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Hideto Asashiba
Osaka City University

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編集：浅芝秀人 (大阪市立大学)

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大阪市立大学
Organizing Committee of The Symposium on
Ring Theory and Representation Theory

The Symposium on Ring Theory and Representation Theory has been held annually in Japan and the Proceedings have been published by the organizing committee. The first Symposium was organized in 1968 by H. Tominaga, H. Tachikawa, M. Harada and S. Endo. After their retirement, a new committee was organized in 1997 for managing the Symposium. The present members of the committee are Y. Hirano (Okayama Univ.), Y. Iwanaga (Shinshu Univ., 1997–2004), S. Koshitani (Chiba Univ.), K. Nishida (Shinshu Univ.) and M. Sato (Yamanashi Univ., 2004–).

The Proceedings of each Symposium is edited by program organizer. Anyone who wants these Proceedings should ask to the program organizer of each Symposium or one of the committee members.

The Symposium in 2005 will be held at Aichi University of Technology in Aichi Prefecture for Sep. 2–4, and the program will be arranged by T. Wakamatsu (Saitama Univ.).

Concerning several information on ring theory group in Japan containing schedules of meetings and symposiums as well as the addresses of members in the group, you should refer the following homepage, which is arranged by M. Sato (Yamanashi Univ.):

civil2.cec.yamanashi.ac.jp/ ring/japan/  (in English)

Yasuo Iwanaga
Nagano, Japan
December, 2004
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PREFACE

The 37th Symposium on Ring Theory and Representation Theory was held at Matsumoto City M-Wing from September 3 to 5 in 2004. The symposium and the proceedings are financially supported by Kenji Nishida (Shinshu University) Grant-in-Aid for Scientific Research (B)(1), No.14340007, JSPS.

This volume consists of eighteen articles presented at the symposium. We would like to thank all speakers and coauthors for their contributions.

We would also like to express our thanks to all the members of the organizing committee for their helpful suggestions concerning the symposium, especially to Masahisa Sato for his administration of the web page and for his kind advice and help to publish the proceedings. Finally, we would like to express our gratitude to Kenji Nishida and all the staffs in Shinshu University for their great effort in the organization of the symposium.

Hideto Asashiba
Osaka, Japan
January, 2005
第37回 環論および表現論シンポジウム（2004）
プログラム

9月3日（金）

09:10/09:40 山形邦夫（東京農工大）
正角ロア被覆をもつフロベニウス多元環の安定性について

09:50/10:20 若松隆義（埼玉大）
一般傾斜加群から導かれる安定同値だが導来同値でない対称多元環

10:30/11:20 Rickard, Jeremy (Univ. of Bristol)
Derived Categories in Representation Theory, I

11:30/12:00 若松隆義，飛田明彦（埼玉大）
Tilting modules and Gorenstein property

12:10/12:40 星野光男（筑波大）
A construction of Auslander-Gorenstein rings

14:00/14:40 奥山樞（北大理）
On the holonomic rank for A-hypergeometric system

14:40/15:10 浅芝秀人（大阪市大理）
Domestic canonical algebras and simple Lie algebras

15:20/16:10 Van den Bergh, Michel (Limburgs Universitair Centrum)
Non-commutative Del Pezzo surfaces, I

16:30/17:00 小池寿俊（沖縄工高専）
Morita duality and ring extensions

17:10/17:40 丸林英俊; Brungs, H., Osmanagic, E.（鳴門教育大; Alberta Univ.）
Non-commutative valuation rings of crossed product algebras

17:50/18:20 藤田尚昌，酒井洋介
Frobenius full matrix algebras and Gorenstein tiled orders

9月4日（土）

09:10/09:40 野々村和晃
Nakayama isomorphisms for the maximal quotient ring of a left Harada ring

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09:50/10:20 本瀬香（弘前大学工）
  Gaussian sums are just characters of the multiplicative group of a finite field

10:30/11:20 Rickard, Jeremy (Univ. of Bristol)
  Derived Categories in Representation Theory, II

11:30/12:00 功刀直子（愛知教育大）
  Some topics on derived equivalent blocks of finite groups

12:10/12:40 宇佐美陽子（お茶の水大理）
  On the principal 3-blocks of the Chevalley groups $G_2(q)$

14:00/14:30 加藤希理子（大阪女子大理）
  Morphisms represented by monomorphisms

14:40/15:10 荒谷智司（岡山大理）
  Remarks on transitivity of exceptional sequences

15:20/16:10 Van den Bergh, Michel (Limburgs Universitair Centrum)
  Non-commutative Del Pezzo surfaces, II

16:30/17:00 中島晴久（城西大理）
  Relative invariants of groups acting on Krull domains

17:10/17:40 吉野雄二（岡山大理）
  Frobenius and quasi-Frobenius property for mod $C$

17:50/18:20 高橋亮（岡山大理）
  On the decomposability of a syzygy of the residue field

9月5日（日）

09:10/09:40 越谷重夫（千葉大理）
  On indecomposability of a module given by Brauer construction

09:50/10:40 Rickard, Jeremy (Univ. of Bristol)
  Derived Categories in Representation Theory, III

10:50/11:40 Van den Bergh, Michel (Limburgs Universitair Centrum)
  Non-commutative crepant resolutions

11:50/12:30 伊山修（兵庫県立大理）
  Higher dimensional Auslander-Reiten theory on maximal orthogonal subcategories

2004-09-03

09:10/09:40 Yamagata, Kunio (Tokyo Univ. of Agr. and Tech.)
Stability of Frobenius algebras with positive Galois coverings

09:50/10:20 Wakamatsu, Takayoshi (Saitama Univ.)
Some examples of stably equivalent but not derived equivalent symmetric algebras
obtained by using generalized tilting modules

10:30/11:20 Rickard, Jeremy (Univ. of Bristol)
Derived Categories in Representation Theory, I

11:30/12:00 Wakamatsu, Takayoshi and Hida, Akihiko (Saitama Univ.)
Tilting modules and Gorenstein property

12:10/12:40 Hoshino, Mitsuo (Univ. of Tsukuba)
A construction of Auslander-Gorenstein rings

14:00/14:30 Okuyama, Go (Univ. of Hokkaido)
On the holonomic rank for A-hypergeometric system

14:40/15:10 Asashiba, Hideto (Osaka City Univ.)
Domestic canonical algebras and simple Lie algebras

15:20/16:10 Van den Bergh, Michel (Limburgs Universitair Centrum)
Non-commutative Del Pezzo surfaces, I

16:30/17:00 Koike, Kazutoshi (Okinawa National College of Tech.)
Morita duality and ring extensions

17:10/17:40 Marubayashi, Hidetoshi; Brungs, H. and Osmanagic, E. (Naruto Univ. of
Education; Albereta Univ.)
Non-commutative valuation rings of crossed product algebras

17:50/18:20 Fujita, Naoaki and Sakai, Yosuke (Univ. of Tsukuba)
Frobenius full matrix algebras and Gorenstein tiled orders

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09:10/09:40 Nonomura, Kazuaki
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09:50/10:20 Motose, Kaoru (Hirosaki Univ.)
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10:30/11:20 Rickard, Jeremy (Univ. of Bristol)
Derived Categories in Representation Theory, II

11:30/12:00 Kunugi, Naoko (Aichi Univ. of Education)
Some topics on derived equivalent blocks of finite groups

12:10/12:40 Usami, Yoko (Ochanomizu Univ.)
On the principal 3-blocks of the Chevalley groups $G_2(q)$

14:00/14:30 Kato, Kiriko (Osaka Women's Univ.)
Morphisms represented by monomorphisms

14:40/15:10 Araya, Tokuji (Univ. of Okayama)
Remarks on transitivity of exceptional sequences

15:20/16:10 Van den Bergh, Michel (Limburgs Universitair Centrum)
Non-commutative Del Pezzo surfaces, II

16:30/17:00 Nakajima, Haruhisa (Josai Univ.)
Relative invariants of groups acting on Krull domains

17:10/17:40 Yoshino, Yuji (Univ. of Okayama)
Frobenius and quasi-Frobenius property for mod $C$

17:50/18:20 Takahashi, Ryo (Univ. of Okayama)
On the decomposability of a syzygy of the residue field

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09:10/09:40 Koshitani, Shigeo (Univ. of Chiba)
On indecomposability of a module given by Brauer construction

09:50/10:40 Rickard, Jeremy (Univ. of Bristol)
Derived Categories in Representation Theory, III

10:50/11:40 Van den Bergh, Michel
Non-commutative crepant resolutions

11:50/12:30 Iyama, Osamu (Univ. of Hyogo)
Higher dimensional Auslander-Reiten theory on maximal orthogonal subcategories
REMARKS ON TRANSITIVITY OF EXCEPTIONAL SEQUENCES

TOKUJI ARAYA

ABSTRACT. Let $k$ be an algebraically closed field of characteristic 0. We denote by $C$ the abelian $k$-category which has enough projectives (or enough injectives), and by $D^b(C)$ the bounded derived category of $C$.

A complex $E^* \in D^b$ is called exceptional if $\text{RHom}(E^*, E^*) \cong k$, and a sequence $\epsilon = (\cdots, E^*_i, E^*_i, \cdots)$ of exceptional complexes is called an exceptional sequence if $\text{RHom}(E^*_i, E^*_j) = 0$ for all $i > j$.

Let $C$ be a category mod$A$ of finitely generated modules of a hereditary $k$-algebra $A$, or a category coh$(X)$ of coherent sheaves of a weighted projective line $X$ over $k$. In this case, for any exceptional sequence $\epsilon$, the length of $\epsilon$ is smaller than or equal to the rank $n$ of Grothendieck group of $C$. An exceptional sequence $\epsilon$ is called complete if the length of $\epsilon$ is equal to $n$. It is shown by W. Crawley-Boevey (in the case of $C = \text{mod}A$) and by H. Meltzer (in the case of $C = \text{coh}(X)$) that the braid group $B_n$ on $n$ strings acts transitively on the set of complete exceptional sequences.

In this talk, we consider exceptional sequences on a translation quiver $\Gamma$.

1. Preliminaries

この講演を通じて $\Gamma = \mathbb{Z}A_n$ を translation quiver とし、$\Gamma_0$ を $\Gamma$ の頂点集合、$\tau$ を translation とする。

定義 1.1 $X, Y \in \Gamma_0$ とする。

1. $X$ から $Y$ へ arrow があるとき、$X \prec Y$ と表す。
2. $X$ から $Y$ へ path があるとき、$X \preceq Y$ と表す。
3. $X$ と $Y$ の間に path がないとき、$X \asymp Y$ と表す。

定義 1.2 頂点集合 $\Gamma_0$ を以下のようにして、$\{ (p, q) \mid 1 \leq p - q \leq n \}$ と同一視する。

---

1The detailed version of this paper will be submitted for publication elsewhere.
1. 一番下の $\tau$-orbit 上の頂点を、$\cdots, (p, p - 1), (p + 1, p), (p + 2, p + 1), \cdots$ とする。

2. $X \in \Gamma_0$ に対し、$X \preceq (p, q), X \neq (p - 1, q)$ のとき、$X = (p, q - 1)$ とする。

$X = (p, q), Y = (p', q') \in \Gamma_0$ に対し、$X$ と $Y$ が同じ $\tau$-orbit にあるための必要十分条件は $p - q = p' - q'$ であることに注意する。

例 1.3 $\Gamma = \mathbb{Z}A_4$ のとき、次のようになる。

```
\begin{array}{c}
\vdots \quad (2, -2) \quad \vdots \\
\vdots \quad (1, -2) \quad \vdots \\
\vdots \quad (0, -1) \quad \vdots \\
\end{array}
```

定義 1.4 syzygy functor $\Omega : \Gamma_0 \to \Gamma_0$ を、$\Omega(p, q) = (q, p - n - 1)$ と定義する。

例 1.5 各 $X \in \Gamma_0$ に対し、$\tau X, \Omega X$ は以下のような位置関係にあることに注意する。

```
\begin{array}{c}
\vdots \quad \tau X \quad \vdots \\
\vdots \quad \Omega X \quad \vdots \\
\vdots \quad X \quad \vdots \\
\end{array}
```

$k$ を標数 0 の代数的閉体とし、$n$ を正の偶数とする。$R = k[x, y]/(y^2 - x^{n+1})$ を 1-次元次数付き環とし、$\text{mod} R$ を有限生成次数付き $R$-加群のなす環で射は次数を保つものとする。さらに CMR を極大 CM 加群全体のなす饱和部分環とする。このとき、CMR の Auslander-Reiten quiver の射影加群ではない極大 CM 加群全体から得られる full subquiver を $\Gamma$ とおくと、$\Gamma = \mathbb{Z}A_n$ である (c.f. [1])。定義 1.4 で定義している $\Omega$ は、この状況での syzygy 加群に対応している。

定義 1.6 各頂点 $X \in \Gamma_0$ に対し、$S^+(X), S^-(X), S^+(X), S^-(X), \Delta(X), \Delta^+(X), \Delta^-(X)$ を以下のように定義する。

1. $S^+(X)$ は slice であり、$Y \in S^+(X)$ ならば $X \preceq Y$ である。
2. $S^{-}(X)$ は slice であり、$Y \in S^{-}(X)$ ならば $Y \leq X$ である。
3. $S^{*}(X) = S^{*} \setminus \{X\}$ とする。(ここで、* = +, - である)
4. $\Delta(X) = \{ Y \in \Gamma_0 \mid X \sim Y \}$ とする。
5. $\Delta^{+}(X) = \bigcup_{\ell \geq 0}(\Delta(\Omega^{-\ell}X) \cup S^{+}(\Omega^{-\ell}X))$ とする。
6. $\Delta^{-}(X) = \bigcup_{\ell \leq 0}(\Delta(\Omega^{-\ell}X) \cup S^{-}(\Omega^{-\ell}X))$ とする。

各 $X \in \Gamma_0$ に対し、$\Delta^{+}(X)$ は以下のような位置関係にあることに注意する。

先程も述べたが、$\Gamma$ は、1 次元次数付き環 $R = k[x, y]/(y^{2} - x^{n+1})$ 上の极大 CM 加群のなす圈 CMR の Auslander-Reiten quiver の full subquiver を意識している。R-上での exceptional sequence の定義は以下の通りである。

定義 1.7 $R = \bigoplus_{i \geq 0} R_i$ を次数付き環で、$R_0 = k$ を標数 0 の代数的閉体とする。このとき、

1. 有限生成 $R$-加群 $E$ が exceptional であるとは、\[
\begin{align*}
\Hom(E, E) &\cong k \\ \Ext^{\ell}(E, E) &= 0 \quad (\ell > 0)
\end{align*}
\]をみたすことである。
2. exceptional 加群の列 $\epsilon = (\cdots, E_i, E_{i+1}, \cdots)$ が exceptional sequence であるとは、\[
\Ext^{\ell}(E_i, E_j) = 0 \quad (i > j, \ell \geq 0)
\]をみたすことである。

$R = k[x, y]/(y^{2} - x^{n+1})$ ($n$ は正の偶数) のときには次のことがわかっている (c.f.[1],[2])。

補題 1.8 $R = k[x, y]/(y^{2} - x^{n+1})$ ($n$ は正の偶数) とし、$\Gamma$ を CMR の Auslander-Reiten quiver の射影加群でない極大 CM 加群全体から得られる full subquiver とする。このとき、
1. 任意の直既約極大 CM 加群は exceptional である。

2. （射影加群でない）直既約極大 CM 加群 X, Y に対し、Hom(X, Y) ≠ 0 であるための必要十分条件は X ≤ Y (in Γ) である。

3. （射影加群でない）直既約極大 CM 加群 X, Y と正の整数 ℓ に対し、次は同値である。
   (a) Ext^ℓ(X, Y) ≠ 0
   (b) Ω^ℓX ≤ Y ≤ τΩ^{ℓ+1}X (in Γ)
   (c) τ^{-1}Ω^{-ℓ+1}Y ≤ X ≤ Ω^{-ℓ}Y (in Γ)

注意 1.9 R, Γ を補題 1.8 の通りとし、X, Y を射影加群でない直既約極大 CM 加群とする。このとき、次は同値である。

1. すべての整数 ℓ に対し、Ext^ℓ(X, Y) = 0 である。

2. X ∈ Δ^+(Y) である。

3. Y ∈ Δ^-(X) である。

これらのことから、一般の translation quiver Γ =ZA_n に対し、exceptional sequence を以下のように定義する。

定義 1.10 頂点 E_1, E_2, ⋯, E_r ∈ Γ_0 に対し、列 ε = (E_1, E_2, ⋯, E_r) が ε exceptional sequence であるとは、次の条件をみたすことである。

\[ E_i ∈ \bigcap_{j<i} Δ^+(E_j) \quad (1 < i ≤ r) \]

この条件は次の条件と同値である。

\[ E_i ∈ \bigcap_{j>i} Δ^-(E_j) \quad (1 ≤ i < r) \]

2. Main results

主結果を述べるためにもう少し準備をする。

定義 2.1 ε = (E_1, E_2, ⋯, E_r) を exceptional sequence とする。E_i, E_j が次の二条件をみたすとき、E_i < E_j と表す。
1. $E_i \in S'(E_j)$ である。

2. もし $E_i \leq E_m \leq E_j$ ならば、$m = i$ または $m = j$ である。

exceptional sequence の定義より次のことが成立していることが容易に確かめられる。

補題 2.2 $\epsilon = (E_1, E_2, \cdots, E_r)$ を exceptional sequence とする。このとき、次のどの列も exceptional sequence になる。

1. $E_{i-1} \sim E_i$ のとき、$(E_1, E_2, \cdots, E_{i-2}, E_i, E_{i-1}, E_{i+1}, \cdots, E_r)$

2. $E_j \not\in S'(E_i) \ (\forall j)$ のとき、$(E_1, E_2, \cdots, E_{i-1}, \Omega E_i, E_{i+1}, \cdots, E_r)$

3. $E_j \not\in S'(E_i) \ (\forall j)$ のとき、$(E_1, E_2, \cdots, E_{i-1}, \Omega^{-1} E_i, E_{i+1}, \cdots, E_r)$

4. $E_i = (p, q)$, $E_j \prec \epsilon E_i$ ならば $j = i - 1$ のとき、

$$ (E_1, E_2, \cdots, E_{i-2}, E_i', E_{i-1}, E_{i+1}, \cdots, E_r) $$

但し、$E'_i = \begin{cases} (q, q') & (E_{i-1} = (p, q') \ のとき) \\ (p', p - n - 1) & (E_{i-1} = (p', q) \ のとき) \end{cases}$

5. $E_i = (p, q)$, $E_i \prec \epsilon E_j$ ならば $j = i + 1$ のとき、

$$ (E_1, E_2, \cdots, E_{i-2}, E_i', E_{i-1}, E_{i+1}, \cdots, E_r) $$

但し、$E'_i = \begin{cases} (q + n + 1, q') & (E_{i+1} = (p, q') \ のとき) \\ (p', p) & (E_{i+1} = (p', q) \ のとき) \end{cases}$

6. $E_{i-2} = (p, q')$, $E_{i-1} = (p', q)$, $E_i = (p, q)$ のとき、

$$ (E_1, E_2, \cdots, E_{i-3}, E_i', E_{i-2}, E_{i-1}, E_{i+1}, \cdots, E_r) $$

但し、$E'_i = (p', q')$

7. $E_i = (p, q)$, $E_{i+1} = (p', q)$, $E_{i+2} = (p, q')$ のとき、

$$ (E_1, E_2, \cdots, E_{i-1}, E_{i+1}, E_{i+2}, E_i', E_{i+3}, \cdots, E_r) $$

但し、$E'_i = (p', q')$

- 5 -
定義 2.3 $\epsilon, \epsilon'$ を exceptional sequence とする。$\epsilon$ に補題 2.2 の変形を有限回行って $\epsilon'$ になるとき、$\epsilon \sim \epsilon'$ と表す。

定理 2.4 $\epsilon$ を exceptional sequence とする。このとき、exceptional sequence $\epsilon'$ と slice $S$ で、$\epsilon \sim \epsilon'$、$\epsilon'$ は $S$ に埋め込むものが存在する。

証明 二段階に分けて証明する。

Step 1. $\epsilon = (E_1, E_2, \cdots, E_r)$ とおくとき、$\epsilon' = (E'_1, E'_2, \cdots, E'_r)$ で、$\epsilon \sim \epsilon'$、$E'_i \in S^-(E'_i) \cup \Delta(E'_i) \cup S^+(E'_i)$ (∀$i$) をみたすものが存在する。

各 $i$ に対し、$\ell_i \in \Delta(\Omega^{i-1}E_1) \cup S^+(\Omega^{i-1}E_1)$ をみたす数として定義する。

exceptional sequence の定義より、$\ell_i \geq 0$ であるので、$\ell = \sum_{i=1}^n \ell_i$ に関する帰納法で示す。$\ell = 0$ ならば $\epsilon' = \epsilon$ とすればよい。$\ell > 0$ のとき、$i = \min\{ j \mid \ell_j > 0 \}$ とおく。

$E_j \notin S^-(E_1)$ (∀$j$) のとき、$\epsilon'' = (E_1, E_2, \cdots, E_{i-1}, \Omega E_i, E_{i+1}, \cdots, E_r)$ とおく。このとき、補題 2.2.2 より、$\epsilon \sim \epsilon''$ であり、$\Omega E_i \in \Delta(\Omega^{-(i-1)}E_1) \cup S^+(\Omega^{-(i-1)}E_1)$ なので、帰納法の仮定より条件をみたす $\epsilon'$ をとることができる。

$E_j \in S^-(E_1)$ なる $j$ が存在するとき、$E_j \ll E_i$ とする。このとき、$E_j$ の取り方より、(必要ならば補題 2.2.1 を使うことで) $j = i - 1$ としてよい。$E_j \ll E_i$ をみたす $j$ が $i - 1$ のときのみの場合には、$\epsilon''$ を補題 2.2.4 のようにとる。$j \neq i - 1$ なる $j$ で $E_j \ll E_i$ をみたすものがあると仮定するとき、(必要ならば補題 2.2.1 を使うことで) $j = i - 2$ とできる。そして、$\epsilon''$ を補題 2.2.6 のようにとる。いずれの場合でも $\epsilon \sim \epsilon''$ である。ここで、$i$ の取り方から $\ell_{i-1} = 0$ であり、$E'_1 < E_{i-1}$ より $E'_1 \in \Delta(E'_1) \cup S^+(E'_1)$ である。よってこの場合も帰納法の仮定より条件をみたす $\epsilon'$ をとることができる。

Step 2. $\epsilon'$ を step 1. のようにとると、$\epsilon'$ はある slice $S$ に埋め込むことができる。

$\epsilon' = (E'_1, E'_2, \cdots, E'_r)$ とおく。exceptional sequence の定義より、任意の $i \neq j$ に対し $E'_i \in S^-(E'_j) \cup \Delta(E'_j) \cup S^+(E'_j)$ をみたすことに注意する。さらに、$i \neq j$ ならば、$E'_i$ と $E'_j$ は異なる $\tau$-orbit にあることに注意する。$E'_i = (p_i, q_i)$ とおき、$p_i - q_i = t_i$ とおく。このとき、$S$ を次のようによると。すべての $E'_i$ は $S$ に属しているとする。$1 \leq t \leq r$, $t \notin \{ t_1, t_2, \cdots, t_r \}$ に対し、$t_i = \max\{ t_i \mid t_i < t \}$, $t_j = \min\{ t_j \mid t_j > t \}$ をとる。さらに、$Y \in S^+(E'_1) \cap S^+(E'_r)$ をとる。そして、$X = (p, q)$ を $p - q = t$, $X \in S^-(Y) \cap (S^+(E'_1) \cup S^+(E'_r))$ ととる。この $X$ が $S$ に属するととすると、$S$ は slice である。□

系 2.5 $\epsilon = (E_1, E_2, \cdots, E_r)$ を exceptional sequence とするとき、$r \leq n$ である。

系 2.6 長さ $n$ の任意の exceptional sequence $\epsilon, \epsilon'$ に対し、$\epsilon \sim \epsilon'$ である。
参考文献


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Let $D$ be a discrete valuation ring with a unique maximal ideal $\pi D$, and let $\Lambda$ be a $D$-order. It is standard to reduce homological properties of $\Lambda$ to those of the factor algebras $\Lambda/\pi \Lambda$ and such factor algebras are deserving of further study. (See [2].)

Let $n$ be an integer with $n \geq 2$. In [1], we introduced an $n \times n$ $A$-full matrix algebra over a field $K$, whose multiplication is determined by a structure system $A$, that is, an $n$-tuple of $n \times n$ matrices with certain properties. A-full matrix algebras are associative, basic, connected $K$-algebras. A prototype of A-full matrix algebras is the class of factor algebras $\Lambda/\pi \Lambda$ of tiled $D$-orders $\Lambda$. Studying representation matrices of certain modules over A-full matrix algebras, Frobenius $A$-full matrix algebras are characterized by the shape of their structure systems $A$. For a Gorenstein tiled $D$-order $\Lambda$, the factor algebra $\Lambda/\pi \Lambda$ is a Frobenius $A$-full matrix algebra. In this paper we study the converse of this fact. Our main result is the following.

**Theorem 1.** (1) For $2 \leq n \leq 7$, all Frobenius $n \times n$ $A$-full matrix algebras are isomorphic to $\Lambda/\pi \Lambda$ for some Gorenstein tiled $D$-orders $\Lambda$. Moreover a list of them (up to isomorphism) is obtained.

(2) For each $n \geq 8$, there is a Frobenius $n \times n$ $A$-full matrix algebra having no corresponding Gorenstein tiled $D$-orders.

1. **A-Full Matrix Algebras**

We begin by recalling A-full matrix algebras. Let $K$ be a field and $n \geq 2$ an integer. Let $A = (A_1, \ldots, A_n)$ be an $n$-tuple of $n \times n$ matrices $A_k = (a_{ij}^{(k)}) \in M_n(K)$ ($1 \leq k \leq n$) satisfying the following three conditions.

(A1) $a_{ij}^{(k)}a_{il}^{(j)} = a_{il}^{(k)}a_{ij}^{(j)}$ for all $i, j, k, l \in \{1, \ldots, n\}$,

(A2) $a_{kj}^{(k)} = a_{kj}^{(k)} = 1$ for all $i, j, k \in \{1, \ldots, n\}$, and

(A3) $a_{ii}^{(k)} = 0$ for all $i, k \in \{1, \ldots, n\}$ such that $i \neq k$.

Let $A = \bigoplus_{1 \leq i < j \leq n} K u_{ij}$ be a $K$-vector space with basis $\{u_{ij} \mid 1 \leq i, j \leq n\}$. Then we define multiplication of $A$ by using $A$, that is,

$$u_{ik}u_{lj} := \begin{cases} a_{ij}^{(k)}u_{lj} & \text{if } k = l \\ 0 & \text{otherwise.} \end{cases}$$

Then $u_{11}, \ldots, u_{nn}$ are orthogonal primitive idempotents such that $u_{11} + \cdots + u_{nn} = 1_A$, an identity of $A$ and $u_iAu_{jj} \cong K$. Hence $A$ is an associative, basic, connected $K$-algebra.

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The detailed version of this paper will be submitted for publication elsewhere.
We call $A$ an $n \times n$ A-full matrix algebra with a structure system $A$. We note that \( \text{gl.dim}A = \infty \), because every entry of the Cartan matrix of $A$ is 1.

In what follows, we assume that $a_{ij}^{(k)} = 0$ or 1 for all $1 \leq i, k, j \leq n$.

2. Tiled orders

Let $D$ be a discrete valuation ring with a unique maximal ideal $\pi D$ and $n \geq 2$ an integer. Let \( \{ \lambda_{ij} \mid 1 \leq i, j \leq n \} \) be a set of integers satisfying

\[
\lambda_{ij} \geq 0, \quad \lambda_{ii} = 0, \quad \lambda_{ik} + \lambda_{kj} \geq \lambda_{ij}, \quad \text{and} \quad \lambda_{ij} + \lambda_{ji} > 0 \quad \text{(if} i \neq j)\]

for all $1 \leq i, j, k \leq n$. Then $A = (\pi \Lambda D)$ is a subring of $M_n(D)$, which we call an $n \times n$ tiled $D$-order.

Example 2. Let $\Lambda = (\pi \Lambda D)$ be an $n \times n$ tiled $D$-order. Put $A := \Lambda/\pi \Lambda$, $K := D/\pi D$ and $u_{ij} := \pi \Lambda e_{ij} + \pi \Lambda \in A$, where $e_{ij}$'s are the matrix units in $M_n(D)$. Define $A_k = (a_{ij}^{(k)}) \in M_n(K)$ ($1 \leq k \leq n$) by

\[
a_{ij}^{(k)} := \begin{cases} 
1 & \text{if } \lambda_{ik} + \lambda_{kj} = \lambda_{ij} \\
0 & \text{otherwise,}
\end{cases}
\]

and set $A := (A_1, \ldots, A_n)$. Then note that

\[
u_{ik}u_{ij} = \begin{cases} 
a_{ij}^{(k)}u_{ij} & \text{if } k = l \\
0 & \text{otherwise.}
\end{cases}
\]

Hence $A$ is an A-full matrix algebra.

3. Representation matrices

Let $A = \bigoplus_{1 \leq i, j \leq n} u_{ij}K$ be an $n \times n$ A-full matrix algebra, where $A = (A_1, \ldots, A_n)$ and $A_k = (a_{ij}^{(k)}) \in M_n(K)$ ($1 \leq k \leq n$). Let $M$ be a right $A$-module with dimension type $\dim M = (1, \ldots, 1)$. Then $M$ has a $K$-basis \( \{ v_i \mid 1 \leq i \leq n \} \) such that $v_iu_i = v_i$ for all $1 \leq i \leq n$. Hence there exists a matrix $S = (s_{ij}) \in M_n(K)$ such that $v_iu_{ij} = s_{ij}v_j$ for all $1 \leq i, j \leq n$. We call $S$ a representation matrix of $M$.

Proposition 3. For each indecomposable projective right $A$-module $u_iA$, $\dim u_iA = (1, \ldots, 1)$ and it has a representation matrix $(a_{ij}^{(k)})_{k,i,j}$, that is, an $n \times n$ matrix whose $(k, j)$-entry is $a_{ij}^{(k)}$. Moreover $u_iA$ is isomorphic to an injective $\text{Hom}_K(Au_i, K)$ if and only if $a_{ii}^{(k)} = 1$ for all $1 \leq i, k \leq n$.

Example 4. Let $A$ be an $A$-full matrix algebra where

\[
A = \begin{pmatrix}
1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}.
\]
Then representation matrices of $u_{11}A, \ldots, u_{44}A$ are given by
\[
\begin{pmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0
\end{pmatrix}.
\]
Hence $u_{11}A, u_{22}A$ and $u_{44}A$ are injective but not $u_{33}A$.

4. FROBENIUS $A$-FULL MATRIX ALGEBRAS

By means of structure systems, we can characterize Frobenius $A$-full matrix algebras. Let $A = \bigoplus_{1 \leq i, j \leq n} u_{ij}K$ be an $n \times n$ $A$-full matrix algebra, where $A = (A_1, \ldots, A_n)$ and $A_k = (a_{ij}^{(k)}) \in M_n(K) (1 \leq k \leq n)$.

**Proposition 5.** The following are equivalent for an $A$-full matrix algebra $A$.

1. $A$ is a Frobenius algebra.
2. There exists a permutation $\sigma$ of the set $\{1, \ldots, n\}$ such that $\sigma(i) \neq i$ for all $1 \leq i \leq n$ and that $a_{\sigma(i)}^{(i)} = 1$ for all $1 \leq i, k \leq n$.

In this case $\sigma$ is a Nakayama permutation of $A$, that is, $\text{soc}(u_{ii}A) \cong \text{top}(u_{\sigma(i)\sigma(i)}A)$ for all $1 \leq i \leq n$. Moreover, for all $1 \leq i, k, j \leq n$, $a_{ij}^{(k)} = a_{kj}^{(i)}$ holds.

Using Proposition 5, we can find structure systems $A$ which define Frobenius $A$-full matrix algebras. Let $\sigma$ be a permutation of the set $\{1, \ldots, n\}$ such that $\sigma(i) \neq i$ for all $1 \leq i \leq n$. Let $T$ be the set of triples $(i, k, j)$ of integers $1 \leq i, k, j \leq n$. Then we have a bijection
\[
\varphi : T \rightarrow T, \quad (i, k, j) \mapsto (k, j, \sigma(i)).
\]
Decompose $T$ into $\varphi$-orbits $\{T_\alpha\}_\alpha$, and put $I := \bigcup\{T_\alpha| (i, k, \sigma(i)) \in T_\alpha\}$, $Z := \bigcup\{T_\alpha| (i, k, i) \in T_\alpha, i \neq k\}$. and $X := \bigcup\{T_\alpha| T_\alpha \not\subseteq I \cup Z\}$. Then we have $T = I \cup Z \cup X$ (disjoint).

**Proposition 6.** (1) Suppose that $A$ is a Frobenius structure system with Nakayama permutation $\sigma$. Then there exists a $\varphi$-invariant subset $Y$ of $X$ such that
\[
a_{ij}^{(k)} = \begin{cases} 
1 & \text{if } (i, k, j) \in I \cup Y \\
0 & \text{otherwise}.
\end{cases}
\]

(2) Let $Y$ be a $\varphi$-invariant subset of $X$, and define $A(Y) = (a_{ij}^{(k)})$ by
\[
a_{ij}^{(k)} := \begin{cases} 
1 & \text{if } (i, k, j) \in I \cup Y \\
0 & \text{otherwise}.
\end{cases}
\]
Then $A(Y)$ is a Frobenius structure system whenever (A1) holds for $A(Y)$.

(3) For the empty subset $\emptyset$ of $X$, $A(\emptyset)$ is a Frobenius structure system.

5. $\varphi$-ORBITS FOR A CYCLIC PERMUTATION

In this section, we clarify the $\varphi$-orbits of $T$ for a cyclic permutation $\sigma = (1 \ 2 \ \cdots \ n)$. First we count the number of $\varphi$-orbits of $T$.

**Proposition 7.** (1) For a $\varphi$-orbit $T_\alpha$ of $T$, the number $|T_\alpha|$ of elements in $T_\alpha$ is $3n$ or $n$. 

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(2) $T$ has a $\varphi$-orbit $T_\alpha$ with $|T_\alpha| = n$ if and only if $n$ is not divisible by 3. In this case, $T$ has a unique $\varphi$-orbit having $n$ elements, which is contained in $X$.

(3) $I$ has $n - 1$ $\varphi$-orbits.

(4) $Z$ has $n - 2$ $\varphi$-orbits.

(5) If $n$ is divisible by 3, then $X$ has $(n - 1)^2/3$ $\varphi$-orbits.

(6) If $n$ is not divisible by 3, then $X$ has $(n - 2)(n - 4)/3 + 1$ $\varphi$-orbits.

Next we clarify the members of each $\varphi$-orbit of $T$.

**Proposition 8.** Let $T_\alpha$ be a $\varphi$-orbit of $T$ and put $T_\alpha^{(r)} := \{(i, k, j) \in T_\alpha \mid k = r\}$ for all $1 \leq r \leq n$.

1. Suppose that $|T_\alpha| = 3n$. Then $|T_\alpha^{(r)}| = 3$ for each $r = 1, \ldots, n$. If $(i, 1, j) \in T_\alpha$, then $T_\alpha^{(1)} = \{(i, 1, j), (\sigma^{-j+1}(1), 1, \sigma^{-j+2}(i)), (\sigma^{-i}(1), 1, \sigma^{-i+1}(1))\}$. 

2. Suppose that $|T_\alpha| = n$. Then $|T_\alpha^{(r)}| = 1$ for each $r = 1, \ldots, n$. If $n = 3t + 1$ then $T_\alpha^{(1)} = \{(t + 1, 1, 2t + 2)\}$. If $n = 3t + 2$ then $T_\alpha^{(1)} = \{(2t + 2, 1, t + 2)\}$.

6. **Minimal Frobenius structure systems**

Let $A$ be a Frobenius $A$-full matrix algebra with Nakayama permutation $\sigma$. Then it follows from Proposition 6 (1) that $A$ is determined by a $\varphi_\sigma$-invariant subset $Y$ of $X$. We call $A$ a minimal Frobenius structure system if $Y$ is minimal among $\varphi_\sigma$-invariant subset of $X$ which define Frobenius full matrix algebras. For a cyclic permutation, minimal Frobenius structure systems are determined by the following theorem.

**Theorem 9.** Let $n$ be an integer with $n \geq 4$, and let $\sigma = (1 \ 2 \ \cdots \ n)$ be a cyclic permutation. Then the following statements hold.

1. Let $n$ be even. Then the $\varphi$-invariant subsets defining minimal Frobenius structure systems are just $\varphi$-orbits contained in $X$.

2. Let $n$ be odd and $n = 2s + 1$ for some $s$. Then the $\varphi$-invariant subsets defining minimal Frobenius structure systems are just $\varphi$-orbits $X_\beta$ contained in $X$ such that $X_\beta$ does not contain any element of the form $(s + 1, 1, k)$ for any $k$ with $k \not\equiv s^2 + 1 \pmod{n}$.

The following example illustrates Theorem 9.

**Example 10.** Let $n = 7$. Then $X$ has 6 $\varphi$-orbits $X_i$ ($1 \leq i \leq 6$) such that

- $X_1^{(1)} = \{(4, 1, 3), (6, 1, 3), (6, 1, 5)\}$
- $X_2^{(1)} = \{(2, 1, 5), (3, 1, 7), (4, 1, 6)\}$
- $X_3^{(1)} = \{(2, 1, 6), (4, 1, 7), (3, 1, 5)\}$
- $X_4^{(1)} = \{(5, 1, 3), (5, 1, 4), (6, 1, 4)\}$
- $X_5^{(1)} = \{(2, 1, 4), (2, 1, 7), (5, 1, 7)\}$
- $X_6^{(1)} = \{(3, 1, 6)\}$

Since $7 = 2 \cdot 3 + 1$ and $3^2 + 1 \equiv 3 \pmod{7}$, there are minimal Frobenius structure systems corresponding to $X_1, X_4, X_5, X_6$, but not to $X_2, X_3$. 

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7. GORENSTEIN TILED ORDERS

A $D$-order $A$ is Gorenstein if $\text{Hom}_D(A, D) = (\pi_A \cdot D)$, $A$ is Gorenstein if $\text{id}_A \cdot \pi_A = \lambda_d \cdot A$, and $A$ is Gorenstein if $\text{id}_A \cdot \pi_A = \lambda_d \cdot A$ for all $1 \leq t, k \leq n$. (See [4].) Since $\pi_A = \text{id}_A \cdot \lambda_d$, it follows from Proposition 5 in our context.

Theorem 11. For every integer $n \geq 8$, there exists a Frobenius $A$-full matrix algebra $D$ with no corresponding Gorenstein tiled orders.

Proof. Let $\sigma$ be a cyclic permutation of $(1, 2, \ldots, n)$. Let $A = \langle \sigma^{-1} \rangle$ define a minimal Frobenius structure system $\Lambda = (\pi_A \cdot D)$. Then, since $n \geq 8$, it follows that there exists a Gorenstein tiled $D$-order $A$. However, the converse is not true.

Example 12. When $n = 8$, the exponent matrix (\lambda_i) of Gorenstein tiled $D$-orders $A = (\pi_A \cdot D)$ is given by

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

where $x := a - b$, $y := a - c$, $z := a - d$, $w := a + b - c - d$, $r := b + c - a$, $s := c + d - a$, $t := 2c - d - b - a$.

For each $n$, $2 \leq n \leq 7$, we can verify that every Frobenius $n \times n$ A-full matrix algebra $A$ has a Gorenstein tiled $D$-order $A$ such that $\lambda_d = A$. The following table shows that for each $n = 2, 3, 4, 5$.
<table>
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<th>n</th>
<th>no. of iso. classes</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
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</tr>
<tr>
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<td>1</td>
</tr>
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<td>21</td>
</tr>
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<td>7</td>
<td>17</td>
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A construction of Auslander-Gorenstein rings

Mitsuo HOSHINO

This is a summary of my joint work with T. Shiba.

Recall that a ring $A$ is said to be an Auslander-Gorenstein ring if it is a left and right noetherian ring and for a minimal injective resolution $E'$ of $A$ we have flat dim $E' \leq n$ for all $n \geq 0$ (cf. [1]). It is well known that the group ring of a finite group over a commutative Gorenstein ring is an Auslander-Gorenstein ring. We refer to [1] for other examples of Auslander-Gorenstein rings. Our main aim is to construct another type of Auslander-Gorenstein rings.

Let $R$ be a ring. In this talk, a ring $A$ is said to be a Frobenius extension of $R$ if it contains $R$ as a subring and satisfies the conditions (F1) $A_R$ and $_RA$ are finitely generated projective, (F2) $A \cong \text{Hom}_R(A_R, R_R)$ and $_RA \cong \text{Hom}_R(A_A, R_R)$, and (F3) the inclusion $R \to A$ is a split monomorphism of $R$-$R$-bimodules (cf. [2]). Again, the group ring of a finite group is a typical example of Frobenius extensions. If $A$ is a Frobenius extension of $R$, then (1) $\text{inj} \dim A_\alpha = \text{inj} \dim R_\alpha$, (2) $\text{inj} \dim A = \text{inj} \dim R$, and (3) $A$ is an Auslander-Gorenstein ring if $R$ is so. Therefore we will provide a way to construct Frobenius extensions of a given ring.

Let $R$ be a ring, $n \geq 2$ an integer and $\nu$ a permutation of $I = \{1, 2, \ldots, n\}$. We will construct a family of Frobenius extensions $A$ of $R$ such that (i) $1_A = \sum_{i \in I} e_i$ with the $e_i$ orthogonal idempotents, (ii) $e_i A \cong e_j A$ unless $i = j$, (iii) $xe_i = e_i x$ for all $i \in I$ and $x \in R$, (iv) $e_i A_\alpha \cong \text{Hom}_R(A_{e_i} R_R, R_R)$ and $_RA_{e_i} \cong \text{Hom}_R(A_{e_i}, R_R)$ for all $i \in I$, and (v) there exists $\eta \in \text{Aut}(A)$ with $\eta(e_i) = e_{\nu(i)}$ for all $i \in I$. Furthermore, the rings $e_i A_{\alpha}$ are local if $R$ is so. In particular, if $R$ is a quasi-Frobenius local ring, then $A$ is a quasi-Frobenius ring with $\text{soc}(e_i A \alpha) \cong e_{\nu(i)} A\alpha_{e_{\nu(i)}} J$ for all $i \in I$, where $J = \text{rad}(A)$. In case $\nu$ has no fixed point, we can construct a desired Frobenius extension $A$ of $R$ as a skew matrix ring over $R$, the notion of which was first introduced in [3] (cf. also [4] and [5]). If $\nu$ has a fixed point, then we can not construct a desired Frobenius extension of $R$ as a skew matrix ring over $R$, but we can construct a desired Frobenius extension $B$ of $R$ which contains an ideal $V$ with $B/V$ a skew matrix ring over $R$, where $V^2 \neq 0$ in general.

Throughout this note, rings are associative rings with identity. For a ring $R$, we denote by $R^e$ the set of units and by $\text{rad}(R)$ the Jacobson radical. We use the notation $X_R$ (resp., $_RX$) to denote that the module $X$ considered is a right (resp., left) $R$-module. Also, we use the notation $X_R$ to denote that $X$ is an $S$-$R$-bimodule. We denote by Mod-$R$ the category of right $R$-modules.

1. Frobenius extension of rings

In this section, we introduce a notion of Frobenius extension of rings (cf. [2]).

**Definition 1.1.** A ring $A$ is said to be a Frobenius extension of a ring $R$ if there exists an injective ring homomorphism $\varphi : R \to A$ satisfying the following conditions:

(F1) $A_R$ and $_RA$ are finitely generated projective;

*The detailed version will be submitted for publication elsewhere.*
(F2) \( A_A \cong \text{Hom}_R(A_R, R) \) and \( _A A \cong \text{Hom}_R(A_A, R) \); and
(F3) \( \varphi \) is a split monomorphism of \( R \)-bimodules.

**Remark 1.2.** Let \( A \) be a Frobenius extension of \( R \). Then for an isomorphism \( \phi : A_A \cong \text{Hom}_R(A_R, R) \) we have a unique ring homomorphism \( \theta : R \to A \) such that \( x\phi(1) = \phi(1)\theta(x) \) for all \( x \in R \). Similarly, for an isomorphism \( \psi : A_A \cong \text{Hom}_R(A_A, A \otimes R) \) we have a unique ring homomorphism \( \eta : R \to A \) such that \( \psi(1)x = \eta(x)\psi(1) \) for all \( x \in R \).

**Definition 1.3 (cf. [1]).** A ring \( R \) is said to be an Auslander-Gorenstein ring if it is left and right noetherian and for a minimal injective resolution \( E \) of \( R \), we have flat dim \( E \leq n \) for all \( n \geq 0 \).

**Definition 1.4.** A ring \( R \) is said to be a quasi-Frobenius ring if it is left and right artinian and left and right selfinjective.

**Proposition 1.5.** Let \( A \) be a Frobenius extension of \( R \). Then the following hold.

1. \( \text{inj dim } A_A = \text{inj dim } R_R \).
2. \( \text{inj dim } _A A = \text{inj dim } R \).
3. \( A \) is an Auslander-Gorenstein ring if \( R \) is so.
4. \( A \) is a quasi-Frobenius ring if \( R \) is so.

**Remark 1.6.** The converse holds in (3) and (4) of Proposition 1.5. However, we do not need this fact in the present note.

In the following, taking Proposition 1.5 into account, we provide a way to construct Frobenius extensions of an arbitrary ring.

2. Skew matrix rings

In the following, we fix a ring \( R \) and a pair of \( \sigma \in \text{Aut}(R) \) and \( c \in R \) such that

\[
\sigma(c) = c \quad \text{and} \quad xc = c\sigma(x) \quad \text{for all } x \in R.
\]

In this section, we develop the construction of skew matrix rings given in [3], [4] and [5]. Let \( n \geq 2 \) be an integer and \( I = \{1, 2, \ldots, n\} \). Let \( \omega : I \times I \to \mathbb{Z} \) be a mapping and set

\[
\lambda(i, j, k) = \omega(i, j) + \omega(j, k) - \omega(i, k)
\]

for \( i, j, k \in I \). We assume the following conditions are satisfied:

\[
\omega(i, i) = 0 \quad \text{for all } i \in I \quad \text{and} \quad \lambda(i, j, k) \geq 0 \quad \text{for all } i, j, k \in I.
\]

**Definition 2.1.** Let \( A \) be a free right \( R \)-module with a basis \( \{e_i\}_{i \in I} \), and define the multiplication on \( A \) subject to the following axioms:
(A1) $xe_{ij} = e_y e^{x(i,j),x} e_{ij} (x)$ for all $i, j \in I$ and $x \in R$.
(A2) $e_i e_k = 0$ unless $j = k$; and
(A3) $e_i e_j = e_k e^{(i,j),k}$ for all $i, j, k \in I$.

