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第 39 回 環論および表現論シンポジウム報告集

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Organizing Committee of The Symposium on Ring Theory and Representation Theory

The Symposium on Ring Theory and Representation Theory has been held annually in Japan and the Proceedings have been published by the organizing committee. The first Symposium was organized in 1968 by H. Tominaga, H. Tachikawa, M. Harada and S. Endo. After their retirement, a new committee was organized in 1997 for managing the Symposium. The present members of the committee are Y. Hirano (Naruto Univ. of Education), S. Koshitani (Chiba Univ.), K. Nishida (Shinshu Univ.) and M. Sato (Yamanashi Univ.).

The Proceedings of each Symposium is edited by program organizer. Anyone who wants these Proceedings should ask to the program organizer of each Symposium or one of the committee members.

The Symposium in 2007 will be held at Tokyo as the joint Symposium with the 5th China-Japan-Korea international Symposium on ring theory for Sep. 10 – 15.

Concerning several information on ring theory group in Japan containing schedules of meetings and symposiums as well as the addresses of members in the group, you should refer the following homepage, which is arranged by M. Sato (Yamanashi Univ.):

http://fuji.cec.yamanashi.ac.jp/~ring/ (in Japanese)

civil2.cec.yamanashi.ac.jp/~ring/japan/ (in English)

Kenji Nishida
Matsumoto, Japan
December, 2006

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PREFACE

The 39th Symposium on Ring Theory and Representation Theory was held at Hiroshima University on September 16th - 18th, 2006. The symposium and the proceedings are financially supported by Iku Nakamura (Hokkaido University) JSPS Grant-in-Aid for Scientific Research (A), No.16204001, and Kiyoichi Oshiro (Yamaguchi University) JSPS Grant-in-Aid for Scientific Research (B), No.18340011.

This volume consists of the articles presented at the symposuim. We would like to thank all speakers and coauthors for their contributions.

We would also like to express our thanks to all the members of the organizing commitee (Professors Kenji Nishida, Shigeo Koshitani, Masahisa Sato and Yasuyuki Hirano) for their helpful suggestions concerning the symposium. Finally we would like to express our gratitude to Professor Fujio Kubo and his students of Hiroshima University who contributed in the organization of the symposium.

Mamoru Kutami
Yamaguchi, Japan
January, 2007

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Compactly generated triangulated categories

11:30 – 12:10 Jerzy Matczuk（Warsaw University）
On S-Cohn-Jordan extensions

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CHARACTER THEORY OF SEMISIMPLE BI-FROBENIUS ALGEBRAS

YUKIO DOI

ABSTRACT. Bi-Frobenius algebras, or briefly bF algebras, were introduced by the author and Takeuchi in [DT]. They are Frobenius algebras with coalgebra structures, and generalize both finite-dimensional Hopf algebras and scheme rings (Bose-Mesner algebras) of (non-commutative association) schemes. In this paper we discuss the character theory of symmetric Frobenius algebras and its application to our bF algebras. Our approach to the representation theory of symmetric algebras was inspired primarily by the work of Geck-Pfeiffer [GP]. But our methods are very different. We begin with a general discussion of symmetric Frobenius algebras and their properties, and prove the semisimplicity criterion in terms of the volume.

はじめに

バイフロベニウス代数 (簡単に bF 代数) は有限次元ホップ代数の自然な一般化として土井・竹内によって導入された ([DT]). ある種の余代数構造をもつフロベニウス代数のことである. そしてバイフロベニウス代数の重要なクラスに群環的代数 (group-like algebra)([D2]) がある. アソシエーションスキームに付随する Bose-Mesner 代数を抽象化したものである. 本報告では表現論の観点からバイフロベニウス代数についての概説を行う. 予備知識はなるべく仮定しない. その為まず一般のフロベニウス代数の復習から始め (1 節), バイフロベニウス代数の基本性質を解説し (2 節), 双対基底から定まるボリュームの媒介を強調する扱い方により, Geck-Pfeiffer による対称代数の指標理論 ([GP]) の整理改良 (3 節) を行う.

1. フロベニウス代数と対称代数

1.1. フロベニウス代数. k を基礎体とし, Hom や $\otimes$ は k 上でとる. A を (k 上の) 代数とするとき, 双対空間 $A^* = \mathrm{Hom}(A,k)$ は次の作用により両側 A 加群となる:

$$(a \rightharpoonup f)(b) = f(ba), \quad (f \leftharpoonup a)(b) = f(ab) \qquad (a, b \in A).$$

フロベニウス代数とは, 有限次元代数 A と $\phi \in A^*$ の組 (A, ϕ) で写像

$$\theta : A \to A^*,\ a \mapsto (\phi \leftharpoonup a)$$

が全単射になるものをいう. 右 A 加群として A と A^* が同型であるということである. θ の双対写像 $\theta^* : A^{**} \to A^*$ と自然同型 $\iota : A \cong A^{**}$ との合成 $\theta' = \theta \circ \iota$ を計算すると

$$< \theta'(a), b >=< \iota(a), \theta(b) >=< \theta(b), a >=< \phi \leftharpoonup b, a >=< a \rightharpoonup \phi, b > .$$

ただし, 記号 $< f, a >$ は $f(a)$ を表す ($f \in A^*$, $a \in A$). したがって, 写像

$$\theta' : A \to A^*,\ a \mapsto (a \rightharpoonup \phi)$$

The detailed version of this paper will be submitted for publication elsewhere.

が左 A 加群同型を与える．θ は線形同型

$$\Theta : A\otimes A \xrightarrow{\mathrm{id}\otimes\theta} A\otimes A^* \cong \mathrm{End}(A). \tag{1.1}$$

を引き起こし，この同型で id_A に対応する $A\otimes A$ の元 $\sum_i x_i\otimes y_i$ をフロベニウス代数 (A,ϕ) の双対基底とよぶ．定義より

$$\sum_i x_i\phi(y_ia) = a \qquad (a\in A) \tag{1.2}$$

である．$\{x_i\}$ を 1 次独立に選べば $\phi(y_ix_j)=\delta_{ij}$ が成り立つ．また (1.2) は等式

$$\sum_i ax_i\otimes y_i = \sum_i x_i\otimes y_ia \qquad (a\in A), \tag{1.3}$$

を導く．実際，任意の $b\in A$ に対し，

$$\begin{aligned}\Theta(\sum_i ax_i\otimes y_i)(b) &= \sum_i ax_i\phi(y_ib) = ab \qquad (\text{by (1.2)})\\ &= \sum_i x_i\phi(y_iab) = \Theta(\sum_i x_i\otimes y_ia)(b).\end{aligned}$$

よって A の元 $v := \sum_i x_iy_i$ は中心 $Z(A)$ に属す．v をフロベニウス代数 (A,ϕ) のボリュームとよぶ．以上の考察から，

v が可逆元 $\implies$ A は分離的代数 (とくに半単純)

がただちにわかる（$\sum_i v^{-1}x_i\otimes y_i$ がいわゆる分離べき等元になるから）．この逆については 3 節で考察する (ϕ が対称的で k の標数が 0 なら正しい)．

写像 $\overline{\theta} : A^*\to A$ を $\overline{\theta}(f) := \sum_i f(x_i)y_i$ で定義する，ただし $\sum_i x_i\otimes y_i$ は (A,ϕ) の双対基底．このとき，$f\in A^*$，$a\in A$ に対し

$$\begin{aligned}\theta(\overline{\theta}(f))(a) &= (\phi \leftharpoonup \overline{\theta}(f))(a)\\ &= \phi(\sum_i f(x_i)y_ia) = \sum_i f(x_i)\phi(y_ia)\\ &= f(\sum_i x_i\phi(y_ia)) = f(a)\end{aligned}$$

であるから，$\theta\circ\overline{\theta} = \mathrm{id}_{A^*}$．$\theta$ は全単射であるから，$\overline{\theta}$ は θ の逆写像と一致し，等式 $\overline{\theta}\circ\theta = \mathrm{id}_A$ が成り立つ．具体的に書き下して

$$\sum \phi(ax_i)y_i = a \qquad (a\in A). \tag{1.4}$$

を得る．

双対的に，C を (k 上の) 余代数，その余積を $\Delta : C\to C\otimes C$，余単位を $\varepsilon : C\to k$ とする．$c\in C$ に対し，$\Delta(c)$ を

$$\sum_{(c)} c_{(1)}\otimes c_{(2)} \quad \text{または} \quad \sum c_1\otimes c_2$$

で表す．双対空間 C^* は convolution 積 $(f*g)(c)=\sum f(c_1)g(c_2)$ により代数 (ε が単位元) となり，C は次の作用により両側 C^* 加群となる：

$$f \rightharpoonup c := \sum c_1 f(c_2), \quad c \leftharpoonup f := \sum f(c_1)c_2.$$

有限次元余代数 C と C の元 t の組 (C,t) が次の条件をみたすとき，**フロベニウス余代数**であるという：

写像 $\kappa : C^* \to C,\ \kappa(f) = (t \leftharpoonup f)$ が全単射.

これは (C^*,t) がフロベニウス代数であることと同値である．

1.2. **対称フロベニウス代数**．フロベニウス代数 (A,ϕ) に対し，

$$\phi(ab)=\phi(b\mathcal{N}(a)) \qquad (a,b\in A)$$

をみたす代数自己同型 $\mathcal{N}:A\to A$ が一意的に定まる．これを A の ϕ に関する**中山自己同型**という．$\mathcal{N}=id_A$ のとき (すなわち $\phi(ab)=\phi(ba)$)，(A,ϕ) は**対称フロベニウス代数**であるという．ただ単に**対称代数**ということもある．対称性と双対基底の関係について次が成り立つ．

補題 1. (A,ϕ) をフロベニウス代数，$\sum_i x_i\otimes y_i$ をその双対基底とする．(A,ϕ) が対称代数であるための必要十分条件は，

$$\sum_i x_i\otimes y_i = \sum_i y_i\otimes x_i \tag{1.5}$$

が成り立つことである．

証明．(1.5) が成り立つと仮定すれば，任意の $a\in A$ に対し，

$$\mathcal{N}(a) =^{(1.2)} \sum_i x_i\phi(y_i\mathcal{N}(a)) = \sum_i \phi(ay_i)x_i =^{(1.5)} \sum_i \phi(ax_i)y_i =^{(1.4)} a$$

となり，(A,ϕ) は対称代数となる．逆に $\mathcal{N}=\mathrm{id}$ とすれば，

$$\sum_i y_i\phi(x_ia) = \sum_i y_i\phi(ax_i) =^{(1.4)} a$$

であり，これは $\Theta(\sum_i y_i\otimes x_i)=\mathrm{id}$ を意味する．よって双対基底の定義より (1.5) が成り立つ． □

代数 A の右正則表現の指標を χ_A で表す．すなわち $R_a : A\to A,\ b\mapsto ba$ とおき，$\mathrm{Tr}:\mathrm{End}(A)\to k$ を通常のトレース写像としたとき，$\chi_A(a)=\mathrm{Tr}(R_a)$ である．

次の補題 2 は対称代数の指標理論 (3 節) においてキーとなるものである．次のよく知られた事実を用いて証明される：有限次元ベクトル空間 M に対して同一視 $M^*\otimes M\cong \mathrm{End}(M)$，$(f\otimes m)(n):=f(n)m$ を行ったとき，$\mathrm{Tr}(f\otimes m)=f(m)$ が成り立つ．

補題 2. (A,ϕ) を対称代数とし，v をそのボリュームとする．このとき

$$\chi_A(a)=\phi(va) \qquad (a\in A)$$

が成り立つ．とくに，$\phi(v)=\dim A$.

証明. $R_a(b) = ba = \sum_i x_i\phi(y_iba)$ であるから

$$R_a = \sum_i (a \rightharpoonup \phi \leftharpoonup y_i) \otimes x_i \quad \text{in } A^* \otimes A.$$

したがって $\chi_A(a) = \sum_i \phi(y_ix_ia) =^{(1.5)} \sum_i \phi(x_iy_ia) = \phi(va)$. □

2. バイフロベニウス代数

2.1. H を有限次元代数とし，余代数構造 $\Delta: H \to H\otimes H$, $\varepsilon: H \to k$ をもつとする．ただし Δ, ε の代数射は仮定しない．元 $\phi \in H^*$, $t \in H$ を与えたとき，写像 $S: H \to H$ を

$$S(h) := t \leftharpoonup (h \rightharpoonup \phi) = \sum \phi(t_1h)t_2$$

で定義する．次の (BF1) から (BF6) をみたすとき，4組 (H, ϕ, t, S) を**バイフロベニウス代数** (bi-Frobenius algebra) または短く **bF 代数**とよぶ．
(BF1) ε は代数射，すなわち $\varepsilon(hh') = \varepsilon(h)\varepsilon(h')$, $\varepsilon(1) = 1$.
(BF2) H の乗法単位元 1 は $\Delta(1) = 1\otimes 1$ をみたす．
(BF3) (H, ϕ) はフロベニウス代数．
(BF4) (H, t) はフロベニウス余代数．
(BF5) S は反代数射，すなわち $S(hh') = S(h')S(h)$, $S(1) = 1$.
(BF6) S は反余代数射，すなわち $\Delta(S(h)) = \sum S(h_{(2)}) \otimes S(h_{(1)})$, $\varepsilon(S(h)) = \varepsilon(h)$.

S を bF 代数のアンチポードとよぶが，ホップ代数のアンチポードの条件 $S * id = u\circ\varepsilon = id * S$ はみたしていない．$S(h) = t \leftharpoonup (h \rightharpoonup \phi) = (\kappa\circ\theta')(h)$ であり，κ, θ' が全単射であるので S も全単射となる．よって $h = \sum S^{-1}(t_2)\phi(t_1h)$ が任意の $h \in H$ に対していえ，

$$\sum S^{-1}(t_2) \otimes t_1$$

が (H, ϕ) の双対基底を与えることがわかった．．

bF 代数 $H = (H, \phi, t, S)$ は有限次元であるから，その双対空間 H^* はまた代数かつ余代数になっている．積は convolution 積で，余積 $\Delta(f) = \sum f_1 \otimes f_2 \in H^* \otimes H^* = (H\otimes H)^*$ は

$$\Delta f(h \otimes l) = \sum f_1(h)f_2(l) = f(hl), \qquad (f \in H^*,\ h, l \in H)$$

で与えられる．そして4組 (H^*, t, ϕ, S^*) が bF 代数をなすことが容易に確かめられる．これを H の**双対** bF 代数という．

2.2. **bF 代数の例.** (1) 群環. 有限群 G の群環 kG を考える．通常の方法 $\Delta(x) = x\otimes x$, $\varepsilon(x) = 1$ $(x \in G)$ で余代数となる．$\phi \in (kG)^*$ を $\phi(g) = \delta_{1g}$ で定義し，$t := \sum_{g\in G} g \in kG$ とする．この ϕ と t から作られた写像 $S: h \mapsto \sum \phi(t_1h)t_2$ は実は逆元対応 $G \ni x \mapsto x^{-1} \in G$ と一致する．これより (kG, ϕ, t, S) が bF 代数であることがわかる．条件 BF1,2,5,6 は問題ない．BF3,4 のチェックは S が全単射であることからでる．なぜなら，一般の有限次元代数かつ余代数 H があって，ある $\phi \in H^*$, $t \in H$ から作られた写像 $S = \kappa\circ\theta'$ が全単射とすると，θ' κ はともに全単射となる ($\dim H = \dim H^*$ だから)．よって BF3,4 が成り立つ．(kG, ϕ) の双対基底は $\sum_{x\in G} x^{-1}\otimes x$, ボリュームは $|G|\cdot 1$ である．明らかに $\sum_{x\in G} x^{-1}\otimes x = \sum_{x\in G} x\otimes x^{-1}$ であるから，または直接 $\phi(xy) = \phi(yx)$, $(x, y \in G)$ が確かめられて，(kG, ϕ) は対称代数となる．

(2) ホップ代数. H を有限次元ホップ代数とする. H^* および H の右積分の対 (ϕ, t) で $\phi(t) = 1$ をみたすものを選ぶ (選べる). これから得られた写像 S はホップ代数のアンチポード自身と一致する ([DT, 2.4 Example]). よく知られているように有限次元ホップ代数のアンチポードは全単射, したがって BF3,4 をみたし, また BF5,6 もみたしているから, (H, ϕ, t, S) は bF 代数となる. (H, ϕ) の双対基底は bF 代数と同じ $\sum S^{-1}(t_2) \otimes t_1$ で与えられ, ボリュームはアンチポードの性質から

$$v = \sum S^{-1}(t_2)t_1 = S^{-1}(\sum t_1 S(t_2)) = \varepsilon(t)1$$

となる. (注意 : bF 代数のボリュームは $v = \sum S^{-1}(t_2)t_1$ までで, これ以上簡単にならない.)

(3) 群環的代数. A を代数射 $\varepsilon : A \to k$ をもつ有限次元 (k-) 代数とし, 単位元 1 を含む (k-) 基底 $\mathbf{B} = \{b_0 = 1, b_1, \ldots, b_d\}$ が指定されているとする. さらに $S^2 = \mathrm{id}$ をみたす基底要素間の置換 $S : \mathbf{B} \to \mathbf{B}$ が与えられていて次の 3 条件をみたすとき, 4 組 $(A, \varepsilon, \mathbf{B}, S)$ を**群環的代数** (group-like algebra) という. (添え字 i, j, k は 0 から d までを動く)
(G1) $\varepsilon(b_{i^*}) = \varepsilon(b_i) \neq 0$;　ただし b_{i^*} は $S(b_i)$ を表す.
(G2) $p_{ij}^k = p_{j^*i^*}^{k^*}$. ただし p_{ij}^k は $\mathbf{B}$ に関する構造定数, i.e., $b_i b_j = \sum_k p_{ij}^k b_k$.
(G3) $p_{ij}^0 = \delta_{ij^*} \varepsilon(b_i)$.

このとき, 各 b_i に対し $\Delta(b_i) = \frac{1}{\varepsilon(b_i)} b_i \otimes b_i$ とおいて, A は余代数になる (余単位元は初めに与えられた ε). また ϕ, t は群環をまねて $\phi(b_i) = \delta_{0i}$, $t := \sum_i b_i$ とおく. このとき,

$$\sum \phi(t_{(1)} b_j) t_{(2)} = \sum_{i=0}^{d} \frac{1}{\varepsilon(b_i)} \phi(b_i b_j) b_i = \sum_{i=0}^{d} \frac{1}{\varepsilon(b_i)} p_{ij}^0 b_i =^{(G3)} = b_{j^*}$$

となり, (A, ϕ, t, S) は bF 代数になる. (A, ϕ) は双対基底 $\sum_i \frac{1}{\varepsilon(b_i)} b_{i^*} \otimes b_i$ をもち, ボリュームは $v = \sum_i \frac{1}{\varepsilon(b_i)} b_{i^*} b_i$ で明らかに対称代数である.

(4) スキーム環. $k = \mathbb{C}$ (複素数体) とし, $X = \{x_1, \ldots, x_n\}$ を有限集合とする. 直積集合 $X \times X$ の部分集合 (すなわち X の関係) R に対し, その転置 ${}^tR$ を ${}^tR = \{(y, x) | (x, y) \in R\}$ と定義する. また $x \in X$ に対し, $R(x) := \{z \in X \mid (x, z) \in R\}$ とおく. $X \times X$ の (直和) 分割

$$X \times X = R_0 \cup \cdots \cup R_d$$

が以下の 3 条件 (AS1), (AS2), (AS3) をみたすとき, 組 $(X, \{R_i\}_{0 \leq i \leq d})$ を**アソシエーションスキーム** (association scheme) または単に**スキーム**という ([BI],[Z]).

(AS1) $R_0 = \{(x, x) | x \in X\}$.
(AS2) 各 R_i に対し, ${}^t(R_i) = R_j$ をみたす $j \in \{0, \ldots, d\}$ が存在する.
　($j = i^*$ で表す. したがって ${}^t(R_i) = R_{i^*}$, $0^* = 0$ である.)
(AS3) 各 R_i, R_j, R_k および $(x, y) \in R_k$ に対し, $|R_i(x) \cap R_{j^*}(y)|$ が R_k の元 (x, y) の取り方によらず一定値になる. この値を p_{ij}^k で表す.

R_k の**隣接行列** (adjacency matrix) を b_k で表す. すなわち b_k は $n = |X|$ 次正方行列で, その (i, j) 成分は $(x_i, x_j) \in R_k$ のとき 1, $(x_i, x_j) \notin R_k$ のとき 0 である. $b_0 = I$ (単位行

列) であり (AS1), b_i の転置行列 ${}^t(b_i)$ は b_{i^*} である (AS2). また (AS3) から

$$b_i b_j = \sum_{k=0}^{d} p_{ij}^k b_k \qquad (i, j = 0, \ldots, d)$$

が成り立つ. したがって $b_0, \ldots, b_d$ の複素 1 次結合全体 A は全行列環 $M_n(\mathbb{C})$ の部分代数をなす. この代数 A をアソシエーションスキーム $(X, \{R_i\}_{0 \le i \le d})$ に付随する Bose-Mesner 代数または簡単に**スキーム環** (scheme ring) という. $\mathbf{B} = \{b_0, \ldots, b_d\}$ は A の基底であり, さらに行列 b_i の各行にある 1 の個数は一定値 $p_{ii^*}^0$ をとる. この値を通常 a_i で表し, 関係 R_i の**分岐指数** (valency) とよぶ.

$$\varepsilon : A \to k, \ \varepsilon(b_i) = a_i, \quad S : A \to A, \ S(b_i) = b_{i^*} = {}^t(b_i)$$

とおくことで, スキーム環 A は ($\mathbb{C}$ 上の) 群環的代数になることが証明できる. $t = b_0 + \cdots + b_d$ はすべての成分が 1 の行列 J であり, $\varepsilon(t) = |X| = n$ である. A のボリュームは

$$v = I + \frac{1}{a_1} {}^t(b_1) b_1 + \cdots + \frac{1}{a_d} {}^t(b_d) b_d$$

である. 埋め込み $A \hookrightarrow M_n(\mathbb{C})$ をスキーム環 A の**標準表現**とよぶ. その指標は $\varepsilon(t)\phi$ である, ただし $\phi(b_i) = \delta_{0i}$.

一番簡単なスキームの例は, $X \times X$ を対角集合 $\{(x,x) | x \in X\}$ とその補集合に分割するものである (**自明なスキーム**という). 対応する隣接行列は単位行列 I と $b := J - I$ である, ただし J はすべての成分が 1 の (n 次) 正方行列. $(J-I)^2 = J^2 - 2J + I = nJ - 2J + I = (n-1)I + (n-2)(J-I)$ だから, 付随するスキーム環の乗積表は

	1	b
1	1	b
b	b	$(n-1)+(n-2)b$

$\varepsilon(b) = n-1$, $S = id$, $v = 2 + \frac{n-2}{n-1} b$ である.

2.3. **積分について.**. $H = (H, \phi, t, S)$ を一般の bF 代数とする. ホップ代数と同様にして, H に対して積分の概念が定義できる. すなわち H の**右積分**とは $\Gamma h = \varepsilon(h)\Gamma$, $\forall h \in H$ をみたす元 $\Gamma \in H$ をいう. 同様に $h\Lambda = \varepsilon(h)\Lambda$, $\forall h \in H$ をみたす元 $\Lambda \in H$ を H の**左積分**という. H の右 (左) 積分全体の作る部分空間を $\int_H^r$ ($\int_H^l$) で表す. 一般には一致しない. もし $\int_H^r = \int_H^l$ が成り立つとき, H は**単調** (unimodular) であるという. このときは左右の区別がなくなり, 積分空間は単に $\int_H$ で表す.

双対 bF 代数 $H^* = (H^*, t, \phi, S^*)$ の右積分空間は

$$\int_{H^*}^r = \{\gamma \in H^* \mid \sum \gamma(h_1) h_2 = \gamma(h) 1, \ \forall h \in H\}$$

となる. 実際, $\gamma \in \int_{H^*}^r \Leftrightarrow \gamma * f = f(1)\gamma \ (\forall f \in H^*) \Leftrightarrow$
$\sum \gamma(h_1) f(h_2) = f(1)\gamma(h) \ (\forall f \in H^*, \ \forall h \in H) \Leftrightarrow \sum \gamma(h_1) h_2 = \gamma(h) 1 \ (\forall h \in H)$.
同様に, $\int_{H^*}^l = \{\lambda \in H^* \mid \sum h_1 \lambda(h_2) = \lambda(h) 1, \ \forall h \in H\}$.

bF 代数の基本性質を定理 1 およびその系 1 としてまとめる. 証明は一部改良されている.

定理 1. ([DT], [D1], [D2]) (H,ϕ,t,S) を一般の bF 代数とする.
(1) 次が成り立つ.

$$\phi \leftharpoonup t = \varepsilon = S^{-1}(t) \rightharpoonup \phi, \quad \phi(t) = 1 = \phi(S^{-1}(t)), \tag{2.1}$$

$$\sum hS^{-1}(t_2) \otimes t_1 = \sum S^{-1}(t_2) \otimes t_1 h \qquad (h \in H), \tag{2.2}$$

$$\sum \phi(hl_1)S(l_2) = \sum \phi(h_1 l)h_2 \qquad (h,\ l \in H) \tag{2.3}$$

(2) 積分の一意性が成り立つ. くわしくは

$$\int_H^r = kt, \quad \int_H^l = kS^{-1}(t), \quad \int_{H^*}^r = k\phi, \quad \int_{H^*}^l = k(\phi \circ S^{-1}).$$

証明. (1) $\sum S^{-1}(t_2) \otimes t_1$ が (H,ϕ) の双対基底であるから, (1.2), (1.4) より

$$\sum S^{-1}(t_2)\phi(t_1 h) = h = \sum \phi(hS^{-1}(t_2))t_1 \qquad (h \in H).$$

この式に ε を施して $\phi(th) = \varepsilon(h) = \phi(hS^{-1}(t))$. したがって $\phi \leftharpoonup t = \varepsilon = S^{-1}(t) \rightharpoonup \phi$ であり, $h = 1$ を代入すれば $\phi(t) = 1 = \phi(S^{-1}(t))$ を得る.

(2.2) は (1.3) から従う. 次に (2.3) を示す. 双対 bF 代数 H^* に (2.2) を適用すると

$$\sum f * S^{*-1}(\phi_2) \otimes \phi_1 = \sum S^{*-1}(\phi_2) \otimes \phi_1 * f \qquad (f \in H^*).$$

したがって

$$\sum f(x_1)\phi_2(S^{-1}(x_2))\phi_1(y) = \sum \phi_2(S^{-1}(x))\phi_1(y_1)f(y_2) \qquad (f \in H^*,\ x,y \in H)$$

したがって

$$\sum x_1\phi(yS^{-1}(x_2)) = \sum \phi(y_1 S^{-1}(x))y_2 \qquad (x,\ y \in H).$$

ここで $l = S^{-1}(x)$, $h = y$ とおけば (2.3) が得られる. ((2.3) の直接証明については [DT, Proposition 3.2(a)] 参照.)

(2) 任意の右 H 加群 M, 左 H 加群 N に対し,

$$M^H := \{m \in M | m \cdot h = m\varepsilon(h),\ \forall\, h \in H\}, \quad {}^H N := \{n \in N | h \cdot n = \varepsilon(h)n,\ \forall\, h \in H\}$$

とおく. 右積分空間 $\int_H^r$ は右正則加群 H に対する H^H のことである. $M = H^*$ の場合, $(H^*)^H = k\varepsilon$ であることは容易にわかる. この事実と等式 (2.1) の $(\phi \leftharpoonup t) = \varepsilon$, および H と H^* の間の右 H 加群同型 $\theta : H \cong H^*$, $h \mapsto (\phi \leftharpoonup h)$ とを組み合わせることで, $\int_H^r = kt$ を得る. 同様に $\varepsilon = S^{-1}(t) \rightharpoonup \phi$ から $\int_H^l = kS^{-1}(t)$ を得る. この結果を双対 bF 代数 H^* に適用することで $\int_{H^*}^r = k\phi$, $\int_{H^*}^l = k(\phi \circ S^{-1})$ を得る. □

bF 代数 H の任意の元 h に対し, ht はまた右積分になる. よって積分の一意性 (定理 1 の (2)) から, $ht = \alpha(h)t$ をみたす代数射 $\alpha \in \mathrm{Alg}(H,k)$ が存在する. この α を H に対する**右調節関数** (right modular function) という. H が単調であるとは $\alpha = \varepsilon$ のことである. $\phi(t) = 1$ (2.1) より, $\alpha(h) = \phi(\alpha(h)t) = \phi(ht) = (t \rightharpoonup \phi)(h)$. よって $\boxed{\alpha = t \rightharpoonup \phi}$ となる.

双対 bF 代数 H^* の右調節関数は $\Delta(\boldsymbol{a}) = \boldsymbol{a} \otimes \boldsymbol{a}$ をみたす H の元 $\boldsymbol{a} \neq 0$ で,

$$\sum h_1\phi(h_2) = \phi(h)\boldsymbol{a}, \qquad (h \in H)$$

で定義される．具体的には $\boxed{a=\phi\rightharpoonup t}$ である．実際，$a=\phi(t)a=\sum t_1\phi(t_2)=\phi\rightharpoonup t$.

系 1. (H,ϕ,t,S) を bF 代数，α を H の右調節関数とする．
(1) 次の (a)-(c) は同値である．

$$\text{(a) } H \text{ が単調} \qquad \text{(b) } t\in\textstyle\int_H^l, \qquad \text{(c) } t=S(t).$$

(2) H が半単純 $\Longrightarrow$ $\varepsilon(t)\neq 0$ $\Longrightarrow$ H が単調.
(3) H の ϕ に関する中山自己同型は

$$\mathcal{N}(h)=S^{-2}\Big(\sum\alpha(h_1)h_2\Big) \qquad (h\in H).$$

(4) $\phi(hl)=\phi(lh),\ \forall\, h,\ l\in H \iff H$ が単調かつ $S^2=\mathrm{id}$.
(5) $\sum t_1\otimes t_2=\sum t_2\otimes t_1 \iff \phi\in\int_{H^*}^l$ かつ $S^2=\mathrm{id}$.

証明. (1) $(a)\Rightarrow(b)$ は自明. $(b)\Rightarrow(c)$：t が左積分なら，$S^{-1}(t)$ は右積分である (S の反乗法性). よって積分の一意性より，$S^{-1}(t)=\lambda t$ なる $\lambda\in k$ が存在する．この両辺に ϕ をほどこすと，(2.1) より $\lambda=1$ を得る．したがって $S^{-1}(t)=t$ すなわち $S(t)=t$ となる．$(c)\Rightarrow(a)$：$\int_H^r=kt=kS^{-1}(t)=\int_H^l$.

(2) 任意の $h\in H$ に対し,

$$t(ht)=tt\alpha(h)=t\varepsilon(t)\alpha(h),\ (th)t=tt\varepsilon(h)=t\varepsilon(t)\varepsilon(h)$$

が成り立つ．よって $\varepsilon(t)\neq 0$ なら，$\alpha=\varepsilon$ となり H は単調である．次に H が半単純なら $\varepsilon(t)\neq 0$ であることを対偶で示す．もし $\varepsilon(t)=0$ なら，$t^2=t\varepsilon(t)=0$ であるから，kt はゼロでないべき零イデアルとなる．よって H は半単純でない.

(3) (2.3) が使われる.

$$\begin{aligned}
\mathcal{N}(h)&=\sum\phi(t_1\mathcal{N}(h))S^{-1}(t_2) \quad (by\ (1.2))\\
&=\sum\phi(ht_1)S^{-1}(t_2)\\
&=S^{-2}(\sum\phi(ht_1)S(t_2))\\
&=S^{-2}(\sum\phi(h_1t)h_2) \quad (by\ (2.3))\\
&=S^{-2}(\sum\alpha(h_1)h_2) \quad (by\ \phi(t)=1)
\end{aligned}$$

(4) $\Leftarrow$) は (3) から明らか. $\Rightarrow$)：$h=S^{-2}\Big(\sum_{(h)}\alpha(h_1)h_2\Big)$ とすると，$S^2(h)=\sum\alpha(h_1)h_2$ である．これに ε をほどこして，$\varepsilon=\alpha$ がでる．したがって $S^2=\mathrm{id}$ もでる.

(5) は (4) の双対である (H^* に (4) を適用). □

3. 対称フロベニウス代数の指標理論

Geck-Pfeiffer の書物 [GP] の 7 章で一般の対称代数に関する表現論が解説されている．その前半部分である通常表現 (半単純性や指標の直交性など) については我々の方法により著しく改良できることを述べる.

(A,ϕ) を一般の対称 (フロベニウス) 代数とし，$\sum x_i\otimes y_i$ をその双対基底とする．基礎体 k は代数閉体とし，さらに A は半単純であると仮定する．A の既約指標の集合を

Irr(A) で表す．対称代数の定義より，$\phi(ab)=\phi(ba)$ が任意の $a,b\in A$ に対して成り立っている．したがってよく知られているように ϕ は既約指標の 1 次結合で表せる：

$$\phi=\sum_{\chi\in\mathrm{Irr}(A)} n_\chi\chi \qquad (n_\chi\in k). \tag{3.1}$$

（A は行列環の直和であり，各直和成分上の線形写像 $\tau: M_n(k)\to k$ で $\tau(ab)=\tau(ba)$ をみたすものは通常のトレース写像のスカラー倍の形しかないから．）

$\chi\in\mathrm{Irr}(A)$ に対応する中心的 (原始) べき等元を e_χ で表す．任意の χ，$\psi\in\mathrm{Irr}(A)$，$a\in A$ に対して，$\psi(ae_\chi)=\psi(a)\delta_{\chi\psi}$ が成り立つことに注意する．$\phi(ve_\chi)$ を 2 種類の方法で計算していこう，ここで $v:=\sum x_iy_i$ は (A,ϕ) のボリューム．まず，

$$\phi(ve_\chi) =^{(3.1)} \sum_{\psi\in\mathrm{Irr}(A)} n_\psi\psi(ve_\chi)=n_\chi\chi(v).$$

一方，補題 2 より $\phi(ve_\chi)=\chi_A(e_\chi)=\chi(1)^2$ である．これからただちに次の結果を得る：

定理 2. k を代数閉体，(A,ϕ) を半単純な対称代数とする．このとき任意の $\chi\in\mathrm{Irr}(A)$ に対し，

$$n_\chi\chi(v)=\chi(1)^2 \tag{3.2}$$

が成り立つ．

ボリューム $v:=\sum S^{-1}(t_2)t_1$ が可逆なら H は分離代数とくに半単純であった．この逆を考える．

系 2. (A,ϕ) を標数 0 の体 k 上の対称代数とし，v をそのボリュームとする．もし A が半単純なら，v は可逆元になる．k が代数閉体なら，

$$v=\sum_{\chi\in\mathrm{Irr}(A)}\frac{\chi(1)}{n_\chi}e_\chi,\quad v^{-1}=\sum_{\chi\in\mathrm{Irr}(A)}\frac{n_\chi}{\chi(1)}e_\chi. \tag{3.3}$$

証明．係数体を拡大することにより，k は代数閉体と仮定してよい．標数は 0 であるから，(3.2) より $\chi(v)\neq 0$，$n_\chi\neq 0$ である．ボリューム v は中心元であるから，$v=\sum_{\chi\in\mathrm{Irr}(A)}\alpha_\chi e_\chi$ の形にかける，ただし $\alpha_\chi\in k$．このとき，$\chi(v)=\alpha_\chi\chi(e_\chi)=\alpha_\chi\chi(1)$ である．よって (3.2) を用いて $\alpha_\chi=\frac{\chi(v)}{\chi(1)}=\frac{\chi(1)}{n_\chi}$ である．したがって，$v=\sum_{\chi\in\mathrm{Irr}(A)}\frac{\chi(1)}{n_\chi}e_\chi$ を得る．各係数 $\frac{\chi(1)}{n_\chi}$ は 0 でないから，v の可逆性が示された．逆元は $v^{-1}=\sum_{\chi\in\mathrm{Irr}(A)}\frac{n_\chi}{\chi(1)}e_\chi$ で与えられる．□

定理 3. k を標数 0 の代数閉体とする．(A,ϕ) を半単純な対称代数とし，$\sum_i x_i\otimes y_i$ をその双対基底とする．

(1)

$$e_\chi=n_\chi\sum_i\chi(x_i)y_i=\frac{\chi(1)^2}{\chi(v)}\sum_i\chi(x_i)y_i. \tag{3.4}$$

(2)(指標の直交性) χ，$\psi\in\mathrm{Irr}(A)$ に対し，

$$\frac{n_\chi}{\chi(1)}\sum_i\chi(x_i)\psi(y_i)=\frac{\chi(1)}{\chi(v)}\sum_i\chi(x_i)\psi(y_i)=\delta_{\chi\psi} \tag{3.5}$$

証明. (2.4) を使って

$$e_\chi =^{(2.4)} \sum_i \phi(e_\chi x_i)y_i = \sum_i \Big(\sum_{\psi\in \mathrm{Irr}(A)} n_\psi \psi(e_\chi x_i)\Big) y_i = n_\chi \sum_i \chi(x_i)y_i$$

を得る. これに ψ を施して

$$\chi(1)\delta_{\chi\psi} = \psi(e_\chi) = n_\chi \sum_i \chi(x_i)\psi(y_i).$$

□

注意. (1) n_χ の逆数 $\frac{1}{n_\chi}$ を χ に付随する Schur element といい, [GP] では c_χ で表している. (3.2) より, $c_\chi = \frac{\chi(v)}{\chi(1)^2}$ となる.
(2) 等式 $(\chi|\psi) = \sum_i \chi(v^{-1}x_i)\psi(y_i)$ が成り立つ. 実際, (3.3) を用いて

$$\sum_i \chi(v^{-1}x_i)\psi(y_i) = \sum_i \chi\Big(\sum_\rho \frac{n_\rho}{\rho(1)} e_\rho x_i\Big)\psi(y_i) = \frac{n_\chi}{\chi(1)} \sum_i \chi(x_i)\psi(y_i) = (\chi|\psi).$$

最初はこの表示を使って指標の直交性を示した [D2, Theorem 1.5].

例. k を標数 0 の代数閉体とする. 定理 3 において, $(\chi|\psi) = \frac{\chi(1)}{\chi(v)}\sum_i \chi(x_i)\psi(y_i)$ とすれば, $(\chi|\psi) = \delta_{\chi\psi}$ となる.
(1) $A = kG$ (群環) の場合, $v = |G|1$ だから

$$(\chi|\psi) = \frac{1}{|G|}\sum_{x\in G} \chi(x)\psi(x^{-1}).$$

(2) 有限次元半単純ホップ代数 H の場合, 標数 0 の仮定のもと必ず $S^2 = \mathrm{id}$ が成り立つ (Larson-Radford の定理! 難解な論文. 改良された証明が [M] にある). したがって系 1 の (2),(4) より, 任意の $0 \neq \phi \in \int_H^r$ に対して (H, ϕ) は対称代数となる. また $v = \varepsilon(t)1$ だから

$$(\chi|\psi) = \frac{1}{\varepsilon(t)}\sum \chi(S(t_2))\psi(t_1) = \frac{1}{\varepsilon(t)}\sum \chi(t_1)\psi(S(t_2)).$$

(3) 半単純 bF 代数 (H, ϕ, t, S) の場合, ホップ代数のように $S^2 = id$ がいえるかどうかわからない. とりあえず, $S^2 = \mathrm{id}$ すなわち ϕ の対称性を仮定しておく. $v = \sum S^{-1}(t_2)t_1$ であり,

$$(\chi|\psi) = \frac{\chi(1)}{\chi(v)}\sum \chi(S^{-1}(t_2))\psi(t_1).$$

(4) 群環的代数 $(A, \varepsilon, \mathbf{B} = \{b_0, \ldots, b_d\}, S)$ の場合, A が半単純であることと $v = \sum_{i=0}^d \frac{1}{\varepsilon(b_i)} b_{i^*} b_i$ が可逆元であることは同値で,

$$(\chi|\psi) = \frac{\chi(1)}{\chi(v)}\sum_{i=0}^d \frac{1}{\varepsilon(b_i)}\chi(b_{i^*})\psi(b_i).$$

C 上の群環的代数 $(A, \varepsilon, \mathbf{B} = \{b_0, \ldots, b_d\}, S)$ が, あるスキームに付随するスキーム環であるとき, A は**スキーム型** (scheme type) であるとよぶことにする. スキーム型になるための判定条件を求めることは大変重要な問題であると思われる. スキーム型であるためには構造定数 p_{ij}^k がすべて非負整数 ($p_{ii^*}^0 = \varepsilon(b_i)$ は正整数) であることが必要であること

は明らか．しかしこれだけでは十分でない．次の定理の後半はスキーム型であるための有力な必要条件を与える．

定理 4. $(A, \varepsilon, \mathbf{B}, S)$ は $\mathbb{C}$ 上の群環的代数で，各 $\varepsilon(b_i)$ がすべて正実数とする．このとき，A は半単純で，したがって A のボリュームは可逆元である (系 2)．もし A がスキーム型なら，任意の既約指標 χ に対し $m_\chi := \varepsilon(t)n_\chi = \frac{\chi(1)^2}{\chi(v)}$ は正整数である．とくに $\chi(v)$ は有理数．

証明．A の任意の元 $x = \sum_{i=0}^{d} \lambda_i b_i$ に対し，$\overline{x} := \sum_{i=0}^{d} \overline{\lambda_i} b_i$ とおく．ただし，$\overline{\lambda_i}$ は λ_i の複素共役．まず，$x \neq 0$ なら $xS(\overline{x}) \neq 0$ であることを証明する．

$$\phi(xS(\overline{x})) = \sum_{i,j} \lambda_i \overline{\lambda_j} p_{ij^*}^0 = \sum_{i,j} \lambda_i \overline{\lambda_j} \varepsilon(b_i)\delta_{ij} = \sum_i \lambda_i \overline{\lambda_i} \varepsilon(b_i) > 0$$

であり，これからとくに $xS(\overline{x}) \neq 0$ を得る．さて A が半単純でないと仮定すると，(Jacobson) 根基 J はゼロでないべき零イデアルである．$x \neq 0 \in J$ を選び，$y := xS(\overline{x}) \in J$ とおく．上の観察より，$y \neq 0$ であり，しかも

$$S(\overline{y}) = S(\overline{xS(\overline{x})}) = S(\overline{x}S(x)) = S^2(x)S(\overline{x}) = xS(\overline{x}) = y$$

であるから，$y^2 = yS(\overline{y}) \neq 0$ である．この議論を繰り返すことで $y^m \neq 0$ が任意の正整数 m に対して成立する．これは J がべき零イデアルであることに矛盾する．したがって A は半単純になる．

後半は，A がスキーム型とするとき，A の標準表現 (前述) の指標が $\varepsilon(t)\phi = \sum_{\chi \in \mathrm{Irr}(A)} \varepsilon(t) n_\chi \chi$ であることからただちに得られる． □

応用例. 3 次元で $S = \mathrm{id}$ なる群環的代数の構造は決定されており ([D2])，パラメーター $p, q, \beta \in \mathbb{C}$ により $A_{p,q}^{\beta}(3)$ で表される．

$A_{p,q}^{\beta}(3)$	1	b_1	b_2
1	1	b_1	b_2
b_1	b_1	$p + (p - 1 - \beta q)b_1 + \beta p b_2$	$\beta q b_1 + (p - \beta p)b_2$
b_2	b_2	$\beta q b_1 + (p - \beta p)b_2$	$q + (q - \beta q)b_1 + (q - 1 - p + \beta p)b_2$

$\varepsilon(b_1) = p,\ \varepsilon(b_2) = q,\ v = 3 + \frac{2p-1-\beta(p+q)}{p} b_1 + \frac{q-p-1+\beta(p+q)}{q} b_2.$

$\beta = 0$ の場合は，指標の計算は比較的やさしく，指標表は次のようになる：

$A_{p,q}^{0}(3)$	1	b_1	b_2	$\chi(v)$	m_χ
ε	1	p	q	$p+q+1$	1
χ_1	1	-1	0	$\frac{p+1}{p}$	$\frac{p(p+q+1)}{p+1}$
χ_2	1	p	$-1-p$	$\frac{(p+1)(p+q+1)}{q}$	$\frac{q}{p+1}$

$p,\ q \in \mathbb{N},\ p \leq q$ とするとき，定理 4 より，$A_{p,q}^{0}(3)$ がスキーム型なら $p + 1 \mid q$ が必要である．逆にこれが成り立つとき，グループわけスキーム $\mathrm{GD}(\frac{q}{p+1} + 1, p + 1)$ が考えられる ([B, Example 2.2])：一クラスが $p + 1$ 人の生徒からなるクラスが全部で $\frac{q}{p+1} + 1$ 組あるとする．生徒全体の集合を X とする ($|X| = p + q + 1$ である)．$X \times X$ の 3 分割 $R_0 = \{(x, x) | x \in X\}$，$R_1 = \{(x, y) | x \neq y,\ x, y$ は同じクラス$\}$，$R_2 = (R_0 \cup R_1)^c$ (補集合)

はスキームとなる．これに付随するスキーム環は $A^0_{p,q}(3)$ に等しいことが確かめられる．このようにして

$$A^0_{p,q}(3) \text{ がスキーム型} \iff p+1 \mid q.$$

終わりに

このあと，有限群の Frobenius-Schur の定理が標数 0 の一般の代数閉体上の bF 代数に対して拡張できること (ホップ代数に対してはすでに拡張されている [LM])；小さい次元の群環的代数の構造および指標表の決定；アソシエーションスキームへの応用；などの話題が続くのだが，別の機会にゆずることとする．

References

[B] R. A. Bailey, "Association Schemes, -Designed Experiments, Algebra and Combinatorics", Cambridge studies in advanced mathematics 84, Cambridge University Press 2004.

