R}$ of $\overline{R}$ is decomposable, we can apply Theorem 2.5 and see that $\dim \overline{S} = 2$ and $I\overline{S} = xy\overline{S}$ for some $x, y \in \mathfrak{n}$ whose images in $\overline{S}$ form a regular system of parameter of $\overline{S}$. Hence $\overline{R} = \overline{S}/xy\overline{S}$ is a hypersurface, in particular a complete intersection, of dimension one. Therefore R is a complete intersection of dimension two by [8, Theorem 2.3.4(a)]. Since S is a regular local ring of dimension three with regular system of parameter x, y, z, the ideal I is generated by an S-sequence by [8, Theorem 2.3.3(c)]. Noting $\operatorname{ht} I = \dim S - \dim R = 1$, we see that I is a principal ideal. Write $I = (l)$ for some $l \in I$. There is an element $f \in S$ such that $l = xy - zf$. Assume that $f \notin \mathfrak{n}$. Then f is a unit of S, and we see that $zR \subseteq xyR$. Hence $\mathfrak{m} = (x, y)R$, and $\operatorname{edim} R = \dim R = 2$. This implies that R is regular, which is a contradiction. It follows that $f \in \mathfrak{n}$. □

Combining Theorem 2.7 with Theorem 2.9 gives birth to the following corollary. Compare it with Corollary 2.6.

Corollary 2.13. *Let $(R, \mathfrak{m}, k)$ be a complete local ring. Suppose that $\mathfrak{m}$ is indecomposable. Then the following conditions are equivalent:*

1. *$\Omega_R^2 k$ has a non-zero proper direct summand of finite G-dimension;*
2. *R is Gorenstein, and $\Omega_R^2 k$ is decomposable;*
3. *There are a complete regular local ring $(S, \mathfrak{n})$ of dimension three, a regular system of parameters x, y, z of S, and $f \in \mathfrak{n}$ such that $R \cong S/(xy - zf)$.*

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ON THE PRINCIPAL 3-BLOCKS OF THE CHEVALLEY GROUPS $G_2(q)$

YOKO USAMI

ABSTRACT. Assume that q is even and

$$q \equiv 4 \text{ or } 7 \pmod 9.$$

Then the principal 3-block of $G_2(q)$ and the principal 3-block of $G_2(4)$ are Morita equivalent. Here a $\Delta(P)$-projective trivial source $G_2(4) \times G_2(q)$-module and its $\mathcal{O}$-dual induce this Morita equivalence as bimodules, where P is a common Sylow 3-subgroup of $G_2(4)$ and $G_2(q)$ and $\Delta(P) = \{(x, x) \in G_2(4) \times G_2(q) \mid x \in P\}$.

1. 序

1.1. G を有限群とする。$\mathcal{O}$ を完備離散付値環とし、K は標数が 0 であるその商体、k は標数が $p > 0$ であるその剰余体とし、またどちらも十分大きいとする。k および K が G の全ての部分群に対して十分大きいとき、$(K, \mathcal{O}, k)$ は G の全ての部分群に対して splitting p-modular system という。群環 $\mathcal{O}G$ を直既約両側イデアル分解したとき出てくるそれぞれの直既約両側イデアルを p -ブロックと呼ぶ 。(群環 kG で考えることもある。) 直既約 $\mathcal{O}G$-module はどれかの p-ブロックの module とみなせてそのブロックに属すといわれる。任意の KG -module は $\mathcal{O}$-free $\mathcal{O}G$-module から係数環拡張で得られる。trivial $\mathcal{O}G$-module の属す p-ブロックは主 (p-) ブロックと呼ばれる。また p-ブロックには、defect group と呼ばれる p-群 D が定まり、特に主 p-ブロックのときは、D はもとの群の Sylow-p 部分群となる。

定義 1.2. ふたつの有限群 G と H が p-local structure を同じくするというのは次の条件が成り立つときにいう。

(i) G と H は Sylow-p 部分群 P を共有する。

(ii) Q_1 と Q_2 は P の部分群で $f : Q_1 \to Q_2$ は同型写像とする。すると すべての Q_1 の元 x に対して $f(x) = x^g$ を満たす G の元 g が存在するときかつそのときに限ってすべての Q_1 の元 x に対して $f(x) = x^h$ を満たす H の元 h が存在する。

1.3. ここでは　p-ブロックの module の圏どうしの同値つまり森田同値に関して類別をすることを考える。この問題設定は次の 2 つの予想と関連して発生している。Puig 予想はもともと source algebra と定義されるものによる同値類別について述べたもので Puig 同値と呼ばれるが、それは Donovan 予想を拡張したものととらえられる。また森田同値を与える bimodule がある良い性質のものであれば、Puig 同値となることも Puig、Scott によって知られている。(Remark 7.5 [3], Theorem 1.6 [1])

The detailed version of this paper will be submitted for publication elsewhere.

Donovan 予想 1.4. (1980 [2]) 与えられた素数 p と defect group D に対して、k 上の群環の D を defect group として持つ p-ブロックの森田同値類は有限個しかないのではないか？

Puig 予想 (の変形版) 1.5. (1999 [3]) Donovan 予想の中の森田同値をより強い Puig 同値に置き換えたもの。

1.6. Lie 型の単純群は、有限体 $GF(q)$ 上に定義され q を動かすことで、無限列となり、位数は q の多項式として表される。p がこの多項式の同じ因子（例えば、$q-1$ など）を割る素数で、p　の同じべきであれば、p-local structure は q に依存せず似たものになっていることが多い。それらの主 p-ブロックどうしは森田同値になりそうなのである。q の値に依存しないと証明できれば、Donovan 予想の部分的裏付けが得られたことになる。またここで森田同値が言えれば、一つの無限列での表現の考察を小さい q の小さい群の場合に帰着できて都合がよい。一方 Hiss, Kessar は $GU_n(q)$ などの族で q を固定し次元 n を動かして unipotent ブロックの森田同値類別（実は Puig 同値類別）を試みている。[4]

1.7. q を動かすことにして有限体 $GF(q)$ 上の Chevalley 群 $G_2(q)$ で 3-local structure を共有するような無限個の群の族を考えその主 3-ブロック達の類似性を研究することにする。いま、$M(3)$ を位数 27 exponent 3 の extra special group と呼ばれるものとする。q が　mod 9 で 2,4,5 または 7 に合同のときは、$G_2(q)$ 達は $M(3)$ と同型な共通の Sylow 3-部分群 P を持ち P の正規化群は $M(3)$ に位数 16 の semidihedral group SD_{16} が忠実に作用した形の半直積と同型になっていて、同じ 3-local structure を持っている。さて主定理は以下の定理 1.8 である。実際、良い性質の bimodule での森田同値なので Puig 同値である。以下定理 1.8 およびその証明のところでは $p=3$ ということになる。

Theorem 1.8. *Assume that q is even and*

$$q \equiv 4 \text{ or } 7 \pmod 9.$$

Then the principal 3-block of $G_2(q)$ and the principal 3-block of $G_2(4)$ are Morita equivalent. Here a $\Delta(P)$-projective trivial source $G_2(4)\times G_2(q)$-module and its $\mathcal{O}$-dual induce this Morita equivalence as bimodules, where P is a common Sylow 3-subgroup of $G_2(4)$ and $G_2(q)$ and $\Delta(P) = \{(x,x) \in G_2(4) \times G_2(q) \mid x \in P\}$.

Cor. 1.9. 定理 1.8 の条件下では Hiss の論文 [5] Table 1 における分解行列中の未知パラメーター γ は 1 とわかる。

2. 定理 1.8 証明方針

2.1. 以下 q は Theorem 1.8 の条件を満たしているとする。以下 $G_2(q)$ を $G(q)$ と略記する。さらに $B(G(q))$ は $G(q)$ の主 3-ブロックとする。まず $B(G(q))$ と $B(G(4))$ の stable equivalence of Morita type（Definition 2.2 を見よ。）を先に証明してから、simple modules の行き先を考察し、以下の Linckelmann の定理（Theorem 2.3）を使って最終的に森田同値であることを証明する。

Definition 2.2. ([6]) Let A and B be $\mathcal{O}$-algebras, $M\,(= {}_BM_A)$ a (B,A)-bimodule, and $N\,(= {}_AN_B)$ an (A,B)-bimodule. We say that M and N induce a *stable equivalence of Morita type* between A and B, if

(i) M is projective as a left B-module and as a right A-module,
(ii) N is projective as a left A-module and as a right B-module,

(iii) $M \underset{A}{\otimes} N = B \oplus X$ for a projective (B, B)-bimodule X and $N \underset{B}{\otimes} M = A \oplus Y$ for a projective (A, A)-bimodule Y.

For k-algebras we define a stable equivalence of Morita type similarly.

Theorem 2.3. (Linckelmann [6]) *Let G and H be two finite groups and b and b' central idempotents of $\mathcal{O}G$ and $\mathcal{O}H$ respectively. Set*

$$A = \mathcal{O}Gb,\ B = \mathcal{O}Hb',\ \overline{A} = k \otimes_{\mathcal{O}} A \text{ and } \overline{B} = k \otimes_{\mathcal{O}} B.$$

Let M be a (B, A)-bimodule which is projective as both a left and a right module, such that the functor $M \otimes_A -$ induces a stable equivalence of Morita type between A and B. Then the following hold.

(i) *Up to isomorphism, M has a unique indecomposable non-projective direct summand M' as a (B, A)-bimodule and $k \otimes_{\mathcal{O}} M'$ is, up to isomorphism, the unique indecomposable non-projective direct summand of $k \otimes_{\mathcal{O}} M$ as a $(\overline{B}, \overline{A})$-bimodule.*

(ii) *If M is indecomposable as a (B, A)-bimodule, then for any simple A-module S, the B-module $M \otimes_A S$ is indecomposable and non-projective as a $\overline{B}$-module.*

(iii) *If for any simple A-module S, the B-module $M \otimes_A S$ is simple, then the functor $M \otimes_A -$ induces a Morita equivalence.*

2.4. ここでは実は $B(G(q))$ と $B(G(4))$ をいきなり比較するのでなく、$G(q)$ の構造の特殊性を利用する。つまり $G(q)$ は $\mathrm{SL}(3,q)$ と同型な部分群を持ちさらにその指数 2 の拡大 ($N(q)$ と略記する。) をまさに P の中心 $Z(P)$ の正規化群として持つことを利用する。ここで主 3-ブロック達 $B(N(q))$, $B(N(4))$ は森田 (Puig) 同値であることは主 3-ブロック達 $B(\mathrm{PSL}(3,q))$ と $B(\mathrm{PSL}(3,4))$ が森田 (Puig) 同値であるという功刀の定理 (定理 2.5) から導いておく。そしてもっぱら $B(G(q))$ と $B(N(q))$ の stable eqivalence of Morita type を与える良い性質をもつ bimodule M_q を探すことにする。(そのとき以下 Broué 定理 (Theorem 2.7) を利用する。) そうすれば、$B(G(q))$ と $B(N(q))$ 、$B(N(q))$ と $B(N(4))$、$B(N(4))$ と $B(G(4))$ の stable equivalence of Morita type をそれぞれ与える bimodules を合成して $B(G(q))$ と $B(G(4))$ の stable equivalence of Morita type を引き出せる。Broué の定理を述べる前に必要な Brauer morphism と呼ばれるものを Definition 2.6 で導入しておく。

Theorem 2.5. (Kunugi [7]) *Let G be the projective special linear group* $\mathrm{PSL}(3, q)$ *for a power q of a prime such that $q \equiv 4$ or $7 \pmod 9$ (so that a Sylow 3-subgroup Q of G is elementary abelian of order 9). Let $(K, \mathcal{O}, k)$ be a splitting 3-modular system for all subgroups of G. Then the principal 3-block A of $\mathcal{O}G$ is Morita equivalent to the principal 3-block A_0 of $\mathcal{O}[\mathrm{PSL}(3, 4)]$. This equivalence is defined by an (A_0, A)-bimodule M which is a $\Delta(Q)$-projective 3-permtation $\mathcal{O}[\mathrm{PSL}(3, 4) \times G]$-module.*

Definition 2.6. ([8,9]) For an $\mathcal{O}G$-module V and a p-subgroup P of G, we set

$$\mathrm{Br}_P(V) = V^P / (\sum_Q Tr_Q^P(V^Q) + \mathcal{P}V^P) \tag{2.1}$$

where V^P denotes the set of fixed points of V under P and Q runs over all proper subgroups of P and

$$\mathrm{Tr}_Q^P(v) = \sum_{x \in P/Q} xv \tag{2.2}$$

for a subgroup Q of P and $v \in V^Q$. ($\mathcal{P}$ is the maximal ideal of $\mathcal{O}$.)