**Proposition 2.2.** The following hold.
(1) $A$ is an associative ring with $1_A = \sum_{i,j} e_{ij}$, where the $e_{ij}$ are orthogonal idempotents.
(2) We have an injective ring homomorphism $\varphi : R \to A, x \mapsto \sum_{i,j} e_{ij} x$, which is a split monomorphism of $R$-$R$-bimodules.
(3) $e_i A_k \cong e_k A_i$ for all $i, j \in I$ with $\lambda(i, j, i) = 0$. In case $c \in R^\times$, the converse holds.

In the following, we consider $R$ as a subring of $A$ via $\varphi : R \to A$. Note that $A$ is a free left $R$-module with a basis $\{ e_{ij} \}_{i,j \in I}$. Also, for any $i \in I$, since $xe_{ij} = e_{ij} x$ for all $x \in R$, $A e_{ij}$ is an $A$-$R$-bimodule and $e_{ij} A$ is an $R$-$A$-bimodule.

**Remark 2.3.** Denote by $M_n(R)$ the $n \times n$ full matrix ring over $R$. Then for any $i \in I$ there exists a ring homomorphism of the form
$$\zeta : A \to M_n(R), \sum_{j,k \in I} e_{jk} x_{jk} \mapsto (e^{x_{ij},x} e_{ij})_{j,k \in I},$$
which is an isomorphism if either $c \in R^\times$ or $\lambda = 0$. Also, if $c$ is regular, then $\zeta$ is injective.

In the following, taking Remark 2.3 into account, we assume $c \notin R^\times$. However, for the sake of convenience, we do not exclude the case where $\lambda = 0$.

**Definition 2.4.** There exists a basis $\{ \alpha_{ij} \}_{i,j \in I}$ for $\text{Hom}_R(A_R, R_R)$ such that $a = \sum_{i,j} e_{ij} \alpha_{ij}(a)$ for all $a \in A$. Similarly, there exists a basis $\{ \beta_{ij} \}_{i,j \in I}$ for $\text{Hom}_R(A_R, A_R)$ such that $a = \sum_{i,j} e_{ij} \beta_{ij}(a)$ for all $a \in A$.

**Lemma 2.5.** For any $i, k \in I$ the following are equivalent.
(1) $\lambda(i, j, k) = 0$ for all $j \in I$.
(2) There exist isomorphisms of the form
$$\phi_{ik} : e_i A_k \to \text{Hom}_R(A e_{ik}, R), a \mapsto \alpha_{ik} e_{ik},$$
$$\psi_{ik} : A e_{ij} \to \text{Hom}_R(A e_{ij} A_k, A), a \mapsto \alpha_{ik} e_{ij}.$$
(3) Either $e_i A_k \cong \text{Hom}_R(A e_{ik}, R)$ or $A e_{ik} \cong \text{Hom}_R(A e_{ij} A_k, R)$.

**Proposition 2.6.** Assume $R$ is a local ring. Then the following hold.
(1) Every $e_i$ is a local idempotent. In particular, $A$ is a semiperfect ring.
(2) Assume either $A_A \cong \text{Hom}_R(A_R, R)$ or $A_A \cong \text{Hom}_R(A_R, A_R)$. Then there exists $\nu \in \text{Aut}(I)$ such that $\lambda(i, j, \nu(i)) = 0$ for all $i, j \in I$.

In the next Proposition, we refer to [6] for derived equivalence of rings.
Proposition 2.7. Assume \( c \) is regular. Then for any \( i \in I \) the following hold.

\( (1) \) \( A \) is derived equivalent to a generalized triangular matrix ring

\[
\begin{bmatrix}
R & \text{Ext}_A^1(Ae_iA, e_{\mu}A) \\
0 & A/ Ae_iA
\end{bmatrix}
\]

\( (2) \) Assume there exists \( j \in N(i) \) such that \( \lambda(j, k, i) = 0 \) for all \( k \in I \). Then \( \text{Ext}_A^1(A/ Ae_iA, e_{\mu}A) \equiv e_{\mu}(A/ Ae_iA) \) as \( R/ A/ Ae_iA \)-bimodules.

In the following, we do not fix \( \omega \) and use the notation \( A_\omega \) to denote that the multiplication of \( A \) is defined by \( \omega \).

3. Classification of \( \omega \)

In the following, with each \( \omega : I \times I \to \mathbb{Z} \) we associate \( \lambda : I \times I \times I \to \mathbb{Z} \) such that

\[
\lambda(i, j, k) = \omega(i, j) + \omega(j, k) - \omega(i, k)
\]

for all \( i, j, k \in I \).

Lemma 3.1. For any \( \omega, \omega' : I \times I \to \mathbb{Z} \), the following are equivalent.

\( (1) \) \( \lambda = \lambda' \).

\( (2) \) There exists \( \chi : I \to \mathbb{Z} \) such that \( \omega'(i, j) = \omega(i, j) - \chi(i) + \chi(j) \) for all \( i, j \in I \).

Definition 3.2. For \( \omega, \omega' : I \times I \to \mathbb{Z} \), we set \( \omega \equiv \omega' \) if the equivalent conditions of Lemma 3.1 are satisfied. Also, for \( \omega : I \times I \to \mathbb{Z} \), we write \( \omega \geq 0 \) if \( \omega(i, j) \geq 0 \) for all \( i, j \in I \).

Lemma 3.3. Let \( \omega, \omega' : I \times I \to \mathbb{Z} \) with \( \omega \equiv \omega' \) and assume there exists \( i_0 \in I \) such that \( \omega(i_0, j) = \omega'(i_0, j) \) for all \( j \in I \). Then \( \omega = \omega' \).

Definition 3.4. We denote by \( \Omega \) the set of \( \omega : I \times I \to \mathbb{Z} \) such that \( \omega(i, i) = 0 \) for all \( i \in I \) and \( \lambda(i, j, k) \geq 0 \) for all \( i, j, k \in I \). Also, for each \( \nu \in \text{Aut}(I) \), we set \( I(\nu) = \{ i \in I \mid \nu(i) = i \} \)

and denote by \( \Omega(\nu) \) the set of \( \omega \in \Omega \) such that

\( (1) \) \( \lambda(i, j, \nu(i)) = 0 \) for all \( i \in N(\nu) \) and \( j \in I \), and

\( (2) \) there exists \( \tau_\omega \geq 0 \) such that \( \lambda(i, j, i) = \tau_\omega \) for all \( i \in I(\nu) \) and \( j \in N(i) \).

Lemma 3.5. For any \( \omega \in \Omega \) and \( i_0 \in I \) the following hold.

\( (1) \) If \( \lambda(i_0, j, i_0) = 0 \) for all \( j \in I \), then \( \omega = 0 \).

\( (2) \) If \( \omega(i_0, j) = 0 \) for all \( j \in I \), then \( \omega \geq 0 \).

\( (3) \) There exists \( \omega' \in \Omega \) such that \( \omega \equiv \omega' \), \( \omega' \geq 0 \) and \( \omega'(i_0, j) = 0 \) for all \( j \in I \).

Definition 3.6. We denote by \( \Omega_\omega \) the set of \( \omega \in \Omega \) such that \( \lambda(i, j, i) > 0 \) for all \( i, j \in I \).
with $i \neq j$. Also, we set $\Omega_\nu(\nu) = \Omega_\nu \cap \Omega(\nu)$ for $\nu \in \text{Aut}(I)$.

**Proposition 3.7.** For any $\nu \in \text{Aut}(I)$ we have $\Omega_\nu(\nu) \neq \emptyset$.

4. Automorphisms of $A_\omega$

In this section, we show that for any $\nu \in \text{Aut}(I)$ and $\omega \in \Omega$ there exists $\eta \in \text{Aut}(A_\omega)$ with $\eta(e_\nu) = e_{\omega \cup \nu}$, for all $i, j \in I$.

**Lemma 4.1.** For any $\omega, \omega' \in \Omega$ the following hold.

1. If there exists $\nu \in \text{Aut}(I)$ such that $\omega' = \omega \circ (\nu \times \nu)$, then there exists a ring isomorphism of the form

$$
\xi : A_\omega \to A_{\omega'}, \sum_{i,j \in I} e_{i}x_{ij} \mapsto \sum_{i,j \in I} e_{i\omega'}x_{\omega'j}.
$$

2. If there exists $\chi : I \to \mathbb{Z}$ such that $\omega'(i, j) = \omega(i, j) - \chi(i) + \chi(j)$ for all $i, j \in I$, then there exists a ring isomorphism of the form

$$
\xi : A_\omega \to A_{\omega'}, \sum_{i,j \in I} e_{i}x_{ij} \mapsto \sum_{i,j \in I} e_{i\omega'}\sigma^{\chi}(x_{ij}).
$$

**Proposition 4.2.** For any $\nu \in \text{Aut}(I)$ and $\omega \in \Omega(\nu)$ the following hold.

1. Define $\chi : I \to \mathbb{Z}$ as follows:

$$
\chi(i) = \begin{cases} 
\omega(i, \nu(i)) & \text{if } i \not\in I(\nu), \\
\tau_\omega & \text{if } i \in I(\nu).
\end{cases}
$$

Then $\omega(\nu(i), \nu(j)) = \omega(i, j) - \chi(i) + \chi(j)$ for all $i, j \in I$.

2. There exist ring automorphisms of the form

$$
\theta : A_\omega \to A_{\omega'}, \sum_{i,j \in I} e_{i}x_{ij} \mapsto \sum_{i,j \in I} e_{i\omega'}\sigma^{\chi}(x_{\omega'j}),
$$

$$
\eta : A_\omega \to A_{\omega'}, \sum_{i,j \in I} e_{i}x_{ij} \mapsto \sum_{i,j \in I} e_{i\omega}\sigma^{\chi}(x_{ij})
$$

which are mutually inverse.

3. Let $e_0 = \sum_{i \in I(\nu)} e_{i\nu} + \sum_{i \in I(\mu)} e_{i\mu} c'$, where $t = t_\omega$ if $I(\nu) \neq \emptyset$. Then $\eta(e_0) = e_0$ and $ae_0 = e_0\eta(a)$ for all $a \in A_\omega$.

5. The case of $\nu$ with $I(\nu) = \emptyset$

In this section, we deal with the case of $\nu \in \text{Aut}(I)$ with $I(\nu) = \emptyset$. Let $\omega \in \Omega_\nu(\nu)$ and $\nu = A_\omega$. We set $\alpha_0 = \sum_{i \in I} \alpha_{i\omega}$ and $\beta_0 = \sum_{i \in I} \beta_{i\omega}$ (see Definition 2.4). By Proposition 4.2, we have ring automorphisms.
\[ \theta : A \rightarrow A, \sum_{i,j \in I} e_i x_j \mapsto \sum_{i,j \in I} e_i \sigma^{(0)}(x_{ui}, x_j), \]
\[ \eta : A \rightarrow A, \sum_{i,j \in I} e_i x_j \mapsto \sum_{i,j \in I} e_i \sigma^{(0)}(x_{ui}, x_j) \]

which are mutually inverse.

Lemma 5.1. There exist isomorphisms of the form

\[ \phi : A \rightarrow \text{Hom}_R(Ae_{\sigma}, R), a \mapsto \alpha_a a, \]
\[ \psi : aA \rightarrow \text{Hom}_R(e_{\sigma}A_{\sigma}, R), a \mapsto a\beta_a. \]

Remark 5.2. The following hold.

(1) \( \theta \) is the unique ring automorphism of \( A \) such that \( \theta(e_{ui}, x_j) = e_i \) for all \( i, j \in I \) and \( x \alpha_0 = \alpha_0 \theta(x) \) for all \( x \in R \).

(2) \( \eta \) is the unique ring automorphism of \( A \) such that \( \eta(e_i) = e_{ui}, x_0 \) for all \( i, j \in I \) and \( \beta_0 x = \eta(x) \beta_0 \) for all \( x \in R \).

Theorem 5.3. The ring \( A \) is a Frobenius extensin of \( R \).

Proposition 5.4. Assume \( R \) is a quasi-Frobenius local ring. Then \( A \) is a quasi-Frobenius ring with \( \text{soc}(e_{\sigma}A_{\sigma}) \cong e_{ui}/A/\text{soc}(e_{ui}) \) for all \( i \in I \), where \( J = \text{rad}(A) \).

6. Another base ring

In this section, we prepare another base ring \( S \) which we need in the next section. We fix an integer \( t > 0 \).

Definition 6.1. Let \( S \) be a free right \( R \)-module with a basis \( \{ e, v \} \) and define the multiplication on \( S \) subject to the following axioms:

(S1) \( e^2 = e, v^2 = -v e \) and \( e v = v e = e v \); and

(S2) \( x e = e x \) and \( x v = v \sigma(x) \) for all \( x \in R \).

Lemma 6.2. The following hold.

(1) \( S \) is an associative ring with \( 1_S = e \).

(2) We have ring homomorphisms \( \varphi : R \rightarrow S, x \mapsto e x \) and \( \pi : S \rightarrow R, (e x + v y) \mapsto x \) with \( \pi \varphi = \text{id} \).

(3) \( S \) is a local ring if \( R \) is so.

In the following, we consider \( R \) as a subring of \( S \) via \( \varphi : R \rightarrow S \). Note that \( S \) is a free left \( R \)-module with a basis \( \{ e, v \} \).

Definition 6.3. There exists a basis \( \{ \alpha, \mu \} \) for \( \text{Hom}_R(S, \text{Hom}_R(S, R)) \) such that \( b = e \alpha(b) + v \mu(b) \) for all \( b \in S \). Similarly, there exists a basis \( \{ \beta, \rho \} \) for \( \text{Hom}_R(S, \text{Hom}_R(S, R)) \) such that \( b = \beta(b) e + \rho(b) v \) for all \( b \in S \).
Lemma 6.4. There exist isomorphisms of the form

\[ \phi : S_2 \rightarrow \text{Hom}_\mathbb{R}(S_\mathbb{R}, R_\mathbb{R}), \quad b \mapsto \mu b, \]
\[ \psi : S \rightarrow \text{Hom}_\mathbb{R}(S_\mathbb{R}, R_\mathbb{R}), \quad b \mapsto b \rho. \]

Theorem 6.5. The ring S is a Frobenius extension of R.

Proposition 6.6. There exist ring automorphisms of the form

\[ \theta : S \rightarrow S, \quad x + vy \mapsto e\sigma'(x) + v\sigma'(y), \]
\[ \eta : S \rightarrow S, \quad x + vy \mapsto e\sigma(x) + v\sigma(y) \]

which are mutually inverse.

Remark 6.7. The following hold.
(1) \( \theta \) is the unique ring automorphism of S such that \( \theta(v) = v \) and \( x \mu = \mu \theta(x) \) for all \( x \in R \).
(2) \( \eta \) is the unique ring automorphism of S such that \( \eta(v) = v \) and \( \alpha x = \eta(x) \rho \) for all \( x \in R \).

7. The case of \( \nu \) with \( I(\nu) \neq \emptyset \)

In this and the next sections, we deal with the case of \( \nu \) with \( I(\nu) \neq \emptyset \). Let \( \omega \in \Omega_\nu(v) \) and \( t = t_\alpha \). We construct a Frobenius extension B of R which contains an ideal V with \( B/V \cong A_{\sigma} \) where \( V^2 \neq 0 \) in general.

Remark 7.1. It may happen that \( \omega \in \Omega_\nu(\tau) \) for some \( \tau \) with \( I(\tau) = \emptyset \). If this is the case, \( A_\omega \) itself is a Frobenius extension of R.

Definition 7.2. Let B be a free right R-module with a basis \( \{e_i\}_{i \in I} \cup \{v_i\}_{i \in I \setminus \mathbb{N}} \) and define the multiplication on B subject to the following axioms:

(B1) \( xe_i = e_i \sigma(x) \) for all \( i, j \in I \) and \( x \in R \);
(B2) \( e_i e_j = 0 \) unless \( j = k \);
(B3) \( e_i e_{k} = e_i \sigma^{K(i,j)} \) unless \( i = k \in I(\nu) \) and \( j \in I(\nu) \);
(B4) \( e_i e_j = 0 \) for all \( i \in I(\nu) \) and \( j \in I(\nu) \);
(B5) \( x v_i = v_i \sigma(x) \) for all \( i \in I(\nu) \) and \( x \in R \);
(B6) \( v_i v_k = 0 \) unless \( i = j \);
(B7) \( v_i \nu_j = v_i \nu_j \) for all \( i \in I(\nu) \);
(B8) \( v_i \nu_j = 0 \) unless \( i = j \);
(B9) \( v_i \nu_j = -v_i \sigma(x) \) for all \( i \in I(\nu) \).

Proposition 7.3. The following hold.
(1) \( B \) is an associative ring with \( 1_B = \sum_i e_i \), where the \( e_i \) are orthogonal idempotents.
(2) We have an injective ring homomorphism \( \varphi : R \rightarrow B, \ x \mapsto \sum_i e_i x \) which is a split.
monomorphism of $R$-$R$-bimodules.

(3) $e_i B e_j \equiv e_j B e_i$ only when $i = j$.

(4) $V = \sum_{a, \omega} \nu R$ is an ideal of $B$ with $B V = A_\omega$.

In the following, we consider $R$ as a subring of $B$ via $\varphi : R \to B$. Note that $B$ is a free left $R$-module with a basis $\{e_i\}_{i \in I} \cup \{v\}_{i \in \nu}$, Also, for any $i \in I$, since $x e_i = e_i x$ for all $x \in R$, $B e_i$ is a $B$-$R$-bimodule and $e_i B$ is an $R$-$B$-bimodule.

Definition 7.4. There exists a basis $\{\alpha_i\}_{i \in \nu} \cup \{\mu_i\}_{i \in \nu R}$ for \(\text{Hom}_R(B_{R'} R_{R'})\) such that $b = \sum_{i \in \nu} e_i \alpha_i(b) + \sum_{i \in \nu R} \nu \beta_i(b)$ for all $b \in B$. Similarly, we have a basis $\{\beta_i\}_{i \in \nu} \cup \{\rho_i\}_{i \in \nu R}$ for \(\text{Hom}_R(B_{R'} R_{R'})\) such that $b = \sum_{i \in \nu} \beta_i(b)e_i + \sum_{i \in \nu R} \rho_i(b)v_i$ for all $b \in B$. We set

\[
\mu_0 = \sum_{i \in \nu} \alpha_i \chi_{\nu} + \sum_{i \in \nu R} \mu_i \chi_{\nu R} \quad \text{and} \quad \rho_0 = \sum_{i \in \nu R} \beta_i \chi_{\nu R} + \sum_{i \in \nu R} \rho_i \chi_{\nu R}.
\]

Lemma 7.5. The following hold.

1. For any $i \in I(\nu)$ there exists isomorphisms of the form

\[
\phi_i : e_i B e_i \cong \text{Hom}_R(B_{e_i R_{e_i R}}, R_R), \quad b \mapsto \mu_i b,
\]

\[
\psi_i : B e_i \cong \text{Hom}_R(B e_i R_{(e_i R)}, R_R), \quad b \mapsto b \rho_i.
\]

2. For any $i \in N(\nu)$ there exists isomorphisms of the form

\[
\phi_i : e_i B e_i \cong \text{Hom}_R(B_{e_i R_{e_i R}}, R_R), \quad b \mapsto \alpha_i \chi_{\nu 0} b,
\]

\[
\psi_i : B e_i \cong \text{Hom}_R(B e_i R_{(e_i R)}, R_R), \quad b \mapsto b \beta_i \chi_{\nu 0}.
\]

3. There exist isomorphisms of the form

\[
\phi : B _{e_i} \cong \text{Hom}_R(B_{e_i R_{e_i R}}, R_R), \quad b \mapsto \mu_0 b,
\]

\[
\psi : B_{e_i} \cong \text{Hom}_R(B_{e_i R_{e_i R}}, R_R), \quad b \mapsto b \rho_0.
\]

Theorem 7.8. The ring $B$ is a Frobenius extension of $R$.

Proposition 7.9. Assume $R$ is a local ring. Then the following hold.

1. Every $e_i$ is a local idempotent. In particular, $B$ is a semiperfect ring.

2. Assume further that $R$ is a quasi-Frobenius local ring. Then $B$ is a quasi-Frobenius ring with $\text{soc}(e_i B_{e_i}) \equiv e_{\nu_0, \nu_0} B e_{\nu_0, \nu_0} I_{e_0, e_0}$ for all $i \in I$, where $J = \text{rad}(B)$.

In the next section, we do not fix $\omega \in \Omega(\nu)$ and use the notation $B_\omega$ to denote that the multiplication of $B$ is defined by $\omega$.

8. Automorphisms of $B_\omega$
In this section, we show that for any $\omega \in \Omega_b(v)$ there exists $\eta \in \text{Aut}(B_\omega)$ such that $\eta(e_i) = e_{\omega(i), \omega(j)}$ for all $i, j \in I$ and $\eta(v_i) = v_i$ for all $i \in I(v)$.

**Lemma 8.1.** For any $\omega, \omega' \in \Omega_b(v)$ the following hold.

1. If there exists $\tau \in \text{Aut}(I)$ with $\tau v = v \tau$ such that $\omega' = \omega \circ (\tau \times \tau)$, then there exists a ring isomorphism of the form

$$\xi : B_\omega \rightarrow B_{\omega'}, b \mapsto \sum_{i,j \in I} e_{\omega(i), \omega(j)}(b) + \sum_{i \in I(v)} v_i \mu_i(b).$$

2. If there exists $\chi : I \rightarrow \mathbb{Z}$ such that $\omega'(i, j) = \omega(i, j) - \chi(i) + \chi(j)$ for all $i, j \in I$, then there exists a ring isomorphism of the form

$$\xi : B_\omega \rightarrow B_{\omega'}, b \mapsto \sum_{i,j \in I} e_{\omega(i), \omega(j)}(b) + \sum_{i \in I(v)} v_i \sigma_{\omega(i)}(\mu_i(b)).$$

**Proposition 8.2.** For any $\omega \in \Omega_b(v)$ the following hold.

1. Define $\chi : I \rightarrow \mathbb{Z}$ as follows:

$$\chi(i) = \begin{cases} \omega(i, v(i)) & \text{if } i \notin I(v), \\ t_o & \text{if } i \in I(v). \end{cases}$$

Then $\omega(v(i), v(j)) = \omega(i, j) - \chi(i) + \chi(j)$ for all $i, j \in I$.

2. There exist ring automorphisms of the form

$$\theta : B_\omega \rightarrow B_\omega, b \mapsto \sum_{i,j \in I} e_{\chi(i), \chi(j)}( \sigma_{\omega(i)}(\mu_i(b))) + \sum_{i \in I(v)} v_i \sigma_{\omega(i)}(\mu_i(b)),$$

$$\eta : B_\omega \rightarrow B_\omega, b \mapsto \sum_{i,j \in I} e_{\chi(i), \chi(j)}( \sigma_{\omega(i)}(\mu_i(b))) + \sum_{i \in I(v)} v_i \sigma_{\omega(i)}(\mu_i(b))$$

which are mutually inverse.

3. Let $e_0 = \sum_{i \in I(v)} e_{i, v_i} e_i'$ and $v_0 = \sum_{i \in I(v)} v_i$, where $t = t_o$. Then $\eta(e_0) = e_0$ and $\eta(v_0) = v_0$. Also, $be_0 = e_0 \eta(b)$ and $bv_0 = v_0 \eta(b)$ for all $b \in B_\omega$.

**Remark 8.3.** For any $\omega \in \Omega_b(v)$ the following hold.

1. $\theta$ is the unique ring automorphism of $B_\omega$ such that $\theta(e_{\omega(i), \omega(j)}) = e_{\omega(i), \omega(j)}$ for all $i, j \in I$ and $x \mu_0 = \mu_0 \theta(x)$ for all $x \in R$.

2. $\eta$ is the unique ring automorphism of $B_\omega$ such that $\eta(e_{\omega(i), \omega(j)}) = e_{\omega(i), \omega(j)}$ for all $i, j \in I$ and $\rho_0 x = \eta(x) \rho_0$ for all $x \in R$.

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HIGHER DIMENSIONAL AUSLANDER-REITEN THEORY ON MAXIMAL ORTHOGONAL SUBCATEGORIES

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ABSTRACT. Auslander-Reiten theory, especially the concept of almost split sequences and their existence theorem, is fundamental to study categories which appear in representation theory, for example, modules over artin algebras $[ARS][GR][R]$, their functorially finite subcategories $[AS][S]$, their derived categories $[H]$, Cohen-Macaulay modules over Cohen-Macaulay rings $[Y]$, lattices over orders $[A2,3][RS]$, and coherent sheaves on projective curves $[AR][GL]$. In this Auslander-Reiten theory, the number '2' is quite symbolic. For one thing, almost split sequences give minimal projective resolutions of simple objects of projective dimension '2' in functor categories. For another, Cohen-Macaulay rings and orders of Krull-dimension '2' have fundamental sequences and provide us one of the most beautiful situations in representation theory $[A4][E][RS][Y]$, which is closely related to McKay's observation on simple singularities $[M]$. In this sense, usual Auslander-Reiten theory should be '2-dimensional' theory, and it would have natural importance to search a domain of higher Auslander-Reiten theory from the viewpoint of representation theory and non-commutative algebraic geometry (e.g. $[V1,2][Ar][GL]$). In this paper, we introduce $(n-1)$-orthogonal subcategories as a natural domain of $(n+1)$-dimensional Auslander-Reiten theory. We show that higher Auslander-Reiten translation and higher Auslander-Reiten duality can be defined quite naturally for such categories. Using them, we show that our categories have $n$-almost split sequences, which are completely new generalization of usual almost split sequences and give minimal projective resolutions of simple objects of projective dimension 'n+1' in functor categories. We also show the existence of higher dimensional analogy of fundamental sequences for Cohen-Macaulay rings and orders of Krull-dimension 'n+1'. We show that an invariant subgroup (of Krull-dimension 'n+1') corresponding to a finite subgroup $G$ of $GL_{n+1}(k)$ has a natural maximal $(n-1)$-orthogonal subcategory.

1 From Auslander-reiten theory

1.1 Let us recall M. Auslander's classical theorem [A1] below, which introduced a completely new insight to representation theory of algebras (see 2.3 for dom.dim $\Gamma$).

Theorem A (Auslander correspondence) There exists a bijection between the set of Morita-equivalence classes of representation-finite finite-dimensional algebras $\Lambda$ and that of finite-dimensional algebras $\Gamma$ with $gl.dim \Gamma \leq 2$ and $dom.dim \Gamma \geq 2$. It is given by $\Lambda \mapsto \Gamma := \text{End}_\Lambda(M)$ for an additive generator $M$ of mod $\Lambda$.

In this really surprising theorem, the representation theory of $\Lambda$ is encoded in the structure of the homologically nice algebra $\Gamma$ called an Auslander algebra. Since the category $\text{mod} \Gamma$ is equivalent to the functor category on $\text{mod} \Lambda$, Auslander correspondence

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The detailed version [I2,3] of this paper have been submitted for publication elsewhere.
I.4. A theorem of Auslander-Reiten theory is 2-dimensional-like. Now we
propose to connect with the McKay graph \( \gamma \) of \( G \).

Then \( \gamma \) is representation-finite with \( \text{CDIM} = 2 \) and the Auslander-Reiten quiver
is important subtending. Assume that does not contain pseudo-representation except the identity.

1.3. Let \( q \) be a quiver embedding of \( \text{CDIM} = 2 \) and \( \gamma \) be

1.2. Let \( A \) be an Auslander-Reiten quiver of \( \gamma \).

1.1. For any \( X \in \text{CDIM} \), the case is 2 is very nice. Using
 Auslander-Reiten theory, we can define the

1.0. In this section, we can see that Auslander-Reiten theory for

Theorem 1.1. (Australander-Reiten translation) There exists an equivalence \( \text{CDIM} \).

Theorem 1.2. (Australander-Reiten translation) There exists an equivalence \( \text{CDIM} \).

Theorem 1.3. (Australander-Reiten translation) There exists an equivalence \( \text{CDIM} \).

Theorem 1.4. (Australander-Reiten translation) There exists an equivalence \( \text{CDIM} \).

Theorem 1.5. (Australander-Reiten translation) There exists an equivalence \( \text{CDIM} \).

Theorem 1.6. (Australander-Reiten translation) There exists an equivalence \( \text{CDIM} \).
Auslander-Reiten theory. Namely, find natural categories \( C \) such that Theorems above replaced ‘2’ and \( \text{CM} \Lambda \) by ‘\( n+1 \)’ and \( C \) respectively hold.

2 Main results

2.1 Definition Let \( A \) be an abelian category, \( B \) a full subcategory of \( A \) and \( n \geq 0 \). For a functorially finite \([AS]\) full subcategory \( C \) of \( B \), we put

\[
\mathcal{C}^{\perp n} := \{ X \in B \mid \text{Ext}^i_A(C,X) = 0 \text{ for any } i (0 < i \leq n) \},
\]

\[
\mathcal{C}^{\perp n} := \{ X \in B \mid \text{Ext}^i_A(X,C) = 0 \text{ for any } i (0 < i \leq n) \}.
\]

We call \( C \) a maximal \( n \)-orthogonal subcategory of \( B \) if

\[
C = \mathcal{C}^{\perp n} = \mathcal{C}^{\perp n}
\]

holds. By definition, \( B \) is a unique maximal 0-orthogonal subcategory of \( B \).

2.2 Example Let \( \Lambda \) be a simple singularity of type \( \Delta \) and dimension \( d = 2 \), \( A := \text{mod}^g \Lambda \) the category of graded \( \Lambda \)-modules and \( B := \text{CM}^g \Lambda \) the category of graded Cohen-Macaulay \( \Lambda \)-modules. Then the number of maximal 1-orthogonal subcategories of \( B \) is given as follows:

<table>
<thead>
<tr>
<th>( \Delta ) number</th>
<th>( A_m )</th>
<th>( B_m,C_m )</th>
<th>( D_m )</th>
<th>( E_6 )</th>
<th>( E_6 )</th>
<th>( E_8 )</th>
<th>( F_4 )</th>
<th>( G_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{m+2} \frac{2m+2}{m+1} )</td>
<td>( \frac{2m}{m} )</td>
<td>( \frac{3m-2}{m} \frac{2m-2}{m-1} )</td>
<td>833</td>
<td>4160</td>
<td>25080</td>
<td>105</td>
<td>8</td>
<td></td>
</tr>
</tbody>
</table>

This is obtained by showing that maximal 1-orthogonal subcategories of \( B \) correspond bijectively to clusters of the cluster algebra of type \( \Delta \) \([2,3]\). See Fomin-Zelevinsky \([FZ1,2]\) and Buan-Marsh-Reineke-Reiten-Teleor [BMRRT]. See also Geiss-Leclerc-Schröer [GLS].

2.3 For a finite-dimensional algebra \( \Gamma \), we denote by \( 0 \rightarrow \Gamma \rightarrow I_0 \rightarrow I_1 \rightarrow \cdots \) a minimal injective resolution of the \( \Gamma \)-module \( \Gamma \). Put \( \text{dom.dim} \Gamma := \inf\{i \geq 0 \mid I_i \text{ is not projective} \} \[T\]. The following theorem gives a higher dimensional version of Theorem A.

**Theorem A’** ((\( n+1 \))-dimensional Auslander correspondence) For any \( n \geq 1 \), there exists a bijection between the set of equivalence classes of maximal \((n-1)\)-orthogonal subcategories \( C \) of \( \text{mod} \Lambda \) with additive generators \( M \) and finite-dimensional algebras \( \Lambda \), and the set of Morita-equivalence classes of finite-dimensional algebras \( \Gamma \) with \( \text{gl.dim} \Gamma \leq n+1 \) and \( \text{dom.dim} \Gamma \geq n+1 \). It is given by \( C \mapsto \Gamma \) := \( \text{End}_A(M) \).

2.4 In the rest of this section, let \( R \) be a complete regular local ring of dimension \( d \), \( \Lambda \) an \( R \)-order which is an isolated singularity, \( A := \text{mod} \Lambda \) and \( B := \text{CM} \Lambda \). For \( n \geq 1 \), we define functors \( \tau_n \) and \( \tau_n^- \) by

\[
\tau_n := \tau \circ \Omega^{n-1} : \text{CM} \Lambda \rightarrow \text{CM} \Lambda \quad \text{and} \quad \tau_n^- := \tau^- \circ \Omega^{-(n-1)} : \text{CM} \Lambda \rightarrow \text{CM} \Lambda,
\]

where \( \Omega : \text{CM} \Lambda \rightarrow \text{CM} \Lambda \) is the syzygy functor and \( \Omega^- : \text{CM} \Lambda \rightarrow \text{CM} \Lambda \) is the cosyzygy functor. For a subcategory \( C \) of \( \text{CM} \Lambda \), we denote by \( \mathcal{C} \) and \( \mathcal{C} \) the corresponding subcategories of \( \text{CM} \Lambda \) and \( \text{CM} \Lambda \) respectively.

**Theorem B’** Let \( C \) be a maximal \((n-1)\)-orthogonal subcategory of \( \text{CM} \Lambda \) \((n \geq 1)\).
(1) (n-Auslander-Reiten translation) For any $X \in \mathcal{C}$, $\tau_n X \in \mathcal{C}$ and $\tau_n^{-1} X \in \mathcal{C}$ hold. Thus $\tau_n : \mathcal{C} \to \mathcal{C}$ and $\tau_n^{-1} : \mathcal{C} \to \mathcal{C}$ are mutually quasi-inverse equivalences.

(2) (n-Auslander-Reiten duality) There exist functorial isomorphisms $\mathcal{C}(Y, \tau_n X) \cong D \text{Ext}^n_{\Lambda}(X, Y) \cong \mathcal{C}(\tau_n^{-1} Y, X)$ for any $X, Y \in \mathcal{C}$.

2.5 Definition Let $\mathcal{C}$ be a full subcategory of $\text{CM} \Lambda$ and $J_\mathcal{C}$ the Jacobson radical of $\mathcal{C}$. We call an exact sequence

$$0 \to Y \xrightarrow{f_0} C_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} C_0 \xrightarrow{f_0} X \to 0$$

(resp. $0 \to Y \xrightarrow{f_0} C_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} C_0 \xrightarrow{f_0} X$) with terms in $\mathcal{C}$ an $n$-almost split sequence (resp. pseudo $n$-almost split sequence) if $f_i \in J_\mathcal{C}$ holds for any $i$ and the following sequences are exact.

$$0 \to \mathcal{C}(\cdot, Y) \xrightarrow{f_0} \mathcal{C}(\cdot, C_{n-1}) \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} \mathcal{C}(\cdot, C_0) \xrightarrow{f_0} \mathcal{J}_\mathcal{C}(\cdot, X) \to 0$$

$$0 \to \mathcal{C}(X, \cdot) \xrightarrow{f_0} \mathcal{C}(X, C_0) \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} \mathcal{C}(X, C_{n-1}) \xrightarrow{f_0} \mathcal{J}_\mathcal{C}(Y, \cdot) \to 0$$

We call $f_0 : C_0 \to X$ a sink map and $f_n : Y \to C_{n-1}$ a source map. We say that $\mathcal{C}$ has $n$-almost split sequences if, for any non-projective $X \in \text{ind} \mathcal{C}$ (resp. non-injective $Y \in \text{ind} \mathcal{C}$), there exists an $n$-almost split sequence $0 \to Y \to C_{n-1} \to \cdots \to C_0 \to X \to 0$. Similarly, we say that $\mathcal{C}$ has pseudo $n$-almost split sequences if, for any projective $X \in \mathcal{C}$ (resp. injective $Y \in \mathcal{C}$), there exists a pseudo $n$-almost split sequence $0 \to Y \to C_{n-1} \to \cdots \to C_0 \to X \to 0$.

Theorem $\mathcal{C}'$ Let $\mathcal{C}$ be a maximal $(n - 1)$-orthogonal subcategory of $\text{CM} \Lambda$ $(n \geq 1)$.

1. $\mathcal{C}$ has $n$-almost split sequences.

2. If $d = n + 1$, then $\mathcal{C}$ has pseudo $n$-almost split sequences.

Consequently, almost all simple objects in the functor category $\text{mod} \mathcal{C}$ have projective dimension $n + 1$. If $d = n + 1$, then all simple objects in the functor category $\text{mod} \mathcal{C}$ have projective dimension $n + 1$. In this sense, we can say that $(n + 1)$-dimensional Auslander-Reiten theory for the case $d = n + 1$ is very nice.

2.6 We will define the Auslander-Reiten quiver $\mathfrak{A}(\mathcal{C})$ of $\mathcal{C}$. For simplicity, we assume that the residue field $k$ of $R$ is algebraically closed. The set of vertices of $\mathfrak{A}(\mathcal{C})$ is $\text{ind} \mathcal{C}$. For $X, Y \in \text{ind} \mathcal{C}$, we denote by $d_{XY}$ be the multiplicity of $X$ in $\mathcal{C}$ for the sink map $C \to Y$, which equals to the multiplicity of $Y$ in $\mathcal{C}'$ for the source map $X \to \mathcal{C}'$. Draw $d_{XY}$ arrows from $X$ to $Y$.

Theorem $\mathcal{D}'$ Let $G$ be a finite subgroup of $\text{GL}_d(\mathbb{C})$, $\Omega := \mathbb{C}[x_1, \ldots, x_d]$ and $\Lambda := \Omega^G$ the invariant subring. Assume that $G$ does not contain pseudo-reflection except the identity, and that $\Lambda$ is an isolated singularity. Then $\mathcal{C} := \text{add}_\Lambda \Omega$ is a maximal $(d - 2)$-orthogonal subcategory of $\text{CM} \Lambda$. Moreover, the Auslander-Reiten quiver $\mathfrak{A}(\mathcal{C})$ of $\mathcal{C}$ coincides with the McKay quiver $\mathfrak{M}(G)$ of $G$, i.e. there exists a bijection $\mathfrak{H} : \text{irr} G \to \text{ind} \mathcal{C}$ such that $d_{XY} = d_{\mathfrak{H}(X), \mathfrak{H}(Y)}$ for any $X, Y \in \text{irr} G$.

3 Non-commutative crepant resolution and representation dimension

3.1 Let us generalize the concept of Van den Bergh's non-commutative crepant resolution [V1,2] of commutative normal Gorenstein domains to our situation.
Again let $\Lambda$ be an $R$-order which is an isolated singularity. We call $M \in \text{CM} \Lambda$ a NCC resolution of $\Lambda$ if $\Lambda \oplus D_d \Lambda \in \text{add} \ M$ and $\Gamma := \text{End}_R(M)$ is an $R$-order with $\text{gl.dim} \Gamma = d$. Our definition is slightly stronger than original non-commutative crepant resolutions in [V2] where $M$ is assumed to be reflexive (not Cohen-Macaulay) and $\Lambda \oplus D_d \Lambda \in \text{add} \ M$ is not assumed. But all examples of non-commutative crepant resolutions in [V1,2] satisfy our condition. For the case $d \geq 2$, we have the remarkable relationship below between NCC resolutions and maximal $(d - 2)$-orthogonal subcategories.

**Theorem** Let $d \geq 2$. Then $M \in \text{CM} \Lambda$ is a NCC resolution of $\Lambda$ if and only if $\text{add} \ M$ is maximal $(d - 2)$-orthogonal subcategory of $\text{CM} \Lambda$.

### 3.2 Conjecture

It is interesting to study relationship among all maximal $(n - 1)$-orthogonal subcategories of $\text{CM} \Lambda$. Especially, we conjecture that their endomorphism rings are derived equivalent. It is suggestive to relate this conjecture to Van den Bergh’s generalization [V2] of Bondal-Orlov conjecture [BO], which asserts that all (commutative or non-commutative) crepant resolutions of a normal Gorenstein domain have the same derived category. Since maximal $(n - 1)$-orthogonal subcategories are analogy of non-commutative crepant resolutions from the viewpoint of 3.1, our conjecture is an analogy of Bondal-Orlov-Van den Bergh conjecture. We have the following partial solution.

**Theorem** (1) Let $C_i = \text{add} M_i$ be a maximal 1-orthogonal subcategory of $\text{CM} \Lambda$ and $\Gamma_i := \text{End}_R(M_i)$ $(i = 1, 2)$. Then $\Gamma_1$ and $\Gamma_2$ are derived equivalent. In particular, $\# \text{ind} C_1 = \# \text{ind} C_2$ holds.

(2) If $d \leq 3$, then all NCC resolutions of $\Lambda$ have the same derived category.

### 3.3 Let us generalize the concept of Auslander’s representation dimension [A1] to relate it to non-commutative crepant resolutions. For $n \geq 1$, define the $n$-th representation dimension $\text{rep.dim}_n \Lambda$ of an $R$-order $\Lambda$ which is an isolated singularity by

$$\text{rep.dim}_n \Lambda := \inf \{ \text{gl.dim} \text{End}_R(M) \mid M \in \text{CM} \Lambda, \ \Lambda \oplus D_d \Lambda \in \text{add} M, \ M \downarrow_{n-1} M \}.$$  

Obviously $d \leq \text{rep.dim}_n \Lambda \leq \text{rep.dim}_{n'} \Lambda$ holds for any $n \leq n'$. For the case $d = 0$, $\text{rep.dim}_1 \Lambda$ coincides with the representation dimension defined in [A1]. We call $\Lambda$ representation-finite if $\# \text{ind}(\text{CM} \Lambda) < \infty$. In the sense of (2) below, $\text{rep.dim}_1 \Lambda$ measures how far $\Lambda$ is from being representation-finite.

**Theorem** (1) Assume $d \leq n + 1$. Then $\text{CM} \Lambda$ has a maximal $(n - 1)$-orthogonal subcategory $C$ with $\# \text{ind} C < \infty$ if and only if $\text{rep.dim}_n \Lambda \leq n + 1$.

(2) Assume $d \leq 2$. Then $\Lambda$ is representation-finite if and only if $\text{rep.dim}_1 \Lambda \leq 2$.

(3) $\Lambda$ has a NCC resolution if and only if $\text{rep.dim}_{\max \{1, d-1\}} \Lambda = d$.

### 3.4 Conjecture

It seems that no example of a maximal $(n - 1)$-orthogonal subcategory $C$ of $\text{CM} \Lambda$ with $\# \text{ind} C = \infty$ is known. This suggests us to study

$$o(\text{CM} \Lambda) := \sup_{C \subseteq \text{CM} \Lambda} \# \text{ind} C.$$  

We conjecture that $o(\text{CM} \Lambda)$ is always finite. If $\Lambda$ is a preprojective algebra of Dynkin type $\Delta$, then Geiss-Schröer [GS] proved that $o(\text{mod} \Lambda)$ equals to the number of positive roots of $\Delta$. It would be interesting to find a geometric interpretation of $o(\text{CM} \Lambda)$ for more
general CMΛ. For some classes of CMΛ, one can calculate o(CMΛ) by using the theorem below. Especially, (1) seems to be interesting in the connection with known results for algebras with representation dimension at most 3 [IT][EHIS].

Theorem (1) rep.dim1 Λ ≤ 3 implies o(CMΛ) < ∞.

(2) If CMΛ has a maximal 1-orthogonal subcategory C, then o(CMΛ) = # ind C.

3.5 Concerning our conjecture, let us recall the well-known proposition below which follows by a geometric argument due to Voigt’s lemma ([P;4.2]). It is interesting to ask whether it is true without the restriction on R. If it is true, then any 1-orthogonal subcategory of CMΛ is ‘discrete’, and our conjecture asserts that it is finite. It is interesting to study the discrete structure of 1-orthogonal objects in CMΛ and the relationship to whole structure of CMΛ.

Proposition Assume that R is an algebraically closed field. For any n > 0, there are only finitely many isoclasses of 1-orthogonal Λ-modules X with dimRX = n.

References


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MORPHISMS REPRESENTED BY MONOMORPHISMS

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ABSTRACT. We answer a question posed by Auslander and Bridger. Every homomorphism of modules is projective-stably equivalent to an epimorphism but is not always to a monomorphism. We prove that a map is projective-stably equivalent to a monomorphism if and only if its kernel is torsionless, that is, a first syzygy. If it occurs although, there can be various monomorphisms that are projective-stably equivalent to a given map. But in this case there uniquely exists a "perfect" monomorphism to which a given map is projective-stably equivalent.

1 Introduction

Let $R$ be a commutative noetherian ring. Linear maps $f : A \to B$ and $f' : A' \to B'$ of finite $R$-modules are said to be projective-stably equivalent (pse for short) if the following diagram is commutative

$$
\begin{array}{ccc}
A \oplus P' & \xrightarrow{(f, s)} & B \oplus Q' \\
\downarrow \cong & & \downarrow \cong \\
A' \oplus P & \xrightarrow{(f', t')} & B' \oplus Q
\end{array}
$$

with some projective modules $P, Q, P', Q'$ and $R$-linear maps $s, t, u, s', t', u'$. We say a morphism $f$ is represented by monomorphisms ("rbm" for short) if there exists a monomorphism that is pse to $f$.

For any homomorphism $f : A \to B$ of $R$-modules, $(f, \rho_B) : A \oplus P_B \to B$ is surjective with a projective cover $\rho_B : P_B \to B$. Thus every morphism is represented by epimorphisms. The choice of epimorphism is unique; if an epimorphism $f'$ is pse to $f$, then two sequences $0 \to \text{Ker} f' \to A' \xrightarrow{f'} B' \to 0$ and $0 \to \text{Ker}(f \rho_B) \to A \oplus P_B \xrightarrow{(f, \rho_B)} B \to 0$ becomes isomorphic after splitting off common projective summands.

The formal analogy to the representations by monomorphisms fails both in existence and in uniqueness. Every morphism is not always represented by monomorphisms (Example 1). Even if a morphism $f$ is rbm, the choice of monomorphism is not unique;

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1The detailed version of this paper has been submitted for publication elsewhere.
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there may be two monomorphisms \( f' \) and \( f'' \) both pse to \( f \) and that \( 0 \to A' \xrightarrow{f'} B' \to \text{Cok } f' \to 0 \) and \( 0 \to A'' \xrightarrow{f''} B'' \to \text{Cok } f'' \to 0 \) are not isomorphic by splitting off common projective summands (Example 2).

The purpose of the paper is finding a condition of a given map to be rbm. Roughly speaking, our problem is to know when an exact sequence of modules

\[
0 \to A \to B \xrightarrow{f} C \to 0
\]

can be modified into an exact sequence

\[
0 \to B \xrightarrow{f} C \to A' \to 0.
\]

Of course the projective stabilization \( \text{mod } R \) of \( \text{mod } R \) is not triangulated in general. So the obstruction for a given map to be rbm should be the obstruction for \( \text{mod } R \) to be triangulated. Our first focus is an analogy to the homotopy category \( \text{K(mod } R) \) of \( R \)-complexes. In [5, Theorem 2.6], the author showed a category equivalence between \( \text{mod } R \)

and a subcategory of \( \text{K(mod } R) \). Due to this equivalence, we describe the obstruction of being rbm with a homology of a complex associated to the given map.