[BI] E. Bannai and T. Ito, "Algebraic Combinatorics I: Association Schemes", Benjamin-Cummings, Menlo Park CA, 1984.

[D1] Y. Doi, *Substructures of bi-Frobenius algebras*, J. Algebra 256 (2002), 568-582.

[D2] Y. Doi, *Bi-Frobenius Algebras and Group-like Algebras*, Lecture notes in pure and applied Mathematics, vol.237 "Hopf Algebras", (2004) 143-155 .

[DT] Y. Doi and M. Takeuchi, *BiFrobenius algebras*, in: Contemporary Mathematics 267 "New trends in Hopf algebras", (2000), 67-97.

[GP] M. Geck and G. Pfeiffer, "Characters of Finite Coxeter Groups and Iwahori-Hecke Algebras", London Mathematical Society Monographs New Series 21, 2000.

[LM] V. Linchenko and S. Montgomery, *A Frobenius-Schur Theorem for Hopf Algebras*, Algrbra and representation Theory 3: 347-377, 2000.

[M] S. Montgomery, *Representation Theory of Semisimple Hopf Algebras*, Roggenkamp and Stefanescu (eds.) Algrbra- Representation Theory, 189-218, 2001.

[Z] P.-H. Zieschang, "Theory of Association Schemes", (Springer Monographs in Mathematics), Springer, 2005.

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LINEARITY DEFECT OF GRADED MODULES OVER KOSZUL ALGEBRAS

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ABSTRACT. We study the *regularities* and *linearity defects* of graded modules over a Koszul algebra. These invariants are closely related to Koszul duality. We mainly consider a Koszul *commutative* algebra A and its dual $A^!$. We also introduce results on monomial ideals in an exterior algebra $E := \bigwedge \langle y_1, \dots, y_n \rangle$, which is a primary example of a Koszul algebra. The linearity defects of monomial ideals in E have combinatorial interest, and the results in this part belong to joint work with R. Okazaki.

1. INTRODUCTION

Let $A = \bigoplus_{i \in \mathbb{N}} A_i$ be a Koszul algebra over a field $K := A_0$, and $^*\mathrm{mod}\, A$ the category of finitely generated graded left A-modules. The Koszul duality is a certain derived equivalence between A and its Koszul dual algebra $A^! := \mathrm{Ext}^\bullet_A(K, K)$.

For $M \in {}^*\mathrm{mod}\, A$, set $\beta_{i,j}(M) := \dim_K \underline{\mathrm{Ext}}^i_A(M, K)_{-j}$. If $P_\bullet : \cdots \to P_1 \to P_0 \to M \to 0$ is a minimal graded free resolution of M, then $P_i \cong \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{i,j}(M)}$. We call

$$\mathrm{reg}_A(M) := \sup\{\, j - i \mid i \in \mathbb{N}, j \in \mathbb{Z} \text{ with } \beta_{i,j}(M) \neq 0 \,\}$$

the *(Castelnuovo-Mumford) regularity* of M. When A is a polynomial ring, $\mathrm{reg}_A(M)$ has been studied by many authors from both geometric and computational interest. Even for a general Koszul algebra A, $\mathrm{reg}_A(M)$ is still an interesting invariant closely related to Koszul duality (see Theorem 5 below).

Let $P_\bullet$ be a minimal graded free resolution of $M \in {}^*\mathrm{mod}\, A$. The *linear part* $\mathrm{lin}(P_\bullet)$ of $P_\bullet$ is the chain complex such that $\mathrm{lin}(P_\bullet)_i = P_i$ for all i and its differential maps are given by erasing all the entries of degree ≥ 2 from the matrices representing the differentials of $P_\bullet$. According to Herzog-Iyengar [7], we call

$$\mathrm{ld}_A(M) := \sup\{\, i \mid H_i(\mathrm{lin}(P_\bullet)) \neq 0 \,\}$$

the *linearity defect* of M. This invariant is related to the regularity via Koszul duality (see Theorem 7 below).

In §4, we study the regularities and linearity defects of modules over a Koszul commutative algebra A or its dual $A^!$. Even in this case, it can occur $\mathrm{ld}_A(M) = \infty$ for some $M \in {}^*\mathrm{mod}\, A$, while Avramov-Eisenbud [1] showed that $\mathrm{reg}_A(M) < \infty$ for all $M \in {}^*\mathrm{mod}\, A$. On the other hand, Herzog-Iyengar [7] proved that if A is complete intersection or Golod then $\mathrm{ld}_A(M) < \infty$ for all $M \in {}^*\mathrm{mod}\, A$. Initiated by these results, we will show the following. Since the results in §4 have not been written elsewhere, we will also give precise proofs.

The detailed versions of this paper will be submitted for publication elsewhere.

Theorem A. *For a Koszul commutative algebra A and $N \in {}^*\mathrm{mod}\, A^!$, we have;*

(1) If $\mathrm{reg}_{A^!}(N) < \infty$, *then* $\mathrm{ld}_{A^!}(N) < \infty$.

(2) If A is a complete intersection, then $\mathrm{reg}_{A^!}(N) < \infty$ *for all* $N \in {}^*\mathrm{mod}\, A^!$.

(3) If A is Golod and N has a finite presentation, then $\mathrm{reg}_{A^!}(N) < \infty$.

Let $E := \bigwedge\langle y_1, \dots, y_n\rangle$ be the exterior algebra. It is a primary example of a Koszul algebra. Eisenbud et. al [5] showed that $\mathrm{ld}_E(N) < \infty$ for all $N \in {}^*\mathrm{mod}\, E$ (now this is a special case of Theorem A). If $n \geq 2$, then $\sup\{\,\mathrm{ld}_E(N) \mid N \in {}^*\mathrm{mod}\, E\} = \infty$. On the other hand, we can prove that there is a uniform bound C such that

$$\mathrm{ld}_E(E/J) < C \qquad \text{for all graded ideals } J \text{ of } E.$$

While we know little about the actual value of C, we can treat $\mathrm{ld}_E(E/J)$ very precisely if $J \subset E$ is a *monomial* ideal. In §5, we collect results in this direction. Most results in this section belong to joint work with R. Okazaki of Osaka university.

For a simplicial complex $\Delta \subset 2^{[n]}$ (here $[n] := \{1, 2, \dots, n\}$), set $J_\Delta := (\prod_{i \in F} y_i \mid F \subset [n], F \not\in \Delta\,)$ to be a monomial ideal of E. Note that any monomial ideal of E is of the form J_Δ for some Δ. Recently, J_Δ has become an important tool of Combinatorial Commutative Algebra.

Theorem B. (Okazaki-Y [10]) *With the above notation, we have the following.*

(1) $\mathrm{ld}_E(E/J_\Delta) \leq \max\{1, n-2\}$.

(2) $\mathrm{ld}_E(J_\Delta)$ *only depends on the topology of the geometric realization* $|\Delta^\vee|$ *of the Alexander dual* $\Delta^\vee$ *of* Δ *(and* $\mathrm{char}(K)$*).*

(3) *If* $n \geq 4$, *we have* $\mathrm{ld}(E/J_\Delta) = n - 2 \iff \Delta$ *is an n-gon.*

2. Koszul Algebras and Koszul Duality

Let $A = \bigoplus_{i \in \mathbb{N}} A_i$ be a graded algebra over a field $K := A_0$ with $\dim_K A_i < \infty$ for all $i \in \mathbb{N}$, ${}^*\mathrm{Mod}\, A$ the category of graded left A-modules, and ${}^*\mathrm{mod}\, A$ the full subcategory of ${}^*\mathrm{Mod}\, A$ consisting of finitely generated modules. We say $M = \bigoplus_{i \in \mathbb{Z}} M_i \in {}^*\mathrm{Mod}\, A$ is *quasi-finite*, if $\dim_k M_i < \infty$ for all i and $M_i = 0$ for $i \ll 0$. If $M \in {}^*\mathrm{mod}\, A$, then it is clearly quasi-finite. We denote the full subcategory of ${}^*\mathrm{Mod}\, A$ consisting of quasi-finite modules by $\mathrm{qf}\, A$. Clearly, $\mathrm{qf}\, A$ is an abelian category with enough projectives. For $M \in {}^*\mathrm{Mod}\, A$ and $j \in \mathbb{Z}$, $M(j)$ denotes the shifted module of M with $M(j)_i = M_{i+j}$. For $M, N \in {}^*\mathrm{Mod}\, A$, set $\underline{\mathrm{Hom}}_A(M, N) := \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{{}^*\mathrm{Mod}\, A}(M, N(i))$ to be a graded k-vector space with $\underline{\mathrm{Hom}}_A(M, N)_i = \mathrm{Hom}_{{}^*\mathrm{Mod}\, A}(M, N(i))$. Similarly, we also define $\underline{\mathrm{Ext}}^i_A(M, N)$.

Set $\mathfrak{m} := \bigoplus_{i > 0} A_i$, and regard $K = A/\mathfrak{m}$ as a graded left A-module. For $M \in \mathrm{qf}\, A$, $i \in \mathbb{N}$ and $j \in \mathbb{Z}$, set

$$\beta_{i,j}(M) := \dim_K \underline{\mathrm{Ext}}^i_A(M, K)_{-j}.$$

Note that $M \in \mathrm{qf}\, A$ has a minimal graded free resolution $P_\bullet : \cdots \to P_1 \to P_0 \to M \to 0$, which is unique up to isomorphism. In this situation, we have $P_i \cong \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{i,j}(M)}$. It is easy to see that $\beta_{i,j}(M) < \infty$ for all i, j. But, if A is not left noetherian, then $\beta_i(M) := \sum_{j \in \mathbb{Z}} \beta_{i,j}(M)$ can be infinite even for $M \in {}^*\mathrm{mod}\, A$.

We say A is *Koszul*, if $\beta_{i,j}(K) \neq 0$ implies $i = j$, in other words, the left A-module K has a graded free resolution of the form

$$\cdots \longrightarrow A(-i)^{\oplus \beta_i} \longrightarrow \cdots \longrightarrow A(-2)^{\oplus \beta_2} \longrightarrow A(-1)^{\oplus \beta_1} \longrightarrow A \longrightarrow K \to 0.$$

Even if we regard K as a right A-module, we get an equivalent definition.

The polynomial ring $K[x_1, \dots, x_n]$ and the exterior algebra $\bigwedge \langle y_1, \dots, y_n \rangle$ are primary examples of Koszul algebras. Of course, there are many other important Koszul algebras. In the noncommutative case, many Koszul algebras are not noetherian.

Koszul duality is a derived equivalence between a Koszul algebra A and its dual $A^!$. A standard reference of this subject is Beilinson et.al [2]. But, in this paper, we follow the convention of Mori [9].

Recall that Yoneda product makes $A^! := \bigoplus_{i \in \mathbb{N}} \underline{\mathrm{Ext}}^i_A(K, K)$ a graded K-algebra. If A is Koszul, then so is $A^!$ and we have $(A^!)^! \cong A$. The Koszul dual of the polynomial ring $S := K[x_1, \dots, x_n]$ is the exterior algebra $E := \bigwedge \langle y_1, \dots, y_n \rangle$. In this case, since S is regular and noetherian, the Koszul duality is very simple. It states an equivalence $\mathcal{D}^b({}^*\mathrm{mod}\, S) \cong \mathcal{D}^b({}^*\mathrm{mod}\, E)$ which is sometimes called *Bernstein-Gel'fand-Gel'fand correspondence* (*BGG correspondence* for short). In the general case, the description of the Koszul duality is slightly technical.

Let $\mathcal{C}(\mathrm{qf}\, A)$ be the homotopy category of cochain complexes in $\mathrm{qf}\, A$, and $\mathcal{C}^{\uparrow}(\mathrm{qf}\, A)$ its full subcategory consisting of complexes $X^\bullet$ satisfying

$$X^i_j = 0 \quad \text{for } i \gg 0 \text{ or } i + j \ll 0.$$

We denote for $\mathcal{D}^{\uparrow}(\mathrm{qf}\, A)$ the localization of $\mathcal{C}^{\uparrow}(\mathrm{qf}\, A)$ at quasi-isomorphisms.

We denote V^* for the dual space of a K-vector space V. Note that if $M \in {}^*\mathrm{Mod}\, A$ then $M^* := \bigoplus_{i \in \mathbb{Z}} (M_{-i})^*$ is a graded right A-module. And we fix a basis $\{x_\lambda\}$ of A_1 and its dual basis $\{y_\lambda\}$ of $(A_1)^* (= (A^!)_1)$. Let $(X^\bullet, \partial) \in \mathcal{C}^{\uparrow}(\mathrm{qf}\, A)$. In this notation, we define the contravariant functor $F_A : \mathcal{C}^{\uparrow}(\mathrm{qf}\, A) \to \mathcal{C}^{\uparrow}(\mathrm{qf}\, A^!)$ as follows.

$$F_A(X^\bullet)^p_q = \bigoplus A^!_{q+j} \otimes_K (X^{j-p}_{-j})^*$$

with the differential $d = d' + d''$ given by

$$d' : A^!_{q+j} \otimes_K (X^{j-p}_{-j})^* \ni a \otimes m \longmapsto (-1)^p \sum y_\lambda a \otimes m x_\lambda \in A^!_{q+j+1} \otimes_K (X^{j-p}_{-j-1})^*$$

and

$$d'' : A^!_{q+j} \otimes_K (X^{j-p}_{-j})^* \ni a \otimes m \longmapsto a \otimes \partial^*(m) \in A^!_{q+j} \otimes_K (X^{j-p-1}_{-j})^*.$$

The contravariant functor $F_{A^!} : \mathcal{C}^{\uparrow}(\mathrm{qf}\, A^!) \to \mathcal{C}^{\uparrow}(\mathrm{qf}\, A)$ is given by the similar way. They induce the contravariant functors $\mathcal{F}_A : \mathcal{D}^{\uparrow}(\mathrm{qf}\, A) \to \mathcal{D}^{\uparrow}(\mathrm{qf}\, A^!)$ and $\mathcal{F}_{A^!} : \mathcal{D}^{\uparrow}(\mathrm{qf}\, A^!) \to \mathcal{D}^{\uparrow}(\mathrm{qf}\, A)$.

Theorem 1. *The contravariant functors $\mathcal{F}_A$ and $\mathcal{F}_{A^!}$ give an equivalence*

$$\mathcal{D}^{\uparrow}(\mathrm{qf}\, A) \cong \mathcal{D}^{\uparrow}(\mathrm{qf}\, A^!)^{\mathrm{op}}.$$

The next result easily follows from Theorem 1 and the fact that $\mathcal{F}_A(K) = A^!$.

Lemma 2 (cf. [9, Lemma 2.8]). *For $M \in \mathrm{qf}\, A$, we have*

$$\beta_{i,j}(M) = \dim H^{i-j}(\mathcal{F}_A(M))_j.$$

3. Castelnuovo-Mumford Regularity and Linearity Defect

Throughout this section, $A = \bigoplus_{i \in \mathbb{N}} A_i$ is a Koszul algebra.

Definition 3. For $M \in \operatorname{qf} A$, we call

$$\operatorname{reg}_A(M) := \sup\{\, j - i \mid i \in \mathbb{N}, j \in \mathbb{Z} \text{ with } \beta_{i,j}(M) \neq 0 \,\}$$

the *(Castelnuovo-Mumford) regularity* of M. For convenience, we set the regularity of the 0 module to be $-\infty$.

If $M \notin {}^*\mathrm{mod}\, A$, then $\beta_{0,j}(M) \neq 0$ for arbitrary large j and $\operatorname{reg}_A(M) = \infty$. So $\operatorname{reg}_A(M)$ is essentially an invariant of $M \in {}^*\mathrm{mod}\, A$. But we regard it as an invariant of $M \in \operatorname{qf} A$ for later convenience. The following is clear.

Lemma 4. *(1) For $M \in \operatorname{qf} A$, we have*

$$\operatorname{reg}_A(M) < \infty \implies \beta_i(M) < \infty \textit{ for all } i \implies M \textit{ has a finite presentation.}$$

(2) Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence in $\operatorname{qf} A$*. If two of M, M' and M'' have finite regularity, so does the third.*

(3) If $M \in {}^*\mathrm{mod}\, A$ has finite length, then $\operatorname{reg}_A(M) = \sup\{\, i \mid M_i \neq 0 \}$.

If A is a polynomial ring $K[x_1, \ldots, x_n]$ (more generally, A is *AS regular*), then $\operatorname{reg}_A(M)$ of $M \in {}^*\mathrm{mod}\, A$ can be defined in terms of the local cohomology modules $H^i_{\mathfrak{m}}(M)$, see [6, 8, 15]. If A is commutative, it is known that $\operatorname{reg}_A(M) < \infty$ for all $M \in {}^*\mathrm{mod}\, A$ (see Theorem 8 below). But this need not be true in the non-commutative case. In fact, if A is not left noetherian, then A has a graded left ideal I such that $\beta_1(A/I) = \infty$. In particular, if A is not left noetherian, then $\operatorname{reg}_A(M) = \infty$ for some $M \in {}^*\mathrm{mod}\, A$. The author does not know any example $M \in {}^*\mathrm{mod}\, A$ such that $\beta_i(M) < \infty$ for all i but $\operatorname{reg}_A(M) = \infty$.

The next result directly follows from Lemma 2.

Theorem 5 (Eisenbud et al [5], Mori [9]). *For $M \in \operatorname{qf} A$, we have*

$$\operatorname{reg}_A(M) = -\inf\{\, i \mid H^i(\mathcal{F}_A(M)) \neq 0 \,\}.$$

Let $P_\bullet : \cdots \to P_1 \to P_0 \to M \to 0$ be a minimal graded free resolution of $M \in \operatorname{qf} A$. The *linear part* $\operatorname{lin}(P_\bullet)$ of $P_\bullet$ is the chain complex such that $\operatorname{lin}(P_\bullet)_i = P_i$ for all i and its differential maps are given by erasing all the entries of degree ≥ 2 from the matrices representing the differentials of $P_\bullet$. It is easy to check that $\operatorname{lin}(P_\bullet)$ is actually a complex, but it is not acyclic in general.

Definition 6 (Herzog-Iyengar [7]). Let $M \in \operatorname{qf} A$ and $P_\bullet$ its minimal graded free resolution. We call

$$\operatorname{ld}_A(M) := \sup\{\, i \mid H_i(\operatorname{lin}(P_\bullet)) \neq 0 \,\}$$

the *linearity defect* of M.

We say $M \in {}^*\mathrm{mod}\, A$ has a *linear free resolution* if there is some $l \in \mathbb{Z}$ such that $\beta_{i,j}(M) \neq 0$ implies that $j - i = l$. It is easy to see that

$$\operatorname{reg}_A(M) = \inf\{\, i \mid M_{\geq i} := \bigoplus_{j \geq i} M_j \text{ has a linear free resolution}\}.$$

For $i \in \mathbb{Z}$ and $M \in \mathrm{qf}\,A$, $M_{(i)}$ denotes the submodule of M generated by the degree i component M_i. We say $M \in \mathrm{qf}\,A$ is *componentwise linear*, if $M_{(i)}$ has a linear free resolution for all $i \in \mathbb{Z}$. For example, if M has a linear free resolution, then it is componentwise linear. Note that M can be componentwise linear even if it is not finitely generated. For example, $\bigoplus_{i\in\mathbb{N}} K(-i)$ is componentwise linear. It is easy to see that $\mathrm{ld}_A(M) = \inf\{\, i \mid \Omega_i(M) \text{ is componentwise linear}\,\}$, here $\Omega_i(M)$ is the ith syzygy of M.

Clearly, we have $\mathrm{ld}_A(M) \leq \mathrm{proj.dim}_A(M)$. The inequality is strict quite often. For example, we have $\mathrm{proj.dim}_A(M) = \infty$ and $\mathrm{ld}_A(M) < \infty$ for many M. On the other hand, sometimes $\mathrm{ld}_A(M) = \infty$.

The next result connects the linearity defect with the regularity via Koszul duality.

Theorem 7 (cf. [15, Theorem 4.7]). *For $M \in \mathrm{qf}\,A$, we have*

$$\mathrm{ld}_A(M) = \sup\{\, \mathrm{reg}_A(H^i(\mathcal{F}_A(M))) + i \mid i \in \mathbb{Z}\}.$$

Proof. For a complex $X^\bullet$, $\mathcal{H}(X^\bullet)$ denotes the complex such that $\mathcal{H}(X^\bullet)^i = H^i(X^\bullet)$ for all i and all differentials are 0. Let $P_\bullet$ be a minimal graded free resolution of M. Then $\mathrm{lin}(P_\bullet)$ is isomorphic to $\mathcal{F}_{A^!}(\mathcal{H}(\mathcal{F}_A(M)))$ (this is proved in [15] under the assumption that A is selfinjective, but the assumption is clearly irrelevant). So the assertion follows from Theorem 5. □

4. Koszul Commutative Algebras and their Quadratic Dual

In this section, we study a Koszul commutative algebra A and its dual $A^!$.

Theorem 8 (Avramov-Eisenbud [1]). *If A is a Koszul commutative algebra, then we have $\mathrm{reg}_A(M) < \infty$ for all $M \in {}^*\mathrm{mod}\,A$.*

On the other hand, even if A is Koszul and commutative, $\mathrm{ld}_A(M)$ can be infinite for some $M \in {}^*\mathrm{mod}\,A$, as pointed out in [7]. In fact, if $\mathrm{ld}_A(M) < \infty$ then the Poincaré series $P_M(t) = \sum_{i\in\mathbb{N}} \beta_i(M) \cdot t^i$ is rational. But there exists a Koszul commutative algebra A such that $P_M(t)$ is not rational for some $M \in {}^*\mathrm{mod}\,A$. But we have the following.

Theorem 9 (Herzog-Iyengar [7]). *Let A be a Koszul commutative algebra.*

(1) *If A is complete intersection, then $\mathrm{ld}_A(M) < \infty$ for all $M \in {}^*\mathrm{mod}\,A$, while $\sup\{\, \mathrm{ld}_A(M) \mid M \in {}^*\mathrm{mod}\,A\,\} = \infty$ in most cases.*
(2) *If A is Golod, then $\mathrm{ld}_A(M) \leq 2 \cdot \dim_K A_1 < \infty$ for all $M \in {}^*\mathrm{mod}\,A$.*

Now we are interested in $\mathrm{reg}_{A^!}(N)$ and $\mathrm{ld}_{A^!}(N)$ for a Koszul commutative algebra A.

Theorem 10. *If A is a Koszul commutative algebra, we have the following.*

(1) *Let $N \in {}^*\mathrm{mod}\,A^!$. If $\mathrm{reg}_{A^!}(N) < \infty$, then $\mathrm{ld}_{A^!}(N) < \infty$.*
(2) *The following conditions are equivalent.*
    (a) *$\mathrm{ld}_A(M) < \infty$ for all $M \in {}^*\mathrm{mod}\,A$.*
    (b) *If $N \in {}^*\mathrm{mod}\,A^!$ has a finite presentation, then $\mathrm{reg}_{A^!}(N) < \infty$.*
(3) *Let $N \in \mathrm{qf}\,A^!$. If there is some $c \in \mathbb{N}$ such that $\dim_K N_i \leq c$ for all $i \in \mathbb{Z}$, then $\mathrm{ld}_{A^!}(N) < \infty$.*

Proof. (1) The complex $F_{A^!}(N)$ is always bounded above. Hence if $\operatorname{reg}_{A^!}(N) < \infty$ then $H^i(\mathcal{F}_{A^!}(N)) \neq 0$ for only finitely many i by Theorem 5. Thus the assertion follows from Theorems 7 and 8.

(2) The implication $(a) \Rightarrow (b)$: First assume that $N \in {}^*\mathrm{mod}\, A^!$ has a finite presentation of the form $A^!(-1)^{\oplus\beta_1} \to A^{!\oplus\beta_0} \to N \to 0$. Then there is $M \in {}^*\mathrm{mod}\, A$ with $M = \bigoplus_{i=0,1} M_i$ such that $\mathcal{F}_A(M)$ gives this presentation. Since $\operatorname{ld}_A(M) < \infty$, we have $\operatorname{reg}_{A^!}(N) < \infty$ by Theorem 7.

If $N \in {}^*\mathrm{mod}\, A^!$ has a finite presentation, then for a sufficiently large s, $N_{\geq s} := \bigoplus_{i \geq s} N_i$ has a presentation of the form $A^!(-s-1)^{\oplus\beta_1} \to A^!(-s)^{\oplus\beta_0} \to N_{\geq s} \to 0$. (To see this, consider the short exact sequence $0 \to N_{\geq s} \to N \to N/N_{\geq s} \to 0$, and use the fact that $\operatorname{reg}_{A^!}(N/N_{\geq s}) < s$.) We have shown that $\operatorname{reg}_{A^!}(N_{\geq s}) < \infty$. So $\operatorname{reg}_{A^!}(N) < \infty$ by the above short exact sequence.

The implication $(b) \Rightarrow (a)$: Set $s := \min\{\, i \mid M_i \neq 0 \,\}$. Then we have a finite presentation $A^! \otimes_K (M_{s+1})^* \to A^! \otimes_K (M_s)^* \to H^{-s}(\mathcal{F}_A(M)) \to 0$. Hence $\operatorname{reg}_{A^!}(H^{-s}(\mathcal{F}_A(M))) < \infty$ by the assumption. Let ∂^i be the differential map of the complex $F_A(M)$. By the exact sequence

$$0 \longrightarrow \operatorname{Im} \partial^{-s-1} \longrightarrow A^! \otimes_K (M_s)^* \longrightarrow H^{-s}(\mathcal{F}_A(M)) \longrightarrow 0,$$

we have $\operatorname{reg}_{A^!}(\operatorname{Im} \partial^{s-1}) < \infty$. Similarly, by the short exact sequence

$$0 \longrightarrow \operatorname{Ker} \partial^{-s-1} \longrightarrow A^! \otimes (M_{s+1})^* \longrightarrow \operatorname{Im} \partial^{-s-1} \longrightarrow 0,$$

we have $\operatorname{reg}_{A^!}(\operatorname{Ker} \partial^{-s-1}) < \infty$. Consider the short exact sequence

$$0 \longrightarrow \operatorname{Im} \partial^{-s-2} \longrightarrow \operatorname{Ker} \partial^{-s+1} \longrightarrow H^{-s-1}(\mathcal{F}_A(M)) \longrightarrow 0.$$

Since there is a surjection $A^! \otimes_K (M_{s+2})^* \to \operatorname{Im} \partial^{-s-2}$, $\operatorname{Im} \partial^{-s-2}$ is finitely generated. Hence $\operatorname{reg}_{A^!}(H^{-s-1}(\mathcal{F}_A(M)))$ has a finite presentation, and its regularity is finite by the assumption. So we also have $\operatorname{reg}_{A^!}(\operatorname{Im} \partial^{-s-2}) < \infty$. Repeating this argument, we can show that $\operatorname{reg}_{A^!}(H^i(\mathcal{F}_A(M))) < \infty$ for all i. On the other hand, by Theorem 5 and Theorem 8, $H^i(\mathcal{F}_A(M)) \neq 0$ for only finitely many i. So the assertion follows from Theorem 7.

(3) Let X be the set of all graded submodules of $A^{\oplus c}$ which are generated by elements of degree 1. By Brodmann [3], there is some $C \in \mathbb{N}$ such that $\operatorname{reg}_A(M) < C$ for all $M \in X$. To prove the theorem, it suffices to show that $\operatorname{reg}_A(H^i(\mathcal{F}_{A^!}(N))) + i < C$ for all i. We may assume that $i = 0$. Note that $H^0(\mathcal{F}_{A^!}(N))$ is the cohomology of the sequence

$$A \otimes_K (N_1)^* \xrightarrow{\partial^{-1}} A \otimes_K (N_0)^* \xrightarrow{\partial^0} A \otimes_K (N_{-1})^*.$$

Since $\operatorname{Im}(\partial^0)(-1)$ is a submodule of $A^{\oplus \dim_K N_{-1}} \subset A^{\oplus c}$ generated by elements of degree 1, we have $\operatorname{reg}_A(\operatorname{Im}(\partial^0)) < C$. Consider the short exact sequence

$$0 \longrightarrow \operatorname{Ker}(\partial^0) \longrightarrow A \otimes_K (N_0)^* \longrightarrow \operatorname{Im}(\partial^0) \longrightarrow 0.$$

Since $\operatorname{reg}_A(A \otimes_K (N_0)^*) = 0$, we have $\operatorname{reg}_A(\operatorname{Ker}(\partial^0)) \leq C$. Similarly, we have $\operatorname{reg}_A(\operatorname{Im}(\partial^{-1})) < C$. By the short exact sequence

$$0 \longrightarrow \operatorname{Im}(\partial^{-1}) \longrightarrow \operatorname{Ker}(\partial^0) \longrightarrow H^0(\mathcal{F}_{A^!}(N)) \longrightarrow 0,$$

we are done. □

Corollary 11. *If A is a Koszul complete intersection, then* $\operatorname{reg}_{A^!}(N) < \infty$ *and* $\operatorname{ld}_{A^!}(N) < \infty$ *for all* $N \in {}^*\mathrm{mod}\, A^!$.

Proof. If $N \in {}^*\mathrm{mod}\, A^!$ has a finite presentation, we have $\mathrm{reg}_{A^!}(N) < \infty$ and $\mathrm{ld}_{A^!}(N) < \infty$ by Theorem 9 (1) and Theorem 10. On the other hand, it is know that $A^!$ is noetherian. Hence all $N \in {}^*\mathrm{mod}\, A^!$ has a finite presentation. So we are done. □

Proposition 12. *Let A be a Koszul commutative algebra which is Golod. If $N \in {}^*\mathrm{mod}\, A^!$ has a presentation of the form $A^!(-1)^{\oplus \beta_1} \to A^{!\,\oplus \beta_0} \to N \to 0$, then* $\mathrm{reg}_{A^!}(N) \leq 2{\cdot}\dim_K A_1$.

Proof. Follows from Theorem 9 (2) and the argument similar to the proof of Theorem 10 (2). □

In the situation of the above proposition, A is not necessarily noetherian. So it can occur $\mathrm{reg}_{A^!}(N) = \infty$ for some $N \in {}^*\mathrm{mod}\, A^!$ even if A is Golod.

5. Linearity Defects of Face Rings

Let $S := K[x_1, \ldots, x_n]$ be the polynomial ring, and $E := \bigwedge \langle y_1, \ldots, y_n \rangle$ the exterior algebra. The next result is now a special case of Theorem 10, but it initiated the study on linearity defect.

Theorem 13 (Eisenbud et. al. [5]). *We have* $\mathrm{ld}_E(M) < \infty$ *for all* $M \in {}^*\mathrm{mod}\, E$.

If $n \geq 2$, there is no uniform bound for $\mathrm{ld}_E(M)$, that is, $\sup\{\, \mathrm{ld}_E(M) \mid M \in {}^*\mathrm{mod}\, E\,\} = \infty$. On the other hand, we have

$$\mathrm{ld}_E(M) \leq c^{n!} \cdot 2^{(n-1)!} \quad (\, c := \max\{\, \dim_K M_i \mid i \in \mathbb{Z}\,\}\,)$$

for $M \in {}^*\mathrm{mod}\, E$. This bound follows from Brodmann's bound for the regularity of $M \in {}^*\mathrm{mod}\, S$. We also remark that the above bound seems very far from sharp. For example, the author does not know a graded ideal $J \subset E$ with $\mathrm{ld}_E(E/J) > n-1$. When J is a monomial ideal, we can actually prove that $\mathrm{ld}_E(E/J) \leq n-1$.

Set $[n] := \{1, 2, \ldots, n\}$. We say $\Delta \subset 2^{[n]}$ is an *(abstract) simplicial complex*, if $F \in \Delta$ and $G \subset F$ imply $G \in \Delta$. For a simplicial complex $\Delta \subset 2^{[n]}$, we have monomial ideals

$$I_\Delta := (\, \prod_{i \in F} x_i \mid F \subset [n], F \not\in \Delta\,) \quad \text{of } S,$$

and

$$J_\Delta := (\, \prod_{i \in F} y_i \mid F \subset [n], F \not\in \Delta\,) \quad \text{of } E.$$

We call $K[\Delta] := S/I_\Delta$ the *Stanley-Reisner ring* of Δ, and $K\langle\Delta\rangle := E/J_\Delta$ the *exterior face ring* of Δ. Both are very important in Combinatorial Commutative Algebra, see [4, 12]. In this section, we introduce the results on the linearity defects of $K[\Delta]$ and $K\langle\Delta\rangle$. See [10] for detail.

Theorem 14 (Okazaki-Y [10]). *For a simplicial complex $\Delta \subset 2^{[n]}$, we have*

$$\mathrm{ld}_E(K\langle\Delta\rangle) = \mathrm{ld}_S(K[\Delta]).$$

There might exists a direct proof of the above result. But, in [10], we use the fact that BGG correspondence $\mathcal{D}^b(^*\mathrm{mod}\, S) \cong \mathcal{D}^b(^*\mathrm{mod}\, S)$ has special meaning for $K[\Delta]$ and $K\langle\Delta\rangle$ (this is the author's previous result, see [13]). From this, we can show that both $\mathrm{ld}_S(K(\Delta))$ and $\mathrm{ld}_E(K\langle\Delta\rangle)$ equal

$$\max\{\, i - \mathrm{depth}_S(\, \mathrm{Ext}_S^{n-i}(I_{\Delta^\vee}, S)\,) \mid 0 \le i \le n \,\}. \tag{5.1}$$

Here

$$\Delta^\vee := \{\, F \subset [n] \mid [n] \setminus F \not\in \Delta \,\}$$

is the *Alexander dual* of Δ (it is easy to check that $\Delta^\vee$ is a simplicial complex again). We also remark that the number in (5.1) is closely related to the notion of *sequentially Cohen-Macaulay modules* (c.f. [12, Theorem 2.11]).

Theorem 14 suggests that we may set

$$\mathrm{ld}(\Delta) := \mathrm{ld}_S(K[\Delta]) = \mathrm{ld}_E(K(\langle\Delta\rangle)).$$

A simplicial complex Δ gives the topological space $|\Delta|$ which is called the *geometric realization* of Δ. In other words, Δ is a "triangulation" of $|\Delta|$. It is well-known that many homological/ring theoretical invariants of $K[\Delta]$ only depend on the topological space $|\Delta|$ (and $\mathrm{char}(K)$). But, for $\mathrm{ld}_S(K[\Delta])$, the Alexander dual $\Delta^\vee$ is essential.

Theorem 15 (Okazaki-Y [10]). *If $\Delta \neq 2^T$ for any $T \subset [n]$, $\mathrm{ld}(\Delta)$ is a topological invariant of the geometric realization $|\Delta^\vee|$ of the Alexander dual $\Delta^\vee$.*

The above result follows from the fact that $\mathrm{ld}(\Delta)$ equals the number given in (5.1) and "sheaf method" in the Stanley-Reisner ring theory, which was introduced by the author ([14]).

As a remark, $\mathrm{ld}(\Delta)$ depends on the characteristic $\mathrm{char}(K)$ of K. In fact, when $|\Delta^\vee|$ is homeomorphic to a real projective plane $\mathbb{P}^2\mathbb{R}$, we have

$$\mathrm{ld}(\Delta) = \begin{cases} 3 & \text{if } \mathrm{char}(K) = 2 \\ 1 & \text{otherwise.} \end{cases}$$

The earlier (and slightly weaker) version of the next result was first given in the thesis of T. Römer [11], and treats $\mathrm{ld}_E(K\langle\Delta\rangle)$. Later, it was improved by the author in [15]. The original proofs were slightly complicated. But, now we can give a simple proof which uses Theorem 14 and the fact that if a free module $S(-i)$ appears in the minimal graded free resolution of $K[\Delta]$ then $i \le n$.

Theorem 16 (Herzog-Römer, Y [15]). *For a simplicial complex $\Delta \subset 2^{[n]}$, we have*

$$\mathrm{ld}(\Delta) \le \max\{\, 1,\, n-2 \,\}.$$

So it is natural to ask which simplicial complex attains the equality $\mathrm{ld}(\Delta) = n-2$. For an answer, the following holds.

Theorem 17 (Okazaki-Y [10]). *If $n \ge 4$, we have $\mathrm{ld}(\Delta) = n-2 \iff \Delta$ is an n-gon (i.e., $|\Delta|$ is a circle).*

To prove the theorem, we use $\mathrm{ld}(\Delta) = \mathrm{ld}_S(K[\Delta])$. If Δ is an n-gon, then $\beta_{n-1}(K[\Delta]) = 0$, $\beta_{n-2,n}(K[\Delta]) \neq 0$ and $\beta_{n-3,n-1}(K[\Delta]) = 0$. Hence we have $[H_{n-2}(\mathrm{lin}(P_\bullet))]_n \neq 0$, where $P_\bullet$ is the minimal graded free resolution of $K[\Delta]$. The proof of the converse can be reduced to the case when $\dim \Delta = 1$ (i.e., Δ is essentially a simple graph). Regarding Δ as a graph, we say a subgraph C of Δ is a *minimal cycle*, if it is a cycle with no chords. In this terminology, Δ is an n-gon if and only if Δ itself is a minimal cycle. Anyway, the assertion essentially comes from the following fact: If $\dim \Delta = 1$, $H_1(\Delta; K)$ is generated by $H_1(C; K)$ for minimal cycles C of Δ, in other words, we have a surjection

$$(5.2) \qquad \bigoplus_{C:\text{minimal cycle}} H_1(C;K) \longrightarrow H_1(\Delta;K) \longrightarrow 0.$$

Example 18. The Alexander dual of the 5-gon is homeomorphic to the Möbius band. So the above theorem states that if $|\Delta^\vee|$ is homeomorphic to the Möbius band then $\mathrm{ld}(\Delta) = 3$, and any triangulation of the Möbius band requires at least 5 points. In this sense, the problem on " a simplicial complex $\Delta \subset 2^{[n]}$ with small $n - \mathrm{ld}(\Delta)$" is weakly related to the classical combinatorial problem on "a triangulation with small number of vertices". For example, if $|\Delta^\vee|$ is homeomorphic to the cylinder or the real projective plane and $\mathrm{char}(K) = 2$, then $\mathrm{ld}(\Delta) = 3$. In both cases, there is a triangulation with 6 vertices (this is the smallest possible number), and then we have $\mathrm{ld}(\Delta) = 3 = n - 3$.

For a simplicial complex $\Delta \subset 2^{[n]}$ and $F \subset [n]$, the restriction $\Delta|_F := \{ G \in \Delta \mid G \subset F \}$ is a simplicial complex again. If $\dim \Delta = 1$ and $\mathrm{ld}(\Delta) \geq 2$, then we have

$$\mathrm{ld}(\Delta) \geq \min\{ \#F - 2 \mid \Delta|_F \text{ is a } \#F\text{-gon} \}.$$

But the inequality can be strict.

References

[1] L. Avramov and D. Eisenbud, *Regularity of modules over a Koszul algebra*, J. Algebra **153** (1992), 85-90.

[2] A. Beilinson, V. Ginzburg and W. Soergel, *Koszul duality patterns in representation theory*, J. Amer. Math. Soc. **9** (1996), 473–5Aesop

[3] M. Brodmann, *Castelnuovo-Mumford regularity and degrees of generators of graded submodules*, Illinois J. Math. **47** (2003), 749–767.