Theorem 2.7. (Broué [9]) *Let G be a finite group with a Sylow p-subgroup P and H a subgroup of G containing $N_G(P)$. Assume that G and H have the same fusion on p-subgroups contained in P (that is, the same p-local structure). Let b and b' be central primitive idempotents of $\mathcal{O}G$ and $\mathcal{O}H$ respectively for the principal blocks*

$$A = \mathcal{O}Gb \text{ and } B = \mathcal{O}Hb'$$

(having a common defect group P.) For a subgroup R of P, set

$$\bar{b}_R = \mathrm{Br}_R(b) \text{ and } \bar{b'}_R = \mathrm{Br}_R(b').$$

Let M be a (B, A)-bimodule and N be an (A, B)-bimodule. For each subgroup R of P set

$$\overline{M}_R = \mathrm{Br}_{\Delta(R)}(M) \text{ and } \overline{N}_R = \mathrm{Br}_{\Delta(R)}(N).$$

Assume that

(i) *M is a direct summand of the restriction of A from $G \times G$ to $H \times G$.*
(ii) *For each non-trivial subgroup R of P, $\overline{M}_R$ and $\overline{N}_R$ induce a Morita equivalence between $kC_G(R)\bar{b}_R$ and $kC_H(R)\bar{b'}_R$.*

Then M and N induce a stable equivalence of Morita type between A and B.

3. $B(G(q))$ と $B(N(q))$ の STABLE EQUIVALENCE OF MORITA TYPE を与える BIMODULE

3.1. 一般に H が G の部分群のとき $\mathcal{O}G$ は　左 $\mathcal{O}H$-加群かつ右 $\mathcal{O}G$-加群とみてテンソルすることで $\mathcal{O}G$-加群を H に制限したり $\mathcal{O}H$-加群を G へ持ち上げたりする働きをする。$\mathcal{O}G$ の両側に $\mathcal{O}H$、$\mathcal{O}G$ のそれぞれ主ブロックべき等元をつけると、主ブロックに属するものを、H へ制限して主ブロックに属するもののみ残すとか、G へ持ち上げてやはり主-ブロックに属するもののみ残す働きをすることになる。ここで $G(q)$ とその部分群 $N(q)$ の間でその主 3-ブロックべき等元 e と f を使って $fOG(q)e$ を考察する。R を P の 1 でない任意の部分群として、$\Delta(R)$ に関する Brauer morphism を施してどう分解しているか調べると Broué の定理 (Theorem 3.2 [8]) によって $fOG(q)e$ 自身がどんな vertex の直既約因子達の直和に分解するかということに対する情報が得られる。(ちなみにこれは vertex ΔP の直既約因子 M_q を持っていて、それがのぞみの bimodule となる。) $fOGe$ は $\Delta(R)$ に対する Brauer morphism を施すと $\bar{f}_R kC_{G(q)}(R)\bar{e}_R$ となる。ここで $\bar{e}_R$, $\bar{f}_R$ はそれぞれ $kC_{G(q)}(R)$, $kC_{N(q)}(R)$ の主 3-ブロックのブロックべき等元である。いろいろの R で試して、$fOG(q)e$ は　nonprojective 因子として M_q ともうひとつ vertex の小さい因子を持つことがわかる。

3.2. $fOG(q)e$ はどういう制限や持ち上げをしているか、片方で、いろいろの permutation module を作りつつ、制限したり持ち上げたりしながら、その中で M_q 自身は何を何に写しているか調べる。(このとき vertex $\Delta(R)$ の permutation module が vertex R' の permutation module をどんな vertex の直既約加群達の直和にするかは Mackey 分解で様子を知ることができる。また $G(q)$ と $N(q)$ の間ではいくつか複数の 3-subgroup に関して Green 対応のある状況であることも有効に使える。)

4. SIMPLE MODULES の行き先

4.1. ここで3.2に続けてさらに2.4で説明したように合成して作るbimoduleが何を何に写すか調べる。このときこれがexact sequenceをexact sequenceに写すことを有効に利用する。だからまずtrivial moduleのように簡単なsimple moduleや簡単な構造のpermutation moduleの行き先を決めておいて、その組成因子の行き先を決めるといったやり方を繰り返すことができる。

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ON THE $\mathbb{Z}D_\infty$ CATEGORY [1]

MICHEL VAN DEN BERGH

ABSTRACT In this paper we give a direct proof of the properties of the $\mathbb{Z}D_\infty$ category which was introduced in the classification of noetherian, hereditary categories with Serre duality by Idun Reiten and the author.

1. INTRODUCTION

Below k is a field. All categories will be k-linear. An abelian or triangulated category $\mathcal{A}$ is Ext-finite if for all objects $A, B \in \mathcal{A}$ one has that $\oplus_i \operatorname{Ext}^i(A,B)$ is finite dimensional. If $\mathcal{A}$ is triangulated and Ext-finite then we say that $\mathcal{A}$ satisfies Serre duality [1] if there exists an auto-equivalence F of $\mathcal{A}$ together with isomorphisms

$$\operatorname{Hom}_{\mathcal{A}}(A,B) \to \operatorname{Hom}_{\mathcal{A}}(B,FA)^*$$

natural in A, B (where $(-)^*$ is the k-dual). If $\mathcal{A}$ is abelian and Ext-finite then we say that $\mathcal{A}$ satisfies Serre duality if this is the case for $D^b(\mathcal{A})$. The following result can be extracted from [5, Ch. 1].

Theorem 1.1. *Assume that $\mathcal{C}$ is an* Ext*-finite hereditary category without injectives or projectives. Then the following are equivalent*

(1) *$\mathcal{C}$ has almost split sequences.*
(2) *$\mathcal{C}$ satisfies Serre duality.*
(3) *There is an auto-equivalence $V : \mathcal{C} \to \mathcal{C}$ together with natural isomorphisms*

$$\operatorname{Hom}_{\mathcal{C}}(A,B) \to \operatorname{Ext}^1_{\mathcal{C}}(B,VA)^* \tag{1.1}$$

Furthermore the functor V coincides with the Auslander-Reiten translate τ when evaluated on objects.

In the classification of noetherian Ext-finite hereditary categories with Serre duality in [5] we considered a category $\mathcal{C}$ defined by the following pullback diagram

$$\begin{tikzcd} \operatorname{mod}(k) \oplus \operatorname{mod}(k) \arrow[r] & \operatorname{mod}(k) \\ \mathcal{C} \arrow[u] \arrow[r] & \operatorname{gr}(k[x]) \arrow[u] \end{tikzcd} \tag{1.2}$$

where the horizontal map sends (V_1, V_2) to $V_1 \oplus V_2$ and the vertical map is localizing at x followed by restricting to degree zero. It was shown by a rather indirect argument that $\mathcal{C}$ is a noetherian, Ext-finite, hereditary abelian category without injectives or projectives

[1]The paper is in a final form and no version of it will be submitted for publication elsewhere.

which satisfies Serre duality. It was also shown that the AR-quiver of $\mathcal{C}$ has two components, one equal to $\mathbb{Z}A_\infty$ (a "wing") and the other equal to $\mathbb{Z}D_\infty$. For this reason $\mathcal{C}$ was called the "$\mathbb{Z}D_\infty$-category".

The aim of this paper is to give a direct proof of the above facts. In addition we will also establish a link with one-dimensional A_n singularities.

2. Elementary properties

It is easy to see that $\mathcal{C}$, as defined in the introduction is a noetherian abelian category. It will be convenient to consider the locally noetherian Grothendieck category $\tilde{\mathcal{C}}$ associated to $\mathcal{C}$. It follows for example by [3, Prop. 2.14] that $D^b(\mathcal{C})$ and $D^b_{\mathcal{C}}(\tilde{\mathcal{C}})$ are equivalent. Hence the Ext-groups between objects in $\mathcal{C}$ may be computed in $\tilde{\mathcal{C}}$.

The objects of $\tilde{\mathcal{C}}$ are quadruples (M, V_0, V_1, ϕ) where M is a graded $k[x]$-module, V_0, V_1 are k-vector spaces and ϕ is an isomorphism of $k[x]$ modules $M_x \to (V_0 \oplus V_1) \otimes_k k[x, x^{-1}]$. Objects in $\mathcal{C}$ are given by the quadruples (M, V_0, V_1, ϕ) in which M is finitely generated.

Sending (M, V_0, V_1, ϕ) to M defines an faithful exact functor $\tilde{\mathcal{C}} \to \mathrm{Gr}(k[x])$ which we call the restriction functor and which we denote by $(-)_{k[x]}$.

We write $M(n) = (M(n), V_0, V_1, \phi)$ where we have identified $M(n)_x$ with M_x through multiplication with x^n. Furthermore we define $\sigma(M) = (M, V_1, V_0, \phi)$.

We define $\tilde{\mathcal{T}} \subset \tilde{\mathcal{C}}$ and $\tilde{\mathcal{F}} \subset \tilde{\mathcal{C}}$ respectively as the inverse images of the x-torsion and x-torsion free modules in $\mathrm{Gr}(k[x])$. $\mathcal{T}$ and $\mathcal{F}$ are defined similarly, but starting from $\mathcal{C}$.

By $\tilde{\mathcal{C}}_x$ we denote the full subcategory of $\tilde{\mathcal{C}}$ with objects the quadruples (M, V_0, V_1, ϕ) in which x acts invertibly on M.

We denote by $(-)_x$ the functor $\tilde{\mathcal{C}} \to \tilde{\mathcal{C}}_x$ which sends (M, V_0, V_1, ϕ) to (M_x, V_0, V_1, ϕ). Clearly if $M \in \tilde{\mathcal{C}}$ and $N \in \tilde{\mathcal{C}}_x$ then the canonical maps

$$\mathrm{Hom}_{\tilde{\mathcal{C}}}(M, N) \to \mathrm{Hom}_{\tilde{\mathcal{C}}_x}(M, N) \to \mathrm{Hom}_{\tilde{\mathcal{C}}_x}(M_x, N) \tag{2.1}$$

are isomorphisms. We list a few other obvious facts.

(O1) $(\tilde{\mathcal{T}}, \tilde{\mathcal{F}})$ forms a torsion pair in $\tilde{\mathcal{C}}$. That is $\mathrm{Hom}(\tilde{\mathcal{T}}, \tilde{\mathcal{F}}) = 0$ and for any $M \in \tilde{\mathcal{C}}$ there exists an exact sequence (necessarily unique)

$$0 \to T \to M \to F \to 0$$

with $T \in \tilde{\mathcal{T}}$ and $F \in \tilde{\mathcal{F}}$.