The problem was originally posed by Auslander and Bridger [1]. They proved that a map is rbm if and only if it is pse to a "perfect" monomorphism. An exact sequence of \( R \)-modules is called perfect if its \( R \)-dual is also exact. A perfect monomorphism refers to a monomorphism whose \( R \)-dual is an epimorphism. This is our next focal point. In the case that a map is rbm, the choice of a monomorphism is not unique, but then a perfect monomorphism pse to the given map is uniquely determined up to direct sum of projective modules. (Theorem 3.6.)

Looking at Theorem 3.6, we see that when a morphism is rbm, its pseudo-kernel is always the first syzygy of its pseudo-cokernel. So it is tempting to ask if the equivalent condition of rbm property is that the kernel is a submodule of a free module. This is our third point. Actually, we need to assume the total ring of fractions \( Q(R) \) is Gorenstein: the condition is satisfied for instance if \( R \) is a domain.

Theorem 4.8: Suppose the total ring of fractions \( Q(R) \) of a ring \( R \) is Gorenstein. A morphism \( f \) is rbm if and only if Ker \( f \) is a submodule of a free module.

Let us give easy examples:

**Example 1** Set \( R = k[[X,Y]]/(XY) \) with any field \( k \), \( g : R^2/(\xi)R \to R/(X) \oplus R/(Y) \) with \( g((x) \mod (\xi))r = (a \mod (X), b \mod (Y)) \). Since Ker \( g \cong R/(X,Y) \) is not a first syzygy, \( g \) is not rbm due to Theorem 4.8.

**Example 2** Set \( R = k[[X,Y]]/(XY) \) with any field \( k \), \( f : R^2/(\xi)R \to R^2/(\zeta_2^3)R \) with \( f((x) \mod (\xi))r = (x_0^2) \mod (\zeta_2^3)R \). The map \( f \) is not a monomorphism; Ker \( f \cong R/(X) \oplus R/(Y) \) is a first syzygy. By Theorem 4.8, \( f \) is rbm. In fact, let \( f' : R^2/(\xi)R \to R^2/(\zeta_2^3)R \oplus R^2 \) be defined as \( f'((x) \mod (\xi))r = ((x_0^2) \mod (\zeta_2^3)R, (x_0^2)) \). Obviously \( f' \)
is a monomorphism that is pse to $f$. On the other hand, $f'' : R^2/(\xi_1) R \to R^2/(\xi_2^2) R \oplus R^2$ with $f''((\xi_1) \mod(\xi_1) R) = ((\xi_2^2) \mod(\xi_2^2) R, (\xi_2^2) \mod(\xi_2^2) R)$ is also a monomorphism and pse to $f$. We have two exact sequences

$$\theta_f : 0 \to R^2/(\xi_1) R \xrightarrow{f} R^2/(\xi_2^2) R \oplus R^2 \to R^2/(\xi_1) R \oplus R^2/(\xi_1) R \to 0,$$

and

$$\sigma : 0 \to R^2/(\xi_1) R \xrightarrow{f''} R^2/(\xi_2^2) R \oplus R^2 \to R^2/(\xi_2) R \oplus R^2/(\xi_2) R \to 0,$$

that are not isomorphic. We see $\theta_f$ is perfect but $\sigma$ is not.

## 2 Stable module category and homotopy category

Throughout the paper, $R$ is a commutative noetherian ring, By an "$R$-module" we mean a finitely generated $R$-module. For an $R$-module $M$, $\rho_M : P_M \to M$ denotes a projective cover of $M$.

**Definition 2.1** The projective stabilization $\text{mod} \ R$ is defined as follows.

- Each object of $\text{mod} \ R$ is an object of $\text{mod} \ R$.
- For objects $A, B$ of $\text{mod} \ R$, a set of morphisms from $A$ to $B$ is $\text{Hom}_R(A, B) = \text{Hom}_R(A, B) / \mathcal{P}(A, B)$ where $\mathcal{P}(A, B) := \{ f \in \text{Hom}_R(A, B) | f \text{ factors through some projective module} \}$. Each element is denoted as $\underline{f} = f \mod \mathcal{P}(A, B)$.

A morphism $\text{mod} \ R$ is called a stable isomorphism if $f$ is an isomorphism in $\text{mod} \ R$. If two $R$-modules $A$ and $A'$ are isomorphic in $\text{mod} \ R$, we say $A$ and $A'$ are stably isomorphic and write $A \overset{\text{st}}{\cong} A'$.

**Definition 2.2** Morphisms $f : A \to B$ and $f' : A' \to B'$ in $\text{mod} \ R$ are said to be projective-stably equivalent (pse for short) and denoted as $f \overset{\text{st}}{=} f'$ if there exist stable isomorphisms $\alpha : A \to A'$ and $\beta : B \to B'$ such that $\beta \circ f = f' \circ \alpha$.

Let $\mathcal{L}$ be a full subcategory of $\mathcal{K}(\text{mod} \ R)$ defined as

$$\mathcal{L} = \{ F^* \in \mathcal{K}(\text{proj} \ R) | H^i(F^*) = 0 (i < 0), \quad H_j(F^*) = 0 (j \geq 0) \}.$$

**Lemma 2.3** ( [5] Proposition 2.3, Proposition 2.4 )

1) For $A \in \text{mod} \ R$, there exists $F_A^* \in \mathcal{L}$ that satisfies

$$H^0(\tau_{\leq 0} F_A^*) \overset{\text{st}}{=} A.$$

Such an $F_A^*$ is uniquely determined by $A$ up to isomorphisms. We fix the notation $F_A^*$ and call this a standard resolution of $A$.
2) For \( f \in \text{Hom}_R(A, B) \), there exists \( f^* \in \text{Hom}_{K(\text{proj}) R}(F_A^*, F_B^*) \) that satisfies

\[
H^0(\tau_{\leq 0} f^*) \cong f.
\]

Such an \( f^* \) is uniquely determined by \( f \) up to isomorphisms, so we use the notation \( f^* \) to describe a chain map with this property for given \( f \).

Theorem 2.4 ([5] Theorem 2.6) The mapping \( A \to F_A^* \) gives a functor from \( \text{mod} R \) to \( K(\text{mod} R) \), and this gives a category equivalence between \( \text{mod} R \) and \( L \).

For \( f \in \text{Hom}_R(A, B) \), there exists a triangle

\[
C(f)^*-1 \overset{\alpha}{\rightarrow} F_A^* \overset{f^*}{\rightarrow} F_B^* \overset{\beta}{\rightarrow} C(f)^*.
\]

(2.1)

In general, \( C(f)^* \) does not belong to \( L \) but it satisfies the following:

\[ H^i(C(f)^*) = 0 \quad (i < -1), \quad H_j(C(f)^*_*) = 0 \quad (j > -1). \]

Definition and Lemma 2.5 ([5], Definition and Lemma 3.1) As objects of \( \text{mod} R \), \( \text{Ker} f := H^{-1}(\tau_{\leq -1} C(f)^*) \) and \( \text{Cok} f := H^0(\tau_{\leq 0} C(f)^*) \) are uniquely determined by \( f \), up to isomorphisms. We call these the pseudo-kernel and the pseudo-cokernel of \( f \).

For a given map \( f : A \to B \), from (2.1), we have an exact sequence

\[
0 \to \text{Ker} f \to A \oplus P \overset{(f, p)}{\to} B \to 0
\]

(2.2)

with some projective module \( P \). This characterizes the pseudo-kernel.

Lemma 2.6 For a given \( f \in \text{Hom}_R(A, B) \), suppose \( A \oplus P' \overset{(f, p')}{\to} B \) is epimorphism with projective module \( P' \). Then \( \text{Ker} (f, p') \cong \text{Ker} f \) and the sequence

\[
0 \to \text{Ker} (f, p') \to A \oplus P' \overset{(f, p')}{\to} B \to 0
\]

is isomorphic to 2.2 after splitting off some split exact sequence of projective modules.

Lemma 2.7 ([5] Lemma 3.6)

1) There is an exact sequence

\[
0 \to \text{Ker} f \to \Omega_R^1(\text{Cok} f) \to 0.
\]

2) There is an exact sequence

\[
0 \to L \to \text{Cok} f \to \text{Cok} f \to 0
\]

such that \( \Omega_R^1(L) \) is the surjective image of \( \text{Ker} f \).
3 Representation by monomorphisms and perfect exact sequences

Definition 3.1 A morphism $f : A \rightarrow B$ in mod $R$ is said to be represented by monomorphisms (rbm for short) if some monomorphism $f' : A' \rightarrow B'$ in mod $R$ is pse to $f$, that is, there exist stable isomorphisms $\alpha : A \rightarrow A'$ and $\beta : B \rightarrow B'$ such that $\beta \circ f = f' \circ \alpha$.

Each morphism is not always rbm. It was Auslander and Bridger who first defined and studied "represented by monomorphisms" property.

Theorem 3.2 (Auslander-Bridger) The following are equivalent for a morphism $f : A \rightarrow B$ in mod $R$.

1) There exists a monomorphism $f' : A \rightarrow B \oplus P$ with a projective module $P$ such that $f = s \circ f'$ via some split epimorphism $s : B \oplus P \rightarrow B$.

2) There exists a monomorphism $f' : A \rightarrow B \oplus P$ with a projective module $P$ such that $f = s \circ f'$ via some split epimorphism $s : B \oplus P \rightarrow B$, and $f^*$ is an epimorphism.

3) $\text{Hom}_R(B, I) \rightarrow \text{Hom}_R(A, I)$ is surjective if $I$ is an injective module.

The condition 1) of Theorem 3.2 turns out to be equivalent to the rbm condition.

Lemma 3.3 For a morphism $f : A \rightarrow B$ in mod $R$, $f$ is rbm if and only if there exists a monomorphism $f' : A \rightarrow B \oplus P$ with a projective module $P$ such that $f = s \circ f'$ via some split epimorphism $s : B \oplus P \rightarrow B$.

The most remarkable point in Auslander-Bridger's Theorem is that being rbm is equivalent to being represented by "perfect monomorphisms" whose $R$-dual is an epimorphism.

Definition 3.4 An exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ of $R$-modules is called a perfect exact sequence or to be perfectly exact if its $R$-dual $0 \rightarrow \text{Hom}_R(C, R) \rightarrow \text{Hom}_R(B, R) \rightarrow \text{Hom}_R(A, R) \rightarrow 0$ is also exact. A monomorphism $f$ is called a perfect monomorphism if $\text{Hom}_R(f, R)$ is an epimorphism.

Proposition 3.5 ([5] Lemma 2.7) The following are equivalent for an exact sequence

$$\theta : 0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0.$$ 

1) $\theta$ is perfectly exact.

2) $0 \rightarrow F_A^* \xrightarrow{f^*} F_B^* \xrightarrow{g^*} F_C^* \rightarrow 0$ is exact.
3) $F_{C^*} \to F_A^* \xrightarrow{f} F_B^* \xrightarrow{g} F_{C^*}$ is a distinguished triangle in $K(\text{mod } R)$.

For a morphism $f : A \to B$, $A \oplus F_B \xrightarrow{(f, \rho_B)} B$ is an epimorphism with a projective cover $\rho_B : F_B \to B$. Thus each morphism is represented by epimorphisms. And the choice of the representing epimorphism is unique up to direct sum of projective modules, as we have seen in Lemma 2.6.

Unlikely, we already know an example of a morphism that is not rbm. And moreover, even if a given map is represented by a monomorphism, there would be another representing monomorphism. (Example 1 and Example 2.)

However, uniqueness theorem is obtained in this way. Due to Theorem 3.2, a morphism is rbm if and only if it is represented by a perfect monomorphism. And if this is the case, the representing perfect monomorphism is uniquely determined up to direct sum of projective modules.

**Theorem 3.6** Let $f : A \to B$ be a morphism in $\text{mod } R$. Then $f$ is rbm if and only if $H^{-1}(C(f)^\circ)$ vanishes. If this is the case, we have the following:

1) We have a perfect exact sequence

$$\theta_f : 0 \to A \xrightarrow{(f)} B \oplus F_A^1 \xrightarrow{(c^f, \pi)} \text{Cok} f \to 0.$$

2) For any exact sequence of the form

$$\sigma : 0 \to A \xrightarrow{(f)} B \oplus P' \xrightarrow{(g, p)} C \to 0$$

with some projective module $P'$, there is a commutative diagram

$$\begin{array}{ccc}
\theta_f & : & 0 \to A \xrightarrow{(f)} B \oplus F_A^1 \xrightarrow{(c^f, \pi)} \text{Cok} f \to 0 \\
\downarrow\alpha & & \downarrow\beta \\
\sigma & : & 0 \to A \xrightarrow{(f)} B \oplus P' \xrightarrow{(g, p)} C \to 0
\end{array}$$

where $\alpha$ and $\beta$ are stable isomorphisms.

3) There is an exact sequence with some projective module $Q$ and $Q'$

$$0 \to Q' \to \text{Cok} f \oplus Q \xrightarrow{(\gamma, p)} C \to 0.$$

In other words, $Ker f$ is projective.

4) If $\sigma$ is also perfectly exact, then $\sigma$ is isomorphic to $\theta_f$ up to direct sum of split exact sequences of projective modules.
proof. We have a triangle

\[ F_A^* \xrightarrow{f} F_B^* \rightarrow C(f)^* \xrightarrow{\eta_f} F_A^{**} \]  

which induces a term-wise exact sequence of complexes in \( \text{C} \text{(proj \text{R})} \)

\[ 0 \rightarrow F_A^* \rightarrow C(n_f)^* \rightarrow C(f)^* \rightarrow 0 \]  

(3.4)

Applying \( \tau_{\leq 0} \) to the diagram above and taking homology, we get the following exact sequence of modules:

\[ \theta_f: \quad 0 \rightarrow H^{-1}(C(f)^*) \rightarrow A \xrightarrow{(f)} B \oplus F_A^1 \xrightarrow{(c_{-1})} \text{Cok}_f \rightarrow 0 \]  

(3.5)

Suppose that \( H^{-1}(C(f)^*) = 0 \). Then \( C(f)^* \cong F_{\text{Cok}_f} \), and the exact sequence 3.4 shows that \( \theta_f \) is perfectly exact.

Conversely, suppose that \( f \) is rmb; there is an exact sequence

\[ \sigma: 0 \rightarrow A \xrightarrow{(f)} B \oplus P' \xrightarrow{(g,p)} C \rightarrow 0. \]

The maps \( \tilde{f} = (f) \) and \( \tilde{g} = (g,p) \) produce the similar diagram as (3.3):

\[ \begin{array}{c}
F_A^* \ar[r]^{\tilde{f}} & F_{B \oplus P'}^* \ar[r] & C(\tilde{f})^* \ar[r] \ar[d]^* & F_A^{**} \ar[d]^{\tilde{\alpha}^{**}} \\
C(\tilde{g})^{**-1} \ar[r] & F_{B \oplus P'}^{**-*} \ar[r]^{\tilde{g}} & C(\tilde{g})^* 
\end{array} \]  

(3.6)

Since \( A \cong \text{Ker} \tilde{g} \), \( \tau_{\leq 0} \tilde{\alpha}^* = 0 \) is an isomorphism, equivalently \( \tau_{\leq -1} C(\tilde{\alpha})^* = 0 \) hence \( \tau_{\leq -2} C(\tilde{\gamma})^* = 0 \). From the long exact sequence of homology groups \( H^{-2}(C(\tilde{g})^*) \rightarrow H^{-1}(C(\tilde{f})^*) \rightarrow H^{-1}(C(f)^*) \), we get \( H^{-1}(C(\tilde{f})^*) = 0 \). Obviously, \( H^{-1}(C(\tilde{f})^*) \cong H^{-1}(C(f)^*) \) hence \( H^{-1}(C(f)^*) = 0 \). Now it remains to prove 2) - 4) in the case \( H^{-1}(C(f)^*) = 0 \).

2) Applying \( \tau_{\leq 0} \) to the diagram (3.6) and taking homology, we get the following diagram with exact rows:

\[ \begin{array}{c}
\theta_f: \quad 0 \rightarrow A \xrightarrow{(f)} B \oplus P' \oplus F_A^1 \xrightarrow{\text{Cok}_f} 0 \\
\sigma_{\tilde{g}}: \quad 0 \rightarrow \text{Ker}(g,p) \rightarrow B \oplus P' \oplus P_C \xrightarrow{(g,p,p_C)} C \rightarrow 0.
\end{array} \]

Notice that \( \tilde{\alpha}' \) and \( \tilde{\beta}' \) are stable isomorphisms. The upper row is a direct sum of \( \theta_f \)
and a trivial complex, and the lower row is that of \( \sigma \) and a trivial complex. Splitting off trivial complexes we get a desired diagram:

\[ \begin{array}{c}
0 \rightarrow A \xrightarrow{(f)} B \oplus F_A^1 \xrightarrow{(c_{-1})} \text{Cok}_f \rightarrow 0 \\
0 \rightarrow A \xrightarrow{(f)} B \oplus P' \xrightarrow{(g,p)} C \rightarrow 0
\end{array} \]  

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3) As we see above, \( \tilde{\gamma}^* : C(f)^* = F_{\text{Cok}} f^* \rightarrow F_C^* \) has \( \tau_{\leq -2} C(\tilde{\gamma})^* = 0 \). We may consider \( \tilde{\gamma}^* \) as \( \tilde{\gamma}_i = \text{id} \) (\( i \leq -1 \)) hence \( Q' = \text{Ker} \tilde{\gamma} \) is projective;

\[
0 \rightarrow Q' \rightarrow \text{Cok} f^* \oplus P_C (\tilde{\gamma}^* \circ \text{pc}) \rightarrow C \rightarrow 0.
\]

Since \( \text{Cok} f^* \cong \text{Cok} f \) and \( \tilde{\gamma}^* \cong \gamma \), the above sequence is the desired sequence \( 0 \rightarrow Q' \rightarrow \text{Cok} f^* \oplus Q \rightarrow C \rightarrow 0 \) with some projective module \( Q \).

4) Suppose \( \sigma \) is perfect. From Proposition 3.5, \( F_C^{-1} \rightarrow F_A^\bullet \xrightarrow{f^\bullet} F_B^\bullet \xrightarrow{g^\bullet} F_C^\bullet \) is a distinguished triangle, and \( F_C^\bullet \cong C(f)^\bullet \), hence the induced sequence \( \sigma \) is isomorphic to \( \theta_f \). (q.e.d.)

4 Representation by monomorphisms and torsionless modules.

In the previous section, we see that a given map \( f \) is represented by monomorphisms if and only if \( H^{-1}(C(f)^\bullet) = 0 \). If this is the case, \( \text{Ker} f = \text{Cok} d_{C(f)}^{-2} \) is the first syzygy of \( \text{Cok} f = \text{Cok} d_{C(f)}^{-1} \). So it is natural to ask the converse: Is a given map \( f \) represented by monomorphisms if \( \text{Ker} f \) is a first syzygy? This section deals with the problem. As a conclusion, the answer is yes if the total ring of fractions \( Q(R) \) of \( R \) is Gorenstein. What is more, if \( Q(R) \) is Gorenstein, instead of a pseudo-kernel, we can use a (usual) kernel to describe rbm condition.

The next is well known. See [1] and [4] for the proof.

Definition and Lemma 4.1 The following are equivalent for an \( R \)-module \( M \).

1) The natural map \( \phi : M \rightarrow M^{**} \) is a monomorphism.

2) \( \text{Ext}_R^1(\text{Tr} M, R) = 0 \)

3) \( M \) is a first syzygy; there exists a monomorphism from \( M \) to a projective module.

If \( M \) satisfies these conditions, \( M \) is said to be torsionless. \(^1\)

To solve our problem, the special kind of maps is a key. For \( M \in \text{mod} R \), consider a module \( J^2 M = \text{Tr} \Omega_R^2 \text{Tr} \Omega_R M \). Since \( \text{Tr} J^2 M \) is a first syzygy, we have \( \text{Ext}_R^1(J^2 M, R) = 0 \), which means \( H_{-1}(F_{J^2 M}^\bullet) = 0 \) and \( \tau_{\geq -2} F_{J^2 M}^\bullet \) is a projective resolution of \( \text{Tr} \Omega_R^2 M = \text{Cok} (d_{F_{J^2 M}}^{-2})^* = \text{Cok} (d_{M}^{-2})^* \). The identity map on \( \text{Tr} \Omega_R^2 M \) induces a chain map \( (F_M)^* \rightarrow (F_{J^2 M})^* \) and its \( R \)-dual \( \psi_M^* : F_{J^2 M}^* \rightarrow F_M^* \) subsequently.

Lemma 4.2 The map \( \psi_M : J^2 M \rightarrow M \) is rbm if and only if an \( R \)-module \( M \) has \( \text{(Ext}_R^1(M, R))^* = 0 \).

\(^1\)In [1], Auslander and Bridger use the term "1-torsion free" for "torsionless". Usually a module \( M \) is called torsion-free if the natural map \( M \rightarrow M \otimes Q(R) \) is injective.

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proof. From Theorem 3.6, \( \psi_M \) is rmb if and only if \( H^{-1}(C(\psi_M)^*) = 0 \). By definition, \( \psi_M^{-1} \) and \( \psi_M^2 \) are identity maps hence \( \psi_M^i \) are identity maps for \( i \leq -1 \). We may assume \( \tau_{\leq -2}C(\psi_M)^* = 0 \), which implies \( H^{-1}(C(\psi_M)^*) = \text{Ker } d_{C(\psi_M)}^{-1} \). As \( \tau_{\geq -1}C(\psi_M)^* \) is a projective resolution of \( \text{Cok } (d_{\psi_M^{-1}})^* = H_{-1}(C(\psi_M)^*) \), we get \( H^{-1}(C(\psi_M)^*) \cong (H_{-1}(C(\psi_M)^*))^* \). A triangle \( F_{\mathcal{P}M^*} \xrightarrow{\psi_M^*} F_M^* \rightarrow C(\psi_M)^* \rightarrow F_{\mathcal{P}M^*}^{+1} \) induces an R-dual triangle \( F_{\mathcal{P}M^*+1} \rightarrow C(\psi_M)^* \rightarrow F_M^* \rightarrow F_{\mathcal{P}M^*}^* \) which produces an exact sequence of modules

\[
0 \rightarrow H_{-1}(C(\psi_M)^*) \rightarrow H_{-1}(F_M^*) \rightarrow H_{-1}(F_{\mathcal{P}M^*}) \rightarrow 0.
\]

As we see in the discussion above, \( H_{-1}(F_{\mathcal{P}M^*}) = 0 \). Hence \( H_{-1}(C(\psi_M)^*) \cong H_{-1}(F_M^*) = \text{Ext}_R^1(M, R) \), and we get \( H^{-1}(C(\psi_M)^*) \cong (\text{Ext}_R^1(M, R))^* \). (q.e.d.)

The above result is generalized as follows:

**Lemma 4.3** Let \( f : A \rightarrow B \) be a morphism in \( \text{mod } R \). Suppose \( (\text{Ext}_R^1(B, R))^* = 0 \). If \( \text{Ker } f \) is projective, then \( f \) is rmb.

**proof.** We may assume \( \tau_{\leq -2}C(f)^* = 0 \). Similarly as in the proof of Lemma 4.2, we have \( H^{-1}(C(f)^*) = (H_{-1}(C(f)^*))^* \). Since \( \text{Ker } f \) is projective, \( f \) induces a stable isomorphism \( J^A \cong J^B \), and via this stable isomorphism, \( \psi_B \) is projective stably equivalent to \( f \circ \psi_A \), equivalently \( \psi_B^* \cong f^* \circ \psi_A^* \) in \( K(\text{mod } R) \). We have a triangle

\[
C(\psi_A)^* \rightarrow C(\psi_B)^* \rightarrow C(f)^* \rightarrow C(\psi_A)^{+1}
\]

and its R-dual

\[
C(\psi_A)^{+1} \rightarrow C(f)_*^* \rightarrow C(\psi_B)_*^* \rightarrow C(\psi_A)_*^*
\]

which induce an exact sequence of modules

\[
0 \rightarrow H_{-1}(C(f)_*^*) \rightarrow H_{-1}(C(\psi_B)_*^*)
\]

Note that \( H_{-1}(C(\psi_B)_*^*) = \text{Ext}_R^1(B, R) \). The assumption \( (\text{Ext}_R^1(B, R))^* = 0 \) equivalently says \( \text{Ext}_R^1(B, R)_p = 0 \) for any associated prime ideal \( p \) of \( R \). A submodule has the same property; \( H_{-1}(C(f)_*^*)_p = 0 \) for any associated prime ideal \( p \) of \( R \) therefore \( (H_{-1}(C(f)_*^*))^* = 0 \). (q.e.d.)

**Proposition 4.4** Let \( f : A \rightarrow B \) a morphism of \( \text{mod } R \). Suppose \( (\text{Ext}_R^1(B, R))^* = 0 \). Then \( f \) is rmb if and only if \( \text{Ker } f \) is torsionless.

**proof.** We already get the "only if " part and have only to show the "if" part. Adding a projective cover of \( B \) to \( f \), we get an exact sequence

\[
\sigma_f : 0 \rightarrow \text{Ker } f \xrightarrow{(\gamma_f)} A \oplus P_B \xrightarrow{(f, \rho_f)} B \rightarrow 0.
\]
Due to Theorem 3.6, we have a perfect exact sequence $\theta_{n^f}$, because $n^f$ is rbm:

$$
\theta_{n^f} : 0 \to \text{Ker} f \xrightarrow{\pi} A \oplus \text{F}_{\text{Ker} f} \xrightarrow{\alpha} \text{Cok} n^f \to 0
$$

$$
\sigma_{f} : 0 \to \text{Ker} f \xrightarrow{\pi} A \oplus P_B \xrightarrow{(f, p_B)} B \to 0.
$$

From Theorem 3.6 3), we know $\text{Ker} \omega_f$ is projective. With the assumption $(\text{Ext}_R^1(B, R))^* = 0$, we can apply Lemma 4.3 and get that $\omega_f$ is rbm. From the equation $f = \omega_f \circ c_{n^f}$, $f$ is rbm if $c_{n^f}$ is rbm. Since

$$
F_{\text{Ker} f}^* \xrightarrow{n^f} F_A^* \xrightarrow{c_{n^f}} C(n_f)^* \to F_{\text{Ker} f}^{*+1}
$$

is a triangle, $C(c_{n^f})^* \cong F_{\text{Ker} f}^{*+1}$; $H^{-1}(C(c_{n^f})^*) \cong H^0(F_{\text{Ker} f}^*) \cong \text{Ext}_R^1(\text{Tr Ker} f, R)$. Hence $c_{n^f}$ is rbm if and only if $\text{Ker} f$ is torsionless. (q.e.d.)

Lemma 4.5 Let the sequence of $R$-modules $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be exact. Suppose $(\text{Ext}_R^1(C, R))^* = 0$. If $A$ and $C$ are torsionless, then so is $B$.

proof. From the assumption, $A \cong \text{Ker} g$ is torsionless. Due to Proposition 4.4, $g$ is rbm; there exists an exact sequence

$$
\theta_{g} : 0 \to B \xrightarrow{(g)} C \oplus Q \to \text{Cok} g \to 0
$$

with a projective module $Q$ and a map $\epsilon : B \to Q$. Since $C$ is a submodule of some projective module, so is $B$. (q.e.d.)

Proposition 4.6 The following are equivalent for a noetherian ring $R$.

1) $Q(R)$ is Gorenstein.

2) $Q(R)$ is Gorenstein of dimension zero.

3) $(\text{Ext}_R^1(M, R))^* = 0$ for each $M \in \text{mod } R$.

4) $\Psi_M$ is rbm for each $M \in \text{mod } R$.

If $R$ is a local ring with the maximal ideal $m$, the above conditions are also equivalent to the following.

5) $\Psi_{R/m}$ is rbm.
proof. As $Q(R)$ is always of dimension zero, we get 1) $\iff$ 2).

3) $\iff$ 4) is already shown in Lemma 4.2.

4) $\Rightarrow$ 5) is obvious.

5) $\Rightarrow$ 1). The condition 5) is equivalent to $\text{Ext}^1_A(R/m, R) \otimes Q(R) \cong \text{Ext}^1_{Q(R)}(R/m \otimes Q(R), Q(R)) = 0$, which means $Q(R)$ is Gorenstein. (q.e.d.)

In the case $Q(R)$ is Gorenstein, every morphism in mod $R$ satisfies the hypotheses of Proposition 4.4 and Lemma 4.5. Thus with the condition $Q(R)$ is Gorenstein, when discussing rbm property, we can deal with normal kernel as well as pseudo-kernel.

Proposition 4.7 Suppose $Q(R)$ is Gorenstein. For a given morphism $f$, Ker $f$ is torsionless if and only if Ker$f$ is torsionless.

proof. From Lemma 2.7, there is an exact sequence $0 \to \text{Ker } f \to \text{Ker } f \to \Omega^1_R(\text{Cok } f) \to 0$. So the "if" part is obvious, and the "only if" part comes from Lemma 4.5. (q.e.d.)

Theorem 4.8 Suppose $Q(R)$ is Gorenstein. The following are equivalent for a morphism $f: A \to B$ in mod $R$.

1) $f$ is rbm.

2) Ker $f$ is torsionless.

3) Ker$f$ is torsionless.

4) $H^{-1}(C(f)^*) = 0$.

5) $\Omega^1_R(\text{Cok } f) \cong \text{Ker } f$.

6) There exists $f'$ such that $f' \cong f$ and Ker $f'$ is torsionless.

7) For any $f'$ with $f' \cong f$, Ker $f'$ is torsionless.

proof. Implications 5) $\Rightarrow$ 3), 7) $\Rightarrow$ 2) and 7) $\Rightarrow$ 6) are obvious. We already showed 1) $\iff$ 4) in Theorem 3.6, 1) $\iff$ 3) in Proposition ??, and 3) $\iff$ 2) in Corollary 4.7. Implications 3) $\Rightarrow$ 7) and 6) $\Rightarrow$ 3) are obtained from "if" and "only if" part of Corollary 4.7 respectively.

4) $\Rightarrow$ 5). It comes directly from Cok $d_{C(f)}^{-1} = \text{Ker } f$ and Cok $d_{C(f)}^0 = \text{Cok } f$. (q.e.d.)

Remark 4.9 Takashima gives an easy proof for Theorem 4.8 using the torsion theory [7].

Corollary 4.10 The following are equivalent for a noetherian ring $R$.

1) $Q(R)$ is Gorenstein.
2) Every morphism with torsionless kernel is rbm.

proof.
1) $\Rightarrow$ 2). It comes directly from Theorem 4.8.
2) $\Rightarrow$ 1). For every $M \in \text{mod } R$, Ker $\psi_M$ is torsionless. Because Ker $\psi_M$ is projective and Ker $\psi_M$ is a submodule of Ker $\psi_M$ from Lemma 2.7 1). So if 2) holds, $\psi_M$ is rbm for any $M \in \text{mod } R$, which implies 1) from Proposition 4.6. (q.e.d.)

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MORITA DUALITY AND RING EXTENSIONS

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ABSTRACT. Let $A$ be a ring with Morita duality induced by a bimodule $B^AQ_A$ and let $R$ be a ring extension of $A$. Müller proved the fundamental result that if $R_A$ and $\text{Hom}_A(R, Q)_A$ are linearly compact, then $R$ has a Morita duality induced by $s\text{Hom}_A(R, Q)_R$, where $S = \text{End}_R(\text{Hom}_A(R, Q))$. We improve this result by showing the existence of a category equivalence between certain categories of $A$-rings and $B$-rings whenever $A$ and $B$ are two Morita dual rings. We also generalize and unify a result of Fuller-Haack about Morita duality of semigroup rings and a result of Mano about self-duality of finite centralizing extensions.

1 研究の背景

環のMorita dualityやself-dualityが様々な拡大環にどのように遺伝するかについては、多くの研究者によって調べられてきた。基礎となるのは次のMüllerの結果である。

定理A ([1, Proposition 7.3]参照). $A$を両側加群$B^AQ_A$によって定められるMorita dualityをもつ環。$R$を$A$の拡大環とする。$R_A$と$\text{Hom}_A(R, Q)_A$がlinearly compactであれば。$R$は両側加群$s\text{Hom}_A(R, Q)_R$によって定められるMorita dualityをもつ。ただし$S = \text{End}_R(\text{Hom}_A(R, Q))$である。

この定理の特別な場合として、Fuller-Haackは次の定理を示した。

定理B ([1, Corollary 9.4]参照). $G$を有限半群とする。環$A$が環$B$右Morita dualであれば。半群環$AG$は半群環$BG$に右Morita dualである。特に環$A$がself-dualityをもつば。半群環$AG$もself-dualityをもつ。

この半群環のself-dualityを一般化する形で、真野は次の定理を証明した。

定理C ([1, Theorem 9.2]参照). 環$A$は両側加群$AQ_A$によって定められるself-dualityをもつとする。$A$の拡大環$R$が条件

1) $R$は$r_1, \ldots, r_n$を基底とする自由右$A$加群である。

*The detailed version of this note will be submitted for publication elsewhere.
（2）各$r_i$は$A$のすべての元と可換である。

（3）$r_i r_j = \sum_{k=1}^{n} r_k a_{ijk}$ ($a_{ijk} \in A$) と表すとき，各$a_{ijk}$は$Q$のすべての元と可換である。

を満たすならば，$R$もself-dualityをもつ。

今回の研究において，2つの環$A$と$B$がMorita dualのとき，ある種の$A$-環の圈と$B$-環の圈の間に同値が存在することを示すことによって，定理Aを改良した（定理2）。またその応用として，自由であるような有限中心的拡大（定理Cの条件（1），(2)を満たす拡大環）のMorita dualityを決定し，定理Bと定理Cを統合・一般化することがであった（定理4）。

以下この報告集では，すべての環は単位元をもち，すべての加群は単位的であるとする。

2 Morita dualityの定義と両側加群の圈

$A$と$B$を環とする，Mod-$A$，B-Modによって，それぞれ右$A$加群全体，左$B$加群全体の圈を表す。Morita dualityとは，次の条件を満たすMod-$A$の充満部分圏$A$とB-Modの充満部分圏$B$の間のduality（すなわち反変同値）$F: A \cong B : G$である：

（1）$A_A \in A$，$B_B \in B$。

（2）$A$と$B$は部分加群と剩余加群で閉じている。

実際には，$F$, $G$は適当な両側加群$BQ_A$を用いて$F \cong \text{Hom}_A(-,Q)$，$G \cong \text{Hom}_B(-,Q)$として表現可能である。ここでは，$BQ_A$は忠実平衡的両側加群で，$Q_A$と$BQ$は移入的余生成加群となる。逆に，このような両側加群$BQ_A$に対して，双対圏手$\text{Hom}_A(-,Q)$，$\text{Hom}_B(-,Q)$は$Q$-反射的加群（定義は後述）からなる充満部分圏の間のMorita dualityとなるため，両側加群$BQ_A$はMorita dualityを定義するという。このように，Morita dualityは片側加群の圈に対して定義されるが，今回の研究ではある種の両側加群の圈に注目した。

以下，この論文を通して，両側加群$BQ_A$はMorita dualityを定めるとし，双対圏手を$(-)^* = \text{Hom}_A(-,Q)$，$(-)^\# = \text{Hom}_B(-,Q)$とおく。右$A$加群$X$（左$B$加群$Y$）は，標準的な評価写像$X \to X^{*\#}$（$Y \to Y^{*\#}$）が同型写像であるとき，$Q$-反射的（$Q$-reflexive）
であるという、両側加群の圏を次のように定める。

\[ \mathcal{M}_{A-}\mathcal{A} = \left\{ A L_A \mid L_A, L^* A \text{は} Q \text{-反射的} \right\}, \]
\[ b\mathcal{M}_A = \left\{ b M_A \mid M_A, b M \text{は} Q \text{-反射的} \right\}, \]
\[ b\mathcal{B}M = \left\{ b N_b \mid b N, b N^\# \text{は} Q \text{-反射的} \right\}. \]

ただし射は両側単型写像である。よく知られているように、\( bQ_A \)がMorita dualityを定めるとき、右\( A \)加群\( X \)（左\( B \)加群\( Y \)）が\( Q \)-反射的であることと linearly compactであることは同値である。したがって、定理\( A \)の拡大環\( R \)に対する仮定は\( A R_A \in \mathcal{M}_{A-}\mathcal{A} \)であることを意味する。Morita dualityを制限することによって、これらの両側加群の圏について、次の補題が成り立つ。

補題 1. (1) \( A A_A \in \mathcal{M}_{A-}\mathcal{A} \)、\( b Q_A \in \mathcal{B}M_A \)、\( b B_B \in b-BM \)。

(2) \( \mathcal{M}_{A-}\mathcal{A}, \mathcal{B}M_A, b-CBM \)は、部分両側加群、剰余両側加群、両側加群の拡大で閉じている。

(3) 双対圏手の対\((-)^* : \mathcal{M}_{A-}\mathcal{A} \rightleftharpoons \mathcal{B}M_A : (\cdot)^\# \)と\((-) : \mathcal{B}M_A \rightleftharpoons b-CBM : (\cdot)^\#\)はdualityである。したがって、これらの合成\((-)^{**} : \mathcal{M}_{A-}\mathcal{A} \rightleftharpoons b-CBM : (\cdot)^{##}\)は圏同値である。

3 環の圏とその同値

環\( R \)と環準同型写像\( A \rightarrow R \)の対\((R, f)\)を\( A \)-環\((A\text{-ring})\)という。\( A \)の拡大環や剰余環は\( A \)-環である。2つの\( A \)-環\((R, f)\)と\((R', f')\)の間の射\( \phi : (R, f) \rightarrow (R', f') \)を\( \phi \circ f = f' \)を満たす環準同型写像\( \phi : R \rightarrow R' \)として定める。以後\( A \)-環\((R, f)\)を単に\( R \)で表す。\( A \)-環\( R \)は\( (A, A) \)両側加群と見なすことができ、\( A \)-環の射\( R \rightarrow R' \)は\( (A, A) \)両側準同型写像となる。

\( A \)-環の圏と\( B \)-環の圏の冪部分圏を、それぞれ

\[ \mathcal{R}_A = \{ R \mid A R_A \in \mathcal{M}_{A-}\mathcal{A} \}, b\mathcal{R} = \{ S \mid b S_B \in b-CBM \} \]
によって定めれば、定理\( A \)の精密度の次の定理を述べることができる。

定理 2. (1) \((-)^{**} : \mathcal{R}_A \rightleftharpoons b\mathcal{R} : (\cdot)^{##}\)は圏同値を定める。

(2) 各\( R \in \mathcal{R}_A \)に対して、\( R^* \)はMorita dualityを定める\((R^{**}, R)\)両側加群となる。
$R \in \mathcal{R}_A$のとき，$A R_A \in \mathcal{M}_{A-A}$であるから，$R^{**} \in B-BM$であるが，実際には$R^{**}$は$B$-環の構造をもつ。さらに，$(-)^{**}$は$A$-環の射を$B$-環の射に写すことを確かめて，定理1の(1)を得る。また，環として$R^{**} \cong \text{End}_R(R^*)$であるから，定理2の(2)は定理Aそのものである。この種の結果においては，通常(定理A，B，Cでも) $\text{End}_R(R^*)$が用いられる。こちらの方が環構造は明らかであるが，圏手として見なす場合は$R^{**}$の方が扱いやすい。また，後述の定理4でも$R^{**}$の方が計算は容易である。

定理2より，環$A$と$B$がMorita dualのとき，$A$と$B$のある種の拡大環と関係には，Morita dualであるという関係の元で，同型の意味で1対1対応が存在する，次の系は，対応する$A$-環と$B$-環のそれぞれ$A_*$，$B$-部分環全体は一種の annihilator によって1対1に対応することを示している。

系3. $R \in \mathcal{R}_A$に対して，$U = R^*$，$S = U^* = R^{**}$とおく，$R$の$A$-部分環$R'$に対して，

$U' = \{u \in U \mid u(R') = 0\}$，$S' = \{s \in S \mid s(U') = 0\}$

とおけば，$U/U'$はMorita dualityを定める$(S', R')$両側加群となる。また，対応$R' \to S'$は$\mathcal{R}_A$の1-部分環全体と$S$の1-部分環全体との間の1対1対応を与える。

この系より，両側加群$BQ_A$がMorita dualityを定めるとき，例えば$n$次行列環$R = M_n(A)$の$A$-部分環もMorita dualityをもつことや，Morita dualな環やMorita dualityを定める両側加群が annihilator によって具体的に記述できることが分かる。

### 4 有限中心的拡大

$R$を$A$の拡大環とする。$A$の任意の元と可換な$R$の元$r_1, \ldots, r_n$で，$R = \sum_{i=1}^n r_i A$となるようなものが存在するとき，$R$は$A$の有限中心的拡大(finite centralizing extension)と言われる。両側加群$A M_A$が$M = mA = Am$と書けるとき，$M \in \mathcal{M}_{A-A}$であること分かる。$A$の有限中心的拡大環$R$は，$(A, A)$両側加群として，この$A M_A$のような両側加群の有限個の直和の剰余両側加群であるから，補題1より$R$は$\mathcal{R}_A$に属する。したがって定理2より有限中心的拡大環にMorita dualityは遺伝する。この事実はもちろんよく知られているが，次の定理において，有限中心的拡大$R = \sum_{i=1}^n r_i A$において，$r_1, \ldots, r_n$が自由$A$-基底になっているとき，$R$とMorita dualな環を完全に決定した。

なお，有限中心的拡大の一般化として，有限正規拡大と有限三角拡大がある([I, Sections 8-9]参照)。これらの拡大環にMorita dualityが遺伝することもすでに示されているが，上の有限中心的拡大の場合と同じ論法で示すことができる。有限正規拡大や有限三角拡大については，たとえば生成元が自由基底になっていても，Morita dualな環を具体的に計算することは難しいものと思われる。
Morita dualityを定める両側加群 $BQ_A$ は忠実平衡であるから，$A$, $B$ の中心は同型.すなわち，環同型写像 $\alpha : \text{Cen}(A) \to \text{Cen}(B)$ が存在する。ここで $\alpha$ は，任意の $q \in Q$ に対して $q\alpha = \alpha(q)a_0$ によって定められる。また次の定理4の条件 (3) における $a_{ijk}$ が$	ext{Cen}(A)$ に属することが直ちに分かる。条件 (3) は，条件 (3)** と対比するために書いただけで，実際には何の制約も与えておらず，単に $R$ は自由であるような $A$ の有限中心的拡大であるという意味である。以上の注意の下，定理4を次のように記述できる。

定理 4. 両側加群 $BQ_A$ は Morita duality を定めるとする。$R$ は $A$ の拡大環で，条件

(1) $r_1, \ldots, r_n$ は $R$ の $A$-自由基底である。

(2) 各 $r_i$ は $A$ の任意の元と可換である。

(3) $r_i r_j = \sum_{k=1}^{n} r_k a_{ijk} (a_{ijk} \in A)$。

を満たすとする。$U = R^*, S = U^*$ とおく。このとき，$U$ は Morita duality を定める $(S, R)$ 両側加群で，$S$ は条件

(1)** $r_i^*, \ldots, r_n^*$ は $S$ の $B$-自由基底である。

(2)** 各 $r_i^*$ は $B$ の任意の元と可換である。

(3)** $r_i^* r_j^* = \sum_{k=1}^{n} r_k^* \alpha(a_{ijk}) (\alpha(a_{ijk}) \in B)$。

を満たす。ただし $r_i^*(u) = u(r_i) (u \in U)$ によって定義される $S$ の元である。

この定理は，$R$ の積の構造定数 $a_{ijk}$ と $S$ の積の構造定数 $\alpha(a_{ijk})$ はほとんど「同じ」であることを示している。半群環の場合，構造定数には $0$ か $1$ しか現れないが，同型写像 $\alpha$ はそれらを保つから，特別な場合として Fuller-Haack の定理 B を得る。また $A = B$ で $\alpha(a_{ijk}) = a_{ijk}$ の場合として，真野の定理 C を得る。

参考文献

ON INDECOMPOSABILITY OF A MODULE GIVEN BY BRAUER CONSTRUCTION

SHIGEO KOSHITANI (越谷重夫)

Abstract. In representation theory of finite groups one of the most important and well-known conjectures is Broué's abelian defect group conjecture. In this note we introduce a sort of technical result which is useful to prove Broué's abelian defect group conjecture for a few examples which are new.

1. Introduction

In representation theory of finite groups, there has been a very important problem, namely,

How are representations of a finite group $G$ over a field $k$ of prime characteristic $p$ similar to those of its subgroup $H$ containing a Sylow $p$-subgroup $P$ of $G$?

\[ \begin{array}{c}
\text{representations of } G \text{ over } k \\
\downarrow \\
\text{representations of } H \text{ over } k
\end{array} \]

An origin of this problem is, of course, due to Richard Brauer (1901–77). From this point of view, one of the most important and interesting (and also well-known) conjectures in representation theory of finite groups is Broué's abelian defect group conjecture. Actually, Michel Broué conjectures the following.

(1.1) Broué's abelian defect group conjecture, (see [2], [3] and [5]). For any prime $p$, if a block algebra $A$ of $\mathcal{O}G$ has an abelian defect group $P$ then $A$ and its Brauer corresponding block algebra $B$ in $\mathcal{O}N_G(P)$ should be derived equivalent, that is,

\[ D^b(\text{mod-}A) \simeq D^b(\text{mod-}B) \]

equivalent

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The detailed final version of this paper will be published, see [7].
as triangulated categories, where $\mathcal{O}$ is a complete discrete valuation ring whose residue field is an algebraically closed field $k$ of characteristic $p$, and mod-$A$ is the category of finitely generated right $A$-modules, and $D^b(\mathfrak{A})$ is the bounded derived category of an abelian category $\mathfrak{A}$.

As well-known, there is a beautiful result of Jeremy Rickard ([9], [10]), which characterizes such a derived equivalence completely and which is a generalization of a Morita equivalence from modules to complexes. As a matter of fact, in (1.1) above, a stronger conclusion is expected. Namely, derived equivalent should be replaced by splendidly Rickard equivalent, which is due to Jeremy Rickard [11].

In this note we present a theorem which is a sort of technical one, but also useful to prove Broué's abelian defect group conjecture, which is used in a work of the author, Kunugi and Waki [6].

Actually, we have a result which is a joint work with Markus Linckelmann. We present it in the next section.

2. The main result

First we introduce notation and terminology we need to state our main result.