[4] W. Bruns and J. Herzog, Cohen-Macaulay rings, revised edition, Cambridge University Press, 1998.

[5] Eisenbud, G. Fløystad and F.-O. Schreyer, *Sheaf cohomology and free resolutions over exterior algebra*, Trans. Amer. Math. Soc. **355** (2003), 4397-4426.

[6] D. Eisenbud and S. Goto, *Linear free resolutions and minimal multiplicity*, J. Algebra **88** (1984), 89–133.

[7] J. Herzog and S. Iyengar, *Koszul modules*, J. Pure Appl. Algebra **201** (2005), 154–188.

[8] P. Jørgensen, *Linear free resolutions over non-commutative algebras*. Compos. Math. **140** (2004). 1053–1058.

[9] I. Mori, *Rationality of the Poincare series for Koszul algebras*, J. Algebra **276** (2004), 602–624.

[10] R. Okazaki and K. Yanagawa, *Linearity defects of face rings*, preprint (math.AC/0607780).

[11] T. Römer, *On minimal graded free resolutions*, Thesis, University of Essen, 2001.

[12] R. Stanley, Combinatorics and commutative algebra, 2nd ed. PAlcoasscahootath., **41**. Birkhäuser 1996.

[13] K. Yanagawa, *Derived category of squarefree modules and local cohomology with monomial ideal support*, J. Math. Soc. Japan **56** (2004) 289-308.

[14] K. Yanagawa, *Stanley-Reisner rings, sheaves, and Poincaré-Verdier duality,* Math. Res. Lett. **10** (2003) 635–650.
[15] K. Yanagawa, *Castelnuovo-Mumford regularity for complexes and weakly Koszul modules*, J. Pure and Appl. Algebra **207** (2006), 77–97.

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LONG EXACT SEQUENCES COMING FROM TRIANGLES

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ABSTRACT. Suppose we are given a homological functor, from a triangulated to an abelian category. It takes triangles to long exact sequences. It turns out that not every long exact sequence can occur; there are restrictions.

CONTENTS

0. INTRODUCTION

Suppose $\mathcal{A}$ is a sufficiently nice abelian category, so that it has a derived category $\mathbf{D}(\mathcal{A})$. This will happen, for example, if $\mathcal{A}$ has enough projectives, or if it has enough injectives; for details see Hartshorne [2] or Verdier [3, 4]. Given a distinguished triangle in $\mathbf{D}(\mathcal{A})$

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X ,$$

we can form the long exact sequence in cohomology. We deduce in $\mathcal{A}$ a long exact sequence

$$\cdots \longrightarrow H^{-1}(Z) \longrightarrow H^0(X) \longrightarrow H^0(Y) \longrightarrow H^0(Z) \longrightarrow H^1(X) \longrightarrow \cdots$$

We can wonder what long exact sequences can be obtained this way.

It is clear that any sequence of length four is obtainable. If we have an exact sequence in $\mathcal{A}$

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow 0$$

then it is very easy to deal with it; consider B and C as objects of $\mathcal{A} \subset \mathbf{D}(\mathcal{A})$, and complete the morphism $B \longrightarrow C$ into a triangle in $\mathbf{D}(\mathcal{A})$. The reader can easily check that the long exact sequence, obtained from the functor H applied to this triangle, is nothing other than

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow 0 .$$

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The remarkable fact, which I do not fully understand, is what comes next. It turns out that not all sequences of length five

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow E \longrightarrow 0$$

are the long exact sequences of triangles. Any exact sequence of length five defines a class in $\mathrm{Ext}^3_{\mathcal{A}}(E, A)$, or equivalently a morphism $E \longrightarrow \Sigma^3 A$ in $\mathbf{D}(\mathcal{A})$. It turns out that the sequence will be the long exact sequence of a triangle if and only if this class in $\mathrm{Ext}^3_{\mathcal{A}}(E, A)$ vanishes. In this article I will only prove the necessity, but the sufficiency is easy enough.

More is true. Given any distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ in the derived category $\mathbf{D}(\mathcal{A})$, we can look at its long exact sequence in cohomology. It can be chopped into bits of length five, for example

$$0 \longrightarrow K \longrightarrow H^0(X) \xrightarrow{H^0(u)} H^0(Y) \xrightarrow{H^0(v)} H^0(Z) \longrightarrow Q \longrightarrow 0\,,$$

where K is the kernel of $H^0(u) : H^0(X) \longrightarrow H^0(Y)$, while Q is cokernel of $H^0(v) : H^0(Y) \longrightarrow H^0(Z)$. We will prove that, for every such length-five bit, the corresponding element in $\mathrm{Ext}^3_{\mathcal{A}}(Q, K)$ vanishes. I know that this vanishing is necessary, but have no idea whether it suffices. In other words, I do not know whether it characterizes the long exact sequences coming from triangles in $\mathbf{D}(\mathcal{A})$.

In the proof we will be slightly more general. We will start with an arbitrary triangulated category $\mathcal{T}$, possessing a t–structure; the reader is referred to Beilinson, Bernstein and Deligne [1] for the definitions and elementary properties of t–structures. We will let $\mathcal{A}$ be the heart of the t–structure. We will assume that $\mathcal{T}$ is nice enough so that the inclusion $\mathcal{A} \longrightarrow \mathcal{T}$ factors through a triangulated functor $F : \mathbf{D}^b(\mathcal{A}) \longrightarrow \mathcal{T}$; this is a very weak hypothesis, usually satisfied. We recall that, for any pair of objects $A, B \in \mathcal{A}$, we have

$$\mathrm{Hom}_{\mathbf{D}^b(\mathcal{A})}(A, B) = \mathrm{Hom}_{\mathcal{T}}(A, B)\,, \qquad \mathrm{Hom}_{\mathbf{D}^b(\mathcal{A})}(A, \Sigma B) = \mathrm{Hom}_{\mathcal{T}}(A, \Sigma B)\,.$$

The reason for the first equality is that $\mathcal{A}$ embeds fully faithfully in both $\mathbf{D}^b(\mathcal{A})$ and $\mathcal{T}$, and the second equality is because both groups classify extensions $0 \longrightarrow B \longrightarrow E \longrightarrow A \longrightarrow 0$ in $\mathcal{A}$. But it is perfectly possible for a non-zero morphism $\alpha : A \longrightarrow \Sigma^n B$, in the category $\mathbf{D}^b(\mathcal{A})$, to map to zero in $\mathcal{T}$; all we learn, from the discussion above, is that this can only happen if $n \geq 2$.

What we will prove, in the generality of triangulated categories $\mathcal{T}$ with t–structures, is the following. Given any triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ in the category $\mathcal{T}$, we can still look at its long exact sequence in cohomology. It can still be chopped into bits of length five, for example

$$0 \longrightarrow K \longrightarrow H^0(X) \xrightarrow{H^0(u)} H^0(Y) \xrightarrow{H^0(v)} H^0(Z) \longrightarrow Q \longrightarrow 0\,.$$

Every such length-five bit corresponds to an element in $\mathrm{Ext}^3_{\mathcal{A}}(Q, K)$, that is to a morphism $\alpha : Q \longrightarrow \Sigma^3 K$ in $\mathbf{D}^b(\mathcal{A})$. We will prove that the functor $F : \mathbf{D}^b(\mathcal{A}) \longrightarrow \mathcal{T}$ must take α to zero.

1. The proof

Before all else we need to fix our conventions for this section.

Notation 1.1. Let $\mathcal{T}$ be a triangulated category with a t-structure. Let $\mathcal{A}$ be the heart of this t-structure. Assume that the category $\mathcal{T}$ is "natural" enough so that the embedding of $\mathcal{A}$ into $\mathcal{T}$ extends to a triangulated functor $F : \mathbf{D}^b(\mathcal{A}) \longrightarrow \mathcal{T}$. We fix these assumptions throughout the section.

Let us also fix the notation that $H : \mathcal{T} \longrightarrow \mathcal{A}$ will be the homological functor sending an object $X \in \mathcal{T}$ to the truncation $H(X) = \left(X^{\leq 0}\right)^{\geq 0}$. We will let $H^n(X) = H(\Sigma^n X)$.

With these conventions, we are ready to state and prove our main observation:

Lemma 1.2. *Let $X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$ be a triangle in $\mathcal{T}$, and suppose that*

(1) X and Y lie in $\mathcal{T}^{\leq 1} \cap \mathcal{T}^{\geq 0}$.

(2) Z lies in $\mathcal{A} = \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$.

This implies that the functor H sends the triangle to the long exact sequence

$$(*) \qquad 0 \longrightarrow H^0(X) \longrightarrow H^0(Y) \longrightarrow H^0(Z) \longrightarrow H^1(X) \longrightarrow H^1(Y) \longrightarrow 0$$

with all the other terms vanishing. In the abelian category $\mathcal{A}$, this 5-term exact sequence defines a class in $\mathrm{Ext}^3_{\mathcal{A}}\big(H^1(Y), H^0(X)\big)$. This class can also be viewed as a morphism $\alpha : H^1(Y) \longrightarrow \Sigma^3 H^0(X)$, in the derived category $\mathbf{D}^b(\mathcal{A})$.

We assert that, under the functor $F : \mathbf{D}^b(\mathcal{A}) \longrightarrow \mathcal{T}$, the image of α vanishes.

Proof. Consider the commutative square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X^{\geq 1} & \longrightarrow & Y^{\geq 1} \end{array} .$$

We may complete to a 3×3 diagram, were the rows and columns are triangles

$$\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X^{\geq 1} & \longrightarrow & Y^{\geq 1} & \longrightarrow & I & \longrightarrow & \Sigma X^{\geq 1} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma X^{\leq 0} & \longrightarrow & \Sigma Y^{\leq 0} & \longrightarrow & \Sigma K & \longrightarrow & \Sigma^2 X^{\leq 0} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma X & \longrightarrow & \Sigma Y & \longrightarrow & \Sigma Z & \longrightarrow & \Sigma^2 X
\end{array},$$

and the proof is by studying this 3×3 diagram. In the second row, we have that $\Sigma X = H^1(X)$ and $\Sigma Y = H^1(Y)$ are both in $\mathcal{A} \subset \mathcal{T}$, and that the morphism $\Sigma X \longrightarrow \Sigma Y$ is surjective; it is the morphism $H^1(X) \longrightarrow H^1(Y)$ in the long exact sequence (*) of the lemma. The second row reduces to the short exact sequence in $\mathcal{A}$

$$0 \longrightarrow I \longrightarrow H^1(X) \longrightarrow H^1(Y) \longrightarrow 0\ ,$$

and the map $Y^{\geq 1} \longrightarrow I$ is the image, under the functor $F : \mathbf{D}^b(\mathcal{A}) \longrightarrow \mathcal{T}$, of the morphism in $\mathbf{D}^b(\mathcal{A})$ defining the extension $0 \longrightarrow I \longrightarrow H^1(X) \longrightarrow H^1(Y) \longrightarrow 0$.

So much for the second row. Now look at the commutative diagram

$$\begin{array}{ccccc}
Z & \longrightarrow & \Sigma X & & \\
\downarrow & & \downarrow & & \\
I & \longrightarrow & \Sigma X^{\geq 1} & \longrightarrow & \Sigma Y^{\geq 1}
\end{array}.$$

If we apply to it the functor H, we discover the diagram

$$\begin{array}{ccccc}
H^0(Z) & \longrightarrow & H^1(X) & & \\
\downarrow & & \| & & \\
H^0(I) & \longrightarrow & H^1(X) & \longrightarrow & H^1(Y)
\end{array}.$$

Both Z and I lie in the heart $\mathcal{A}$, and the diagram above identifies for us the map $Z \longrightarrow I$ as the factorization of the morphism from $Z = H^0(Z)$ to $H^1(X)$ through the kernel of $H^1(X) \longrightarrow H^1(Y)$, which is the image of $Z \longrightarrow H^1(X)$. Now the column

$$K \longrightarrow Z \longrightarrow I \longrightarrow \Sigma K$$

is a triangle, which reduces to the short exact sequence $0 \longrightarrow K \longrightarrow Z \longrightarrow I \longrightarrow 0$ in $\mathcal{A} \subset \mathcal{T}$. We also learn that the map $I \longrightarrow \Sigma K$ is the image, under the functor $F : \mathbf{D}^b(\mathcal{A}) \longrightarrow \mathcal{T}$, of the morphism in $\mathbf{D}^b(\mathcal{A})$ corresponding to the extension.

Next consider the commutative square

$$\begin{array}{ccc} Y^{\leq 0} & \longrightarrow & K \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array} \quad .$$

If we apply the functor H we learn that the map $Y^{\leq 0} \longrightarrow K$, which is a map between objects in $\mathcal{A}$, is just the factorization through K of the morphism $H^0(Y) \longrightarrow H^0(Z) = Z$. The triangle

$$X^{\leq 0} \longrightarrow Y^{\leq 0} \longrightarrow K \longrightarrow \Sigma X^{\leq 0}$$

is therefore nothing fancy; it is simply the exact sequence $0 \longrightarrow H^0(X) \longrightarrow H^0(Y) \longrightarrow K \longrightarrow 0$ in $\mathcal{A}$. Moreover, the map $K \longrightarrow \Sigma X^{\leq 0}$ is just exactly the image, under the functor $F : \mathbf{D}^b(\mathcal{A}) \longrightarrow \mathcal{T}$, of the morphism in $\mathbf{D}^b(\mathcal{A})$ corresponding to the extension.

What we have learned so far is that three of the six triangles, in our 3×3 diagram, amount to short exact sequences in $\mathcal{A}$

$$\begin{array}{l} 0 \longrightarrow H^0(X) \longrightarrow H^0(Y) \longrightarrow K \longrightarrow 0 \\ 0 \longrightarrow K \longrightarrow Z \longrightarrow I \longrightarrow 0 \\ 0 \longrightarrow I \longrightarrow H^1(X) \longrightarrow H^1(Y) \longrightarrow 0 \end{array} \quad .$$

Moreover, the differentials of these triangles are the classes of the three extensions, and are also part of our 3×3 diagram. The composite of these three differentials is the map

$$\begin{array}{ccc} Y^{\geq 1} & \longrightarrow & I \\ & & \downarrow \\ & & \Sigma K & \longrightarrow & \Sigma^2 X^{\leq 0} \end{array} \quad ,$$

which the reader will find in our diagram. The commutativity of

$$\begin{array}{ccccc} Y^{\geq 1} & \longrightarrow & I & \longrightarrow & \Sigma X^{\geq 1} \\ & & \downarrow & & \downarrow \\ & & \Sigma K & \longrightarrow & \Sigma^2 X^{\leq 0} \end{array} \quad ,$$

coupled with the vanishing of $Y^{\geq 1} \longrightarrow I \longrightarrow \Sigma X^{\geq 1}$, tells us that this composite vanishes. In the category $\mathcal{T}$ the three extensions compose to zero. □

Proposition 1.3. *Let the conventions be as in Notation 1.1. Suppose $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ is a triangle in $\mathcal{T}$. Complete $H^0(X) \longrightarrow H^0(Y) \longrightarrow H^0(Z)$ to an exact sequence*

$$0 \longrightarrow K \longrightarrow H^0(X) \xrightarrow{H^0(u)} H^0(Y) \xrightarrow{H^0(v)} H^0(Z) \longrightarrow Q \longrightarrow 0 \ ,$$

where K must be the kernel of $H^0(u) : H^0(X) \longrightarrow H^0(Y)$, while Q is forced to be the cokernel of $H^0(v) : H^0(Y) \longrightarrow H^0(Z)$. The sequence defines an element in $\mathrm{Ext}^3_{\mathcal{A}}(Q, K)$, or equivalently a morphism $\alpha : Q \longrightarrow \Sigma^3 K$ in $\mathbf{D}^b(\mathcal{A})$.

We assert that the functor $F : \mathbf{D}^b(\mathcal{A}) \longrightarrow \mathcal{T}$ takes α to zero.

Proof. Consider the commutative square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X^{\geq 0} & \longrightarrow & Y^{\geq 0} \end{array} .$$

It may be extended to a morphism of triangles, which we will write

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X' \end{array} .$$

That is $X' = X^{\geq 0}$ and $Y' = Y^{\geq 0}$. We have

(1) X' and Y' belong to $\mathcal{T}^{\geq 0}$, while Z' belongs to $\mathcal{T}^{\geq -1}$.

(2) The three maps

$$H^0(X) \longrightarrow H^0(X'), \qquad H^0(Y) \longrightarrow H^0(Y'), \qquad H^0(Z) \longrightarrow H^0(Z')$$

are all isomorphisms. For $X' = X^{\geq 0}$ and $Y' = Y^{\geq 0}$ this is obvious, by the definition of the functor H in terms of truncations. For Z' consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} H^0(X) & \longrightarrow & H^0(Y) & \longrightarrow & H^0(Z) & \longrightarrow & H^1(X) & \longrightarrow & H^1(Y) \\ {\scriptstyle\rho}\downarrow & & {\scriptstyle\sigma}\downarrow & & {\scriptstyle\tau}\downarrow & & {\scriptstyle\Sigma\rho}\downarrow & & {\scriptstyle\Sigma\sigma}\downarrow \\ H^0(X') & \longrightarrow & H^0(Y') & \longrightarrow & H^0(Z') & \longrightarrow & H^1(X') & \longrightarrow & H^1(Y') \end{array} .$$

We know that ρ, σ, $\Sigma\rho$ and $\Sigma\sigma$ are isomorphisms. The 5–lemma permits us to conclude that so is τ.

Now apply the dual construction; consider the commutative square

$$\begin{array}{ccc} (Y')^{\leq 0} & \longrightarrow & (Z')^{\leq 0} \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Z' \end{array}$$

We can extend to a morphism of triangles

$$\begin{array}{ccccccc} X'' & \longrightarrow & Y'' & \longrightarrow & Z'' & \longrightarrow & \Sigma X'' \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X' \end{array} \quad ;$$

as before, this means $Y'' = (Y')^{\leq 0}$ and $Z'' = (Z')^{\leq 0}$. We leave it as an exercise to the reader to check that

(1) X'' belongs to $\mathcal{T}^{\leq 1} \cap \mathcal{T}^{\geq 0}$, and Y'' belong to $\mathcal{A} = \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$, while Z'' belongs to $\mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq -1}$.

(2) The three maps

$$H^0(X'') \longrightarrow H^0(X'), \qquad H^0(Y'') \longrightarrow H^0(Y'), \qquad H^0(Z'') \longrightarrow H^0(Z')$$

are all isomorphisms.

The proposition now follows from Lemma 1.2, applied to the triangle $\Sigma^{-1}Z'' \longrightarrow X'' \longrightarrow Y'' \longrightarrow Z''$. □

References

1. Alexander A. Beilinson, Joseph Bernstein, and Pierre Deligne, *Analyse et topologie sur les éspaces singuliers*, Astérisque, vol. 100, Soc. Math. France, 1982 (French).
2. Robin Hartshorne, *Residues and duality*, Lecture Notes in Mathematics, vol. 20, Springer-Verlag, 1966.
3. Jean-Louis Verdier, *Catégories dérivées, état 0*, SGA 4.5, Lec. Notes in Math., vol. 569, Springer-Verlag, 1977, pp. 262–308 (French).
4. ______, *Des catégories dérivées des catégories abeliennes*, Asterisque, vol. 239, Société Mathématique de France, 1996 (French).

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ON S-COHN-JORDAN EXTENSIONS

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ABSTRACT. Let a monoid S act on a ring R by injective endomorphisms. A series of results relating various algebraic properties of R and that of the S-Cohn-Jordan ring extension $A(R;S)$ of R are presented. For example: primeness, Goldie conditions and other finiteness conditions are considered. Some problems and possible applications will be also discussed.

1. INTRODUCTION

Let R be an associative unital ring and $\sigma\colon R \to R$ an injective endomorphism of R. Jordan [2] constructed a minimal, in a sense that $A = \sum_{i=0}^{\infty} \sigma^{-i}(R)$, over-ring A of R such that σ extends to an automorphism of A. Then he began systematic studies of relations between various algebraic properties of R and that of A. The motivation for such studies was the observation that this knowledge can often be used for reducing the investigation of a skew polynomial ring $R[x;\sigma]$ of endomorphism type, to the case of the skew polynomial ring $A[x;\sigma]$ of automorphism type, which is much easier to handle. Examples of such approach one can also find in [6] and [7].

Instead of looking at the action of a single endomorphism σ on R one can consider the action of a monoid. Let S denote a monoid which acts on R by injective endomorphisms. That is, a homomorphism $\phi\colon S \to \mathrm{End}(R)$ is given, such that $\phi(s)$ is a monomorphism, for any $s \in S$. We say that an over-ring $A(R;S)$ of R is an S-Cohn-Jordan extension of R if it is a minimal over-ring of R such that the action of S on R extends to the action of S on $A(R;S)$ by automorphisms (Cf. Definition 1).

A classical result of Cohn (see Theorem 7.3.4 [1]) says that if the monoid S possesses a group $S^{-1}S$ of left quotients, then $A(R;S)$ exists, moreover it is uniquely determined up to an R-isomorphism.

The above mentioned theorem of Cohn was originally formulated in much more general context of Ω-algebras, not just rings. The construction of $A(R;S)$ was given as a limit of a suitable directed system.

The possibility of enlarging an object and replacing the action of endomorphisms by the action of automorphisms is a powerful tool, similar to a localization. Perhaps this was the reason that the theorem of Cohn was formulated and reproved in various algebraic contexts (see for example [3], [9], [10], [11], [12]).

The aim of the paper is to present a series of results relating various algebraic properties of R and that of the S-Cohn-Jordan extension $A(R;S)$ of R. For example: primeness, Goldie conditions and other finiteness conditions are considered. Most of the presented

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results can be found in full details in [7]. Some problems, questions and applications are also discussed.

2. Properties of S-Cohn-Jordan Extensions

Henceforth R stands for an associative ring, $\phi\colon S \to \mathrm{End}(R)$ denotes the action of a monoid S on R by injective endomorphisms. For any $s \in S$, the endomorphism $\phi(s) \in \mathrm{End}(R)$ will be denoted by ϕ_s.

Definition 1. An over-ring $A(R;S)$ of R is called an S-Cohn-Jordan extension of R if:

(1) the action of S on R extends to an action of S (also denoted by ϕ) on $A(R;S)$ by automorphisms, i.e. ϕ_s is an automorphism of $A(R;S)$, for any $s \in S$.
(2) every element $a \in A(R;S)$ is of the form $a = \phi_s^{-1}(b)$, for some suitable $b \in R$ and $s \in S$.

Henceforth, as in the above definition, ϕ_s will also denote the automorphism $\phi(s)$ of $A(R;S)$, where $s \in S$.

As it was mentioned in the introduction, the extension $A(R;S)$ exists provided the monoid S possesses a group of left quotients. Recall that this is the case exactly when the monoid S is left and right cancellative and satisfies the left Ore condition, that is, for any $s_1, s_2 \in S$, there exist $t_1, t_2 \in S$ such that $t_1 s_1 = t_2 s_2$.

In the case $S = \langle\sigma\rangle$ is a cyclic monoid, Jordan recognized $A(R;S)$ as a subring of the left localization of the Ore extension $R[x;\sigma]$ with respect to the set of all powers of the indeterminate x.

When the monoid S possesses a group of left quotients, then one can construct $A(R;S)$ in a similar way as Jordan did. Namely, let us consider the skew semigroup ring $R\#_\phi S$. One can check that elements of S are regular in $R\#_\phi S$ and S is a left Ore set in $R\#_\phi S$. In particular, we can consider the left localization $T = S^{-1}(R\#_\phi S)$ of $R\#_\phi S$. For any $s \in S$ and $r \in R$, we have $sr = \phi_s(r)s$ in $R\#_\phi S$. Thus one can think of $s^{-1}Rs$ as the preimage $(\phi_s)^{-1}(R)$ of R in T. The Goldie condition implies that $A = \bigcup_{s\in S} s^{-1}Rs \subseteq T$ is a subring of T. In fact it is easy to see that $A = A(R;S)$ (Cf. Lemma 2.1 [7]), in this case.

Suppose that $A(R;S)$ exists and the action of S on R is faithful, in the sense that ϕ is an injection. Then the action of S on $A(R;S)$ is also faithful. This means that the monoid S embeds in a group (the group of automorphisms of $A(R;S)$). However even in this case, conditions for existence of $A(R;S)$ seem to be not clear and we may formulate the following:

Problem 2. Suppose that S acts faithfully on R.

(1) What are the necessary and sufficient conditions for the existence of $A(R;S)$?
(2) Assume that $A(R;S)$ exists. What are the necessary and sufficient conditions for uniqueness of $A(R;S)$?
(3) Let T be a submonoid of S. Suppose that $A(R;S)$ exists. When does $A(R;T)$ exist? If so, is it naturally embedded in $A(R;S)$?

The Definition 1 reminds somehow the definition of a left localization of R with respect to a multiplicatively closed set. In this way, the analogue of a common left denominator for a finite set X of elements of $A(R;S)$, should be an element ϕ_s, for some $s \in S$, such that $\phi_s(X) \subseteq R$. It is easy to see that, for any finite subset X of $A(R;S)$, such element ϕ_s do exists. This suggests that the relations between some algebraic properties of R and its S-Cohn-Jordan extension $A(R;S)$ should be similar to those between R and its localization. This is indeed the case. In particular we have:

Proposition 3. (Cf. [7])

(1) *Let $\mathcal{T}$ denote one of the following classes of rings: the class of all division, simple, von Neumann regular, prime, semiprime rings, rings having finite block theory. If $R \in \mathcal{T}$ then $A(R;S) \in \mathcal{T}$.*

(2) *Let $\mathcal{P}$ denote one of the following classes of rings: the class of all domains, reduced rings, $n \times n$ matrix rings, commutative or, more generally, rings satisfying a fixed polynomial identity. Then $A(R;S) \in \mathcal{P}$ if and only if $R \in \mathcal{P}$.*

The properties listed in the statement (1) of the above proposition do not pass down from $A(R;S)$ to R. Indeed, the following easy example shows that $A(R;S)$ can be a field with R being not simple.

Example 4. Let $A = K(x_i \mid i \in \mathbb{Z})$ be the field of rational functions over a field K in commuting indeterminates $\{x_i\}_{i \in \mathbb{Z}}$ and $S = \langle \sigma \rangle$, where σ is the K-automorphism of A given by $\sigma(x_i) = x_{i+1}$, for $i \in \mathbb{Z}$. Let us set $R = K(x_i \mid i \geq 1)[x_0] \subseteq A$. Then S acts in a natural way on R and for any $a \in A$, there exists $n \geq 1$ such that $\sigma^n(a) \in K(x_i \mid i \geq 1) \subseteq R$. This means that $A = A(R;S)$.

Example 1.10 [7] offers a prime ring R such that $A(R;S)$ is not semiprime. Example 1.15 [7] shows that there exists a ring R having infinitely many central orthogonal idempotents, while $A(R;S)$ has no nontrivial central idempotents, i.e. $A(R;S)$ has a finite block theory but R does not.

Theorem 5. (Cf. [7]) *Suppose $A(R;S)$ exists. Then:*

(1) *$A(R;S)$ is semiprime if and only if for any nonzero left ideal I of R, there exists $s \in S$ such that $(R\phi_s(I))^2 \neq 0$.*

(2) *Suppose that R is left noetherian. Then R is prime (semiprime) if and only if $A(R;S)$ is prime (semiprime).*

When R is one-sided noetherian, then there exists a finite common bound on the cardinality of sets of orthogonal idempotents of R (as otherwise R would have infinite left Goldie dimension). Thus Proposition 1.14 [7] yields immediately the following:

Theorem 6. *Suppose that R is a one-sided noetherian ring and $A(R;S)$ exists. Then R has finite block theory if and only if $A(R;S)$ has finite block theory. Moreover if $\bigoplus_{i=1}^n e_iR$ is a decomposition of R into indecomposable blocks, then $\bigoplus_{i=1}^n e_iA(R;S)$ is a block decomposition of $A(R;S)$.*

Much more can be said about the relations of R and that of $A(R;S)$, provided the monoid S has a group of left quotients. The idea, which goes back to Jordan [2], is to compare left ideals I of $A(R;S)$ with its orbits $\{\phi_s(I) \cap R \mid s \in S\}$ in R. An important role is also played by S-closed left ideals J of R, i.e. left ideals J such that $A(R;S)J \cap R = J$.

The following theorem (Cf. Theorem 2.19 and Corollary 2.20 of [7]) offers complete characterization of artinian property of $A(R;S)$.

Theorem 7. *Suppose that S possesses a group of left quotients. Then:*

(1) *The ring $A(R;S)$ is left artinian if and only if there exists a finite bound on lengths of chains of S-closed left ideals of R. Moreover, if one of the equivalent conditions holds, then the length of $A(R;S)$ as a left $A(R;S)$-module is equal to the length of the longest chain of S-closed left ideals of R.*
(2) *If R is left artinian then so is $A(R;S)$.*

In the case S is a cyclic monoid, the above theorem was proved in [2]. Surprisingly, the proof of the theorem in the general case seems to be easier than the arguments used in the case S is a cyclic monoid.

Making use of Theorems 5, 6, 7 and some localizations technics one can prove the following two results (Cf. [7]):

Theorem 8. *Suppose S possesses a group of left quotients. If the ring R is left artinian, then:*

(1) *R is a semisimple ring if and only if $A(R;S)$ is a semisimple ring.*
(2) *If $R = \bigoplus_{i=1}^{k} e_iR$, with $e_iR = M_{n_i}(B_i)$, is a block decomposition of the semisimple ring R, then $A(R;S) = \bigoplus_{i=1}^{k} e_iA(R;S)$ is a block decomposition of $A(R;S)$ and $e_iA(R;S) = M_{n_i}(D_i)$ for some division ring. Moreover, for $1 \leq i \leq k$, the division ring D_i is an extension of B_i.*

In the case $S = \langle\sigma\rangle$ being a cyclic monoid the above theorem was known in special cases. Namely, the first statement was proved in [2], the second one appeared in [4].

From now on $Q(R)$ will denote the classical left quotient ring of a semiprime left Goldie ring R and udim R will stand for the left uniform dimension of R.

Theorem 9. *Suppose S possesses a group of left quotients. Let R be a semiprime left Goldie ring. Then $A(R;S)$ is also a semiprime left Goldie ring. Moreover $Q(A(R;S)) = A(Q(R);S)$ and* udim $R =$ udim $A(R;S)$.

Contrary to the artinian property, the situation with the noetherian property of $A(R;S)$ seems to be not clear at all. Even when S is a cyclic monoid, one can find examples of rings R and $A(R;S)$ showing that one of those rings is left noetherian but the other is not left noetherian. Nevertheless Jordan [2] succeeded to give necessary and sufficient conditions for $A(R;S)$ to be left noetherian, in the case S is a cyclic monoid. The characterization was given in terms of properties of the lattice of S-closed left ideals of R.

Problem 10. To characterize the left noetherian property of the S-Cohn-Jordan extension $A(R;S)$ in terms of properties of R and the action of S.

If $G = S^{-1}S$ is the group of left quotients of S, then we have seen that $A(R;S)$ can be considered as a subring of the left localization $S^{-1}(R\#_\phi S)$ of $R\#_\phi S$. Using this approach, one can see that there is a natural isomorphism between $A(R;S)\#_\phi G$ and $S^{-1}(R\#_\varphi S)$. Since the left noetherian property of a ring is preserved under left localization with respect to a left Ore set, we have:

Proposition 11. *Suppose S possesses a group of left quotients. If the ring $R\#_\phi S$ is left noetherian, then $A(R;S)$ is also left noetherian.*

3. Examples of Applications

As it was briefly mentioned at the end of the previous section, when S has the group G of left quotients, then $R\#_\phi S \subseteq A(R;S)\#_\phi S \subseteq A(R;S)\#_\phi G = S^{-1}(R\#_\phi S)$. This means that problems concerning the skew semigroup rings $R\#_\phi S$ can often be reduced to the skew group ring $A(R;S)\#_\phi S$. The following theorem is an example of such application.

Theorem 12. (Cf. [7]) *Let S be a monoid having a poly-infinite cyclic group of left quotients. Suppose that S acts on a semiprime (prime) left Goldie ring R by injective endomorphisms. Then the skew semigroup ring $R\#_\phi S$ is a semiprime (prime) left Goldie ring and* udim $(R\#_\phi S) =$ udim R.

The idea of the proof of the above theorem is as follows. By Theorem 9, $A(R;S)$ is a semiprime left Goldie ring and the assumption imposed on the group $G = S^{-1}S$ of left quotients of the monoid S imply that $A(R;S)\#_\phi G$ is a semiprime left Goldie ring. $R\#_\phi S$ is a subring of $A(R;S)\#_\phi G$ such that $S^{-1}(R\#_\phi S) = A(R;S)\#_\phi G$. Thus the localization $S^{-1}(R\#_\phi S)$ is a semiprime left Goldie ring. Hence the ring $R\#_\phi S$ is also semiprime left Goldie.

It was proved in [5] that the property of being a semiprime left Goldie ring lifts from a ring R to its Ore extension $R[x;\sigma,\delta]$, where σ is an automorphism and δ a σ-derivation of R. This result was extended in [4] to the following theorem.

Theorem 13. *Let R be a semiprime left Goldie ring, σ, δ an injective endomorphism and a σ-derivation of R, respectively. Then $R[x;\sigma,\delta]$ is also a semiprime left Goldie ring and* udim $R[x;\sigma,\delta] =$ udim $R =$ udim $A(R;\langle\sigma\rangle)$.

One of the key ingredient in the proof of the above theorem was the use of the $\langle\sigma\rangle$-Cohn-Jordan extension $A(R;\langle\sigma\rangle)$ and Theorem 9.

Mushrub in [9] investigated the left uniform dimension of skew polynomial rings $R[x;\sigma]$, where σ denotes an injective endomorphism of the ring R. He proved, in particular, that udim $R[x;\sigma] =$ udim $A(R;\langle\sigma\rangle)$ (for a short proof see Lemma 3.2 [4]). He also constructed examples showing that:

1. For any $n \in \mathbb{N}$, there is a commutative ring R (not semiprime) with an injective endomorphism σ, such that udim $R = n$ and udim $R[x;\sigma] = 1$.
2. There exists a domain R of infinite left uniform dimension and an injective endomorphism σ of R such that udim $R[x;\sigma] = 1$.

The following question comes from [9].

Question 14. (Mushrub) Let R be a semiprime ring of finite left Goldie dimension. Suppose that σ is an injective endomorphism of R. Is udim $R =$ udim $R[x;\sigma]$?

As we recorded earlier, udim $R[x;\sigma] =$ udim $A(R;\langle\sigma\rangle)$, for any injective endomorphism σ of R. Thus the above question of Mushrub can be be read as a question: Is udim $R =$ udim $A(R;\langle\sigma\rangle)$? Therefore, the following question can be viewed as a generalization of Question 14.

Question 15. Suppose that R is a semiprime ring of finite left Goldie dimension acted by a monoid S which has a group of left quotients. Is udim $R =$ udim $A(R;S)$?

The left uniform dimension is preserved under left localizations with respect to Ore sets of regular elements (Cf. Lemma 2.2.12 [8]). This implies that udim $R\#_\phi S =$ udim $S^{-1}(R\#_\phi S) =$ udim $A(R;S)\#_\phi G$. Thus Theorem 12 yields that Question 15 has a positive answer if the group $G = S^{-1}S$ is poly-infinite cyclic and R satisfies the ACC on left annihilators. This also means that Question 14 has a positive answer if one additionally assume that R has the ACC on left annihilators. The last fact was observed earlier in [4].

References

[1] P.M. Cohn, *Universal Algebra,* Harper and Row, (1965).

[2] D.A. Jordan, *Bijective Extensions of Injective Rings Endomorphisms,* J. London Math. Soc. **25** (1982), 435–448.

[3] N.S. Larsen, *Crossed Products by Semigroups of Endomorphisms and Groups of Partial Automorphisms,* Bull. Canad. Math. **46(1)** (2003), 98–112.

[4] A. Leroy, J. Matczuk, *Goldie Conditions for Ore Extensions over Semiprime Rings,* Algebras and Representation Theory **8** (2005), 679–688.

[5] J. Matczuk, *Goldie Rank of Ore Extensions,* Comm. Algebra **23(4)** (1995), 1455–1471.

[6] J. Matczuk, *A Characterization of σ-Rigid Rings,* Comm. Algebra **32(11)** (2004), 4333–4336.

[7] J. Matczuk, *S-Cohn-Jordan Extensions,* Comm. Algebra, to appear.

[8] J.C. McConnell, J.C. Robson. *Noncommutative Noetherian Rings,* John Willey & Sons Ltd, (1987).

[9] V.A. Mushrub, *On The Goldie Dimension of Ore Extensions With Several Variables.* Fundam. Prikl. Mat. **7(4)** (2001), 1107–1121.

[10] D. Pask, T. Yeend, *Voltage Graphs,*Voltage Graphs. Proc. Centre Math. and Appl., Austral. Nat. Univ, (1999).

[11] D. Pask , I. Raeburn , T. Yeend, *Actions of Semigroups on Directed Graphs and their C^*-algebras,* Pure Appl. Alg. **159** (2001), 297–313

[12] G. Picavet, *Localization with Respect to Endomorphisms,* Semigroup Forum **67** (2003), 76–96.

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A CONSTRUCTION OF LOCAL QF-RINGS WITH RADICAL CUBED ZERO

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ABSTRACT. The purpose of this paper is to give a construction of local QF-rings with Jacobson radical cubed zero. From our construction, we can foresee that there are many QF-rings which are not finite dimensional algebras over fields. Needless to say, local QF-rings together with local Nakayama rings are important artinian rings in the sense that these rings are parts of QF-rings and Nakayama rings. Furthermore, as we mention, local QF-rings are important for the study on the Faith conjecture, since the Faith conjecture is not solved even for local semiprimary one-sided selfinjective rings with Jacobson radical cubed zero.

1. INTRODUCTION

There are many open problems on QF-rings. The two most famous, longstanding, unsolved problems are the Nakayama conjecture and the Faith conjecture. One may refer to Nicholson-Yousif [16] for the Faith conjecture, as well as for several more recent questions on QF-rings.

The Faith conjecture. Is a semiprimary right self-injective ring a QF-ring? Faith conjectured "no" in his book [6].

The Faith conjecture is not solved even for a local semiprimary ring with radical cubed zero. Thus we record:

Problem 1. Is a semiprimary local right self-injective ring with radical cubed zero a QF-ring?

The following result gives some information on this question.

Fact 1. (Baba-Oshiro [2]) *If R is a semiprimary ring, then R is a right self-injective ring if and only if R is a right simple-injective ring. In particular, if R is a local semiprimary ring with radical J cubed zero, then R is right self-injective if and only if ${}_RJ^2, J^2_R$ are simple and, for any maximal right submodule M of J, there exists $a \in J \setminus J^2$ satisfying $aM = 0$.*

We now provide a careful analysis of Problem 1 and translate this problem into a problem on two-sided vector spaces over division rings.

In order to do so, let R be a local semiprimary ring with $J^2 \neq 0$ and $J^3 = 0$, where $J := J(R)$ denotes the radical of R. Let D denote the division ring R/J and put $\overline{J} = J/J^2$.

The detailed version of this paper will be submitted for publication elsewhere.

Then, $\overline{J}$ and J^2 are (D, D)-bispaces. We denote by $Soc^{\ell}(R)$ and $Soc^{r}(R)$ the left and the right socle of R, respectively and by $l_R(A)$ and $r_R(A)$ the left and the right annihilator of a subset A of R, respectively.

We now record some properties on R.

Fact 2. (1) *If ${}_RSoc^{\ell}(R)$ and $Soc^{r}(R)_R$ are simple, then $r_Rl_R(A) = A$ and $l_Rr_R(B) = B$ for any finitely generated right submodule A and for any finitely generated left submodule B of J, $J^2 = Soc^{\ell}(R) = Soc^{r}(R)$, and ${}_DJ^2$ and J^2_D are one-dimensional spaces.*

(2) *If J_R is finitely generated and ${}_RSoc^{\ell}(R)$ and $Soc^{r}(R)_R$ are simple, then R is QF. For this QF-ring R, we can make a new QF-ring T of graded type as follows: Consider the (D, D)-bispace $T = D \times \overline{J} \times J^2$. In T, we define a multiplication by setting*

$$t_1t_2 = (d_1d_2, d_1\overline{a_2} + \overline{a_1}d_2, d_1s_2 + s_1d_2 + a_1a_2)$$

for $t_1 = (d_1, \overline{a_1}, s_1)$ and $t_2 = (d_2, \overline{a_2}, s_2) \in T$, where $\overline{a_i} = a_i + J^2 \in J/J^2 =: \overline{J}$. Then, T is a QF-ring with $J(T) = 0 \times \overline{J} \times J^2$, $J(T)^2 = 0 \times 0 \times J^2$ and $J(T)^3 = 0$. (In general, $R \ncong T$.)

Fact 3. *Assume that R_R is (simple-)injective. Then*

(1) *${}_RSoc^{\ell}(R)$ and $Soc^{r}(R)_R$ are simple.*

(2) *For any maximal submodule M of J_R, $aM = 0$ for some $a \in J \setminus S$.*

(3) *$r_Rl_R(A) = A$ for any finitely generated submodule A of J_R and $l_Rr_R(B) = B$ for any finitely generated submodule B of ${}_RJ$.*

(4) *Put $J^* = \mathrm{Hom}_R(\overline{J}_R, J^2_R)$. Then, for any $a \in J$, the map $a \to (a)_L$ (left multiplication) gives an (R, R)-bimodule isomorphism and a (D, D)-bispace isomorphism: ${}_R\overline{J}_R \cong {}_RJ^*_R$ and ${}_D\overline{J}_D \cong {}_DJ^*_D$.*

(5) *Put $\alpha = \dim(\overline{J}_D)$. If α is finite, then R is QF, while if α is infinite, then $\dim({}_D\overline{J}) = (\#D)^{\alpha} = \#R > \alpha$; in particular, if $\alpha = \aleph_0$ and $\#D = \aleph$, then $\dim({}_D\overline{J}) = \aleph$, where $\#A$ denotes the cardinal number of a set A.*

Most known information on R emanates from Fact 2 and Fact 3. In particular, (4) in Fact 3 is important for investigating Problem 1.

2. Local QF-rings

We now give a construction of local QF-rings.