(O2) If $T \in \tilde{\mathcal{T}}$ and $M \in \tilde{\mathcal{C}}$ then

$$\mathrm{Hom}_{\tilde{\mathcal{C}}}(T, M) = \mathrm{Hom}_{k[x]}(T, M)$$
$$\mathrm{Hom}_{\tilde{\mathcal{C}}}(M, T) = \mathrm{Hom}_{k[x]}(M, T)$$

(O3) The restriction functor defines an equivalence between $\tilde{\mathcal{T}}$ and $\mathrm{Tors}(k[x])$ where $\mathrm{Tors}(k[x])$ denotes the x-torsion modules is $\mathrm{Gr}(k[x])$.

(O4) The functor $\tilde{\mathcal{C}}_x \to \mathrm{Mod}(k) \oplus \mathrm{Mod}(k)$ which sends (M, V_0, V_1, ϕ) to $V_0 \oplus V_1$ is an equivalence of categories.

Combining (O4) with (2.1) yields in particular

(O5) If $N \in \tilde{\mathcal{C}}_x$ then $\mathrm{Hom}_{\mathcal{C}}(-, N)$ is exact. Hence the objects in $\tilde{\mathcal{C}}_x$ are injective in $\tilde{\mathcal{C}}$.

We now describe the indecomposable injectives in $\tilde{\mathcal{C}}$. For $n \in \mathbb{Z}$ let E_n be the graded injective $k[x]$-module given by $k[x, x^{-1}]/x^{n+1}k[x]$. Since $E_n \in \mathrm{Tors}(k[x])$ there exists by (O3) a corresponding object in $\tilde{\mathcal{T}}$ which we denote by the same symbol. From (O2) it follows that $\mathrm{Hom}(-, E_n)$ is exact and hence E_n is injective in $\tilde{\mathcal{C}}$.

To construct other injectives we note that by (O5) we know that the objects in $\tilde{\mathcal{C}}_x$ are injective in $\tilde{\mathcal{C}}$. Since by (O4) $\tilde{\mathcal{C}}_x$ is equivalent to $\mathrm{Mod}(k) \oplus \mathrm{Mod}(k)$ there must be two corresponding indecomposable injectives in $\tilde{\mathcal{C}}$. They are given by

$$E^0 = (k[x, x^{-1}], k, 0, \mathrm{id}_{k[x,x^{-1}]})$$
$$E^1 = (k[x, x^{-1}], 0, k, \mathrm{id}_{k[x,x^{-1}]})$$

Proposition 2.1. (1) $(E_n)_n, E^0, E^1$ *forms a complete list of indecomposable injectives in* $\tilde{\mathcal{C}}$.

(2) *Every object in* $\tilde{\mathcal{C}}$ *has injective dimension one (and hence* $\tilde{\mathcal{C}}$ *and* $\mathcal{C}$ *are hereditary* [5, Prop. A.3]*)*.

(3) $\mathcal{C}$ *is* Ext*-finite.*

Proof. Since the listed injectives are clearly indecomposable (1) follows if we can show that any indecomposable object can be embedded in a direct sum of them [4].

We prove (1) and (2) together by showing that every object $M \in \tilde{\mathcal{C}}$ has a resolution of length at most two whose terms consist of direct sums of the injectives given in (1). By (O1) it is clearly sufficient to prove this claim separately in the cases $M \in \tilde{\mathcal{T}}$ and $M \in \tilde{\mathcal{F}}$.

Assume first that $M \in \tilde{\mathcal{T}}$. Then M has an injective resolution

$$0 \to M \to I_0 \to I_1 \to 0 \tag{2.2}$$

in $\mathrm{Tors}(k[x])$. By (O3) this resolution corresponds to one in $\tilde{\mathcal{C}}$. Furthermore by the structure of the injectives in $\mathrm{Gr}(k[x])$ the I_i are direct sums of the E_n in $\mathrm{Gr}(k[x])$. Again by (O3) the same is true in $\tilde{\mathcal{C}}$.

Now assume that $M \in \tilde{\mathcal{F}}$. Consider the short exact sequence

$$0 \to M \to M_x \to M_x/M \to 0 \tag{2.3}$$

M_x lies in $\tilde{\mathcal{C}}_x$ and hence by (O4) is a direct sum of copies of E^0 and E^1. M_x/M is x-divisible and lies in $\tilde{\mathcal{T}}$ and so by (O3) M_x/M is a direct sum of copies of E_n. Whence (2.3) is the kind of resolution we were looking for.

To prove (3) we note that if E, F are indecomposable injectives as in (1) then $\dim \mathrm{Hom}_{\tilde{\mathcal{C}}}(E, F) \leq 1$. Thus it suffices to show that every $M \in \mathcal{C}$ has an injective resolution consisting in every degree of a finite number of indecomposable injectives. This follows easily from the construction. □

Proposition 2.2. *If* $F \in \tilde{\mathcal{F}}$ *and* $T \in \tilde{\mathcal{T}}$ *then* $\mathrm{Ext}^1_{\tilde{\mathcal{C}}}(F, T) = 0$. *In particular every object in* $\mathcal{C}$ *is of the form* $F \oplus T$ *with* $F \in \mathcal{F}$ *and* $T \in \mathcal{T}$.

Proof. It follows from (O2) that $\mathrm{Hom}(F, -)$ is exact on $\mathcal{T}$. Since by the proof of the previous proposition T has a $\tilde{\mathcal{C}}$ injective resolution inside $\tilde{\mathcal{T}}$, we are done. □

Now we describe the Ext-groups between objects in $\tilde{\mathcal{F}}$.

Lemma 2.3. *Assume that* $F = (F, V_0, V_1, \phi)$, $F' = (F', V_0', V_1', \phi')$ *are objects in* $\tilde{\mathcal{F}}$. *Then there exists an exact sequence of the form*

(2.4)

$$0 \to \mathrm{Hom}_{\tilde{\mathcal{C}}}(F, F') \to \mathrm{Hom}_{k[x]}(F, F') \to \mathrm{Hom}_k(V_0, V_1') \oplus \mathrm{Hom}_k(V_1, V_0') \to \mathrm{Ext}^1_{\tilde{\mathcal{C}}}(F, F') \to 0$$

Proof. We start with the short exact sequence

$$0 \to F' \to F'_x \to F'_x/F' \to 0$$

which according to the proof of lemma 2.1 is an injective resolution of F', both in $\tilde{\mathcal{C}}$ and in $\mathrm{Gr}(k[x])$.

Applying $\mathrm{Hom}_{\tilde{\mathcal{C}}}(F,-)$, $\mathrm{Hom}_{\mathrm{Gr}(k[x])}(F,-)$ and comparing yields a commutative diagram with exact rows and columns.

$$\begin{tikzcd}
 & & 0 \arrow[d] & & & \\
0 \arrow[r] & \mathrm{Hom}_{\tilde{\mathcal{C}}}(F,F') \arrow[r] \arrow[d] & \mathrm{Hom}_k(V_0,V_0') \oplus \mathrm{Hom}_k(V_1,V_1') \arrow[r] \arrow[d] & \mathrm{Hom}_{\tilde{\mathcal{C}}}(F,F'_x/F') \arrow[r] \arrow[d, equal] & \mathrm{Ext}^1_{\tilde{\mathcal{C}}}(F,F') \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Hom}_{\mathrm{Gr}(k[x])}(F,F') \arrow[r] & \mathrm{Hom}_k(V_0 \oplus V_1, V_0' \ominus V_1') \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathrm{Gr}(k[x])}(F,F'_x/F') \arrow[r] & 0 & \\
 & & \mathrm{Hom}(V_0,V_1') \oplus \mathrm{Hom}(V_1,V_0') \arrow[d] & & & \\
 & & 0 & & &
\end{tikzcd}$$

(2.4) now follows from the previous diagram through an easy diagram chase. □

Proposition 2.4. *$\mathcal{C}$ has neither injectives nor projectives.*

Proof. Since $\mathcal{T}$ is equivalent to the x-torsion modules in $\mathrm{gr}(k[x])$, it is easy to see that $\mathcal{T}$ does not contain any injective or projectives.

If $0 \neq (F, V_0, V_1, \phi)$ in $\mathcal{F}$ then by considering the faithful restriction functor to $\mathrm{gr}(k[x])$ we see that $\mathrm{Hom}_{\mathcal{C}}(F, \sigma F(-n)) = 0$ for $n \gg 0$. On the other hand V_0 or $V_1 \neq 0$. It follows from the previous lemma that $\mathrm{Ext}^1_{\mathcal{C}}(F, \sigma F(-n)) \neq 0$ for $n \gg 0$. Hence F is not projective. A similar argument shows that F is not injective. □

Remark 2.5. The reason why we called this section "Elementary properties" is that the stated results hold in greater generality. For example, suitably adapted versions would be valid for the pullback of

$$\begin{tikzcd}
\mathrm{mod}(k)^{\oplus m} \arrow[r] & \mathrm{mod}(k) \\
 & \mathrm{gr}(k[x]) \arrow[u]
\end{tikzcd}$$

for any m. By contrast, the results in the next section require $m = 2$.

3. Serre duality

§ Our next aim is to prove that $\mathcal{C}$ satisfies Serre duality. First we construct a Serre functor on $\mathcal{F}$. Put $VM = \sigma(M)(-1)$.

The first step in proving Serre duality is constructing a "trace map" $\eta_M : \mathrm{Ext}^1_{\mathcal{C}}(M, VM) \to k$ for $M \in \mathcal{F}$ which should corresponds to the identity map in $\mathrm{Hom}_{\mathcal{C}}(M, M)$ under the isomorphism (1.1).

We now use (2.4) to construct the trace map η_F for $F = (F, V_0, V_1, \phi) \in \mathcal{F}$. In this case $VF = (F(-1), V_1, V_0, \phi)$ and we have an exact sequence

$$\mathrm{Hom}_{k[x]}(F, VF) \to \mathrm{Hom}_k(V_0, V_0) \oplus \mathrm{Hom}_k(V_1, V_1) \to \mathrm{Ext}^1_{\mathcal{C}}(F, VF) \to 0$$

Lemma 3.1. *The composition*

$$\mathrm{Hom}_{k[x]}(F, VF) \to \mathrm{Hom}_k(V_0, V_0) \oplus \mathrm{Hom}_k(V_1, V_1) \xrightarrow{\mathrm{Tr}_{V_0} + \mathrm{Tr}_{V_1}} k \tag{3.1}$$

is the zero map.