(2.1) Notation and assumption. Let $G$ be a finite group, and let $\mathcal{O}$ and $k$ be as in (1.1). We call an $\mathcal{O}G$-lattice of finite rank a finitely generated $\mathcal{O}G$-module. For any subgroup $L$ of $G$ let $\Delta L = \{(l, \ell) \in L \times L \mid \ell \in L\}$, and we consider $\mathcal{O}G$ as a right $\mathcal{O}([\Delta L])$-module via $a(l, \ell) = l^{-1}a_l$ for $a \in \mathcal{O}G$ and $\ell \in L$. Then, we next define $(\mathcal{O}G)^{\Delta L}$ as $(\mathcal{O}G)^{\Delta L} = \{a \in \mathcal{O}G \mid a(l, \ell) = a_{l\ell} \text{ for all } l \in L\} = \{a \in \mathcal{O}G \mid l^{-1}a_l = a_{l\ell} \text{ for all } \ell \in L\}$. For a $p$-subgroup $R$ of $G$ let $Br_{\Delta R}$ be the Brauer map (homomorphism) with respect to $R$. Namely, $Br_{\Delta R}$ is defines as

$$Br_{\Delta R} : (\mathcal{O}G)^{\Delta R} \rightarrow kC_G(R), \quad \sum_{g \in G} \alpha_g g \mapsto \sum_{g \in C_G(R)} \tilde{\alpha}_g g,$$

where $\alpha_g \in \mathcal{O}$ and $\tilde{\alpha}_g$ is its image in $k$

Note the Brauer map $Br_{\Delta R}$ is a surjective algebra-homomorphism. For a $p$-subgroup $R$ which is contained in both of $G$ and $L$ and an $(\mathcal{O}G, \mathcal{O}L)$-bimodule $Y$ we define the Brauer construction (quotient) $Y(\Delta R)$ by

$$Y(\Delta R) = \left( Y^{\Delta R} / \sum_{R' \leq R} \text{Tr}_{\Delta R}^{\Delta R'}(Y^{\Delta R'}) \right) \otimes_{\mathcal{O}} k$$

where Tr is the transfer (trace) map. Clearly, the Brauer construction $?(\Delta R)$ induces an additive functor from the category $\mathcal{O}G$-mod-$\mathcal{O}L$ of the finitely generated $(\mathcal{O}G, \mathcal{O}L)$-bimodules to the category $kC_G(R)$-mod-$kC_L(G)$. In particular if $Y = \mathcal{O}G$ we get $Y(\Delta R) = (\mathcal{O}G)(\Delta R) \cong kC_G(R)$, and we identify these since we have a commutative diagram

$$(\mathcal{O}G)^{\Delta R} \xrightarrow{Br_{\Delta R}} kC_G(R)$$
canonical epimorphism

\[\begin{array}{ccc}
\mathcal{O}G(\Delta R) & \xrightarrow{=} & kC_G(R)
\end{array}\]
First, let \((P, e)\) be a Brauer pair in \(G\), that is, \(P\) is a \(p\)-subgroup of \(G\) and \(e\) is a block of \(kC_G(P)\). Then, by results of Alperin-Broué [1, Theorem 3.4] and Broué-Puig [4, Theorem 1.8], it follows that, for any subgroup \(Q\) of \(P\) there uniquely exists a block \(e_Q\) of \(kC_G(Q)\) such that \((Q, e_Q) \leq (P, e)\). For other notation and terminology, see books of Nagao-Tsushima [8] and Thévenaz [12].

Now, we can state the main result of this note; that is,

(1.2) Theorem (Koshitani-Linckelmann, see [7]). Let \(b\) be a block (block idempotent) of a block algebra \(A\) of \(OG\) (so that \(A = bOG\)), and let \((P, e)\) be a maximal \(b\)-Brauer pair in \(G\), and hence \(P\) is a defect group of \(b\) and \(e\) is a block of \(kC_G(P)\) such that \(Br_{\Delta P}(b)e = e\). Next, let \(H = N_G(P, e) = \{g \in N_G(P) | g^{-1}eg = e\}\) be the inertial group of \(e\) in \(N_G(P)\). For each subgroup \(Q\) let \(e_Q\) be the same as above, and let \(f_Q\) be a block of \(kC_H(Q)\) such that \((Q, f_Q) \leq (P, e)\) since \(C_G(P) = C_H(P)\). Moreover, let \(f\) be a primitive idempotent of \((bOG)_{\Delta H}\) such that \(Br_{\Delta P}(f)e = e\), and let \(X = OGf\). Then we have the following.

(i) \(X\) is an indecomposable \(O[G \times H]\)-module with vertex \(\Delta P\).

(ii) If \(Q\) is a subgroup of \(Z(P)\), the center of \(P\), then \(e_Q \cdot X(\Delta Q) \cdot f_Q\) is a unique indecomposable direct summand of \(e_Q kC_G(Q) f_Q\) with vertex \(\Delta P\) as a \((kC_G(Q), kC_H(Q))\)-bimodule.

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SOME TOPICS ON DERIVED EQUIVALENT BLOCKS
OF FINITE GROUPS

NAOKO KUNUGI

1. INTRODUCTION

Let \( G \) be a finite group. Let \( k \) be an algebraically closed field of characteristic \( \ell > 0 \). We denote the principal block of \( kG \) by \( B_0(G) \).

We say that two finite groups \( G \) and \( H \) have the same \( \ell \)-local structure if \( G \) and \( H \) have a common Sylow \( \ell \)-subgroup \( P \) such that whenever \( Q_1 \) and \( Q_2 \) are subgroups of \( P \) and \( f : Q_1 \rightarrow Q_2 \) is an isomorphism, then there is an element \( g \in G \) such that \( f(x) = x^g \) for all \( x \in Q_1 \) if and only if there is an element \( h \in H \) such that \( f(x) = x^h \) for all \( x \in Q_1 \).

There is a well known conjecture due to Broué.

Conjecture 1.1 (Broué [1, 2]). Let \( G \) and \( H \) be finite groups having the same \( \ell \)-local structure with common Sylow \( \ell \)-subgroup \( P \). If \( P \) is abelian then the principal blocks of \( G \) and \( H \) would be derived equivalent.

If \( P \) is not abelian, then there is a counterexample to this conjecture. However, there are some examples that \( P \) is not abelian and there is a derived equivalence between the principal blocks of \( G \) and \( H \). We will give such examples in §3.

2. GENERAL THEORY

In this section, let \( G \) and \( H \) be finite groups having the same \( \ell \)-local structure with common Sylow \( \ell \)-subgroup \( P \). We say that a complex of \( (B_0(G), B_0(H)) \)-bimodules is splendid if each indecomposable summand of each term of the complex is a direct summand of a module of the form \( kG \otimes_{kQ} kH \) for a subgroup \( Q \) of \( P \).

Definition 2.1. Let \( X^\bullet \) be a splendid complex of \( (B_0(G), B_0(H)) \)-bimodules. We say that \( X^\bullet \) induces a splendid stable equivalence if we have isomorphisms

\[
X^\bullet \otimes_{B_0(H)} X^{**} \cong B_0(G) \oplus Z_1, \quad X^{**} \otimes_{B_0(G)} X^\bullet \cong B_0(H) \oplus Z_2
\]

where \( Z_1 \) and \( Z_2 \) are homotopy equivalent to complexes of projective bimodules.

Definition 2.2. Let \( X^\bullet \) be a splendid complex of \( (B_0(G), B_0(H)) \)-bimodules. We say that \( X^\bullet \) induces a splendid equivalence if we have isomorphisms

\[
X^\bullet \otimes_{B_0(H)} X^{**} \cong B_0(G) \oplus Z_1, \quad X^{**} \otimes_{B_0(G)} X^\bullet \cong B_0(H) \oplus Z_2
\]

where \( Z_1 \) and \( Z_2 \) are homotopy equivalent to 0. The complex \( X^\bullet \) is called a splendid tilting complex.

The detailed version of this paper will be submitted for publication elsewhere.
By the definition, splendid equivalences induce derived equivalences and homotopy equivalences.

**Theorem 2.1** (Rouquier [12]). Let $X^*$ be a splendid complex of $(B_0(G), B_0(H))$-bimodules. Then the following are equivalent.

1. The complex $X^*$ induces a splendid stable equivalence between $B_0(G)$ and $B_0(H)$.
2. For every non-trivial subgroup $Q$ of $P$, the complex $X^*(\Delta(Q))$ induces a splendid equivalence between $B_0(C_G(Q))$ and $B_0(C_H(Q))$, where $\Delta(Q)$ is a diagonal subgroup and

$$X(Q) = X^{\Delta(Q)}/\sum_{R \leq Q} \text{Tr}_R^Q X^{\Delta(R)}.$$ 

In our example in §3 we will use the following method when we prove splendid equivalences.

**Step 1** Construct a splendid tilting complex between $B_0(C_G(Q))$ and $B_0(C_H(Q))$ for every non-trivial subgroup $Q$ of $P$.

**Step 2** Construct a splendid stable equivalence $F$ from $B_0(G)$ to $B_0(H)$ by gluing the splendid tilting complexes obtained in Step 1 (by using the above theorem).

**Step 3** Calculate $F(S)$ for the simple $B_0(G)$-modules.

**Step 4** Lift the stable equivalence in Step 2 to a splendid equivalence by looking at the modules calculated in Step 3.

### 3. General Linear Groups and Unitary Groups

Let $q$ be a power of a prime. Assume that $\ell$ is odd and $\ell^\varepsilon$ divides $q + 1$ but $\ell^{\varepsilon + 1}$ does not divide $q + 1$ for some $\varepsilon > 0$. Under this condition, we consider representations of the general linear group $GL(n, q^2)$ and the unitary group $GU(n, q^2)$ for small $n$. Note that if $\ell > n$ then the principal $\ell$-block of $GL(n, q^2)$ is Morita equivalent to its Brauer correspondent by Puig’s result (see [8]).

#### 3.1. $GL(2, q^2)$ and $GU(2, q^2)$

We have isomorphisms

$$B_0(GL(2, q^2)) \cong k\mathbb{Z}_{\ell^e} \otimes B_0(SL(2, q^2)), \quad B_0(GU(2, q^2)) \cong k\mathbb{Z}_{\ell^e} \otimes B_0(SU(2, q^2)).$$

The blocks $B_0(SL(2, q^2))$ and $B_0(SU(2, q^2))$ have cyclic defect groups, and they are splendid equivalent by Rouquier’s result in [11]. Therefore the principal blocks $B_0(GL(2, q^2))$ and $B_0(GU(2, q^2))$ are splendid equivalent.

#### 3.2. $GL(3, q^2)$ and $GU(3, q^2)$ in characteristic $\ell > 3$

In this case, Sylow $\ell$-subgroups of $GL(3, q^2)$ and $GU(3, q^2)$ are abelian. As in case $n = 2$, we have isomorphisms

$$B_0(GL(3, q^2)) \cong k\mathbb{Z}_{\ell^e} \otimes B_0(SL(3, q^2)), \quad B_0(GU(3, q^2)) \cong k\mathbb{Z}_{\ell^e} \otimes B_0(SU(3, q^2)).$$

In [5], Waki and the author showed that $B_0(SU(3, q^2))$ and its Brauer correspondent, which is isomorphic to the Brauer correspondent of $B_0(SL(3, q^2))$, are splendid equivalent. Therefore $B_0(SL(3, q^2))$ and $B_0(SU(3, q^2))$ are splendid equivalent since as we mentioned above $B_0(SL(3, q^2))$ and its Brauer correspondent are Morita equivalent to Puig’s result. Hence we also have $B_0(GL(3, q^2))$ and $B_0(GU(3, q^2))$ are splendid equivalent.

#### 3.3. $GL(3, q^2)$ and $GU(3, q^2)$ in characteristic 3

In this case Sylow 3-subgroups of $GL(3, q^2)$ and $GU(3, q^2)$ are not abelian. Our main result in this paper is the following theorem.
Theorem 3.1 (with T. Okuyama). Assume that $3^e$ divides $q + 1$ but $3^{e+1}$ does not divide $q + 1$ for $e > 0$. Then
(1) The principal 3-blocks $B_0(PSL(3,q^2))$ and $B_0(PSU(3,q^2))$ are splendid equivalent.
(2) The principal 3-blocks $B_0(SL(3,q^2))$ and $B_0(SU(3,q^2))$ are splendid equivalent.
(3) The principal 3-blocks $B_0(PGL(3,q^2))$ and $B_0(PGU(3,q^2))$ are splendid equivalent.
(4) The principal 3-blocks $B_0(GL(3,q^2))$ and $B_0(GU(3,q^2))$ are splendid equivalent.

Remark 3.1. If $e = 1$, then the result for (1) has been obtained by [6, 4, 3] and the result for (3) has been obtained by Usami and the author.

4. OUTLINE OF PROOF OF THEOREM

In this section, we give an outline of a proof of Theorem 3.1 (1) and (2). Let $G = SL(3,q^2)$, $H = SU(3,q^2)$, $\overline{G} = PSL(3,q^2)$ and $\overline{H} = PSU(3,q^2)$. Let $P$ be a common Sylow 3-subgroup of $G$ and $H$. We denote the image of a subgroup $L$ of $G$ (or $H$) in $\overline{G}$ (or $\overline{H}$) by $\overline{L}$. For each subgroup $R$ of $\overline{P}$, let $\overline{G}_R := C_{\overline{G}}(R)$, $\overline{H}_R := C_{\overline{H}}(R)$, and let $\overline{M}_R$ be the Scott module of $\overline{G}_R \times \overline{H}_R$ with vertex $\Delta(\overline{R})$, where $\overline{R}$ is a Sylow 3-subgroup of $\overline{G}_R$ and $\overline{H}_R$.

(Step 1). There is essentially one subgroup of $P$ (up to conjugate), which we denote by $Q$, containing $Z(P)$ such that $B_0(C_G(Q))$ and $B_0(C_H(Q))$ are not Morita equivalence. Then $C_G(Q) \cong GL(2,q^2)$ and $C_H(Q) \cong GU(2,q^2)$. Let $\overline{M}_Q \rightarrow k_{\overline{G}_Q \times \overline{H}_Q}$ be a $\Delta(\overline{Q})$-projective cover of $k_{\overline{G}_Q \times \overline{H}_Q}$ and $\overline{N}_Q \rightarrow \Omega_{\Delta(\overline{Q})}(k_{\overline{G}_Q \times \overline{H}_Q})$ be a $\Delta(\overline{Q})$-projective cover of $\Omega_{\Delta(\overline{Q})}(k_{\overline{G}_Q \times \overline{H}_Q})$. Then we have a splendid tilting complex for $B_0(\overline{G}_Q)$ and $B_0(\overline{H}_Q)$ of the form

$$0 \rightarrow \overline{N}_Q \rightarrow \overline{M}_Q \rightarrow 0.$$  

For a subgroup $\overline{R}$ of $\overline{P}$ not contained in $\overline{Q}$, the blocks $B_0(\overline{G}_R)$ and $B_0(\overline{H}_R)$ are Morita equivalent and the Scott module $\overline{M}_R$ gives a splendid tilting complex for these two blocks.

(Step 2). Let $M$ be the Scott module of $G \times H$ with vertex $\Delta(P)$. Let $M \rightarrow k_{G \times H}$ be a $\Delta(P)$-projective cover of $k_{G \times H}$ and $N \rightarrow \Omega_{\Delta(P)}(k_{G \times H})$ be a $\Delta(Q)$-projective cover of $\Omega_{\Delta(P)}(k_{G \times H})$. Consider the following complex

$$M^* : \quad 0 \rightarrow N \rightarrow M \rightarrow 0.$$  

and set $M^* = \text{Inv}_{Z(P)\times 1}(M^*)$. Then the complex $M^*$ is a splendid complex, and for each non-trivial subgroup $\overline{R}$ of $\overline{P}$, the complex $\overline{M}^*(\Delta(\overline{R}))$ coincides with the complex in (Step 1). Therefore by Rouquier's theorem (Theorem 2.1) we can see that the complex $M^*$ induces a splendid stable equivalence between $B_0(\overline{G})$ and $B_0(\overline{H})$.

(Step 3). Let $F = - \otimes_{B_0(\overline{G})} \overline{M}^*$. The principal block of $B_0(\overline{G})$ has 5 simple modules $k, S, T_1, T_2$ and $T_3$ and the principal block of $B_0(\overline{H})$ has 5 simple modules $k, \varphi, \theta_1, \theta_2$ and $\theta_3$. Then we have the following lemma.

Lemma 4.1. There exist exact sequences

$$0 \rightarrow \Omega^{-1}(U(k, \varphi)) \rightarrow \Omega(F(S)) \rightarrow k \oplus k \rightarrow 0$$

and

$$0 \rightarrow \Omega^{-1}(U(k, \varphi, \theta_i)) \rightarrow \Omega^2(F(T_i)) \rightarrow k \rightarrow 0$$

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for $i = 1, 2$ and $3$, where $U(k, v)$ is a uniserial module of length $2$ with top $k$, and $U(k, v, \theta_i)$ is a uniserial module of length $3$ with top $k$ and socle $\theta_i$.

(Step 4). It follows from Lemma 4.1 that the tilting complex defined by a sequence \{\theta_1, \theta_2, \theta_3\}, \{v, \theta_1, \theta_2, \theta_3\} and \{\theta, \theta_1, \theta_2, \theta_3\} of subsets of the set of simple modules (see [6]) gives a derived equivalence between $B_0(\mathcal{G})$ and $B_0(\mathcal{H})$. The equivalence is a lift of the stable equivalence given by $F$ (see [7]), and therefore $B_0(G)$ and $B_0(H)$ are splendid equivalent.

Now we have the splendid tilting complex for $B_0(\mathcal{G})$ and $B_0(\mathcal{H})$ of the form

\[ X^* : 0 \rightarrow \mathcal{Q}_3 \rightarrow \mathcal{Q}_2 \rightarrow \mathcal{Q}_1 \oplus \mathcal{N} \rightarrow \mathcal{M} \rightarrow 0 \]

where $\mathcal{M} = \text{Inv}_{Z(P) \times 1}(M)$ and $\mathcal{N} = \text{Inv}_{Z(P) \times 1}(N)$ and $\mathcal{Q}_1$, $\mathcal{Q}_2$, and $\mathcal{Q}_3$ are projective bimodules. Since $\text{Inv}_{Z(P) \times 1}(\cdot)$ induces a one to one correspondence between the set of trivial source $k[G \times H]$-modules with vertex $\Delta(Z(P))$ and the set of projective $k[G \times H]$-modules, we have a tilting complex of the form

\[ X^* : 0 \rightarrow Q_3 \rightarrow Q_2 \rightarrow Q_1 \oplus N \rightarrow M \rightarrow 0 \]

for $B_0(G)$ and $B_0(H)$, where $Q_1$, $Q_2$ and $Q_3$ are direct sums of trivial source $k[G \times H]$-modules with vertex $\Delta(Z(P))$ and $\text{Inv}_{Z(P) \times 1}(X^*) = X^*$ (see [12, A.4]).

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GAUSS SUMS ARE JUST CHARACTERS OF MULTIPLICATIVE GROUPS OF FINITE FIELDS

KAORU MOTOSE

This paper is a summary of some papers [4,5,6] such that using special commutative group algebras, we could prove alternatively some reciprocity theorems, prime decompositions of Gauss sums and Lenstra's primality test.

1. Group Algebra \( \text{Map}(F, K) \)

Let \( A = \text{Map}(F, K) \) be the set of all mappings from a finite field \( F = F_q \) of order \( q \) to a field \( K \) where \( q \) is a power of a prime \( p \). Then we define the convolution product in \( A \) by the following

\[
(f * g)(c) = \sum_{a+b=c} f(a)g(b)
\]

for \( f, g \in A \) and \( c \in F \). This product together with the usual sum and the scalar product gives the structure of a commutative algebra over \( K \). If there is no chance of confusion we shall denote the product \( f * g \) by the usual notation \( fg \).

Let \( u_a \) be the characteristic function of \( a \in F \), namely, \( u_a \) is defined by the following

\[
u_a(b) := \begin{cases} 1 & \text{if } b = a \\ 0 & \text{if } b \neq a. \end{cases}
\]

Then we have the following equations.

\[
u_a u_b = u_{a+b} \text{ and } f = \sum_{a \in F} f(a)u_a \text{ for } f \in A.
\]

Thus \( \{u_a \mid a \in F\} \) forms a basis of the group algebra \( A \) of the additive group of \( F \) over \( K \). We denote by \( \hat{F} \) the set of all characters of the multiplicative group \( F^* = F \setminus \{0\} \), by \( \chi^{(k)} \) \( k \)-th power of \( \chi \in \hat{F} \) with respect to the convolution product and by \( \zeta \) the trivial character. We set \( \zeta(0) = 1 \) and \( \chi(0) = 0 \) for \( \chi \neq \zeta \in \hat{F} \). Thus we have \( \hat{F} \subset A \). We set \( J(f_1, f_2, \ldots, f_n) = (f_1f_2\cdots f_n)(1) \) for \( f_1, f_2, \ldots, f_n \in A \) which is

\footnote{This paper is a summary of some papers [4,5,6] that was already published. This paper was financially supported by Fund for the Promotion of International Scientific Research B-2, 2004, Aomori, Japan.}
usually called the Jacobi sum.

2. Gauss sums and Jacobi sums

It is easy to see that $\epsilon * \epsilon = q\epsilon$ and $\lambda * \epsilon = 0$ for nontrivial $\lambda \in \hat{F}$. We have the following another relations which are important to our object.

**Lemma 1.** Assume that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are nontrivial elements in $\hat{F}$ and $q - 1 \neq 0$ in $K$. Then we have the next equations in each case.

1. In case $\lambda_1 \lambda_2 \ldots \lambda_n \neq \epsilon$, we have

   $$\lambda_1 * \lambda_2 * \cdots * \lambda_n = J(\lambda_1, \lambda_2, \ldots, \lambda_n)\lambda_1 \lambda_2 \cdots \lambda_n.$$ 

2. In case $\lambda_1 \lambda_2 \cdots \lambda_n = \epsilon$, we have

   $$\lambda_1 * \lambda_2 * \ldots * \lambda_n = \lambda_n(-1)J(\lambda_1, \lambda_2, \ldots, \lambda_{n-1})(qu_0 - \epsilon)$$

   where $J(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}) = 1$ if $n = 2$.

For $\chi \in \hat{F}$, we can write $\chi = \Sigma_{a \in F_\chi} \chi(a)u_a$. On the other hand Gauss sums is defined by

$$g(\chi) = \sum_{a \in F} \chi(a)z_p^{tr(a)}$$

where $z_p := e^{2\pi i/p}$, $q = p^r$ and $tr(a) = a + a^p + \cdots + a^{p^{r-1}}$ for $a \in F$. Hence, in case $K = C$ the complex number field, a map $\chi \mapsto g(\chi)$ ($u_a \mapsto z_p^{tr(a)}$) is the natural homomorphism from $A$ to $C$. Therefore, it is natural to think of $\chi$ as Gauss sum $g(\chi)$. It is easy to see $\hat{F}$ forms a basis of $A$ because $u_a = \frac{1}{q-1} \Sigma_{\chi \in \hat{F}} \chi(a^{-1})\chi$ if $q - 1 \neq 0$ in $K$.

3. Quadratic characters for odd primes

In this section, we shall have evaluation of the quadratic character $\eta \in A$ for an odd prime $q$. Using the character table and a permutation $b \mapsto b^{-1}$ on $F^*$, we can see easily the next proposition.

**Proposition 2.**

1. $\det[u_{ab-1}]_{a,b} = (\epsilon - u_0) * \prod_{\chi \neq \epsilon}^* \chi$ where $\prod^*$ means the product of all nontrivial multiplicative characters with respect to the convolution product.

2. $\det[u_{ab}]_{a,b} = (-1)^\frac{q-1}{2} q^{\frac{q+1}{2}} \eta$ where $q$ is odd.

The next needs for evaluation of $\eta$. This follows from Proposition 2.

**Lemma 3.**

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(1) \( \prod_{k=1}^{s-1} (u_0 - u_1^k) = qu_0 - \epsilon. \)

(2) \( \eta = u_1^{(s^2-1)\xi} \prod_{k=1}^{s-1} (u_0 - u_1^k). \)

(3) \( \eta = (-1)^{s-1} v^{(s^2-1)\xi} \prod_{k=1}^{s-1} (v^k - v^{-k}) \) where \( v = u_2 u_1^s. \)

We can see the evaluation of ordinary Gauss sum

\[ g(\eta) = i^{\frac{\tau - 1}{2}} \sqrt{q} \]

from Lemma 3 and the equation \( \prod_{k=1}^{n} 2\sin\left(\frac{k\pi}{n}\right) = \sqrt{n} \) for an odd \( n. \)

4. Prime decompositions of Gauss sums

In this section, using commutative group algebras, we shall give an alternative proof of theorem about the prime decomposition of the Gauss sum which was essentially used in the proof of Stickelberger relation (see [1]).

Let \( m \) be a natural number, let \( p \) be a prime which does not divide \( m \), let \( f \) be the order of \( p \) mod \( m \), and \( q = p^f \). Moreover let \( O \) be the ring of algebraic integers in \( Q(\zeta_{q-1}) \) and let \( P \) be a prime ideal containing \( p \), where \( \zeta_{q-1} \) is a primitive \( (q-1) \)-th root of 1. Then it is well known that \( q \) is the order of a finite field \( F = O/P. \)

We consider the Gauss sum \( g_\alpha = g(\chi^\alpha) = \sum_{\alpha \in F} \chi^\alpha \zeta_p^{\text{tr}(\alpha)} \) where \( \chi \) is a generator of \( \hat{F} \) and \( \text{tr}(\alpha) \) is the trace of \( \alpha \). Let \( P \) be the ideal generated by \( P \) and \( \{1 - \zeta_p^k \mid 0 < k < p\} \) in the ring of algebraic integers \( O \) of \( Q(\zeta_{q-1})_p. \) It is easy to see \( P \) is the prime ideal generated by \( P \) and \( 1 - \zeta_p \). We set \( a^* = b_0 + b_1 + \cdots + b_{f-1} \) for a positive integer \( a = b_0 + b_1 p + \cdots + b_{f-1} p^{f-1} \) where \( 0 < a < q \) and \( 0 \leq b_k < p. \)

The next follows essentially from [3, Proposition 3.2] and this was used essentially for the Stickelberger relation (see [1]).

**Theorem 4.** \( \text{ord}_P(g_\alpha) = a^* \) for \( 0 < a < q, \) namely, \( P^{a^*} \) divides exactly \( g_\alpha. \)

*Proof.* Let \( \nu \) be a natural homomorphism from \( \text{Map}(F, O) \) to \( \text{Map}(F, O/P) \) and let \( J \) be the ideal generated by \( P \) and \( \{u_0 - u_\alpha \mid \alpha \in F\} \). Since \( \nu(\chi^\alpha) = 0 \) for \( \chi^\alpha \neq 1, \) we obtain that \( \nu(\chi^\alpha) \) is contained in \( \nu(J), \) the radical of the group algebra \( \text{Map}(F, O/P), \) and so \( \chi^\alpha \in J. \) [3, Proposition 3.2] together with this implies that \( \gamma \chi^\alpha \in J^{a^*} \) for the Jacobi sum \( \gamma \in O \setminus P. \) The character \( u_\gamma \mapsto \zeta_p^{\text{tr}(\alpha)} \) induces the epimorphism \( \phi : \text{Map}(F, O) \to O \) with \( \phi(J) = P \) and \( \phi(\gamma \chi^\alpha) = \gamma g_\alpha. \) Thus we have \( \text{ord}_P(g_\alpha) \geq a^*. \)
On the other hand, \( \text{ord}_p(g_a) + \text{ord}_p(g_{q-1-a}) = f(p-1) = a^* + (q-1-a)^* \) follows from \( g_a g_{q-1-a} = \chi^*(1)q \) and \( \text{ord}_p(p) = p-1 \). This completes our proof.

**Remark 5.** [3, Proposition 3.3] shows that \( \{\chi^a|a^* = k\} \) forms a basis of \( \nu(J)^k/\nu(J)^{k+1} \) and so \( \nu(J)(\chi^a) = a^* \), namely, \( a^* \) is the maximum integer \( s \) such that \( \chi^a \in J^s \).

Loewy series of \( \text{Map}(F,O/P) \) are computed from this. (\( a^* \) is the maximum integer \( s \) with \( \chi^a \in J^s \))

5. Reciprocity theorems and Lenstra’s primality test

The next lemma is essential in proving quadratic, cubic and biquadratic reciprocity theorems, and Lenstra’s primality test.

**Lemma 6.** Let \( \ell \) be the order of \( \chi \in \hat{F} \), let \( n \) be a prime number with \( (n,q) = 1 \) and let \( e \) and \( s \) be natural numbers with \( n^e \equiv s \mod \ell \). Then

\[
\chi^{-es}(n) \equiv (jq)^{\frac{e-1}{p-1}} \chi^{[q]}(1) \mod n \quad \text{where} \quad j = \chi(-1) \chi^{[e]}(1). 
\]

**Theorem 7 (Lenstra).** Let \( n \) be an odd integer and let \( r \) be a prime divisor of \( n \). Let \( T \) be a finite set consisting of 2 and odd primes \( p \) satisfying \( (n,p) = 1 \) and \( n^{p-1} \not\equiv 1 \mod p^2 \). We set \( t = \prod_{p \in T} p \). Let \( S \) be the set of primes \( q \) satisfying \( (n,q) = 1 \) and \( (q-1) \mid t \). We set \( s = \prod_{q \in S} q \).

We assume there exists an integer \( c \) such that \( c^{\frac{n}{r-1}} \equiv -1 \mod n \), and \( (jq)^{\frac{n^{p-1}-1}{r}} \equiv \chi_q(n) \mod n \) for every \( p \in T \), \( q \in S \) and \( \chi_q \in \hat{F} \) with order \( p \). Then we have \( r \equiv n^i \mod s \) for some \( i < t \).

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RELATIVE INVARIENTS OF GROUPS ACTING ON KRULL DOMAINS AND APPLICATIONS

HARUHISA NAKAJIMA

Abstract. In this paper, we will study on relative invariants of a group $G$ consisting of automorphisms of a Krull domain $R$ and give a criterion for $R_x$ to be a free $R^G$-module of rank one for a 1-cocycle $\gamma$ of $G$ in the unit group $U(R)$, in terms of local 1-cocycles which is similar to one in [N1, S1, S2]. On the other hand, suppose that $G$ is equal to the centralizer of an algebraic torus $G^0$. Consider an affine factorial $G$-variety $X$ with trivial units, over an algebraically closed field $k$ of characteristic zero. Let $V$ be the $K$-dual of a finite dimensional generating $G$-submodule $V'$ of $O(X)$ having a $K$-basis $\Omega$ consisting of weight vectors of $G^0$ such that $\Omega$ does not degenerate under $V\setminus\{0\} \to P(V)$. Suppose that the action $(X,G)$ is stable (cf. [K1, P]) and consider the $G$-submodule $W$ of $V$ such that $G$ is diagonal on $V/W$ and $W \ni x \mapsto x+w \in V$ induces $W//G_w \cong V//G_1$, for some $w \in V$. As an application of the result on modules of relative invariants mentioned above, we show that, for a minimal $W$, $\cl(X/G) \hookrightarrow \eta (G_w/s_W(G_w))$ whose cokernel is finite. Here $\eta (G_w/s_W(G_w))$ is a group of rational characters of $G_w$ modulo its largest pseudo-reflection subgroup on $W$. Some related results are discussed.

1. Introduction.

Let $Q(R)$, $U(R)$ and $H_1(R)$ denote the total quotient ring of a commutative ring $R$, the unit group of $R$ and the set consisting of all prime ideals $p$ of $R$ of height one, respectively. For a ring extension $S \hookrightarrow R$, put

$$H_1(R, S) := \{ p \in H_1(R) \mid p \cap S \in H_1(S) \}$$

and, moreover for any $\Omega \in H_1(S)$, put

$$\Omega(R) := \{ p \in H_1(R) \mid p \cap S = \Omega \}.$$ 

In the case where $R$ is a Krull domain, let $v_{R,p}$ be the discrete valuation of $R$ defined by $p \in H_1(R)$ and, for a subset $\Omega$ of $Q(R)$, let div$_R(\Omega)$ be the divisor on $R$ associated to the divisorialization $\Omega \cdot R$ of $\Omega \cdot R$, if it is a fractional ideal of $R$. Moreover, supp$_R(\Omega)$ stands for the set

$$\{ p \in H_1(R) \mid v_{R,p}(\Omega \cdot R) \neq 0 \}.$$ 

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This is an expository article on some results of the author which shall be published in the forthcoming papers with detailed proofs.
If $R$ is a Krull domain, then a subring $S$ of $R$ satisfying the condition that $Q(S) \cap R$ is so and we can define the reduced ramification index $e(p, p \cap S)$ of $p \in \text{Ht}_1(R, S)$ over $p \cap S$ (cf. [N4]). In this case, the ring $R$ is said to be unramified over $S$ at a subset $\Gamma \subseteq \text{Ht}_1(R, S)$, if $R$ is unramified over $S$ at each $p \in \Gamma$ (cf. [N4]). Let us consider an action of a group $G$ on $R$ as ring automorphisms. For a 1-cocycle $\chi$ of $G$ in the unit group $U(R)$ of $R$, we denote by $R \chi$ the $R^G$-module of $\chi$-invariants or invariants relative to $\chi$, where $R^G$ denotes the subring of $R$ of invariants of $G$.

(1.1) Suppose that $X$ is an affine normal variety over an algebraically closed field $K$ of characteristic $p \geq 0$, whose structure sheaf is denoted by $\mathcal{O}$, with a regular action of an affine algebraic $K$-group $G$. Suppose that $Q(\mathcal{O}(X)^G) = Q(\mathcal{O}(X))^G$ and that $\mathcal{O}(X)^{R_u(G)}$ is noetherian, where $R_u(G)$ is the unipotent radical of $G$. Regard $Z^1(G, U(\mathcal{O}(X)))$, the group of 1-cocyles of $G$, as an additive group and also $\chi(G)$, the rational character group of $G$, as an additive subgroup. We can choose a finite subset $\Gamma$ of $\text{Ht}_1(\mathcal{O}(X), \mathcal{O}(X)^G)$ in such a way that

$$\Gamma \supseteq \text{supp}_{\mathcal{O}(X)}(\mathcal{O}(X) \chi) \cap \text{Ht}_1(\mathcal{O}(X), \mathcal{O}(X)^G)$$

$$(\forall \chi \in Z^1(G, U(\mathcal{O}(X)))) \text{ with } \mathcal{O}(X) \chi \neq \{0\})$$

and $X_{p \cap \mathcal{O}(X)^G}(\mathcal{O}(X)) \subseteq \Gamma$ for all $p \in \Gamma$. For any $p \in \Gamma$, let $I_G(p \cap \mathcal{O}(X)^{G^0})$ denote the inertia group of $G$ at $p \cap \mathcal{O}(X)^{G^0}$ (cf. [N4]). Set $U = \cap_{p \in \Gamma}(\mathcal{O}(X) \chi)$ and let $\pi_p \in p \cap \mathcal{O}(X)^{G^0}$ ($p \in \Gamma$) be a relative invariant of $G^0$ whose associated 1-cocycle is denoted to $\delta_{\pi_p} \in Z^1(G^0, U(\mathcal{U}^{-1}\mathcal{O}(X)))$ such that $\pi_p$ generates $p U^{-1}\mathcal{O}(X)$. Let $\Delta_p$ be the 1-cocycle in $Z^1(I_G(p \cap \mathcal{O}(X)^{G^0}), U(\mathcal{U}^{-1}R))$ defined by $\prod_{q \in I_G(p \cap \mathcal{O}(X)^{G^0})} \pi_q$. For any $\chi \in Z^1(G, U(\mathcal{O}(X)))$ with $\mathcal{O}(X) \chi \neq \{0\}$, $s_p(\chi)$ stands for the smallest $a \in \mathbb{Z}_0$ satisfying

$$\chi|_{I_G(p \cap \mathcal{O}(X)^{G^0})} \equiv a \cdot \Delta_p \mod B^1(I_G(p \cap \mathcal{O}(X)^{G^0}), U(\mathcal{U}^{-1}\mathcal{O}(X)))$$

We can choose a nonnegative integer $b_p(\chi)$ ($p \in \Gamma$) such that

$$\chi|_{G^0} - \sum_{p \in \Gamma} s_p(\chi) \cdot \delta_{\pi_p} \equiv \sum_{p \in \Gamma} b_p(\chi) \cdot \delta_{\pi_p} \mod B^1(G^0, U(\mathcal{U}^{-1}\mathcal{O}(X)))$$

and, for any minimal subset $\Delta$ of $\Gamma$ satisfying

$$\left\{ \sum_{p \in \Delta} N \cdot \delta_{\pi_p} \right\} \cap B^1(G^0, U(\mathcal{U}^{-1}R)) \neq \emptyset,$$

$$\{ b_p(\chi) \mid p \in \Delta \} \text{ contains 0.}$$

As an application of our main result in Sect. 2, we obtain

**Theorem 1.2.** Under the circumstances as in (1.1), for $\chi \in Z^1(G, U(\mathcal{O}(X)))$ such that $\mathcal{O}(X) \chi \neq \{0\}$, the following conditions are equivalent:

1. $\mathcal{O}(X) \chi \cong \mathcal{O}(X)^G$ as $\mathcal{O}(X)^G$-modules.
2. $\dim_{Q(\mathcal{O}(X)^G)}(\mathcal{O}(X) \chi \otimes_{\mathcal{O}(X)^G} Q(\mathcal{O}(X)^G)) = 1$, and the intersection of

$$\sum_{p \in \Gamma} (b_p(\chi) + s_p(\chi)) \cdot \text{div}_R(p) + \sum_{p \in \text{Ht}_1(\mathcal{O}(X)) \setminus \text{Ht}_1(\mathcal{O}(X)^{G^0})} Z \cdot \text{div}_{\mathcal{O}(X)}(p)$$

and $\{ \text{div}_{\mathcal{O}(X)}(f) \mid f \in \mathcal{O}(X) \chi \setminus \{0\} \}$ is nonempty.
If these conditions are satisfied, then $O(X)_x$ is generated by the element $f$ as an $O(X)^G$-module such that $\{\text{div}_{O(X)}(f)\}$ is equal to the non-empty set of divisors stated in (2).

This can be applied to establishing a reduction of invariants of certain reductive algebraic groups to ones of their stabilizers, which is similar to slice method.

(1.3) Furthermore suppose that $G^0$ is an algebraic torus. Let $O(X)_{st}$ be the largest $G$-invariant affine $K$-subalgebra of $O(X)$ such that the action of $G$ on $\text{Spec}(O(X)_{st})$ is stable (e.g. [K1, P]) and $X_{st}$ the affine variety defined by $O(X_{st}) = O(X)_{st}$ (cf. [W1, N3]). Consider a finite dimensional rational $G$-submodule $V^V$ of $O(X)$ which generates $O(X)$ as a $K$-algebra. Let $V$ denote the dual space of $V^V$ on which $G$ acts naturally. We have a canonical $G$-equivariant closed embedding $X \hookrightarrow V$ and identify the dual of $V$ with $V^V$. A pair $(W, w)$ is defined to be a parallelled linear hull of $(X, G)$ through $V^V$ or, simply, of $(V, G)$, if $W$ is a $G$-submodule of $V_{st}$ such that $G$ is diagonalizable on $V_{st}/W$, $w$ is a nonzero vector of $V_{st}$ such that $W \cap <Gw>_K = \{0\}$ and the morphism

$$(\bullet + w) : W \ni x \mapsto x + w \in V_{st}$$

induces the isomorphism

$$\pi_{V_{st}/G_{st}/V^V} \circ (\bullet + w)//G_{w} : W//G_{w} \overset{\sim}{\longrightarrow} V_{st}///G.$$ 

Here $(\bullet + w)//G_{w} : W//G_{w} \longrightarrow V_{st}///G_{w}$ is the algebraic quotient of $(\bullet + w)$ and

$$\pi_{V_{st}/G_{st}/V^V} : V_{st}///G_{w} \longrightarrow V_{st}///G$$

is induced by $O(V_{st})^G \hookrightarrow O(V_{st})_{Gw}$. Under the assumption that $G = G^0$, the pair $(W, w)$ seems to be initially defined and used by H. P. Kraft and D. H. Wehlau (cf. §1 of [K2] and [W1, W2]). A paralleled linear hull $(W_0, w_0)$ of $(V, G)$ is said to be minimal, if $W_0$ is minimal in the subspaces $W$ which admit paralleled linear hulls $(W, w)$ of $(V, G)$ for some $w$'s. A element of $\sigma \in G$ is said to be a pseudo-reflection on $V$ (resp. a generalized-reflection on $O(X)$), if $\dim(\sigma - 1)(V) = \text{ht}(\sigma - 1)(V) \cdot O(V) \cap O(V)^G = 1$ (resp. $\text{ht}((\sigma - 1)(O(X)) \cdot O(X)) = \text{ht}((\sigma - 1)(O(X))) \cdot O(X) \cap O(X)^G = 1$). We denote by $R_V(G)$ (resp. by $R_O(X)(G)$) the subgroup of $G$ generated by all pseudo-reflections (resp. generalized-reflections) of $G$ on $V$ (resp. $O(X)$).

**Theorem 1.4.** Under the circumstances as in (1.3), moreover suppose that $p = 0$ and $X$ is an affine conical factorial variety with a regular conical action of $G$. Let $V^V$ be the rational homogeneous $G$-submodule of $O(X)$ minimally generating $O(X)$ as a $K$-algebra. Suppose that $Z_G(G^0) = G$. Let $(W, w)$ be a minimal paralleled linear hull of $(V, G)$ through $V^V$, where $V$ denotes the $K$-dual $(V^V)^V$ of $V^V$. Then

1. The quotient morphism $\pi_{V^V/G_{w}}$ (resp. $\pi_{X_{st}/G_{w}}$) is no-blowing-up of codimension one, and for any $p \in \text{Ht}_1(O(X)_{st}/G_{w})$ and $q \in \text{Ht}_1(O(W)/G_{w})$, the actions of $G_{w}$ on $X_p(O(X)_{st})$ and $X_q(O(W))$ are transitive.
2. $\text{Cl}(O(X)^G) \cong \text{Cl}(O(X)_{st}/G_{w}) \cong \text{Cl}(O(V)^G) \cong \text{Ht}_1(O(V) / R_O(X)(G_{w}))$.
3. If $X_{st}/G_{w}$ is an affine space, then so is $V//G$.
4. Suppose that $G_{w}$ is solvable. Then $V//G$ is an affine space if and only if $O(X)^G$ is factorial.
The last assertion can be regarded as a generalization of [W2] and Proposition 3.6 of [N1]. Similarly by the use of paralleled linear hulls, we can determine representations of diagonalizable groups with the algebra of invariants which are hypersurfaces. Further applications to invariants of non-semisimple reductive groups shall be given by the author elsewhere.

The notations \( \mathbb{N} \) and \( \mathbb{Z} \) are standard and \( \mathbb{Z}_0 \) stands for the set of all nonnegative integers.

2. Modules of relative invariants.

From now on to the end of Proposition 2.10, suppose that \( R \) is a Krull domain with an action of a group \( G \) through the automorphism group \( \text{Aut}(R) \) of \( R \). The group of cocycles of degree 1 including character groups are represented as additive groups.

**Proposition 2.1.** For \( \chi \in Z^1(G, U(R)) \), \( R_\chi \cong R^G \) as \( R^G \)-modules if and only if

\[
\dim_{Q(R^G)}(Q(R_\chi) \otimes_{R^G} R_\chi) = 1 \quad \text{and} \quad \frac{1}{f} R \cap Q(R^G) = R^G \text{ for some } f \in R_\chi.
\]

We derive the next theorem from Proposition 2.1.

**Theorem 2.2.** For \( \chi \in Z^1(G, U(R)) \), the \( R^G \)-module \( R_\chi \) is free and of rank one if and only if the following two conditions are satisfied:

1. \( \dim_{Q(R^G)}(Q(R_\chi) \otimes_{R^G} R_\chi) = 1 \)
2. There exists a nonzero element \( f \) satisfying

\[
(2.2.1) \quad \forall q \in \text{Ht}_1(R^G) \quad \Rightarrow \quad \exists \mathfrak{p} \in \text{X}_q(R) \text{ such that } v_{R, \mathfrak{p}}(f) < e(\mathfrak{p}, q).
\]

If these conditions are satisfied, then \( R_\chi = R^G \cdot f \) for any nonzero \( f \in R_\chi \) such that (3.3.1) holds for \( f \). \( \square \)

The following two propositions play an important role in defining the local datum which shall be introduced in (2.5).

**Proposition 2.3.** Let \( \chi \in Z^1(G, U(R)) \) and let \( H \) be a normal subgroup of \( G \) of a finite index. Suppose that \( R^H \) is unramified over \( R^G \) at \( \text{supp}_R(R_\chi) \cap \text{Ht}_1(R, R^G) \). Then the following conditions are equivalent:

1. \( R_\chi \) is an \( R^G \)-free module of rank one and

\[
\dim_{Q(R^H)}(Q(R^H) \otimes_{R^H} R_{\chi|H}) = 1.
\]

2. There are nonzero \( g \in R_{\chi|H} \) and \( u \in U(R^H) \) such that \( R_{\chi|H} = R^H \cdot g \) and \( R_\chi \ni g \cdot u \).

If these conditions are satisfied, then \( R_\chi = R^G \cdot gu \) for the elements \( g \) and \( u \) in (2).

**Proposition 2.4.** Suppose that \( Q(R^G) = Q(R)^G \) and let \( \chi \in Z^1(G, U(R)) \) such that \( R_\chi \neq \{0\} \). Let \( U \) be a multiplicative system of \( R \) invariant under the action of \( G \) such that

\[
p \cap U = \emptyset \quad (\forall \mathfrak{p} \in \bigcup_{p \in \text{supp}(R_\chi) \cap \text{Ht}_1(R, R^G)} X_{p \cap R^G}(R)),
\]

\[
p \cap U \neq \emptyset \quad (\forall \mathfrak{p} \in \text{supp}(R_\chi) \text{ satisfying } \text{ht}(p \cap R^G) \geq 2).
\]
Then we have

(1) For any \( q \in \text{supp}(R_X) \cap \text{Ht}_1(R, R^G) \) and \( p \in X_{q \cap R^G}(R) \), the following congruence holds:

\[
\nu_{R, p}(R_X) \equiv \nu_{U^{-1}R, pU^{-1}R}(U^{-1}R)_X \mod (p, q \cap R^G).
\]

(2) If \( R_X \cong R^G \) as \( R^G \)-modules, \( (U^{-1}R)_X \cong (U^{-1}R)^G \) as \( (U^{-1}R)^G \)-modules and \( (U^{-1}R)_X = R_X \cdot (U^{-1}R)^G \).