In this section, let D be a division ring and let ${}_DV_D$ be a (D, D)-bispace. We put

$$T = D \times V \times (V \otimes_D V).$$

Then, T is a (D, D)-bispace. In T, we define a multiplication as follows:

$$t_1t_2 = (d_1d_2, d_1v_2 + v_1d_2, d_1x_2 + x_1d_2 + v_1 \otimes v_2)$$

for $t_1 = (d_1, v_1, x_1)$ and $t_2 = (d_2, v_2, x_2) \in T$. It is easy to see that T is a local semiprimary ring with radical cubed zero and that

$$J(T) = 0 \times V \times V \otimes_D V \quad \text{and} \quad J(T)^2 = 0 \times 0 \times V \otimes_D V.$$

We identify $D \times 0 \times 0$, $0 \times V \times 0$ and $0 \times 0 \times V \otimes_D V$ with D, V and $V \otimes_D V$, respectively.
We note the following:

Proposition 4. (1) *Assume that there exists a (D, D)-bisubspace I of $V \otimes_D V$ with $\dim((V \otimes_D V)/I_D) = \dim((V \otimes_D V)/_D I) = 1$ and $vD \otimes_D V \not\subset I$ and $V \otimes_D Dv \not\subset I$ for any $0 \neq v \in V$. Then, I is an ideal of T, $J(T/I)^2 = Soc^r(T/I) = Soc^\ell(T/I)$, and $J(T/I)^2$ is simple as a left T/I-module and as a right T/I-module.*

(2) *Assume that $\dim(V_D)$ is finite and such a (D, D)-bisubspace I in (1) exists. Then, T/I is a local QF-ring with radical cubed zero.*

Proof. (1) is easily seen and (2) follows from Fact 1. □

Let pD be a one-dimensional right vector space and let $\rho \in \mathrm{Aut}(D)$. Then, pD becomes a one-dimensional left vector space by defining $dp = p\rho(d)$ for $d \in D$. We denote such a (D, D)-bispace by pD^ρ. We also put

$$V^* = \mathrm{Hom}_D(V_D, pD^\rho_D).$$

Then, V^* is canonically a (D, D)-bispace. Here we assume the following:

Assumption A: There exists a (D, D)-bispace isomorphism $\theta : V \to V^*$.

Since the map $(V, V) \to pD^\rho$ given by $(v, w) \mapsto \theta(v)(w)$ is a bilinear (D, D) onto map, the map

$$\lambda : V \otimes_D V \to pD^\rho \quad \text{by} \quad \sum_i v_i \otimes w_i \mapsto \sum_i \theta(v_i)(w_i)$$

is a (D, D)-bispace onto homomorphism. As is easily seen, Ker λ is an ideal of the ring $T = D \times V \times (V \otimes_D V)$. We put

$$D\langle V, \theta, \rho, pD^\rho\rangle = T/\operatorname{Ker}\lambda.$$

Let $w \in \lambda^{-1}(p)$ be fixed and put $s = w + \operatorname{Ker}\lambda \in (V \otimes_D V)/\operatorname{Ker}\lambda$. Then we can show the following result.

Theorem 5. *Let R be the ring $D\langle V, \theta, \rho, pD^\rho\rangle$ above. Then the following hold.*

(1) *$J := J(R) = (V \times V \otimes_D V)/\operatorname{Ker}\lambda$, $J^2 = (V \otimes_D V)/\operatorname{Ker}\lambda = Rs = sR$, $J^3 = 0$ and $Soc^r(R) = Soc^\ell(R) = J^2$.*

(2) *${}_R J^2$ and J^2_R are simple.*

(3) *R is a right self-injective ring.*

(4) *R is QF if and only if $\dim(V_D)$ is finite.*

Proof. (1), (2) and (4) follow from Proposition 4. To show (3), let I be a maximal submodule of J_R. By Fact 1, it suffices to show that there exists $a \in J \setminus J^2$ satisfying $aI = 0$. Let X be a subspace of V_D with $X/\operatorname{Ker}\lambda = I$. Then, as X is a proper subspace of V, we can take $0 \neq v^* \in V^*$ such that $v^*(X) = 0$. Put $a = \theta^{-1}(v^*) + \operatorname{Ker}\lambda$. Then, $a \in J \setminus J^2$ and $aI = 0$, as desired. □

By Theorem 5, we can translate Problem 1 into the following:

Problem 2. Does there exist a division ring D and a (D, D)-bispace V such that $\dim(V_D) = \infty$ and ${}_DV_D \cong {}_DV_D^*$ ((D, D)-isomorphism)?

If such a space ${}_DV_D$ exists, Theorem 5 asserts the Faith conjecture is true, that is, we can construct a semiprimary right self-injective ring which is not QF. However, this problem is very difficult. In fact, if we try to solve this, we immediately encounter pathologies.

However, as a biproduct of the study on Problem 2, we can obtain an important way of constructing local QF-rings. We shall state this construction.

Lemma 6. *Let V be a bispace over a division ring D with $n = \dim({}_DV) = \dim(V_D) < \infty$. Then we can take $x_1, \ldots, x_n \in V$ satisfying $V = Dx_1 \oplus \cdots \oplus Dx_n = x_1D \oplus \cdots \oplus x_nD$.*

Proof. Let $x_1, \ldots, x_k, y, z \in V$ such that $x_1, \ldots, x_k, y$ and $x_1, \ldots, x_k, z$ are left and right independent over D, respectively. If $Dz \cap \sum_{i=1}^k Dx_i = 0$ or $yD \cap \sum_{i=1}^k x_iD = 0$, then $x_1, \ldots, x_k, z$ or $x_1, \ldots, x_k, y$ are left and right independent, respectively. If otherwise, i.e., $Dz \subset \sum_{i=1}^k Dx_i$ and $yD \subset \sum_{i=1}^k x_iD$, then $x_1, \ldots, x_k, y+z$ are left and right independent. By continuing this procedure, the statement is shown. □

Now, henceforth, let

$$V_D = x_1D \oplus \cdots \oplus x_nD$$

be a finite dimensional right vector space over a division ring D and let

$$\sigma = (\sigma_{ij}) : D \to (D)_n \ \text{ by } \ d \mapsto \sigma(d) = \begin{pmatrix} \sigma_{11}(d) & \cdots & \sigma_{1n}(d) \\ & \cdots & \\ \sigma_{n1}(d) & \cdots & \sigma_{nn}(d) \end{pmatrix}$$

be a ring homomorphism, where $(D)_n$ is the ring of all $n \times n$ matrices over D. By using σ, we define a left D-operation on V as follows: For $d \in D$, $dx_i = \sum_{j=1}^n x_j\sigma_{ji}(d)$, namely,

$$d(x_1, \ldots, x_n) = (x_1, \ldots, x_n) \begin{pmatrix} \sigma_{11}(d) & \cdots & \sigma_{1n}(d) \\ & \cdots & \\ \sigma_{n1}(d) & \cdots & \sigma_{nn}(d) \end{pmatrix}.$$

Then, V_D becomes a (D, D)-bispace. We denote this bispace by $V\langle x_1, \ldots, x_n; \sigma\rangle$ or simply V^σ. We note that pD^ρ mentioned above is $pD\langle p; \rho\rangle$.

Proposition 7. *The following are equivalent:*

(1) $V^\sigma = Dx_1 \oplus \cdots \oplus Dx_n$.

(2) *There is a ring homomorphism* $\xi = (\xi_{ij}) : D \to (D)_n$ *such that for* $1 \le i, k \le n$, *the following formulas hold:*

$$\sum_{j=1}^{n} \sigma_{kj}(\xi_{ij}(d)) = \begin{cases} d & k = i \\ 0 & k \neq i \end{cases}$$

and

$$\sum_{j=1}^{n} \xi_{jk}(\sigma_{ji}(d)) = \begin{cases} d & k = i \\ 0 & k \neq i. \end{cases}$$

Proof. (1) $\Rightarrow$ (2). Since $x_i d = \sum_j \xi_{ij}(d) x_j = \sum_j (\sum_k x_k \sigma_{kj}(\xi_{ij}(d)))$, we see that

$$\sum_{j} \sigma_{kj}(\xi_{ij}(d)) = \begin{cases} d & k = i \\ 0 & k \neq i. \end{cases}$$

Similarly, the second formula is obtained.

(2) $\Rightarrow$ (1). Since $\sum_j \xi_{ij}(d) x_j = \sum_j (\sum_k x_k \sigma_{kj}(\xi_{ij}(d))) = x_i \sum_j \sigma_{ij}(\xi_{ij}(d)) = x_i d$, we see that for any $d \in D$,

$$\xi(d) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} d,$$

from which $V^\sigma = Dx_1 + \cdots + Dx_n$.

Next, to show that $\{Dx_1, \ldots, Dx_n\}$ is an independent set, assume $d_1 x_1 + \cdots + d_n x_n = 0$ for $d_1, \ldots, d_n \in D$. Since $\sum_i (\sum_j x_j \sigma_{ji}(d_i)) = 0$, we see that $\sum_i \sigma_{ji}(d_i) = 0$ for $j = 1, \ldots n$ and hence

$$\begin{aligned} \sigma_{11}(d_1) + \sigma_{12}(d_2) + \cdots + \sigma_{1n}(d_n) &= 0 \quad \cdots (1) \\ \sigma_{21}(d_1) + \sigma_{22}(d_2) + \cdots + \sigma_{2n}(d_n) &= 0 \quad \cdots (2) \\ \cdots \quad \cdots \quad \cdots & \\ \sigma_{n1}(d_1) + \sigma_{n2}(d_2) + \cdots + \sigma_{nn}(d_n) &= 0 \quad \cdots (n). \end{aligned}$$

Thus, $0 = \xi_{1j} \times (1) + \xi_{2j} \times (2) + \cdots + \xi_{nj} \times (n) = \sum_i \xi_{ij}(\sigma_{ij}(d_j)) = d_j$; hence $d_j = 0$, as desired. □

By Proposition 7, we see that, if there is a ring homomorphism $\xi = (\xi_{ij}) : D \to (D)_n$ satisfying the formulas of the proposition, then $V^\sigma = Dx_1 \oplus \cdots \oplus Dx_n$. For this situation, we use $V\langle x_1, \ldots, x_n; \sigma, \xi\rangle$ instead of $V\langle x_1, \ldots, x_n; \sigma\rangle$. Moreover, we construct the ring R above for this bispace under Assumption A and denote it by $D\langle V, \sigma, \xi, \theta, \rho, pD^\rho\rangle$. Combining the proposition with Lemma 6, we have the following:

Theorem 8. *Assume that* $\dim({}_D V) = n < \infty$. *Then,* $R = D\langle V, \sigma, \xi, \theta, \rho, pD^\rho\rangle$ *is a local QF-ring with radical cubed zero.*

Now, we return to our Assumption A and construct such a (D, D)-bispace isomorphism $\theta : V \to V^*$ under some condition. Let $\xi = (\xi_{ij}) : D \to (D)_n$ be a ring homomorphism satisfying the formulas of Proposition 7(2) such that $\rho\xi_{ij} = \sigma_{ij}$ for all i, j. For each i, let $\alpha_i \in V^* = \mathrm{Hom}_D(V_D, pD^{\rho}_D)$ be defined by

$$\alpha_i : x_1 d_1 + \cdots + x_n d_n \mapsto pd_i,$$

where $d_1, \ldots, d_n \in D$. Then, ${}_DV^* = D\alpha_1 \oplus \cdots \oplus D\alpha_n$ and the map

$$\theta^* : {}_DV_D \to {}_DV^*_D \ \text{ by } \ d_1x_1 + \cdots + d_nx_n \to d_1\alpha_1 + \cdots + d_n\alpha_n$$

is a (D, D)-bispace isomorphism. Therefore, in this case, we can make a local QF-ring $D\langle V, \sigma, \xi, \theta^*, \rho, pD^{\rho}\rangle$. In particular, setting $\rho = id_D$, we can take σ as ξ above . Hence, we obtain the following, which is useful for making local QF-rings with radical cubed zero.

Theorem 9. *Let $V_D = x_1D \oplus \cdots \oplus x_nD$ be an n-dimensional vector space over a division ring D and let $\sigma = (\sigma_{ij}) : D \to (D)_n$ be a homomorphism satisfying the formulas: For $1 \le i, k \le n$,*

$$\sum_{j=1}^{n} \sigma_{kj}(\sigma_{ij}(d)) = \begin{cases} d & k = i \\ 0 & k \neq i \end{cases}$$

and

$$\sum_{j=1}^{n} \sigma_{jk}(\sigma_{ji}(d)) = \begin{cases} d & k = i \\ 0 & k \neq i. \end{cases}$$

Then we can make a local QF-ring $D\langle V, \sigma, \sigma, \theta^*, id_D, 1D^{id_D}\rangle$.

3. Examples of local QF-rings

Example 1. Let $V = x_1D \oplus \cdots \oplus x_nD$ be an n-dimensional vector space over a division ring D and let π be a ring automorphism of D. Consider a ring homomorphism

$$\sigma : D \to (D)_n \ \text{ by } \ d \mapsto \begin{pmatrix} \pi(d) & & 0 \\ & \ddots & \\ 0 & & \pi(d) \end{pmatrix}.$$

Then, by σ, V becomes a (D, D)-bispace and

$$\begin{pmatrix} \pi^{-1}(d) & & 0 \\ & \ddots & \\ 0 & & \pi^{-1}(d) \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} d.$$

The map $\xi = (\xi_{ij}) : D \to (D)_n$ is given by

$$d \mapsto \begin{pmatrix} \pi^{-1}(d) & & 0 \\ & \ddots & \\ 0 & & \pi^{-1}(d) \end{pmatrix}.$$

It then follows from the argument above Theorem 9 that we can construct a local QF-ring $D\langle V, \sigma, \xi, \theta^*, \pi^2, pD^{\pi^2}\rangle$.

Example 2. Let $\mathbb{C}$ be the field of complex numbers, let $V = x_1\mathbb{C} \oplus x_2\mathbb{C}$ be a 2-dimensional vector space over $\mathbb{C}$ and consider a ring homomorphism

$$\sigma : \mathbb{C} \to (\mathbb{C})_2 \text{ by } a + bi \mapsto \begin{pmatrix} a & bi \\ bi & a \end{pmatrix}.$$

Then the map σ satisfies the formulas in Theorem 9. Hence we can make a local *QF*-ring $\mathbb{C}\langle V, \sigma, \sigma, \theta^*, id_{\mathbb{C}}, 1\mathbb{C}^{id_{\mathbb{C}}}\rangle$.

This example can be slightly generalized as the following:

Example 3. Let k be a commutative field and let $f(x) = x^n - a \in k[x]$ be irreducible with α a root, $D = k(\alpha)$ and $V = \sum_{i=1}^{n} \oplus x_i D$. Let a map

$$\sigma : D \to (D)_n \text{ by } \sum_{i=0}^{n-1} a_i \alpha^i \mapsto \begin{pmatrix} a_0 & a_1\alpha & a_2\alpha^2 & \cdots & a_{n-1}\alpha^{n-1} \\ a_{n-1}\alpha^{n-1} & a_0 & a_1\alpha & \cdots & a_{n-2}\alpha^{n-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_2\alpha^2 & \ddots & \ddots & \ddots & a_1\alpha \\ a_1\alpha & a_2\alpha^2 & \cdots & a_{n-1}\alpha^{n-1} & a_0 \end{pmatrix}.$$

Then, σ is a ring homomorphism satisfying the formulas in Theorem 9. Hence, for a given n-dimensional vector space V over D, we can make a local *QF*-ring $D\langle V, \sigma, \sigma, \theta^*, id_D, 1D^{id_D}\rangle$.

Example 4. Let $\mathbb{H}$ be the quaternion algebra, let $V = x_1\mathbb{H} \oplus x_2\mathbb{H} \oplus x_3\mathbb{H} \oplus x_4\mathbb{H}$ be a 4-dimensional vector space over $\mathbb{H}$ and consider a ring homomorphism

$$\sigma : \mathbb{H} \to (\mathbb{H})_4 \text{ by } a + bi + cj + dk \mapsto \begin{pmatrix} a & bi & cj & dk \\ bi & a & dk & cj \\ cj & dk & a & bi \\ dk & cj & bi & a \end{pmatrix}.$$

Then, σ satisfies the formulas in Theorem 9. Hence we can make a local *QF*-ring $\mathbb{H}\langle V, \sigma, \sigma, \theta^*, id_{\mathbb{H}}, 1\mathbb{H}^{id_{\mathbb{H}}}\rangle$.

Further, using Theorem 9, we shall show two constructing ways of local *QF*-algebras R with radical cubed zero, one of which gives an example of a local *QF*-algebra which is not a finite dimensional algebra.

Example 5. Let E be a field and let π be an automorphism of E satisfying

(1) $\pi^2 = id_E$.

(2) $\alpha\pi(\alpha) + \beta\pi(\beta) = 0 \Rightarrow \alpha = 0$ and $\beta = 0$ for $\alpha, \beta \in E$.

Define a 2-dimensional vector space over E : Let $D = E \oplus Ei = \{\alpha + \beta i \mid \alpha, \beta \in E\}$ with the product $i\alpha = \pi(\alpha)i$ for any $\alpha \in E$ (the addition, as well as the multiplication between elements of E being the natural ones). Then, D is a division ring, as it can be checked; see the product:

$$(\alpha + \beta i)(\pi(\alpha) - \beta i) = \alpha\pi(\alpha) + \beta\pi(\beta)$$

and if $\alpha + \beta i \neq 0$, then we have $\alpha\pi(\alpha) + \beta\pi(\beta) \neq 0$ by (3). Also, the center of D is $K := \{a \in E \mid \pi(a) = a\}$.

Let $V = x_1D \oplus x_2D$ be a 2-dimensional vector space over D and consider a ring homomorphism

$$\sigma : D \to (D)_2 \text{ by } \alpha + \beta i \mapsto \begin{pmatrix} \alpha & \beta i \\ \beta i & \alpha \end{pmatrix}.$$

Then we see that σ satisfies the formulas in Theorem 9. Hence we can make a local QF-ring $R = D\langle V, \sigma, \sigma, \theta^*, id_D, 1D^{id_D}\rangle$.

We shall give some examples of fields E satisfying (1) and (2) above.

(i) Let $E = \mathbb{C}$ or an arbitrary imaginary quadratic field (e.g. $\mathbb{Q}(\sqrt{-3})$) and the map $\pi : E \to E$ defined by $\pi(\alpha) = \overline{\alpha}$, where $\overline{\alpha}$ denotes the conjugate of α.

(ii) Let K be a field and π an automorphism of K satisfying the conditions (1) and (2). Moreover, let $E = K(x)$ be the field of rational functions in x over K. For $f = a_nx^n + \cdots + a_1x + a_0 \in K[x]$, we put $\overline{f} = \pi(a_n)x^n + \cdots + \pi(a_1)x + \pi(a_0)$. Then the map $\overline{\pi} : E \to E$ given by $\overline{\pi}(f/g) = \overline{f}/\overline{g}$ is an automorphism of E. We see that the fixed field of $\overline{\pi}$ in E is $F(x)$, where F is the fixed field of π in K, and E and $\overline{\pi}$ satisfies (1) and (2) again.

Example 6. Let E be a division ring such that E is infinite dimensional over its center K and $x^2 \neq -1$ holds for any element $x \in E$.

Define a 2-dimensional vector space over E : Let $D = E \oplus Ei = \{\alpha + \beta i \mid \alpha, \beta \in E\}$. Define the products $i^2 = -1$ and $i\alpha = \alpha i$ for any $\alpha \in E$. Then, D becomes a ring (the addition, as well as the multiplication between elements of E being the natural ones). Furthermore, D is a division ring. Actually, let $d = \alpha + \beta i$ be a non-zero element in D. If $\beta = 0$, then clearly $d^{-1} = \alpha^{-1}$. In case $\beta \neq 0$, it is easily checked that $(\alpha+\beta i)\cdot(\beta^{-1}\alpha - i)\beta^{-1}((\alpha\beta^{-1})^2+1)^{-1} = 1$ and $((\beta^{-1}\alpha)^2+1)^{-1}(\beta^{-1}\alpha - i)\beta^{-1}\cdot(\alpha+\beta i) = 1$. This means that d is invertible.

Next, let $V = x_1D \oplus x_2D$ be a 2-dimensional vector space over D and consider a ring homomorphism

$$\sigma : D \to (D)_2 \text{ by } \alpha + \beta i \mapsto \begin{pmatrix} \alpha & \beta i \\ \beta i & \alpha \end{pmatrix}.$$

Then we see that σ satisfies the formulas in Theorem 9. Hence we can make a local QF-ring $R = D\langle V, \sigma, \sigma, \theta^*, id_D, 1D^{id_D}\rangle$ and we can see that R is an infinite dimensional algebra with K (its center).

We shall give an example of a division ring E in Example 6. Consider the functional field $L = \mathbb{R}(x)$ over the field $\mathbb{R}$ of real numbers and let σ be an into monomorphism of L given by $f(x)/g(x) \to f(x^2)/g(x^2)$. Let $L[y;\sigma]$ be a skew-polynomial ring associated with σ. Although $L[y;\sigma]$ is a non-commutative domain, it has the quotient ring which is a division ring. We denote it by E. As is easily seen, the center of E is $\mathbb{R}$ and E is infinite dimensional over $\mathbb{R}$ and it holds that $a^2 \neq -1$ for any non-zero element $a \in E$.

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REFERENCES

[1] E.P. Armendariz and J.K. Park, *Self-injective rings with restricted chain conditions*, Arch. Math. **58** (1992), 24-53.

[2] Y. Baba and K. Oshiro, *On a Theorem of Fuller*, J. Algebra **154, 1** (1993), 86-94.

[3] Changchang Xi, *On the finitistic conjecture*, Advances in Ring Theory, Proceedings of the 4th China-Japan-Korea international Conference, 282-294

[4] J. Clark and D.V. Huynh, *A note on perfect self-injective rings*, Quart. J. Math. (Oxford) **45** (1994), 13-17.

[5] ———, *When is a self-injective semiperfect ring quasi-Frobenius?*, J. Algebra **165** (1994), 531-542.

[6] C. Faith, "Algebra II. Ring Theory", Grundlehren Math. Wiss. 192, Springer-Verlag, Heidelberg/New York/Berlin (1976).

[7] C. Faith and D.V. Huynh, *When self-injective rings are QF: A report on a problem*, J. Algebra Appl. **1** (2002), 72-105.

[8] I. Kikumasa, K. Oshiro and H. Yoshimura, *A construction of local QF-rings with radical cubed zero*, preprint.

[9] I. Kikumasa and H. Yoshimura, *Commutative QF-algebras with radical cube zero*, Comm. Algebra **31** (2003), 1837-1858.

[10] ———, *Isomorphism classes of QF-algebras with radical cube zero*, "Advances in Ring Theory", Proceedings of the 4-th China-Japan-Korea International Symposium on Ring Theory, World Scientific (2005), 106-117.

[11] K. Koike, *On self-injective semiprimary rings*, Comm. Algebra **28** (2000), 4303-4319.

[12] A.I. Kostrikin and I.R. Shafarevich, "Algebra II", Encyclopaedia of Mathematical Sciences 18, Springer-Verlag, Berlin/Heidelberg/New York (1991).

[13] T. Nakayama, *On algebras with complete homology*, Abh.Math.Sem, Univ.Hanburg **22** (1958), 300-307.

[14] W.K. Nicholson and M.F. Yousif, *Mininjective rings*, J. Algebra **187** (1997), 548-578.

[15] ———, *On perfect simple-injective rings*, Proc. Amer. Math. Soc. **125** (1997), 979-985.

[16] ———, "Quasi-Frobenius Rings", Cambridge Tracts in Mathematics 158, Cambridge University Press, Cambridge (2003).

[17] B. Osofsky, *A generalization of quasi-Frobenius rings*, J. Algebra **4** (1966), 373-387.

[18] ———, *Two examples of local artinian rings*, Proc. Amer. Math. Soc. **107** (1989), 63-65.

[19] ———, "Rings with Morita Duality", Lect. Notes Math. 1523, Springer-Verlag, New York/Heidelberg/Berlin (1992).

[20] ———, *A note on perfect self-injective rings*, Comm. Algebra **24** (1996), 749-755.

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DUAL OJECTIVITY OF QUASI-DISCRETE MODULES AND LIFTING MODULES

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ABSTRACT. In [3], K.Oshiro and his students introduced "ojectivity (generalized injectivity)", a new concept of relative injectivity, and using this injectivity we obtained some results for direct sums of extending modules. Afterward, S.H.Mohamed and B.J.Müller [9] defined a dual concept of ojectivity as follows:

Definition. M is said to be *N-dual ojective (or generalized N-projective)* if, for any epimorphism $g : N \to X$ and any homomorphism $f : M \to X$, there exist decompositions $N = N_1 \oplus N_2$, $M = M_1 \oplus M_2$, a homomorphism $h_1 : M_1 \to N_1$ and an epimorphism $h_2 : N_2 \to M_2$, such that $gh_1 = f|_{M_1}$ and $fh_2 = g|_{N_2}$.

The concept of relative dual ojectivity is a generalization of relative projectivity and this projectivity has an important meaning for the study of direct sums of lifting modules (cf. [6], [9]).

In this paper we introduce some results on "dual ojectivity" and apply it to direct sums of quasi-discrete modules.

1. INTRODUCTION

A module M is said to be *lifting* if, it satisfies the following property: For any submodule X of M, there exists a decomposition $M = X^* \oplus X^{**}$ such that $X^* \subseteq X$ and the kernel X/X^* of the canonical epimorphism $M/X^* \to M/X$ is a small submodule of M/X^*, equivalently, $X \cap X^{**}$ is a small submodule of X^{**}. In [9], S.H.Mohamed and B.J.Müller defined dual ojective module. This projectivity plays an important role in the study of direct sums of lifting modules (cf. [6], [9]). Since the structure of dual ojectivity is complicated, it is difficult to see whether dual ojectivity pass to a (finite) direct sum. This problem is not easy even in the case each module is quasi-discrete.

In this paper we consider this problem and apply it to direct sums of quasi-discrete modules.

Throughout this paper R is a ring with identity and all modules considered are unitary right R-modules. A submodule S of a module M is said to be a *small* submodule, if $M \neq K + S$ for any proper submodule K of M and we write $S \ll M$ in this case. Let M be a module and let N and K be submodules of M with $K \subseteq N$. K is said to be a *co-essential* submodule of N in M if $N/K \ll M/K$ and we write $K \subseteq_c N$ in M in this case. Let X be a submodule of M. X is called *co-closed* submodule in M if X has not a proper co-essential submodule in M. X' is called a *co-closure* of X in M if X' is a co-closed submodule of M with $X' \subseteq_c X$ in M.

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A module M has the *finite internal exchange property* if, for any finite direct sum decomposition $M = M_1 \oplus \cdots \oplus M_n$ and any direct summand X of M, there exists $\overline{M_i} \subseteq M_i$ $(i = 1, \cdots, n)$ such that $M = X \oplus \overline{M_1} \oplus \cdots \oplus \overline{M_n}$.

A module M is said to be a *lifting* module if, for any submodule X, there exists a direct summand X^* of M such that $X^* \subseteq_c X$ in M.

Let $\{M_i \mid i \in I\}$ be a family of modules and let $M = \oplus_I M_i$. M is said to be a *lifting module for* the decomposition $M = \oplus_I M_i$ if, for any submodule X of M, there exist $X^* \subseteq M$ and $\overline{M_i} \subseteq M_i$ $(i \in I)$ such that $X^* \subseteq_c X$ in M and $M = X^* \oplus (\oplus_I \overline{M_i})$, that is, M is a lifting module and satisfies the internal exchange property in the direct sum $M = \oplus_I M_i$.

Let X be a submodule of a module M. A submodule Y of M is called a *supplement* of X in M if $M = X + Y$ and $X \cap Y \ll Y$, if and only if Y is minimal with $M = X + Y$. Note that supplement Y of X in M is co-closed in M. A module M is $(\oplus-)$*supplemented* if, for any submodule X of M, there exists a submodule (direct summand) Y of M such that Y is supplement of X in M. A module M is called *amply supplemented* if, X contains a supplement of Y in M whenever $M = X + Y$. We note that

$$\text{lifting} \Rightarrow \text{amply supplemented} \Rightarrow \text{supplemented}.$$

Now we consider the following condition:

($\sharp$) Any submodule of M has a co-closure in M.

Note that a module M is amply supplemented if and only if M is supplemented with a condition ($\sharp$) (cf. [2], [5]).

The reader can refer to [1], [4], [8], [11] and [12] for research on lifting modules, quasi-discrete modules and exchange properties.

2. Generalized Projectivity

A module A is said to be *B-dual ojective* (*generalized B-projective*) if, for any homomorphism $f : A \to X$ and any epimorphism $g : B \to X$, there exist decompositions $A = A_1 \oplus A_2$, $B = B_1 \oplus B_2$, a homomorphism $h_1 : A_1 \to B_1$ and an epimorphism $h_2 : B_2 \to A_2$ such that $g \circ h_1 = f|_{A_1}$ and $f \circ h_2 = g|_{B_2}$ (cf. [9]). Note that every B-projective modules is B-dual ojective.

Now we introduce some properties of the dual ojectivity.

Proposition 2.1 (cf. [9]). Let B^* be a direct summand of B. If A is B-dual ojective, then A is B^*-dual ojective.

Proposition 2.2 (cf. [6, Proposition 2.2]). Let A be a module with the finite internal exchange property and let A^* be a direct summand of A. If A is B-dual ojective, then A^* is B-dual ojective.

Proposition 2.3 (cf. [6, Proposition 2.3]). Let $M = A \oplus B$ be supplemented with ($\sharp$) and let A^* be a direct summand of A. If A is B-dual ojective, then A^* is B-dual ojective.

A ring R is said to be *right perfect* if any right R-module has projective cover. By [10, Theorem 1.3], any submodule N of a module M over a right perfect ring has co-closure of N in M. Thus the following is immediate from Proposition 2.3.

Corollary 2.4. Let R be a right perfect ring, A, B be R-modules and A^* be a direct summand of A. If A is B-dual ojective, then A^* is B-dual ojective.

A module A is said to be *im-small B-projective* if, for any epimorphism $g : B \to X$ and any homomorphism $f : A \to X$ with $\mathrm{Im} f \ll X$, there exists a homomorphism $h : A \to B$ such that $g \circ h = f$ (cf. [5]).

Proposition 2.5. (1) Let A be a module and let $\{B_i \mid i = 1, \cdots, n\}$ be a family of modules. Then A is im-small $\oplus_{i=1}^n B_i$-projective if and only if A is im-small B_i-projective $(i = 1, \cdots, n)$.

(2) Let I be any set and let $\{A_i \mid i \in I\}$ be a family of modules. Then $\oplus_I A_i$ is im-small B-projective if and only if A_i is im-small B-projective for all $i \in I$.

Proposition 2.6 (cf. [6, Proposition 2.5]). Let A be any module and let B be a lifting module. If A is B-dual ojective, then A is im-small B-projective.

The concept of relative dual ojectivity has an important meaning for the study of direct sums of lifting modules.

Theorem 2.7 (cf. [6, Theorem 3.7]). Let $M_1, \cdots, M_n$ be lifting modules with the finite internal exchange property and put $M = M_1 \oplus \cdots \oplus M_n$. Then the following conditions are equivalent.

(1) M is lifting with the finite internal exchange property.

(2) M is lifting for $M = M_1 \oplus \cdots \oplus M_n$.

(3) M_i and $\oplus_{j \neq i} M_j$ are relative dual ojective.

3. Direct sums of quasi-discrete modules

A lifting module M is said to be *quasi-discrete* if M satisfies the following condition (D):

(D) If M_1 and M_2 are direct summands of M such that $M = M_1 + M_2$, then $M_1 \cap M_2$ is a direct summand of M.

Any quasi-discrete module has the internal exchange property [10, Theorem 3.10].

Lemma 3.1 (cf. [7]). Let N be a quasi-discrete module and let $M = M_1 \oplus \cdots \oplus M_n$ be lifting for $M = M_1 \oplus \cdots \oplus M_n$. Assume that M_i is generalized N-projective $(i = 1, \cdots, n)$. Then, for any epimorphism $f : M \to X$ with $\ker f \ll M$ and any epimorphism $g : N \to X$ with $\ker g \ll N$, there exist decompositions $M = \overline{M} \oplus \overline{\overline{M}}$, $N = \overline{N} \oplus \overline{\overline{N}}$ and epimorphisms $\varphi : \overline{\overline{M}} \to \overline{N}$, $\psi : \overline{\overline{N}} \to \overline{M}$ such that $f|_{\overline{\overline{M}}} = g \circ \varphi$ and $g|_{\overline{\overline{N}}} = f \circ \psi$.

By the using lemma above, we can obtain the following propositions.

Proposition 3.2 (cf. [7]). Let N be a quasi-discrete module and $M = M_1 \oplus \cdots \oplus M_n$ be lifting for $M = M_1 \oplus \cdots \oplus M_n$. If M_i is N-dual ojective $(i = 1, \cdots, n)$, then M is N-dual ojective.

Proposition 3.3 (cf. [7]). Let M be a quasi-discrete module and $N = N_1 \oplus \cdots \oplus N_m$ be lifting for $N = N_1 \oplus \cdots \oplus N_m$. If N_i and M are relative dual ojective $(i = 1, \cdots, m)$, then M is N-dual ojective.

The following is immediate from Propositions 3.2, 3.3, Theorem 2.7 and induction.

Theorem 3.4. Let $M_1, \cdots, M_n$ be quasi-discrete modules and put $M = M_1 \oplus \cdots \oplus M_n$. Then the following conditions are equivalent.

(1) M is lifting with the (finite) internal exchange property.

(2) M is lifting for $M = M_1 \oplus \cdots \oplus M_n$.

(3) M_i is M_j-dual ojective $(i \neq j)$.

A module H is said to be *hollow* if it is an indecomposabe lifting module.

Corollary 3.5. Let $H_1, \cdots, H_n$ be hollow modules and put $M = H_1 \oplus \cdots \oplus H_n$. Then the following conditions are equivalent.

(1) M is lifting with the (finite) internal exchange property.

(2) M is lifting for $M = H_1 \oplus \cdots \oplus H_n$.

(3) H_i is H_j-dual ojective $(i \neq j)$.

REFERENCES

[1] J. Clark, C. Lomp, N. Vanaja and R. Wisbauer, Lifting modules, Frontiers in Math., Birkhauser (2006).

[2] L. Ganesan and N. Vanaja, *Modules for which every submodule has a unique coclosure*, Comm. Algebra **30** (2002), 2355–2377.

[3] K. Hanada, Y. Kuratomi and K. Oshiro, *On direct sums of extending modules and internal exchange property*, J.Algebra **250** (2002), 115–133.

[4] M. Harada, Factor categories with applications to direct decomposition of modules, LN Pure Appl. Math. 88, Dekker, New York (1983).

[5] D. Keskin, *On lifting modules*, Comm. Algebra **28** (2000), 3427-3440.

[6] Y. Kuratomi, *On direct sums of lifting modules and internal exchange property*, Comm. Algebra **33** (2005), 1795–1804.

[7] Y. Kuratomi, *Generalized projectivity of quasi-discrete modules*, preprint.

[8] S.H. Mohamed and B.J. Müller, Continuous and Discrete Modules, London Math. Soc., LN **147**, Cambridge Univ. Press, Cambridge (1990).

[9] S.H. Mohamed and B.J. Müller, *Cojective modules*, J.Egyptian Math. **12** (2004), 83–96.

[10] K. Oshiro, *Semiperfect modules and quasi-semiperfect modules*, Osaka J. Math. **20** (1983), 337-372.

[11] K. Oshiro, *Theories of Harada in Aritinian Rings and applications to classical Artinian Rings*, International Symposium on Ring Theory, Trend in Math. Birkhäuser, (2001), 279-301.

[12] R. Wisbauer, Foundations of Module and Ring Theory, Gordon and Breach Science Publications (1991).

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PERFECT ISOMETRIES AND THE ALPERIN-MCKAY CONJECTURE

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ABSTRACT. We give a brief survey of results and conjectures concerning the local determination of invariants of Brauer p-blocks of finite groups. We highlight the connections between the various conjectures, in particular those of Alperin-McKay and of Broué, and identify where further conjectures have to be made. We focus on the problem of generalising Broué's conjecture, and suggest a generalisation of the idea of a perfect isometry. Finally we present evidence that such a generalised perfect isometry should exist in certain cases.

CONTENTS

1. INTRODUCTION

The purpose of this survey, closely based on the author's series of lectures given at the Symposium, is to give motivation for the generalisation of a conjecture of Broué, and to present one possibility for such a generalisation. As such, we are quite selective in the material presented, giving only those results, examples and conjectures which illuminate our chosen path. Hence we apologise in advance for omitting Dade's conjectures and those related to it, as well as some of the excellent work which has been done on Broué's conjectures and on fusion systems.

One of the main parts of the modular representation theory of finite groups concerns *local determination*, which is the determination of invariants of a block of a group by examining so-called local subgroups, with respect to a fixed prime p. Many of the main results in the area may be phrased in this way, for example the Green correspondence and Brauer's first and second main theorems, as well as many conjectures, including the Alperin-McKay conjecture of the title. The Alperin-McKay conjecture predicts a straightforward equality between the number of *height zero* irreducible characters in a

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block and the number of such irreducible characters in a uniquely determined block of a certain subgroup. We describe how Broué's conjecture explains the Alperin-McKay conjecture in restricted cases, resulting in particular in a structured bijection of irreducible characters rather than just an equality of numbers.

In the last part of the survey we will discuss the generalisation of the weaker of Broué's conjectures, and propose such a generalisation.

2. Background in block theory

Excellent references for this section are [5], [10], [12] and [13].

Let G be a finite group and p a prime. In order to study the characteristic zero representations of G in relation to the prime p, we consider a p-modular system $(K, \mathcal{O}, k)$ relating fields K and k of characteristic zero and p respectively via a ring $\mathcal{O}$. The conditions which we take on $(K, \mathcal{O}, k)$ are not intended to be in anyway minimal. Briefly, we let $\mathcal{O}$ be a complete local discrete valuation ring containing a primitive $|G|^3$-root of unity, such that $k = \mathcal{O}/J(\mathcal{O})$ is algebraically closed with $char(k) = p$ and K is the field of fractions of $\mathcal{O}$ (we take $|G|^3$th roots of unity rather than $|G|$th roots because at some stage we may need to take a central extension of G of order dividing $|G|^3$).

Our approach will mostly be motivated by the study of characters, so our first task is to partition the set $\mathrm{Irr}(G)$ of irreducible characters (with respect to K) of G into blocks. The advantage of studying representations one block at a time is that representations associated to the same block share some properties which we can take advantage of.

2.1. Characters in blocks. Decompose the group algebra $\mathcal{O}G$ into indecomposable two-sided ideals:

$$\mathcal{O}G = B_1 \oplus \cdots \oplus B_n.$$

This corresponds to a decomposition of $1 \in Z(\mathcal{O}G)$ into primitive idempotents of $Z(\mathcal{O}G)$, say $1 = e_1 + \cdots + e_n$, with $e_i\mathcal{O}G = B_i$.

Similarly we may decompose kG. For $a = \sum_{g \in G} a_g g \in \mathcal{O}G$, write

$$\bar{a} = \sum_{g \in G} (a_g + J(\mathcal{O}))g \in kG.$$

Then $\bar{1} = \bar{e}_1 + \cdots + \bar{e}_n$ is a decomposition into primitive idempotents of $Z(kG)$, and

$$kG = \bar{e}_1 kG \oplus \cdots \oplus \bar{e}_n kG$$

is a decomposition into indecomposable two-sided ideals. Write $\overline{B}_i = \bar{e}_i kG$. (Note that each such decomposition of $\bar{1}$ lifts to a decomposition 1 into primitive idempotents of $Z(\mathcal{O}G)$). Essential in this is our choice of $\mathcal{O}$ complete.

We call the B_i (and $\overline{B}_i$) *blocks* of G, and the e_i (and $\bar{e}_i$) block idempotents.

Now let M be an $\mathcal{O}G$-module. Then

$$M = B_1 M \oplus \cdots \oplus B_n M.$$

Hence if M is indecomposable, then $M = B_i M$ for some unique block B_i, and we say that M *belongs* to B_i. The same argument holds for kG-modules.

If $\chi \in \mathrm{Irr}(G)$ (the set of irreducible (K-)characters of G), then χ is afforded by some irreducible KG-module V. There is an indecomposable $\mathcal{O}G$-lattice M such that $V =$

$K \otimes_{\mathcal{O}} M$. We say that χ belongs to the block to which M belongs. This is independent of the choice of M.

So we can partition $\mathrm{Irr}(G)$ into sets $\mathrm{Irr}(B_i)$ of characters belonging to B_i.

Alternatively, $\chi \in \mathrm{Irr}(B_i)$ if $\chi(e_j) = \chi(1)$ for $j = i$ and $\chi(e_j) = 0$ otherwise, giving the same partition. In much of block theory there are several different ways of making any definition, and they are usually equivalent.

We can determine the block idempotents quite explicitly from the values of the characters in a block.

The primitive idempotent of $Z(KG)$ corresponding to χ is

$$e_\chi = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g^{-1})g.$$

Fixing a block B, with block idempotent e_B, we have

$$e_B = \sum_{\chi \in \mathrm{Irr}(B)} e_\chi.$$

2.2. **Brauer characters and decomposition matrices.** Our aim here is to give the characters of the (projective) indecomposable summands of $\mathcal{O}G$ as a left $\mathcal{O}G$-module. To do this we use Brauer characters. These are a way of assigning class functions with values in K to simple kG-modules, in order that we may more directly compare the simple kG-modules to $\mathrm{Irr}(G)$.

Let S be a simple kG-module, and let $\rho : G \to GL_t(k)$ be an associated representation.

Let $g \in G_{p'}$, the set of p-regular elements of G, and let m be the p'-part of the exponent of G. Then $\rho(g)$ has eigenvalues which are m-roots of unity. Let ω be a primitive m-th root of unity in K, and note that the groups of m-th roots of unity of K and of k are isomorphic. Say $\omega \to \overline{\omega}$ under such an isomorphism.

$Trace(\rho(g))$ is a sum of m-th roots of unity, say $\sum_i \overline{\omega}^{r_i}$. Define $\varphi(g) = \sum_i \omega^{r_i}$. We call φ the irreducible Brauer character associated to S. This is a class function defined on p-regular elements.

We assign φ to the same block as S, and write $\mathrm{IBr}(B)$ for the set of irreducible Brauer characters belonging to B.

The number of distinct irreducible Brauer characters equals the number of p-regular conjugacy classes of G. Further, the irreducible Brauer characters $\mathrm{IBr}(G)$ span the space of class functions defined on p-regular conjugacy classes of G.