Proof. To see this note that $\mathrm{Hom}_{k[x]}(F, VF) = \mathrm{Hom}(F, F(-1))$ and furthermore that (3.1) can be extended to a commutative diagram.

$$\begin{tikzcd}
\mathrm{Hom}_{k[x]}(F, F(-1)) \arrow[r] & \mathrm{Hom}_k(V_0 \oplus V_1, V_0 \oplus V_1) \arrow[r] \arrow[d, "\mathrm{Tr}"] & \mathrm{Hom}_k(V_0, V_0) \oplus \mathrm{Hom}_k(V_1, V_1) \arrow[dl, "\mathrm{Tr}_{V_0} + \mathrm{Tr}_{V_1}"] \\
& k &
\end{tikzcd}$$

By choosing a basis for F as graded $k[x]$-module one easily sees that every element of $\mathrm{Hom}(F, F(-1)) \subset \mathrm{Hom}(F, F)$ is nilpotent. Since nilpotent elements have zero trace it follows that the composition

$$\mathrm{Hom}_{k[x]}(F, F(-1)) \to \mathrm{Hom}_k(V_0 \oplus V_1, V_0 \oplus V_1) \xrightarrow{\mathrm{Tr}} k$$

is zero. This proves what we want. $\square$

From lemma 3.1 together with (2.4) there exists a unique map $\eta_F : \mathrm{Ext}^1_{\mathcal{C}}(F, VF) \to k$ which makes the following diagram commutative.

$$\begin{tikzcd}
\mathrm{Hom}_{k[x]}(F, VF) \arrow[r] & \mathrm{Hom}(V_0, V_0) \oplus \mathrm{Hom}(V_1, V_1) \arrow[r] \arrow[dr, "\mathrm{Tr}_{V_0} + \mathrm{Tr}_{V_1}"'] & \mathrm{Ext}^1_{\mathcal{C}}(F, VF) \arrow[r] \arrow[d, "\eta_F"] & 0 \\
& & k &
\end{tikzcd}$$

To continue it will be convenient to use the Yoneda multiplication on $\mathrm{Ext}^*_{\mathcal{C}}(-,-)$. In order to have compatibility with the notation for compositions of maps we will write the Yoneda multiplication as a pairing

$$\mathrm{Ext}^*_{\mathcal{C}}(B, C) \times \mathrm{Ext}^*_{\mathcal{C}}(A, B) \to \mathrm{Ext}^*_{\mathcal{C}}(A, C)$$

We extend η_F to a map $\mathrm{Ext}^*(F, VF) \to k$ by letting it act trivially on $\mathrm{Hom}(F, VF)$.

Lemma 3.2. *Let $F, G \in \mathcal{F}$ and assume that $f \in \mathrm{Ext}^*_{\mathcal{C}}(F, G)$ and $g \in \mathrm{Ext}^*_{\mathcal{C}}(G, VF)$. Then we have $\eta_F(gf) = \eta_G(V(f)g)$.*

Proof. We may assume that f and g are homogeneous. Furthermore the cases where f,g are both of degree 0 or of degree 1 are trivial. Hence we may assume that $(\deg f, \deg g) = (0, 1)$ or $(\deg f, \deg g) = (1, 0)$.

Let us consider the first possibility. We check that $\eta_F(-f) = \eta_G(V(f)-)$ as maps $\mathrm{Ext}^1(G, VF) \to k$. This amounts to the commutativity of

$$\begin{tikzcd}
\mathrm{Ext}^1(G, VF) \arrow[r] \arrow[d] & \mathrm{Ext}^1(G, VG) \arrow[d, "\eta_G"'] \\
\mathrm{Ext}^1(F, VF) \arrow[r, "\eta_F"] & k
\end{tikzcd} \tag{3.2}$$

Assume that $F = (F, V_0, V_1, \phi)$, $G = (G, W_0, W_1, \theta)$. Then f induces maps $f_0 : V_0 \to W_0$ and $f_1 : V_1 \to W_1$. Elementary linear algebra yields that we have a commutative diagram

$$\begin{array}{ccc} \operatorname{Hom}(W_0, V_0) \oplus \operatorname{Hom}(W_1, V_1) & \longrightarrow & \operatorname{Hom}(W_0, W_0) \oplus \operatorname{Hom}(W_1, W_1) \\ \big\downarrow & & \big\downarrow{\scriptstyle \operatorname{Tr}_{W_0} + \operatorname{Tr}_{W_1}} \\ \operatorname{Hom}(V_0, V_0) \oplus \operatorname{Hom}(V_1, V_1) & \xrightarrow{\operatorname{Tr}_{V_0} + \operatorname{Tr}_{V_1}} & k \end{array}$$

This diagram, together with the definition of η yields the commutativity of (3.2).

Now we consider the possibility $(\deg f, \deg g) = (1, 0)$. Since we trivially have

$$\eta_{VX} \circ V = \eta_X \tag{3.3}$$

it is sufficient to prove that $\eta_{VF}(V(g)V(f)) = \eta_G(V(f)g)$. Replacing (Vf, g) by (g, f) this reduces to the previous case. □

We are now in a position to prove Serre duality for objects in $\mathcal{F}$. We will show that the pairing

$$\operatorname{Hom}_{\mathcal{C}}(F, G) \times \operatorname{Ext}^1_{\mathcal{C}}(G, VF) \to \operatorname{Ext}^1(F, VF) \xrightarrow{\eta_F} k : (f, g) \mapsto \eta_F(gf) \tag{3.4}$$

is non-degenerate. By lemma 3.2 the non-degeneracy of (3.4) for all F, G is equivalent to the non-degeneracy of the pairing

$$\operatorname{Ext}^1_{\mathcal{C}}(F, G) \times \operatorname{Hom}_{\mathcal{C}}(G, VF) \to \operatorname{Ext}^1(F, VF) \xrightarrow{\eta_F} k : (f, g) \to \eta_F(gf) \tag{3.5}$$

for all F, G. It follows also easily from lemma 3.2 that (3.4) and (3.5) are natural in F and G.

Lemma 3.3. *If we have and exact sequence*

$$0 \to F_1 \to F \to F_2 \to 0 \tag{3.6}$$

in $\mathcal{F}$ and if we have non-degeneracy of (3.4) *and* (3.5) *for two out of the three pairs $(F_1, G), (F, G), (F_2, G)$ then we also have it for the third one. A similar statement holds for an exact sequence*

$$0 \to G_1 \to G \to G_2 \to 0 \tag{3.7}$$

Proof. Assume that we have an exact sequence of the form (3.6). We claim that the following diagram with exact rows is commutative.

$$\begin{array}{ccccccccccccc} 0 \to & \operatorname{Hom}(F_2, G) & \to & \operatorname{Hom}(F, G) & \to & \operatorname{Hom}(F_1, G) & \to & \operatorname{Ext}^1(F_2, G) & \to & \operatorname{Ext}^1(F, G) & \to & \operatorname{Ext}^1(F_1, G) & \to 0 \\ & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \beta_2} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta_1} & \\ 0 \to & \operatorname{Ext}^1(G, VF_2)^* & \to & \operatorname{Ext}^1(G, VF)^* & \to & \operatorname{Ext}^1(G, VF_1)^* & \to & \operatorname{Hom}(G, VF_2)^* & \to & \operatorname{Hom}(G, VF)^* & \to & \operatorname{Hom}(G, VF_1)^* & \to 0 \end{array}$$

Here the maps labeled by α are obtained from (3.4) whereas those labeled by β are obtained from (3.5).

The commutativity of this diagram follows easily from lemma 3.2 together with the observation that all horizontal arrows are obtained by Yoneda multiplying with elements of suitable Ext-groups. For example the connecting maps are obtained from multiplying with the element of $\operatorname{Ext}^1(F_2, F_1)$ representing the exact sequence (3.6).

If we now have non-degeneracy for two out of the three pairs (F_1, G), (F, G), (F_2, G) then we also have it for the third pair because of the five-lemma.

The case where we have an exact sequence as in (3.7) is treated similarly. □

To continue we define a some canonical objects in $\mathcal{F}$. Let $a \in \mathbb{N}$. Then we write

$$F^0_{0a} = (x^{-a}k[x], k, 0, \mathrm{id}_{k[x,x^{-1}]})$$
$$F^1_{0a} = (x^{-a}k[x], 0, k, \mathrm{id}_{k[x,x^{-1}]})$$

(the reason for this notation will become clear in Section §5).

Lemma 3.4. *Every object F in $\mathcal{F}$ has a finite filtration $0 = F_0 \subset \cdots \subset F_n = F$ such that the corresponding subquotients are among the F^i_{0a}.*

Proof. By the structure of C_x there must be a surjective map $\phi : F_x \to E^i$ where $i = 0$ or $i = 1$. Hence $\operatorname{im}\phi$ is a non-trivial quotient. Since it is easy to see that the subobjects of E^i in C are of the form F^i_{0a} we are done. □

Using (2.4) we can compute the Hom and Ext-groups between the F^i_{0a}. The results are given in the next lemma.

Lemma 3.5. *One has*

$$\mathrm{Hom}(F^i_{0a}, F^j_{0b}) = \begin{cases} k & \text{if } i = j \text{ and } a \le b \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{Ext}^1(F^i_{0a}, F^j_{0b}) = \begin{cases} k & \text{if } i = 1 - j \text{ and } a > b \\ 0 & \text{otherwise} \end{cases}$$

Proof. The claim for Hom is trivial, so we concentrate on Ext.

We use (2.4). This immediately yields that $\mathrm{Ext}^1(F^i_{0a}, F^j_{0b}) = 0$ if $j \neq 1 - i$. If $j = 1 - i$ then we have the following exact sequence.

$$\mathrm{Hom}_{k[x]}(x^{-a}k[x], x^{-b}k[x]) \to k \to \mathrm{Ext}^1_C(F^i_{0a}, F^j_{0b}) \to 0 \tag{3.8}$$

This yields that $\mathrm{Ext}^1_C(F^i_{0a}, F^j_{0b}) = k$ if and only if $\mathrm{Hom}_{k[x]}(x^{-a}k[x], x^{-b}k[x]) = 0$, i.e. if and only if $a > b$. □

We are now in a position to prove the main result of this section.

Theorem 3.6. *C satisfies Serre duality.*

Proof. We show first that C satisfies Serre duality for objects F, G in $\mathcal{F}$. We will show the non-degeneracy of (3.4) and (3.5) by induction of $\mathrm{rk}_{k[x]}(F)$, $\mathrm{rk}_{k[x]}(G)$. This reduces us to the case where $F = F^i_{0a}$, $G = F^j_{0b}$. So we need to check the non-degeneracy of

$$\mathrm{Hom}_C(F^i_{0a}, F^j_{0b}) \times \mathrm{Ext}^1_C(F^j_{0b}, F^{i'}_{0,a-1}) \to \mathrm{Ext}^1_C(F^i_{0a}, F^{i'}_{0,a-1}) \to k \tag{3.9}$$

$$\mathrm{Ext}^1_C(F^i_{0a}, F^j_{0b}) \times \mathrm{Hom}_C(F^j_{0b}, F^{i'}_{0,a-1}) \to \mathrm{Ext}^1_C(F^i_{0a}, F^{i'}_{0,a-1}) \to k \tag{3.10}$$

where $i' = 1 - i$. We will concentrate ourselves on (3.10). (3.9) is similar. By (3.5) the only non-trivial case is given by $j = 1 - i$ and $a > b$. In that case all vector spaces involved are equal to k and what we want to prove follows from inspecting (3.8).

Now we show that C has almost split sequences. By Theorem 1.1 this implies that C satisfies Serre duality.

By Proposition 2.2 it is clearly sufficient to construct almost split sequences ending in indecomposable objects in $\mathcal{F}$ and $\mathcal{T}$. First let $F \in \mathcal{F}$ be indecomposable. Since $\mathrm{Ext}^1_C(F, VF) \cong \mathrm{Hom}_C(F, F)^*$, $\mathrm{Ext}^1_C(F, VF)$ has a simple socle as (left or right) $\mathrm{Hom}_C(F, F)$

module. Let ζ be a non-zero element this socle. It is well-known, and easy to see that ζ defines the almost split sequence in $\mathcal{F}$ ending in F.

$$0 \to VF \to M \to F \to 0$$

But this is also an almost split sequence in $\mathcal{C}$ since if $T \in \mathcal{T}$ then $\mathrm{Hom}(T, F) = 0$.

Now let $T \in \mathcal{T}$. Then there exists an almost split sequence in $\mathcal{T}$

$$0 \to T'' \to T' \to T \to 0 \tag{3.11}$$

(since $\mathcal{T}$ is equivalent to the category of x-torsion modules over $k[x]$).