(2.5) We explain our notations and circumstances which shall be considered as follows. Suppose that \( Q(R^G) = Q(R) \). Let \( \chi \in Z^1(G, U(R)) \) be a 1-cocycle such that \( R_X \neq \{0\} \) and \( \Gamma \) a finite subset of \( \text{Ht}_1(R, R^G) \) such that

\[
\Gamma \supseteq \text{supp}_R(R_X) \cap \text{Ht}_1(R, R^G)
\]

and \( \Gamma \supseteq X_{p \cap R^G}(R) \) for any \( p \in \Gamma \). Let \( H \) be a normal subgroup of \( G \) of a finite index stabilizing each \( p \) in \( \Gamma \). Put \( U = \cap_{p \in \Gamma} (R \setminus p) \). Let \( \pi_p \) be an element which generates \( pU^{-1}R \) and, for any subgroup \( N \) of \( G \), set

\[
\pi_{Np} = \prod_{q \in Np} \pi_q.
\]

Let \( \Delta_{N\pi_p} \in Z^1(N, U(U^{-1}R)) \) be the cocycle defined by

\[
N \ni \tau \longmapsto \Delta_{N\pi_p}(\tau) = \frac{\tau(\pi_{Np})}{\pi_{Np}} \in U(U^{-1}R)
\]

and \( \delta_{\pi_p} \in Z^1(H, U(U^{-1}R)) \) the cocycle defined by

\[
H \ni \sigma \longmapsto \delta_{\pi_p}(\sigma) = \frac{\sigma(\pi_p)}{\pi_p} \in U(U^{-1}R).
\]

Moreover we donate by \( s_{I_G(p \cap R^H)p}(\chi) \) the infimum of

\[
\{ a \in Z_0 \mid \chi|_{I_G(p \cap R^H)} \equiv a \cdot \Delta_{I_G(p \cap R^H)p} \mod B^1(I_G(p \cap R^H), U(U^{-1}R)) \}
\]

and denote by \( \chi^H \) the cocycle

\[
\chi|_H - \sum_{p \in \Gamma} s_{I_G(p \cap R^H)p}(\chi) \cdot \delta_p \in Z^1(H, U(U^{-1}R)).
\]

The independence of the number \( s_{I_G(p \cap R^H)p}(\chi) \) on the choice of the subgroup \( H \) is guaranteed in the next lemma.
Lemma 2.6. Under the same circumstances as in (2.5), the following properties hold for each \( p \in \Gamma \):

1. \( e(q, q \cap R^G) = e(p, p \cap R^G) \) for all \( q \in I_G(p \cap R^H) \).
2. \( \text{ord}(\Delta_{I_G(p \cap R^H)} p) \mod B^1(I_G(p \cap R^H), U(U^{-1}R)) = e(p, p \cap R^G) \).
3. \( 0 \leq s_{I_G(p \cap R^H)}(\chi \chi) < e(p, p \cap R^G) \) and

\[
 s_{I_G(p \cap R^H)}(\chi \chi) \equiv v_{R^p}(R_{\chi}) \mod e(p, p \cap R^G).
\]

In order to define the characteristic divisor \( D_\chi \), we need

Lemma 2.7. Under the same circumstances as in (2.5), there exist non-negative integers \( a_p(\chi) \) \((\forall p \in \Gamma)\) such that \( \{a_p(\chi) \mid p \in X_\Delta(R)\} \) contains 0, for any restriction \( \Omega \in \{p \cap R^H \mid p \in \Gamma\} \) and the following congruence holds;

\[
\chi^H \equiv \sum_{p \in \Gamma} a_p(\chi) e(p, p \cap R^G) \cdot \delta_{\pi_p} \mod B^1(H, U(U^{-1}R)).
\]

Furthermore:

1. \( (a_p(\chi) \mid p \in \Gamma) \) is uniquely determined by \( \chi \).
2. Extending \( a_p(\chi) = 0 \) for any \( p \in Ht_1(R, R^G) \setminus \Gamma \), we see that \( (a_p(\chi) \mid p \in Ht_1(R, R^G)) \) is independent on the choice of \( \Gamma \).
3. There is an element \( u \in U(U^{-1}R) \) such that the \((U^{-1}R)^G\)-module \((U^{-1}R)_\chi\) is generated by the element

\[
u \cdot \prod_{p \in \Gamma} \pi_{\delta_p} \cdot \frac{a_p(\chi)e(p, p \cap R^G) + s_{I_G(p \cap R^H)}(\chi \chi)}{s_{I_G(p \cap R^H)}(\chi \chi)}.
\]

(2.8) Under the same circumstances as in (2.5) and Lemma 2.7, let \( D_\chi \) denote the divisor

\[
\sum_{p \in \sigma(R_\chi) \cap Ht_1(R, R^G)} \{a_p(\chi)e(p, p \cap R^G) + s_{I_G(p \cap R^H)}(\chi \chi)\} \cdot \text{div}_R(p)
\]

in \( \text{Div}(R) \) and we call \( D_\chi \) the characteristic divisor of \( \chi \). Let \( D_G \) denote the subgroup

\[
\sum_{p \in Ht_1(R) \setminus Ht_1(R, R^G)} \mathbb{Z} \cdot \text{div}_R(p)
\]

of \( \text{Div}(R) \).

Using the residue class of the characteristic divisor, we obtain the main result of this section which can be regarded as a generalization of the Stanley criterion (cf. [S1, S2]) for \( R_\chi \) to be a free \( R^G \)-module of rank one:
Theorem 2.9. Under the circumstances as in (2.5) and Lemma 2.7, \( R_X \cong R^G \) as \( R^G \)-modules if and only if \( \dim_{Q(R^G)}(R_X \otimes_{R^G} Q(R^G)) = 1 \) and
\[
(D_X + D_G) \cap \{ \text{div}_R(f) \mid f \in R_X \setminus \{0\} \} \neq \emptyset.
\]
In the case where these equivalent conditions are satisfied,
\[
(D_X + D_G) \cap \{ \text{div}_R(f) \mid f \in R_X \setminus \{0\} \} = \{ \text{div}_R(f_X) \}
\]
for some \( f_X \in R_X \) and \( R_X = R^G \cdot f_X \).

The criterion in the case of algebraic group actions is reduced to Theorem 2.9, by the aid of the following two results.

Proposition 2.10. Suppose that \( Q(R^G) = Q(R)^G \). Let \( H \) be a normal subgroup of \( G \) of a finite index and \( \chi \in Z^1(G, U(R)) \). If \( R_X \cong R^G \) as \( R^G \)-modules, then
\[
\text{supp}_R(R_X) \cap Ht_1(R, R^G) \subseteq \bigcap_{f \in R_X \setminus \{0\}} \text{supp}_R(f) \cup \{ p \in Ht_1(R, R^G) \mid I_G(p \cap R^H) |_{R^H} \neq \{1\} \}.
\]

Corollary 2.11. Suppose that \( G \) is an affine algebraic group over an algebraically closed field \( K \). Suppose that \( R \) is an affine normal domain over \( K \) on which \( G \) acts \( K \)-rationally as \( K \)-algebra automorphisms. Suppose that \( Q(R^G) = Q(R)^G \) and \( R^{R_u(G^0)} \) is noetherian. Then
\[
\left\{ \bigcup_{\chi \in \{ x \in Z^1(G, U(R)) \mid R_X \cong R^G \}} \text{supp}_R(R_X) \right\} \cap Ht_1(R, R^G)
\]
is a finite set.

(2.12) Proof of Theorem 1.2. By Corollary 2.11, we see that the finite set \( \Gamma \) in (1.1) exists. If \( \Delta \) is a minimal subset of \( \Gamma \) such that there exist \( a_p \in \mathbb{N} \) (\( p \in \Delta \)) and a unit \( u \) of \( U^{-1} \mathcal{O}(X) \), satisfying
\[
u \cdot \prod_{p \in \Delta} \pi_{p^a_p}^u \in (U^{-1} \mathcal{O}(X))^G,
\]
then, for \( p \in \Delta \),
\[
\{ pU^{-1} \mathcal{O}(X) \mid p \in \Delta \} = X_{pU^{-1} \mathcal{O}(X) \cap (U^{-1} \mathcal{O}(X))^G}(U^{-1} \mathcal{O}(X)) \]
For any subset \( \Delta \) of \( \Gamma \), the equality (2.12.1) holds for a prime ideal \( p \in \Delta \) if and only if \( \Delta \) is minimal in the subsets \( \Delta \)'s satisfying
\[
\sum_{p \in \Delta} N \cdot \delta_{a_p} \cap B^1(G^0, U(U^{-1} R)) \neq \emptyset.
\]
Hence, by (2.5), such a subset \( \Delta \) as in (2.11.1) is identical with \( X_{p \cap \mathcal{O}(X)^G}(\mathcal{O}(X)) \) for a prime ideal \( p \in \Delta \). Consequently the assertion of Theorem 1.2 follows from Theorem 2.9. \( \square \)
3. Applications.

In this section, suppose that \( R \) is an affine factorial domain with trivial units and \( G^0 \) is an algebraic torus over an algebraically closed field \( K \) of characteristic zero. A subset \( \{f_1, \ldots, f_n\} \) of \( n \) elements in \( R \) is said to be \((R, G)\)-basic, if each \( f_i \) is a relative invariant of \( G \) which is prime in \( R \) and \( R \cdot f_i \neq R \cdot f_j \) for any \( 1 \leq i < j \leq n \). Moreover, a subset \( \{f_1, \ldots, f_n\} \) of \( n \) elements is said to be \((R, G)\)-free, if it is \((R, G)\)-basic and, for any \( a_i \in \mathbb{Z}_0 \), there exists a character \( \chi \in \mathcal{X}(G) \) such that

\[
R_{\chi} = R^G \cdot \prod_{i=1}^{n} f_i^{a_i}.
\]

We further say that \( \{f_1, \ldots, f_n\} \) of \( n \) elements is \((R, G)\)-afforded, if it is \((R, G^0)\)-basic, the \( K \)-subspace \( < f_1, \ldots, f_n >_K \) of \( R \) is \( G \)-invariant and \( \{f_1, \ldots, f_n\} \) forms a \( K \)-basis of this space. There exists a \( G \)-afforded generating system \( \{g_1, \ldots, g_n\} \), which is denoted to \( \Omega \) for a convenience sake, of \( R \) of \( n \) elements as a \( K \)-algebra. We denote by \( (K^x \circ \Omega)_{R, \text{st}}^G \) the set consisting of all \( g_i \)'s such that \( R \cdot g_i \cap R^G \neq \{0\} \) and \( g_i \) are relative invariants of \( G \). The subring \( R_{\text{st}} \) is an affine factorial \( K \)-domain on which \( G \) acts naturally (cf. Sect. 1 and [N3]) and \( (K^x \circ \Omega)_{R, \text{st}}^G \subseteq R_{\text{st}} \). Let \( \Omega_{R, G} \) denote the set

\[
\{g_i \in (K^x \circ \Omega)_{R, \text{st}}^G \mid \text{ht}(R \cdot g_i \cap R^G) = 1 \text{ and } \mathcal{X}_{R, g_i \cap R^G}(R)^G = \mathcal{X}_{R, g_i \cap R^G}(R)\}.
\]

For any pair \( g_{i_1}, g_{i_2} \) of elements in \( \Omega_{R, G} \), define the relation \( g_{i_1} \equiv_G g_{i_2} \), if \( R \cdot g_{i_1} \cap R^G = R \cdot g_{i_2} \cap R^G \). Then we denote by \( \Omega_{R, G}^{\equiv_{G}} \) the set consisting of all complete systems of representatives of \( \Omega_{R, G} \) modulo \( \equiv_G \).

As an application of Theorem 2.9, we have

**Theorem 3.1.** Let \( \{f_1, \ldots, f_m\} \) be a subset of \( R_{\text{st}} \) of \( m \) elements. Then \( \{f_1, \ldots, f_m\} \) is \((R, G)\)-free if and only if there exists a system \( \Upsilon \in \Omega_{R, G}^{\equiv_{G}} \) such that

\[
(K^x \circ \Omega)_{R, \text{st}}^G \setminus \Upsilon \cap K^x \circ f_i \neq \emptyset \quad (1 \leq i \leq m).
\]

Let \( V \) be the dual space of \( < \Omega >_K \) and denote by \( \rho \) the canonical \( K \)-morphism \( \mathcal{O}(V) \to R \) defined by the inclusion \( \Omega \subseteq R \). We easily see that \((K^x \circ \Omega)_{R, \text{st}}^G = (K^x \circ \Omega)_{\mathcal{O}(V), \text{st}}^G \).

**Corollary 3.2.** The restriction of the mapping

\[
2^{\mathcal{O}(V_{\text{st}})} \ni \Gamma \mapsto \rho(\Gamma) \in 2^{R_{\text{st}}}
\]

is a one-to-one correspondence between the set consisting of all finite \((\mathcal{O}(V), G)\)-free subsets of \( \mathcal{O}(V_{\text{st}}) \) and the set consisting of all finite \((R, G)\)-free subsets of \( R_{\text{st}} \). Moreover, the equality \( \Omega_{R, G}^{\equiv_{G}} = \Omega_{\mathcal{O}(V), G}^{\equiv_{G}} \) holds.

**Proof.** Thanks to Theorem 3.1, the proof of the first assertion is reduced to show that, for a subset \( \Gamma \) of \( \Omega \), the condition that \( \Gamma \) is \((\mathcal{O}(V), G^0)\)-free is equivalent to the one that \( \Gamma \) is \((R, G^0)\)-free. The last equality follows easily from Theorem 3.1 and the first assertion. \( \square \)
Proposition 3.3. Suppose that \( \{f_1, \ldots, f_m\} \) is a maximal \((R,G)\)-free subset of \( R \) of \( m \) elements. Then there is not a non-empty \((R, \cap_{i=1}^m G_{f_i})\)-free subset consisting relative invariants of \( G \). Especially in the case where \( G \) is connected, we have \( \text{ht}(R \cdot f \cap R \cap \cap_{i=1}^m G_{f_i}) = 1 \) and \( \sharp(X_{R,f \cap R \cap \cap_{i=1}^m G_{f_i}}(R)) = 1 \), for any prime element \( f \in R \) such that \( R \cdot f \cap R \cap \cap_{i=1}^m G_{f_i} \neq \{0\} \).

Proposition 3.4. For a subspace \( W \) of \( V \), the following conditions are equivalent:

1. \((W, w)\) is a paralleled linear hull of \((V, G)\) for a nonzero \( w \in V_{st} \).
2. There exists a system \( \Upsilon \in \Omega^{\equiv \mathcal{G}}(V, G) \) and a subset \( \mathcal{F}_W \) of \((K^x \circ \Omega)^{\mathcal{G}}(V)_{st} \setminus \Upsilon \) such that

\[
W = \{ x \in V_{st} \mid F(x) = 0 \quad (\forall F \in \mathcal{F}_W) \},
\]

where \( \Omega \) is regarded as a subset of \( \mathcal{O}(V) \).

Proof. (2) ⇒ (1) : By Theorem 3.1, we see that the set \( \mathcal{F}_W \) is \((\mathcal{O}(V_{st}), G)\)-free. For each \( z \in \mathcal{F}_W \), let \( \chi_z \in \mathcal{X}(G) \) to satisfy \( \mathcal{O}(V_{st})_{\chi_z} \ni z \). Since \( (V_{st}^V)^{\chi_z} = K \cdot z \), for each \( z \in \mathcal{F}_W \), we can choose a unique vector \( z^V \) from \((V_{st}^V)_{\chi_z} \) in such a way that \( z^V(z) = 1 \). Putting \( w = \sum_{z \in \mathcal{F}_W} z^V \), we infer that \((W, w)\) is a paralleled linear hull of \((V, G)\).

(1) ⇒ (2) : There are weight vectors \( v_i (1 \leq i \leq m) \) of \( G \) in \( V_{st} \) whose reduction modulo \( W \) form a \( K \)-basis of \( V_{st}/W \) such that \( w = \sum_{i=1}^m v_i \). Let \( \{v_1^V, \ldots, v_m^V\} \) be a basis of \( W^\perp = (V_{st}/W)^V \) dual to \( \{v_1 \mod W, \ldots, v_m \mod W\} \). Then, since

\[
\pi_{V_{st}/G_{v_i}, V/G, \mathcal{O} \circ (\cdot + w)}/G_{w} : W//G_{w} \twoheadrightarrow V_{st}//G,
\]

the set \( \{v_1^V, \ldots, v_m^V\} \) is \((\mathcal{O}(X), G)\)-free. Thus the assertion follows from Theorem 3.1. \( \square \)

Consequently \((W, w)\) is a minimal paralleled linear hull of \((V, G)\) for some \( w \in V \) if and only if

\[
W = \{ x \in V_{st} \mid F(x) = 0 \quad (\forall F \in (K^x \circ \Omega)^{\mathcal{G}}(V)_{st} \setminus \Upsilon) \},
\]

for some system \( \Upsilon \in \Omega^{\equiv \mathcal{G}}(V, G) \).

(3.5) Suppose that \( g \in R_X \) a fixed relative invariant of \( G \) such that \( \{g\} \) is \((R, G)\)-free and, for a convenience sake, put \( H = \text{Ker} \chi_S \) and \( S = R_H \). Furthermore, suppose that the action \((\text{Spec}(R), G)\) is stable and \( Z_G(G^0) = G \). As in [M] (e.g., [N2]), we similarly introduce the following subgroups of \( \text{Div}(S) \):

\[
E^*(G, S) = \bigoplus_{p \in H_{1}(S^0)} \mathbb{Z} \cdot \left( \sum_{q \in \mathcal{X}_{p}(S)} e(q, p) \cdot \text{div}_S(q) \right)
\]

\[\oplus \bigoplus_{q \in H_{1}(S), \text{ht}(q \cap S^0) \geq 2} \mathbb{Z} \cdot \text{div}_S(q),\]

\[
\Delta_{S,g}(G, S) = \bigoplus_{q \in H_{1}(S) \setminus \mathcal{Y}_{S,g}(S), q \cap S^0 \neq \{0\}} \mathbb{Z} \cdot \text{div}_S(q),
\]

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where \( Y_{S\cdot g}(S) = X_{S\cdot g \cap S^G}(S) \), if \( \text{ht}(S \cdot g \cap S^G) = 1 \), and \( Y_{S\cdot g}(S) = \{S \cdot g\} \), otherwise. Let \( \Phi^* : E^*(G, S) \rightarrow \text{Div}(S^G) \) be a homomorphism defined by

\[
\sum_{p \in H^1(S^G)} a_p \cdot D_p + \sum_{q \in H^1(S), \text{ht}(q \cap S^G) \geq 2} b_q \cdot \text{div}_S(q) \quad \longmapsto \quad \sum_{p \in H^1(S^G)} a_p \cdot \text{div}_{S^G}(p) \quad (a_p, b_q \in \mathbb{Z}),
\]

where \( D_p = \sum_{q \in X_p(S)} e(q, p) \cdot \text{div}_S(q) \).

Under the same circumstances as in (3.5), we have Lemma 3.6, Proposition 3.6 and Proposition 3.7 as follows:

**Lemma 3.6.** The canonical homomorphism \( \text{Div}(S^G) \rightarrow \text{Cl}(S) \) is an epimorphism.

**Proof.** For any \( p \in H^1(S) \), choose a prime \( f \) from \( R \) in such a way that \( R \cdot f \cap R^H = p \). Then, expressing \( H \cdot Rf = \{R \cdot f_1, \ldots, R \cdot f_{(H \cdot Rf)}\} \), we see

\[
a := \prod_{1 \leq i \leq \delta(H \cdot Rf)} f_i
\]

is an invariant relative to some \( \psi \in \mathcal{X}(H) \) such that \( R \cdot a \cap R^H = R \cdot a \cdot R_{-\psi} \). As \( Z_G(G^0) = G \), there is a character \( \delta \in \mathcal{X}(G^0 \cdot H) \) such that \( \delta|_H = \delta \) and \( \delta(\ker((G^0 \cdot H) \rightarrow \text{Aut}(R))) \) is trivial. Since \( \delta \) is regarded as a character of the group \( G^0 \cdot H|_R \), the module \( R_\delta \) contains a nonzero element \( b \). So \( R^H \cdot \frac{b}{a} \cdot (R \cdot a)^H = (R \cdot b)^H \). By the stability of the action of \( H \) on \( R \) (cf. [P, N3]), we see \( \frac{a}{b} \in \mathcal{Q}(R^H) = \mathcal{Q}(R^H) \), which implies that

\[
\text{div}_{R^H}((R \cdot b) \cap R^H) \equiv \text{div}_{R^H}(p) \mod \text{Prin}(R^H).
\]

From this, we infer that the homomorphism

\[
\text{Div}(R^H)^{G^0 \cdot H} \hookrightarrow \text{Div}(R^H) \rightarrow \text{Cl}(R^H)
\]

is surjective. \( \square \)

The next two propositions are based on Lemma 3.6, in which the assumption that \( G \) is a central extension of \( G^0 \) is essential.

**Proposition 3.7.** Suppose that \( \text{ht}((S \cdot g \cap S^G) \geq 2 \). Then the sequence

\[
0 \rightarrow \text{Prin}(S) \cap \Delta_{S \cdot g}(G, S) \rightarrow \Delta_{S \cdot g}(G, S) \rightarrow \text{Cl}(S) \rightarrow 0
\]

is exact and

\[
\Phi^*(\text{Prin}(S) \cap \Delta_{S \cdot g}(G, S)) = \text{Prin}(S^G).
\]

Moreover, \( \Phi^\ast \) induces the isomorphism \( \text{Cl}(S) \cong \text{Cl}(S^G) \).
Proposition 3.8. Suppose that \( \text{ht}(S \cdot g \cap S^G) = 1 \). Let \( q_0 \in \text{Ht}_1(S) \) satisfy the equality \( X_{S,g \cap S^G}(S) \backslash \{S \cdot g\} = \{q_0\} \). Set

\[
\nabla(G, S) := \Delta_{S,g}(G, S) \oplus \mathbb{Z} \cdot \text{div}_S(q_0) \subseteq \text{Div}(S)
\]

and let \( \Theta : E^*(G, S) \to \nabla(G, S) \) be the composite

\[
E^*(G, S) \hookrightarrow \text{Div}(S) \xrightarrow{\Theta} \nabla(G, S).
\]

Then the square

\[
\begin{array}{ccc}
\text{Cl}(S^G) & \xrightarrow{\sim} & \text{Cl}(S) \\
\text{can.} & \downarrow & \text{can.} \\
\text{Div}(S^G) & \xrightarrow{\sim} & \nabla(G, S)
\end{array}
\]

is a commutative diagram with horizontal isomorphisms.

Applying Proposition 3.7 and Proposition 3.8 inductively to a maximal \((R, G)\)-free set, by the aid of Theorem 3.1, we now establish

Theorem 3.9. Suppose that \( Z_G(G^0) = G \). Let \((W, w)\) be a minimal paralleled linear hull of \((V, G)\) through \( \Omega > K \). Then

1. The inclusions \( R^{G_w}_{st} \hookrightarrow R_{st} \) and \( \mathcal{O}(W)^{G_w} \hookrightarrow \mathcal{O}(W) \) are no-blowing-up of codimension one and, for each \( p \in \text{Ht}_1(R^{G_w}_{st}) \) and \( q \in \text{Ht}_1(\mathcal{O}(W)^{G_w}) \), the actions of \( G_w \) on \( X_p(R_{st}) \) and \( X_q(\mathcal{O}(W)) \) are, respectively, transitive.

2. We have the monomorphism

\[
\text{Cl}(R^G) \cong \text{Cl}(R^{G_w}_{st}) \cong \mathcal{X}(G_w/R_{st}(G_w))
\]

\[
\xrightarrow{\varphi} \mathcal{X}(G_w/R_{st}(G_w)) \cong \text{Cl}(\mathcal{O}(V)^G)
\]

with its cokernel, isomorphic to a subgroup of the abelianization of the finite group \( R_{R_{st}}(G_w)/R_{st}(G_w) \).

Proof of Theorem 1.4. The first and second assertions follow from Theorem 3.9. Suppose that \( \text{Cl}(R^G) \) is finite. Then, the assertion (2) of Theorem 3.9 implies that \( G_w \) is also finite. Hence the third assertion follows from [C, R1] and, in the case where \( G_w \) is solvable, applying Proposition 3.6 of [N1] to the \( G_w \)-module \( W \), we see that \( \mathcal{O}(V)^G \) is a polynomial ring over \( K \) if and only if it is factorial. Thus the last assertion follows from this observation and Theorem 3.9. \( \square \)

As in [D] and [N1], it shall be expected that Theorem 2.9 and Theorem 3.9 imply the detailed description of \( \text{Cl}(\mathcal{O}(X)^G) \) in terms of group theory. Finally, we point out that these theorems seem somewhat useful in studying invariants of non-semisimple reductive algebraic groups which are treated in [P] for semisimple groups.
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NAKAYAMA ISOMORPHISMS FOR THE MAXIMAL QUOTIENT RING
OF A LEFT HARADA RING

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ABSTRACT. From several results of Kado and Oshiro, we see that if the maximal quotient ring of a given left Harada ring $R$ of type (•) has a Nakayama automorphism, then $R$ has a Nakayama isomorphism. This result poses a question whether if the maximal quotient ring of a given left Harada ring $R$ has a Nakayama isomorphism, then $R$ has a Nakayama isomorphism. In this paper, we shall show that a basic ring of the maximal quotient ring of a given Harada ring has a Nakayama isomorphism if and only if its Harada ring has a Nakayama isomorphism.

INTRODUCTION

Let $R$ be a basic left Harada ring. Then we have a complete set
\[ \{e_{11}, \ldots, e_{1n(1)}, \ldots, e_{m1}, \ldots, e_{mn(m)}\} \]
of primitive idempotents for $R$ such that for each $i = 1, \ldots, m$

(a) $e_{ii}R$ is injective as a right $R$-module;

(b) $J(e_{i,k-1}R) \cong e_kR$ for each $k = 2, \ldots, n(i)$.

We call $R$ a ring of type (•) if there exists an unique $g_i$ in $\{e_{im(i)}\}_{i=1}^{m(i)}$ for each $i = 1, \ldots, m$ such that the socle of $e_{i1}R$ is isomorphic to $g_iR/J(g_iR)$ and the socle of $Rg_i$ is isomorphic to $Re_{i1}/J(Re_{i1})$.

Oshiro [10] showed the following:

Result A ([10, Theorem 2]). Suppose that $R$ is a left Harada ring which is not of type (•). Then there exists a series of left Harada rings and surjective ring homomorphisms:
\[ T_1 \xrightarrow{\phi_1} T_2 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_{n-1}} T_n \xrightarrow{\phi_n} R \]
such that

1. $T_1$ is of type (•), and
2. $\text{Ker } \phi_i$ is a simple ideal of $T_i$ for any $i \in \{1, \ldots, n\}$.

Kado and Oshiro [7] showed the following results;

Result B ([7, Proposition 5.3]). If every basic QF rings has a Nakayama automorphism, then every basic left Harada ring of type (•) has a Nakayama isomorphism.

The detailed version of this paper will be submitted for publication elsewhere.
Result C ([7, Proposition 5.4]). Let S be a two-sided ideal of R that is simple as a left ideal and as a right ideal. If R has a Nakayama isomorphism, then R/S has a Nakayama isomorphism.

Moreover Kado showed the following;

Result D ([6, Corollary]). The maximal quotient ring of a left Harada ring of type (∗) is a QF ring.

Using these four results, we see that if the maximal quotient ring of a given left Harada ring R of type (∗) has a Nakayama automorphism, then R has a Nakayama isomorphism. So this result poses a question whether if the maximal quotient ring of a given left Harada ring R has a Nakayama isomorphism, then R has a Nakayama isomorphism. In this paper, we shall show that the maximal quotient ring of a given left Harada ring R has a Nakayama isomorphism iff R has a Nakayama isomorphism.

Throughout this paper, we assume that all rings are associative rings with identity and all modules are unitary. By $M_R$ (resp. $R M$), we means that M is a right (resp. left) R-module, respectively. We denote the set of primitive idempotents of R by $\Pi(R)$, and denote a complete set of primitive idempotents of R by $\pi(R)$.

We call a one-sided artinian ring R right (resp. left) QF-3 ring if $E(R R)$ (resp. $E(R R)$) is projective, respectively.

We denote the maximal left (resp. right) quotient ring of R by $Q_l(R)$ (resp. $Q_r(R)$), respectively, and denote the maximal left and maximal right quotient ring of R by $Q(R)$. If a ring is QF-3, its maximal left quotient ring and its right quotient ring coincide by [16, Theorem 1.4].

1. Maximal Quotient Ring

We list some basic results, which several authors showed, for our main result in this paper. Recall that for $e, f \in \Pi(R)$, we say that the pair $(e R : R f)$ is an i-pair if $S(e R) \cong f R J(f R)$ and $S(R f) \cong R e J(R e)$.

Lemma 1 ([5]). Let R be a one-sided artinian ring, and let $e \in \Pi(R)$. Then the following conditions are equivalent:

1. $e R$ is injective.
2. There exists some $f \in \Pi(R)$ such that $(e R : R f)$ is an i-pair.

In this case, $R f$ is also injective.

Let R be a left perfect ring. Then R has a primitive idempotent e with $S(R e) \neq 0$. If R is QF-3, then the primitive idempotent e with $S(R e) \neq 0$ are characterized as follows;

Lemma 2 ([4, Theorem 2.1]). Let R be a one-sided artinian QF-3 ring, and let $e \in \Pi(R)$. Then $R e R$ is injective if and only if $S(R e) \neq 0$.

We call $e \in \Pi(R)$ right (resp. left) S-primitive if $S(R e) \neq 0$ (resp. $e S(R e) \neq 0$), respectively.

The following statement, which Storrer [15, Proposition 4.8] showed, is helpful in this paper.

Lemma 3 ([15, Proposition 4.8]). Let R and Q = Q(R) be left perfect. Then

1. If e is a right S-primitive idempotent for R, then so is it for Q.
(2) if $e_1, e_2$ are right $S$-primitive idempotents for $R$, then $e_1 R \cong e_2 R$ if and only if $e_1 Q \cong e_2 Q$.
(3) If $e$ is a right $S$-primitive idempotent for $Q$, then there exists a right $S$-primitive idempotent $e' \in R$ such that $e Q \cong e' Q$.

A ring $R$ is called a left Harada ring if it is left artinian and its complete set $\pi(R)$ of orthogonal primitive idempotents is arranged as follows:

$$\pi(R) = \bigcup_{i=1}^{m} \{ e_{i,j} \}^{n(i)}_{j=1},$$

where
(a) each $e_{i,R}$ is an injective module for each $i = 1, 2, \ldots, m$.
(b) $e_{i,k-1} R \cong e_{i,k} R$, or $J(e_{i,k-1} R) \cong e_{i,k} R$ for each $i$ and each $k = 2, 3, \ldots, n(i)$.
(c) $e_{i,k} R \not\cong e_{i,j} R$ for $i \not= j$.

**Remark 1.** Let $R$ be a left Harada ring. Then $Q(R)$ is also a left Harada ring (See [6, Theorem 4]) and a complete set $\pi(Q)$ of orthogonal primitive idempotents for $Q$ coincides with $\pi(R)$ (See [6, p.248]).

Using Remark 1, Kado showed the following;

**Proposition 4 ([6, Proposition 2]).** Let $R$ be a left Harada ring, and let $(e R : R f)$ be an $i$-pair for $e, f \in \pi(R)$. Then $(e Q(R) : Q(R) f)$ is an $i$-pair.

Recall the following notation [6, p.249]. Let $\theta : f R \to e R$ be an $R$-monomorphism such that $\text{Im} \theta = J(e R)$, where $e, f \in \pi(R)$. Then by [15, Proposition 4.3], $\theta$ can be uniquely extended to a $Q, (R)$-homomorphism $\theta^* : f Q(R) \to e Q(R)$.

We shall need the following results.

**Lemma 5 ([6, Proposition 3]).** Let $R$ be a basic and left Harada ring, and $Q = Q(R)$ and $\theta$ as above. Then the following hold.

(1) if $e$ is not right $S$-primitive, then the extension $\theta^* : f Q \to e Q$ is an isomorphism.
(2) if $e$ is right $S$-primitive, then the extension $\theta^* : f Q \to e Q$ is a monomorphism such that $\text{Im} \theta^* = J(e Q)$.

**Remark 2** (cf. [15, Lemma 4.2]). Let $\{ g_i \} \cup \{ f_j \}$ be a complete set of orthogonal primitive idempotents for $R$, where the $g_i$ are right $S$-primitive and the $f_j$ are not right $S$-primitive. We denote $g_0$ by $g_0 = \sum g_i$. Then $Q(R)g_0 = Rg_0$ and $Q(R)g = Rg$ for every right $S$-primitive idempotent $g$ of $R$.

Let $R$ be a basic left artinian ring, and let $\{ e_1, e_2, \ldots, e_n \}$ be a complete set of orthogonal primitive idempotents for $R$ and let

$$S = \text{End}_R(\oplus_{i=1}^{n} E(Re_i/J(Re_i)))$$

be the endomorphism ring of a minimal injective cogenerator for $R$-mod. Let $f_i$ be the primitive idempotent for $S$ corresponding to the projection

$$\oplus_{i=1}^{n} E(Re_i/J(Re_i)) \to E(Re_i/J(Re_i)).$$

Then we call a ring isomorphism $\tau : R \to S$ a Nakayama isomorphism if $\tau(e_i) = f_i$ for each $i = 1, 2, \ldots, n$. By [3, p.42], the existence of a Nakayama isomorphism does not depend on the choice of the complete set $\{ e_1, e_2, \ldots, e_n \}$ of orthogonal primitive idempotents. (See [7, Remark on p.387].)
It is important whether the maximal quotient ring of a basic artinian ring is basic since a Nakayama isomorphism is defined on a basic ring. Here we shall study the case that the maximal quotient ring of a given left Harada ring is basic.

Theorem 6 (cf. [2, Corollary 22]). Let $R$ be a basic and left Harada ring and $Q = Q(R)$. Then $Q$ is a basic ring if and only if $R$ either is QF or satisfies the following; $n(i) = 1$ or 2 and $Re_{i1}$ is injective for any $i$. In this case $R = Q$.

Proof. Note that both $R$ and $Q$ are artinian QF-3. Let $\pi(R) = \bigcup_{i=1}^{m} \{e_{ij}\}_{j=1}^{n(i)}$ be a complete set of orthogonal primitive idempotent for $R$ satisfying the following conditions:

(a) $e_{i1}R_{R}$ is injective for each $i = 1, 2, \ldots, m$,
(b) $e_{ij+1}R_{R} \cong J(e_{i}R_{R})$ for $j = 1, 2, \ldots, n(i) - 1$.

We have a complete set $\{R_{g_{1}}, \ldots, R_{g_{m}}\}$ of pairwise non-isomorphic indecomposable injective projective left $R$-modules, such that the $(e_{i1}R : R_{g_{i}})$ are $i$-pair for each $i = 1, 2, \ldots, m$ since $R$ is basic and artinian QF-3.

Assume that $Q$ is basic. Let $e_{ik+1}, e_{ik} \in \{e_{ij}\}_{j=2}^{n(i)}$. Then we have an $R$-monomorphism $\theta_{ik} : e_{ik+1}R_{R} \to e_{ik}R_{R}$ such that $\text{Im} \theta = J(e_{ik}R_{R})$. If $e_{ik}$ is not right $S$-primitive, then $e_{ik}R_{Q} \cong e_{ik}Q_{R}$ by Lemma 5. This contradicts that $Q$ is basic. Hence $e_{ik}$ is right $S$-primitive for $k = 1, 2, \ldots, n(i) - 1$. Since the $Re_{ik}$ are injective for each $k = 1, 2, \ldots, n(i) - 1$ by Lemma 2, there exists some $R_{g}$ in $\{R_{g_{1}}, \ldots, R_{g_{m}}\}$ such that $Re_{ik} \cong R_{g}$. However $R$ is basic, so we see that $n(i) \leq 2$ and $e_{i1}$ is right $S$-primitive.

In case $n(i) = 1$ for every $i = 1, \ldots, m$, then $R$ is QF.

In case $n(i) = 2$ for some $i \in \{1, \ldots, m\}$. If $e_{in(i)}$ is right $S$-primitive, then $RRe_{in(i)}$ is injective by Lemma 2. Hence $e_{in(i)}$ is not right $S$-primitive since $RRe_{i1}$ is injective and so $\{R_{g_{1}}, \ldots, R_{g_{m}}\} = \{Re_{i1}, \ldots, Re_{m1}\}$.

Conversely, first, assume that $R$ is QF. Since $RRe$ is injective for any $e \in \pi(R)$, $e$ is right $S$-primitive by Lemma 2. Thus, $eQ \neq fQ$ for any $e, f \in \pi(R) = \pi(Q)$ by Lemma 3. Therefore $Q$ is basic. Next, assume that $R$ satisfies $n(i) = 1$ or 2 and $Re_{i1}$ is injective for any $i$. Then $e_{i1}$ is left $S$-primitive and so $eQ = eR$ by Remark 2. Hence $J(eQ) = J(eR)$. Therefore it is also clear to see that $R = Q$. \hfill \Box

Example 1. We shall give a basic left Harada ring $R$ with $J(R)^{5} = 0$, which is not QF. Let $R$ be an algebra over a field $K$ defined by the following quiver;

$$
\begin{array}{ccc}
\gamma & 1 & \gamma' \\
3 & \alpha & 4 \\
\beta & 2 & \beta'
\end{array}
$$

with the relations $\gamma\beta = \gamma'\beta'$, $\alpha\gamma\beta = 0$, and $\beta'\alpha\gamma = 0$.

The composition diagrams of the Loewy factors of the indecomposable projective modules of $R_{R}$ is the following.

$$
\begin{array}{cccc}
eR/eJ & 1 & 2 & 3 & 4 \\
eJ/eJ^{2} & 2 & 3 & 4 & 1 \\
eJ^{2}/eJ^{3} & 3 & 4 & 1 & 2 & 2 \\
eJ^{4} & 1 & 4 & 3
\end{array}
$$
Then $R$ is a left Harada ring which is not QF since $e_1 R_R$, $e_3 R_R$ and $e_4 R_R$ are injective and $e_2 R_R \cong J(e_1 R)$. Moreover $e_1, e_3, e_4$ are right $S$-primitive. Hence $e_1 Q(R) = e_1 R, e_3 Q(R) = e_3 R$ and $e_4 Q(R) = e_4 R$ are injective and $e_2 Q(R) \cong J(e_1 Q(R))$. Therefore $R = Q(R)$.

**Example 2.** We shall give a basic Harada ring $R$ with $J(R)^6 = 0$, but $Q(R)$ is not basic. Let $R$ be an algebra over a field $K$ defined by the following quiver:

![Quiver Diagram]

with the relations $0 = \beta \alpha \gamma \beta = \beta' \alpha \gamma' \beta' = \beta \alpha \gamma = \beta' \alpha \gamma'$, and $\gamma \beta = \gamma' \beta'$. Then the composition diagrams of the Loewy factors of the indecomposable projective modules of $R_R$ is the following.

$$
\begin{array}{cccccc}
e_{i_1} R / e_{i_1} J & 1 & 2 & 3 & 4 \\
e_{i_2} J / e_{i_2} J^2 & 2 & 3 & 4 & 1 & 1 \\
e_{i_3} J^2 / e_{i_3} J^3 & 3 & 4 & 1 & 2 & 2 \\
e_{i_4} J^4 / e_{i_4} J^5 & 1 & 2 & 4 & 3 \\
e_{i_5} J^5 & 2 &
\end{array}
$$

Then since $e_1 R_R$, $e_3 R_R$ and $e_4 R_R$ are injective and $e_2 R_R \cong J(e_1 R)$, $R$ is a left Harada ring which is not QF. Hence $e_2 Q(R) \cong e_1 Q(R)$ since $e_1$ is not right $S$-primitive. Therefore $Q(R)$ is not basic.

2. **NAKAYAMA ISOMORPHISM**

In this section, we study the Nakayama isomorphisms for the representative matrix ring of a basic left Harada ring and its maximal quotient ring. Let $R$ be a basic left Harada ring, and let $\pi(R) = \bigcup_{i=1}^{m} \{ e_{ij} \}_{j=1}^{n(i)}$ be a complete set of orthogonal primitive idempotents as in Theorem 6. Furthermore, let $R^*$ be the representative matrix ring of $R$. $R^*$ is represented as block matrices as follows:

$$R^* = \begin{pmatrix} R_{11}^* & \cdots & R_{1m}^* \\ \vdots & \ddots & \vdots \\ R_{m1}^* & \cdots & R_{mm}^* \end{pmatrix},$$

where $R_{ij}^* = P_{ij}$ for $j \neq \sigma(i)$ and $R_{i\sigma(i)}^* = P_{i\sigma(i)}^*$ (See [7, Section 4]).

Here, adding one row and one column to $R^*$, we make an extended matrix ring $W_i(R)$ of $R$ as follows:

$$
\begin{pmatrix}
R_{11} & \cdots & R_{1,i} & Y_1 & R_{1,i+1} & \cdots & R_{1,m} \\
\vdots & \ddots & \vdots & \vdots & \vdots \\
R_{i1} & \cdots & R_{i,i} & Y_i & R_{i,i+1} & \cdots & R_{im} \\
X_1 & \cdots & X_{i-1} & X_i & Q & X_{i+1} & \cdots & X_m \\
R_{i+1,1} & \cdots & R_{i+1,i} & Y_{i+1} & R_{i+1,i+1} & \cdots & R_{i+1,m} \\
\vdots & \ddots & \vdots & \vdots & \vdots \\
R_{m1} & \cdots & R_{mi} & Y_m & R_{m,i+1} & \cdots & R_{mm} 
\end{pmatrix},
$$

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where $X_k$ is the last row of $R^*_k$ ($k = 1, \ldots, m$, $k \neq i$), $Y_k$ is the last column of $R^*_k$ ($k = 1, \ldots, m$), $X_i = (P^*_{in(i),1} \cdots P^*_{in(i),in(i)-1} J(P^*_{in(i),in(i)}), and Q = P^*_{in(i),in(i)}$.

Then $W_i(R)$ naturally becomes a ring by operations of $R^*$. We call this the $i$-th extended ring of $R$.

**Proposition 7** ([7, Proposition 5.11]). If $W_i(R)$ has a Nakayama isomorphism, then $R$ also has a Nakayama isomorphism.

Let $R$ be a basic and left Harada ring, and let

$$\text{pi}(R) = \bigcup_{i=1}^{m} \{e_{ij}\}_{j=1}^{n(i)}$$

be a complete set of orthogonal primitive idempotents of $R$ satisfying the following;

1. $e_{ii} R R$ is injective for each $i = 1, 2, \ldots, m$.
2. $e_{ij} R \cong J(e_{ij-1} R)$ for each $j = 2, \ldots, n(i)$.

Then (See [7, p.388]), for any $e_{ij}$ in $\text{pi}(R)$, there exists some $g_i$ in $\text{pi}(R)$ with $Rg_i$ injective such that $E(Re_{ij}/J(Re_{ij})) \cong Rg_i/S_{j-1}(Rg_i)$, where $S_j(Rg_i)$ is the $j$-th socle of $Rg_i$. We denote the generator $g_i + S_{j-1}(Rg_i)$ of $Rg_i/S_{j-1}(Rg_i)$ by $g_{ij}$ for each $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n(i)$. Then by [7, Proposition 3.2], a minimal injective cogenerator $G = \otimes e_{ij} Rg_{ij}$ is finitely generated. Therefore we note that $R$ is left Morita dual to $\text{End}_R(G)$ by [1, Theorem 30.4]. We call this $\text{End}_R(G)$ the dual ring of $R$. We denote the dual ring of $R$ by $T(R)$.

For the proof of proposition 8 below, we denote

$$\begin{pmatrix} 0 & \cdots & 0 \\
0 & \cdots & R^*_{ij} & 0 & \cdots & 0 \\
0 & \cdots & 0 
\end{pmatrix} \subseteq R^*$$

by $[R^*_i]_j$ and

$$\begin{pmatrix} 0 & \cdots & 0 \\
0 & \cdots & R^*_{ij} & 0 & \cdots & 0 \\
0 & \cdots & 0 
\end{pmatrix} \subseteq W_i(R)$$

by $[R^*_i]_j$.

By using the result that Kado and Oshiro [7, Proposition 5.11] showed, we shall show the following proposition. The proposition is essential in this paper.