If χ is a character of G, write $\chi_{p'}$ for the restriction of χ to the p-regular conjugacy classes. In fact $\chi_{p'}$ is a non-negative integer linear combination of irreducible Brauer characters. If $\chi \in \mathrm{Irr}(G)$, then write

$$\chi = \sum_{\varphi \in \mathrm{IBr}(G)} d_{\chi\phi},$$

where the $d_{\chi\phi}$ are non-negative integers.

The $d_{\chi\varphi}$ are called the *decomposition numbers* of G, and we call the matrix $D = (d_{\chi\varphi})$ the *decomposition matrix* of G.

If χ and φ are in different blocks, then $d_{\chi\varphi} = 0$, and so we can define the decomposition matrix D_B of a block B.

We obtain the Cartan matrix C by $C = D^T D$, where D^T denotes the transpose of D. Recall that C is the matrix recording the occurrence of the simple kG-modules as composition factors of the projective covers of all the simple kG-modules. Again we may take the Cartan matrix C_B of a block B.

We may now write down the character of a projective indecomposable $\mathcal{O}G$-module P. Now $P/J(\mathcal{O})P$ is a projective indecomposable kG-module, which is the projective cover of a simple kG-module, say S. Let φ be the irreducible Brauer character associated to S. Then P has character

$$\Phi = \Phi_\varphi = \sum_{\chi \in \mathrm{Irr}(G)} d_{\chi\varphi}\chi.$$

Two important facts concerning the characters Φ are that

(i) $\Phi(g) = 0$ whenever g is p-singular (i.e., g has order divisible by p);

(ii) if χ is a character of G such that $\chi(g) = 0$ for all p-singular g, then χ is a $\mathbb{Z}$-linear combination of characters Φ_φ for $\varphi \in \mathrm{IBr}(G)$.

2.3. **Defect groups.** We present one (of several equivalent) characterisations of the defect groups of a block. As we mentioned earlier, we will usually consider local determination of invariants of blocks from invariants of normalisers of p-subgroups. The defect groups of a block are a G-conjugacy class of p-subgroups associated to B, and local determination will occur via normalisers of subgroups of the defect groups.

Let S be an indecomposable kG-module and let $H \leq G$. We say S is H-projective if there is a kH-module T such that $S \mid \mathrm{Ind}_H^G(T)$.

Now if $P \in \mathrm{Syl}_p(G)$, then S is P-projective. Further, if $J \leq H$ and S is J-projective, then S is H-projective.

Hence there are subgroups Q which are minimal such that S is Q-projective, and these must be p-groups. Call these the *vertices* of S.

Using the Mackey decomposition, we can see that the vertices of S form a G-conjugacy class of p-subgroups.

Let B be a block. Define the *defect groups* of B to be the p-subgroups of G maximal amongst the vertices of the simple modules in B.

The defect groups of B form a conjugacy class of p-subgroups of G. If D is a defect group of B and $|D| = p^d$, then we say B has *defect d*. The defect is related to the degrees of the irreducible characters in B (in fact we may also determine the defect groups themselves from the irreducible characters, but we do not describe that here).

For $\chi \in \mathrm{Irr}(G)$, define the defect of χ to be the integer $d(\chi)$ such that $|G|_p = p^{d(\chi)}\chi(1)_p$. Then

$$d = max\{d(\chi) : \chi \in \mathrm{Irr}(B)\}.$$

Examples

(a) We call the block containing the trivial character the *principal block*. The Sylow p-subgroups are the defect groups.

(b) The blocks of defect zero, where the trivial group is the only defect group, are of particular importance. These are simple algebras, and we do not expect to obtain any information about them from local subgroups. A block B has defect zero if and only if $k(B) = 1$. The unique irreducible character χ in a block of defect zero satisfies $\chi(1)_p = |G|_p$.

2.4. **Brauer correspondence.** If we are to compare blocks of G with blocks of subgroups of G, then we need a way of naturally associating them.

Let P be a p-subgroup of G and let $H \leq G$ such that $C_G(P) \leq H \leq N_G(P)$. Define $Br_P : Z(kG) \to Z(kH)$ by defining $Br_P(\hat{C}) = 0$ if $C \cap C_G(P) = \emptyset$ and $Br_P(\hat{C}) = \sum_{g \in C \cap C_G(P)} g$ otherwise, where C is any conjugacy class of G and $\hat{C} = \sum_{g \in C} g$. Then Br_P is an algebra homomorphism, called the *Brauer homomorphism*.

Let b be a block of H, with defect group Q, and suppose that $C_G(Q) \leq H \leq N_G(Q)$. Then there is a unique block B of G such that $Br_Q(\bar{e}_B)\bar{e}_b = \bar{e}_b$. We write $b^G = B$, and call B the *Brauer correspondent* of b in G. We also sometimes call B the *induced block*.

If we fix a p-subgroup D of G, then the Brauer correspondence gives a bijection between blocks of G with defect group D and blocks of $N_G(D)$ with defect group D (this is Brauer's first main theorem of block theory). Further (Brauer's third main theorem), the principal blocks correspond under this bijection.

Note that this is a slightly simplified definition, but one which suffices for our purposes. The Brauer correspondence may be defined in greater generality than this. Note also that there are several different definitions, which are not necessarily equivalent unless we put restrictions on H similar to those above.

3. Local determination

We would like to obtain information about representations of a block B of a group G from information about subgroups. So we have two questions: "what sort of subgroups?", and "what sort of information?"

What sort of subgroups?

One immediate restriction on our choice of subgroup is that we would like the Brauer correspondence to be defined, i.e., we would like there to exist blocks of subgroups with Brauer correspondent B, and we would like the blocks of our subgroups to have a Brauer correspondent. We saw in the previous section conditions for the existence of Brauer correspondents.

Further, Clifford's theorem tells us that $O_p(G) \leq \ker(S)$ for every simple kG-module S. This and other results tells us that in some respects we the presence of normal p-subgroups allows us to obtain information from smaller groups still.

Existing results and conjectures involve the extraction of information from *local subgroups*, which may mean:

- normalisers $N_G(Q)$ of p-subgroups Q
- stablisers (under conjugation) of chains of p-subgroups
- centralisers $C_G(Q)$ of p-subgroups
- subgroups H with $O_p(H) \neq 1$ (see the very nice paper by Thévenaz [16])
- slightly altered versions of the above (e.g., normalisers of subpairs)

What sort of information?

This ranges from numerical information, e.g.,

- $k(B) = |\operatorname{Irr}(B)|$
- $l(B) = |\operatorname{IBr}(B)|$
- $k_0(B) = |\{\chi \in \operatorname{Irr}(B) : d(\chi) = d(B)\}|$ (irreducible characters of *height* zero) elementary divisors of the Cartan matrix

to categorical information, e.g.,

- the derived category $\mathcal{D}^b(B)$
- stable module category $\underline{mod}(B)$

There are many examples of theorems in local determination, but we concentrate here on the conjectures.

Throughout, let B be a block of G with defect group D. One of the earliest conjectures is

Conjecture 1 (Alperin-McKay). *Let b be the unique block of $N_G(D)$ with Brauer correspondent B. Then*

$$k_0(B) = k_0(b).$$

Here, local determination is particularly straightforward. However, we do not always expect to obtain our information from just one source. For example, Alperin's weight conjecture gives $l(B)$ in terms of information from many $N_G(Q)$, for p-subgroups Q. Let $\mathcal{P}_0(G)$ be the set of p-subgroups of G.

Conjecture 2 (Alperin's weight conjecture).

$$l(B) = \sum_{Q \in \mathcal{P}_0(G)} f_0^{(B)}(N_G(Q)/Q),$$

where $f_0^{(B)}(N_G(Q)/Q)$ is the number of Q-projective simple $kN_G(Q)$-modules in blocks with Brauer correspondent B.

The Knörr-Robinson reformulation of Alperin's weight conjecture is even more complicated in terms of the number of local subgroups used, and gives $k(B)$ in terms of an alternating sum over stabilisers of chains of p-subgroups.

Note that in Alperin's conjecture, using properties of the Brauer correspondence it suffices to consider subgroups Q contained in a defect group for B.

If D is abelian, then Alperin's weight conjecture predicts that $l(B) = l(b)$, and the Knörr-Robinson reformulation predicts that $k(B) = k(b)$, i.e., local determination comes from one subgroup. To see this for Alperin's weight conjecture, consider $Q \leq D$, and let S be a simple $kN_G(Q)$-module in a block b with Brauer correspondent B. Let R be a defect group of b with $R \leq D$ (replacing D by a conjugate containing Q if necessary). Then $Q \leq R$ and by [9] we have $R = C_R(Q) \leq Q \leq R$. So b has defect group Q, and so by Brauer's first main theorem b^G has defect group Q. But $b^G = B$, so $D = Q$ after all. But every simple $kN_G(D)$-module is D-projective, so we are done.

Alperin's weight conjecture (and its reformulations by Knörr and Robinson are just two of a wide array of conjectures concerning ever more detailed numerical invariants. These would take a long time to state, so we have only presented those which are relevant to out story.

4. Broué's conjecture

We would like to understand the numerical conjectures introduced in the last section more deeply, for example as consequences of results about the module categories. However, we also saw that in general local determination of numerical invariants involves comparing a number of groups at once. Since we know best how to compare two categories, we start by looking at situations where we expect local determination (of a block B of G) to use just one block of one subgroup. We saw that one such case is where the defect group D is abelian.

Throughout this section, let b be the unique block of $N_G(D)$ with Brauer correspondent B. An excellent reference for this section is [10].

Conjecture 3 (Broué). *Suppose D is abelian. Then the derived categories $\mathcal{D}^b(B)$ and $\mathcal{D}^b(b)$ are equivalent (as triangulated categories).*

Remark 4. Actually, more recent versions of Broué's conjecture state that we should have a splendid equivalence (also known as a Rickard equivalence). This places additional restrictions on the tilting complex giving the derived equivalence, which amongst other things ensure that we also have a family of compatible derived equivalences between various subgroups.

Broué's conjecture is very hard to verify for a given block, but it is known in many cases. For reasons of space we do not attempt to list these here.

We relate Broué's conjecture to numerical conjectures such as Alperin-McKay's

Suppose that $\mathcal{D}^b(B)$ and $\mathcal{D}^b(b)$ are equivalent as triangulated categories (with no restrictions on D). Then $\mathcal{D}^b(\overline{B})$ and $\mathcal{D}^b(\overline{b})$ are also equivalent as triangulated categories . We have:

- *mod*(B) and *mod*(b) (and *mod*$(\overline{B})$ and *mod*$(\overline{b})$) have isomorphic Grothendieck groups (see [K-Z,6.3.3])
- B and b (and $\overline{B}$ and $\overline{b}$) have isomorphic centres (see [K-Z,6.3.2])

In particular,

- $k(B) = k(b)$, $l(B) = l(b)$
- also $k_0(B) = k_0(b)$, although this takes more work to prove.

Hence, in the abelian defect group case, Broué's conjecture gives the Alperin, Knörr-Robinson, Alperin-McKay (and Dade) conjectures.

Now suppose that D is non-abelian. Then in general we do not have $k(B) = k(b)$, $l(B) = l(b)$ (although we do expect $k_0(B) = k_0(b)$). Hence there cannot be a derived equivalence in general.

Even when we do have equality of numerical invariants, e.g., $k(B) = k(b)$, $l(B) = l(b)$, etc., there is sometimes no derived equivalence:

We say that D is a *trivial intersection* (TI) subgroup of G if for each $g \in G - N_G(D)$, we have $D^g \cap D = 1$. If B is a block with TI defect group D, then Alperin's conjecture states that $l(B) = l(b)$, and the Knörr-Robinson reformulation states that $k(B) = k(b)$. Actually

Theorem 5 (An-Eaton [2]). *Suppose B is a block with TI defect groups. Then Alperin's, Alperin-McKay's (and Dade's, Isaacs-Navarro's, Uno's) conjectures all hold for B.*

The principal 2-block of $Sz(8) = {}^2B_2(8)$ has TI defect groups. However, it has long been known that B and b are not derived equivalent in this case. This was first observed by Thompson, but see also Cliff [4], which shows that $Z(B)$ and $Z(b)$ are not isomorphic, and Robinson [15], which we will discuss later.

To summarise , we have numerical conjectures which may be applied to *all* blocks, and in a very restricted case (abelian defect groups) we have a deep structural explanation for them, albeit a conjectural one!

A big problem is how to explain the numerical coincidences in general.

One approach would be to attempt to generalise Broué's conjecture directly, e.g., to generalise the concept of a derived equivalence. Alternatively, we could use invariants of derived categories lying somewhere between the simplest numerical ones (number of irreducible characters, etc.) and the derived equivalence class of a category.

So we try to formulate conjectures implying those of Alperin, Alperin-McKay, Dade's, etc., which hold in some non-abelian defect cases. This should give evidence for possible generalisations of Broué's conjecture.

We begin by looking at some consequences of Broué's conjecture in more depth.

5. Perfect isometries

Excellent references for this section are [3], [7] and [10].

For a block (or sum of blocks) B of a group G, denote by

$$\mathcal{R}(G, B)$$

the additive group of characters generated by $\mathrm{Irr}(B)$. We may identify this with the Grothendieck group of $mod(K \otimes_{\mathcal{O}} B)$. We may consider $\mathcal{R}(G, B)$ as lying in $CF(G, B, K) \subset CF(G, K)$, the space of K-valued class functions spanned by $\mathrm{Irr}(G, B)$.

Let b be a block of another group H. Note that $B \otimes b^\circ$ is a block of $G \times H^\circ$, where b°, H° denote the opposite algebra, group respectively

Given

$$\mu \in \mathcal{R}(G \times H^\circ, B \otimes b^\circ),$$

we define maps

$$I_\mu : CF(H, b, K) \to CF(G, B, K)$$

$$R_\mu : CF(G, B, K) \to CF(H, b, K)$$

where I_μ and R_μ are adjoint linear maps with respect to the usual scalar product on characters, as follows:

Let $\alpha \in CF(H, b, K)$, $\beta \in CF(G, B, K)$, $h \in H$, $g \in G$. Define

$$I_\mu(\alpha)(g) = \frac{1}{|H|} \sum_{h \in H} \mu(g, h^{-1})\alpha(h),$$

$$R_\mu(\beta)(h) = \frac{1}{|G|}\sum_{g\in G}\mu(g^{-1},h)\beta(g).$$

Actually, if we have a linear map $I : \mathcal{R}(H,b) \to \mathcal{R}(G,B)$ then defining

$$\mu = \sum_{\theta\in \mathrm{Irr}(H,b)} I(\theta)\theta$$

gives $I_\mu = I$. This follows from the orthogonality relations for ordinary characters.

Let $\mu \in \mathcal{R}(G \times H^\circ, B \otimes b^\circ)$. So far, the maps R_μ and I_μ induced by μ tell us nothing which relates the structures of B and b. They become more interesting when we require that μ is *perfect*. Before defining perfect characters, we motivate them.

Suppose that $\mathcal{D}^b(A)$ and $\mathcal{D}^b(B)$ are equivalent as triangulated categories.

Then this equivalence may be induced by a bounded complex

$$M : \cdots \to M_{-r} \to M_{-r+1} \to \cdots \to M_s \to \cdots$$

of B-b-bimodules such that each M_r is projective as a B-module and as a b-module (see [10]).

Let μ_r be the character afforded by $K \otimes_{\mathcal{O}} M_r$. Then the generalised character

$$\mu = \sum_r (-1)^r \mu_r$$

gives an isometry I_μ. In particular, if Broué's conjecture holds, then we get an isometry $CF(N_G(D), b, K) \to CF(G,B,K)$ related to the complex inducing the equivalence of categories.

A complex M of B-b-bilmodules whose terms are projective as B-modules and as b-modules is called a *perfect complex*, and the definition of a perfect generalised character is related to this.

Denote by $CF_{p'}(G,B,K)$ the subspace of class functions $\alpha \in CF(G,B,K)$ such that if $g \in G - G_{p'}$, then $\alpha(g) = 0$.

Definition 6. $\mu \in \mathcal{R}(G \times H^\circ, B \otimes b^\circ)$ is *perfect* if

(a) I_μ gives a map $CF(H,b,\mathcal{O}) \to CF(G,B,\mathcal{O})$ and R_μ gives a map $CF(G,B,\mathcal{O}) \to CF(H,b,\mathcal{O})$

(b) I_μ gives a map $CF_{p'}(H,b,\mathcal{O}) \to CF_{p'}(G,B,\mathcal{O})$ and R_μ gives a map $CF_{p'}(G,B,\mathcal{O}) \to CF_{p'}(H,b,\mathcal{O})$.

Proposition 7 (Broué). $\mu \in \mathcal{R}(G \times H^\circ, B \otimes b^\circ)$ *is* perfect *if and only if*

(a') for all $(g,h) \in G \times H$, we have $\mu(g,h)/|C_G(g)| \in \mathcal{O}$ and $\mu(g,h)/|C_H(h)| \in \mathcal{O}$,

(b') if $\mu(g,h) \neq 0$, then both g and h are p-singular or both g and h are p-regular.

Remark 8. Suppose that a character μ of $G \times H^\circ$ is afforded by an $\mathcal{O}(G \times H^\circ)$-module which is projective as G- and H°-modules. Then μ is perfect.

Example 9. Suppose $H \leq G$, and let μ be the character of the KG-KH-bimodule KG. Then μ is perfect. Here I_μ is induction and R_μ is restriction of characters. Explicitly,

$$\mu(g,h) = \sum_{\chi\in\mathrm{Irr}(G)}\sum_{\theta\in\mathrm{Irr}(H)} (Res^G_H(\chi),\theta)\chi(g)\theta(h).$$

Similarly, $K \otimes B$ may be considered as a $K \otimes B$-$K \otimes b$-bimodule in this way, to give blockwise induction and restriction, which means induction and restriction, but only taking only components in B or b.

We define a *perfect isometry* to be a map I_μ which is an isometry, such that μ is perfect. The inverse map is R_μ. This gives a 'bijection with signs' between $\mathrm{Irr}(B)$ and $\mathrm{Irr}(b)$.

Broué conjectures that:

Conjecture 10 (Broué's isometry conjecture). *Let B be a block with abelian defect group D, and let b be the unique block of $N_G(D)$ with $b^G = B$. Then there is a perfect isometry*

$$I_\mu : CF(N_G(D), b, K) \to CF(G, B, K).$$

Remark 11. When discussing Conjecture 3, we mentioned splendid equivalences, which give families of derived equivalences. This is in part motivated by a stronger form of the above conjecture, which predicts an *isotypy*. This is a family of compatible perfect isometries. However, we will not discuss these in detail here, although they are very important to the subject. Actually, in some sense they aid the search for a perfect isometry between B and b. We should further remark however, that perfect isometries arising from stable equivalences in the TI defect group situation (in a similar way to property (P+) later) automatically give isotypys.

5.1. **Invariants preserved by perfect isometries.** Suppose that I_μ is a perfect isometry. Define

$$I_\mu^0 : Z(KHe_b) \to Z(KGe_B)$$

by

$$I_\mu^0(a) = \left(\frac{1}{H} \sum_{g \in G} \sum_{h \in H} \mu(g^{-1}, h) a_h \right) g,$$

where $a = \sum_{h \in H} a_h h$.

Since μ is perfect, this also defines an invertible $\mathcal{O}$-linear map

$$Z(\mathcal{O}He_b) \to Z(\mathcal{O}Ge_B).$$

Write R_μ^0 for the analogous map $Z(KGe_B) \to Z(KHe_b)$. Then $a \to I_\mu^0(aR_\mu^0(e_B))$ defines an algebra isomorphism

$$Z(\mathcal{O}He_b) \to Z(\mathcal{O}Ge_B).$$

The calculations used to show the algebra isomorphism can also be used to show that for each $\theta \in \mathrm{Irr}(b)$,

$$\frac{|G|/I_\mu(\theta)(1)}{|H|/\theta(1)} \in \mathcal{O}$$

and is invertible in $\mathcal{O}$. Hence I_μ preserves the defects of the ordinary irreducible characters. Since $d(B) = max\{d(\chi) : \chi \in \mathrm{Irr}(B)\}$, this means that the defect of a block is preserved. (It is not known - to the authors knowledge - that a perfect isometry, or even Morita equivalence preserves the isomorphism class of a defect group, although neither is the author aware of a counterexample).

It is also the case that, modulo p,

$$\frac{|G|/I_\mu(\theta)(1)}{|H|/\theta(1)}$$

is independent (up to sign) of the choice of $\theta \in \mathrm{Irr}(b)$.

Now suppose further that $H = N_G(D)$, that B (and so b) is the principal block, and that $I_\mu(1_H) = \pm 1_G$. Then D is a Sylow p-subgroup, and

$$[G : N_G(D)] \equiv 1 \ mod\ p.$$

Then

$$I_\mu(\theta)(1) \equiv \pm\theta(1) \ mod\ p.$$

This is a motivation for the following strengthening of the Alperin-McKay conjecture (although we do not claim that it was the original motivation).

Let B be a block of a group G with defect group D. Let b be the unique block of $N_G(D)$ with $b^G = B$. Let r be an integer. Write

$$\mathrm{Irr}(B, [r]) = \{\chi \in \mathrm{Irr}(B) : \frac{|G|}{\chi(1)_p} \equiv \pm r \ mod\ p\}$$

and $k(B, [r]) = |\mathrm{Irr}(B, [r])|$.

Conjecture 12 (Isaacs-Navarro). *For each integer r, we have $k_0(B, [r]) = k_0(b, [r])$.*

So in the above situation, for the principal block, the Isaacs-Navarro conjecture is a consequence of a perfect isometry.

Remark 13. (a) Uno has announced a generalisation of the Isaacs-Navarro conjecture to arbitary character defects, which is also a strengthening of Dade's conjecture.

(b) Just as with the other numerical conjectures, when the defect group is TI, a straightforward equality is predicted, with all information coming from just one local subgroup, $N_G(D)$. I regard this as evidence that there should be a generalisation of a perfect isometry which at least holds in the TI defect group case.

Other invariants which are preserved by perfect isometries are $l(B)$ and the elementary divisors of the Cartan matrix.

Further evidence that the TI defect group case should be similar to the abelian defect group case is the following consequence of Theorem 5 (see [6]):

Proposition 14. *Suppose that B has TI defect group D, and let b be the unique block of $N_G(D)$ with Brauer correspondent B. Then the Cartan matrices of B and b have the same elementary divisors.*

5.2. **Existence and non-existence of perfect isometries.** As mentioned earlier, Cliff in [4] has proved that if G is $Sz(8)$ and B is the principal 2-block, then $Z(B)$ is not isomorphic to $Z(b)$, where b is the Brauer correspondent of B in $N_G(D)$. Hence there can be no perfect isometry in this case.

Robinson in [15] gives general conditions for the non-existence of a perfect isometry, based on the block having many irreducible characters constant on p-singular conjugacy classes when $N_G(D)/O_{p'}(N_G(D))$ is a Frobenius group. Such a condition can be checked easily for, e.g., the Suzuki groups.

As before, we do not attempt to list the cases for which Broué isometry conjecture is known. However, we draw the reader's attention to what may be considered the high point of work on the conjecture, which is [7], where it is proved that the conjecture holds for the principal block for $p = 2$.

6. Generalising perfect isometries

If we believe the numerical conjectures, then in general we expect local determination to be complicated, because we expect information to come from a number of subgroups simultaneously, as in Alperin's weight conjecture. However, in some cases, the numerical conjectures suggest that we may find information from just one subgroup. An example is when $N_G(D)$ controls fusion in D, which includes the case D is abelian. This also includes the case that D is TI.

We present here an observation of some very compelling behaviour in the TI defect group case, which leads to a generalisation of Conjecture 10. Most of the results in this section are taken from [6].

Throughout, let B be a block of G with defect group D, and let b be the unique block of $N_G(D)$ with Brauer correspondent B.

Definition 15. We say that B satisfies property (P) if there is perfect $\mu \in \mathcal{R}(G \times N_G(D)^{\circ}, B \otimes b^{\circ})$ such that for each $\theta \in Irr_0(N_G(D), b)$, the map

$$I_\mu : CF(N_G(D), b, K) \to CF(G, B, K)$$

induced by μ satisfies

$$I_\mu(\theta) = \epsilon\chi + \Delta$$

for some $\chi \in \mathrm{Irr}_0(G, B)$, where $\epsilon \in \{-1, 1\}$ and no constituent of Δ has height zero, and for each $\chi \in \mathrm{Irr}_0(G, B)$, the map

$$R_\mu : CF(G, B, K) \to CF(N_G(D), b, K)$$

satisfies

$$R_\mu(\chi) = \epsilon\theta + \Theta$$

for some $\theta \in \mathrm{Irr}_0(N_G(D), b)$ where $\epsilon \in \{-1, 1\}$ and no constituent of Θ has height zero.

Remark 16. (a) Property (P) gives rise to a bijection 'with signs' between $\mathrm{Irr}_0(B)$ and $\mathrm{Irr}_0(b)$, just as a perfect isometry does.

(b) If I_μ is a perfect isometry, then μ gives (P).

(c) If D is abelian, then Brauer's abelian defect group conjecture predicts that $\mathrm{Irr}_0(B) = \mathrm{Irr}(B)$, and we already know that $\mathrm{Irr}_0(b) = \mathrm{Irr}(b)$. Hence if Brauer's conjecture is true and D is abelian, then I_μ is a perfect isometry if and only if μ gives (P).

(P) holds for every example of a block with TI defect groups so far checked. Unfortunately, (P) does not hold for all blocks, for example:

Proposition 17. *Suppose that B is the principal block of $G = PSL_3(2)$. Then no choice of μ can give (P).*

Proof. $D \cong D_8$ and $N_G(D) = D$. By checking the short list of possibilities, we cannot have $\mu(1, h) = 0$ for each nontrivial $h \in D$. □

Note that $G = PSL_3(2)$ has non-TI Sylow 2-subgroups. Further, $D = N_G(D)$ does not control fusion in D.

However, we will see in the first example that often something stronger than (P) actually holds.

Now suppose that $H = N_G(D)$ and that $b^G = B$. Consider 'blockwise induction and restriction', which is given by $\Phi = \sum_{\chi \in \mathrm{Irr}(B)} \sum_{\theta \in \mathrm{Irr}(b)} (\mathrm{Res}^G_{N_G(D)}(\chi), \theta)\chi\theta$. The map I_Φ is then 'induction, taking terms in B,' and R_Φ is 'restriction, taking terms in b.' Just as with induction/restriction, this is a perfect character.

Definition 18. We say that B satisfies (P+) if there is $\mu \in \mathcal{R}(G \times N_G(D)^\circ, B \otimes b^\circ)$ giving (P) of the form $\mu = \Phi + \sum_{s,t} a_{s,t}\Gamma_s\Phi_t$, where each $a_{s,t}$ is an integer and Γ_s resp. Φ_t is the character of a projective indecomposable module of B resp. b.

A generalised character μ of this form is necessarily perfect.

Remark 19. If B satisfies property (P+), then it is immediate that Conjecture 12 holds for that block.

We give the following example in full as an illustration. Note that the block we use does not have TI defect groups, but (P+) holds anyway.

Example 20. Let $G = S_5$ and $p = 2$. Let B be the principal block of G, so B has defect group $D \cong D_8$, a Sylow 2-subgroup of G. We have $N_G(D) = D$. Now the irreducible characters of G are $\chi_1 \ldots, \chi_7$, with degrees $1, 1, 4, 4, 5, 5, 6$ respectively. We have $\mathrm{Irr}(B) = \{\chi_1, \chi_2, \chi_5, \chi_6, \chi_7\}$. The irreducible characters of $N_G(P)$ are $\theta_1, \ldots, \theta_5$, with degrees $1, 1, 1, 1, 2$ respectively. We will need the following characters of projective indecomposable modules: $\Gamma = \chi_5 + \chi_6 + \chi_7$ and $\Phi_1 = \theta_1 + \cdots + \theta_4 + 2\theta_5$.

The restrictions of the irreducible characters of B to $N_G(D)$ are as follows:

$$\begin{array}{lclcl}
\mathrm{Res}^G_{N_G(D)}(\chi_1) & = & \theta_1 & = & \theta_1 \\
\mathrm{Res}^G_{N_G(D)}(\chi_2) & = & \theta_3 & = & \theta_3 \\
\mathrm{Res}^G_{N_G(D)}(\chi_5) & = & \theta_1 + \theta_2 + \theta_3 + \theta_5 & = & \Phi_1 - \theta_4 - \theta_5 \\
\mathrm{Res}^G_{N_G(D)}(\chi_6) & = & \theta_1 + \theta_3 + \theta_4 + \theta_5 & = & \Phi_1 - \theta_2 - \theta_5 \\
\mathrm{Res}^G_{N_G(D)}(\chi_7) & = & \theta_2 + \theta_4 + 2\theta_5 & = & \Phi_1 - \theta_1 - \theta_3
\end{array}$$

Hence $\mu = \Phi - \Gamma\Phi_1$ gives the bijection with signs

$$\begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_5 \\ \chi_6 \end{pmatrix} \leftrightarrow \begin{pmatrix} \theta_1 \\ \theta_3 \\ -\theta_4 \\ -\theta_2 \end{pmatrix}.$$

(P) is partially motivated by the following, from [14]:

Theorem 21 (Navarro). *Let G be a p-solvable group such that $N_G(P) = P$ for a Sylow p-subgroup P. Then*

(a) for each $\theta \in \mathrm{Irr}(P)$ with $\theta(1) = 1$, we have $\mathrm{Ind}^G_P(\theta) = \chi + \Delta$ where $\theta \in \mathrm{Irr}(G)$ with $p \nmid \chi(1)$ and $p|\delta(1)$ for each irreducible constituent δ of Δ, and

(b) for each $\chi \in \mathrm{Irr}(G)$ with $p \nmid \chi(1)$, we have $\mathrm{Res}^G_P(\chi) = \theta + \Theta$, where $\theta \in \mathrm{Irr}(P)$ with $\theta(1) = 1$ and $p|\gamma(1)$ for each irreducible constituent γ of Θ.

This means that for the principal block of a p-solvable group with $N_G(P) = P$, the character for Φ for induction/restriction gives property (P+).

6.1. Controlled blocks. The principal block is a *controlled block* if, for P a Sylow p-subgroup of G, if $Q \leq P$ and $g \in G$ such that $Q^g \in P$, then $g = cn$ for some $c \in C_G(G)$ and $h \in N_G(P)$. There are examples of controlled blocks which do not have TI defect groups. E.g., the principal 3-blocks of J_2 and J_3, and also the principal 5-block of Co_3.

(P+) holds for J_2 and J_3, but not for Co_3. However it is not clear whether (P) holds for Co_3.

6.2. Conjectures. We feel confident that the following holds:

Conjecture 22. *Let B be a block with TI defect groups. Then (P+) holds for B.*

We speculate that, if $N_G(D)$ controls fusion in D, then (P) holds.

6.3. Checking the conjectures. The following is an important example, since it is the original example of a block with TI defect groups such that the conclusions of Broué's conjecture fail.

Example 23. Let $G = {}^2B_2(8)$ and $p = 2$. Let B be the principal block and $P \in \mathrm{Syl}_p(G)$.

The irreducible characters of $N_G(P)$ are $\theta_1, \ldots, \theta_{10}$, with degrees $1, 1, 1, 1, 1, 1, 1, 7, 14, 14$ respectively. These all lie in the principal block b. The irreducible characters of G are $\chi_1, \ldots, \chi_{11}$, with degrees $1, 14, 14, 35, 35, 35, 64, 65, 65, 65, 91$ respectively. All but χ_7 lie in B.

The characters of the projective indecomposable modules of $N_G(P)$ are $\Phi_i = \theta_i + \theta_8 + 2\theta_9 + 2\theta_{10}$, for $1 \leq i \leq 7$. The characters of the relevant projective indecomposable modules of B are

$\Gamma_2 = \chi_2 + \chi_3 + \chi_4 + 2\chi_5 + \chi_6 + 2\chi_8 + 2\chi_9 + 3\chi_{10} + 3\chi_{11}$,

$\Gamma_3 = \chi_2 + \chi_3 + \chi_4 + \chi_5 + 2\chi_6 + 3\chi_8 + 2\chi_9 + 2\chi_{10} + 3\chi_{11}$,

$\Gamma_4 = \chi_2 + \chi_3 + 2\chi_4 + \chi_5 + \chi_6 + 2\chi_8 + 3\chi_9 + 2\chi_{10} + 3\chi_{11}$,

$\Gamma_5 = \chi_5 + \chi_8 + \chi_{10} + \chi_{11}$,

$\Gamma_6 = \chi_6 + \chi_8 + \chi_9 + \chi_{11}$,

$\Gamma_7 = \chi_4 + \chi_9 + \chi_{10} + \chi_{11}$.

We give the restrictions of the χ_i below, along with constituents of the images in R_μ in $\mathrm{Irr}_0(b)$ (which we write as R_μ^0), where

$$\mu = \Phi - (\Gamma_4 - \Gamma_5 - \Gamma_6 - \Gamma_7)\Phi_2 - (\Gamma_2 - \Gamma_5 - \Gamma_6 - \Gamma_7)\Phi_3 - (\Gamma_3 - \Gamma_5 - \Gamma_6 - \Gamma_7)\Phi_4,$$

and Φ is as before.

χ_i	$\mathrm{Res}^G_{N_G(P)}(\chi_i)$	$R^0_\mu(\chi_i)$
χ_1	θ_1	θ_1
χ_2	$\theta_9 = \Phi_2 + \Phi_3 + \Phi_4 - \theta_2 - \theta_3 - \theta_4 - 3\theta_8 - 5\theta_9 - 6\theta_{10}$	$-\theta_2 - \theta_3 - \theta_4 - 3\theta_8$
χ_3	$\theta_{10} = \Phi_2 + \Phi_3 + \Phi_4 - \theta_2 - \theta_3 - \theta_4 - 3\theta_8 - 6\theta_9 - 5\theta_{10}$	$-\theta_2 - \theta_3 - \theta_4 - 3\theta_8$
χ_4	$\theta_8 + \theta_9 + \theta_{10} = -\theta_2 - \theta_9 - \theta_{10} + \Phi_2$	$-\theta_2$
χ_5	$\theta_8 + \theta_9 + \theta_{10} = -\theta_3 - \theta_9 - \theta_{10} + \Phi_3$	$-\theta_3$
χ_6	$\theta_8 + \theta_9 + \theta_{10} = -\theta_4 - \theta_9 - \theta_{10} + \Phi_4$	$-\theta_4$
χ_8	$\theta_4 + \theta_5 + \theta_8 + 2\theta_9 + 2\theta_{10} = \theta_5 + \Phi_4$	θ_5
χ_9	$\theta_2 + \theta_7 + \theta_8 + 2\theta_9 + 2\theta_{10} = \theta_7 + \Phi_2$	θ_7
χ_{10}	$\theta_3 + \theta_6 + \theta_8 + 2\theta_9 + 2\theta_{10} = \theta_6 + \Phi_3$	θ_6
χ_{11}	$\theta_8 + 3\theta_9 + 3\theta_{10}$	θ_8

We are able to verify that (P+) holds when B is the principal 2-block of any ${}^2B_2(2^{2m+1})$, and when B is any p-block of $SU_3(p^m)$. We are also able to prove the following:

Theorem 24. *Let p be 5 or a prime such that $3 \nmid (p+1)$. Let B be a block with TI non-abelian defect group D such that $|D| \leq p^5$. Then (P) holds for B.*

Remark 25. Further, (P+) holds if G is quasisimple.

Outline of proof: We use Clifford-theoretic methods similar to those in [2] to reduce to non-abelian simple groups, their automorphism groups and their covering groups. In [2] certain Morita equivalences are constructed to achieve a similar reduction, and we use the fact that Morita equivalences give perfect isometries.

It suffices to consider blocks with TI defect groups of central p'-extensions of automorphism groups of non-abelian simple groups.

These have been classified in [1], and it suffices to check the following cases:

(a) $D \cong 3^{1+2}_-$ and G is $\mathrm{Aut}({}^2G_2(3)') = {}^2G_2(3)$;

(b) $D \cong 5^{1+2}_+$ and G is $3.McL$, $\mathrm{Aut}(McL)$, $SU_3(5)$, $GU_3(5)$, $PSU_3(5).2$ or $PGU_3(5).2$, where the extension is by the unique field automorphism of order 2;

(c) $D \cong 5^{1+2}_-$ and G is $\mathrm{Aut}({}^2B_2(32))$;

(d) $D \cong p^{1+2}_+$ and G is $PSU_3(p)$ or $PSU_3(p).2$, where the extension is by the unique field automorphism of order 2 and $3 \nmid p+1$.

Finally, we have checked all of these cases.

6.4. **Other generalisations.** (I) The problem of generalising perfect isometries has also been studied by Jean-Baptiste Gramain in [8].

He uses the definition of a perfect isometry in Külshammer-Olsson-Robinson's paper [11] on generalised blocks of symmetric groups. The generalisation does not include Broué's conjecture on perfect isometries, but does give an isometry involving *all* irreducible characters. Gramain verifies the conjecture for various classes of blocks with TI defect groups. It is not clear whether a counterexample exists when the defect group is not TI.

(II) In the main part of this section, we have been attempting to generalise the idea of a perfect isometry by generalising from an isometry, whilst still considering perfect characters. We may also attempt to find isometries with strong structural properties so that we may generalise Broué's conjecture. In examples of blocks with TI defect groups tested, the following occurs:

There exists an isometry

$$I_\mu : CF(N_G(D), b, K) \to CF(G, B, K)$$

where μ satisfies

(*) Suppose $\mu(g,h) \neq 0$. Let g_p be the (uniquely defined) part of g, and h_p the p-part of h. Then either g_p and h_p are both conjugate to an element of the derived subgroup D', or neither are.

In the case that D is abelian this is one of the conditions for a perfect isometry.

However, there is little evidence for this phenomenon, and there is no analogue for the other condition for a perfect isometry. Also, there are counterexamples when D is not TI.

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References

[1] J. An and C. W. Eaton, *Blocks with trivial intersection defect groups*, Math. Z. **247** (2004), 461–486.
[2] J. An and C. W. Eaton, *Modular representation theory of blocks with trivial intersection defect groups*, Algebras and Representation Theory **8** (2006), 427–448.
[3] M. Broué, *Isométries parfaites, types de blocs, catégories dérivées*, Astérisque **181-182** (1990), 61–92.
[4] G. Cliff, *On centers of 2-blocks of Suzuki groups*, J. Algebra **226** (2000), 74–90.
[5] C. W. Curtis and I. Reiner, *Methods of Representation Theory, I and II*, Wiley (1986).
[6] C. W. Eaton, *Perfect generalized characters inducing the Alperin-McKay conjecture*, in preparation.
[7] P. Fong and M. Harris, *On perfect isometries and isotypies in finite groups*, Invent. Math. **114** (1993), 139–191.
[8] J-B. Gramain, *Generalized perfect isometries in some groups of Lie rank one*, J. Algebra **299** (2006), 820–840.
[9] R. Knörr, *On the vertices of irreducible modules*, Annals of Math. **110** (1979), 487–499.
[10] S. König and A. Zimmermann, *Derived Equivalences for Group Rings*, Springer LNM 1685 (1998).
[11] B. Külshammer, J. B. Olsson and G. R. Robinson, *Generalized blocks for symmetric groups*, Invent. Math. **151** (2003), 513–552.
[12] H. Nagao and Y. Tsushima, *Representations of Finite Groups*, Academic Press (1987).
[13] G. Navarro, *Characters and Blocks of Finite Groups*, CUP (1998).
[14] G. Navarro, *Linear characters of Sylow subgroups*, J. Algebra **269** (2003), 589-598.
[15] G. R. Robinson, *A note on perfect isometries*, J. Algebra **226** (2000), 71–73.
[16] J. Thévenaz, *Locally determined functions and Alperin's conjecture*, J. London Mathematical Society **45** (1992), 446–468.

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MAXIMAL ORDERS AND VALUATION RINGS IN NONASSOCIATIVE QUATERNION ALGEBRAS

JOHN S. KAUTA

ABSTRACT. A *nonassociative quaternion algebra* over a field F is a four-dimensional F-algebra A whose nucleus is a separable quadratic extension field of F. We define the notion of a valuation ring for A, and we also define a value function on A with values from a totally ordered group. We determine the structure of the set on which a value function assumes non-negative values. The main result of this paper states that, given a valuation ring of a quaternion algebra A, there is a value function associated to it if and only if the valuation ring is invariant under proper F-automorphisms of A and is integral over its center. We later restrict our attention to the case when the nucleus is a tamely ramified and defectless extension of F. With this assumption, we determine the precise connection between value functions, valuation rings, and maximal orders in A – the latter in the event F is discretely valued. We give various examples that illustrates the difference between the associative valuation theory and the nonassociative one.

Key Words: Value functions, valuation rings, maximal orders, nonassociative quaternion algebras.

1. INTRODUCTION

A ring will have a multiplicative unit element and, unless the context demands otherwise, will be assumed to be nonassociative. Let A be an algebra over a field F. The *nucleus* $N = N(A)$ of A is the set of elements of A which associate with every pair of elements of A, that is,

$$(ab)c = a(bc)$$

when one of the elements is in N. It is an associative subalgebra of A. The center $Z(A)$ of A is

$$\{z \in N \mid za = az \forall a \in A\}.$$

The algebra A is said to be *simple* in case 0 and A are the only ideals of A. It is called *central simple* if $A \otimes_F L$ is simple for every field extension L of F. It is said to be a *division algebra* if it is not the zero ring and the equations

$$ax = b, \quad ya = b$$

have unique solutions $x, y \in A$ for all $a \neq 0, b \in A$. *We shall always assume that A is a finite dimensional division algebra over F in this paper.* By [7, Theorem 2.1], $Z(A)$ is a field extension of F and A is a central simple algebra over $Z(A)$. Since A is a division algebra, a routine argument shows that N is an associative division algebra. If R is a ring, let

$$U(R) = \{a \in R \mid \ ba = ac = 1 \text{ for some } b, c \in R\}.$$

The detailed version of this paper has been submitted for publication elsewhere.

Observe that if R is associative, then $U(R)$ is simply the group of multiplicative units of R. If R is an associative ring, then $J(R)$ will denote its Jacobson radical and $\overline{R} = R/J(R)$.