We have to prove that any pullback for $C \to T$ of (3.11) with C indecomposable is split. Clearly we only have to consider the case $C \in \mathcal{F}$. But then it follows from Proposition 2.2 that the pullback is split. This finishes the proof. □

4. Relation with one-dimensional graded type A_n-singularities

Let $\mathcal{C}$ be the hereditary category which was described in the previous section. We will now show that $\mathcal{C}$ can be considered as a limit of certain graded simple singularities.

If $m \in \mathbb{N}$ then the graded simple A_{2m-1}-singularity of dimension one is by definition the graded subring R_m of $k[x] \oplus k[x]$ generated by $u = (x, x)$ and $v = (x^m, 0)$. It is easy to see that $R \cong k[u,v]/(u^m v - u^{2m})$ and hence this is equivalent to the classical definition (see for example [2]). We put $\mathcal{C}_m = \mathrm{mod}(R_m)$.

Let us consider $k[x]$ as being diagonally embedded in $k[x] \oplus k[x]$. That is we identify x with (x, x). Clearly we have

$$(R_m)_x = k[x, x^{-1}] \oplus k[x, x^{-1}]$$

Hence if $M \in \mathrm{Mod}(R_m)$ then M_x is canonically a sum of two $k[x, x^{-1}]$-modules which we denote by M_x^0 and M_x^1 respectively. This allows us to define the following functor.

$$U_m : \mathcal{C}_m \to \mathcal{C} : M \mapsto (M, (M_x^0)_0, (M_x^1)_0, \mathrm{id})$$

Clearly U_m is faithful. We have inclusions

$$k[x] \subset \cdots \subset R_{m+1} \subset R_m \subset \cdots R_0 = k[x] \oplus k[x]$$

Dualizing these yield restriction functors

$$\mathcal{C}_0 \to \cdots \to \mathcal{C}_m \to \mathcal{C}_{m+1} \to \cdots \to \mathrm{mod}(k[x])$$

It is clear that these restriction functors are compatible with the functors $(U_m)_m$. Define $\mathcal{C}_\infty$ as the 2-direct limit of the $\mathcal{C}_m$. That is the objects in $\mathcal{C}_\infty$ are the objects in $\coprod_m \mathcal{C}_m$ and we put

$$\mathrm{Hom}_{\mathcal{C}_\infty}(M, N) = \mathrm{inj\,lim}\,\mathrm{Hom}_{\mathcal{C}_m}(M, N)$$

The functors $(U_m)_m$ define a functor $U_\infty : \mathcal{C}_\infty \to \mathcal{C}$.

Proposition 4.1. *The functor U_∞ defined above is an equivalence.*

Proof. From the definition it is clear that U_∞ is faithful. So we only have to show that it is full and essentially surjective.

We will first show that U_∞ is full. Let $M, N \in \mathcal{C}_m$ and let $f : M \to N$ be a homomorphism in $\mathcal{C}$. So f is in fact a $k[x]$-linear homomorphism $f : M \to N$ such that the localization f_x is $k[x, x^{-1}] \oplus k[x, x^{-1}]$-linear.

Let $y = (x, 0)$. Then $y^n \in R_m$ for $n \geq m$ and R_n as subring of R_m is generated by x and y^n. To prove fullness of U_∞ it is sufficient that f is y^n-linear for $n \gg 0$. Let

T be the torsion submodule of N and consider the $k[x]$ linear map $M \to N$ given by $f^{(n)} = f(y^n -) - y^n f(-)$. Since after localizing at x, f is y linear, it follows that the image of $f^{(n)}$ lies in T. Since T is rightbounded it is clear that $f^{(n)}$ must be zero if $n \gg 0$. This proves what we want.

Now we prove essential surjectivity. First let $F \in \mathcal{F}$. Then we claim that $F \subset F_x$ is stable under multiplication by y^n for $n \gg 0$. First note that yF is a finitely generated $k[x]$-submodule of F_x. Hence $x^n yF \subset F$. Since $x^n y = y^{n+1}$ this proves what we want.

Now let $T \in \mathcal{T}$. Then as graded $k[x]$-module T has right bounded grading and since $k[x]_{<m} = (R_n)_{<m}$ for $n \geq m$ it follows that for $n \gg 0$ we may consider T as a graded R_n-module. This proves what we want. □

5. Representation theory

In section we construct the AR-quiver of $\mathcal{C}$. From the above discussion it follows that the components of the AR-quiver of $\mathcal{C}$ lie either in $\mathcal{T}$ or in $\mathcal{F}$. Since $\mathcal{T}$ is equivalent to the x-torsion modules in $\mathrm{gr}(k[x])$ it has a unique component which is $\mathbb{Z}A_\infty$. So the main difficulty is represented by the component(s) in $\mathcal{F}$.

We now describe the indecomposable torsion free objects in $\mathcal{C}$ as well as the associated Auslander-Reiten quiver (see [5]). Using Proposition 4.1 this could be easily obtained by using a graded version of the results in [2]. However for completeness we give an independent proof here.

For $m > 0$ denote by F_{ma} the unique indecomposable projective R_m-module in with grading starting in degree $-a$ (thus $F_{ma} = F_{m0}(a)$). For $m = 0$ we let F^0_{00}, F^1_{00} be the two indecomposable R_0 modules whose gradings starts exactly at 0. We also put $F^i_{0a} = F^i_{00}(a)$ (as in Section §3).

Finally to simplify the notation we will write F^i_{ma} ($i = \emptyset$, if $m \neq 0$) for $U_m(F^i_{ma})$.

Proposition 5.1. *The indecomposable objects in $\mathcal{F}$ are given by F^i_{ma}. Furthermore the associated Auslander-Reiten quiver is given by Figure 1*

Proof. By Serre duality it follows that $\mathrm{Ext}^1_{\mathcal{C}}(F^i_{ma}, VF^i_{ma})$ is one dimensional. Therefore its unique (up to scalar multiplication) non-zero element represents the almost split sequence ending in F^i_{ma}.

Let us now explicitly construct non-split extensions between F^i_{ma} and VF^i_{ma}. First note

$$VF_{ma} = F_{m,a-1}$$

where for simplicity we have written $F_{0a} = F^0_{0a} \oplus F^1_{1a}$, and

$$VF^i_{0a} = F^{1-i}_{0a-1}$$

To construct the extension associated to $F_{m,a}$ we note that $F_{m-1,a-1}$ and $F_{m+1,a}$ are naturally submodules of $F_{m,a}$ whose sum is $F_{m,a}$ and whose intersection is $F_{m,a-1}$. Hence the exact sequence

$$0 \to F_{m,a-1} \to F_{m-1,a-1} \oplus F_{m+1,a} \to F_{m,a} \to 0 \tag{5.1}$$

yields the sought extension.

To construct the extension associated to F^i_{0a} we note that F_{1a} maps surjectively to F^i_{0a} with kernel F^{1-i}_{0a-1}. Thus in this case the sought extension is

$$0 \to F^{1-i}_{0,a-1} \to F_{1a} \oplus F^i_{0a} \to F_{m,a} \to 0 \tag{5.2}$$

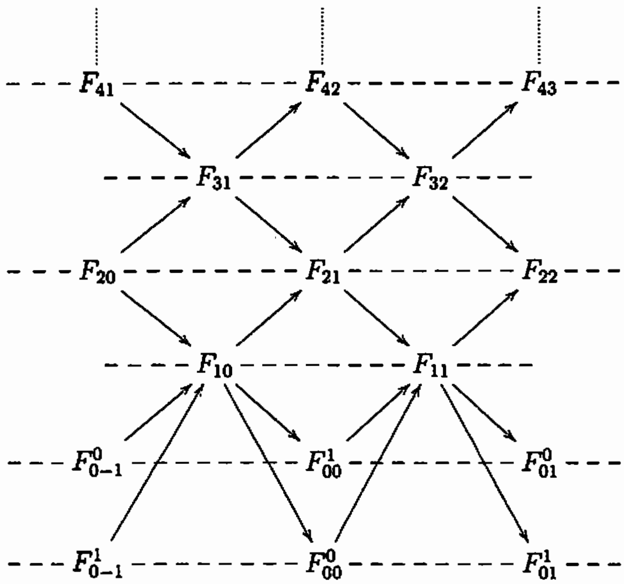

FIGURE 1. The Auslander-Reiten quiver of $\mathcal{F}$

It is now easy to assemble the almost split sequences given by (5.1) and (5.2) into the translation quiver given by Figure 1.

To show that Figure 1 is the entire AR-quiver of $\mathcal{F}$ (and not just a component) we have to show that there are no other indecomposable objects.

So assume that F is an indecomposable object in $\mathcal{F}$, not occurring among the F^i_{ma}. By lemma 3.4 there exist a non-zero map $F^i_{0a} \to F$ for some i, a. Using the defining property of AR-sequences we may use this to construct a non-zero map F^i_{mb} for some i (possibly $\emptyset$) and m, and for b arbitrarily large.

Now note that the only non-trivial torsion free quotients of F^i_{mb} are F^i_{mb} itself and $F^{0,1}_{0b}$ (if $m \neq 0$). Since all these quotients possess a non-trivial element in degree $-b$ it follows that $\operatorname{Hom}_{\mathcal{C}}(F^i_{mb}, F) = 0$ for $b \gg 0$. This finishes the proof. $\square$

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STABLE EQUIVALENCES INDUCED FROM GENERALIZED TILTING MODULES II [1]

TAKAYOSHI WAKAMATSU

1. INTRODUCTION

Let A and B be finite dimensional algebras over a field K. We suppose that ${}_BT_A$ is a generalized tilting module and (φ,ψ) an admissible system for a symmetric algebra, where ${}_AM\otimes_A M_A \xrightarrow{\varphi} {}_AM_A$ and ${}_AM\otimes_A M_A \xrightarrow{\psi} {}_A\mathrm{D}A_A$. Then the transformed system (φ^T,ψ^T) is defined over the bimodule ${}_BM^T_B = {}_BT\otimes_A \mathrm{Hom}_A(T,M)_B$ and we have two symmetric algebras $\Lambda(\varphi,\psi) = A\oplus M\oplus \mathrm{D}A$ and $\Lambda(\varphi^T,\psi^T) = B\oplus M^T\oplus \mathrm{D}B$ under the assumption (1) the canonical map ${}_BT\otimes_A\mathrm{Hom}_A(T,M)_B \xrightarrow{\theta} {}_B\mathrm{Hom}_A(T,T\otimes_A M)_B$ defined by $\theta(t\otimes f)(t') = t\otimes f(t')$ is bijective. In the previous note[2], we have shown the existence of a stable equivalence

$$\mathcal{S}:\underline{\mathrm{mod}}-\Lambda(\varphi,\psi)\ \approx\ \underline{\mathrm{mod}}-\Lambda(\varphi^T,\psi^T)$$

by using the assumptions (2) the class $\mathcal{C}(T_A) = \mathrm{gen}^*(T_A)\cap\bigcap_{n>0}\mathrm{KerExt}^n_A(T,?)$ is contravariantly finite in mod$-A$ and, dually, $\mathcal{D}(\mathrm{D}T_B) = \mathrm{cog}^*(\mathrm{D}T_B)\cap\bigcap_{n>0}\mathrm{KerExt}^n_B(?,\mathrm{D}T)$ covariantly finite in mod$-B$, and (3) the modules M_A and $T\otimes_A M_A$ are in the class $\mathcal{C}(T_A)$. Those assumptions (1) to (3) are satisfied if we suppose

(a) the module ${}_AM_A$ is of the form $\bigoplus_{(X,Y)} {}_AX\otimes_K Y_A$ with all Y_A's are in the class $\mathcal{C}(T_A)$, and

(b) one of the algebras A and B is representation-finite.