**Proposition 8.** $W_i(R)$ has a Nakayama isomorphism if and only if so does $R$.

**Proof.** ($\Rightarrow$). By Proposition 7 ([7, Proposition 5.11]). ($\Leftarrow$). As [7, Proposition 5.11], let $e_{ij}$ be the matrix of $R^*$ such that the $(ij, ij)$-component is the unity and other components are zero, and let $w_{ij}$ be the matrix of $W_i(R)$ such that the $(ij, ij)$-component is the unity and other components are zero. Note that the size of the columns in $W_i(R)$ is $n(i) + 1$. Let $\Psi$ be the natural embedding homomorphism;

$$\begin{pmatrix} R^*_{i1} & \cdots & R^*_{im} \\
\vdots \\
R^*_{m1} & \cdots & R^*_{mm} 
\end{pmatrix} \downarrow \Psi$$

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\[
\begin{pmatrix}
R_{i1}^* & \cdots & R_{ii}^* & 0 & R_{i,i+1}^* & \cdots & R_{im}^* \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
R_{i1}^* & \cdots & R_{ii}^* & 0 & R_{i,i+1}^* & \cdots & R_{im}^* \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
R_{i+1,1}^* & \cdots & R_{i+1,i}^* & 0 & R_{i+1,i+1}^* & \cdots & R_{i+1,m}^* \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
R_{m1}^* & \cdots & R_{mi}^* & 0 & R_{m,i+1}^* & \cdots & R_{mm}^* \\
\end{pmatrix}^{i+1}
\]

where $R_{ij}^* \rightarrow R_{ij}^*$ are identity maps for all $i, j$. Moreover let $h_{ij}$ be the matrix of $T(R)$ such that the $(ij, ij)$-component is the unity and other components are zero, and let $v_{ij}$ be the matrix of $W_i(T(R))$ such that the $(ij, ij)$-component is the unity and other components are zero. Note that the size of the columns in $W_i(T(R))$ is $n(i) + 1$. Let

\[
\begin{pmatrix}
T(R)_{11} & \cdots & T(R)_{1m} \\
\vdots & \ddots & \vdots \\
T(R)_{m1} & \cdots & T(R)_{mm}
\end{pmatrix}
\]

be the representative matrix ring $T(R)^*$ of $T(R)$, and let $T(W_i(R))$ be the dual ring of $W_i(R)$ as follows;

\[
\begin{pmatrix}
T(R)_{11} & \cdots & T(R)_{i1} & \dagger Y_1 & T(R)_{i,i+1} & \cdots & T(R)_{im} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
T(R)_{i1} & \cdots & T(R)_{ii} & \dagger Y_i & T(R)_{i,i+1} & \cdots & T(R)_{im} \\
\dagger X_1 & \cdots & \dagger X_i & \dagger Q & \dagger X_{i+1} & \cdots & \dagger X_m \\
T(R)_{i+1,1} & \cdots & T(R)_{i+1,i} & \dagger Y_{i+1} & T(R)_{i+1,i+1} & \cdots & T(R)_{i+1,m} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
T(R)_{m1} & \cdots & T(R)_{mi} & \dagger Y_m & T(R)_{m,i+1} & \cdots & T(R)_{mm}
\end{pmatrix}
\]

Letting $\Psi_{T(R)}$ be the natural embedding homomorphism;

\[
\begin{pmatrix}
T(R)_{11} & \cdots & T(R)_{1m} \\
\vdots & \ddots & \vdots \\
T(R)_{m1} & \cdots & T(R)_{mm}
\end{pmatrix}
\]

\[
\downarrow \Psi_{T(R)}
\]

\[
\begin{pmatrix}
T(R)_{11} & \cdots & T(R)_{i1} & 0 & T(R)_{i,i+1} & \cdots & T(R)_{im} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
T(R)_{i1} & \cdots & T(R)_{ii} & 0 & T(R)_{i,i+1} & \cdots & T(R)_{im} \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
T(R)_{i+1,1} & \cdots & T(R)_{i+1,i} & 0 & T(R)_{i+1,i+1} & \cdots & T(R)_{i+1,m} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
T(R)_{m1} & \cdots & T(R)_{mi} & 0 & T(R)_{m,i+1} & \cdots & T(R)_{mm}
\end{pmatrix}^{i+1}
\]

where $T(R)_{ij} \rightarrow T(R)_{ij}$ are identity maps for all $i, j$. We note that $T(W_i(R)) = W_i(T(R))$ (See [7, Proposition 5.11]).
Assume that $\varphi : R^* \to T(R)^*$ is a Nakayama isomorphism with $\varphi(e_{ij}) = h_{ij}$. (i.e., $\varphi([r_{kl}]) \in [T(R)_{kl}]$ for any $[r_{kl}] \in [R]_{kl}$, where $(k, l)$-componentwise of $R$ corresponds to $(k, l)$-componentwise of $T(R)$.) We consider the following diagram:

$$
\begin{array}{ccc}
W_i(R) & \rightarrow & W_i(T(R)) \\
\psi \uparrow & & \psi_{T(R)} \\
R^* & \xrightarrow{\varphi} & T(R).
\end{array}
$$

Here we define a map $\overline{\varphi} : W_i(R) \to W_i(T(R))$ as follows:

(a) $\overline{\varphi}([r_{kl}])^w = [\varphi([r_{kl}])]^w \in [T(R)_{kl}]^w$ for any $[r_{kl}]^w \in [R]_{kl}^w, 1 \leq k \leq m, 1 \leq l \leq m$;

(b) $\overline{\varphi}(x)^w \in [X_k]^w$ for any $x^w \in [X_k], k = 1, \ldots, m$;

(c) $\overline{\varphi}(y)^w \in [Y_i]^w$ for any $y^w \in [Y_i], l = 1, \ldots, m$;

(d) $\overline{\varphi}(q)^w \in [Q]^w$ for any $q^w \in [Q]^w$.

Since $\varphi(e_{ij}) = h_{ij}$, $\overline{\varphi}$ is well-defined. Moreover it is satisfied $\overline{\varphi}(w_{i,n(i)+1}) = f_{i,n(i)+1}$. Then we can easily check that $\overline{\varphi}$ is a Nakayama isomorphism. □

**Remark 3.** We shall define a special case of an extended ring for a given ring $R$. Let $\{e_1, e_2, \ldots, e_n\}$ be a complete set of orthogonal primitive idempotents for $R$. Then for primitive idempotent $e_i$ in $R$, we define $R_{e_i}$ as follows:

$$
\begin{pmatrix}
e_iRe_1 & \cdots & e_iRe_i & Y_1 & e_iRe_{i+1} & \cdots & e_iRe_n \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
e_iRe_1 & \cdots & e_iRe_i & Y_i & e_iRe_{i+1} & \cdots & e_iRe_n \\
X_1 & \cdots & X_i & U & X_{i+1} & \cdots & X_n \\
e_{i+1}Re_1 & \cdots & e_{i+1}Re_i & Y_{i+1} & e_{i+1}Re_{i+1} & \cdots & e_{i+1}Re_n \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
e_nRe_1 & \cdots & e_nRe_i & Y_n & e_nRe_{i+1} & \cdots & e_nRe_n
\end{pmatrix},
$$

where the $X_j$ are $e_iRe_i$ for $j = 1, \ldots, i-1, i+1, \ldots, n$, $X_i$ is $J(e_iRe_i)$, the $Y_k$ are $e_kRe_i$ for $k = 1, \ldots, n$ and $U$ is $e_iRe_i$. Then $R_{e_i}$ is a ring by usual matrix operations.

**Remark 4.** Proposition 8 says that a basic left Harada ring $R$ has a Nakayama isomorphism if and only if so does $R_e$ for $e \in \pi(R) = \bigcup_{i=1}^m \{e_{ij}\}_{j=1}^{n(i)}$.

We denote a basic ring of $Q(R)$ by $Q^b(R)$.

**Remark 5.** If $R$ is a one-sided artinian QF-3 ring, the number of right $S$-primitive idempotents for $R$ coincides with that of left $S$-primitive idempotents for $R$.

**Theorem 9.** Let $R$ be a basic and left Harada ring and let $Q = Q(R)$. Then $Q$ has a Nakayama isomorphism if and only if so does $R$.

**Proof.** If $Q$ is basic, then $R = Q$ by Theorem 6. Hence we may assume that $Q$ is not basic. Let $\pi(R) = \bigcup_{i=1}^m \{e_{ij}\}_{j=1}^{n(i)}$ be a complete set of primitive idempotents for $R$ as given in the proof of Theorem 6. Then if $\{e_{ij}\}_{j=1}^{n(i)}$ has no right $S$-primitive idempotents, then $e_iQ \cong e_jQ$ for $j = 2, \ldots, n(i)$ by Lemma 5. If $\{e_{ij}\}_{j=1}^{n(i)}$ has only one right $S$-primitive
idempotent, say $e_{ik}$, then

$$
\begin{align*}
& e_{i1}Q \cong e_{ij}Q & \text{for } j = 2, \ldots, k; \\
& e_{i,k+1}Q \cong J(e_{ik}Q) & \text{and} \\
& e_{i,k+1}Q \cong e_{ij}Q & \text{for } j = k + 2, \ldots, n(i).
\end{align*}
$$

Moreover, if $\{e_{ij}\}_{j=1}^{n(i)}$ has two right $S$-primitive idempotents, say, $e_{ik}, e_{it} \ (k < t)$, then

$$
\begin{align*}
& e_{i1}Q \cong e_{ij}Q & \text{for } j = 2, \ldots, k; \\
& e_{i,k+1}Q \cong J(e_{ik}Q) & \text{and} \\
& e_{i,k+1}Q \cong e_{ij}Q & \text{for } j = k + 2, \ldots, t; \\
& e_{i,t+1}Q \cong J(e_{it}Q) & \text{and} \\
& e_{i,t+1}Q \cong e_{ij}Q & \text{for } j = t + 2, \ldots, n(i).
\end{align*}
$$

Repeating the same argument and Remark 5, we have the following sequences for $i = 1, \ldots, m$;

$$
\begin{align*}
e_{i1}Q & > e_{i1}J(Q) \\
& \uparrow \\
e_{i,k1+1}Q & > J(e_{i,k1+1}Q) \\
& \uparrow \\
e_{i,k2+1}Q & \ldots,
\end{align*}
$$

where $e_{ik}$ is right $S$-primitive. Hence the complete set of the primitive idempotents $\pi(Q^b)$ for $Q^b$ is $\bigcup_{i=1}^{m} \{e_{i1}, e_{i,k1+1}\}_{i \geq 1} \subseteq \pi(R) = \pi(Q)$ and $e_{i1}Q^b$ is injective. Since $e_{i1}$ is left $S$-primitive, $e_{i1}R = e_{i1}Q$ by Remark 2 and so $e_{i1}Re_{i1} = e_{i1}Qe_{i1}$. Hence we have a ring isomorphism from $Q^b$ to a subring of $R$.

(i) We choose $\{e_{h1}\}_{h=1}^{n(h)} \subset \pi(R)$ with $e_{hm(h)}$ right $S$-primitive. We put $e_h = e_{h1} + \cdots + e_{hm(h)}$. Then by Lemma 3 and Lemma 5, we $e_hR = e_hQ$. (ii) We choose $\{e_{h1}\}_{h=1}^{n(h)} \subset \pi(R)$ without right $S$-primitive. By Remark 3, $Q^b_{e_{h1}}$ is isomorphism to a ring with the complete set $\bigcup_{i \neq h} \{e_{i1}, e_{i,k1+1}\}_{i \geq 1} \cup \{e_{h1}, e_{h2}\}$ of primitive idempotents. Similarly repeating $n(h) - 2$ times, we can make an extended ring with the complete set $\bigcup_{i \neq h} \{e_{i1}, e_{i,k1+1}\}_{i \geq 1} \cup \{e_{hj}\}_{j=1}^{n(h)}$ of primitive idempotents.
Letting,

\[ Q^b = \begin{pmatrix}
  * & e_{11}R_{e1} & * \\
  e_{h1}R_{e1} & \ldots & e_{h1}R_{e1} & \ldots & e_{h1}R_{e_m} & \ldots \\
  \vdots & & \vdots & & \vdots & & \vdots \\
  * & e_{m1}R_{e1} & * \\
  \vdots & & \vdots & & \vdots & & \vdots \\
  * & e_{11}R_{e1} & e_{11}R_{e1} & * \\
  e_{h1}R_{e1} & \ldots & e_{h1}R_{e1} & e_{h1}R_{e1} & \ldots & e_{h1}R_{e_m} & \ldots \\
  \vdots & & \vdots & & \vdots & & \vdots \\
  e_{h1}R_{e1} & \ldots & J(e_{h1}R_{e1}) & e_{h1}R_{e1} & \ldots & e_{h1}R_{e_m} & \ldots \\
  e_{m1}R_{e1} & e_{m1}R_{e1} & * \\
  \vdots & & \vdots & & \vdots & & \vdots \\
  * & e_{m1}R_{e1} & * \\
  \vdots & & \vdots & & \vdots & & \vdots \\
\end{pmatrix}, \]

\[ Q_{e_{h1}}^b = \begin{pmatrix}
  * & e_{11}R_{e1} & * \\
  e_{h1}R_{e1} & \ldots & e_{h1}R_{e1} & \ldots & e_{h1}R_{e_m} & \ldots \\
  \vdots & & \vdots & & \vdots & & \vdots \\
  * & e_{m1}R_{e1} & * \\
  \vdots & & \vdots & & \vdots & & \vdots \\
  * & e_{11}R_{e1} & e_{11}R_{e1} & * \\
  e_{h1}R_{e1} & \ldots & e_{h1}R_{e1} & e_{h1}R_{e1} & \ldots & e_{h1}R_{e_m} & \ldots \\
  \vdots & & \vdots & & \vdots & & \vdots \\
  e_{h1}R_{e1} & \ldots & J(e_{h1}R_{e1}) & e_{h1}R_{e1} & \ldots & e_{h1}R_{e_m} & \ldots \\
  e_{m1}R_{e1} & e_{m1}R_{e1} & * \\
  \vdots & & \vdots & & \vdots & & \vdots \\
  * & e_{m1}R_{e1} & * \\
  \vdots & & \vdots & & \vdots & & \vdots \\
\end{pmatrix}. \]

For two submodules

\[ A = h_1 > \begin{pmatrix}
  0 & \ldots & \ldots & \ldots & 0 \\
  e_{h1}R_{e1} & \ldots & e_{h1}R_{e1} & e_{h1}R_{e1} & \ldots & e_{h1}R_{e_m} & \ldots \\
  0 & \ldots & \ldots & 0 \\
  \end{pmatrix}, \]

\[ B = h_1 > \begin{pmatrix}
  0 & \ldots & \ldots & 0 \\
  e_{h1}R_{e1} & \ldots & J(e_{h1}R_{e1}) & e_{h1}R_{e1} & \ldots & e_{h1}R_{e_m} & \ldots \\
  0 & \ldots & \ldots & 0 \\
  \end{pmatrix}. \]

of \( Q^b \), \( J(A) \cong B \) by [13, Theorem 1].
Hence as a ring isomorphism,

\[
\begin{pmatrix}
* & e_{11}Re_{h_1} & e_{11}Re_{h_2} & * \\
\vdots & \vdots & \vdots & \vdots \\
e_{h_1}Re_{11} & \cdots & e_{h_1}Re_{h_1} & \cdots & e_{h_1}Re_{m_1} & \cdots \\
e_{h_2}Re_{11} & \cdots & e_{h_2}Re_{h_1} & \cdots & e_{h_2}Re_{m_1} & \cdots \\
* & e_{m_1}Re_{h_1} & e_{m_1}Re_{h_2} & * \\
\vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}
\]

\(\cong\)

\[
\begin{pmatrix}
* & e_{11}Re_{h_1} & e_{11}Re_{h_1} & * \\
\vdots & \vdots & \vdots & \vdots \\
e_{h_1}Re_{11} & \cdots & e_{h_1}Re_{h_1} & \cdots & e_{h_1}Re_{m_1} & \cdots \\
e_{h_1}Re_{11} & \cdots & J(e_{h_1}Re_{h_1}) & \cdots & e_{h_1}Re_{m_1} & \cdots \\
* & e_{m_1}Re_{h_1} & e_{m_1}Re_{h_1} & * \\
\vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}
\]

by [13, Theorem 1] again.

(iii) We choose \(\{e_{h_1}\}_{h=1}^{n(h)} \subset \pi(R)\) with some right \(S\)-primitive idempotents. Then we denote a right \(S\)-primitive idempotent of \(\{e_{h_1}\}_{h=1}^{n(h)}\) by \(e_{h_k}\). We reset

\[
\{e_{h_1}\}_{h=1}^{n(h)} = \{e_{h_1}, \ldots, e_{h_{k_1}}, \ldots, e_{h_{k_i}}\}.
\]

Then the complete set \(\pi(Q^b)\) of \(Q^b\) is \(\bigcup_{i=1}^{m} \{e_{i_{1}}, e_{i_{k_1+1}}\}_{i \geq 1}\). First by the same argument above for \(e_{i_1}, e_{i_{k_1+1}}\), we have a ring isomorphic to a ring with the complete set \(\{e_{i_{1}}, \ldots, e_{i_{k_1+1}}\} \subset \pi(R)\). Next, by [13, Theorem 1], repeating the same argument like (ii), for \(e_{i_{k_1+1}}, e_{i_{k_2+1}}\), we have a ring isomorphism to a ring with the complete set \(\{e_{i_{1}}, \ldots, e_{i_{k_1}}, e_{i_{k_1+1}}, \ldots, e_{i_{k_2}}, e_{i_{k_2+1}}\}\). Hence the suitable extended ring of \(Q^b\) is isomorphic to \(R\). Therefore, by Proposition 8, \(Q^b\) has a Nakayama isomorphism if and only if so does \(R\). \(\square\)

### 3. Another question

Oshiro’s result (Result A) in the introduction also poses another question whether there exist surjective ring homomorphisms:

\[
\begin{pmatrix}
Q(T_1) & \phi_1 & Q(T_2) & \phi_2 & \cdots & \phi_{n-1} & Q(T_n) & \phi_n & Q(R) \\
\vee & \vee & \vee & \vee & \vee & \vee & \vee & \vee & \vee \\
T_1 & \phi_1 & T_2 & \phi_2 & \cdots & \phi_{n-1} & T_n & \phi_n & R
\end{pmatrix}
\]

However K. Koikey informed the author the following examples;

**Example 3.** Let \(Q\) be a local serial ring, and \(J(Q) \neq 0, J(Q)^2 = 0\). Then \(J(Q) = S(Q)\). We put

\[
R = \begin{pmatrix} Q & Q \\ J & J \end{pmatrix} / \begin{pmatrix} 0 & J \\ 0 & J \end{pmatrix},
\]

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where $J = J(Q)$. Then $R$ is a serial ring of an admissible sequence $(3, 2)$ and so we see that $R = Q(R)$. Also

$$T_1 = \begin{pmatrix} Q & Q \\ J & Q \end{pmatrix}, \quad T_2 = \begin{pmatrix} Q & Q \\ J & Q \end{pmatrix} / \begin{pmatrix} 0 & J \\ 0 & 0 \end{pmatrix},$$

$$Q(T_1) = \begin{pmatrix} Q & Q \\ Q & Q \end{pmatrix}, \quad Q(T_2) = T_2.$$

$\begin{pmatrix} J & J \\ J & J \end{pmatrix}$ is a unique non-trivial ideal of $Q(T_1)$. Hence there does not exist a surjective ring homomorphism $Q(T_1)$ to $Q(T_2)$.

**Example 4.** We put

$$T = \begin{pmatrix} K & K & K \\ 0 & K & K \\ 0 & 0 & K \end{pmatrix}, \quad I = \begin{pmatrix} 0 & 0 & K \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $K$ is a field, and $R = T/I$. Then $R$ is a serial ring of an admissible sequence $(2, 2, 1)$ and we have a natural map

$$T = T_1 \rightarrow R.$$

However the maximal quotient ring $Q(T)$ of $T$ is the full matrix algebra with degree 3 over a field $K$ and $Q(R) = R$. Since $Q(T)$ is semisimple, there does not exist a surjective ring homomorphism $Q(T)$ to $Q(R)$.

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THE HOLONOMIC RANK FORMULA FOR $A$-HYPERGEOMETRIC SYSTEM

GO OKUYAMA

1. INTRODUCTION

Given a finite set $A$ of $d$-dimensional integral vectors which belong to one hyperplane off the origin in $\mathbb{Q}A$ and a parameter vector $\beta \in \mathbb{C}^d$, Gel'fand, Kapranov and Zelevinsky [5] defined a system of differential equations, called an $A$-hypergeometric system $M_A(\beta)$. They proved that the holonomic rank of $M_A(\beta)$ equals the normalized volume of the convex hull of $A$ and the origin (denote by $\text{vol}(A)$) for any $\beta$ when the semigroup ring $\mathbb{C}[NA]$ determined by $A$ is Cohen-Macaulay. In general, the rank is not less than the volume (see [1], [13], Theorem 3.5.1). Meanwhile Adolphson [1] showed that even when $\mathbb{C}[NA]$ is not Cohen-Macaulay, the holonomic rank equals $\text{vol}(A)$, as long as $\beta$ is generic in a certain sense. After Strumfels and Takayama showed that the holonomic rank can actually be greater than $\text{vol}(A)$ for non-generic parameters $\beta$, Cattani, D'Andrea and Dickenstein showed that if the convex hull of $A$ is a segment, then there exists a rank-jumping parameter whenever $\mathbb{C}[NA]$ is not a Cohen-Macaulay ring. Saito, who generalized this result by using different methods, showed that there exist rank-jumping parameters for any non-Cohen-Macaulay simplicial semigroup ring $\mathbb{C}[NA]$. Matushevich [6] showed that, if the toric ideal defined by $A$ is generic in a certain sense and non-Cohen-Macaulay, then there exists a rank-jumping parameter. However, when we fix a parameter $\beta$, it is not well-known how the holonomic rank is described explicitly except when the convex hull of $A$ is simplicial (see [10], Theorem 6.3). In this paper, using combinatorial notion, we provide a rank formula in the case where the rank of $A$ is three.

1.1. Definition of $A$-hypergeometric system. Let $A = (a_1, \ldots, a_n) = (a_{ij})$ be a $d \times n$-matrix of rank $d$ with coefficients in $\mathbb{Z}$. Let $k$ be a field of characteristic zero and $N$ the set of nonnegative integers. We denote the set $\{a_1, \ldots, a_n\}$ by $A$ as well. Let $\mathcal{F}_A$ denote the face lattice of the cone

$$\mathbb{Q}_{\geq 0}A := \left\{ \sum_{j=1}^{n} c_j a_j \mid c_j \in \mathbb{Q}_{\geq 0} \right\}.$$  

Let $NA$ denote the semigroup generated by $A$ and by $k[NA]$ its semigroup ring contained in the Laurent polynomial ring $k[t_1^{\pm}, \ldots, t_d^{\pm}]$. For a face $\sigma$ in $\mathcal{F}_A$, we denote by $N(A \cap \sigma)$ the semigroup generated by $A \cap \sigma$, and by $\mathbb{Z}(A \cap \sigma)$ the group generated by

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1The detailed version of this paper has been submitted for publication elsewhere.
$A \cap \sigma$. When $A \cap \sigma = \emptyset$, we agree that $N(A \cap \sigma) = \mathbb{Z}(A \cap \sigma) = 0$. We consider the $k$-algebra homomorphism $\phi_A : k[\partial_1, \ldots, \partial_n] \to k[\mathbb{N}A]$ defined by

$$\phi_A \left( \sum_{u \in \mathbb{N}^n} c_u \partial^u \right) := \sum_{u \in \mathbb{N}^n} c_u t^{Au},$$

where $c_u \in k$, $\partial^u := \partial_1^{u_1} \cdots \partial_n^{u_n}$, and $t^{Au} := t_1^{Au_1} \cdots t_d^{Au_d}$. We denote by $I_A(\partial)$ the kernel of $\phi_A$ and call it the toric ideal of $A$. Since $\phi_A$ is an epimorphism, we have

$$k[\partial]/I_A(\partial) \cong k[\mathbb{N}A].$$

Given a column parameter vector $\beta = (\beta_1, \ldots, \beta_d) \in k^d$, let $H_A(\beta)$ denote the left ideal of the $n$-th Weyl algebra

$$D = k(x_1, \ldots, x_n, \partial_1, \ldots, \partial_n)$$

generated by $I_A(\partial)$ and $\sum_{j=1}^n a_{ij} \theta_j - \beta_i$ $(i = 1, \ldots, d)$, where $\theta_j := x_j \partial_j$. We call the quotient $D$-module $M_A(\beta) := D/H_A(\beta)$ the $A$-hypergeometric system with parameter $\beta$. This system was introduced in the late eighties by Gel’fand, Graev, and Zelevinski (see [4]); its systematic study was started by Gel’fand, Zelevinski, and Kapranov (see, e.g. [5]).

1.2. Known results on the holonomic rank of $M_A(\beta)$. In this note, we define the holonomic rank of the $A$-hypergeometric system $\text{rank}(M_A(\beta))$ as follows:

$$\text{rank}(M_A(\beta)) := \dim_{k(\beta)}(k(x) \otimes_{k[\beta]} M_A(\beta)).$$

Here $k(x) = k(x_1, \ldots, x_n)$ is the field of rational functions. One of the results shown in [5] about the holonomic rank of $A$-hypergeometric system is that $\text{rank}(M_A(\beta)) = \text{vol}(A)$ for any $\beta \in k^d$ when the semigroup ring $k[\mathbb{N}A]$ is a Cohen-Macaulay ring. Here $\text{vol}(A)$ means the normalized volume of the convex hull in $\mathbb{Q}^d$ of $A$ and the origin. This equality can fail if $k[\mathbb{N}A]$ is not a Cohen-Macaulay ring. However, even if we drop the assumption that $k[\mathbb{N}A]$ is a Cohen-Macaulay ring, we have

$$\text{rank}(M_A(\beta)) \geq \text{vol}(A)$$

for any $\beta \in k^d$, and the equality holds for generic $\beta$. So we write $j_A(\beta)$ for the gap between the holonomic rank and the volume in this talk.

Moreover, in fact, Matussevich, Miller and Walther [7] completely showed that the rank of $M_A(\beta)$ is independent of $\beta$, that is, $j_A(\beta) = 0$ for any $\beta$ if and only if $\mathbb{C}[\mathbb{N}A]$ is Cohen-Macaulay. However, given a parameter $\beta$, we do not know the formula of the rank of $M_A(\beta)$ except when the convex hull of $A$ is simplicial.

2. Main Result

2.1. Combinatorial term $F_A(\beta)$. As in the previous section, in order to compute the gap $j_A(\beta)$, we introduce a combinatorial term as follows. First, for $\lambda \in \mathbb{Z}A$ and $\beta$, we define the subset $\mathcal{J}(\lambda; \beta)$ of $\mathcal{F}_A$ by

$$\mathcal{J}(\lambda; \beta) := \{ \sigma \in \mathcal{F}_A : \lambda \not\in \mathbb{N}A + \mathbb{Z}(A \cap \sigma), \beta - \lambda \in k(A \cap \sigma) \}.$$ 

Second, we define a preorder on $\mathbb{Z}A$ as follows:

$$\lambda < \mu \iff \exists \sigma \in \mathcal{J}(\lambda; \beta), \lambda + \mathbb{Z}(A \cap \sigma) = \mu + \mathbb{Z}(A \cap \sigma).$$
Then we have the following proposition on this set.

**Proposition 2.1.**

1. The set $\mathcal{J}(\lambda; \beta)$ does not contain $\mathbb{Q}_{\geq 0} A$.
2. If $\lambda \in \mathbb{N} A$, then we have $\mathcal{J}(\lambda; \beta) = 0$.
3. If $\lambda < \mu$, then we have $\mathcal{J}(\lambda; \beta) \subset \mathcal{J}(\mu; \beta)$.

Now, we consider the subset of $\mathbb{Z} A \setminus \mathbb{N} A$:

$$E_A(\beta) = \{ \lambda \in \mathbb{Z} A \setminus \mathbb{N} A | \mathcal{J}(\lambda; \beta) \neq \emptyset \}.$$

We denote by $F_A(\beta)$ the inductive limit of the set $(E_A(\beta), <)$ which we regard as an inductive system. In other words, $F_A(\beta)$ coincides with the set of maximal elements in $((\mathbb{Z} A \setminus \mathbb{N} A)/ \sim, <)$, where $\sim$ means the equivalence relation defined by

$$\lambda \sim \mu \overset{\text{def}}{\iff} \lambda < \mu \text{ and } \lambda > \mu.$$

Since $\lambda \in \beta + \bigcup_{\tau \in FA} k(A \cap \tau)$ for any $\lambda \in F_A(\beta)$ and $[\mathbb{Z} A \cap Q \tau : \mathbb{Z} (A \cap \tau)] < \infty$ for any face $\tau$, we see that $F_A(\beta)$ is a finite set.

2.2. Main result. Let $d = 3$ to the end of this note. First assume that the cardinality of $F_A(\beta)$ is one. Let $F_A(\beta) = \{ \lambda \}$ and $\mathcal{J}(\lambda; \beta)$ denote the set of maximal elements in $\mathcal{J}(\lambda; \beta)$. Then the sets $\mathcal{J}(\lambda; \beta)$ can be classified into four cases:

1. $\mathcal{J}(\lambda; \beta) = 0$,
2. $\mathcal{J}(\lambda; \beta)$ consists of one proper face $\sigma$,
3. $\mathcal{J}(\lambda; \beta)$ consists of all facets,
4. none of the above.

For each case, we have the following theorem:

**Theorem 2.2.** Let $d = 3$. Assume that the cardinality of $F_A(\beta)$ is one. Then we have the following.

1. $\mathcal{J}(\lambda; \beta)$ satisfies the case (1) $\Rightarrow j_A(\beta) = 0$,
2. $\mathcal{J}(\lambda; \beta)$ satisfies the case (2) $\Rightarrow$

$$j_A(\beta) = \begin{cases} 0 & \text{if } \sigma \text{ is a facet,} \\ \text{vol}(A \cap \sigma) & \text{if } \sigma \text{ is an edge,} \\ 2 & \text{if } \sigma = \{0\}, \end{cases}$$

3. $\mathcal{J}(\lambda; \beta)$ satisfies the case (3) $\Rightarrow j_A(\beta) = 0$,
4. $\mathcal{J}(\lambda; \beta)$ satisfies the case (4) $\Rightarrow j_A(\beta) = \sum_{\sigma \in \mathcal{J}(\lambda; \beta)_{\text{edges}}} (\text{vol}(A \cap \sigma)) + m - 1$.

Here $m$ means the number of connected components of the finite graph $G_\lambda = \{ \sigma \in FA | \{0\} \neq \sigma \subset \tau \text{ for some } \tau \in \mathcal{J}(\lambda; \beta) \}$ with respect to the inclusion relation.

Second not assume that the cardinality of $F_A(\beta)$ is one. In this case, it suffices that for each $\lambda \in F_A(\beta)$ we compute the number determined by the previous theorem, that is, we compute the right hand side of the equality in the theorem, regarding $F_A(\beta)$ as the singleton set $\{ \lambda \}$. For each $\lambda \in F_A(\beta)$, let $l_\lambda$ denote the number in this meaning.

Then we have the rank formula as desired:

**Theorem 2.3.** Let $d = 3$. Then we have $j_A(\beta) = \sum_{\lambda \in F_A(\beta)} l_\lambda$. 

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3. EXAMPLES

Example 1. Let \( A_1 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 5 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \). Then we have \( \text{vol}(A_1) = 7 \).

First we consider the case where \( \beta = (1, 2, 0) \). Then we have \( F_{A_1}(\beta) = \{ \beta \} \) and \( \tilde{J}(\beta; \beta) = \{ Q_{\geq 0}a_1, Q_{\geq 0}a_4 \} \). Hence we have \( j_{A_1}(\beta) = 1 \).

Second we consider the case where \( \beta = (2/5, 1, 0) \). Then we have \( F_{A_1}(\beta) = \{ (1, 1, 0), (1, 4, 0) \} \) and \( \tilde{J}(\{ (1, 1, 0); \beta \} = \{ Q_{\geq 0}a_1 \} \) and \( \tilde{J}(\{ (1, 4, 0); \beta \} = \{ Q_{\geq 0}a_4 \} \). Hence \( F_{A_1}(\beta) \) is semisimple. Since \( Q_{\geq 0}a_1 \) and \( Q_{\geq 0}a_4 \) are both edges, we have \( j_{A_1}(\beta) = 1 + 1 = 2 \).

\( \text{Figure 1. The set } A_1 \)

Example 2. Let \( A_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 2 & 2 \end{pmatrix} \). Then we have \( \text{vol}(A_2) = 6 \).

Let \( \beta = (1, 1, 1) \). Then we have \( F_{A_2}(\beta) = \{ \beta \} \) and \( \tilde{J}(\beta; \beta) = \{ \{ 0 \} \} \). Hence we have \( j_{A_2}(\beta) = 2 \).

\( \text{Figure 2. The set } A_2 \)

Example 3. Let \( A_3 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \). Then we have \( \text{vol}(A_3) = 6 \).

Let \( \beta = (0, 1, 0) \). Then we have \( F_{A_3}(\beta) = \{ \beta \} \) and \( \tilde{J}(\beta; \beta) = \{ Q_{\geq 0}a_1 + Q_{\geq 0}a_4, Q_{\geq 0}a_3 + Q_{\geq 0}a_6 \} \). Hence we have \( j_{A_3}(\beta) = 1 \).

\( \text{Figure 3. The set } A_3 \)
Example 4. Let $A_4 = \begin{pmatrix} 3 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$; not homogeneous. Then we have $\text{vol}(A_4) = 12$.

First we consider the case where $\beta = (1, 1, 0)$. Then we have $F_{A_4}(\beta) = \{\beta\}$ and $\bar{J}(\beta; \beta) = \{\mathbb{Q}_{2^0}a_4\}$. Hence we have $j_{A_4}(\beta) = \text{vol}(A_4 \cap \mathbb{Q}_{2^0}a_4) = 3$.

Second we consider the case where $\beta = (2, 2, 0)$. Then $F_{A_4}(\beta) = \{\beta\}$ and $\bar{J}(\beta; \beta) = \{\{0\}\}$. Hence we have $j_{A_4}(\beta) = 2$.

![Figure 4. The set $A_4$](image)

References


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ON THE DECOMPOSABILITY OF A SYZYGY OF THE RESIDUE FIELD

Ryo Takahashi

1. INTRODUCTION

Throughout the present paper, we assume that all rings are commutative noetherian local rings and all modules are finitely generated modules.

Dutta [10] proved the following theorem in his research into the homological conjectures:

**Theorem 1.1** (Dutta). Let \((R, m, k)\) be a local ring. Suppose that the \(n\)th syzygy module of \(k\) has a non-zero direct summand of finite projective dimension for some \(n \geq 0\). Then \(R\) is regular.

Since \(G\)-dimension is similar to projective dimension, this theorem naturally leads us to the following question:

**Question 1.2.** Let \((R, m, k)\) be a local ring. Suppose that the \(n\)th syzygy module of \(k\) has a non-zero direct summand of finite \(G\)-dimension for some \(n \geq 0\). Then is \(R\) Gorenstein?

It is obviously seen from the indecomposability of \(k\) that this question is true if \(n = 0\). Hence this question is worth considering just in the case where \(n \geq 1\).

We are able to answer in this paper that the above question is true if \(n \leq 2\). Furthermore, we can even determine the structure of a ring satisfying the assumption of the above question for \(n = 1, 2\).

2. MAIN RESULTS

For a local ring \(R\), we denote by \(\text{mod} R\) the category of finitely generated \(R\)-modules. First of all, we recall the definition of \(G\)-dimension.

**Definition 2.1.** (1) We denote by \(\mathcal{G}(R)\) the full subcategory of \(\text{mod} R\) consisting of all \(R\)-modules \(M\) satisfying the following three conditions:

- (i) \(M\) is reflexive,
- (ii) \(\text{Ext}^i_M(M, R) = 0\) for every \(i > 0\),
- (iii) \(\text{Ext}^i_R(M^*, R) = 0\) for every \(i > 0\).

The final version of this paper has been submitted for publication elsewhere.

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(2) Let $M$ be an $R$-module. If $n$ is a non-negative integer such that there is an exact sequence

$$0 \to G_n \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$$

of $R$-modules with $G_i \in \mathcal{G}(R)$ for every $i$, then we say that $M$ has $G$-dimension at most $n$, and write $\text{G-dim}_R M \leq n$. If such an integer $n$ does not exist, then we say that $M$ has infinite $G$-dimension, and write $\text{G-dim}_R M = \infty$.

For properties of $G$-dimension, we refer to [3] or [9].

**Proposition 2.2.** Let $(R, m, k)$ be a local ring. Suppose that there is a direct sum decomposition $m = I \oplus J$ where $I, J$ are non-zero ideals of $R$. Let $M$ be a non-free indecomposable module in $\mathcal{G}(R)$. Then there exist elements $x, y \in m$ such that

1. $I = (x)$ and $J = (y)$,
2. $(0 : x) = (y)$ and $(0 : y) = (x)$,
3. $M$ is isomorphic to either $(x)$ or $(y)$.

Hence the minimal free resolution of $k$ is as follows:

$$\cdots \xrightarrow{(y \ 0) \ \ x} R^2 \xrightarrow{(x \ 0) \ \ y} R^2 \xrightarrow{(0 \ \ 0) \ \ z} R \xrightarrow{(x \ y)} R \xrightarrow{k} 0.$$

**Proof.** The modules $M^*$ and $\Omega M$ are also non-free indecomposable modules in $\mathcal{G}(R)$. There are isomorphisms

$$M^* \cong \text{Hom}_R(M, m) = \text{Hom}_R(M, I \oplus J) \cong \text{Hom}_R(M, I) \oplus \text{Hom}_R(M, J).$$

The indecomposability of $M^*$ implies that either $\text{Hom}_R(M, I) = 0$ or $\text{Hom}_R(M, J) = 0$. We may assume that

(2.2.1) \hspace{1cm} $\text{Hom}_R(M, J) = 0.$

There is an exact sequence

(2.2.2) \hspace{1cm} $0 \to \Omega M \to R^n \to M \to 0.$

Dualizing this by $J$, we obtain another exact sequence

$$\text{Hom}_R(M, J) \to J^n \to \text{Hom}_R(\Omega M, J).$$

We have $\text{Hom}_R(\Omega M, J) \neq 0$ by (2.2.1). Applying the above argument to the module $\Omega M$ yields

(2.2.3) \hspace{1cm} $\text{Hom}_R(\Omega M, I) = 0.$

Also, dualizing (2.2.2) by $I$, we get an exact sequence

$$0 \to \text{Hom}_R(M, I) \to I^n \to \text{Hom}_R(\Omega M, I),$$

and hence $M^* \cong \text{Hom}_R(M, I) \cong I^n$. The indecomposability of $M^*$ implies that $n = 1$ (i.e. $M$ is cyclic), and $M^* \cong I$.

We also have

$$M \cong M^{**} \cong \text{Hom}_R(M^*, m) \cong \text{Hom}_R(M^*, I) \oplus \text{Hom}_R(M^*, J).$$
Note that $\text{Hom}_R(M^*, I)$ is isomorphic to $\text{Hom}_R(I, I)$, which contains the identity map of $I$. Hence $\text{Hom}_R(M^*, I) \neq 0$ and therefore

$$\text{Hom}_R(M^*, J) = 0.$$  

Applying the above argument to the module $M^*$, we see that $M^*$ is also cyclic and $M \cong M^{**} \cong I$. Thus, we have shown that $M \cong M^* \cong I$ and these modules are cyclic. Noting (2.2.3) and applying the above argument to the module $\Omega M$, we see that $\Omega M \cong (\Omega M)^* \cong J$ and these modules are cyclic.

Now, writing $I = (x)$ and $J = (y)$, we can prove $(x) = (0 : y)$ and $(0 : x) = (y)$. Thus we obtain the minimal free resolutions of $(x)$ and $(y)$:

\[
\begin{aligned}
\cdots & \to R \xrightarrow{\delta} R \xrightarrow{\nu} R \to (x) \to 0, \\
\cdots & \to R \xrightarrow{\delta} R \xrightarrow{\nu} R \to (y) \to 0.
\end{aligned}
\]

Taking the direct sum of these exact sequences, we get

\[
\cdots \xrightarrow{\delta} R^2 \xrightarrow{\delta} R^2 \xrightarrow{\delta} R^2 \to m \to 0.
\]

Joining this to the natural exact sequence $0 \to m \to R \to k \to 0$ constructs the minimal free resolution of $k$ in the assertion.

We denote by edim $R$ the embedding dimension of a local ring $R$. When a homomorphic image of a regular local ring is given, we can choose a minimal presentation of the ring in the following sense:

**Proposition 2.3.** Let $R$ be a homomorphic image of a regular local ring. Then there exist a regular local ring $(S, n)$ and an ideal $I$ of $S$ contained in $n^2$ such that $R \cong S/I$.

Here we introduce a famous result due to Tate [17, Theorem 6]. See also [5, Remarks 8.1.1(3)].

**Lemma 2.4 (Tate).** Let $(S, n, k)$ be a regular local ring, $I$ an ideal of $S$ contained in $n^2$, and $R = S/I$ a residue class ring. Suppose that the complexity of $k$ over $R$ is at most one. (In other words, the set of all the Betti numbers of the $R$-module $k$ is bounded.) Then $I$ is a principal ideal.

We denote by $\beta^R(M)$ the $i$th Betti number of a module $M$ over a local ring $R$. Handling the above results, we can determine the structure of a local ring with decomposable maximal ideal having a non-free module of G-dimension zero, as follows:

**Theorem 2.5.** Let $(S, n, k)$ be a regular local ring, $I$ an ideal of $S$ contained in $n^2$, and $R = S/I$ a residue class ring. Suppose that there exists a non-free $R$-module in $G(R)$.

Then the following conditions are equivalent:

1. The maximal ideal of $R$ is decomposable;
2. $\dim S = 2$ and $I = (xy)$ for some regular system of parameter $x, y$ of $S$.

**Proof.** Let $m = n/I$ be the maximal ideal of $R$.

(2) $\Rightarrow$ (1): It is easy to see that $m = xR \oplus yR$ and that $xR, yR$ are non-zero.

(1) $\Rightarrow$ (2): First of all, note from the condition (1) that $R$ is not an integral domain, hence is not a regular local ring.

Proposition 2.2 says that $m = xR \oplus yR$ for some $x, y \in n$, and that $\beta^R(k) = 2$ for every $i \geq 2$. It follows from Lemma 2.4 that $I$ is a principal ideal. Hence $R$ is a hypersurface.
We write $I = (f)$ for some $f \in \mathfrak{n}^2$. Since $m$ is decomposable, the local ring $R$ is not artinian. (Over an artinian Gorenstein local ring, the intersection of non-zero ideals is also non-zero; cf. [8, Exercise 3.2.15].) Hence we have $0 < \text{dim} R < \text{edim} R = 2$, which says that $\text{dim} R = 1$ and $\text{dim} S = 2$.

Note that $n = (x, y, f)$. Because $\text{edim} S = \text{dim} S = 2$, one of the elements $x, y, f$ belongs to the ideal generated by the other two elements. Noting that the images of elements $x, y$ in $m$ form a minimal system of generators of $m$, we see that $f \in (x, y)$, and hence $x, y$ is a regular system of parameters of $S$.

On the other hand, noting $xR \cap yR = 0$, we get $xy \in I = (f)$. Write $xy = cf$ for some $c \in S$. Since the associated graded ring $\text{gr}_n(S)$ is a polynomial ring over $k$ in two variables $\bar{x}, \bar{y} \in n/n^2$, we especially have $\bar{x}\bar{y} \neq 0$ in $n^2/n^3$, namely, $xy \not\in n^3$. It follows that $c \not\in n$ because $f \in n^2$. Therefore the element $c$ is a unit of $S$, and thus $I = (xy)$.

Using Theorem 2.5 and Cohen's structure theorem, we obtain the following corollary.

**Corollary 2.6.** Let $(R, m)$ be a complete local ring. The following conditions are equivalent:

1. There is a non-free module in $\mathcal{G}(R)$, and $m$ is decomposable;
2. $R$ is Gorenstein, and $m$ is decomposable;
3. There are a complete regular local ring $S$ of dimension two and a regular system of parameters $x, y$ of $S$ such that $R \cong S/(xy)$.

The finiteness of $G$-dimension is independent of completion. Thus, Corollary 2.6 not only gives birth to a generalization of [15, Proposition 2.3] but also guarantees that Question 1.2 is true if $n = 1$.

As far as here, we have observed a local ring whose maximal ideal is decomposable. From here to the end of this paper, we will observe a local ring such that the second syzygy module of the residue class field is decomposable. We begin with the following theorem, which implies that Question 1.2 is true if $n = 2$.

**Theorem 2.7.** Let $(R, m, k)$ be a local ring. Suppose that $m$ is indecomposable and that $\Omega^2_R k$ has a non-zero proper direct summand of finite $G$-dimension. Then $R$ is a Gorenstein ring of dimension two.

**Proof.** Replacing $R$ with its $m$-adic completion, we may assume that $R$ is a complete local ring. In particular, note that $R$ is Henselian.

We have $\Omega^2_R k = M \oplus N$ for some non-zero $R$-modules $M$ and $N$ with $G\text{-dim}_R M < \infty$. There is an exact sequence

$$0 \rightarrow M \oplus N \xrightarrow{(f, g)} R^e \rightarrow m \rightarrow 0$$

of $R$-modules, where $e = \text{edim} R$. Setting $A = \text{Coker} f$ and $B = \text{Coker} g$, we get exact sequences

(2.7.1) \[
\begin{cases}
0 \rightarrow M \xrightarrow{f} R^e \xrightarrow{\alpha} A \rightarrow 0,
0 \rightarrow N \xrightarrow{g} R^e \xrightarrow{\beta} B \rightarrow 0.
\end{cases}
\]

It is easily observed that there are exact sequences

(2.7.2) \[
0 \rightarrow R^e \xrightarrow{(f)} A \oplus B \rightarrow m \rightarrow 0
\]
and

\[
\begin{align*}
0 \to M \xrightarrow{\rho_l} B \to m \to 0, \\
0 \to N \xrightarrow{\alpha} A \to m \to 0.
\end{align*}
\]  

(2.7.3)

Using (2.7.1), (2.7.2) and (2.7.3), we can prove that \(\text{Ext}_R^2(k, R) \neq 0\). (Hence depth \(R \leq 2\).)

Fix a non-free indecomposable module \(X \in \mathcal{G}(R)\). Applying the functor \(\text{Hom}_R(X, -)\) to (2.7.2) gives an exact sequence

\[
0 \to (X^*)^c \to \text{Hom}_R(X, A) \oplus \text{Hom}_R(X, B) \to \text{Hom}_R(X, m) \to 0
\]

and an isomorphism

\[(2.7.4) \quad \text{Ext}_R^2(X, A) \oplus \text{Ext}_R^1(X, B) \cong \text{Ext}_R^1(X, m).\]

We have \((X^*)^c \in \mathcal{G}(R)\) and \(\text{Hom}_R(X, m) \in \mathcal{G}(R)\), hence

\[
\text{Hom}_R(X, A) \in \mathcal{G}(R).
\]

Take the first syzygy module of \(X\); we have an exact sequence

\[
0 \to \Omega X \to R^n \to X \to 0.
\]

Dualizing this sequence by \(A\), we obtain an exact sequence

\[
0 \to \text{Hom}_R(X, A) \to A^n \to \text{Hom}_R(\Omega X, A) \to \text{Ext}_R^1(X, A) \to 0.
\]

Divide this into two short exact sequences

\[
(2.7.5) \quad \begin{align*}
0 \to \text{Hom}_R(X, A) \to A^n & \to C \to 0, \\
0 \to C \to \text{Hom}_R(\Omega X, A) & \to \text{Ext}_R^1(X, A) \to 0
\end{align*}
\]

of \(R\)-modules. Since \(\Omega X\) is also a non-free indecomposable module in \(\mathcal{G}(R)\), applying the above argument to \(\Omega X\) instead of \(X\) shows that the module \(\text{Hom}_R(\Omega X, A)\) also belongs to \(\mathcal{G}(R)\). We have \(G\text{-dim}_R(A^n) < \infty\) by the first sequence in (2.7.1). Hence it follows from (2.7.5) that \(G\text{-dim}_R C < \infty\), and

\[(2.7.6) \quad G\text{-dim}_R(\text{Ext}_R^1(X, A)) < \infty.\]

On the other hand, applying the functor \(\text{Hom}_R(X, -)\) to the natural exact sequence

\[
0 \to m \to R \to k \to 0,
\]

we get an exact sequence

\[
0 \to \text{Hom}_R(X, m) \to X^* \to \text{Hom}_R(X, k) \to \text{Ext}_R^1(X, m) \to 0.
\]

There is an isomorphism \(\text{Hom}_R(X, k) \cong \text{Ext}_R^1(X, m)\), hence \(\text{Ext}_R^1(X, m)\) is a \(k\)-vector space. Since \(\text{Ext}_R^1(X, A)\) is contained in \(\text{Ext}_R^1(X, m)\) by (2.7.4),

\[(2.7.7) \quad \text{Ext}_R^1(X, A) \text{ is a } k\text{-vector space.}\]

Using (2.7.6) and (2.7.7), we can prove that the local ring \(R\) is Gorenstein.