This paper is organized in the following manner. In Section 1 we define the notions of a valuation ring and value functions on arbitrary nonassociative finite dimensional division algebras, and we also state some elementary general results. The rest of the paper, however, is entirely devoted to how the valuation rings and value functions thus defined relate to nonassociative quaternion algebras. In Section 2, we determine the structure of the set on which a value function assumes non-negative values. The main result of this paper is Theorem 14 in Section 3, which states that, given a valuation ring of a quaternion algebra A, there is a value function associated to it if and only if the valuation ring is invariant under proper F-automorphisms of A and is integral over its center. In Section 4, we restrict our attention to the case when the nucleus is a tamely ramified and defectless extension over F. With this assumption, we determine the precise connection between value functions, valuation rings, and maximal orders in A – the latter in the event F is discretely valued. Finally in Section 5, we give various examples that illustrates the difference between the associative valuation theory and the nonassociative one. Also, the examples demonstrate the necessity of certain assumptions made earlier on in the paper. The reader interested in learning more about nonassociative quaternion algebras is referred to articles [3, 10].

If A is associative, recall that a subring B of A is called a (Dubrovin) valuation ring of A if there is an ideal I of B such that:

(a) B/I is simple Artinian,

(b) if $x \in A \setminus B$, then there are $b_1, b_2 \in B$ with $b_1x, xb_2 \in B \setminus I$

(see [1, 5, 9]). Note that since A is finite dimensional over its center, B/I is a PI-ring for any ideal I of B hence, if I is a maximal ideal of B, then B/I must be Artinian. Therefore condition (a) can be replace by the weaker:

(a′) I is a maximal ideal of B.

We therefore make the following definition in the nonassociative setting, leaving out the Artinianness condition.

Definition 1. Let A be a division algebra finite dimensional over its center F. If B is a subring of A and I is a maximal ideal of B such that, if $x \in A \setminus B$, then there are $b_1, b_2 \in B \cap N$ with $b_1x, xb_2 \in B \setminus I$, then we shall call (B, I) a valuation ring pair of A.

If (B, I) a valuation ring pair of A, we shall sometimes simply refer to B as a valuation ring of A if there is no danger of confusion. We set $\overline{B} = B/I$. Observe that, if A were associative, then our definition of a valuation ring above would become that of a Dubrovin valuation ring.

Now let Γ be a totally ordered group, written additively for convenience although it is not assumed to be abelian.

Definition 2. A value function on A with value group Γ is a surjection $w : A \mapsto \Gamma \cup \{\infty\}$ such that for all $a, b \in A$ we have:

(1) $w(a) = \infty$ if and only if $a = 0$,

(2) $w(a + b) \geq \min\{w(a), w(b)\}$,

(3) $w(ab) \geq w(a) + w(b)$.

(4) $w(a^{-1}) = -w(a) \; \forall \, a \in U(N)$,
(5) $im(w) = w(U(N)) \cup \{\infty\}$

(cf [5, Definition 2.1]).

The following lemma is now self-evident, or can be proved in the same manner as the statements in [5, Lemma 2.2].

Lemma 3. *Suppose A has a value function w. Then*
(1) $w(ab) = w(a) + w(b), w(ba) = w(b) + w(a) \; \forall \, a \in U(N)$.
(2) *$w|_N$ is a valuation on N.*
(3) *If $w(a) \neq w(b)$, then $w(a \pm b) = \min\{w(a), w(b)\}$.*
(4) *$B_w = \{a \in A \mid w(a) \geq 0\}$ is a subring of A and $J_w = \{a \in A \mid w(a) > 0\}$ is a two-sided ideal of B_w.*

We will denote B_w/J_w by $\overline{B}_w$.

Remark 4. Given a valuation w on an associative division algebra A, it does satisfy [5, Definition 2.1]. In particular, condition (4) of [5, Definition 2.1] is satisfied, that is,

$$im(w) = w(st(w)) \cup \{\infty\}, \quad \text{where } st(w) = \{s \in U(A) \mid w(s^{-1}) = -w(s)\}.$$

Notice that if w is a valuation then the stabilizer of w, $st(w)$, coincides with $U(N)$ in the associative case. In fact, by [5, Lemma 2.2], a value function in the sense of [5] is a valuation if and only if $st(w) = U(A)$ $(= U(N))$.

If A is nonassociative, we split condition (4) of [5, Definition 2.1] into two parts, namely (4) and (5) of Definition 2. As a result, and in view of Lemma 3(2) above, our value functions generalize valuations on associative division algebras, rather than value functions of [5]. Further, the nucleus plays a role synonymous to that played by the stabilizer in the associative case. Consequently, the value group of A, which coincides with $im(st(w))$ in the associative case, is now equal to the value group of N.

Finally, given a value function w on A, just as in the proof of [5, Theorem 2.4], for each $x \in A \setminus B_w \; \exists \; b_1, b_2 \in B \cap N$ such that $b_1x, xb_2 \in B_w \setminus J_w$: choose $t \in U(N)$ such that $w(t) = w(x)$ and set $b_1 = b_2 = t^{-1}$. Hence (B_w, J_w) is a valuation ring pair if and only if J_w is a maximal ideal of B_w.

Proposition 5. *We have the following:*
(1) *Γ is abelian.*
(2) $aB_wa^{-1} = B_w \forall a \in U(N)$.
(3) *One-sided ideals of the form aB_w $(= B_wa), a \in N$, are actually two-sided, and are totally ordered by inclusion.*

2. Value Functions on Nonassociative Quaternion Algebras

A nonassociative quaternion algebra over F is a four-dimensional F-algebra A, with a unit element, whose nucleus N is a separable quadratic extension field of F. If $x \mapsto \hat{x}$ denotes the F-involution on N, then by [10]

$$A = N \oplus NJ$$

where $Jx = \hat{x}J \ \forall \ x \in N$ and $J^2 = b \in N \setminus F$.

Incidentally, given a four-dimensional F-algebra of the form $A' = N \oplus NJ'$ where $J'x = \hat{x}J' \ \forall \ x \in N$ and $J'^2 = b' \in N$, then one has the following classification: if $b' = 0$, then A' is not simple, since NJ' is a proper ideal; otherwise if $0 \neq b' \in F$, then one gets the usual cyclic F-algebras of degree 2; if $b' \in N \setminus F$, then one obtains the nonassociative quaternion algebras now under discussion - which are always division algebras (see [10]), and $Z(A') = F$.

If A has a value function, then by Lemma 3(2) F is a valued field. On the other hand, suppose (F, V) is a valued field and let A be a finite dimensional associative division algebra with center F. In [8, Theorem] it was shown that the valuation on F extends to a valuation on A if and only if V is indecomposed in each field K such that $F \subseteq K \subseteq A$. In [4, Theorem 2], it was shown that the valuation on F extends to a valuation on A if and only if $A \otimes_F F_h$ is a division algebra, where (F_h, V_h) is the Henselization of (F, V) (see [2] for the definition and properties of Henselization). We now have the following analogous results in this nonassociative setting:

Proposition 6. *Suppose F is a valued field with valuation ring V. Then the following are equivalent:*

(1) *A has a value function w with $F \cap B_w = V$.*
(2) *V is indecomposed in N.*
(3) *$A \otimes_F F_h$ is a division algebra, where (F_h, V_h) is the Henselization of (F, V).*

Example 7. Let $F = \mathbb{Q}$, $N = \mathbb{Q}(i)$, $b = i \in N \setminus F$, and let v be a valuation on N extending the 3-adic valuation on F. If one defines w on A by

$$w(x + yJ) = \min(v(x), v(y)) \ \forall \ x, y \in N,$$

then it is easily seen that w is a value function on A and

$$B_w = \mathbb{Z}[i]_{(3\mathbb{Z}[i])} \oplus \mathbb{Z}[i]_{(3\mathbb{Z}[i])}J \ \text{ and } \ J_w = 3\mathbb{Z}[i]_{(3\mathbb{Z}[i])} \oplus 3\mathbb{Z}[i]_{(3\mathbb{Z}[i])}J.$$

We will see later (Theorem 17(2)) that (B_w, J_w) is actually a valuation ring pair of A. □

For the rest of this section, A will have a value function w defined on it. Then $w|_N$ is a valuation on N, which for now we will also denote by w. Let S be its corresponding valuation ring and let $V = S \cap F$. Then S is the integral closure of V in N, since V is indecomposed in N. We have $S = N \cap B_w$. We will say that w is a *normalized value function* if $w(J) = 0$. Given an arbitrary value function w, we know there is a $t \in U(N)$ such that $w(t) = w(J)$. Since

$$A = N \oplus NJ = N \oplus N(\frac{1}{t}J),$$

upon replacing J by $\frac{1}{t}J$ if necessary, we may and will assume that w is normalized in this section. Since $J^2 = b$, we see that $w(b) \geq 2w(J) = 0$, hence we will always have $b \in S$ in this section.

By [10], there are only two types of F-automorphisms on A: for the first type, the automorphism ϕ is given by

$$\phi(x + yJ) = x + \gamma yJ \ \text{ where } \ \gamma\hat{\gamma} = 1.$$

Such a map fixes N element-wise and is called a *proper automorphism* of A. We will see in Section 3 that valuation rings that are integral and invariant under proper F-automorphisms of A are precisely those arising from value functions on A. On the other hand, we will make no use of automorphisms of the second type, which occur only when $\hat{b} = -b$ and there is a $\gamma \in N$ satisfying $\gamma\hat{\gamma} = -1$. An automorphism of this type is given by $\phi(x + yJ) = \hat{x} + \gamma\hat{y}J$.

Proposition 8. *If ϕ is a proper automorphism of A, then $w(\phi(z)) = w(z)\ \forall\ z \in A$. In particular, $\phi(B_w) = B_w$ and $\phi(J_w) = J_w$.*

Proposition 9. $U(B_w) = \{x + yJ \in B_w \mid x\hat{x} - y\hat{y}b \in U(S)\}$.

Let $T = T_{N/F} : N \mapsto F$ be the usual trace map, i.e., $T(x) = x + \hat{x}$. As an F-linear mapping, it is known that right multiplication by an element $z = x + yJ$ of A has characteristic polynomial

$$c_z(t) = \{t^2 - T(x)t + x\hat{x} - y\hat{y}b\}\{t^2 - T(x)t + x\hat{x} - y\hat{y}\hat{b}\} \in F[t].$$

If we agree to interpret $c_z(x + yJ)$ as

$$\{(x + yJ)^2 - T(x)(x + yJ) + x\hat{x} - y\hat{y}b\}\{(x + yJ)^2 - T(x)(x + yJ) + x\hat{x} - y\hat{y}\hat{b}\},$$

which is an unambiguous expression, then we have $c_z(x + yJ) = 0$. Given a subring R of F, we will say that $z = x + yJ \in A$ is *integral over R* if $c_z(t) \in R[t]$. A subring B of A will be called *integral* if each one of its elements is integral over $B \cap F$. A valuation ring pair (B, I) of A will be called *integral* if B is integral. A subring of A will be called an *order* in A if it contains an F-basis of A. It will be called an *R-order* if it is an integral order containing R and the field of fractions of R is F. If an R-order is maximal among the R-orders of A with respect to inclusion, it will be called a *maximal R-order* (or just a maximal order if the context is clear). Clearly, every R-order is contained in a maximal order. Note that, if an order B containing R is finitely generated over R and $z \in B$ then, as in the associative case, by computing $c_z(t)$ using an F-basis of A contained in B, one readily sees that $c_z(t) \in R[t]$ and hence B is an R-order if F is the field of fractions of R. Conversely, if R is Noetherian and B is an R-order in A, then the proof of [6, Theorem 10.3] shows that B is finitely generated over R: if $\{u_1, u_2, u_3, u_4\} \subseteq B$ is an F-basis for A and $\alpha = \det(T(u_iu_j)) \in F$, then $\alpha \neq 0$ as was pointed out in the paragraph before [3, Proposition 1.4], and B is a submodule of the finitely generated R-module $\alpha^{-1}(Ru_1 + Ru_2 + Ru_3 + Ru_4)$.

Proposition 10. *B_w is a V-order in A.*

We will encounter more V-orders in §4.

Proposition 11. *We have the following:*

(1) *If $w(x + yJ) = \min(w(x), w(y))\ \forall\ x, y \in N$, then $B_w = S \oplus SJ$ and $J_w = J(S) \oplus J(S)J$.*

(2) *If $B_w = S \oplus SJ$, then $w(x+yJ) = \min(w(x), w(y))\ \forall\, x, y \in N$ and $\overline{B}_w = \overline{S} \oplus \overline{S}\,\overline{J}$, where $\overline{J}\overline{s} = \bar{\overline{s}}\,\overline{J}$ and $\overline{J}^2 = \bar{b}$.*

(3) *If $w(b) > 0$, then $w(x + yJ) = \min(w(x), w(y))\ \forall\, x, y \in N$ and (B_w, J_w) is <u>not</u> a valuation ring pair of A.*

Since (B_w, J_w) cannot be a valuation ring pair when $w(b) > 0$, we turn our attention to the case where we may have $w(b) = 0$. To handle the general situation, we will make use of the following notation: by definition of a value function, for each $u \in U(S)$ there is a $\lambda_u \in S \setminus \{0\}$ such that $w(\lambda_u) = w(1 + uJ)$. Let

$$B_u = S \oplus \lambda_u^{-1} S(1 + uJ),$$

a free S-submodule of B_w.

Theorem 12. *With the notation described above, we have the following:*

(1) $w(\lambda_{u_1}) \leq w(\lambda_{u_2})$ *if and only if* $B_{u_1} \subseteq B_{u_2}$. *In particular, the set* $\{B_u \mid u \in U(S)\}$ *is linearly ordered by inclusion.*

(2) $S \oplus SJ \subseteq B_u \; \forall \, u \in U(S)$, $B_w = \cup_{u \in U(S)} B_u$, *and* $J_w = \cup_{u \in U(S)} [J(S) \oplus \lambda_u^{-1} J(S)(1 + uJ)]$.

(3) *For each* $u \in U(S)$, B_u *is a subring of* B_w *and* $T(\frac{1}{\lambda_u} S) \subseteq V$.

(4) B_w *is finitely generated over* S *if and only if* $B_w = B_u$ *for some* $u \in U(S)$.

3. Valuation Rings in Nonassociative Quaternion Algebras

Let (B, I) be a valuation ring pair of A. In this section, we are going to determine the precise conditions that will guarantee the existence of a value function w on A such that $(B, I) = (B_w, J_w)$.

A subring B of A will be called *invariant* if $\phi(B) = B$ for every proper F-automorphism ϕ of A. A valuation ring pair (B, I) will be called *invariant* if B is invariant.

A valuation ring pair (B, I) of A will be called *normalized* if $J \in B \setminus I$. Without loss of generality, we may assume that (B, I) is normalized: if $J \notin B$, then we know there is a $t \in N$ such that $tJ \in B \setminus I$; in this case, replace J by tJ. If $J \in I$, then $\frac{1}{b} J \notin B$, otherwise we would have $1 = (\frac{1}{b} J) J \in I$. So there is a $t \in N$ such that $\frac{t}{b} J \in B \setminus I$, in which case we replace J by $\frac{t}{b} J$.

Lemma 13. *If* (B, I) *is normalized, integral, and invariant, then*

(1) $S = B \cap N$ *is a valuation ring of* N.

(2) *If* $u \in U(S)$, *then there is a* $\sigma_u \in S \setminus \{0\}$ *such that* $\frac{\sigma_u}{1 - u\hat{u}b}(1 + uJ) \in B \setminus I$.

Further, for any $t \in N$,

(3) $v(t) = v(\sigma_u)$ *if and only if* $\frac{t}{1 - u\hat{u}b}(1 + uJ) \in B \setminus I$, *where* v *is a valuation on* N *corresponding to* S.

If (B, I) is normalized, integral, and invariant and if Γ is the value group of the valuation v, we define a map $w : A \mapsto \Gamma \cup \{\infty\}$ by

$$w(x + yJ) = \begin{cases} \infty & \text{if } x + yJ = 0, \\ \min(v(x), v(y)) & \text{if } v(x) \neq v(y), \\ v(x) + v(1 - u\hat{u}b) - v(\sigma_u) & \text{otherwise, where } u = \frac{y}{x}. \end{cases}$$

By Lemma 13, σ_u exists for each $u \in U(S)$ and $v(\sigma_u)$ depends only on u. Hence w is well defined. This w turns out to be a value function corresponding to (B, I) in the following theorem.

In the associative setting, given a Dubrovin valuation ring of a finite-dimensional division algebra, then in [9, Theorem G & Corollary G] we learn that there is a valuation on the division algebra giving rise to the Dubrovin valuation ring if and only if the Dubrovin valuation ring is invariant under inner automorphisms of the division algebra. In [5], certain value functions are defined on central simple algebras. Given a Dubrovin valuation ring of such an algebra, there is one such value function giving rise to the Dubrovin valuation ring if and only if the Dubrovin valuation ring is integral [5, Corollary 2.5]. We have the following analogue of these two results, but here we need both the invariance and the integralness assumptions.

Theorem 14. *Given a valuation ring pair (B, I) of A, there is a value function w such that $(B, I) = (B_w, J_w)$ if and only if (B, I) is integral and invariant.*

Note that the condition is clearly necessary, by Proposition 10 and Proposition 8.

Corollary 15. *Let (B, I) be a valuation ring pair of A that is invariant and integral. Then:*

(1) *I is the unique maximal ideal of B such that, if $z \in A \setminus B$, then there are $b_1, b_2 \in B \cap N$ with $b_1 z, z b_2 \in B \setminus I$.*
(2) *$\phi(I) = I$ for every proper F-automorphism ϕ.*
(3) *$U(B) = \{x + yJ \in B \mid x\hat{x} - y\hat{y}b \in U(S)\}$.*
(4) *If in addition (B, I) is normalized, then $B = \cup_{u \in U(S)}[S \oplus \frac{\sigma_u}{1 - u\hat{u}b} S(1 + uJ)]$ and $I = \cup_{u \in U(S)}[J(S) \oplus \frac{\sigma_u}{1 - u\hat{u}b} J(S)(1 + uJ)]$.*

By Remark 4, we immediately have:

Corollary 16. *Given a value function w on A, if J_w is a maximal ideal of B_w, then it is the unique maximal ideal of B_w satisfying the condition that, if $z \in A \setminus B_w$, then there are $b_1, b_2 \in N \cap B_w$ with $b_1 z, z b_2 \in B_w \setminus J_w$.*

4. The case when N/F is Tamely Ramified and Defectless

All undefined terminology used in this section relating to valuations on fields can be found in [2]. Let us once and for all fix some notation for this section. The quaternion algebra A will have a normalized value function w defined on it. Let $S = N \cap B_w$, a valuation ring of N, and let $V = S \cap F$, a valuation ring of F. We also know that V is indecomposed in N, and so S is the integral closure of V in N.

Let v be a valuation on N with valuation ring S. Let e (resp. f) be the ramification index (resp. residue degree) of S over F. In our case, it is well known that

$$ef \leq 2.$$

If we have equality $ef = 2$, then we say N/F is *defectless*. We call N/F *tamely ramified* if the characteristic of $\overline{V}$ does not divide e and $\overline{S}$ is separable over $\overline{V}$. When N/F is tamely ramified and defectless then, in our situation, there are exactly two cases: N/F is an *inertial* extension if $f = 2$; it is *tamely and totally ramified* when $e = 2$. In the latter case, the characteristic of $\overline{V}$ is not 2, of course.

In this section, we will assume that N/F is tamely ramified and defectless. Under this assumption, B_w has a particulary desirable form and we will determine precise conditions for (B_w, J_w) to be a valuation ring pair of A. This section also shows that there are abundant examples of valuation ring pairs of A when N/F is tamely ramified and defectless.

Theorem 17. *Suppose N/F is tamely ramified and defectless. Then:*

(1) $B_w = S \oplus SJ$.

(2) *If N/F is inertial, then (B_w, J_w) is a valuation ring pair of A if and only if $w(b)=0$. When this occurs, J_w is the unique maximal ideal of B_w and $\overline{B}_w$ is a central simple $\overline{V}$-algebra, which is a division algebra unless $\overline{b}$ is a norm from $\overline{S}$ to $\overline{V}$.*

(3) *If N/F is tamely and totally ramified, then (B_w, J_w) is a valuation ring pair of A if and only if $\overline{b}$ is not a square in $\overline{S}$. When this occurs, then J_w is the unique maximal ideal of B_w and $\overline{B}_w$ is a separable quadratic extension field of $\overline{V}$.*

For the rest of this section, we shall assume that V is a DVR, hence N/F is defectless by [2, Corollary 18.7]. Let $J(S) = \pi S$, and let v be the $J(S)$-adic valuation on N.

The set $\{t \in N \mid v(t\hat{t}b - 1) \geq 0\}$ is clearly non-empty. Let $k \in \{0,1\}$ be the largest integer such that there is a $u \in N$ with $v(u\hat{u}b - 1) \geq 2k$. If we assume N/F is a tamely and totally ramified extension, then by [3, Proposition 2.5], if $k = 0$, then $B = S \oplus S(1 + uJ)$ is the unique maximal V-order containing S, while if $k = 1$, then there are exactly two maximal orders containing S, namely $B_1 = S \oplus \pi^{-1} S(1 + uJ)$ and $B_2 = S \oplus \pi^{-1} S(1 + u\frac{\hat{\pi}}{\pi} J)$.

Corollary 18. *Suppose V is a DVR and N/F is tamely ramified. Then we have*

(1) *If N/F is inertial, then (B_w, J_w) is a valuation ring pair of A if and only if B_w is a maximal order and $w(b) = 0$.*

(2) *Otherwise if N/F is tamely and totally ramified, then:*

  (a) *If $w(b) = 0$, then B_w is the intersection of (at most two) maximal orders.*

  (b) *(B_w, J_w) is a valuation ring pair of A if and only if B_w is a maximal order and $\overline{b}$ is not a square in $\overline{S}$.*

5. Examples

Example 19. *A subring B_1 of A that is invariant but not integral, a subring B_2 of A that is integral but not invariant, and a valuation ring pair (B, I) that is neither integral nor invariant.*

Let $F = \mathbb{Q}, V = \mathbb{Z}_{(5)}, N = \mathbb{Q}(i)$. Then $J(V)$ splits completely in N. Let $W = \mathbb{Z}[i]_{(2+i)}$, one of the two extensions of V to N. Let S be the integral closure of V in N, that is, $S = \mathbb{Z}[i]_{(2+i)} \cap \mathbb{Z}[i]_{(2-i)}$.

Then $B_1 = W$ is an invariant subring, but not integral. (If A was an associative division algebra, then any subring that is invariant under F-automorphisms of A is integral.)

Now let $b = i \in S \setminus F$. Then $B_2 = S \oplus SJ$ is integral, but not invariant under the proper automorphism $\phi(x + yJ) = x + y(\frac{2-i}{2+i})J$.

Therefore, in general, being integral and being invariant are mutually independent phenomena.

Finally, let $b = i \in W \setminus F$ and let $B = W \oplus WJ$. Note that B is not invariant under the proper automorphism $\phi(x+yJ) = x+y(\frac{2-i}{2+i})J$. Let $I = J(W) \oplus J(W)J$. Then (B, I) is a valuation ring pair of A. It is neither integral nor invariant.

Therefore, unlike in the associative setting, valuation rings over a DVR need not be maximal orders.

Example 20. *An invariant maximal order over a DVR that is not a valuation ring.*

Let $F = \mathbb{Q}$, $V = \mathbb{Z}_{(3)}$, and $N = \mathbb{Q}(i)$. Then $S = \mathbb{Z}[i]_{(3\mathbb{Z}[i])}$ is a valuation ring of N lying over V which is inertial over F. Let $b = 3 + 9i \in S \setminus F$ and let $B = S \oplus SJ$. Then B is a maximal order but, for any maximal ideal I of B, (B, I) is not a valuation ring pair of A.

Therefore, unlike in the associative case, maximal orders over a DVR need not be valuation rings.

The condition that $w(b) = 0$ is necessary in part 2(a) of Corollary 18, as the following example shows. Keeping the notation of Section 4, we have:

Example 21. *A B_w which is not an intersection of maximal orders, but V is a DVR and N/F is tamely and totally ramified.*

Suppose V is a DVR and N/F is tamely and totally ramified. Let $J(S) = \pi S$ and let v be the $J(S)$-adic valuation on N. Let $b = (\bar{\pi}\pi)\pi \in S \setminus F$. If $w(x + yJ) = \min(v(x), v(y))\ \forall\, x, y \in N$, it is easily seen that w is a value function on A and $B_w = S \oplus SJ$. But B_w is not the intersection of maximal orders in A.

REFERENCES

[1] N. I. Dubrovin, *Noncommutative valuation rings*, Trudy Moskov. Mat. Obshch. **45** (1982), 265–280; English transl., Trans. Moscow Math. Soc. **45** (1985), 273–287.

[2] O. Endler, Valuation Theory, Springer-Verlag, New York, 1972.

[3] H. J. Lee and W. C. Waterhouse, *Maximal Orders in Nonassociative Quaternion Algebras*, J. Algebra **146** (1992), 441–453.

[4] P. J. Morandi, *The Henzelization of a valued division algebra*, J. Algebra **122** (1989), 232–243.

[5] ———, *Value functions on central simple algebras*, Trans. Amer. Math. Soc. **315** (1989), 605–622.

[6] I. Riener, Maximal Orders, Academic Press, London, 1975.

[7] R. D. Schafer, An Introduction to Nonassociative Algebras, Dover Inc., New York, 1995.

[8] A.R. Wadsworth, *Extending valuations to finite-dimensional division algebras*, Proc. Amer. Math. Soc. **98** (1986), no. 1, 20–22.

[9] ———, *Dubrovin valuation rings and Henselization*, Math. Ann. **283** (1989), 301–328.

[10] W. C. Waterhouse, *Nonassociative Quaternion Algebras*, Algebras Groups Geom. **4** (1987) 365–378.

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COHEN-MACAULAY MODULES AND HOLONOMIC MODULES ON FILTERED GORENSTEIN RINGS

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ABSTRACT. This paper is a joint work with K.Nishida. We will study about filtered Gorenstein rings, then Cohen-Macaulay modules and holonomic modules are difined and studied.

1. INTRODUCTION AND PRELIMINARIES

Definition 1. Let Λ be a (not necessarily commutative) ring. A family of additive subgroups $\{ \mathcal{F}_p\Lambda \mid p \in \mathbb{N} \}$ of Λ, where $\mathbb{N}$ is the set of all non-negative integers, is called a *filtration* of Λ, if

(1) $1 \in \mathcal{F}_0\Lambda$
(2) $\mathcal{F}_p\Lambda \subset \mathcal{F}_{p+1}\Lambda$
(3) $(\mathcal{F}_p\Lambda)(\mathcal{F}_q\Lambda) \subset \mathcal{F}_{p+q}\Lambda$
(4) $\Lambda = \bigcup_{p\in\mathbb{N}} \mathcal{F}_p\Lambda$.

A pair $(\Lambda, \mathcal{F})$ is called a *filtered ring* if Λ has a filtration. Let $\sigma_p : \mathcal{F}_p\Lambda \longrightarrow \mathcal{F}_p\Lambda/\mathcal{F}_{p-1}\Lambda$ ($\mathcal{F}_{-1}\Lambda = 0$) be a natural homomorphism, then $\mathrm{gr}\Lambda := \bigoplus_{p\in\mathbb{N}} \mathcal{F}_p\Lambda/\mathcal{F}_{p-1}\Lambda$ is a graded ring with multiplication

$$\sigma_p(a)\sigma_q(b) = \sigma_{p+q}(ab), \quad a \in \mathcal{F}_p\Lambda,\ b \in \mathcal{F}_q\Lambda$$

Through the paper, we assume that $\mathrm{gr}\,\Lambda$ is a commutative Noetherian ring. Therefore Λ is a left and right Noetherian ring.

Let Λ be a filtered ring and M a (left) Λ-module. A family of additive subgroups $\{ \mathcal{F}_p\Lambda \mid p \in \mathbb{Z} \}$ of M is called a *filtration* of M, if

(1) $\mathcal{F}_pM \subset \mathcal{F}_{p+1}M$
(2) $\mathcal{F}_pM = 0$ for $p \ll 0$
(3) $(\mathcal{F}_p\Lambda)(\mathcal{F}_qM) \subset \mathcal{F}_{p+q}M$
(4) $M = \bigcup_{p\in\mathbb{Z}} \mathcal{F}_pM$.

A pair $(M, \mathcal{F})$ is called a *filtered Λ-module* if M has a filtration. Let $\tau_p : \mathcal{F}_pM \longrightarrow \mathcal{F}_pM/\mathcal{F}_{p-1}M$ be a natural homomorphism, then $\mathrm{gr}M := \bigoplus_{p\in\mathbb{Z}} \mathcal{F}_pM/\mathcal{F}_{p-1}M$ is a graded $\mathrm{gr}\Lambda$-module by

$$\sigma_p(a)\tau_q(x) = \tau_{p+q}(ax), \quad a \in \mathcal{F}_p\Lambda, x \in \mathcal{F}_qM$$

Let Λ be a filtered ring and let $(M, \mathcal{F})$ be a filtered Λ-module. A filtration $\mathcal{F}$ is called *good* if $\mathrm{gr}M$ is a finitely generated $\mathrm{gr}\Lambda$-module. The module M has a good filtration if and only if M is finitely generated.

The detailed version of this paper will be submitted for publication elsewhere.

Definition 2. A Λ-module M is said to have *Gorenstein dimension zero*, denoted by $\text{G-dim}_\Lambda M = 0$, if $M^{**} \cong M$ and $\text{Ext}^k_\Lambda(M,\Lambda) = \text{Ext}^k_{\Lambda^{op}}(M^*,\Lambda) = 0$, where $M^* = \text{Hom}_\Lambda(M, \Lambda)$ for $k > 0$. Then, G-dim $M = 0$ if and only if $\text{Ext}^k_\Lambda(M,\Lambda) = \text{Ext}^k_{\Lambda^{op}}(\text{Tr}M,\Lambda) = 0$ for $k > 0$.

For a positive integer k, M is said to have *Gorenstein dimension less than or equal to* k, denoted by G-dim $M \leq k$, if there exists an exact sequence $0 \to G_k \to \cdots \to G_0 \to M \to 0$ with G-dim $G_i = 0$ for $0 \leq i \leq k$. G-dim $M \leq k$ if and only if G-dim $\Omega^k M = 0$. Also, if G-dim $M < \infty$ then G-dim $M = \sup\{k : \text{Ext}^k_\Lambda(M,\Lambda) \neq 0\}$.

2. Homological property on filtered rings

In this section, we will talk about Gorenstein dimension and grade of filtered modules. The following fact and the lemma are important to prove our main results.

Fact. *Let Λ be a filtered ring and M a filtered Λ-module with a good filtation. Then* $\text{gr}\,\text{Ext}^i_\Lambda(M,\Lambda)$ *is a subfactor of* $\text{Ext}^i_{\text{gr}\Lambda}(\text{gr}M, \text{gr}\Lambda)$ *for* $i \geq 0$.

Lemma 3. *Let Λ be a filtered ring and M a filtered Λ-module with a good filtation. Then there exists an epimorphism* $\alpha : \text{Tr}_{\text{gr}\Lambda}(\text{gr}M) \to \text{gr}(\text{Tr}_\Lambda M)$.

Moreover, if $\text{G-dim}\,\text{gr}M = 0$, *then α is an isomorphism.*

Combining the above fact and the lemma, we get the following theorem.

Theorem 4. *Let Λ be a filtered ring M a filtered Λ-module with a good filtration, and let k be a non-negative integer. Then* $\text{G-dim}\,\text{gr}M \leq k$ *implies* $\text{G-dim}M \leq k$.

Corollary 5. *If* $\text{G-dim}\,\text{gr}M < \infty$, *then* G-dim $M = \text{G-dim}\,\text{gr}M$.

On the other hand, we get the relation between grade of modΛ and grade of mod(grΛ)

Theorem 6. *Let Λ be a filtered ring such that grΛ is a commutative Gorenstein ring and M a filtered Λ-module with a good filtration. Then* $\text{grade}_\Lambda M = \text{grade}_{\text{gr}\Lambda}\text{gr}M$ *holds, where* $\text{grade}M = \inf\{\, i \mid \text{Ext}^i_\Lambda(M, \Lambda) \neq 0\,\}$.

Remark 7. The above theorem is proved under the assumption that grΛ is Gorenstein, but we can have the equality under a more general condition about a module. We shall study it in another paper.

3. Introduction to filtered Gorenstein rings

A commutative graded ring R is called *local ring if R has a unique maximal graded ideal (*maximal ideal). We assume that grΛ is a commutative Gorenstein *local ring (with unique *maximal ideal $\mathcal{M}$) satisfying the following condition (P):

(P) : There exists an element of positive degree in $\text{gr}\Lambda - \mathfrak{p}$ for any graded prime ideal $\mathfrak{p} \neq \mathcal{M}$

Fact. *Let $(R, \mathcal{M})$ be a commutative *local Gorenstein ring with the condition (P), and let A be a finite graded R-module. Then we have the following* :

$$\text{G-dim}\,A + {}^*\text{depth}A = {}^*\text{depth}R \qquad ({}^*\text{depth}A := \text{depth}(\mathcal{M}, A))$$
$$\text{grade}A + {}^*\text{dim}A = {}^*\text{dim}R \qquad ({}^*\text{dim}A := \text{ht}\ \mathcal{M}/\text{Ann}_R(A)).$$

Proposition 8. *Let Λ be a filtered ring such that* $\mathrm{gr}\Lambda$ *is a commutative* **local Gorenstein ring with the condition (P), and let M be a filtered Λ-module with a good filtration. Then the following holds* :

$$G\text{-}dim_{\Lambda} M + {}^*\mathrm{depth}\,\mathrm{gr}M = n \quad (n := {}^*\dim \mathrm{gr}\Lambda)$$
$$\mathrm{grade}_{\Lambda} M + {}^*\dim \mathrm{gr}M = n.$$

Corollary 9. *Let Λ be a filtered ring such that* $\mathrm{gr}\Lambda$ *is a commutative* **local Gorenstein ring with the condition (P). Then,* $\mathrm{id}_{\Lambda}\Lambda = \mathrm{id}_{\Lambda^{op}}\Lambda \le n$. *Therefore, let* $\mathrm{id}\Lambda = d$, *then* ${}^*\dim \mathrm{gr}M \ge n - d$ *for all filtered Λ-module M with a good filtration.*

Definition 10. We call a filtered ring Λ a *filtered Gorenstein ring* if $\mathrm{gr}\Lambda$ is a commutative Gorenstein *local ring with the condition (P)

We can naturally get the following .

Definition 11. Let Λ be a filtered Gorenstein ring. We call filtered Λ-module M with a good filtration a *CM Λ-module*, if $\mathrm{gr}M$ is a graded CM $\mathrm{gr}\Lambda$-module.

The following proposition and corollary are well known for the case commutative rings.

Proposition 12. *Let Λ be a filtered Gorenstein ring and M a filtered Λ-module with a good filtration. Then M is CM if and only if* $\mathrm{grade}M = \mathrm{G\text{-}dim}M$

Corollary 13. *Let Λ be a filtered Gorenstein ring, and put*
$C_k(\Lambda) = \{ M \in \mathrm{mod}\Lambda \mid M$ is CM with $G\text{-}dim\, M = k \}$.
Then, the functor $\mathrm{Ext}^k_{\Lambda}(-, \Lambda)$ *induces a duality between the categories* $\underline{C}_k(\Lambda)$ *and* $\underline{C}_k(\Lambda^{op})$.

Definition 14. Let Λ be a filtered Gorenstein ring. We call filtered Λ-module M with a good filtration a *holonomic*, if ${}^*\dim \mathrm{gr}M = n - d$, where $n = {}^*\dim \mathrm{gr}\Lambda$, $d = \mathrm{id}\Lambda$.

Finally, we will show the basic properties of holonomic modules on filtered Gorenstein rings.

Proposition 15. *Let Λ be a filtered Gorenstein ring, and let* $d = \mathrm{id}\Lambda$. *Then a Λ-module M is holonomic if and only if* $\mathrm{grade}M = d$. *Therefore, any holonomic module is CM*

Proposition 16. *Let Λ be a filtered Gorenstein ring, M a holonomic Λ-module, and N a submodule of M. Then N and M/N are holonomic.*

Proposition 17. *A holonomic module is artinian. Therefore, it is of finite length.*

References

[1] Li Huishi and F. van Oystaeyen, *Zariskian Filtrations*, K-Monographs in Mathematics, 2, (1996)
[2] H.Miyahara and K.Nishida, *Cohen-macaulay modules and holonomic modules on filtered Noetherian rings*, preprint

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SYMMETRY IN THE VANISHING OF EXT-GROUPS

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ABSTRACT. In this note, we will find a class of rings R satisfying the following property: for every pair of finitely generated right R-modules M and N, $\mathrm{Ext}^i_R(M,N)=0$ for all $i \gg 0$ if and only if $\mathrm{Ext}^i_R(N,M)=0$ for all $i \gg 0$. In particular, we will show that such a class of rings includes a group algebra of a finite group and the exterior algebra of odd degree.

1. MOTIVATION

Throughout, we always assume that k is a field, R is a (right and left) noetherian ring, $\operatorname{mod} R$ is the category of finitely generated right R-modules, and $M, N \in \operatorname{mod} R$.

If R is a commutative local ring, then Serre [15] defined the intersection multiplicity of $M, N \in \operatorname{mod} R$ by

$$\chi(M,N) := \sum_{i=0}^{\infty}(-1)^i \operatorname{length} \mathrm{Tor}_i^R(M,N).$$

If R is not commutative, then $\mathrm{Tor}_i^R(M,N)$ do not make sense, but $\mathrm{Ext}^i_R(M,N)$ do, so Smith and I [14] defined a new intersection multiplicity of $M, N \in \operatorname{mod} R$ by

$$M \cdot N := (-1)^{\operatorname{codim} M} \sum_{i=0}^{\infty}(-1)^i \operatorname{length} \mathrm{Ext}^i_R(M,N)$$

in order to develop an intersection theory over a noncommutative ring. (Note that if R is not commutative, then $\mathrm{Ext}^i_R(M,N)$ are no longer R-modules, so we defined the above intersection multiplicity in [14] only over a k-algebra R, replacing $\operatorname{length} \mathrm{Ext}^i_R(M,N)$ by $\dim_k \mathrm{Ext}^i_R(M,N)$.) Fortunately, these two definitions of the intersection multiplicity agree over reasonably nice commutative rings.

Theorem 1. [5, Theorem 4, Theorem 5] *If R is a commutative local complete intersection ring, or a commutative local Gorenstein ring of* $\operatorname{Kdim} R \le 5$, *then*

$$M \cdot N = \chi(M,N)$$

for all $M, N \in \operatorname{mod} R$ such that

- $\operatorname{length}(M \otimes_R N) < \infty$,
- $\operatorname{pd}(M) < \infty$, $\operatorname{pd}(N) < \infty$, *and*
- $\operatorname{Kdim} M + \operatorname{Kdim} N \le \operatorname{Kdim} R$.

This note is basically a summary of [13] which has been accepted for publication in *J. Algebra*.

Three conditions on $M, N \in \operatorname{mod} R$ in the above theorem guarantee that both intersection multiplicities $\chi(M, N)$ and $M \cdot N$ are well-defined. In order to justify our new intersection theory, the following questions are natural over more general rings.

Question. Let R be an algebra or a commutative ring, and $M, N \in \operatorname{mod} R$.

(1) $M \cdot N = N \cdot M$ if both sides are well-defined?
(2) $M \cdot N$ is well-defined if and only if $N \cdot M$ is well-defined?

Over a commutative Gorenstein local ring, the first question above is equivalent to Serre's vanishing conjecture by [9]. In this note, we will focus on the second question above. Note that $M \cdot N$ is well-defined if and only if

- $\operatorname{length} \operatorname{Ext}^i_R(M, N) < \infty$ for all i, and
- $\operatorname{Ext}^i_R(M, N) = 0$ for all $i \gg 0$,

so we can split the second question above into the following two questions:

Question. Let R be an algebra or a commutative ring, and $M, N \in \operatorname{mod} R$.

(1) $\operatorname{length} \operatorname{Ext}^i_R(M, N) < \infty$ for all i if and only if $\operatorname{length} \operatorname{Ext}^i_R(N, M) < \infty$ for all i?
(2) $\operatorname{Ext}^i_R(M, N) = 0$ for all $i \gg 0$ if and only if $\operatorname{Ext}^i_R(N, M) = 0$ for all $i \gg 0$?

The first question above was answered affirmatively over a commutative ring.

Theorem 2. [9, Corollary 3.2] *Let R be a commutative local ring. Then, for all $M, N \in \operatorname{mod} R$,*

$$\operatorname{length} \operatorname{Ext}^i_R(M, N) < \infty \textit{ for all } i \Leftrightarrow \operatorname{length} \operatorname{Ext}^i_R(N, M) < \infty \textit{ for all } i.$$

For the second question above, we will make the following definition.

Definition 3. We say that a ring R satisfies (**ee**) if, for all $M, N \in \operatorname{mod} R$,

$$\operatorname{Ext}^i_R(M, N) = 0 \text{ for all } i \gg 0 \Leftrightarrow \operatorname{Ext}^i_R(N, M) = 0 \text{ for all } i \gg 0.$$

First, we will make an easy observation.

Example 4. If R is regular, that is, $\operatorname{gldim} R < \infty$, then, for all $M, N \in \operatorname{mod} R$, $\operatorname{Ext}^i_R(M, N) = 0$ for all $i > \operatorname{gldim} R$, so R satisfies (**ee**).

Conversely, if R is a commutative local ring satisfying (**ee**), then $\operatorname{Ext}^i_R(R, k) = 0$ for all $i \geq 1$ where k is the residue field of R, so $\operatorname{Ext}^i_R(k, R) = 0$ for all $i \gg 0$, hence R is Gorenstein, that is, $\operatorname{id}(R) < \infty$.

It follows that the class of commutative local rings satisfying (**ee**) is somewhere between regular rings and Gorenstein rings. In commutative ring theory, there is a nice class of rings between them, namely complete intersection rings.

Theorem 5. [2] *Every commutative locally complete intersection ring satisfies (**ee**).*

It is not very difficult to find an example of non complete intersection ring which satisfies (**ee**). Very recently, Jorgensen and Sega [8] found an example of a commutative Gorenstein ring that does not satisfy (**ee**), so the class of commutative rings satisfying (**ee**) is strictly between complete intersection rings and Gorenstein rings.

2. Conjecture of Auslander

We will define another technical condition on a ring.