The purpose of the present note is to give an example of a couple of an admissible system (φ,ψ) and a generalized tilting module ${}_BT_A$ for which the symmetric algebras $\Lambda(\varphi,\psi)$ and $\Lambda(\varphi^T,\psi^T)$ are stably equivalent but not derived equivalent. Such an example means that our stable equivalence $\mathcal{S}$ is not induced from Morita theory of derived categories.

2. AN EXAMPLE

Define an algebra A by the quiver

$$\mathrm{Q}(A):\quad \begin{array}{ccc} & \beta & \\ 1 & \rightarrow & 2 \\ \circlearrowleft & & \circlearrowright \\ \alpha & & \gamma \end{array}$$

with the relations $\alpha^2=0$, $\gamma^2=0$, $\beta\cdot\alpha=0$ and $\gamma\cdot\beta=0$. It is checked that the algebra A is representation-finite with only eight non-isomorphic indecomposable modules. We also

[1]The detailed version of this paper will be submitted for publication elsewhere.

have

$$A_A = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \oplus \begin{pmatrix} & 2 & \\ 1 & & 2 \end{pmatrix} \quad \text{and} \quad \mathrm{D}A_A = \begin{pmatrix} 1 & & 2 \\ & 1 & \end{pmatrix} \oplus \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

Choose a generalized tilting module as $T_A = \begin{pmatrix} 1 & & 2 & \\ & 1 & & 2 \end{pmatrix} \oplus \begin{pmatrix} 2 \\ 2 \end{pmatrix}$. Then, the quiver $\mathrm{Q}(B)$ of $B = \mathrm{End}(T_A)$ is given by $\underset{\circlearrowleft}{1} \leftrightarrows 2$ and we have

$$B_B = \begin{pmatrix} & 1 & & \\ 1 & & 2 & \\ & & & 1 \end{pmatrix} \oplus \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathrm{D}B_B = \begin{pmatrix} 2 & & & & \\ & 1 & & & \\ & & 2 & & 1 \\ & & & 1 & \end{pmatrix} \oplus \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}.$$

Now, we set ${}_AM_A = {}_AAe_1 \otimes_K e_1\mathrm{D}A_A = \begin{pmatrix} & 1 & \\ 1 & & 2 \end{pmatrix} \oplus \begin{pmatrix} 1 & & 2 \\ & 1 & \end{pmatrix}$ and $\varphi = 0$ the zero map from ${}_AM \otimes_A M_A$ to ${}_AM_A$. Since our module ${}_AM_A$ is canonically isomorphic to its dual ${}_A\mathrm{D}M_A$, we have a map $\psi : {}_AM \otimes_A M_A \to {}_A\mathrm{D}A_A$ and (φ, ψ) becomes an admissible system for a symmetric algebra. Then, the assumptions (a) and (b) are satisfied and, therefore, the symmetric algebras $\Lambda = \Lambda(\varphi, \psi)$ and $\Gamma = \Lambda(\varphi^T, \psi^T)$ are stably equivalent.

In order to prove that the algebras Λ and Γ are not derived equivalent, we use the following well-known result. The proof can be seen in the paper [1] by Usami.

Lemma 1. *If the algebras Λ and Γ are derived equivalent, there exists a regular matrix $P \in \mathrm{Mat}_n(\mathbb{Z})$ and their Cartan matrices satisfy the equation ${}^tP \cdot \mathrm{C}_\Lambda \cdot P = \mathrm{C}_\Gamma$.*

We have $\mathrm{C}_\Lambda = \begin{pmatrix} 8 & 3 \\ 3 & 5 \end{pmatrix}$ since

$$e_1\Lambda_A = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \oplus \left\{ \begin{pmatrix} 1 & & 2 \\ & 1 & \end{pmatrix} \oplus \begin{pmatrix} 1 & & 2 \\ & 1 & \end{pmatrix} \right\} \oplus \begin{pmatrix} 1 & & 2 \\ & 1 & \end{pmatrix}$$

and

$$e_2\Lambda_A = \begin{pmatrix} & 2 & \\ 1 & & 2 \end{pmatrix} \oplus \left\{ \begin{pmatrix} 1 & & 2 \\ & 1 & \end{pmatrix} \right\} \oplus \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

Similarly, we have $\mathrm{C}_\Gamma = \begin{pmatrix} 10 & 3 \\ 3 & 4 \end{pmatrix}$ from

$$f_1\Gamma_B = \begin{pmatrix} & 1 & & \\ 1 & & 2 & \\ & & & 1 \end{pmatrix} \oplus \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \oplus \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \oplus \begin{pmatrix} & & & & 2 \\ & & & 1 & \\ 1 & & 2 & & \\ & 1 & & & \end{pmatrix}$$

and

$$f_2\Gamma_B = \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix} \oplus \{0\} \oplus \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix},$$

where e_i (resp. f_i) stands for the primitive idempotent element in the algebra A (resp. B) corresponding to the vertex in $\mathrm{Q}(A)$ (resp. $\mathrm{Q}(B)$) indexed by i and n is the common number of non-isomorphic simple A- or B-modules.

Put $P = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{Z})$, then we have

$$ {}^tP \cdot \mathrm{C}_\Lambda \cdot P = \begin{pmatrix} 8a^2 + 6ab + 5b^2 & 8ac + 3(ad + bc) + 5bd \\ 8ac + 3(ad + bc) + 5bd & 8c^2 + 6cd + 5d^2 \end{pmatrix}. $$

Hence, ${}^tP \cdot \mathrm{C}_\Lambda \cdot P = \mathrm{C}_\Gamma$ implies that

$$ 5c^2 + 3(c + d)^2 + 2d^2 = 8c^2 + 6cd + 5d^2 = 4, $$

and this is impossible for integers $c, d \in \mathbb{Z}$. Therefore, the algebras Λ and Γ are not derived equivalent by the previous lemma.

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STABILITY OF FROBENIUS ALGEBRAS WITH POSITIVE GALOIS COVERINGS [1]

KUNIO YAMAGATA [2]

ABSTRACT. A finite dimensional self-injective algebra will be determined when it is stably equivalent to a positive self-injective algebra of Dynkin, tilted or quasi-tilted type.

1. INTRODUCTION

In a series of joint work with A. Skowroński, we have been studying self-injective algebras with Galois coverings by repetitive algebras [10] [11] [12] [13], and [4]. One of the aims of the work is to characterize self-injective algebras ring theoretically or module categorically which have Galois coverings by the repetitive algebras. A ring theoretical criterion theorem was found in [11] for a self-injective algebra to have a Galois covering by a repetitive algebra. On the other hand, study of the module category over a self-injective algebra depends on the type of a repetitive algebra which defines a Galois covering. In this survey paper, some of main theorems related to module categories in the joint work are arranged into two theorems (Theorems 2, 3). The proofs, however, are not unified (and refer the proofs in the references mentioned above), but a common idea of the proofs is to find an ideal so that a criterion theorem is applicable to conclude the existence of a Galois covering by a repetitive algebra.

Throughout this paper, K will be a fixed (commutative) field, and by an algebra we mean a basic and associative K-algebra which is not necessarily finite dimensional, but with a complete set of orthogonal primitive idempotents, that is, with a set of orthogonal primitive idempotents, say $\{e_i\}_{i\in I}$ of an algebra Λ, such that $\Lambda = \bigoplus_{i\in I} \Lambda e_i = \bigoplus_{i\in I} e_i\Lambda$. A K-category R is a category whose hom-sets are K-spaces and composition of morphisms are K-bilinear. Then, K-categories (finite K-categories, respectively) are in one-to-one correspondence with algebras having fixed complete set of orthogonal primitive idempotents (finite dimensional algebras with identity, respectively). We freely identify K-categories with K-algebras. A will be a finite dimensional, connected *self-injective* algebra, and B and Λ will be finite dimensional algebras. Two finite dimensional self-injective algebras A_1, A_2 are said to be *socle equivalent* if $A_1/\operatorname{soc} A_1 \cong A_2/\operatorname{soc} A_2$ where soc is the socle of an algebra, and *stably equivalent* if their stable categories $\underline{\operatorname{mod}}A_1$ and $\underline{\operatorname{mod}}A_2$ of the categories of finite dimensional modules are isomorphic.

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2. Repetitive categories

2.1. Let B be a K-category, and $B_n = B$, $DB_n = DB$ the copies of B and DB, respectively. The *repetitive category* $\widehat{B}$ of B is the direct sum of K-spaces $\widehat{B} = \bigoplus_{n \in \mathbb{Z}}(B_n \oplus DB_n)$ with multiplication:

$$\left(\sum_i (b_i, f_i)\right) \cdot \left(\sum_j (c_j, g_j)\right) = \sum_i (b_i c_i, b_i g_i + f_i c_{i+1})$$

for $b_i, c_j \in B_i$ and $f_i, g_j \in DB_i$ for $i \in \mathbb{Z}$. The category may be written as a matrix algebra without identity in the following way:

$$\widehat{B} = \begin{pmatrix} \ddots & \ddots & & & 0 \\ & B_{n-1} & DB_{n-1} & & \\ & & B_n & DB_n & \\ & & & B_{n+1} & \ddots \\ 0 & & & & \ddots \end{pmatrix}$$

where all B_n lie on the diagonal, and $\widehat{B}$ is the set of those matrices that all but finitely many entries are zero. Summation and multiplication are defined as ones of matrices with $DB \otimes DB \to 0$ zero map. $\widehat{B}$ is a K-category whose object set is the disjoint union of copies of $Obj(B)$ in each B_n.

2.2. A group G of automorphisms of a K-category $\widehat{B}$ is said to be *admissible* if any automorphism of G acts freely on $Obj(\widehat{B})$, and the G-orbit sets of $Obj(\widehat{B})$ is finite. The category $\widehat{B}/G$ is then naturally defined by the G-orbits of $\widehat{B}$-objects and $\widehat{B}$-morphisms. See [1] for details. The orbit category $\widehat{B}/G$ is clearly finite dimensional and self-injective.

An automorphism $\nu_{\widehat{B}}$ of $\widehat{B}$ is called the *Nakayama automorphism* of $\widehat{B}$ if the restriction to any $B_n \oplus DB_n$ is identity onto $B_{n+1} \oplus DB_{n+1}$. An automorphism φ of $\widehat{B}$ is said to be *positive* if $\varphi(B_n) \subseteq \sum_{i \geq n}(B_i \oplus DB_i)$ $(n \in \mathbb{Z})$ or, equivalently, $\varphi(Obj(B_n)) \subseteq \bigcup_{i \geq n} Obj(B_i)$ $(n \in \mathbb{Z})$, and *strictly positive* if φ is positive and $\varphi(B_n) \neq B_n$ for all n.

Importance of the orbit categories by repetitive categories may be suggested by the classification theorem of representation-finite self-injective algebras over an algebraically closed field, which was proved by C. Riedtmann [8] [9] (and also by D. Hughes - J. Waschbüsch [3]). Also, D. Happel (1991) showed that the bounded category $D^b(\operatorname{mod} B)$ of a finite dimensional algebra B is embedded into the stable category $\underline{\operatorname{mod}}\widehat{B}$ of the category of finite dimensional modules over $\widehat{B}$. Moreover, $D^b(\operatorname{mod} B)$ is isomorphic to $\underline{\operatorname{mod}}\widehat{B}$ if and only if $\operatorname{gldim} B < \infty$.

2.3. (1) A typical example of the orbit algebras by repetitive algebras is $\widehat{B}/\langle \nu_{\widehat{B}} \rangle$ which is isomorphic to the trivial extension algebra $B \ltimes DB$.