Since the only number \(i\) such that \(\text{Ext}_R^i(k, R) \neq 0\) is the Krull dimension of \(R\) if \(R\) is Gorenstein, it follows from the above two claims that \(R\) is a Gorenstein local ring of dimension two, which completes the proof of the theorem. \(\square\)
The above theorem interests us in the observation of a Gorenstein local ring of dimension two such that the second syzygy module of the residue class field is decomposable. We introduce here a related result due to Yoshino and Kawamoto.

A homomorphic image of a convergent power series ring over a field \( k \) is called an analytic ring over \( k \). Any complete local ring containing a field is an analytic ring over its coefficient field, and it is known that any analytic local ring is Henselian; see [14, Chapter VII]. Yoshino and Kawamoto observed the decomposability of the fundamental module of an analytic normal domain.

**Theorem 2.8 (Yoshino-Kawamoto).** Let \( R \) be an analytic normal local domain of dimension two. Suppose that the residue class field of \( R \) is algebraically closed and has characteristic zero. Then the following conditions are equivalent:

1. The fundamental module of \( R \) is decomposable;
2. \( R \) is an invariant subring of a regular local ring by a cyclic group. (In other words, \( R \) is a cyclic quotient singularity.)

For the details of this theorem, see [21, Theorem (2.1)] or [19, Theorem (11.12)].

With the notation of the above theorem, suppose in addition that \( R \) is a complete Gorenstein ring such that \( \Omega^2_R k \) is decomposable. Then \( R \) satisfies the condition (1) in the above theorem. Hence the proof of the above theorem shows that \( R \) is of finite Cohen-Macaulay representation type (i.e. there exist only finitely many non-isomorphic maximal Cohen-Macaulay \( R \)-modules); see [21] or [19]. Therefore it follows from a theorem of Herzog [12] that \( R \) is a hypersurface. Thus the local ring \( R \) is a rational double point of type \((A_n)\) for some \( n \geq 1 \) by [21, Proposition (4.1)], namely, \( R \) is isomorphic to

\[
k[[X,Y,Z]]/(XY - Z^{n+1}).
\]

From a more general viewpoint, we can give a characterization as follows:

**Theorem 2.9.** Let \((S, n, k)\) be a regular local ring, \( I \) an ideal of \( S \) contained in \( n^2 \), and \( R = S/I \) a residue class ring. Suppose that \( R \) is a Henselian Gorenstein ring of dimension two. Then the following conditions are equivalent:

1. \( \Omega^2_R k \) is decomposable;
2. \( \dim S = 3 \) and \( I = (xy - zf) \) for some regular system of parameters \( x, y, z \) of \( S \) and \( f \in n \).

It is necessary to prepare three elementary lemmas to prove this theorem. The first one is both well-known and easy to check, and we omit the proof.

**Lemma 2.10.** Let \((S, n, k)\) be a regular local ring of dimension three and \( R = S/(f) \) a hypersurface with \( f \in n^2 \). Then \( f = x f_x + y f_y + z f_z \) for some \( f_x, f_y, f_z \in n \), and the minimal free resolution of \( k \) over \( R \) is as follows:

\[
\cdots \longrightarrow C \longrightarrow R^4 \xrightarrow{D} R^4 \xrightarrow{C} R^4 \xrightarrow{D} R^4 \xrightarrow{C} R^4 \xrightarrow{B} R^3 \xrightarrow{A} R \longrightarrow k \longrightarrow 0,
\]

where

\[
A = \begin{pmatrix} z & y & z \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -x & f_x \\ z & 0 & -x f_y \\ -y & z & 0 \end{pmatrix},
\]

\[
C = \begin{pmatrix} 0 & -f_x & f_y & z \\ -f_y & 0 & -f_x y & z \\ -f_z & f_y & 0 & x \\ f_x & 0 & f_y & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & -y & f_x \\ z & 0 & -x f_y \\ -y & z & 0 \end{pmatrix}.
\]
Lemma 2.11. Let \((R, m, k)\) be a local ring and \(x \in m - m^2\) an \(R\)-regular element. Then we have a split exact sequence
\[
0 \rightarrow k \xrightarrow{\theta} m/\langle x \rangle \xrightarrow{\pi} m/xR \rightarrow 0,
\]
where \(\theta\) is defined by \(\theta(\overline{a}) = \overline{xa}\) for \(\overline{a} \in R/m = k\) and \(\pi\) is the natural surjection.

**Proof.** Let \(x_1, x_2, \ldots, x_n\) be a minimal system of generators of \(m\) with \(x_1 = x\). Define a homomorphism \(\varepsilon : m/\langle x \rangle \rightarrow k\) by \(\varepsilon(\sum_{i=1}^{n} \overline{x_i}a_i) = \overline{a_1}\). We easily see that the composite map \(\varepsilon\theta\) is the identity map of \(k\), which means that \(\theta\) is a split-monomorphism. \(\square\)

Lemma 2.12. Let \((R, m, k)\) be a Cohen-Macaulay local ring of dimension one. Then the following conditions are equivalent:

1. \(R\) is a discrete valuation ring;
2. \(m^*\) is a cyclic \(R\)-module.

**Proof.** (1) \(\Rightarrow\) (2): This implication is obvious since the maximal ideal \(m\) is a free \(R\)-module of rank one.

(2) \(\Rightarrow\) (1): We have \(m^* \cong R/I\) for some ideal \(I\) of \(R\). Dualizing the natural exact sequence \(0 \rightarrow m \rightarrow R \rightarrow k \rightarrow 0\), we obtain an exact sequence
\[
\text{Hom}_R(k, R) \rightarrow R \rightarrow m^*.
\]
Since \(\text{Hom}_R(k, R) = 0\) by the assumption that \(R\) is Cohen-Macaulay, there is an injective homomorphism \(R \rightarrow R/I\). We easily observe that \(I = 0\), equivalently, \(m^* \cong R\). This implies the condition (1). \(\square\)

Let \(R\) be a local ring and \(I\) an ideal of \(R\). We recall that the **grade** of \(I\) is defined to be the infimum of the integers \(n\) such that \(\text{Ext}^n_R(R/I, R) \neq 0\), and is denoted by \(\text{grade} I\). As is well-known, it coincides with the length of any maximal \(R\)-sequence in \(I\). Now let us prove Theorem 2.9.

**Proof of Theorem 2.9.** (2) \(\Rightarrow\) (1): We have \(xz - zf = x \cdot 0 + y \cdot x + z \cdot (-f)\). Lemma 2.10 gives a finite free presentation
\[
R^4 \xrightarrow{C} R^4 \rightarrow \Omega^2_{Rk} \rightarrow 0
\]
of the \(R\)-module \(\Omega^2_{Rk}\), where \(C = \begin{pmatrix} 0 & f & \frac{y}{z} & z \\ -f & 0 & 0 & 0 \\ 0 & 0 & y & -z \\ 0 & 0 & -1 & 0 \end{pmatrix}\). Putting \(P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\) and \(Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\), we obtain
\[
PCQ = \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}
\]
where \(U = \begin{pmatrix} \frac{y}{z} & f \\ -f & 0 \end{pmatrix}\). It is easily seen that the matrices \(P, Q\) are invertible. Denoting by \(M\) (resp. \(N\)) the cokernel of the homomorphism defined by the matrix \(U\) (resp. \(U\)), we get an isomorphism \(\Omega^2_{Rk} \cong M \oplus N\).

(1) \(\Rightarrow\) (2): First of all, note that the local ring \(R\) is not regular. We denote by \(m\) the maximal ideal \(n/I\) of \(R\).

We can choose an element \(z \in n - n^2\) whose image in \(m\) is an \(R\)-regular element and that the module \(m/zR\) is decomposable. Put \((-) = (-) \otimes_{S} S/(z)\). Note that \(\overline{S}\) is also a regular local ring because \(z\) is a minimal generator of the maximal ideal \(n\) of \(S\) (see the proof of Proposition 2.3). Since the maximal ideal \(m\overline{R}\) of \(\overline{R}\) is decomposable,
we can apply Theorem 2.5 and see that \( \dim S = 2 \) and \( IS = xyS \) for some \( x, y \in \mathfrak{n} \) whose images in \( \overline{S} \) form a regular system of parameter of \( \overline{S} \). Hence \( R = S/xyS \) is a hypersurface, in particular a complete intersection, of dimension one. Therefore \( R \) is a complete intersection of dimension two by [8, Theorem 2.3.4(a)]. Since \( S \) is a regular local ring of dimension three with regular system of parameter \( x, y, z \), the ideal \( I \) is generated by an \( S \)-sequence by [8, Theorem 2.3.3(c)]. Noting \( \text{ht } I = \dim S - \dim R = 1 \), we see that \( I \) is a principal ideal. Write \( I = (l) \) for some \( l \in I \). There is an element \( f \in S \) such that \( l = xy - zf \). Assume that \( f \not\in \mathfrak{n} \). Then \( f \) is a unit of \( S \), and we see that \( zR \subseteq xyR \). Hence \( m = (x, y)R \), and \( \text{edim } R = \dim R = 2 \). This implies that \( R \) is regular, which is a contradiction. It follows that \( f \in \mathfrak{n} \).

Combining Theorem 2.7 with Theorem 2.9 gives birth to the following corollary. Compare it with Corollary 2.6.

**Corollary 2.13.** Let \((R, m, k)\) be a complete local ring. Suppose that \( m \) is indecomposable. Then the following conditions are equivalent:

1. \( \Omega_H^2 k \) has a non-zero proper direct summand of finite \( G \)-dimension;
2. \( R \) is Gorenstein, and \( \Omega_H^2 k \) is decomposable;
3. There are a complete regular local ring \((S, n)\) of dimension three, a regular system of parameters \( x, y, z \) of \( S \), and \( f \in \mathfrak{n} \) such that \( R \cong S/(xy - zf) \).

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ON THE PRINCIPAL 3-BLOCKS OF
THE CHEVALLEY GROUPS $G_2(q)$

YOKO USAMI

ABSTRACT. Assume that $q$ is even and

$q \equiv 4$ or $7 \pmod{9}$.

Then the principal 3-block of $G_2(q)$ and the principal 3-block of $G_2(4)$ are Morita equivalent. Here a $\Delta(P)$-projective trivial source $G_2(4) \times G_2(q)$-module and its $\mathcal{O}$-dual induce this Morita equivalence as bimodules, where $P$ is a common Sylow 3-subgroup of $G_2(4)$ and $G_2(q)$ and $\Delta(P) = \{(x, z) \in G_2(4) \times G_2(q) \mid z \in P\}$.

1. 序

1.1. $G$ を有限群とする。$\mathcal{O}$ を完備離散付值環とし、$K$ は標数が 0 であるその商体、$k$ は標数が $p > 0$ であるその剰余体とし、またどちらも十分大きいとする。$k$ および $K$ が $G$ の全ての部分群に対して十分大きいとき、$(K, \mathcal{O}, k)$ は $G$ の全ての部分群に対して splitting $p$-modular system という。群環 $\mathcal{O}G$ を直既約両側イデアル分解したとき出てくるそれぞれの直既約両側イデアルを $p$-ブロックと呼ぶ。（群環 $kG$ で考えることもある。）

1.2. 仮言的有限群 $G$ と $H$ が $p$-local structure 同じくするというのは次の条件が成り立つときにいう。

(i) $G$ と $H$ は Sylow-$p$ 部分群 $P$ を共有する。

(ii) $Q_1$ と $Q_2$ は $P$ の部分群で $f : Q_1 \rightarrow Q_2$ は同型写像とする。するとすべての $Q_1$ の元 $x$ に対して $f(x) = x^g$ を満たす $G$ の元 $g$ が存在するときかつそのときに限ってすべての $Q_1$ の元 $x$ に対して $f(x) = x^h$ を満たす $H$ の元 $h$ が存在する。

1.3. ここでは $p$-ブロックの module の圈どうしの同値つまり森田同値に関して類別することを考える。この問題設定は次の 2 つの予想と関連して発生している。Puig 予想はもともと source algebra と定義されるものによる同値類別について述べたもので Puig 同値と呼ばれるが、それは Donovan 予想を拡張したものととらえられる。また森田同値を与える bimodule がある良性質のものであれば、Puig 同値とならることも Puig、Scott によって知られている。( Remark 7.5 [3], Theorem 1.6 [1] )

The detailed version of this paper will be submitted for publication elsewhere.
Donovan予想 1.4. (1980 [2]) 与えられた素数 \( p \) と defect group \( D \) に対して、\( k \) 上の群環の \( D \) を defect group として持つ \( p \)-ブロックの森田同値類は有限個しかないのではないか？

Puig予想 (の変形版) 1.5. (1999 [3]) Donovan予想の中の森田同値をより強いPuig同値に置き換えたもの。

1.6. Lie型の単純群は、有限体 \( GF(q) \) 上に定義され \( q \) を動かすことで、無限列となり、位数は \( q \) の多項式として表される。\( p \) がこの多項式の同じ因子（例えば、\( q-1 \)など）を割る素数で、\( p \) の同じべきであれば、\( p \)-local structure は \( q \) に依存せず似たものになっていることが多い。それらの \( p \)-ブロックどうしは森田同値になりそうなのである。\( q \) の値に依存しないと証明できれば、Donovan予想の部分的裏付けが得られたことになる。またここで森田同値が言えれば、一つの無限列での表現の考察を小さい \( q \) の小さい群の場合に帰着できて都合がよい。

1.7. \( q \) を動かすことにして有限体 \( GF(q) \) 上の Chevalley 群 \( G_2(q) \) で 3-local structure を共有するような無限個の群の族を考えその主3ブロック類の類似性を研究することにする。いま、\( M(3) \) を位数 27 exponent 3 の extra special group と呼ぶものとする。\( q \) が mod 9 で 2, 4, 5 または 7 に合同のときは、\( G_2(q) \) は \( M(3) \) と同型な共通の Sylow 3-部分群 \( P \) を持つ \( P \) の正規化群は \( M(3) \) に位数 16 の semidihedral group \( SD_{16} \) が定数に作用した形の半直積と同型になっていて、同じ 3-local structure を持っている。さて主定理は以下の定理 1.8 である。実際、良い性質の bimodule での森田同値なので Puig同値である。以下定理 1.8 およびその証明のところでは \( p = 3 \) ということになる。

Theorem 1.8. Assume that \( q \) is even and

\[
q \equiv 4 \text{ or } 7 \pmod{9}.
\]

Then the principal 3-block of \( G_2(q) \) and the principal 3-block of \( G_2(4) \) are Morita equivalent. Here a \( \Delta(P) \)-projective trivial source \( G_2(4) \times G_2(q) \)-module and its \( \mathcal{O} \)-dual induce this Morita equivalence as bimodules, where \( P \) is a common Sylow 3-subgroup of \( G_2(4) \) and \( G_2(q) \) and \( \Delta(P) = \{(x, x) \in G_2(4) \times G_2(q) \mid x \in P\} \).

Cor. 1.9. 定理 1.8 の条件下では Hiss の論文 [5] Table 1 における分解行列中の未知パラメーター \( \gamma \) は 1 とわかる。

2. 定理 1.8 証明方針

2.1. 以下 \( q \) はTheorem 1.8 の条件を満たしているとする。以下 \( G_2(q) \) を \( G(q) \) と略記する。さらに \( B(G(q)) \) は \( G(q) \) の主3ブロックとする。まず \( B(G(q)) \) と \( B(G(4)) \) のstable equivalence of Morita type ( Definition 2.2 を見よ。) を先に証明してから、simple modules の行き先を考察し、以下の Linckelmann の定理 ( Theorem 2.3 ) を使って最終的に森田同値であることを見証する。

Definition 2.2. ([6]) Let \( A \) and \( B \) be \( \mathcal{O} \)-algebras, \( M (= _BM_A) \) a \( (B, A) \)-bimodule, and \( N (= _AN_B) \) an \( (A, B) \)-bimodule. We say that \( M \) and \( N \) induce a stable equivalence of Morita type between \( A \) and \( B \), if

(i) \( M \) is projective as a left \( B \)-module and as a right \( A \)-module,
(ii) \( N \) is projective as a left \( A \)-module and as a right \( B \)-module,
(iii) $M \otimes_A N = B \otimes X$ for a projective $(B, B)$-bimodule $X$ and $N \otimes_B M = A \otimes Y$

for a projective $(A, A)$-bimodule $Y$.

For $k$-algebras we define a stable equivalence of Morita type similarly.

**Theorem 2.3.** (Linckelmann [6]) Let $G$ and $H$ be two finite groups and $b$ and $b'$ central idempotents of $OG$ and $OH$ respectively. Set

$$A = OGb, \quad B = OHb', \quad \overline{A} = k \otimes_O A \quad \text{and} \quad \overline{B} = k \otimes_O B.$$ 

Let $M$ be a $(B, A)$-bimodule which is projective as both a left and a right module, such that the functor $M \otimes_A -$ induces a stable equivalence of Morita type between $A$ and $B$. Then the following hold.

(i) Up to isomorphism, $M$ has a unique indecomposable non-projective direct summand $M'$ as a $(B, A)$-bimodule and $k \otimes_O M'$ is, up to isomorphism, the unique indecomposable non-projective direct summand of $k \otimes_O M$ as a $(B, A)$-bimodule.

(ii) If $M$ is indecomposable as a $(B, A)$-bimodule, then for any simple $A$-module $S$, the $B$-module $M \otimes_A S$ is indecomposable and non-projective as a $\overline{B}$-module.

(iii) If for any simple $A$-module $S$, the $B$-module $M \otimes_A S$ is simple, then the functor $M \otimes_A -$ induces a Morita equivalence.

2.4. ここでは実は $B(G(q))$ と $B(G(4))$ をいきなり比較するのでなく、$G(q)$ の構造の特殊性を利用とする。つまり $G(q)$ は $SL(3, q)$ と同型な部分群を持ちさらにその指数 2 の拡大（$N(q)$ と略記する。）をまさに $P$ の中心 $Z(P)$ の正規化群として持つことも利用する。そこで主 3-ブロック達 $B(N(q))$, $B(N(4))$ は森田（Fuipa）同値であることは主 3-ブロック達 $B(PSL(3, q))$ と $B(PSL(3, 4))$ が森田（Fuipa）同値であるという労力の定理（定理 2.5）から導いておく。そしてもっぱら $B(G(q))$ と $B(N(q))$ のłable equivalence of Morita type を与える良い性質をもつ bimodule $M_q$ を探すことにする。（そのとき以下 BROUE 定理（Theorem 2.7）を利用する。）そうすれば、$B(G(q))$ と $B(N(q))$, $B(N(q))$ と $B(N(4))$, $B(N(4))$ と $B(G(4))$ のłable equivalence of Morita type をそれぞれ与える bimodules を合成して $B(G(q))$ と $B(G(4))$ のłable equivalence of Morita type を引き出せる。BROUE の定理を述べる前に必要な Brauer morphism と呼ばれるものを Definition 2.6 で導入しておく。

**Theorem 2.5.** (Kunugi [7]) Let $G$ be the projective special linear group $PSL(3, q)$ for a power $q$ of a prime such that $q \equiv 4 \lor 7 \pmod 9$ (so that a Sylow 3-subgroup $Q$ of $G$ is elementary abelian of order 9). Let $(K, O, k)$ be a splitting 3-modular system for all subgroups of $G$. Then the principal 3-block $A$ of $OG$ is Morita equivalent to the principal 3-block $A_0$ of $O[PSL(3, 4)]$. This equivalence is defined by an $(A_0, A)$-bimodule $M$ which is a $(\Delta Q)$-projective 3-permutation $O[PSL(3, 4) \times G]$-module.

**Definition 2.6.** ([8,9]) For an $OG$-module $V$ and a $p$-subgroup $P$ of $G$, we set

$$\text{Br}_P(V) = V^P / \left( \sum_Q \text{Tr}_Q^P (V^Q) + \mathcal{P} V^P \right)$$

where $V^P$ denotes the set of fixed points of $V$ under $P$ and $Q$ runs over all proper subgroups of $P$ and

$$\text{Tr}_Q^P(v) = \sum_{x \in P/Q} x v$$

for a subgroup $Q$ of $P$ and $v \in V^Q$. ($\mathcal{P}$ is the maximal ideal of $O$. )
Theorem 2.7. (Broué [9]) Let $G$ be a finite group with a Sylow $p$-subgroup $P$ and $H$ a subgroup of $G$ containing $N_G(P)$. Assume that $G$ and $H$ have the same fusion on $p$-subgroups contained in $P$ (that is, the same $p$-local structure). Let $b$ and $b'$ be central primitive idempotents of $OG$ and $OH$ respectively for the principal blocks

$$A = OGb$$
$$B = OHb'$$

(having a common defect group $P$.) For a subgroup $R$ of $P$, set

$$b_R = Br_R(b)$$
$$b'_R = Br_R(b').$$

Let $M$ be a $(B, A)$-bimodule and $N$ be an $(A, B)$-bimodule. For each subgroup $R$ of $P$ set

$$M_R = Br_{Δ(R)}(M)$$
$$N_R = Br_{Δ(R)}(N).$$

Assume that

(i) $M$ is a direct summand of the restriction of $A$ from $G \times G$ to $H \times G$.

(ii) For each non-trivial subgroup $R$ of $P$, $M_R$ and $N_R$ induce a Morita equivalence between $kC_G(R)b_R$ and $kC_H(R)b'_R$.

Then $M$ and $N$ induce a stable equivalence of Morita type between $A$ and $B$.

3. $B(G(q))$ と $B(N(q))$ の STABLE EQUIVALENCE OF MORITA TYPE を与える BIMODULE

3.1. 一般に $H$ が $G$ の部分群のとき $OG$ は 左 $OH$-加群かつ右 $OG$-加群とみてテンソルすることで $OG$-加群を $H$ に制限したり $OH$-加群を $G$ へ持ち上げたりする働きをする。$OG$ の両側に $OH$, $OG$ のそれぞれ主ブロックべき等元をつけると、主ブロックに属するものを、$H$ へ制限して主ブロックに属するもののみ残すとか、$G$ へ持ち上げて余り主-ブロックに属するもののみ残す働きをすることになる。ここで $G(q)$ とその部分群 $N(q)$ の内でその主 3-ブロックべき等元 $e$ と $f$ を使って $fOG(q)e$ を考察する。$R$ を $P$ の 1 でない任意の部分群として、$Δ(R)$ に関する Brauer morphism を施してどう分解しているか調べると Broué の定理 (Theorem 3.2 [8]) によって $fOG(q)e$ がどのように vertex の直既約因子選の直和に分解するかということに対する情報が得られる。（ちなみにこれは vertex $ΔP$ の直既約因子 $M_q$ を持っていて、それがのぞみの bimodule となる。) $fOGe$ は $Δ(R)$ に対する Brauer morphism を施すと $fRkC_{G(q)}(R)eR$ となる。これが $eR$, $fR$ はそれぞれ $kC_{G(q)}(R), kCN(q)(R)$ の主 3-ブロックのブロックべき等元である。いろいろの $R$ で試して、$fOG(q)e$ は nonprojective 因子として $M_q$ ともうひと vertex の小さい因子を持つことがわかる。

3.2. $fOG(q)e$ はどういう制限や持ち上げをしているか、片方で、いろいろの permutaion module を作りつつ、制限したり持ち上げたりしながら、その中で $M_q$ 自身は何を何に写しているか調べる。（このとき vertex $Δ(R)$ の permutation module が vertex $R'$ の permutation module をどんな vertex の直既約因子選の直和にするかは Mackey 分解で様子を知ることができる。また $G(q)$ と $N(q)$ の間ではいくつか複数の 3-subgroup に関して Green 対応のある状況であることも有効に使える。)
4. SIMPLE MODULES の行き先

4.1. ここで 3.2 に続けてさらに 2.4 で説明したように合成して作る bimodule が何を何に写すか調べる。このときこれが exact sequence を exact sequence に写すことを有効に利用する。だからまず trivial module のように簡単な simple module や簡単な構造の permutation module の行き先を決めておいて、その組成因子の行き先を決めるといったやり方を繰り返すことができる。

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ON THE $\mathcal{ZD}_\infty$ CATEGORY

MICHEL VAN DEN BERGH

Abstract In this paper we give a direct proof of the properties of the $\mathcal{ZD}_\infty$ category which was introduced in the classification of noetherian, hereditary categories with Serre duality by Idun Reiten and the author.

1. Introduction

Below $k$ is a field. All categories will be $k$-linear. An abelian or triangulated category $\mathcal{A}$ is Ext-finite if for all objects $A, B \in \mathcal{A}$ one has that $\oplus_i \text{Ext}^i(A, B)$ is finite dimensional. If $\mathcal{A}$ is triangulated and Ext-finite then we say that $\mathcal{A}$ satisfies Serre duality [1] if there exists an auto-equivalence $F$ of $\mathcal{A}$ together with isomorphisms

$$\text{Hom}_{\mathcal{A}}(A, B) \to \text{Hom}_{\mathcal{A}}(B, FA)^*$$

natural in $A, B$ (where $(-)^*$ is the $k$-dual). If $\mathcal{A}$ is abelian and Ext-finite then we say that $\mathcal{A}$ satisfies Serre duality if this is the case for $D^b(\mathcal{A})$. The following result can be extracted from [5, Ch. 1].

Theorem 1.1. Assume that $\mathcal{C}$ is an Ext-finite hereditary category without injectives or projectives. Then the following are equivalent

1. $\mathcal{C}$ has almost split sequences.
2. $\mathcal{C}$ satisfies Serre duality.
3. There is an auto-equivalence $V: \mathcal{C} \to \mathcal{C}$ together with natural isomorphisms

$$(1.1) \quad \text{Hom}_{\mathcal{C}}(A, B) \to \text{Ext}^1_{\mathcal{C}}(B, VA)^*$$

Furthermore the functor $V$ coincides with the Auslander-Reiten translate $\tau$ when evaluated on objects.

In the classification of noetherian Ext-finite hereditary categories with Serre duality in [5] we considered a category $\mathcal{C}$ defined by the following pullback diagram

$$\begin{align*}
\text{mod}(k) \oplus \text{mod}(k) & \to \text{mod}(k) \\
\uparrow & \uparrow \\
\mathcal{C} & \to \text{gr}(k[x])
\end{align*}$$

(1.2)

where the horizontal map sends $(V_1, V_2)$ to $V_1 \oplus V_2$ and the vertical map is localizing at $x$ followed by restricting to degree zero. It was shown by a rather indirect argument that $\mathcal{C}$ is a noetherian, Ext-finite, hereditary abelian category without injectives or projectives.

\footnote{The paper is in a final form and no version of it will be submitted for publication elsewhere.}
which satisfies Serre duality. It was also shown that the AR-quiver of \( \mathcal{C} \) has two components, one equal to \( \mathbb{Z}A_\infty \) (a “wing”) and the other equal to \( \mathbb{Z}D_\infty \). For this reason \( \mathcal{C} \) was called the “\( \mathbb{Z}D_\infty \)-category”.

The aim of this paper is to give a direct proof of the above facts. In addition we will also establish a link with one-dimensional \( A_n \) singularities.

2. Elementary properties

It is easy to see that \( \mathcal{C} \), as defined in the introduction, is a noetherian abelian category. It will be convenient to consider the locally noetherian Grothendieck category \( \tilde{\mathcal{C}} \) associated to \( \mathcal{C} \). It follows for example by [3, Prop. 2.14] that \( D^b(\mathcal{C}) \) and \( D^b(\tilde{\mathcal{C}}) \) are equivalent. Hence the Ext-groups between objects in \( \mathcal{C} \) may be computed in \( \tilde{\mathcal{C}} \).

The objects of \( \tilde{\mathcal{C}} \) are quadruples \((M, V_0, V_1, \phi)\) where \( M \) is a graded \( k[z] \)-module, \( V_0, V_1 \) are \( k \)-vector spaces and \( \phi \) is an isomorphism of \( k[z] \) modules \( M_z \to (V_0 \oplus V_1) \otimes_k k[z, x^{-1}] \). Objects in \( \mathcal{C} \) are given by the quadruples \((M, V_0, V_1, \phi) \) in which \( M \) is finitely generated.

Sending \((M, V_0, V_1, \phi) \) to \( M \) defines an faithful exact functor \( \tilde{\mathcal{C}} \to \text{Gr}(k[z]) \) which we call the restriction functor and which we denote by \((-)_{k[z]} \).

We write \( M(n) = (M(n), V_0, V_1, \phi) \) where we have identified \( M(n)_z \) with \( M_z \) through multiplication with \( x^n \). Furthermore we define \( \sigma(M) = (M, V_1, V_0, \phi) \).

We define \( \tilde{T} \subset \tilde{\mathcal{C}} \) and \( \tilde{\mathcal{F}} \subset \tilde{\mathcal{C}} \) respectively as the inverse images of the \( x \)-torsion and \( x \)-torsion free modules in \( \text{Gr}(k[z]) \). \( T \) and \( \mathcal{F} \) are defined similarly, but starting from \( \mathcal{C} \).

By \( \tilde{\mathcal{C}}_x \) we denote the full subcategory of \( \tilde{\mathcal{C}} \) with objects the quadruples \((M, V_0, V_1, \phi) \) in which \( x \) acts invertibly on \( M \).

We denote by \((-)_x \) the functor \( \tilde{\mathcal{C}} \to \tilde{\mathcal{C}}_x \) which sends \((M, V_0, V_1, \phi) \) to \((M_x, V_0, V_1, \phi) \). Clearly if \( M \in \tilde{\mathcal{C}} \) and \( N \in \tilde{\mathcal{C}}_x \) then the canonical maps

\[
(2.1) \quad \text{Hom}_\mathcal{C}(M, N) \to \text{Hom}_{\tilde{\mathcal{C}}_x}(M, N) \to \text{Hom}_{\tilde{\mathcal{C}}_x}(M_x, N)
\]

are isomorphisms. We list a few other obvious facts.

(O1) \((\tilde{T}, \tilde{\mathcal{F}}) \) forms a torsion pair in \( \tilde{\mathcal{C}} \). That is \( \text{Hom}(\tilde{T}, \tilde{\mathcal{F}}) = 0 \) and for any \( M \in \tilde{\mathcal{C}} \) there exists an exact sequence (necessarily unique)

\[
0 \to T \to M \to F \to 0
\]

with \( T \in \tilde{T} \) and \( F \in \tilde{\mathcal{F}} \).

(O2) If \( T \in \tilde{T} \) and \( M \in \tilde{\mathcal{C}} \) then

\[
\text{Hom}_{\tilde{\mathcal{C}}}(T, M) = \text{Hom}_{k[z]}(T, M)
\]

\[
\text{Hom}_{\tilde{\mathcal{C}}}(M, T) = \text{Hom}_{k[z]}(M, T)
\]

(O3) The restriction functor defines an equivalence between \( \tilde{T} \) and \( \text{Tors}(k[z]) \) where \( \text{Tors}(k[z]) \) denotes the \( x \)-torsion modules is \( \text{Gr}(k[z]) \).

(O4) The functor \( \tilde{\mathcal{C}}_x \to \text{Mod}(k) \oplus \text{Mod}(k) \) which sends \((M, V_0, V_1, \phi) \) to \( V_0 \oplus V_1 \) is an equivalence of categories.

Combining (O4) with (2.1) yields in particular

(O5) If \( N \in \tilde{\mathcal{C}}_x \) then \( \text{Hom}_\mathcal{C}(-, N) \) is exact. Hence the objects in \( \tilde{\mathcal{C}}_x \) are injective in \( \tilde{\mathcal{C}} \).

We now describe the indecomposable injectives in \( \tilde{\mathcal{C}} \). For \( n \in \mathbb{Z} \) let \( E_n \) be the graded injective \( k[z] \)-module given by \( k[z, x^{-1}]/x^{n+1}k[z] \). Since \( E_n \in \text{Tors}(k[z]) \) there exists by (O3) a corresponding object in \( \tilde{T} \) which we denote by the same symbol. From (O2) it follows that \( \text{Hom}(-, E_n) \) is exact and hence \( E_n \) is injective in \( \tilde{\mathcal{C}} \).
To construct other injectives we note that by (O5) we know that the objects in \( \tilde{\mathcal{C}}_x \) are injective in \( \tilde{\mathcal{C}} \). Since by (O4) \( \tilde{\mathcal{C}}_x \) is equivalent to \( \text{Mod}(k) \oplus \text{Mod}(k) \) there must be two corresponding indecomposable injectives in \( \tilde{\mathcal{C}} \). They are given by

\[
E^0 = (k[x, x^{-1}], k, 0, \text{id}_{k[x, x^{-1}]}) \\
E^1 = (k[x, x^{-1}], 0, k, \text{id}_{k(x, x^{-1})})
\]

Proposition 2.1.  
(1) \((E_n)_n, E^0, E^1\) forms a complete list of indecomposable injectives in \( \tilde{\mathcal{C}} \).

(2) Every object in \( \tilde{\mathcal{C}} \) has injective dimension one (and hence \( \tilde{\mathcal{C}} \) and \( \mathcal{C} \) are hereditary [5, Prop. A.3]).

(3) \( C \) is Ext-finite.

Proof. Since the listed injectives are clearly indecomposable (1) follows if we can show that any indecomposable object can be embedded in a direct sum of them [4].

We prove (1) and (2) together by showing that every object \( M \in \tilde{\mathcal{C}} \) has a resolution of length at most two whose terms consist of direct sums of the injectives given in (1). By (O1) it is clearly sufficient to prove this claim separately in the cases \( M \in \tilde{T} \) and \( M \in \tilde{F} \).

Assume first that \( M \in \tilde{T} \). Then \( M \) has an injective resolution

\[
0 \rightarrow M \rightarrow I_0 \rightarrow I_1 \rightarrow 0
\]

in \( \text{Tors}(k[x]) \). By (O3) this resolution corresponds to one in \( \tilde{\mathcal{C}} \). Furthermore by the structure of the injectives in \( \text{Gr}(k[x]) \) the \( I_i \) are direct sums of the \( E_n \) in \( \text{Gr}(k[x]) \). Again by (O3) the same is true in \( \tilde{\mathcal{C}} \).

Now assume that \( M \in \tilde{F} \). Consider the short exact sequence

\[
0 \rightarrow M \rightarrow M_z \rightarrow M_z/M \rightarrow 0
\]

(2.3)

\( M_z \) lies in \( \tilde{\mathcal{C}}_x \) and hence by (O4) is a direct sum of copies of \( E^0 \) and \( E^1 \). \( M_z/M \) is \( x \)-divisible and lies in \( \tilde{T} \) and so by (O3) \( M_z/M \) is a direct sum of copies of \( E_n \). Whence (2.3) is the kind of resolution we were looking for.

To prove (3) we note that if \( E, F \) are indecomposable injectives as in (1) then \( \dim \text{Hom}_C(E, F) \leq 1 \). Thus it suffices to show that every \( M \in \mathcal{C} \) has an injective resolution consisting in every degree of a finite number of indecomposable injectives. This follows easily from the construction.

Proposition 2.2. If \( F \in \tilde{F} \) and \( T \in \tilde{T} \) then \( \text{Ext}_C^1(F, T) = 0 \). In particular every object in \( \mathcal{C} \) is of the form \( F \oplus T \) with \( F \in \mathcal{F} \) and \( T \in \mathcal{T} \).

Proof. It follows from (O2) that \( \text{Hom}(F, -) \) is exact on \( \mathcal{T} \). Since by the proof of the previous proposition \( T \) has a \( \tilde{\mathcal{C}} \) injective resolution inside \( \tilde{T} \), we are done.

Now we describe the \( \text{Ext} \)-groups between objects in \( \tilde{F} \).

Lemma 2.3. Assume that \( F = (F, V_0, V_1, \phi) \), \( F' = (F', V'_0, V'_1, \phi') \) are objects in \( \tilde{F} \). Then there exists an exact sequence of the form

\[
0 \rightarrow \text{Hom}_k(F, F') \rightarrow \text{Hom}_{k[x]}(F, F') \rightarrow \text{Hom}_k(V_0, V'_0) \oplus \text{Hom}_k(V_1, V'_0) \rightarrow \text{Ext}_C^1(F, F') \rightarrow 0
\]

(2.4)

Proof. We start with the short exact sequence

\[
0 \rightarrow F' \rightarrow F'_x \rightarrow F'_x/F' \rightarrow 0
\]
which according to the proof of lemma 2.1 is an injective resolution of $F'$, both in $\mathcal{C}$ and in $\text{Gr}(k[[x]])$.

Applying $\text{Hom}_C(F, -)$, $\text{Hom}_{\text{Gr}(k[[x]])}(F, -)$ and comparing yields a commutative diagram with exact rows and columns.

\[
\begin{array}{ccccccccc}
0 & \to & \text{Hom}_C(F, F') & \to & \text{Hom}_k(V_0, V'_0) \oplus \text{Hom}_k(V_1, V'_1) & \to & \text{Hom}_C(F, F'/F') & \to & \text{Ext}_C^1(F, F') & \to & 0 \\
0 & \to & \text{Hom}_{\text{Gr}(k[[x]])}(F, F') & \to & \text{Hom}_k(V_0 \oplus V_1, V'_0 \oplus V'_1) & \to & \text{Hom}_{\text{Gr}(k[[x]])}(F, F'/F') & \to & 0 \\
& & & & & & \text{Hom}(V_0, V'_0) \oplus \text{Hom}(V_1, V'_1) & \to & 0
\end{array}
\]

(2.4) now follows from the previous diagram through an easy diagram chase. □

**Proposition 2.4.** $C$ has neither injectives nor projectives.

**Proof.** Since $T$ is equivalent to the $x$-torsion modules in $\text{gr}(k[[x]])$, it is easy to see that $T$ does not contain any injective or projectives.

If $0 \neq (F, V_0, V_1, \phi)$ in $\mathcal{F}$ then by considering the faithful restriction functor to $\text{gr}(k[[x]])$ we see that $\text{Hom}_C(F, \sigma F(-n)) = 0$ for $n \gg 0$. On the other hand $V_0$ or $V_1 \neq 0$. It follows from the previous lemma that $\text{Ext}_C^1(F, \sigma F(-n)) \neq 0$ for $n \gg 0$. Hence $F$ is not projective. A similar argument shows that $F$ is not injective. □

**Remark 2.5.** The reason why we called this section “Elementary properties” is that the stated results hold in greater generality. For example, suitably adapted versions would be valid for the pullback of

\[
\begin{array}{cccc}
\text{mod}(k)^{\otimes m} & \to & \text{mod}(k) \\
& & \uparrow \\
& & \text{gr}(k[[x]])
\end{array}
\]

for any $m$. By contrast, the results in the next section require $m = 2$.

### 3. Serre Duality

§ Our next aim is to prove that $C$ satisfies Serre duality. First we construct a Serre functor on $\mathcal{F}$. Put $VM = \sigma(M)(-1)$.

The first step in proving Serre duality is constructing a “trace map" $\eta_M : \text{Ext}_C^1(M, VM) \to k$ for $M \in \mathcal{F}$ which should corresponds to the identity map in $\text{Hom}_C(M, M)$ under the isomorphism (1.1).

We now use (2.4) to construct the trace map $\eta_F$ for $F = (F, V_0, V_1, \phi) \in \mathcal{F}$. In this case $VF = (F(-1), V_1, V_0, \phi)$ and we have an exact sequence

\[
\text{Hom}_{k[[x]]}(F, VF) \to \text{Hom}_k(V_0, V_0) \oplus \text{Hom}_k(V_1, V_1) \to \text{Ext}_C^1(F, VF) \to 0
\]
Lemma 3.1. The composition

\[ \text{Hom}_{k[z]}(F, VF) \to \text{Hom}_k(V_0, V_0) \oplus \text{Hom}_k(V_1, V_1) \xrightarrow{\text{Tr}_0 + \text{Tr}_1} k \]

is the zero map.

Proof. To see this note that \( \text{Hom}_{k[z]}(F, VF) = \text{Hom}(F, F(-1)) \) and furthermore that (3.1) can be extended to a commutative diagram.

\[
\begin{array}{ccc}
\text{Hom}_{k[z]}(F, F(-1)) & \longrightarrow & \text{Hom}_k(V_0 \oplus V_1, V_0 \oplus V_1) \\
& & \downarrow \text{Tr} \\
& & \text{Hom}_k(V_0, V_0) \oplus \text{Hom}_k(V_1, V_1) \\
\end{array}
\]

By choosing a basis for \( F \) as graded \( k[z] \)-module one easily sees that every element of \( \text{Hom}(F, F(-1)) \subset \text{Hom}(F, F) \) is nilpotent. Since nilpotent elements have zero trace it follows that the composition

\[ \text{Hom}_{k[z]}(F, F(-1)) \to \text{Hom}_k(V_0 \oplus V_1, V_0 \oplus V_1) \xrightarrow{\text{Tr}} k \]

is zero. This proves what we want. \( \square \)

From lemma 3.1 together with (2.4) there exists a unique map \( \eta_F : \text{Ext}_C^1(F, VF) \to k \) which makes the following diagram commutative.

\[
\begin{array}{ccc}
\text{Hom}_{k[z]}(F, VF) & \longrightarrow & \text{Hom}(V_0, V_0) \oplus \text{Hom}(V_1, V_1) \\
& & \downarrow \text{Tr}_0 + \text{Tr}_1 \\
& & \text{Ext}_C^1(F, VF) \\
& & \downarrow \eta_F \\
& & 0 \\
& & k \\
\end{array}
\]

To continue it will be convenient to use the Yoneda multiplication on \( \text{Ext}_C^*(-, -) \). In order to have compatibility with the notation for compositions of maps we will write the Yoneda multiplication as a pairing

\[ \text{Ext}_C^*(B, C) \times \text{Ext}_C^*(A, B) \to \text{Ext}_C^*(A, C) \]

We extend \( \eta_F \) to a map \( \text{Ext}_C^*(F, VF) \to k \) by letting it act trivially on \( \text{Hom}(F, VF) \).

Lemma 3.2. Let \( F, G \in \mathcal{F} \) and assume that \( f \in \text{Ext}_C^*(F, G) \) and \( g \in \text{Ext}_C^*(G, VF) \). Then we have \( \eta_F(gf) = \eta_G(V(f)g) \).

Proof. We may assume that \( f \) and \( g \) are homogeneous. Furthermore the cases where \( f, g \) are both of degree 0 or of degree 1 are trivial. Hence we may assume that \( (\deg f, \deg g) = (0, 1) \) or \( (\deg f, \deg g) = (1, 0) \).

Let us consider the first possibility. We check that \( \eta_F(-f) = \eta_G(V(f)--) \) as maps \( \text{Ext}_C^1(G, VF) \to k \). This amounts to the commutativity of

\[
\begin{array}{ccc}
\text{Ext}_C^1(G, VF) & \longrightarrow & \text{Ext}_C^1(G, VG) \\
& & \downarrow \eta_G \\
& & \text{Ext}_C^1(F, VF) \\
& & \eta_F \downarrow k \\
& & k \\
\end{array}
\]

\[ \text{Ext}_C^1(G, VF) \to \text{Ext}_C^1(G, VG) \]
Assume that $F = (F, V_0, V_1, \phi)$, $G = (G, W_0, W_1, \theta)$. Then $f$ induces maps $f_0 : V_0 \to W_0$ and $f_1 : V_1 \to W_1$. Elementary linear algebra yields that we have a commutative diagram

\[
\begin{array}{ccc}
\text{Hom}(V_0, V_0) \oplus \text{Hom}(V_1, V_1) & \longrightarrow & \text{Hom}(W_0, W_0) \oplus \text{Hom}(W_1, W_1) \\
\downarrow & & \downarrow \\text{Tr}_{V_0} + \text{Tr}_{V_1} & k
\end{array}
\]

This diagram, together with the definition of $\eta$ yields the commutativity of (3.2).

Now we consider the possibility $(\deg f, \deg g) = (1, 0)$. Since we trivially have

\[
(3.3) \quad \eta_{V,F} \circ V = \eta_X
\]

it is sufficient to prove that $\eta_{V,F}(V(f)V(g)) = \eta_G(V(f)g)$. Replacing $(Vf, g)$ by $(g, f)$ this reduces to the previous case. \hfill \square

We are now in a position to prove Serre duality for objects in $F$. We will show that the pairing

\[
(3.4) \quad \text{Hom}_c(F, G) \times \text{Ext}^1_c(G, VF) \to \text{Ext}^1(F, VF) \overset{\eta_F}{\rightarrow} k : (f, g) \mapsto \eta_F(gf)
\]

is non-degenerate. By lemma 3.2 the non-degeneracy of (3.4) for all $F, G$ is equivalent to the non-degeneracy of the pairing

\[
(3.5) \quad \text{Ext}^1_c(F, G) \times \text{Hom}_c(G, VF) \to \text{Ext}^1(F, VF) \overset{\eta_F}{\rightarrow} k : (f, g) \mapsto \eta_F(gf)
\]

for all $F, G$. It follows also easily from lemma 3.2 that (3.4) and (3.5) are natural in $F$ and $G$.

**Lemma 3.3.** If we have and exact sequence

\[
(3.6) \quad 0 \to F_1 \to F \to F_2 \to 0
\]

in $F$ and if we have non-degeneracy of (3.4) and (3.5) for two out of the three pairs $(F_1, G), (F, G), (F_2, G)$ then we also have it for the third one. A similar statement holds for an exact sequence

\[
(3.7) \quad 0 \to G_1 \to G \to G_2 \to 0
\]

**Proof.** Assume that we have an exact sequence of the form (3.5). We claim that the following diagram with exact rows is commutative.

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \text{Hom}(F_2, G) & \longrightarrow & \text{Hom}(F, G) & \longrightarrow & \text{Hom}(F_1, G) & \longrightarrow & \text{Ext}^1(F_2, G) & \longrightarrow & \text{Ext}^1(F, G) & \longrightarrow & \text{Ext}^1(F_1, G) & \longrightarrow & 0 \\
\downarrow \alpha_2 & & \downarrow \alpha & & \downarrow \alpha_1 & & \downarrow \beta & & \downarrow \beta_2 & & \downarrow \beta_1 & & \downarrow \beta_1 \\
0 & \longrightarrow & \text{Ext}^1(G, VF_2)^* & \longrightarrow & \text{Ext}^1(G, VF_1)^* & \longrightarrow & \text{Hom}(G, VF_2)^* & \longrightarrow & \text{Hom}(G, VF_1)^* & \longrightarrow & \text{Hom}(G, VF)^* & \longrightarrow & \text{Hom}(G, VF_1)^* & \longrightarrow & 0
\end{array}
\]

Here the maps labeled by $\alpha$ are obtained from (3.4) whereas those labeled by $\beta$ are obtained from (3.5).

The commutativity of this diagram follows easily from lemma 3.2 together with the observation that all horizontal arrows are obtained by Yoneda multiplying with elements of suitable Ext-groups. For example the connecting maps are obtained from multiplying with the element of $\text{Ext}^1(F_2, F_1)$ representing the exact sequence (3.6).