Definition 6. We say that a ring R satisfies (**ac**) if, for each $M \in \operatorname{mod} R$, there exists $n_M \in \mathbb{N}$ such that, for all $N \in \operatorname{mod} R$,

$$\operatorname{Ext}^i_R(M,N) = 0 \text{ for all } i \gg 0 \Rightarrow \operatorname{Ext}^i_R(M,N) = 0 \text{ for all } i > n_M.$$

There was a conjecture in representation theory of finite dimensional algebras.

Conjecture. (Auslander) Every artinian algebra satisfies (**ac**).

The above conjecture was important since it implies the famous conjecture below.

Conjecture. (Finitistic dimension conjecture) If R is an artinian algebra, then there exists $n_R \in \mathbb{N}$ such that, for all $M \in \operatorname{mod} R$,

$$\operatorname{pd}(M) < \infty \Rightarrow \operatorname{pd}(M) \leq n_R.$$

Although the above conjecture was raised in representation theory of finite dimensional algebras, it became also interested in commutative ring theory due to the following result.

Theorem 7. [6, Theorem 4.1], [13, Theorem 3.2] *Let R be a commutative local Gorenstein ring. If R satisfies* (**ac**)*, then R satisfies* (**ee**)*.*

Although the condition (**ac**) is interesting, it is not easy to find non-trivial examples of algebras satisfying (**ac**). In fact, there had been very few examples of algebras satisfying (**ac**) until recently.

Theorem 8. [4, Theorem 2.4] *Every group algebra of a finite group satisfies* (**ac**)*.*

Theorem 9. [2, Theorem 4.7, Proposition 6.2] *Every commutative locally complete intersection ring satisfies* (**ac**)*.*

Due to the above theorem, the following is a natural question.

Question. If R is a noncommutative analogue of a commutative complete intersection ring, then does R satisfy (**ac**) and/or (**ee**)?

On the positive side, we have the following result.

Theorem 10. [13, Corollary 2.3] *If R is a regular ring and $\{x_1, \dots, x_n\}$ is a regular central sequence of R, then $R/(x_1, \dots, x_n)$ satisfies* (**ac**)*.*

The above theorem produces a new example of an algebra satisfying (**ac**).

Example 11. Every exterior algebra can be written as

$$\Lambda(k^n) \cong R/(x_1^2, \dots, x_n^2),$$

where

$$R = k\langle x_1, \dots, x_n\rangle/(x_ix_j + x_jx_i)_{1\le i<j\le n}$$

is a regular ring (an anti-commutative polynomial ring), and $\{x_1^2, \dots, x_n^2\}$ is a regular central sequence of R, so $\Lambda(k^n)$ satisfies (**ac**).

Jorgensen and Sega [7] found an example of a commutative Frobenius algebra that does not satisfy (**ac**), so the Auslander conjecture is false. The following theorem also shows that the Auslander conjecture is false. In particular, we cannot replace "central" by "normalizing" in the above theorem.

Theorem 12. [12, Theorem 6.5] *Let $\Lambda = k\langle x_1, \dots, x_n\rangle/(x_ix_j + \alpha_{ij}x_jx_i, x_i^2)$ be a skew exterior algebra where $0 \ne \alpha_{ij} \in k$ for $1 \le i < j \le n$. Then Λ satisfies* (**ac**) *if and only if α_{ij} are roots of unity for all $1 \le i < j \le n$.*

3. Stably Symmetric Algebras

In this section, we will define a stably symmetric algebra, which is a generalization of a symmetric algebra.

Definition 13. Let $\mathcal{C}$ be a k-linear Hom-finite category, that is,

$$\dim_k \operatorname{Hom}_{\mathcal{C}}(M, N) < \infty$$

for all $M, N \in \mathcal{C}$. A Serre functor on $\mathcal{C}$ is an autoequivalence $\mathcal{K} : \mathcal{C} \to \mathcal{C}$ such that

$$\operatorname{Hom}_{\mathcal{C}}(M, N) \cong D \operatorname{Hom}_{\mathcal{C}}(N, \mathcal{K}(M))$$

for all $M, N \in \mathcal{C}$ where $D(-)$ is the functor taking the k-vector space dual.

A Serre functor on $\mathcal{C}$ is unique if it exists. Moreover, if $\mathcal{C}$ is a triangulated category, then a Serre functor $\mathcal{K} : \mathcal{C} \to \mathcal{C}$ is exact, so the following lemma is immediate.

Lemma 14. *Let $\mathcal{C}$ be a k-linear* Hom*-finite triangulated category. Then an exact autoequivalence $\mathcal{K} : \mathcal{C} \to \mathcal{C}$ is a Serre functor on $\mathcal{C}$ if and only if*

$$\operatorname{Ext}^i_{\mathcal{C}}(M, N) \cong D \operatorname{Ext}^{-i}_{\mathcal{C}}(N, \mathcal{K}(M))$$

for all i and all $M, N \in \mathcal{C}$.

The definition of a Serre functor was motivated by the Serre duality.

Example 15. If X is a smooth projective scheme of finite type over k, then the bounded derived category of coherent $\mathcal{O}_X$-modules $\mathcal{D}^b(X)$ has a Serre functor

$$- \otimes_X \omega_X[d] : \mathcal{D}^b(X) \to \mathcal{D}^b(X)$$

where ω_X is the canonical sheaf on X and $d = \dim X$, so that

$$\operatorname{Ext}^i_X(\mathcal{F}, \mathcal{G}) \cong D \operatorname{Ext}^{-i}_X(\mathcal{G}, \mathcal{F} \otimes_X \omega_X[d]) \cong D \operatorname{Ext}^{d-i}_X(\mathcal{G}, \mathcal{F} \otimes_X \omega_X)$$

for all i and all $\mathcal{F}, \mathcal{G} \in \operatorname{coh} X$. In particular, the classical Serre duality

$$\operatorname{H}^i(X, \mathcal{G}) \cong \operatorname{Ext}^i_X(\mathcal{O}_X, \mathcal{G}) \cong D \operatorname{Ext}^{d-i}_X(\mathcal{G}, \omega_X)$$

holds for all i and all $\mathcal{G} \in \operatorname{coh} X$.

We will apply the theory of a Serre functor to the triangulated category defined as follows. Let $\underline{\mathrm{mod}}\, R$ be the stable category of $\mathrm{mod}\, R$ by projective modules. In general, $\underline{\mathrm{mod}}\, R$ is not a triangulated category, but there is a natural way of making it a triangulated category. We define the category $\mathcal{S}(\underline{\mathrm{mod}}\, R)$, called the stabilization of $\underline{\mathrm{mod}}\, R$, whose objects are of the form $\Omega^i M$ where $M \in \underline{\mathrm{mod}}\, R$ and $i \in \mathbb{Z}$ modulo $M \cong N$ in $\mathcal{S}(\underline{\mathrm{mod}}\, R)$ if $\Omega^i M \cong \Omega^i N$ in $\underline{\mathrm{mod}}\, R$ for all $i \gg 0$. It turns out that $\mathcal{S}(\underline{\mathrm{mod}}\, R)$ is a triangulated category with the translation functor

$$\Omega^{-1} : \mathcal{S}(\underline{\mathrm{mod}}\, R) \to \mathcal{S}(\underline{\mathrm{mod}}\, R).$$

We refer to [3] for more details on this construction. If R is a regular algebra, then, for all $M \in \mathrm{mod}\, R$, $\Omega^i M \cong 0$ for all $i > \mathrm{gldim}\, R$, so $\mathcal{S}(\underline{\mathrm{mod}}\, R)$ is trivial. On the other hand, if R is a Frobenius algebra, then $\underline{\mathrm{mod}}\, R$ is already a triangulated category, so $\mathcal{S}(\underline{\mathrm{mod}}\, R) \cong \underline{\mathrm{mod}}\, R$.

Definition 16. Let R be an algebra. We say that R is stably symmetric if

$$\mathcal{K} = \Omega^{-d} : \mathcal{S}(\underline{\mathrm{mod}}\, R) \to \mathcal{S}(\underline{\mathrm{mod}}\, R)$$

is a Serre functor for some $d \in \mathbb{Z}$.

In other words, R is stably symmetric if and only if $\mathcal{S}(\underline{\mathrm{mod}}\, R)$ is Calabi-Yau. However, we will see later that the definition of stably symmetric does not coincide with that of Calabi-Yau in the graded case. Note that if R is a regular ring, then $\mathcal{S}(\underline{\mathrm{mod}}\, R)$ is trivial, so R is stably symmetric. The following result is well known.

Lemma 17. *If R is a Frobenius algebra, then $\mathcal{S}(\underline{\mathrm{mod}}\, R) \cong \underline{\mathrm{mod}}\, R$ has a Serre functor*

$$\mathcal{K} = \Omega\mathcal{N} : \underline{\mathrm{mod}}\, R \to \underline{\mathrm{mod}}\, R$$

where

$$\mathcal{N}(-) = D\,\mathrm{Hom}_R(-, R) : \mathrm{mod}\, R \to \mathrm{mod}\, R$$

is the Nakayama functor.

If R is a symmetric algebra, then R is Frobenius such that the Nakayama functor is the identity, so we have the following.

Corollary 18. *Every symmetric algebra is stably symmetric.*

Example 19. The algebras below are examples of symmetric algebras, so they are stably symmetric by the above corollary.

- A commutative local Frobenius algebra.
- A semi-simple algebra.
- The trivial extension of an artinian algebra.
- The group algebra of a finite group.
- The exterior algebra $\Lambda(k^n)$ when n is odd.

4. Vogel Cohomology

In this section, we will interpret the two conditions (**ac**) and (**ee**) in terms of Vogel cohomologies. For $M, N \in \operatorname{mod} R$, the i-th Vogel cohomology is defined by

$$\widehat{\operatorname{Ext}}^i_R(M,N) := \varinjlim_{n\to\infty} \underline{\operatorname{Hom}}_R(\Omega^{n+i}M, \Omega^n N).$$

Note that $\widehat{\operatorname{Ext}}^i_R(M,N)$ are defined for all integers $i \in \mathbb{Z}$. The below are two main results of this note.

Theorem 20. [13, Theorem 3.2] *Let R be a Gorenstein ring. Then the following conditions are equivalent:*

(1) *R satisfies (**ac**).*
(2) *For all $M, N \in \operatorname{mod} R$,*

$$(*)\quad \widehat{\operatorname{Ext}}^i_R(M,N) = 0 \text{ for all } i \gg 0 \Rightarrow \widehat{\operatorname{Ext}}^i_R(M,N) = 0 \text{ for all } i.$$

Theorem 21. [13, Theorem 4.6] *Let R be a stably symmetric Gorenstein algebra. Then the following conditions are equivalent:*

(1) *R satisfies (**ee**).*
(2) *For all $M, N \in \operatorname{mod} R$,*

$$(**)\quad \widehat{\operatorname{Ext}}^i_R(M,N) = 0 \text{ for all } i \gg 0 \Rightarrow \widehat{\operatorname{Ext}}^i_R(M,N) = 0 \text{ for all } i \ll 0.$$

Since the condition (*) above is stronger than the condition (**) above, the following is immediate.

Corollary 22. [13, Theorem 4.7] *Let R be a stably symmetric Gorenstein algebra. If R satisfies (**ac**), then R satisfies (**ee**).*

The above corollary produces a few more examples of algebras satisfying (**ee**).

Example 23. Every group algebra of a finite group is a symmetric algebra satisfying (**ac**), so it satisfies (**ee**).

Example 24. The exterior algebra $\Lambda(k^n)$ where n is odd is a symmetric algebra satisfying (**ac**), so it satisfies (**ee**).

5. AS-Gorenstein Koszul Algebras

In this last section, we will make similar analysis for AS-Gorenstein Koszul algebras. From now on, we will assume that A is a connected graded algebra over k, $\operatorname{grmod} A$ is the category of finitely generated graded right A-modules, and $M, N \in \operatorname{grmod} A$.

If A is a Koszul algebra, then A is a quadratic algebra, that is, $A = T(V)/(W)$ where $T(V)$ is the tensor algebra on the finite dimensional vector space V over k, $W \subset V \otimes_k V$ is a subspace, and (W) is the two-sided ideal of $T(V)$ generated by W. It is known that its quadratic (Koszul) dual $A^! = T(V^*)/(W^\perp)$ is also Koszul where

$$W^\perp = \{\lambda \in V^* \otimes_k V^* \mid \lambda(w) = 0 \text{ for all } w \in W \subset V \otimes_k V\}.$$

Clearly, $(A^!)^! \cong A$ as graded algebras.

Example 25. An exterior algebra $\Lambda(k^n)$ is a Koszul algebra whose Koszul dual is a polynomial algebra $\Lambda(k^n)^! \cong S(k^n)$.

The class of algebras defined below plays an important role in noncommutative algebraic geometry.

Definition 26. A connected graded algebra A is called AS-Gorenstein if

- $\operatorname{id}(A) = d < \infty$, and
- $\operatorname{Ext}^i_A(k, A) = \begin{cases} k & \text{if } i = d, \\ 0 & \text{if } i \neq d. \end{cases}$

The following are versions of the Koszul duality.

Theorem 27. [10, Proposition 4.5], [11, Theorem 5.3] *If A is a noetherian AS-Gorenstein Koszul algebra such that $A^!$ is noetherian, then there is a duality*

$$E : \mathcal{D}^b(\operatorname{grmod} A) \to \mathcal{D}^b(\operatorname{grmod} A^!),$$

which induces a duality

$$E : \mathcal{S}(\underline{\operatorname{grmod}} A) \to \mathcal{D}^b(\operatorname{Proj} A^!)$$

as triangulated categories.

We refer to [1] for the definition of $\operatorname{Proj} A^!$ when $A^!$ is not commutative. We modify the definition of a stably symmetric algebra in the graded case.

Definition 28. Let A be a connected graded algebra. We say that A is stably symmetric in the graded sense if

$$\mathcal{K} = \Omega^{-d}(-)(\ell) : \mathcal{S}(\underline{\operatorname{grmod}} A) \to \mathcal{S}(\underline{\operatorname{grmod}} A)$$

is a Serre functor for some $d \in \mathbb{Z}$ and $\ell \in \mathbb{Z}$ where $(\ell) : \operatorname{grmod} A \to \operatorname{grmod} A$ is the functor shifting degree by ℓ.

The theorem below produces many examples of stably symmetric graded algebras.

Theorem 29. [13, Corollary 5.7] *Let A be a noetherian AS-Gorenstein Koszul algebra such that $A^!$ is commutative. Then A is stably symmetric in the graded sense if and only if* $\operatorname{Proj} A^!$ *is smooth.*

Example 30. If $\Lambda(k^n)$ is an exterior algebra, then $\Lambda(k^n)$ is a noetherian AS-Gorenstein Koszul algebra such that $\Lambda(k^n)^! \cong S(k^n)$ is a commutative polynomial algebra. Since $\operatorname{Proj} \Lambda(k^n)^! \cong \mathbb{P}^{n-1}$ is a projective space, $\Lambda(k^n)$ is stably symmetric in the graded sense whether n is odd or even.

It follows that $\Lambda(k^n)$ satisfies (ee) in the graded sense, that is, the symmetry in the vanishing of Ext-groups holds for any pair of graded right modules over every exterior algebra.

We can construct many stably symmetric graded algebras which are not even artinian.

Example 31. If

$$A = k\langle x, y, z\rangle/(xz + zx, yz + zy, xy + yx + z^2, x^2, y^2),$$

then A is a noetherian AS-Gorenstein Koszul algebra such that

$$A^! \cong k[x,y,z]/(xy-z^2)$$

is commutative. Since $\operatorname{Proj} A^! \cong \mathbb{P}^1$ is smooth, A is stably symmetric in the graded sense. It is easy to see that A is not artinian.

References

[1] M. Artin and J.J. Zhang, *Noncommutative Projective Schemes*, Adv. Math. **109** (1994), 228-287.

[2] L.L. Avramov and R. Buchweitz, *Support Varieties and Cohomology over Complete Intersections*, Invent. Math. **142** (2000), 285–318.

[3] A. Beligiannis, *The Homological Theory of Contravariantly Finite Subcategories: Auslander-Buchweitz Contexts, Gorenstein Categories and (Co-)Stabilization*, Comm. Algebra **28** (2000), 4547-4596.

[4] D.J. Benson, J.F. Carlson and G.R. Robinson, *On the Vanishing of Group Cohomology*, J. Algebra **131** (1990), 40-73.

[5] C.J. Chan, *An Intersection Multiplicity in Terms of Ext-modules*, Proc. Amer. Math. Soc. **130** (2001), 327-336.

[6] C. Huneke and D.A. Jorgensen, *Symmetry in the Vanishing of Ext over Gorenstein Rings*, Math. Scand. **93** (2003), 161-184.

[7] D.A. Jorgensen and L.M. Sega, *Nonvanishing Cohomology and Classes of Gorenstein Rings*, Adv. Math. **188** (2004), 470-490.

[8] D.A. Jorgensen and L.M. Sega, *Asymmetric Complete Resolutions and Vanishing of Ext over Gorenstein Rings*, Int. Math. Res. Not. **56** (2005), 3459–3477.

[9] I. Mori, *Serre's Vanishing Conjecture for Ext-groups*, J. Pure Appl. Algebra **187** (2004), 207-240.

[10] I. Mori, *Rationality of the Poincaré Series for Koszul Algebras*, J. Algebra **276** (2004), 602-624.

[11] I. Mori, *Riemann-Roch Like Theorem for Triangulated Categories*, J. Pure Appl. Algebra **193** (2004), 263-285.

[12] I. Mori, *Co-point Modules over Koszul Algebras*, J. Lond. Math. Soc., to appear.

[13] I. Mori, *Symmetry in the Vanishing of Ext over Stably Symmetric Algebras*, J. Algebra, to appear.

[14] I. Mori and S.P. Smith, *Bézout's Theorem for Non-commutative Projective Spaces*, J. Pure Appl. Algebra **157** (2001), 279-299.

[15] J.P. Serre, *Algèbre Locale - Multiplicités*, Springer-Verlag, LNM No. **11** (1961).

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ON DERIVED EQUIVALENCES FOR SELFINJECTIVE ALGEBRAS

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ABSTRACT. We show that if A is a representation-finite selfinjective artin algebra then every $P^\bullet \in \mathsf{K}^\mathrm{b}(\mathcal{P}_A)$ with $\mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$ and $\mathrm{add}(P^\bullet) = \mathrm{add}(\nu P^\bullet)$ is a direct summand of a tilting complex, and that if A, B are derived equivalent representation-finite selfinjective artin algebras then there exists a sequence of selfinjective artin algebras $A = B_0, B_1, \cdots, B_m = B$ such that, for any $0 \leq i < m$, B_{i+1} is the endomorphism algebra of a tilting complex for B_i of length ≤ 1.

1. INTRODUCTION

Let A be an artin algebra. Rickard [7, Proposition 9.3] showed that for any tilting complex $P^\bullet \in \mathsf{K}^\mathrm{b}(\mathcal{P}_A)$ the number of nonisomorphic indecomposabele direct summands of $P^\bullet$ coincides with the rank of $K_0(A)$, the Grothendieck group of A, which generalizes earlier results [2, Proposition 3.2] and [6, Theorem 1.19]. He raised a question whether a complex $P^\bullet \in \mathsf{K}^\mathrm{b}(\mathcal{P}_A)$ with $\mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$ is a tilting complex or not if the number of nonisomorphic indecomposable direct summands of $P^\bullet$ coincides with the rank of $K_0(A)$ (see also [6]). In case $P^\bullet$ is a projective resolution of a module $T \in \mathrm{mod}\text{-}A$ with proj dim $T_A \leq 1$, Bongartz [1, Lemma of 2.1] has settled the question affirmatively. More precisely, he showe d that every $T \in \mathrm{mod}\text{-}A$ with proj dim $T_A \leq 1$ and $\mathrm{Ext}^1_A(T,T) = 0$ is a direct summand of a classical tilting module, i.e., a tilting module of projective dimension ≤ 1. Unfortunetely, this is not true in general (see [7, Section 8]). Our first aim is to show that if A is a representation-finite selfinjective artin algebra then every $P^\bullet \in \mathsf{K}^\mathrm{b}(\mathcal{P}_A)$ with $\mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$ and $\mathrm{add}(P^\bullet) = \mathrm{add}(\nu P^\bullet)$, where ν is the Nakayama functor, is a direct summand of a tilting complex (Theorem 4).

Rickard [8, Theorem 4.2] showed that the Brauer tree algebras over a field with the same numerical invariants are derived equivalent to each other. Subsequently, Okuyama pointed out that for any Brauer tree algebras A, B with the same numerical invariants there exists a sequence of Brauer tree algebras $A = B_0, B_1, \cdots, B_m = B$ such that, for any $0 \leq i < m$, B_{i+1} is the endomorphism algebra of a tilting complex for B_i of length ≤ 1. These facts can be formulated as follows. For any tilting complex $P^\bullet \in \mathsf{K}^\mathrm{b}(\mathcal{P}_A)$ associated with a certain sequence of idempotents in a ring A, there exists a sequence of rings $A = B_0, B_1, \cdots, B_m = \mathrm{End}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet)$ such that, for any $0 \leq i < m$, B_{i+1} is the endomorphism ring of a tilting complex for B_i of length ≤ 1 determined by an idempotent (see [4, Proposition 3.2]). We refer to [3], [5] for other examples of derived equivalences which are iterations of derived equivalences induced by tilting complexes of length ≤ 1. Our second aim is to show that for any derived equivalent representation-finite selfinjective artin algebras A, B there exists a sequence of selfinjective artin algebras

The detailed version of this paper has been submitted for publication elsewhere.

$A = B_0, B_1, \cdots, B_m = B$ such that, for any $0 \leq i < m$, B_{i+1} is the endomorphism algebra of a tilting complex for B_i of length ≤ 1 (Theorem 5).

2. Derived equivalences for selfinjective algebras

In the following, R is a commutative artinian ring with the Jacobson radical $\mathfrak{m}$ and A is an artin R-algebra, i.e., A is a ring endowed with a ring homomorphism $R \to A$ whose image is contained in the center of A and is finitely generated as an R-module.

For any artin R-algebra A, we denote by Mod-A the category of right A-modules and by mod-A the full subcategory of Mod-A consisting of finitely generated modules. We denote by $\mathcal{P}_A$ the full subcategory of mod-A consisting of projective modules. Also, we set $D = \mathrm{Hom}_R(-, E(R/\mathfrak{m}))$, where $E(R/\mathfrak{m})$ is an injective envelope of $R/\mathfrak{m}$ in Mod-R, and $\nu = D \circ \mathrm{Hom}_A(-, A)$, which is called the Nakayama functor.

Definition 1. Assume A is selfinjective and let $\{e_1, \cdots, e_n\}$ be a basic set of orthogonal local idempotents in A. Then there exists a permutation ρ of the set $I = \{1, \cdots, n\}$, called the Nakayama permutation, such that $\nu(e_i A) \simeq e_{\rho(i)}A$ for all $i \in I$.

Proposition 2. *Assume A is selfinjective and has a cyclic Nakayama permutation. Let B be a selfinjective artin R-algebra derived equivalent to A. Then B is Morita equivalent to A.*

For a cochain complex $X^\bullet$ over an abelian category $\mathcal{A}$, we denote by $\mathrm{H}^n(X^\bullet)$ the n-th cohomology of $X^\bullet$. For an additive category $\mathcal{B}$, we denote by $\mathsf{K}(\mathcal{B})$ (resp., $\mathsf{K}^+(\mathcal{B})$, $\mathsf{K}^-(\mathcal{B})$, $\mathsf{K}^b(\mathcal{B})$) the homotopy category of complexes (resp., bounded below complexes, bounded above complexes, bounded complexes) over $\mathcal{B}$. As usual, we consider objects of $\mathcal{B}$ as complexes over $\mathcal{B}$ concentrated in degree zero.

Definition 3. For any nonzero $P^\bullet \in \mathsf{K}^-(\mathcal{P}_A)$ we set

$$a(P^\bullet) = \max\{i \in \mathbb{Z} \mid \mathrm{H}^i(P^\bullet) \neq 0\},$$

and for any nonzero $P^\bullet \in \mathsf{K}^+(\mathcal{P}_A)$ we set

$$b(P^\bullet) = \min\{i \in \mathbb{Z} \mid \mathrm{Hom}_{\mathsf{K}(\text{Mod-}A)}(P^\bullet[i], A) \neq 0\}.$$

Then for any nonzero $P^\bullet \in \mathsf{K}^b(\mathcal{P}_A)$ we set $l(P^\bullet) = a(P^\bullet) - b(P^\bullet)$ and call it the length of $P^\bullet$. For the sake of convenience, we set $l(P^\bullet) = 0$ for $P^\bullet \in \mathsf{K}^b(\mathcal{P}_A)$ with $P^\bullet \simeq 0$.

For an object X in an additive category $\mathcal{B}$, we denote by $\mathrm{add}(X)$ the full subcategory of $\mathcal{B}$ whose objects are direct summands of finite direct sums of copies of X and by $X^{(n)}$ the direct sum of n copies of X.

Theorem 4. *Assume A is selfinjective and representation-finite. Let $P^\bullet \in \mathsf{K}^b(\mathcal{P}_A)$ be a complex with $\mathrm{Hom}_{\mathsf{K}(\text{Mod-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$ and $\mathrm{add}(P^\bullet) = \mathrm{add}(\nu P^\bullet)$. Then there exists some $Q^\bullet \in \mathsf{K}^b(\mathcal{P}_A)$ such that $Q^\bullet \oplus P^\bullet$ is a tilting complex. In particular, if the number of nonisomorphic indecomposable direct summands of $P^\bullet$ coincides with the rank of the Grothendieck group $K_0(A)$, then $P^\bullet$ is a tilting complex.*

Theorem 5. *Assume A is selfinjective and representation-finite. Then for any selfinjective artin R-algebra B derived equivalent to A the following hold.*

(1) *There exists a sequence of selfinjective artin R-algebras $A = B_0, B_1, \cdots,$ $B_m = B$ such that for any $0 \leq i < m$, B_{i+1} is the endomorphism algebra of a tilting complex for B_i of length ≤ 1.*

(2) *The Nakayama permutation of B coincides with that of A.*

The proofs of Theorems 4 and 5 follow by induction on the length of $P^\bullet$. But, in Theorem 5, we set $P^\bullet$ to be a tilting complex with $\mathrm{End}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet) \cong B$. The key of the induction is the following Lemma 6.

Lemma 6. *Assume A is selfinjective and representation-finite. Let $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ be a complex of length ≥ 1 with $\mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$ and $\mathrm{add}(P^\bullet) = \mathrm{add}(\nu P^\bullet)$. Then there exists a tilting complex $T^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ of length 1 such that*

(1) $\mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(T^\bullet, P^\bullet[i]) = 0$ *for* $i \geq l(P^\bullet)$,

(2) $\mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet[i], T^\bullet) = 0$ *for* $i < 0$, *and*

(3) $\mathrm{End}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(T^\bullet)$ *is a selfinjective artin R-algebra whose Nakayama permutation coincides with that of A.*

REFERENCES

[1] K. Bongartz, Tilted algebras, in: *Representations of algebras (Puebla, 1980)*, 26–38, Lecture Notes in Math., **903**, Springer, Berlin.

[2] D. Happel and C. M. Ringel, Tilted algebras, Trans. Amer. Math. Soc. **274** (1982), no. 2, 399–443.

[3] D. Happel, J. Rickard and A. Schofield, Piecewise hereditary algebras, Bull. London Math. Soc. **20** (1988), no. 1, 23–28.

[4] M. Hoshino and Y. Kato, Tilting complexes associated with a sequence of idempotents, J. Pure Appl. Algebra **183** (2003), no. 1-3, 105–124.

[5] S. König and A. Zimmermann, Tilting selfinjective algebras and Gorenstein orders, Quart. J. Math. Oxford Ser. (2) **48** (1997), no. 191, 351–361.

[6] Y. Miyashita, Tilting modules of finite projective dimension, Math. Z. **193** (1986), no. 1, 113–146.

[7] J. Rickard, Morita theory for derived categories, J. London Math. Soc. (2) **39** (1989), no. 3, 436–456.

[8] J. Rickard, Derived categories and stable equivalence, J. Pure Appl. Algebra **61** (1989), 303–317.

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AZUMAYA'S CONJECTURE AND HARADA RINGS

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ABSTRACT. Azumaya conjectured that every exact ring has a self-duality. Recently we study self-duality of (quasi-)Harada rings and obtain several results about Azumaya's conjecture and related problems in [10].

1. 研究の背景

東屋は 1983 年に [3] において，次の予想を提示した．

東屋の予想. すべての exact 環は self-duality をもつであろう．

exact 環のクラスは serial 環を含んでいるが，これについては，self-duality の理論において非常に有名な結果

定理 A (Dischinger-Müller [5]). すべての serial 環は weakly symmetric self-duality をもつ．

が知られている．exact 環と serial 環のクラスの間には局所分配的環と呼ばれる環のクラスがあるので，

問題 B. すべての局所分配的環は self-duality をもつか?

が問題となる．東屋の予想が肯定的であれば，これも肯定的であるはずだが，いまだに未解決である．筆者は [10] において原田環や準原田環の研究を行い，それらを応用することにより，東屋の予想や問題 B に関するさまざまな結果を得た．この報告集の主な結果は，論文 [10] (特に第 5 節) によるものであり，証明等は [10] を参照されたい．

以下この報告集では，すべての環は単位元をもち，すべての加群は単位的であるとする．扱う環は主にアルチン環 (両側アルチン環) である．加群 M に対して，radical と socle それぞれ $J(M)$ と $S(M)$ で表す．

2. SELF-DUALITY

まず最初に self-duality の定義を思い出しておこう．アルチン環 R, S 上の両側加群 ${}_SU_R$ に対して，双対関手 $\mathrm{Hom}(-,U)$ が有限生成右 R 加群全体の圏と有限生成左 S 加群全体の圏との間の duality を定めるとき，両側加群 ${}_SU_R$ は Morita duality を定めるという．特に，Morita duality を定める両側加群 ${}_RU_R$ が存在するとき，環 R は self-duality をもつという．QF 環 R において正則両側加群 ${}_RR_R$ は self-duality を定めるので，QF 環は self-duality をもつ環の典型的な例である．

This note is mainly based on Section 5 of [10] and is in a final form.

self-dualityを定める両側加群${}_RU_R$について，Rの任意の原始羃等元eに対して，$S(eU) \cong eR/J(eR)$が成り立つとき，${}_RU_R$は **weakly symmetric self-duality を定める**という．また，Rの任意のイデアルIに対して，$l_R l_U(I) = I$が成り立つとき，${}_RU_R$は **good self-duality を定める**という．ここでl_Rやl_Uは左annihilatorを表す．${}_RU_R$がgood self-dualityを定めれば，weakly symmetric self-dualityを定める．体K上有限次元多元環Rにおいて，(R,R)両側加群$\mathrm{Hom}_K(R,K)$はgood self-dualityを定める．また，定理Aとして述べたように，serial環はweakly symmetric self-duality (実際にはgood self-duality)をもつ．

RをQF環，$e_1, e_2, \ldots, e_n$をRの直交原始羃等元の基本集合とする．このとき$S(e_{\sigma(i)}R) \cong e_iR/J(e_iR)$ $(1 \le i \le n)$を満たす$\{1,2,\ldots,n\}$の置換σが存在する．この置換を **Rの中山置換**という．中山置換が恒等的なQF環を **weakly symmetric** であるという．Rの環自己同型写像ϕが$\phi(e_i) = e_{\sigma(i)}$ $(1 \le i \le n)$を満たすとき，ϕを **QF環Rの中山自己同型写像**という．basicなQF環については，weakly symmetric self-dualityの存在と中山自己同型写像の存在は同値である．

Morita dualityを定める両側加群の列${}_{R_1}U_{1R_2}, {}_{R_2}U_{2R_3}, \ldots, {}_{R_m}U_{mR_{m+1}}$で$R = R_1 = R_{m+1}$となるものが存在するとき，環$R$は **almost self-duality** をもつという．almost self-dualityはself-dualityの一般化である．

このように，さまざまな種類のself-dualityを考えるのは，それぞれ環の変形(剰余環，有限生成射影的加群の自己準同型環，等)への遺伝の状況が異なるからである．端的に言うと，good self-dualityが最も遺伝しやすく，(単なる) self-dualityが最も遺伝しにくい．

3. SERIAL環と局所分配的環，EXACT環

任意の直既約射影的右加群がuniserial (すなわち，部分加群全体がchainをなす)であるようなアルチン環を**右serial環**という．右serialかつ左serialなアルチン環を **serial環**という．定理Aとして述べたように，Dischinger と Müller [5] はすべてのserial環はweakly symmetric self-dualityをもつことを証明した．一方Waschbüsch [11] は，すでにAmdalとRingdal [1] によってserial環におけるself-dualityの存在は主張されていることを指摘し，彼自身も証明を与えている．しかしながら，それらの証明は技巧的である．Haack [6] はserial環におけるself-dualityの存在の一般的な証明には成功しなかったものの，いくつかの部分的な結果を示した．そのうちの一つの「任意の(basicな) serial QF環はweakly symmetric self-duality (中山自己同型写像)をもつ」は，証明が平易であるだけでなく，その結果自体，加戸・大城 [7] によって与えられ，その後同様な方向によりÁnh [2] や筆者(定理8参照)によっても与えられたserial環におけるself-dualityの存在の別証明の基礎ともなるので，ここで述べておこう．

Rをbasicで環として直既約なserial QF環とする．$e_1, e_2, \ldots, e_n$をRの直交原始羃等元の完全集合で，$e_1R, e_2R, \ldots, e_nR$がKupisch series，すなわち，任意の$i = 1,2,\ldots,n$に対して$e_iR \to J(e_{[i+1]}R)$が射影被覆となるものとする．ただし，$[j]$はnを法とするjの最小正剰余を表す．RはQFであるから，$e_1R, e_2R, \ldots, e_nR$は同じ組成列の長さをもつ．特にRの中山置換は，あるmに対して$i \mapsto [i-m]$で与えられる．RがQF環としてweakly symmetricの場合，恒等写像が中山自己同型写像である．

定理 1 (Haack [6]). 以上の設定において，R が QF 環として weakly symmetric でない場合，R の環自己同型写像 ϕ で，$\phi(e_i) = e_{[i-1]}$ $(1 \leq i \leq n)$ を満たすものが存在する．特に ϕ^m は R の中山自己同型写像となる．したがって，任意の (basic な) serial QF 環は weakly symmetric self-duality (中山自己同型写像) をもつ．

uniserial 加群の一般化の 1 つとして，部分加群全体の束が分配的なとき，加群は**分配的 (distributive)** であるという．この条件は少し分かりにくいが，分配的加群は，任意の剰余加群の socle が同型な単純加群の 2 個以上の直和を含まない，という使いやすい条件により特徴付けられることが知られている．

任意の直既約射影的左または右加群が分配的なとき，アルチン環は**局所分配的 (locally distributive)** であるという．一般に，good self-duality を定める両側加群は weakly symmetric self-duality を定めるが，局所分配的環上では逆が成り立つ．以後，局所分配的環においては weakly symmetric self-duality について述べることにする．

例 2. K を体，Q を quiver $\alpha \circlearrowleft 1 \xrightarrow{\beta} 2$ とし，$R = KQ/\langle \alpha^2, \beta\alpha \rangle$ をパス多元環の剰余多元環とするとき，直既約射影的加群の Loewy series は $R_R = \begin{smallmatrix}1\\1\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\end{smallmatrix}$，${}_RR = \begin{smallmatrix}&1&\\1&&2\end{smallmatrix} \oplus 2$ である．したがって，任意の直既約射影的左または右 R 加群は分配的であるから，R は局所分配的環である．

アルチン環 R は，両側イデアルとしての組成列 (すなわち，各組成因子が両側加群として単純なもの)

$$R = I_0 > I_1 > I_2 > \cdots > I_n = 0$$

で，各組成因子 I_i/I_{i+1} は (R, R) 両側加群として平衡的 (すなわち，すべての自己準同型写像は R の元による乗法写像で与えられる) なものが存在するとき，**exact 環**であるという．局所分配的環は exact 環であることが知られている．したがって，東屋の予想の前に局所分配的環の self-duality (問題 B) を考える必要がある．

4. 原田環と準原田環

大城によって導入され，深く研究されている原田環と呼ばれる環のクラスがある．原田環はさまざまな特徴づけをもつが，ここでは，射影的右加群全体のクラスが本質的拡大で閉じているようなアルチン環として**左原田環 (left Harada ring)** を定義することとする．左原田環は QF-3 である．また，任意の直既約射影的右加群が擬移入的 (quasi-injective) であるとき，アルチン環は**左準原田環 (left quasi-Harada ring)** であるという．(左準原田環の概念は左原田環の一般化として，馬場・岩瀬 [4] によって左アルチン環に対して定義されたが，必然的に右アルチンになることを筆者が示した．) 左準原田環 R は**右 QF-2** (すなわち，任意の直既約射影的右 R 加群の socle は単純) であるが，局所分配的環については逆が成り立つ．

補題 3. 局所分配的右 QF-2 環は左準原田環である．

したがって局所分配的右 QF-2 環について，準原田環の研究を応用することができる．

例 4. (1) Rを例2の局所分配的右QF-2環とする．補題3よりRは左準原田環で，行列表現$R=\begin{bmatrix} A & 0 \\ J(A) & A/J(A) \end{bmatrix}$をもつ．ただし，$A=e_1Re_1$で$e_1$は$R$の頂点1に対応する冪等元とする．$A$は$J(A)^2=0$なる局所serial環である．

(2) Aを(1)と同じ局所serial環とするとき，$\begin{bmatrix} A & A/J(A) \\ J(A) & A/J(A) \end{bmatrix}$は左原田環(serial環)である．

大城は原田環の構造を深く研究し，すべての片側原田環はQF環から構成されることを証明した．我々は同様に準原田環もQF環から構成されることを示した．

定理 5. 任意の左準原田環Rに対して，QF環Sが存在して，RはSから出発して，ある種の部分環の剰余環を取るという操作を有限回繰り返すことによって復元できる．

したがって，左準原田環に関するある種の議論はQF環に帰着できる．特に補題3より，局所分配的右QF-2環については，局所分配的QF環に帰着できるのである．

例 6. 例4(1)の左準原田環Rは，QF環$S=\begin{bmatrix} A & A \\ A & A \end{bmatrix}$の部分環$\begin{bmatrix} A & 0 \\ J(A) & A \end{bmatrix}$のイデアル$\begin{bmatrix} 0 & 0 \\ 0 & J(A) \end{bmatrix}$による剰余環と同型である．

原田環はQF環やserial環の一般化と見なすことができる．QF環やserial環はself-dualityをもつため，原田環についてもself-dualityの有無が問題となり，加戸・大城は次の定理を証明した．

定理 7 (加戸・大城 [7])**.** 次は同値である．

(A) 任意の左原田環はself-dualityをもつ．
(B) 任意の左原田環はweakly symmetric self-dualityをもつ．
(C) 任意の(basicな) QF環はweakly symmetric self-duality (中山自己同型写像)をもつ．

この後，筆者は[8]においてself-dualityをもたない左原田環が存在することを示したが，この定理7の手法は有用であり，今回の研究でも重要な役割を果たした．なお，左原田環は必ずしもself-dualityをもたないものの，つねにalmost self-dualityをもつことを，やはり筆者[9]は証明している．

5. 主結果と関連する問題

最後に，東屋の予想や問題Bに関する主結果と関連する問題について述べる．

すでに述べたように，補題3や定理5を用いれば，局所分配的右QF-2環に関するさまざまな議論が局所分配的QF環に関する問題に帰着できる．特に，局所分配的右serial環のweakly symmetric self-dualityの問題はserial QF環における問題に帰着されるが，定理1よりserial QF環はweakly symmetric self-dualityをもつ．したがって，serial環のweakly symmetric self-duality (定理A)は次のように改良できる．

定理 8. すべての局所分配的右 serial 環は weakly symmetric self-duality をもつ.

次の定理は, 直交原始冪等元の個数が高々2個の局所分配的 QF 環は weakly symmetric self-duality をもつことから従う.

定理 9. 局所分配的右 QF-2 環 R について, 右 socle $S(R_R)$ が高々2 個の非同型な単純加群しか含まない場合, R は weakly symmetric self-duality をもつ.

これらの定理は東屋の予想や問題 B の部分的な解答を与えている. 一般の局所分配的右 QF-2 環の self-dualty についてはまだ分からないが, 定理 7 の手法を用いることにより, 次のように言い換えることができた.

定理 10. 次は同値である.

(A) 任意の局所分配的右 QF-2 環は self-duality をもつ.
(B) 任意の局所分配的右 QF-2 環は weakly symmetric self-duality をもつ.
(C) 任意の (basic な) 局所分配的 QF 環は weakly symmetric self-duality (中山自己同型写像) をもつ.

なお, 次のように局所分配的右 QF-2 環における almost self-duality の存在は示すことができた.

定理 11. 任意の局所分配的右 QF-2 環は almost self-duality をもつ.

定理 10 と同様に定理 7 の手法を用いて, 東屋の予想や問題 B も次のように言い換えられる.

定理 12. 次は同値である.

(A) 任意の局所分配的環 (resp. exact 環) は self-duality をもつ.
(B) 任意の局所分配的環 (resp. exact 環) は weakly symmetric self-duality をもつ.

定理 8, 9, 11 から, 東屋の予想や問題 B の前に, 局所分配的右 QF-2 環における self-duality の問題を解決すべきであると思われるが, 定理 10 よりこれは次と同じである.

問題 13. すべての (basic な) 局所分配的 QF 環は weakly symmetric self-duality (中山自己同型写像) をもつか?

basic な serial QF 環が中山自己同型写像をもつことは, 定理 1 で述べたように, serial QF 環の場合は中山置換の形が決まっており扱いやすいからだと考えられる. これから, basic な局所分配的 QF 環における中山自己同型写像の存在を論じるため, まず中山置換に serial QF 環の場合と同様な制限を課すことが考えられる. しかしながら, 局所分配的 QF 環の中山置換は多様である. 実際, 与えられた任意の置換を中山置換としてもつ, 環として直既約な局所分配的 QF 環が存在する ([10, Example 5.10]). したがって, 問題 13 を一般的に解決するためには, 最終的には中山置換に制限を課すことはできない.

REFERENCES

[1] I. K. Amdal and F. Ringdal. Catégories unisérielles. *C. R. Acad. Sci. Paris Sér. A-B*, 267:A85–A87 and A247–A249, 1968.

[2] P. N. Ánh. Selfdualities and serial rings, revisited. *Bull. London Math. Soc.*, 38(3):411–420, 2006.

[3] G. Azumaya. Exact and serial rings. *J. Algebra*, 85(2):477–489, 1983.