(2) Let A be 4-dimensional, local and self-injective algebras such that

$$A \cong \widehat{B}/\langle \varphi \rangle; \; B = k[x]/(x^2), \; \varphi^2 = \psi \nu_{\widehat{B}}$$

for a positive automorphisms φ, ψ, and they are socle equivalent. Those algebras were first presented by T. Nakayama and C. Nesbitt in 1938 [5]. J. Rickard showed that such algebras are isomorphic if they are stably equivalent (see [15] [6]). Thus it should be noted that a socle equivalence does not imply a stable equivalence, in general.

3. Type of self-injective algebras

3.1. An algebra B is called *quasi-tilted algebra* if $B \cong \mathrm{End}_{\mathcal{H}}(T)$ where T is a tilting object in a hereditary abelian K-category $\mathcal{H}$, that is, $\mathrm{Hom}_{\mathcal{H}}(X,Y)$ and $\mathrm{Ext}^1_{\mathcal{H}}(X,Y)$ are finite dimensional and $\mathrm{Ext}^2_{\mathcal{H}}(X,Y) = 0$ for all objects X, Y of $\mathcal{H}$. This is equivalent to the statement that $\mathrm{gldim}(B) \leq 2$ and every finitely generated indecomposable B-module X has $\mathrm{pd}(X) \leq 1$ or $\mathrm{id}(X) \leq 1$. An algebra is said to be *canonical* [7] if its ordinary quiver has a unique source ω and a unique sink o, and consists of paths $p_1, \ldots, p_n$ $(n \geq 2)$ from ω to o, which meet each other only at ω and o. Moreover, in case $n = 2$ there is no non-trivial relation (i.e., the algebra is hereditary), and in case $n \geq 3$ the length of each path is more than 1 and relations are

$$p_1 + p_2 + p_3 = 0, \;\; p_1 + \lambda_i p_2 + p_i = 0 \;\; (i = 4, \ldots, n)$$

for $\lambda_i \neq 0, 1$ and $\lambda_i \neq \lambda_j$ for all $i \neq j$.

Theorem 1. [2] *An algebra B is quasi-tilted if and only if $B \cong \mathrm{End}_{\Lambda}(T)$ where T is a tilting module over a hereditary algebra or a canonical algebra Λ.*

3.2. Let A be a self-injective algebra. A is said to be *positive* (*strictly positive*, respectively) if $A \cong \widehat{B}/(\varphi\nu_{\widehat{B}})$ for some algebra B and *positive* (*strictly positive*, respectively) automorphism φ of $\widehat{B}$. (See [16].) A positive self-injective algebra A is said to be of

(i) (Δ-)*tilted type* if B is a tilted algebra of type Δ,

(ii) *canonical type* if B is the endomorphism algebra of a tilting module over a *canonical* algebra,

(iii) *quasi-tilted type* if B is a quasi-tilted algebra.

Thus, by Theorem 1, a positive self-injective algebra A is of quasi-tilted type if and only if A is of tilted type or canonical type.

4. Main Theorems

4.1. Some of main results on the module categories in [10] [12] [13] [4] are stated as follows.

Theorem 2. *A self-injective algebra A is stably equivalent to a positive self-injective algebra of Δ-tilted type if and only if A is socle equivalent to a positive self-injective algebra of Δ-tilted type. Moreover, in case K is algebraically closed, those statements are equivalent to the statement that A is isomorphic to a positive self-injective algebra of Δ-tilted type.*

Theorem 3. *The followings are equivalent.*

(1) *A is stably equivalent to a strongly positive self-injective algebra of quasi-tilted type.*

(2) *A is isomorphic to a strongly positive self-injective algebra of quasi-tilted type.*

4.2. Theorem 2 was first proved in [10] for algebras of non-Dynkin type, and later in [13] for algebras of Dynkin type ($\Delta = A_n, B_n, C_n, D_n \ (n \geq 4), E_6, E_7, E_8, F_4, G_2$). On the other hand, Theorem 3 was proved in [12] for non-Dynkin type, and recently in [4] for canonical type where a new characterization of a quasi-tilted algebra is given and a precise observation is required for the form of Auslander-Reiten components.

In any case, however, we have to find an ideal I satisfying the annihilator condition $r_A(I) = eI$ for some $e = e^2$, and then, apply the following criterion theorem for the existence of a positive Galois covering.

Criterion Theorem 4 . [11] *A self-injective algebra Λ is positive if there is an ideal I of Λ such that, for some idempotent e of Λ, the following conditions are satisfied:*
(1) $r_\Lambda(I) = eI$,
(2) *The canonical algebra epimorphism $e\Lambda e \to e\Lambda e/eIe$ splits.*
In this case, Λ is isomorphic to $\widehat{B}/(\varphi\nu_{\widehat{B}})$ for $B = \Lambda/I$ and a positive automorphism φ of $\widehat{B}$.

The idea of the application of the criterion theorem is to construct another self-injective algebra Λ by making use of the annihilator condition (1) (see [11]), so that Λ has both conditions (1) and (2), and A and Λ are socle equivalent. Thus we can know the existence of a positive Galois covering of A *up to socle*. The converse of Theorem 4 is also true [14] and refer to the survey paper [16].

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FROBENIUS AND QUASI-FROBENIUS PROPERTY FOR mod $\underline{\mathcal{C}}$

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§1. Introduction

This is a survey report of my recent work [8], and we shall omit every proofs of the result in this monograph. The reader should refer to the original paper [8] [1] for the detail.

In the following R always denotes a commutative Noetherian ring, and $\mathrm{mod}\,R$ is the category of finitely generated R-modules. We are interested in the subcategories $\mathcal{G}$ and $\mathcal{H}$ of $\mathrm{mod}\,R$ that are defined as follows:

Definition 1. $\mathcal{G}$ is defined to be the full subcategory of $\mathrm{mod}\,R$ consisting of all modules $X \in \mathrm{mod}\,R$ that satisfy

$$\mathrm{Ext}^i_R(X, R) = 0 \quad \text{and} \quad \mathrm{Ext}^i_R(\mathrm{Tr}X, R) = 0 \quad \text{for any} \quad i > 0.$$

We also define $\mathcal{H}$ to be the full subcategory consisting of all modules with the first half of the above conditions, therefore a module $X \in \mathrm{mod}\,R$ is an object in $\mathcal{H}$ if and only if

$$\mathrm{Ext}^i_R(X, R) = 0 \quad \text{for any} \quad i > 0.$$

Note that $\mathcal{G} \subseteq \mathcal{H}$ and that $\mathcal{G}$ is called the subcategory of modules of G-dimension zero. See [2] for the G-dimension of modules.

Recently, D.Jorgensen and L.M.Sega [5] reported that they constructed an example of an artinian ring R, on which $\mathcal{G} \neq \mathcal{H}$. However, we still expect that the equality $\mathcal{G} = \mathcal{H}$ holds in many cases.

The main purpose of this paper is to characterize functorially these two subcategories and to get the conditions under which a subcategory $\mathcal{C}$ of $\mathcal{H}$ is contained in $\mathcal{G}$.

First we settle the notation which we shall use later. When we say $\mathcal{C}$ is a subcategory of $\mathrm{mod}\,R$, we always mean the following:

- $\mathcal{C}$ is essential in $\mathrm{mod}\,R$, i.e. if $X \cong Y$ in $\mathrm{mod}\,R$ and if $X \in \mathcal{C}$, then $Y \in \mathcal{C}$.
- $\mathcal{C}$ is full in $\mathrm{mod}\,R$, i.e. $\mathrm{Hom}_{\mathcal{C}}(X, Y) = \mathrm{Hom}_R(X, Y)$ for $X, Y \in \mathcal{C}$.
- $\mathcal{C}$ is additive and additively closed in $\mathrm{mod}\,R$, i.e. for any $X, Y \in \mathrm{mod}\,R$, $X \oplus Y \in \mathcal{C}$ if and only if $X \in \mathcal{C}$ and $Y \in \mathcal{C}$.
- $\mathcal{C}$ contains all projective modules in $\mathrm{mod}\,R$.

Of course $\mathcal{G}$ and $\mathcal{H}$ are subcategories in this sense.

Let $\mathcal{C}$ be any subcategory of $\mathrm{mod}\,R$. As in the general notation we denote the associated stable category by $\underline{\mathcal{C}}$. Of course, there is a natural functor $\mathcal{C} \to \underline{\mathcal{C}}$. We should note that the transpose Tr and the syzygy Ω are well-defined functors over tha stable category $\underline{\mathcal{C}}$:

$$\mathrm{Tr} : (\underline{\mathcal{C}})^{op} \to \underline{\mathrm{mod}\,R} \qquad \Omega : \underline{\mathcal{C}} \to \underline{\mathrm{mod}\,R}.$$

[1] The detailed version [8] of this paper has been submitted for publication elsewhere.

We also note just from the definition that Tr gives dualities on $\underline{\mathcal{G}}$ and also on $\underline{\mathrm{mod}R}$.

For an additive category $\mathcal{A}$, a contravariant additive functor from $\mathcal{A}$ to the category (Ab) of abelian groups is referred to as an $\mathcal{A}$-module, and a natural transform between two $\mathcal{A}$-modules is referred to as an $\mathcal{A}$-module morphism. We denote by $\mathrm{Mod}\mathcal{A}$ the category consisting of all $\mathcal{A}$-modules and all $\mathcal{A}$-module morphisms. Note that $\mathrm{Mod}\mathcal{A}$ is obviously an abelian category. An $\mathcal{A}$-module F is called finitely presented if there is an exact sequence

$$\mathrm{Hom}_{\mathcal{A}}(\ ,X_1) \to \mathrm{Hom}_{\mathcal{A}}(\ ,X_0) \to F \to 0,$$

for some $X_0, X_1 \in \mathcal{A}$. We denote by $\mathrm{mod}\mathcal{A}$ the full subcategory of $\mathrm{Mod}\mathcal{A}$ consisting of all finitely presented $\mathcal{A}$-modules.

It follows easily from Yoneda's lemma that an $\mathcal{A}$-module is projective in $\mathrm{mod}\mathcal{A}$ if and only if it is isomorphic to $\mathrm{Hom}_{\mathcal{A}}(\ ,X)$ for some $X \in \mathcal{A}$. Also note that the functor $\mathcal{A}$ to $\mathrm{mod}\mathcal{A}$ which sends X to $\mathrm{Hom}_{\mathcal{A}}(\ ,X)$ is a full embedding.

Now let $\mathcal{C}$ be a subcategory of $\mathrm{mod}R$ and let $\underline{\mathcal{C}}$ be the associated stable category. Then the category of finitely presented $\mathcal{C}$-modules $\mathrm{mod}\mathcal{C}$ and the category of finitely presented $\underline{\mathcal{C}}$-modules $\mathrm{mod}\underline{\mathcal{C}}$ are defined as in the above course. Note that for any $F \in \mathrm{mod}\mathcal{C}$ (resp. $G \in \mathrm{mod}\underline{\mathcal{C}}$) and for any $X \in \mathcal{C}$ (resp. $\underline{X} \in \underline{\mathcal{C}}$), the abelian group $F(X)$ (resp. $G(\underline{X})$) has naturally an R-module structure, hence F (resp. G) is in fact a contravariant additive functor from $\mathcal{C}$ (resp. $\underline{\mathcal{C}}$) to $\mathrm{mod}R$. As we stated above there is a natural functor $\mathcal{C} \to \underline{\mathcal{C}}$. We can define from this the functor $\iota : \mathrm{mod}\underline{\mathcal{C}} \to \mathrm{mod}\mathcal{C}$ by sending $F \in \mathrm{mod}\underline{\mathcal{C}}$ to the composition functor of $\mathcal{C} \to \underline{\mathcal{C}}$ with F. Then it is well known and is easy to prove that ι gives an equivalence of categories between $\mathrm{mod}\underline{\mathcal{C}}$ and the full subcategory of $\mathrm{mod}\mathcal{C}$ consisting of all finitely presented $\mathcal{C}$-modules F with $F(R) = 0$.