If we now have non-degeneracy for two out of the three pairs $(F_1, G), (F, G), (F_2, G)$ then we also have it for the third pair because of the five-lemma.

The case where we have an exact sequence as in (3.7) is treated similarly. \hfill \square
To continue we define a some canonical objects in $\mathcal{F}$. Let $a \in \mathbb{N}$. Then we write

$$F_{0a}^0 = (x^{-a}k[x], k, 0, \text{id}_{k[x], x^{-1}})$$
$$F_{0a}^1 = (x^{-a}k[x], 0, k, \text{id}_{k[x], x^{-1}})$$

(the reason for this notation will become clear in Section §5).

**Lemma 3.4.** Every object $F$ in $\mathcal{F}$ has a finite filtration $0 = F_0 \subset \cdots \subset F_n = F$ such that the corresponding subquotients are among the $F_{0a}^i$.

**Proof.** By the structure of $C$ there must be a surjective map $\phi : C_i \to E^i$ where $i = 0$ or $i = 1$. Hence $\ker \phi$ is a non-trivial quotient. Since it is easy to see that the subobjects of $E_i$ in $C$ are of the form $F_{0a}^i$, we are done.

Using (2.4) we can compute the Hom and Ext-groups between the $F_{0a}^i$. The results are given in the next lemma.

**Lemma 3.5.** One has

$$\text{Hom}(F_{0a}^i, F_{0b}^j) = \begin{cases} 
k & \text{if } i = j \text{ and } a \leq b \\
0 & \text{otherwise} \end{cases}$$

$$\text{Ext}^1(F_{0a}^i, F_{0b}^j) = \begin{cases} 
k & \text{if } i - j = 1 \text{ and } a > b \\
0 & \text{otherwise} \end{cases}$$

**Proof.** The claim for Hom is trivial, so we concentrate on Ext.

We use (2.4). This immediately yields that $\text{Ext}^1(F_{0a}^i, F_{0b}^j) = 0$ if $j \neq 1 - i$. If $j = 1 - i$ then we have the following exact sequence.

$$(3.8) \quad \text{Hom}_C(x^{-a}k[x], x^{-b}k[x]) \to k \to \text{Ext}^1_C(F_{0a}^i, F_{0b}^j) \to 0$$

This yields that $\text{Ext}^1_C(F_{0a}^i, F_{0b}^j) = k$ if and only if $\text{Hom}_C(x^{-a}k[x], x^{-b}k[x]) = 0$, i.e. if and only if $a > b$.

We are now in a position to prove the main result of this section.

**Theorem 3.6.** $C$ satisfies Serre duality.

**Proof.** We show first that $C$ satisfies Serre duality for objects $F, G$ in $\mathcal{F}$. We will show the non-degeneracy of (3.4) and (3.5) by induction of $rk_{k[x]}(F)$, $rk_{k[x]}(G)$. This reduces us to the case where $F = F_{0a}^i$, $G = F_{0b}^j$. So we need to check the non-degeneracy of

$$(3.9) \quad \text{Hom}_C(F_{0a}^i, F_{0b}^j) \times \text{Ext}^1_C(F_{0b}^j, F_{0,a-1}^i) \to \text{Ext}^1_C(F_{0a}^i, F_{0,a-1}^j) \to k$$

$$(3.10) \quad \text{Ext}^1_C(F_{0a}^i, F_{0b}^j) \times \text{Hom}_C(F_{0b}^j, F_{0,a-1}^i) \to \text{Ext}^1_C(F_{0a}^i, F_{0,a-1}^j) \to k$$

where $i' = 1 - i$. We will concentrate ourselves on (3.10). (3.9) is similar. By (3.5) the only non-trivial case is given by $j = 1 - i$ and $a > b$. In that case all vector spaces involved are equal to $k$ and what we want to prove follows from inspecting (3.8).

Now we show that $C$ has almost split sequences. By Theorem 1.1 this implies that $C$ satisfies Serre duality.

By Proposition 2.2 it is clearly sufficient to construct almost split sequences ending in indecomposable objects in $\mathcal{F}$ and $T$. First let $F \in \mathcal{F}$ be indecomposable. Since $\text{Ext}^2_C(F, VF) = \text{Hom}_C(F, F)^* = \text{Ext}_C(F, VF)$ has a simple socle as (left or right) $\text{Hom}_C(F, F)$
module. Let ζ be a non-zero element this socle. It is well-known, and easy to see that ζ defines the almost split sequence in F ending in F.

$$0 \to VF \to M \to F \to 0$$

But this is also an almost split sequence in C since if T ∈ T then Hom(T, F) = 0.

Now let T ∈ T. Then there exists an almost split sequence in T

$$(3.11) \quad 0 \to T' \to T \to 0$$

(since T is equivalent to the category of x-torsion modules over k[z]).

We have to prove that any pullback for C → T of (3.11) with C indecomposable is split. Clearly we only have to consider the case C ∈ F. But then it follows from Proposition 2.2 that the pullback is split. This finishes the proof. □

4. RELATION WITH ONE-DIMENSIONAL GRADED TYPE $A_n$-SINGULARITIES

Let C be the hereditary category which was described in the previous section. We will now show that C can be considered as a limit of certain graded simple singularities.

If m ∈ N then the graded simple $A_{2m-1}$-singularity of dimension one is by definition the graded subring $R_m$ of $k[x] \oplus k[z]$ generated by $u = (x, x)$ and $v = (x^m, 0)$. It is easy to see that $R \cong k[u, v]/(u^m v - u^2)$ and hence this is equivalent to the classical definition (see for example [2]). We put $C_m = \text{mod}(R_m)$.

Let us consider $k[x]$ as being diagonally embedded in $k[x] \oplus k[z]$. That is we identify x with $(x, x)$. Clearly we have

$$(R_m)_x = k[x, x^{-1}] \oplus k[x, x^{-1}]$$

Hence if $M \in \text{Mod}(R_m)$ then $M_x$ is canonically a sum of two $k[x, x^{-1}]$-modules which we denote by $M_x^0$ and $M_x^1$ respectively. This allows us to define the following functor.

$$U_m : C_m \to C : M \mapsto (M, (M^0_x)_0, (M^1_x)_0, \text{id})$$

Clearly $U_m$ is faithful. We have inclusions

$$k[x] \subset \cdots \subset R_{m+1} \subset R_m \subset \cdots \supset R_0 = k[x] \oplus k[z]$$

Dualizing these yield restriction functors

$$C_0 \to \cdots \to C_m \to C_{m+1} \to \cdots \to \text{mod}(k[x])$$

It is clear that these restriction functors are compatible with the functors $(U_m)_m$. Define $C_\infty$ as the 2-direct limit of the $C_m$. That is the objects in $C_\infty$ are the objects in $\bigsqcup_m C_m$ and we put

$$\text{Hom}_{C_\infty}(M, N) = \text{inj lim} \text{Hom}_{C_m}(M, N)$$

The functors $(U_m)_m$ define a functor $U_\infty : C_\infty \to C$.

**Proposition 4.1.** The functor $U_\infty$ defined above is an equivalence.

**Proof.** From the definition it is clear that $U_\infty$ is faithful. So we only have to show that it is full and essentially surjective.

We will first show that $U_\infty$ is full. Let $M, N \in C_m$ and let $f : M \to N$ be a homomorphism in C. So f is in fact a $k[x]$-linear homomorphism $f : M \to N$ such that the localization $f_x$ is $k[x, x^{-1}] \oplus k[x, x^{-1}]$-linear.

Let $y = (x, 0)$. Then $y^n \in R_m$ for $n \geq m$ and $R_n$ as subring of $R_m$ is generated by $x$ and $y^n$. To prove fullness of $U_\infty$ it is sufficient that $f$ is $y^n$-linear for $n \gg 0$. Let
$T$ be the torsion submodule of $N$ and consider the $k[x]$ linear map $M \to N$ given by 
$f^{(n)} = f(y^n) - y^n f(-)$. Since after localizing at $x$, $f$ is $y$ linear, it follows that the image of $f^{(n)}$ lies in $T$. Since $T$ is right bounded it is clear that $f^{(n)}$ must be zero if $n \gg 0$. This proves what we want.

Now we prove essential surjectivity. First let $F \in \mathcal{F}$. Then we claim that $F \subseteq F_x$ is stable under multiplication by $y^n$ for $n \gg 0$. First note that $yF$ is a finitely generated $k[x]$-submodule of $F_x$. Hence $y^n F \subseteq F$. Since $y^n y = y^{n+1}$ this proves what we want.

Now let $T \in \mathcal{T}$. Then as graded $k[x]$-module $T$ has right bounded grading and since $k[x]_{< m} = (R_n)_{< m}$ for $n \geq m$ it follows that for $n \gg 0$ we may consider $T$ as a graded $R_n$-module. This proves what we want.

5. Representation theory

In section we construct the AR-quiver of $C$. From the above discussion it follows that the components of the AR-quiver of $C$ lie either in $\mathcal{T}$ or in $\mathcal{F}$. Since $\mathcal{T}$ is equivalent to the $x$-torsion modules in $\text{gr}(k[x])$ it has a unique component which is $ZA_\infty$. So the main difficulty is represented by the component(s) in $\mathcal{F}$.

We now describe the indecomposable torsion free objects in $C$ as well as the associated Auslander-Reiten quiver (see [5]). Using Proposition 4.1 this could be easily obtained by using a graded version of the results in [2]. However for completeness we give an independent proof here.

For $m > 0$ denote by $F_{ma}$ the unique indecomposable projective $R_m$-module in with grading starting in degree $-a$ (thus $F_{ma} = F_{ma}(a)$). For $m = 0$ we let $F_{00}, F_{01}$ be the two indecomposable $R_0$ modules whose gradings starts exactly at 0. We also put $F_{0a}^i = F_{00}(a)$ (as in Section 3).

Finally to simplify the notation we will write $F_{ma}^i (i = 0, if m \neq 0)$ for $U_m(F_{ma})$.

**Proposition 5.1.** The indecomposable objects in $\mathcal{F}$ are given by $F_{ma}^i$. Furthermore the associated Auslander-Reiten quiver is given by Figure 1

**Proof.** By Serre duality it follows that $\text{Ext}_{C}^1(F_{ma}^i, V F_{ma}^i)$ is one dimensional. Therefore its unique (up to scalar multiplication) non-zero element represents the almost split sequence ending in $F_{ma}^i$.

Let us now explicitly construct non-split extensions between $F_{ma}^i$ and $VF_{ma}^i$. First note 

$$VF_{ma}^i = F_{m,a-1}$$

where for simplicity we have written $F_{0a} = F_{0a}^0 \oplus F_{1a}$, and  

$$VF_{0a}^i = F_{0a-1}^{1-i}$$

To construct the extension associated to $F_{ma}$ we note that $F_{m-1,a-1}$ and $F_{m+1,a}$ are naturally submodules of $F_{m,a}$ whose sum is $F_{m,a}$ and whose intersection is $F_{m,a-1}$. Hence the exact sequence

$$0 \to F_{m,a-1} \to F_{m-1,a-1} \oplus F_{m+1,a} \to F_{m,a} \to 0$$

yields the sought extension.

To construct the extension associated to $F_{0a}^0$ we note that $F_{1a}$ maps surjectively to $F_{0a}^1$ with kernel $F_{0a-1}^1$. Thus in this case the sought extension is

$$0 \to F_{0a-1}^{1-i} \to F_{1a} \oplus F_{0a}^i \to F_{m,a} \to 0$$
It is now easy to assemble the almost split sequences given by (5.1) and (5.2) into the translation quiver given by Figure 1.

To show that Figure 1 is the entire AR-quiver of $\mathcal{F}$ (and not just a component) we have to show that there are no other indecomposable objects.

So assume that $F$ is an indecomposable object in $\mathcal{F}$, not occurring among the $F_{ma}^i$. By lemma 3.4 there exist a non-zero map $F_{oa}^i \to F$ for some $i, a$. Using the defining property of AR-sequences we may use this to construct a non-zero map $F_{mb}^i$ for some $i$ (possibly $0$) and $m$, and for $b$ arbitrarily large.

Now note that the only non-trivial torsion free quotients of $F_{mb}^i$ are $F_{mb}^b$ itself and $F_{ob}^b$ (if $m \neq 0$). Since all these quotients possess a non-trivial element in degree $-b$ it follows that $\text{Hom}_C(F_{mb}^i, F) = 0$ for $b \gg 0$. This finishes the proof. □

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- 112 -
STABLE EQUIVALENCES INDUCED FROM GENERALIZED TILTING MODULES II

TAKAYOSHI WAKAMATSU

1. INTRODUCTION

Let $A$ and $B$ be finite dimensional algebras over a field $K$. We suppose that $B T_A$ is a generalized tilting module and $(\phi, \psi)$ an admissible system for a symmetric algebra, where $A M \otimes_A M_A \xrightarrow{\phi} A M_A$ and $A M \otimes_A M_A \xrightarrow{\psi} A D A$. Then the transformed system $(\phi^T, \psi^T)$ is defined over the bimodule $B M_B^T = B T \otimes_A \text{Hom}_A(T, M)_B$ and we have two symmetric algebras $\Lambda(\phi, \psi) = A \oplus M \oplus DA$ and $\Lambda(\phi^T, \psi^T) = B \oplus M^T \oplus DB$ under the assumption (1) the canonical map $B T \otimes_A \text{Hom}_A(T, M)_B \xrightarrow{\delta} B \text{Hom}_A(T, T \otimes_A M)_B$ defined by $\theta(t \otimes f)(t') = t \otimes f(t')$ is bijective. In the previous note[2], we have shown the existence of a stable equivalence $S : \text{mod} - \Lambda(\phi, \psi) \cong \text{mod} - \Lambda(\phi^T, \psi^T)$ by using the assumptions (2) the class $C(T_A) = \text{gen}^*(T_A) \cap \bigcap_{n>0} \text{KerExt}_A^n(T, ?)$ is contravariantly finite in $\text{mod} - A$ and, dually, $D(DT_B) = \text{cog}^*(DT_B) \cap \bigcap_{n>0} \text{KerExt}_B^n(?, DT)$ covariantly finite in $\text{mod} - B$, and (3) the modules $M_A$ and $T \otimes_A M_A$ are in the class $C(T_A)$. Those assumptions (1) to (3) are satisfied if we suppose

(a) the module $A M_A$ is of the form $\bigoplus_{(X, Y)} A X \otimes_K Y_A$ with all $Y_A$'s are in the class $C(T_A)$, and

(b) one of the algebras $A$ and $B$ is representation-finite.

The purpose of the present note is to give an example of a couple of an admissible system $(\phi, \psi)$ and a generalized tilting module $B T_A$ for which the symmetric algebras $\Lambda(\phi, \psi)$ and $\Lambda(\phi^T, \psi^T)$ are stably equivalent but not derived equivalent. Such an example means that our stable equivalence $S$ is not induced from Morita theory of derived categories.

2. AN EXAMPLE

Define an algebra $A$ by the quiver

\[ Q(A) : \begin{array}{c} 1 \rightarrow 2 \\ \oplus \oplus \\ \alpha \gamma \end{array} \]

with the relations $\alpha^2 = 0$, $\gamma^2 = 0$, $\beta \cdot \alpha = 0$ and $\gamma \cdot \beta = 0$. It is checked that the algebra $A$ is representation-finite with only eight non-isomorphic indecomposable modules. We also

\footnotesize
\[ \text{The detailed version of this paper will be submitted for publication elsewhere.} \]

\large
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have
\[ A_A = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad DA_A = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix} \oplus \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \]

Choose a generalized tilting module as \( T_A = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix} \oplus \begin{pmatrix} 2 \\ 2 \end{pmatrix} \). Then, the quiver \( Q(B) \) of \( B = \text{End}(T_A) \) is given by \( 1 \leftarrow 2 \) and we have
\[
B_B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \oplus \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad DB_B = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \end{pmatrix} \oplus \begin{pmatrix} 2 \\ 2 \end{pmatrix}. \]

Now, we set \( _A\text{M}_A = _A \Lambda e_1 \otimes_K e_1 \text{D}_A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \oplus \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) and \( \varphi = 0 \) the zero map from \( _A\text{M} \otimes_A \text{M}_A \) to \( _A\text{M}_A \). Since our module \( _A\text{M}_A \) is canonically isomorphic to its dual \( _A\text{D}_A \), we have a map \( \psi : _A\text{M} \otimes_A \text{M}_A \rightarrow _A\text{D}_A \) and \( \psi \psi^T = \varphi \). Then, the assumptions (a) and (b) are satisfied and, therefore, the symmetric algebras \( \Lambda = \Lambda(\varphi, \psi) \) and \( \Gamma = \Lambda(\varphi^T, \psi^T) \) are stably equivalent.

In order to prove that the algebras \( \Lambda \) and \( \Gamma \) are not derived equivalent, we use the following well-known result. The proof can be seen in the paper [1] by Usami.

**Lemma 1.** If the algebras \( \Lambda \) and \( \Gamma \) are derived equivalent, there exists a regular matrix \( P \in \text{Mat}_n(\mathbb{Z}) \) and their Cartan matrices satisfy the equation \( ^tP \cdot C_\Lambda \cdot P = C_\Gamma \).

We have \( C_\Lambda = \begin{pmatrix} 8 & 3 \\ 3 & 5 \end{pmatrix} \) since
\[ e_1\text{A}_A = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \oplus \left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \oplus \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \oplus \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]
and
\[ e_2\text{A}_A = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \oplus \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \oplus \begin{pmatrix} 2 \\ 2 \end{pmatrix}. \]

Similarly, we have \( C_\Gamma = \begin{pmatrix} 10 & 3 \\ 3 & 4 \end{pmatrix} \) from
\[ f_1\Gamma_B = \begin{pmatrix} 1 & 2 \\ 1 \end{pmatrix} \oplus \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \oplus \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \oplus \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]
and
\[ f_2\Gamma_B = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \oplus \left\{ 0 \right\} \oplus \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \]
where \( e_i \) (resp. \( f_i \)) stands for the primitive idempotent element in the algebra \( A \) (resp. \( B \)) corresponding to the vertex in \( Q(A) \) (resp. \( Q(B) \)) indexed by \( i \) and \( n \) is the common number of non-isomorphic simple \( A \)- or \( B \)-modules.
Put \( P = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \text{Mat}_2(\mathbb{Z}) \), then we have
\[
{}^tP \cdot C_\Lambda \cdot P = \begin{pmatrix} 8a^2 + 6ab + 5b^2 & 8ac + 3(ad + bc) + 5bd \\ 8ac + 3(ad + bc) + 5bd & 8c^2 + 6cd + 5d^2 \end{pmatrix}.
\]
Hence, \( {}^tP \cdot C_\Lambda \cdot P = C_\Gamma \) implies that
\[
5c^2 + 3(c + d)^2 + 2d^2 = 8c^2 + 6cd + 5d^2 = 4,
\]
and this is impossible for integers \( c, d \in \mathbb{Z} \). Therefore, the algebras \( \Lambda \) and \( \Gamma \) are not derived equivalent by the previous lemma.

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STABILITY OF FROBENIUS ALGEBRAS WITH POSITIVE GALOIS COVERINGS

Kunio Yamagata

Abstract. A finite dimensional self-injective algebra will be determined when it is stably equivalent to a positive self-injective algebra of Dynkin, tilted or quasi-tilted type.

1. Introduction

In a series of joint work with A. Skowroński, we have been studying self-injective algebras with Galois coverings by repetitive algebras [10] [11] [12] [13], and [4]. One of the aims of the work is to characterize self-injective algebras ring theoretically or module categorically which have Galois coverings by the repetitive algebras. A ring theoretical criterion theorem was found in [11] for a self-injective algebra to have a Galois covering by a repetitive algebra. On the other hand, study of the module category over a self-injective algebra depends on the type of a repetitive algebra which defines a Galois covering. In this survey paper, some of main theorems related to module categories in the joint work are arranged into two theorems (Theorems 2, 3). The proofs, however, are not unified (and refer the proofs in the references mentioned above), but a common idea of the proofs is to find an ideal so that a criterion theorem is applicable to conclude the existence of a Galois covering by a repetitive algebra.

Throughout this paper, $K$ will be a fixed (commutative) field, and by an algebra we mean a basic and associative $K$-algebra which is not necessarily finite dimensional, but with a complete set of orthogonal primitive idempotents, that is, with a set of orthogonal primitive idempotents, say $\{e_i\}_{i \in I}$ of an algebra $\Lambda$, such that $\Lambda = \bigoplus_{i \in I} \Lambda e_i = \bigoplus_{i \in I} e_i \Lambda$. A $K$-category $R$ is a category whose hom-sets are $K$-spaces and composition of morphisms are $K$-bilinear. Then, $K$-categories (finite $K$-categories, respectively) are in one-to-one correspondence with algebras having fixed complete set of orthogonal primitive idempotents (finite dimensional algebras with identity, respectively). We freely identify $K$-categories with $K$-algebras. $A$ will be a finite dimensional, connected self-injective algebra, and $B$ and $\Lambda$ will be finite dimensional algebras. Two finite dimensional self-injective algebras $A_1, A_2$ are said to be socle equivalent if $A_1/\soc A_1 \cong A_2/\soc A_2$ where $\soc$ is the socle of an algebra, and stably equivalent if their stable categories $\mod A_1$ and $\mod A_2$ of the categories of finite dimensional modules are isomorphic.

1The paper is in a final form and no version of it will be submitted for publication elsewhere.

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2. REPETITIVE CATEGORIES

2.1. Let $B$ be a $K$-category, and $B_n = B$, $DB_n = DB$, respectively. The repetitive category $\hat{B}$ of $B$ is the direct sum of $K$-spaces $\hat{B} = \bigoplus_{n \in \mathbb{Z}} (B_n \oplus DB_n)$ with multiplication:

$$\left( \sum_i (b_i, f_i) \right) \cdot \left( \sum_j (c_j, g_j) \right) = \sum_i (b_i c_i, b_i g_i + f_i c_{i+1})$$

for $b_i, c_j \in B_i$ and $f_i, g_j \in DB_i$ for $i \in \mathbb{Z}$. The category may be written as a matrix algebra without identity in the following way:

$$\hat{B} = \begin{pmatrix}
\cdots & \cdots & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
B_n & DB_n & B_{n-1} & \cdots \\
0 & \cdots & \cdots & \cdots \\
\end{pmatrix}$$

where all $B_n$ lie on the diagonal, and $\hat{B}$ is the set of those matrices that all but finitely many entries are zero. Summation and multiplication are defined as ones of matrices with $DB \otimes DB \to 0$ zero map. $\hat{B}$ is a $K$-category whose object set is the disjoint union of copies of $\text{Obj}(B)$ in each $B_n$.

2.2. A group $G$ of automorphisms of a $K$-category $\hat{B}$ is said to be admissible if any automorphism of $G$ acts freely on $\text{Obj}(\hat{B})$, and the $G$-orbit sets of $\text{Obj}(\hat{B})$ is finite. The category $\hat{B}/G$ is then naturally defined by the $G$-orbits of $\hat{B}$-objects and $\hat{B}$-morphisms. See [1] for details. The orbit category $\hat{B}/G$ is clearly finite dimensional and self-injective.

An automorphism $\nu_B$ of $\hat{B}$ is called the Nakayama automorphism of $\hat{B}$ if the restriction to any $B_n \oplus DB_n$ is identity onto $B_{n+1} \oplus DB_{n+1}$. An automorphism $\varphi$ of $\hat{B}$ is said to be positive if $\varphi(B_n) \subseteq \bigcup_{i \geq n} (B_i \oplus DB_i)$ ($n \in \mathbb{Z}$) or, equivalently, $\varphi(\text{Obj}(B_n)) \subseteq \bigcup_{i \geq n} \text{Obj}(B_i)$ ($n \in \mathbb{Z}$), and strictly positive if $\varphi$ is positive and $\varphi(B_n) \neq B_n$ for all $n$.

Importance of the orbit categories by repetitive categories may be suggested by the classification theorem of representation-finite self-injective algebras over an algebraically closed field, which was proved by C. Riedtmann [8] [9] (and also by D. Hughes - J. Waschbüsche [3]). Also, D. Happel (1991) showed that the bounded category $\mathcal{D}^b(\text{mod } B)$ of a finite dimensional algebra $B$ is embedded into the stable category $\text{mod } \hat{B}$ of the category of finite dimensional modules over $\hat{B}$. Moreover, $\mathcal{D}^b(\text{mod } B)$ is isomorphic to $\text{mod } \hat{B}$ if and only if $\text{gldim } B < \infty$.

2.3. (1) A typical example of the orbit algebras by repetitive algebras is $\hat{B}/\langle \varphi \rangle$ which is isomorphic to the trivial extension algebra $B \times DB$.

(2) Let $A$ be 4-dimensional, local and self-injective algebras such that

$$A \cong \hat{B}/\langle \varphi \rangle; \quad B = k[x]/(x^2), \quad \varphi^2 = \psi \nu_B$$

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for a positive automorphisms \( \varphi, \psi \), and they are socle equivalent. Those algebras were first presented by T. Nakayama and C. Nesbitt in 1938 [5]. J. Rickard showed that such algebras are isomorphic if they are stably equivalent (see [15] [6]). Thus it should be noted that a socle equivalence does not imply a stable equivalence, in general.

3. **Type of self-injective algebras**

3.1. An algebra \( B \) is called *quasi-tilted algebra* if \( B \cong \text{End}_K(T) \) where \( T \) is a tilting object in a hereditary abelian \( K \)-category \( \mathcal{H} \), that is, \( \text{Hom}_\mathcal{H}(X,Y) \) and \( \text{Ext}^1_\mathcal{H}(X,Y) \) are finite dimensional and \( \text{Ext}^2_\mathcal{H}(X,Y) = 0 \) for all objects \( X, Y \) of \( \mathcal{H} \). This is equivalent to the statement that \( \text{gldim}(B) \leq 2 \) and every finitely generated indecomposable \( B \)-module \( X \) has \( \text{pd}(X) \leq 1 \) or \( \text{id}(X) \leq 1 \). An algebra is said to be *canonical* [7] if its ordinary quiver has a unique source \( \omega \) and a unique sink \( \sigma \), and consists of paths \( p_1, \ldots, p_n \) \( (n \geq 2) \) from \( \omega \) to \( \sigma \), which meet each other only at \( \omega \) and \( \sigma \). Moreover, in case \( n = 2 \) there is no non-trivial relation (i.e., the algebra is hereditary), and in case \( n \geq 3 \) the length of each path is more than 1 and relations are

\[
p_1 + p_2 + p_3 = 0, \quad p_i + \lambda_i p_2 + p_i = 0 \quad (i = 4, \ldots, n)
\]

for \( \lambda_i \neq 0, 1 \) and \( \lambda_i \neq \lambda_j \) for all \( i \neq j \).

**Theorem 1.** [2] An algebra \( B \) is quasi-tilted if and only if \( B \cong \text{End}_K(T) \) where \( T \) is a tilting module over a hereditary algebra or a canonical algebra \( \Lambda \).

3.2. Let \( A \) be a self-injective algebra. \( A \) is said to be *positive* (strictly positive, respectively) if \( A \cong \tilde{B}/(\varphi v_B) \) for some algebra \( B \) and positive (strictly positive, respectively) automorphism \( \varphi \) of \( \tilde{B} \). (See [16].) A positive self-injective algebra \( A \) is said to be of

(i) \( (\Delta) \)-*tilted type* if \( B \) is a tilted algebra of type \( \Delta \),

(ii) *canonical type* if \( B \) is the endomorphism algebra of a tilting module over a canonical algebra,

(iii) *quasi-tilted type* if \( B \) is a quasi-tilted algebra.

Thus, by Theorem 1, a positive self-injective algebra \( A \) is of quasi-tilted type if and only if \( A \) is of tilted type or canonical type.

4. **Main Theorems**

4.1. Some of main results on the module categories in [10] [12] [13] [4] are stated as follows.

**Theorem 2.** A self-injective algebra \( A \) is stably equivalent to a positive self-injective algebra of \( \Delta \)-tilted type if and only if \( A \) is socle equivalent to a positive self-injective algebra of \( \Delta \)-tilted type. Moreover, in case \( K \) is algebraically closed, those statements are equivalent to the statement that \( A \) is isomorphic to a positive self-injective algebra of \( \Delta \)-tilted type.

**Theorem 3.** The followings are equivalent.

1. \( A \) is stably equivalent to a strongly positive self-injective algebra of quasi-tilted type.
2. \( A \) is isomorphic to a strongly positive self-injective algebra of quasi-tilted type.
4.2. Theorem 2 was first proved in [10] for algebras of non-Dynkin type, and later in [13] for algebras of Dynkin type \(\Delta = A_n, B_n, C_n, D_n\) \((n \geq 4), E_6, E_7, E_8, F_4, G_2\). On
the other hand, Theorem 3 was proved in [12] for non-Dynkin type, and recently in [4] for canonical type where a new characterization of a quasi-tilted algebra is given and a precise observation is required for the form of Auslander-Reiten components.

In any case, however, we have to find an ideal \(I\) satisfying the annihilator condition
\[r_A(I) = eI\text{ for some } e = e^2,\]
and then, apply the following criterion theorem for the existence of a positive Galois covering.

Criterion Theorem 4. [11] A self-injective algebra \(\Lambda\) is positive if there is an ideal
\(I\) of \(\Lambda\) such that, for some idempotent \(e\) of \(\Lambda\), the following conditions are satisfied:
1. \(r_\Lambda(I) = eI,\)
2. The canonical algebra epimorphism \(e\Lambda e \to e\Lambda e/eI e\) splits.
In this case, \(\Lambda\) is isomorphic to \(\overline{B}/(\varphi_{\varphi_B})\) for \(B = \Lambda/I\) and a positive automorphism \(\varphi\)
of \(\overline{B}\).

The idea of the application of the criterion theorem is to construct another self-
injective algebra \(B\) by making use of the annihilator condition (1) (see [11]), so that \(\Lambda\)
has both conditions (1) and (2), and \(A\) and \(\Lambda\) are socle equivalent. Thus we can know the
existence of a positive Galois covering of \(A\) up to socle. The converse of Theorem
4 is also true [14] and refer to the survey paper [16].

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FROBENIUS AND QUASI-FROBENIUS PROPERTY FOR mod $\mathcal{C}$

Yuji YOSHINO

§1. Introduction

This is a survey report of my recent work [8], and we shall omit every proofs of the result in this monograph. The reader should refer to the original paper [8] \(^1\) for the detail.

In the following $R$ always denotes a commutative Noetherian ring, and mod $R$ is the category of finitely generated $R$-modules. We are interested in the subcategories $\mathcal{G}$ and $\mathcal{H}$ of mod $R$ that are defined as follows:

Definition 1. $\mathcal{G}$ is defined to be the full subcategory of mod $R$ consisting of all modules $X \in \text{mod } R$ that satisfy

\[ \text{Ext}_R^i(X, R) = 0 \quad \text{and} \quad \text{Ext}_R^i(\text{Tr}X, R) = 0 \quad \text{for any} \quad i > 0. \]

We also define $\mathcal{H}$ to be the full subcategory consisting of all modules with the first half of the above conditions, therefore a module $X \in \text{mod } R$ is an object in $\mathcal{H}$ if and only if

\[ \text{Ext}_R^i(X, R) = 0 \quad \text{for any} \quad i > 0. \]

Note that $\mathcal{G} \subseteq \mathcal{H}$ and that $\mathcal{G}$ is called the subcategory of modules of $G$-dimension zero. See [2] for the $G$-dimension of modules.

Recently, D.Jorgensen and L.M.Sega [5] reported that they constructed an example of an artinian ring $R$, on which $\mathcal{G} \neq \mathcal{H}$. However, we still expect that the equality $\mathcal{G} = \mathcal{H}$ holds in many cases.

The main purpose of this paper is to characterize functorially these two subcategories and to get the conditions under which a subcategory $\mathcal{C}$ of $\mathcal{H}$ is contained in $\mathcal{G}$.

First we settle the notation which we shall use later. When we say $\mathcal{C}$ is a subcategory of mod $R$, we always mean the following:

- $\mathcal{C}$ is essential in mod $R$, i.e. if $X \cong Y$ in mod $R$ and if $X \in \mathcal{C}$, then $Y \in \mathcal{C}$.
- $\mathcal{C}$ is full in mod $R$, i.e. $\text{Hom}_C(X, Y) = \text{Hom}_R(X, Y)$ for $X, Y \in \mathcal{C}$.
- $\mathcal{C}$ is additive and additively closed in mod $R$, i.e. for any $X, Y \in \text{mod } R$, $X \oplus Y \in \mathcal{C}$ if and only if $X \in \mathcal{C}$ and $Y \in \mathcal{C}$.
- $\mathcal{C}$ contains all projective modules in mod $R$.

Of course $\mathcal{G}$ and $\mathcal{H}$ are subcategories in this sense.

Let $\mathcal{C}$ be any subcategory of mod $R$. As in the general notation we denote the associated stable category by $\mathcal{C}$. Of course, there is a natural functor $\mathcal{C} \rightarrow \mathcal{C}$. We should note that the transpose $\text{Tr}$ and the syzygy $\Omega$ are well-defined functors over the stable category $\mathcal{C}$:

\[ \text{Tr}: (\mathcal{C})^{\text{op}} \rightarrow \text{mod } R \quad \Omega: \mathcal{C} \rightarrow \text{mod } R. \]

\(^1\)The detailed version [8] of this paper has been submitted for publication elsewhere.
We also note just from the definition that $\text{Tr}$ gives dualities on $\mathcal{G}$ and also on $\text{mod} R$.

For an additive category $\mathcal{A}$, a contravariant additive functor from $\mathcal{A}$ to the category (Ab) of abelian groups is referred to as an $\mathcal{A}$-module, and a natural transform between two $\mathcal{A}$-modules is referred to as an $\mathcal{A}$-module morphism. We denote by $\text{Mod}\mathcal{A}$ the category consisting of all $\mathcal{A}$-modules and all $\mathcal{A}$-module morphisms. Note that $\text{Mod}\mathcal{A}$ is obviously an abelian category. An $\mathcal{A}$-module $F$ is called finitely presented if there is an exact sequence

$$\text{Hom}_\mathcal{A}(\_ , X_1) \to \text{Hom}_\mathcal{A}(\_ , X_0) \to F \to 0,$$

for some $X_0, X_1 \in \mathcal{A}$. We denote by $\text{mod}\mathcal{A}$ the full subcategory of $\text{Mod}\mathcal{A}$ consisting of all finitely presented $\mathcal{A}$-modules.

It follows easily from Yoneda's lemma that an $\mathcal{A}$-module is projective in $\text{mod}\mathcal{A}$ if and only if it is isomorphic to $\text{Hom}_\mathcal{A}(\_ , X)$ for some $X \in \mathcal{A}$. Also note that the functor $\mathcal{A}$ to $\text{mod}\mathcal{A}$ which sends $X$ to $\text{Hom}_\mathcal{A}(\_ , X)$ is a full embedding.

Now let $\mathcal{C}$ be a subcategory of $\text{mod} R$ and let $\mathcal{C}$ be the associated stable category. Then the category of finitely presented $\mathcal{C}$-modules $\text{mod}\mathcal{C}$ and the category of finitely presented $\mathcal{C}$-modules $\text{mod}\mathcal{C}$ are defined as in the above course. Note that for any $F \in \text{mod}\mathcal{C}$ (resp. $G \in \text{mod}\mathcal{C}$) and for any $X \in \mathcal{C}$ (resp. $X \in \mathcal{C}$), the abelian group $F(X)$ (resp. $G(X)$) has naturally an $R$-module structure, hence $F$ (resp. $G$) is in fact a contravariant additive functor from $\mathcal{C}$ (resp. $\mathcal{C}$) to $\text{mod} R$. As we stated above there is a natural functor $\mathcal{C} \to \mathcal{C}$. We can define from this the functor $\iota : \text{mod}\mathcal{C} \to \text{mod}\mathcal{C}$ by sending $F \in \text{mod}\mathcal{C}$ to the composition functor of $\mathcal{C} \to \mathcal{C}$ with $F$. Then it is well known and is easy to prove that $\iota$ gives an equivalence of categories between $\text{mod}\mathcal{C}$ and the full subcategory of $\text{mod}\mathcal{C}$ consisting of all finitely presented $\mathcal{C}$-modules $F$ with $F(R) = 0$.

§2. Frobenius property of $\text{mod}\mathcal{G}$

Let $\mathcal{C}$ be a subcategory of $\text{mod} R$. We say that $\mathcal{C}$ is closed under kernels of epimorphisms if it satisfies the following condition:

If $0 \to X \to Y \to Z \to 0$ is an exact sequence in $\text{mod} R$, and if $Y, Z \in \mathcal{C}$, then $X \in \mathcal{C}$.

(In Quillen's terminology, all epimorphisms from $\text{mod} R \in \mathcal{C}$ are admissible.)

We say that $\mathcal{C}$ is closed under extension or extension-closed if it satisfies the following condition:

If $0 \to X \to Y \to Z \to 0$ is an exact sequence in $\text{mod} R$, and if $X, Z \in \mathcal{C}$, then $Y \in \mathcal{C}$.

A subcategory $\mathcal{C}$ is said to be a resolving subcategory if it is extension-closed and closed under kernels of epimorphisms. Also $\mathcal{C}$ is said to be closed under $\Omega$ if it satisfies that $\Omega X \in \mathcal{C}$ whenever $X \in \mathcal{C}$. Similarly to this, the closedness under $\text{Tr}$ is defined.

Note that the categories $\mathcal{G}$ and $\mathcal{H}$ are resolving subcategories and that $\mathcal{G}$ is closed under $\text{Tr}$.

We note that, if a subcategory $\mathcal{C}$ of $\text{mod} R$ is closed under kernels of epimorphisms, then it is closed under $\Omega$. And if a subcategory $\mathcal{C}$ of $\text{mod} R$ is extension-closed and closed under $\Omega$, then it is resolving.

The following proposition is shown straightforward from the definitions. Note that the proof of the proposition is completely similar to that of [7, Lemma (4.17)], in which it is
proved that mod\mathcal{C} is an abelian category when \( R \) is a Cohen-Macaulay local ring and \( \mathcal{C} \) is the category of maximal Cohen-Macaulay modules.

**Proposition 2.** Let \( \mathcal{C} \) be a subcategory of \( \text{mod}R \) which is closed under kernels of epimorphisms. Then \( \text{mod}\mathcal{C} \) is an abelian category with enough projectives.

A category \( \mathcal{A} \) is said to be a Frobenius category if it is an abelian category with enough projectives and with enough injectives, and if the class of projective objects in \( \mathcal{A} \) coincides with the class of injective objects in \( \mathcal{A} \). Likewise, a category \( \mathcal{A} \) is said to be a quasi-Frobenius category if it is an abelian category with enough projectives and all projective objects in \( \mathcal{A} \) are injective.

The following theorem is the first result I have got and that motivated me to the detail study on the category \( \text{mod}\mathcal{C} \).

**Theorem 3.** Let \( \mathcal{C} \) be a subcategory of \( \text{mod}R \) that is closed under kernels of epimorphisms. If \( \mathcal{C} \subseteq \mathcal{H} \), then \( \text{mod}\mathcal{C} \) is a quasi-Frobenius category.

The proof of the theorem is not difficult. It is enough to notice that the injective objects in \( \text{mod}\mathcal{C} \) are nothing but half-exact functors as a functor on \( \mathcal{C} \). See [8, Theorem 3.5].

**Theorem 4.** Let \( \mathcal{C} \) be a subcategory of \( \text{mod}R \). And suppose the following conditions.

1. \( \mathcal{C} \) is a resolving subcategory of \( \text{mod}R \).
2. \( \mathcal{C} \subseteq \mathcal{H} \).
3. The functor \( \Omega: \mathcal{C} \to \mathcal{C} \) yields a surjective map on the set of isomorphism classes of the objects in \( \mathcal{C} \).

Then \( \text{mod}\mathcal{C} \) is a Frobenius category. In particular, \( \text{mod}\mathcal{C} \) is a Frobenius category.

From the third assumption in the theorem, the syzygy functor \( \Omega \) gives an automorphism on the category \( \mathcal{C} \), hence there exists a cosyzygy functor \( \Omega^{-1} \). Using this fact we can easily prove the theorem as in the same course of the proof of the previous theorem.

Now let us consider the following four conditions for a resolving subcategory \( \mathcal{C} \) of \( \text{mod}R \):

A. \( \mathcal{C} \) is a subcategory of \( \mathcal{H} \).
B. \( \text{mod}\mathcal{C} \) is a quasi-Frobenius category.
C. \( \text{mod}\mathcal{C} \) is a Frobenius category.
D. \( \mathcal{C} \) is a subcategory of \( \mathcal{G} \).

Then, the following implications hold:

\[ (A) \implies (B) \iff (C) \iff (D) \]

The first implication follows from Theorem 3 and the third will follow from 4 (under a suitable condition on syzygy functor). Of course it is obvious that second implication always holds. Our program is that we analyze closely the reverse implications. Actually, in the next section we shall show that

- \( (B) \implies (A) \) holds if \( R \) is a henselian local ring.
- \( (C) \implies (D) \) holds under the validity of the Auslander-Reiten conjecture.
- \( (B) \implies (C) \) holds if \( \mathcal{C} \) is of finite type (by Nakayama Theorem).

§3. Main theorems

In this section we always assume that \( R \) is a henselian local ring with maximal ideal \( m \) and with the residue class field \( k = R/m \). In the following, what we shall need from
this assumption is the fact that $X \in \text{mod}R$ is indecomposable only if $\text{End}_R(X)$ is a (noncommutative) local ring. In fact it is easy to see that $\text{mod}\mathcal{C}$ is a Krull-Schmidt category for any subcategory $\mathcal{C} \subseteq \text{mod}R$.

We can prove the converse of Theorem 3 under this assumption.

**Theorem 5.** Let $\mathcal{C}$ be a resolving subcategory of $\text{mod}R$, where $R$ is a henselian local ring. Suppose that $\text{mod}\mathcal{C}$ is a quasi-Frobenius category. Then $\mathcal{C} \subseteq \mathcal{H}$.

In a sense $\mathcal{H}$ is the largest resolving subcategory $\mathcal{C}$ of $\text{mod}R$ for which $\text{mod}\mathcal{C}$ is a quasi-Frobenius category.

The proof of this theorem is not so easy. Essential part of the proof is to show that if $\text{mod}\mathcal{C}$ is quasi-Frobenius, then any object $X \in \mathcal{C}$ satisfies $\text{Ext}_R^1(X, R) = 0$. The reader should refer to the paper [8, Theorem 4.2] for the complete proof.

As to the implication $(C) \implies (D)$ in the last paragraph of the previous section, we can show the following result.

**Theorem 6.** Let $R$ be a henselian local ring as above. Suppose that

1. $\mathcal{C}$ is a resolving subcategory of $\text{mod}R$.
2. $\text{mod}\mathcal{C}$ is a Frobenius category.
3. There is no nonprojective module $X \in \mathcal{C}$ with $\text{Ext}_R^1(\_, X)|_{\mathcal{C}} = 0$.

Then $\mathcal{C} \subseteq \mathcal{G}$.

**Remark 7.** We conjecture that $\mathcal{G}$ should be the largest resolving subcategory $\mathcal{C}$ of $\text{mod}R$ such that $\text{mod}\mathcal{C}$ is a Frobenius category.

Theorem 6 together with Theorem 4 say that this is true modulo Auslander-Reiten conjecture:

(AR) If $\text{Ext}_R^i(X, X \oplus R) = 0$ for any $i > 0$ then $X$ should be projective.

In fact, if the conjecture (AR) is true, then the third assumption of Theorem 6 is automatically satisfied.

The proof of Theorem 6 is not short, and we restrict ourselves to say that the following lemma is essential in its proof.

**Lemma 8.** Let $R$ be a henselian local ring and let $\mathcal{C}$ be an extension-closed subcategory of $\text{mod}R$. For objects $X, Y \in \mathcal{C}$, we assume the following:

1. There is a monomorphism $\varphi$ in $\text{Mod}\mathcal{C}$:
   $$\varphi : \text{Hom}_R(\_, Y)|_{\mathcal{C}} \to \text{Ext}_R^1(\_, X)|_{\mathcal{C}}$$

2. $X$ is indecomposable in $\mathcal{C}$.
3. $Y \not\cong 0$ in $\mathcal{C}$.

Then the module $X$ is isomorphic to a direct summand of $\Omega Y$.

Let $\mathcal{A}$ be any additive category. We denote by $\text{Ind}(\mathcal{A})$ the set of nonisomorphic modules which represent all the isomorphism classes of indecomposable objects in $\mathcal{A}$. If $\text{Ind}(\mathcal{A})$ is a finite set, then we say that $\mathcal{A}$ is a category of finite type. The following theorem is a main theorem of the paper [8], which claims that any resolving subcategory of finite type in $\mathcal{H}$ are contained in $\mathcal{G}$. See [8, Theorem 5.5]

**Theorem 9.** Let $R$ be a henselian local ring and let $\mathcal{C}$ be a subcategory of $\text{mod}R$ which satisfies the following conditions.
(1) $\mathcal{C}$ is a resolving subcategory of $\text{mod}R$.
(2) $\mathcal{C} \subseteq \mathcal{H}$.
(3) $\mathcal{C}$ is of finite type.

Then, $\text{mod}\mathcal{C}$ is a Frobenius category and $\mathcal{C} \subseteq \mathcal{G}$.

We should remark about the proof of Theorem 9. Since we assume that $\mathcal{C}$ is of finite type, the category $\text{mod}\mathcal{C}$ is isomorphic to the module category of certain artinian algebra $A$ that is called the Auslander algebra of $\mathcal{C}$:

$$\text{mod}\mathcal{C} \cong \text{mod}A$$

Note that the ring $A$ is a finite (noncommutative) algebra over a commutative artinian ring. Since we assume that $\mathcal{C} \subseteq \mathcal{H}$, we know that $\text{mod}\mathcal{C}$, hence $\text{mod}A$, is a quasi-Frobenius category. (See Theorem 3.) This means that the artinian ring $A$ is left selfinjective. It is known by Nakayama's Theorem (cf. [6] for example) that $A$ is right selfinjective as well, and therefore, using the duality between $\text{mod}A$ and $\text{mod}A^{\text{op}}$, we can conclude that $\text{mod}A$, hence $\text{mod}\mathcal{C}$, is a Frobenius category.

To prove that $\mathcal{C} \subseteq \mathcal{G}$ in the theorem, we use Theorem 6. Actually, since we have shown that $\text{mod}\mathcal{C}$ is a Frobenious category, it is enough to check the following statement:

(*) If $X \in \mathcal{C}$ such that $X \not\cong 0$ in $\mathcal{C}$, then we have $\text{Ext}_R^2(\ , X)|_\mathcal{C} \neq 0$.

This can be proved by using the same idea of Nakayama which we can see in the monograph of Yamagata [6].

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