§2. Frobenius property of $\mathrm{mod}\underline{\mathcal{G}}$

Let $\mathcal{C}$ be a subcategory of $\mathrm{mod}R$. We say that $\mathcal{C}$ is closed under kernels of epimorphisms if it satisfies the following condition:

> If $0 \to X \to Y \to Z \to 0$ is an exact sequence in $\mathrm{mod}R$, and if $Y, Z \in \mathcal{C}$, then $X \in \mathcal{C}$.

(In Quillen's terminology, all epimorphisms from $\mathrm{mod}R$ in $\mathcal{C}$ are admissible.)

We say that $\mathcal{C}$ is closed under extension or extension-closed if it satisfies the following condition:

> If $0 \to X \to Y \to Z \to 0$ is an exact sequence in $\mathrm{mod}R$, and if $X, Z \in \mathcal{C}$, then $Y \in \mathcal{C}$.

A subcategory $\mathcal{C}$ is said to be a resolving subcategory if it is extension-closed and closed under kernels of epimorphisms. Also $\mathcal{C}$ is said to be closed under Ω if it satisfies that $\Omega X \in \underline{\mathcal{C}}$ whenever $X \in \mathcal{C}$. Similarly to this, the closedness under Tr is defined.

Note that the categories $\mathcal{G}$ and $\mathcal{H}$ are resolving subcategories and that $\mathcal{G}$ is closed under Tr.

We note that, if a subcategory $\mathcal{C}$ of $\mathrm{mod}R$ is closed under kernels of epimorphisms, then it is closed under Ω. And if a subcategory $\mathcal{C}$ of $\mathrm{mod}R$ is extension-closed and closed under Ω, then it is resolving.

The following proposition is shown straightforward from the definitions. Note that the proof of the proposition is completely similar to that of [7, Lemma (4.17)], in which it is proved that mod$\underline{\mathcal{C}}$ is an abelian category when R is a Cohen-Macaulay local ring and $\mathcal{C}$ is the category of maximal Cohen-Macaulay modules.

Proposition 2. *Let $\mathcal{C}$ be a subcategory of* mod$R$ *which is closed under kernels of epimorphisms. Then* mod$\underline{\mathcal{C}}$ *is an abelian category with enough projectives.*

A category $\mathcal{A}$ is said to be a Frobenius category if it is an abelian category with enough projectives and with enough injectives, and if the class of projective objects in $\mathcal{A}$ coincides with the clasee of injective objects in $\mathcal{A}$. Likewise, a category $\mathcal{A}$ is said to be a quasi-Frobenius category if it is an abelian category with enough projectives and all projective objects in $\mathcal{A}$ are injective.

The following theorem is the first result I have got and that motivated me to the detail study on the category mod$\underline{\mathcal{C}}$.

Theorem 3. *Let $\mathcal{C}$ be a subcategory of* mod$R$ *that is closed under kernels of epimorphisms. If $\mathcal{C} \subseteq \mathcal{H}$ then* mod$\underline{\mathcal{C}}$ *is a quasi-Frobenius category.*

The proof of the theorem is not difficult. It is enough to notice that the injective objects in mod$\underline{\mathcal{C}}$ are nothing but half-exact functors as a functor on $\mathcal{C}$. See [8, Theorem 3.5].

Theorem 4. *Let $\mathcal{C}$ be a subcategory of* mod$R$. *And suppose the following conditions.*

(1) *$\mathcal{C}$ is a resolving subcategory of* mod$R$.

(2) $\mathcal{C} \subseteq \mathcal{H}$.

(3) *The functor $\Omega : \underline{\mathcal{C}} \to \underline{\mathcal{C}}$ yields a surjective map on the set of isomorphism classes of the objects in $\underline{\mathcal{C}}$.*

Then mod$\underline{\mathcal{C}}$ *is a Frobenius category. In particular,* mod$\underline{\mathcal{G}}$ *is a Frobenius category.*

From the third assumption in the theorem, the syzygy functor Ω gives an automorphism on the category $\underline{\mathcal{C}}$, hence there exists a cosyzygy functor Ω^{-1}. Using this fact we can easily prove the theorem as in the same course of the proof of the previous theorem.

Now let us consider the following four conditions for a resolving subcategory $\mathcal{C}$ of mod$R$:

(A) $\mathcal{C}$ is a subcategory of $\mathcal{H}$.

(B) mod$\underline{\mathcal{C}}$ is a quasi-Frobenius category.

(C) mod$\underline{\mathcal{C}}$ is a Frobenius category.

(D) $\mathcal{C}$ is a subcategory of $\mathcal{G}$.

Then, the following implications hold:

$$(A) \Longrightarrow (B) \Longleftarrow (C) \Longleftarrow (D)$$

The first implication follows from Theorem 3 and the third will follow from 4 (under a suitable condition on syzygy functor). Of course it is obvious that second implication always holds. Our program is that we analyze closely the reverse implications. Actually, in the next section we shall show that

- $(B) \Rightarrow (A)$ holds if R is a henselian local ring.
- $(C) \Rightarrow (D)$ holds under the validity of the Auslander-Reiten conjecture.
- $(B) \Rightarrow (C)$ holds if $\mathcal{C}$ is of finite type (by Nakayama Theorem).

§3. Main theorems

In this section we always assume that R is a henselian local ring with maximal ideal $\mathfrak{m}$ and with the residue class field $k = R/\mathfrak{m}$. In the following, what we shall need from this assumption is the fact that $X \in \operatorname{mod} R$ is indecomposable only if $\operatorname{End}_R(X)$ is a (noncommutative) local ring. In fact it is easy to see that $\operatorname{mod}\underline{\mathcal{C}}$ is a Krull-Schmidt category for any subcategory $\mathcal{C} \subseteq \operatorname{mod} R$.

We can prove the converse of Theorem 3 under this assumption.

Theorem 5. *Let $\mathcal{C}$ be a resolving subcategory of* $\operatorname{mod} R$*, where R is a henselian local ring. Suppose that* $\operatorname{mod}\underline{\mathcal{C}}$ *is a quasi-Frobenius category. Then $\mathcal{C} \subseteq \mathcal{H}$.*

In a sense $\mathcal{H}$ is the largest resolving subcategory $\mathcal{C}$ of $\operatorname{mod} R$ for which $\operatorname{mod}\underline{\mathcal{C}}$ is a quasi-Frobenius category.

The proof of this theorem is not so easy. Essential part of the proof is to show that if $\operatorname{mod}\underline{\mathcal{C}}$ is quasi-Frobenius, then any object $X \in \mathcal{C}$ satisfies $\operatorname{Ext}^1_R(X, R) = 0$. The reader should refer to the paper [8, Theorem 4.2] for the complete proof.

As to the implication $(C) \Longrightarrow (D)$ in the last paragraph of the previous section, we can show the following result.

Theorem 6. *Let R be a henselian local ring as above. Suppose that*

(1) *$\mathcal{C}$ is a resolving subcategory of* $\operatorname{mod} R$.
(2) $\operatorname{mod}\underline{\mathcal{C}}$ *is a Frobenius category.*
(3) *There is no nonprojective module $X \in \mathcal{C}$ with* $\operatorname{Ext}^1_R(\ \ , X)|_{\underline{\mathcal{C}}} = 0$.

Then $\mathcal{C} \subseteq \mathcal{G}$.

Remark 7. We conjecture that $\mathcal{G}$ should be the largest resolving subcategory $\mathcal{C}$ of $\operatorname{mod} R$ such that $\operatorname{mod}\underline{\mathcal{C}}$ is a Frobenius category.

Theorem 6 together with Theorem 4 say that this is true modulo Auslander-Reiten conjecture:

(AR) If $\operatorname{Ext}^i_R(X, X \oplus R) = 0$ for any $i > 0$ then X should be projective.

In fact, if the conjecture (AR) is true, then the third assumption of Thereom 6 is automatically satisfied.

The proof of Theorem 6 is not short, and we restrict ourselves to say that the following lemma is essential in its proof.

Lemma 8. *Let R be a henselian local ring and let $\mathcal{C}$ be an extension-closed subcategory of* $\operatorname{mod} R$*. For objects $\underline{X}, \underline{Y} \in \underline{\mathcal{C}}$, we assume the following:*

(1) *There is a monomorphism φ in* $\operatorname{Mod}\underline{\mathcal{C}}$*:*

$$\varphi : \underline{\operatorname{Hom}}_R(\ \ , Y)|_{\underline{\mathcal{C}}} \to \operatorname{Ext}^1(\ \ , X)|_{\underline{\mathcal{C}}}$$

(2) *$\underline{X}$ is indecomposable in $\underline{\mathcal{C}}$.*
(3) *$\underline{Y} \not\cong 0$ in $\underline{\mathcal{C}}$.*

Then the module X is isomorphic to a direct summand of ΩY.

Let $\mathcal{A}$ be any additive category. We denote by $\operatorname{Ind}(\mathcal{A})$ the set of nonisomorphic modules which represent all the isomorphism classes of indecomposable objects in $\mathcal{A}$. If $\operatorname{Ind}(\mathcal{A})$ is a finite set, then we say that $\mathcal{A}$ is a category of finite type. The following theorem is a main theorem of the paper [8], which claims that any resolving subcategory of finite type in $\mathcal{H}$ are contained in $\mathcal{G}$. See [8, Theorem 5.5]

Theorem 9. *Let R be a henselian local ring and let $\mathcal{C}$ be a subcategory of* $\operatorname{mod} R$ *which satisfies the following conditions.*

(1) *$\mathcal{C}$ is a resolving subcategory of* $\operatorname{mod} R$.
(2) $\mathcal{C} \subseteq \mathcal{H}$.
(3) *$\mathcal{C}$ is of finite type.*

Then, $\operatorname{mod}\underline{\mathcal{C}}$ *is a Frobenius category and* $\mathcal{C} \subseteq \mathcal{G}$.

We should remark about the proof of Theorem 9. Since we assume that $\mathcal{C}$ is of finite type, the category $\operatorname{mod}\underline{\mathcal{C}}$ is isomorphic to the module category of certain artinian algebra A that is called the Auslander algebra of $\underline{\mathcal{C}}$:

$$\operatorname{mod}\underline{\mathcal{C}} \;\cong\; \operatorname{mod} A$$

Note that the ring A is a finite (noncommutative) algebra over a commutative artinian ring. Since we assume that $\mathcal{C} \subseteq \mathcal{H}$, we know that $\operatorname{mod}\underline{\mathcal{C}}$, hence $\operatorname{mod} A$, is a quasi-Frobenius category. (See Theorem 3.) This means that the artinian ring A is left selfinjective. It is known by Nakayam's Theorem (cf. [6] for example) that A is right selfinjective as well, and therefore, using the duality between $\operatorname{mod} A$ and $\operatorname{mod} A^{op}$, we can conclude that $\operatorname{mod} A$, hence $\operatorname{mod}\underline{\mathcal{C}}$, is a Frobenius category.

To prove that $\mathcal{C} \subseteq \mathcal{G}$ in the theorem, we use Theorem 6. Actually, since we have shown that $\operatorname{mod}\underline{\mathcal{C}}$ is a Frobenious category, it is enough to check the following statement:

(*) If $X \in \mathcal{C}$ such that $\underline{X} \not\cong \underline{0}$ in $\underline{\mathcal{C}}$, then we have $\operatorname{Ext}^1_R(\ \ ,X)|_{\underline{\mathcal{C}}} \neq 0$.

This can be proved by using the same idea of Nakayama which we can see in the monograph of Yamagata [6].

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