[4] Y. Baba and K. Iwase. On quasi-Harada rings. *J. Algebra*, 185(2):544–570, 1996.

[5] F. Dischinger and W. Müller. Einreihig zerlegbare artinsche Ringe sind selbstdual. *Arch. Math. (Basel)*, 43(2):132–136, 1984.

[6] J. K. Haack. Self-duality and serial rings. *J. Algebra*, 59(2):345–363, 1979.

[7] J. Kado and K. Oshiro. Self-duality and Harada rings. *J. Algebra*, 211(2):384–408, 1999.

[8] K. Koike. Examples of QF rings without Nakayama automorphism and H-rings without self-duality. *J. Algebra*, 241(2):731–744, 2001.

[9] K. Koike. Almost self-duality and Harada rings. *J. Algebra*, 254(2):336–361, 2002.

[10] K. Koike. Self-duality of quasi-Harada rings and locally distributive rings. *J. Algebra*, 302(2):613–645, 2006.

[11] J. Waschbüsch. Self-duality of serial rings. *Comm. Algebra*, 14(4):581–589, 1986.

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STABLE EQUIVALENCES INDUCED FROM GENERALIZED TILTING MODULES III

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ABSTRACT. For a generalized tilting module ${}_BT_A$ and a nilpotent symmetric algebra $({}_AM_A, \varphi, \psi)$, under natural assumptions, the stable functors $\mathcal{K}er : \underline{\text{mod}}-\Lambda(\psi,\varphi) \to \underline{\text{mod}}-\Lambda(\varphi^T,\psi^T)$ and $\mathcal{C}oker : \underline{\text{mod}}-\Lambda(\varphi^T,\psi^T) \to \underline{\text{mod}}-\Lambda(\psi,\varphi)$ has been constructed and it was proved that they induce an equivalence $\underline{\text{mod}}-\Lambda(\psi,\varphi) \approx \underline{\text{mod}}-\Lambda(\psi^T,\varphi^T)$ in [2]. In this note, it is proved that those functors $\mathcal{K}er$ and $\mathcal{C}oker$ preserve the distinguished triangles and, therefore, the stable categories $\underline{\text{mod}}-\Lambda(\psi,\varphi)$ and $\underline{\text{mod}}-\Lambda(\psi^T,\varphi^T)$ are equivalent as triangulated categories.

1. INTRODUCTION

Let A and B be finite dimensional algebras over a field K. A bimodule ${}_BT_A$ is called a generalized tilting module if

(1) $B = \text{End}(T_A)$ and $\text{End}({}_BT) = A$, and
(2) $\text{Ext}^n_B(T,T) = 0 = \text{Ext}^n_A(T,T)$ for any $n > 0$.

A system $({}_AM_A, \psi, \varphi)$ consisting of a bimodule ${}_AM_A$ and two homomorphisms $\varphi : {}_AM \otimes_A M_A \to {}_AM_A$ and $\psi : {}_AM \otimes_A M_A \to {}_A\text{D}A_A$ is called a nilpotent symmetric algebra if

(1) the algebra (M, φ) is associative and nilpotent,
(2) the homomorphism ψ satisfies
 (i) $\psi(\varphi(m_1 \otimes m_2) \otimes m_3) = \psi(m_1 \otimes \varphi(m_2 \otimes m_3))$,
 (ii) $\psi(m_1 \otimes m_2)(1_A) = \psi(m_2 \otimes m_1)(1_A)$
 for all elements $m_1, m_2, m_3 \in M$, and
(3) the homomorphism ψ is non-degenerate in the sense that the condition $\psi(m \otimes M) = 0$ implies $m = 0$ for an element $m \in M$,

where D stands for the canonical duality functor $\text{Hom}_K(?, K)$. Let ${}_BT_A$ is a generalized tilting module and $({}_AM_A, \varphi, \psi)$ a nilpotent symmetric algebra. The induced system $({}_BM^T_B, \varphi^T, \psi^T)$ is defined as $M^T = T \otimes_A \text{Hom}_A(T, M)$ and

$$\varphi^T(t_1 \otimes f_1 \otimes t_2 \otimes f_2) = t_1 \otimes \varphi(f_1(t_2) \otimes f_2(?)) \in M^T,$$

$$\psi^T(t_1 \otimes f_1 \otimes t_2 \otimes f_2) = \psi(f_1(t_2) \otimes f_2(?t_1))(1_A) \in \text{D}B$$

for elements $t_1, t_2 \in T$ and $f_1, f_2 \in \text{Hom}_A(T, M)$. Then, the system (φ^T, ψ^T) is again a nilpotent symmetric algebra if the homomorphism

$$\theta_{T,M} : {}_BT \otimes_A \text{Hom}_A(T, M)_B \to {}_B\text{Hom}_A(T, T \otimes_A M)_B$$

defined by $\theta_{T,M}(t \otimes f)(t') = t \otimes f(t')$ for $t, t' \in T$ and $f \in \text{Hom}_A(T, M)$ is bijective. In this case, we have two symmetric algebras

$$\Lambda(\varphi, \psi) = A \oplus M \oplus \text{D}A$$

The detailed version of this paper will be submitted for publication elsewhere.

and

$$\Lambda(\varphi^T, \psi^T) = B \oplus M^T \oplus \mathrm{D}B.$$

The multiplication of the algebra $\Lambda(\varphi, \psi)$ is defined as

$$(a, m, s) \cdot (a', m', s') = (aa', am' + ma' + \varphi(m \otimes m'), as' + sa' + \psi(m \otimes m'))$$

for $a, a' \in A$, $m, m' \in M$ and $s, s' \in \mathrm{D}A$. In the same way, the multiplication of the algebra $\Lambda(\varphi^T, \psi^T)$ is defined by using homomorphisms φ^T and ψ^T. For such symmetric algebras $\Lambda(\varphi, \psi)$ and $\Lambda(\varphi^T, \psi^T)$, assuming several conditions, it is proved that the kernel functor $\mathcal{K}er : \underline{\mathrm{mod}}{-}\Lambda(\varphi, \psi) \to \underline{\mathrm{mod}}{-}\Lambda(\varphi^T, \psi^T)$ and the cokernel functor $\mathcal{C}oker : \underline{\mathrm{mod}}{-}\Lambda(\varphi^T, \psi^T) \to \underline{\mathrm{mod}}{-}\Lambda(\varphi, \psi)$ are defined and that those functors induce a category equivalence $\underline{\mathrm{mod}}{-}\Lambda(\varphi, \psi) \approx \underline{\mathrm{mod}}{-}\Lambda(\varphi^T, \psi^T)$.

It is known by D. Happel [1] that the stable module category $\underline{\mathrm{mod}}{-}\Lambda$ of any self-injective algebra Λ has a natural structure of triangulated category with Ω_Λ^{-1} as the translation functor. In this note, we prove that our functor $\mathcal{K}er$ preserves the distinguished triangles and, therefore, the stable module categories $\underline{\mathrm{mod}}{-}\Lambda(\varphi, \psi)$ and $\underline{\mathrm{mod}}{-}\Lambda(\varphi^T, \psi^T)$ are equivalent as triangulated categories.

2. The stable functor $\mathcal{K}er$

In order to check that the functor $\mathcal{K}er : \underline{\mathrm{mod}}{-}\Lambda(\varphi, \psi) \to \underline{\mathrm{mod}}{-}\Lambda(\varphi^T, \psi^T)$ preserves distinguished triangles in the next section, we recall here its definition.

Let $({}_AM_A, \varphi, \psi)$ be a nilpotent symmetric algebra and ${}_BT_A$ a generalized tilting module. We call an exact sequence

$$\cdots\cdots \to T_1 \to T_0 \to X \to 0$$

a dominant right T_A-resolusion of a module X_A if (1) $T_k \in \mathrm{add}(T_A)$ for all $k \geqslant 0$ and (2) the sequence

$$\cdots\cdots \to \mathrm{Hom}_A(T, T_1) \to \mathrm{Hom}_A(T, T_0) \to \mathrm{Hom}_A(T, X) \to 0$$

is exact again. We denote by $\mathrm{gen}^*(T_A)$ the class of all modules X_A for which there exist dominant right T_A-resolutions. The notion of dominant left $\mathrm{D}T_B$-resolutions of B-modules and the class $\mathrm{cog}^*(\mathrm{D}T_B)$ are defined in the dual manner. To define the stable functors

$$\mathcal{K}er : \underline{\mathrm{mod}}{-}\Lambda(\varphi, \psi) \leftrightarrows \underline{\mathrm{mod}}{-}\Lambda(\varphi^T, \psi^T) : \mathcal{C}oker$$

and to prove that those induce an equivalence $\underline{\mathrm{mod}}{-}\Lambda(\varphi, \psi) \approx \underline{\mathrm{mod}}{-}\Lambda(\varphi^T, \psi^T)$, we suppose that the following four conditions

(A) the map $\theta_{T,M} : T \otimes_A \mathrm{Hom}_A(T, M) \to \mathrm{Hom}_A(T, T \otimes_A M)$ is bijective,
(B) the modules M_A and $T \otimes_A M_A$ are in the class $\mathcal{C}(T_A)$,
(C) the class $\mathcal{C}(T_A)$ is contravariantly finite in the category $\mathrm{mod}{-}A$, and
(D) the class $\mathcal{D}(\mathrm{D}T_B)$ is covariantly finite in the category $\mathrm{mod}{-}B$

are satisfied, where $\mathcal{C}(T_A) = (T_A)^\perp \cap \mathrm{gen}^*(T_A)$ and $\mathcal{D}(\mathrm{D}T_B) = {}^\perp(\mathrm{D}T_B) \cap \mathrm{cog}^*(\mathrm{D}T_B)$.

Let $X_{\Lambda(\varphi,\psi)}$ be a module over the symmetric algebra $\Lambda(\varphi, \psi) = A \oplus M \oplus \mathrm{D}A$. Since A is a subalgebra of $\Lambda(\varphi, \psi)$, X can be seen as a module over A, which we call the underlying module of $X_{\Lambda(\varphi,\psi)}$ and denote by X_A. Then, the multiplication $X \times \Lambda(\varphi, \psi) \to X$ defines two homomorphisms $\alpha_X : X \otimes_A M_A \to X_A$ and $\beta_X : X \otimes_A \mathrm{D}A_A \to X_A$ and they satisfy the four conditions

(M-1) $\beta_X \cdot (\beta_X \otimes \mathrm{D}A) = 0$,
(M-2) $\alpha_X \cdot (\beta_X \otimes M) = 0$,
(M-3) $\beta_X \cdot (\alpha_X \otimes \mathrm{D}A) = 0$, and
(M-4) $\alpha_X \cdot (\alpha_X \otimes M) = \alpha_X \cdot (X \otimes \varphi) + \beta_X \cdot (X \otimes \psi)$.

Conversely, for a module X_A and two homomorphisms $\alpha_X : X \otimes_A M_A \to X_A$ and $\beta_X : X \otimes_A \mathrm{D}A_A \to X_A$ satisfying the four conditions above, we can define a $\Lambda(\varphi,\psi)$-module structure on X by $x \cdot (a,m,s) = xa + \varphi(x \otimes m) + \psi(x \otimes s)$ for elements $x \in X$ and $(a,m,s) \in \Lambda(\varphi,\psi)$. In this way, we may identify any module $X_{\Lambda(\varphi,\psi)}$ with the triple (X_A, α_X, β_X). Similarly, a homomorphism of $\Lambda(\varphi,\psi)$-modules $f : X_{\Lambda(\varphi,\psi)} \to Y_{\Lambda(\varphi,\psi)}$ is a homomorphism of underlying modules $X_A \to Y_A$ which satisfies the following two conditions

(H-1) $f \cdot \alpha_X = \alpha_Y \cdot (f \otimes M)$ and
(H-2) $f \cdot \beta_X = \beta_Y \cdot (f \otimes \mathrm{D}A)$.

Let (X_A, α_X, β_X), (Y_A, α_Y, β_Y) be $\Lambda(\varphi,\psi)$-modules and $f : X_{\Lambda(\varphi,\psi)} \to Y_{\Lambda(\varphi,\psi)}$ a homomorphism. By condition (C), there exist exact sequences of the form

$$0 \to V_X \to W_X \xrightarrow{\gamma_X} X \to 0 \quad \text{and} \quad 0 \to V_Y \to W_Y \xrightarrow{\gamma_Y} Y \to 0$$

such that V_X, $V_Y \in \mathcal{C}(T_A)$ and W_X, $W_Y \in {}^{\perp}\mathcal{C}(T_A)$. Since $\mathrm{Ext}^1_A(W_X, V_Y) = 0$, we get two homomorphisms $W_f : W_X \to W_Y$ and $V_f : V_X \to V_Y$ over A such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_X & \longrightarrow & W_X & \xrightarrow{\gamma_X} & X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle V_f} & & \downarrow{\scriptstyle W_f} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & V_Y & \longrightarrow & W_Y & \xrightarrow[\gamma_Y]{} & Y & \longrightarrow & 0 \end{array}$$

is commutative.

It is checked that there is an isomorphism $\Lambda(\varphi^T,\psi^T) \otimes_B T \cong T \otimes_A \Lambda(\varphi,\psi)$ of K-spaces and this defines a $(\Lambda(\varphi^T,\psi^T), \Lambda(\varphi,\psi))$-bimodule, which we denote by ${}_{\Lambda(\varphi^T,\psi^T)}\Theta_{\Lambda(\varphi,\psi)}$. Then, the $\Lambda(\varphi^T,\psi^T)$-modules $\mathcal{K}er(X)$, $\mathcal{K}er(Y)$ and a $\Lambda(\varphi^T,\psi^T)$-homomorphism $\mathcal{K}er(f) : \mathcal{K}er(X) \to \mathcal{K}er(Y)$ are defined by the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K}er(X) & \longrightarrow & \mathrm{Hom}_B(\Lambda(\varphi^T,\psi^T), W_X \otimes_A \mathrm{D}T) & \xrightarrow{\lambda_X} & \mathrm{Hom}_{\Lambda(\varphi,\psi)}(\Theta, X) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathcal{K}er(f)} & & \downarrow{\scriptstyle \mathrm{Hom}(\Lambda(\varphi^T,\psi^T), W_f \otimes \mathrm{D}T)} & & \downarrow{\scriptstyle \mathrm{Hom}(\Theta, f)} & & \\ 0 & \longrightarrow & \mathcal{K}er(Y) & \longrightarrow & \mathrm{Hom}_B(\Lambda(\varphi^T,\psi^T), W_Y \otimes_A \mathrm{D}T) & \xrightarrow[\lambda_Y]{} & \mathrm{Hom}_{\Lambda(\varphi,\psi)}(\Theta, Y) & \longrightarrow & 0 \end{array}$$

where the homomorphism λ_X is defined as follows: First the underlying module of $\mathrm{Hom}_{\Lambda(\varphi,\psi)}(\Theta, X)$ is $\mathrm{Hom}_A(T, X)$ since $\mathrm{Hom}_{\Lambda(\varphi,\psi)}(\Theta, X) = \mathrm{Hom}_{\Lambda(\varphi,\psi)}(T \otimes_A \Lambda(\varphi,\psi), X) \cong \mathrm{Hom}_A(T, X)$. Second, the underlying module of the $\Lambda(\varphi^T,\psi^T)$-module

$$\mathrm{Hom}_B(\Lambda(\varphi^T,\psi^T), W_X \otimes_A \mathrm{D}T) = \mathrm{Hom}_B(B \oplus T \otimes_A \mathrm{Hom}_A(T, M) \oplus \mathrm{D}B, W_X \otimes_A \mathrm{D}T)$$

is isomorphic to a direct sum of three modules

$$\mathrm{Hom}_B(\mathrm{D}B, W_X \otimes_A \mathrm{D}T) \cong \mathrm{Hom}_B(T \otimes_A \mathrm{D}T, W_X \otimes_A \mathrm{D}T) \cong \mathrm{Hom}_B(T, W_X),$$

$$\begin{aligned} \mathrm{Hom}_B(T \otimes_A \mathrm{Hom}_A(T, M), W_X \otimes_A \mathrm{D}T) &\cong \mathrm{D}(\mathrm{Hom}_A(T, \mathrm{D}\mathrm{Hom}_A(T, M)) \otimes_B \mathrm{Hom}_A(W_X, T)) \\ &\cong \mathrm{D}\mathrm{Hom}_A(W_X, \mathrm{D}\mathrm{Hom}_A(T, M)) \cong W_X \otimes_A \mathrm{Hom}_A(T, M) \end{aligned}$$

and

$$\mathrm{Hom}_B(B, W_X \otimes_A DT) \cong W_X \otimes_A DT.$$

Using those modules, the map λ_X is defined by giving its three components

$$\lambda_{X,1} = \mathrm{Hom}(T, \gamma_X) : \mathrm{Hom}_A(T, W_X) \to \mathrm{Hom}_A(T, X),$$

$$\lambda_{X,2} = \alpha_X^* \cdot (\gamma_X \otimes \mathrm{Hom}_A(T, M)) : W_X \otimes_A \mathrm{Hom}_A(T, M) \to \mathrm{Hom}_A(T, X)$$

and

$$\lambda_{X,3} = \beta_X^* \cdot (\gamma_X \otimes DT) : W_X \otimes_A DT \to \mathrm{Hom}_A(T, X),$$

where $\alpha_X^* : X \otimes_A \mathrm{Hom}_A(T, M) \to \mathrm{Hom}_A(T, X)$ and $\beta_X^* : X \otimes_A DT \to \mathrm{Hom}_A(T, X)$ are the adjoint maps of the structure maps α_X and β_X, respectively.

This defines a K-linear functor $\mathcal{K}er : \text{mod}-\Lambda(\varphi, \psi) \to \underline{\text{mod}}-\Lambda(\varphi^T, \psi^T)$ and it induces a stable functor $\mathcal{K}er : \underline{\text{mod}}-\Lambda(\varphi, \psi) \to \underline{\text{mod}}-\Lambda(\varphi^T, \psi^T)$. Similarly, by using the condition (D), the functor $\mathcal{C}oker : \underline{\text{mod}}-\Lambda(\varphi^T, \psi^T) \to \underline{\text{mod}}-\Lambda(\varphi, \psi)$ is defined. Finally, by the condition (B), it is ckecked that those functors define the stable equivalence $\underline{\text{mod}}-\Lambda(\varphi, \psi) \approx \underline{\text{mod}}-\Lambda(\varphi^T, \psi^T)$.

3. Equivalences of triangulated categories

A distinguished triangle

$$X_1 \xrightarrow{f} X_2 \longrightarrow C_f \longrightarrow \Omega^{-1}_{\Lambda(\varphi,\psi)}(X_1)$$

in the stable module category $\underline{\text{mod}}-\Lambda(\varphi, \psi)$ is given by the push-out diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_1 & \longrightarrow & E(X_1) & \longrightarrow & \Omega^{-1}_{\Lambda(\varphi,\psi)}(X_1) & \longrightarrow & 0 \\ & & {\scriptstyle f}\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & X_2 & \longrightarrow & C_f & \longrightarrow & \Omega^{-1}_{\Lambda(\varphi,\psi)}(X_1) & \longrightarrow & 0 \end{array}$$

in the module category $\text{mod}-\Lambda(\varphi, \psi)$, where $X_1 \hookrightarrow E(X_1)$ is an injection into an injective module $E(X_1)$ and $X \xrightarrow{f} X_2$ an arbitrary homomorphism of $\Lambda(\varphi, \psi)$-modules. We have to prove that the sequence

$$\mathcal{K}er(X_1) \xrightarrow{\mathcal{K}er(f)} \mathcal{K}er(X_2) \longrightarrow \mathcal{K}er(C_f) \longrightarrow \mathcal{K}er(\Omega^{-1}_{\Lambda(\varphi,\psi)}(X_1))$$

is again a distinguished triangle in the category $\underline{\text{mod}}-\Lambda(\varphi^T, \psi^T)$.

We start with the following result:

Lemma 1. *Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be an exact sequence of $\Lambda(\varphi, \psi)$-modules. Then there exist right ${}^{\perp}\mathcal{C}(T_A)$-approximations $W_X \xrightarrow{\gamma_X} X \to 0$, $W_Y \xrightarrow{\gamma_Y} Y \to 0$ and $W_Z \xrightarrow{\gamma_Z} Z \to 0$*

such that all the rows and columns are exact in the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V_X & \xrightarrow{V_f} & V_Y & \xrightarrow{V_g} & V_Z & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & W_X & \xrightarrow{W_f} & W_Y & \xrightarrow{W_g} & W_Z & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \gamma_X} & & \downarrow{\scriptstyle \gamma_Y} & & \downarrow{\scriptstyle \gamma_Z} & & \\
0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Proof. We choose first any right ${}^{\perp}\mathcal{C}(T_A)$-approximations $W_X \xrightarrow{\gamma_X} X$ and $W'_Y \xrightarrow{\gamma'_Y} Y$ and get the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V_X & \xrightarrow{s} & W_X & \xrightarrow{\gamma_X} & X & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle V'_f} & & \downarrow{\scriptstyle W'_f} & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & V'_Y & \xrightarrow[t]{} & W'_Y & \xrightarrow[\gamma'_Y]{} & Y & \longrightarrow & 0
\end{array}$$

In the diagram, W'_f may not be injective, but since $W_X \in {}^{\perp}\mathcal{C}(T_A) \subsetneq \mathrm{cog}^{*}(T_A)$, we can take a left $\mathrm{add}(T_A)$-approximation $0 \to W_X \xrightarrow{u} T_0$ and, by setting

$$V_f = \begin{pmatrix} V'_f \\ u \cdot s \end{pmatrix}, \quad W_f = \begin{pmatrix} W'_f \\ u \end{pmatrix} \quad \text{and} \quad \gamma_Y = (\gamma'_Y, 0), \quad t' = \begin{pmatrix} t & 0 \\ 0 & \mathrm{id}_{T_0} \end{pmatrix},$$

we have the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V_X & \xrightarrow{s} & W_X & \xrightarrow{\gamma_X} & X & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle V_f} & & \downarrow{\scriptstyle W_f} & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & V'_Y \oplus T_0 & \xrightarrow{t'} & W'_Y \oplus T_0 & \xrightarrow{\gamma_Y} & Y & \longrightarrow & 0
\end{array}$$

Here we put $W_Y = W'_Y \oplus T_0$, $V_Y = V'_Y \oplus T_0$, $W_Z = \mathrm{Coker}(W_f)$, $V_Z = \mathrm{Coker}(V_f)$ and denote the cokernels of the maps W_f and V_f by $W_Y \xrightarrow{W_g} W_Z \to 0$ and $V_Y \xrightarrow{V_g} V_Z \to 0$, respectively. Then, by the snake lemma, we get an exact sequence

$$0 \longrightarrow V_Z \longrightarrow W_Z \xrightarrow{\gamma_Z} Z \longrightarrow 0$$

in which $V_Z \in \mathcal{C}(T_A)$ and $W_Z \in {}^{\perp}\mathcal{C}(T_A)$ hold as easily seen. It is now obvious that those modules and homomorphisms make the diagram as stated in the lemma. **q.e.d**

For a short exact sequence of $\Lambda(\varphi,\psi)$-modules

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

we choose three ${}^{\perp}\mathcal{C}(T_A)$-approximations $\gamma_X : W_X \to X$, $\gamma_Y : W_Y \to Y$, $\gamma_Z : W_Z \to Z$ and two homomorphisms $W_f : W_X \to W_Y$, $W_g : W_Y \to W_Z$ as stated in the lemma. By making use of those modules and homomorphisms, the sequence

$$\mathcal{K}er(X) \xrightarrow{\mathcal{K}er(f)} \mathcal{K}er(Y) \xrightarrow{\mathcal{K}er(g)} \mathcal{K}er(Z)$$

is defined in the module category mod$-\Lambda(\varphi^T,\psi^T)$ by the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K}er(X) & \longrightarrow & \mathrm{Hom}_B(\Lambda(\varphi^T,\psi^T), W_X \otimes_A DT) & \xrightarrow{\lambda_X} & \mathrm{Hom}_{\Lambda(\varphi,\psi)}(\Theta, X) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathcal{K}er(f)} & & \downarrow{\scriptstyle \mathrm{Hom}(\Lambda(\varphi^T,\psi^T), W_f\otimes DT)} & & \downarrow{\scriptstyle \mathrm{Hom}(\Theta,f)} & & \\
0 & \longrightarrow & \mathcal{K}er(Y) & \longrightarrow & \mathrm{Hom}_B(\Lambda(\varphi^T,\psi^T), W_Y \otimes_A DT) & \xrightarrow{\lambda_Y} & \mathrm{Hom}_{\Lambda(\varphi,\psi)}(\Theta, Y) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathcal{K}er(g)} & & \downarrow{\scriptstyle \mathrm{Hom}(\Lambda(\varphi^T,\psi^T), W_g\otimes DT)} & & \downarrow{\scriptstyle \mathrm{Hom}(\Theta,g)} & & \\
0 & \longrightarrow & \mathcal{K}er(Z) & \longrightarrow & \mathrm{Hom}_B(\Lambda(\varphi^T,\psi^T), W_Z \otimes_A DT) & \xrightarrow{\lambda_Z} & \mathrm{Hom}_{\Lambda(\varphi,\psi)}(\Theta, Z) & \longrightarrow & 0
\end{array}$$

and we get the following lemma.

Lemma 2. *When we choose right* ${}^{\perp}\mathcal{C}(T_A)$*-approximations* $W_X \xrightarrow{\gamma_X} X$, $W_Y \xrightarrow{\gamma_Y} Y$ *and* $W_Z \xrightarrow{\gamma_Z} Z$ *as in the previous lemma, the sequence*

$$0 \longrightarrow \mathcal{K}er(X) \xrightarrow{\mathcal{K}er(f)} \mathcal{K}er(Y) \xrightarrow{\mathcal{K}er(g)} \mathcal{K}er(Z) \longrightarrow 0$$

is exact in the module category mod$-\Lambda(\varphi^T,\psi^T)$.

Proof. Applying the functor $\mathrm{Hom}_{\Lambda(\varphi,\psi)}(\Theta, ?)$ to the exact sequence

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

we have the following commutative diagram with exact rows.

$$\begin{array}{cccccccc}
0 & \longrightarrow & \mathrm{Hom}_{\Lambda(\varphi,\psi)}(\Theta, X) & \xrightarrow{\mathrm{Hom}(\Theta,f)} & \mathrm{Hom}_{\Lambda(\varphi,\psi)}(\Theta, Y) & \xrightarrow{\mathrm{Hom}(\Theta,g)} & \mathrm{Hom}_{\Lambda(\varphi,\psi)}(\Theta, Z) & \\
 & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow & \\
0 & \longrightarrow & \mathrm{Hom}_A(T, X) & \xrightarrow[\mathrm{Hom}(T,f)]{} & \mathrm{Hom}_A(T, Y) & \xrightarrow[\mathrm{Hom}(T,g)]{} & \mathrm{Hom}_A(T, Z) & \longrightarrow \mathrm{Ext}^1_A(T, X)\cdots
\end{array}$$

Similarly, applying the functor

$$\mathrm{Hom}_B(\Lambda(\varphi^T,\psi^T), ?) \cong \begin{array}{c} \mathrm{Hom}_B(B, ?) \\ \oplus \\ \mathrm{Hom}_B(T \otimes_A \mathrm{Hom}_A(T, M), ?) \\ \oplus \\ \mathrm{Hom}_B(DB, ?) \end{array}$$

to the exact sequence

$$0 \longrightarrow W_X \otimes_A DT \xrightarrow{W_f \otimes DT} W_Y \otimes_A DT \xrightarrow{W_g \otimes DT} W_Z \otimes_A DT \longrightarrow 0$$

we have two exact sequences

$$0 \longrightarrow \mathrm{Hom}_B(B, W_X \otimes_A DT) \longrightarrow \mathrm{Hom}_B(B, W_Y \otimes_A DT) \longrightarrow \mathrm{Hom}_B(B, W_Z \otimes_A DT) \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{Hom}_B(N, W_X \otimes_A DT) \longrightarrow \mathrm{Hom}_B(N, W_Y \otimes_A DT) \longrightarrow \mathrm{Hom}_B(N, W_Z \otimes_A DT) \longrightarrow 0,$$

where $N = T \otimes_A \mathrm{Hom}_A(T, M)$, and the commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Hom}_B(DB, W_X \otimes_A DT) & \longrightarrow & \mathrm{Hom}_B(DB, W_Y \otimes_A rmDT) & \longrightarrow & \mathrm{Hom}_B(DB, W_Z \otimes_A DT) & \\
 & \cong\downarrow & & \cong\downarrow & & \cong\downarrow & \\
0 \longrightarrow & \mathrm{Hom}_A(T, W_X) & \xrightarrow[\mathrm{Hom}(T, W_f)]{} & \mathrm{Hom}_A(T, W_Y) & \xrightarrow[\mathrm{Hom}(T, W_g)]{} & \mathrm{Hom}_A(T, W_Z) & \rightarrow \cdots
\end{array}$$

Then, combining those diagrams, we get the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & 0 & & 0 & & 0 & & & \\
 & \downarrow & & \downarrow & & \downarrow & & & \\
0 \longrightarrow & \mathcal{K}er(X) & \xrightarrow{\mathcal{K}er(f)} & \mathcal{K}er(Y) & \xrightarrow{\mathcal{K}er(g)} & \mathcal{K}er(Z) & & & \\
 & \downarrow & & \downarrow & & \downarrow & & & \\
0 \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \mathrm{Hom}_B(\Lambda(\varphi^T, \psi^T), W_Z \otimes_A DT) & \longrightarrow & \mathrm{Ext}^1_A(T, W_X) & \rightarrow \cdots \\
 & \downarrow\lambda_X & & \downarrow\lambda_Y & & \downarrow\lambda_Z & & & \\
0 \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \mathrm{Hom}_{\Lambda(\varphi,\psi)}(\Theta, Z) & \longrightarrow & \mathrm{Ext}^1_A(T, X) & \rightarrow \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & & \\
 & 0 & & 0 & & 0 & & &
\end{array}$$

On the other hand, from the exact sequence $0 \to V_X \to W_X \xrightarrow{\gamma_X} X \to 0$ with $V_X \in \mathcal{C}(T_A)$, we have an isomorphism $\mathrm{Ext}^1(T, \gamma_X) : \mathrm{Ext}^1_A(T, W_X) \xrightarrow{\approx} \mathrm{Ext}^1_A(T, X)$. Therefore, to prove the surjectivity of the map $\mathcal{K}er(g) : \mathcal{K}er(Y) \to \mathcal{K}er(Z)$, it is enough to show that the diagram

$$\begin{array}{ccc}
\mathrm{Hom}_B(\Lambda(\varphi^T, \psi^T), W_Z \otimes_A DT) & \longrightarrow & \mathrm{Ext}^1_A(T, W_Z) \\
\lambda_Z\downarrow & & \downarrow\mathrm{Ext}^1(T, \gamma_X) \\
\mathrm{Hom}_{\Lambda(\varphi,\psi)}(\Theta, Z) & \longrightarrow & \mathrm{Ext}^1_A(T, X)
\end{array}$$

is commutative. It is easyto see that the commutativity of the above diagram is equivalent to the following two assertions:

(1) The composition maps

$$Z \otimes_A \mathrm{Hom}_A(T, M) \xrightarrow{\alpha_Z^*} \mathrm{Ext}^1_A(T, Z) \xrightarrow{\Delta} \mathrm{Ext}^1_A(T, X)$$

and

$$Z \otimes_A DT \xrightarrow{\beta_Z^*} \mathrm{Hom}_A(T, Z) \xrightarrow{\Delta} \mathrm{Ext}^1_A(T, X)$$

are the zero maps, where $\mathrm{Hom}_A(T, Z) \xrightarrow{\Delta} \mathrm{Ext}^1_A(T, X)$ satands for the connecting homomorphism corresponding to the exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$.

(2) The diagram

$$\begin{array}{ccccc}
\mathrm{Hom}_B(T\otimes_A \mathrm{D}T, W_Z\otimes_A \mathrm{D}T) & \underset{\cong}{\xleftarrow{(?\otimes \mathrm{D}T)}} & \mathrm{Hom}_A(T, W_Z) & \xrightarrow{\Delta} & \mathrm{Ext}^1_A(T, W_X) \\
\zeta \big\uparrow \cong & & & & \cong \big\downarrow \mathrm{Ext}^1(T,\gamma_X) \\
\mathrm{Hom}_A(T, W_Z) & \xrightarrow[\mathrm{Hom}(T,\gamma_Z)]{} & \mathrm{Hom}_A(T, Z) & \xrightarrow[\Delta]{} & \mathrm{Ext}^1_A(T, X)
\end{array}$$

is commutative, where the vertical map ζ in the left hand side is the composition

$$\mathrm{Hom}_A(T, W_Z) \underset{\cong}{\xrightarrow{\mathrm{Hom}(T,\eta^{\mathrm{D}T}_{W_Z})}} \mathrm{Hom}_A(T, \mathrm{Hom}_B(\mathrm{D}T, W_Z\otimes_A \mathrm{D}T) \underset{\cong}{\xrightarrow{\mathrm{can}}} \mathrm{Hom}_B(T\otimes_A \mathrm{D}T, W_Z\otimes_A \mathrm{D}T)$$

and the map $\mathrm{Hom}_A(T, W_Z) \xrightarrow{\Delta} \mathrm{Ext}^1_A(T, W_X)$ stands for the connecting homomorphism corresponding to the exact sequence $0 \to W_X \xrightarrow{W_f} W_Y \xrightarrow{W_g} W_Z \to 0$.

Proof of the assertion (1): For any element $y \in Y$ and $u \in \mathrm{Hom}_A(T, M)$, the element $\Delta(\alpha^*_Z(g(y)\otimes u)) \in \mathrm{Ext}^1_A(T, X)$ is determined by the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \longrightarrow & E & \longrightarrow & T & \longrightarrow & 0 \\
 & & \big\| & & \big\downarrow & & \big\downarrow \alpha^*_Z(g(y)\otimes u) & & \\
0 & \longrightarrow & X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z & \longrightarrow & 0
\end{array}$$

and it is easily verified that the homomorphism $\alpha^*_Z(g(y)\otimes u)$ is lifted to the homomorphism $\alpha^*_Y(y\otimes u)$ through the surjective map g. Therefore, the upper sequence in the diagram splits and we have $\Delta\cdot\alpha^*_Z = 0$. We can prove $\Delta\cdot\beta^*_Z = 0$ in the same way.

Proof of the assertion (2): It is checked that the map ζ coincides with $(?\otimes \mathrm{D}T)$: $\mathrm{Hom}_A(T, W_Z) \to \mathrm{Hom}_B(T\otimes_A \mathrm{D}T, W_Z\otimes_A \mathrm{D}T)$. Hence, the commutativity of the diagram follows from the naturality of the connecting homomorphisms. **q.e.d**

Theorem 3. *The stable equivalence functor $\mathcal{K}er$: $\underline{\mathrm{mod}}\text{-}\Lambda(\varphi,\psi) \to \underline{\mathrm{mod}}\text{-}\Lambda(\varphi^T,\psi^T)$ is an equivalence of triangulated categories.*

Proof. Applying Lemma 2 to the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X_1 & \longrightarrow & E(X_1) & \longrightarrow & \Omega^{-1}_{\Lambda(\varphi,\psi)}(X_1) & \longrightarrow & 0 \\
 & & f\big\downarrow & & \big\downarrow & & \big\| & & \\
0 & \longrightarrow & X_2 & \longrightarrow & C_f & \longrightarrow & \Omega^{-1}_{\Lambda(\varphi,\psi)}(X_1) & \longrightarrow & 0
\end{array}$$

we have the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K}er(X_1) & \longrightarrow & \mathcal{K}er(E(X_1)) & \longrightarrow & \mathcal{K}er(\Omega^{-1}_{\Lambda(\varphi,\psi)}(X_1)) & \longrightarrow & 0 \\
 & & \mathcal{K}er(f)\big\downarrow & & \big\downarrow & & \big\| & & \\
0 & \longrightarrow & \mathcal{K}er(X_2) & \longrightarrow & \mathcal{K}er(C_f) & \longrightarrow & \mathcal{K}er(\Omega^{-1}_{\Lambda(\varphi,\psi)}(X_1)) & \longrightarrow & 0
\end{array}$$

We know that the module $\mathcal{K}er(Q)$ over the algebra $\Lambda(\varphi^T, \psi^T)$ is projective for any projective module Q over the algebra $\Lambda(\varphi, \psi)$ by the construction. Therefore, we see that the equality

$$\mathcal{K}er(\Omega^{-1}_{\Lambda(\varphi,\psi)}(X_1)) = \Omega^{-1}_{\Lambda(\varphi^T,\psi^T)}(\mathcal{K}er(X_1))$$

holds and the sequence

$$\mathcal{K}er(X_1) \xrightarrow{\mathcal{K}er(f)} \mathcal{K}er(X_2) \longrightarrow \mathcal{K}er(C_f) \longrightarrow \mathcal{K}er(\Omega^{-1}_{\Lambda(\varphi,\psi)}(X_1))$$

is again a distinguished triangle in the stable category $\underline{\mathrm{mod}}-\Lambda(\varphi^T, \psi^T)$. This completes the proof. **q.e.d.**

REFERENCES

[1] D. Happel, *Triangulated categories in the representation theory of finite-dimensional algebras*, London Math. Soc. Lect. Notes, vol.**119**, Cambridge University Press, Cambridge 1988

[2] T. Wakamatsu, *Stable equivalences induced from generalized tilting modules*, Proc. Symposium on Ring Theory and Representation Theory **36** (2004), 147-156

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ON A TENSOR PRODUCT OF SQUARE MATRICES IN JORDAN CANONICAL FORMS

RYO IWAMATSU

ABSTRACT. Let K be an algebraically closed field of characteristic $p \geqslant 0$. We shall consider the problem of finding out a Jordan canonical form of $J(a, s) \otimes_K J(b, t)$, where $J(a, s)$ means the Jordan block with eigenvalue $a \in K$ and size s.

1. INTRODUCTION

To construct graded local Frobenius algebras over an algebraically closed field K, it is important to find out a Jordan canonical form (simply, JCF) of tensor product of square matrices. In fact, it is known that any graded local Frobenius algebra is of the form of $\Lambda(\varphi, \gamma) = T(V)/R(\varphi, \gamma)$, where V is a finite dimensional K-vector space, γ an element of $GL(V)$, and $\varphi : V^{\otimes n} \to K$ a K-linear map satisfying several conditions. Further, if we decompose as $(V, \gamma) = \bigoplus_i (V_i, \gamma_i)$, then the conditions of φ can be described in terms of each $\varphi_{i_1 \ldots i_r} : V_{i_1} \otimes \cdots \otimes V_{i_r} \to K$. Then, we have to find out a Jordan canonical form of $\gamma_{i_1} \otimes \cdots \otimes \gamma_{i_r}$ as an element in $GL(V_{i_1} \otimes \cdots \otimes V_{i_r})$. (For detail, refer to T. Wakamatsu [2]).

Let K be an algebraically closed field of characteristic $p \geqslant 0$, and $J(a, s)$, $J(b, t)$ Jordan blocks over K. We shall consider the problem of finding out a JCF of $J(a, s) \otimes J(b, t)$, where $\otimes$ means $\otimes_K$. And then we may assume $s \leqslant t$.

In the case of $ab \neq 0$, our problem is reduced to the problem of finding the indecomposable decomposition of R as a $K[\theta]$-module, where R means the polynomial ring $K[x, y]$ with relation $(x^s = 0 = y^t)$ and $\theta = x + y$. In Theorem 3, we show that we can find out s homogeneous elements $\omega_0, \omega_1, \ldots, \omega_{s-1}$ such that

$$R \cong \bigoplus_{i=0}^{s-1} K[\theta]\omega_i$$

as $K[\theta]$-modules, where the degree of ω_i is i (for each $0 \leqslant i \leqslant s-1$). Applying this result, we show an algorithm for computing a JCF of $J(a, s) \otimes J(b, t)$ in Theorem 15. In the case of $ab = 0$, we give the complete solution of our problem in Theorem 9.

A. Martsinkovsky and A. Vlassov [1] gave the solution of this problem in the case of $p = 0$.

2. MAIN RESULT

2.1. The indecomposable decomposition that gives a JCF of $J(a, s) \otimes J(b, t)$. To find out a JCF of $J(a, s) \otimes J(b, t)$, we have to find its eigenvalues, the number of Jordan

The detailed version of this paper will be submitted for publication elsewhere.

blocks, and the sizes of Jordan blocks. It is clear the eigenvalue of $J(a, s)\otimes J(b, t)$ is only ab.

We consider the indecomposable decomposition of

$$\frac{K[X]}{((X-a)^s)}\otimes\frac{K[Y]}{((Y-b)^t)}$$

as a $K[X\otimes Y]$-module. By replacing variables and so on, we have the following:

(1) $ab\neq 0$:

$$\left(\frac{K[X]}{((X-a)^s)}\otimes\frac{K[Y]}{((Y-b)^t)}\right)_{K[X\otimes Y]}\cong\left(\frac{K[X, Y]}{(X^s, Y^t)}\right)_{K[X+Y]}.$$

(2) $a=0, b\neq 0$:

$$\left(\frac{K[X]}{(X^s)}\otimes\frac{K[Y]}{((Y-b)^t)}\right)_{K[X\otimes Y]}\cong\left(\frac{K[X, Y]}{(X^s, Y^t)}\right)_{K[X]}.$$

(3) $a\neq 0, b=0$:

$$\left(\frac{K[X]}{((X-a)^s)}\otimes\frac{K[Y]}{(Y^t)}\right)_{K[X\otimes Y]}\cong\left(\frac{K[X, Y]}{(X^s, Y^t)}\right)_{K[Y]}.$$

(4) $a=0=b$:

$$\left(\frac{K[X]}{(X^s)}\otimes\frac{K[Y]}{(Y^t)}\right)_{K[X\otimes Y]}\cong\left(\frac{K[X, Y]}{(X^s, Y^t)}\right)_{K[XY]}.$$

We put $x=\bar{X}$, $y=\bar{Y}\in K[X, Y]/(X^s, Y^t)$, and $R=K[x, y]$.

Lemma 1. *Our problem is reduced to the problem of finding the indecomposable decomposition of R as a $K[\theta]$-module, where θ means $x+y$ (if $ab\neq 0$), x ($a=0, b\neq 0$), y ($a\neq 0, b=0$), and xy ($a=0=b$).*

We discuss on the assumption $ab\neq 0$, i.e. $\theta=x+y$, unless otherwise stated.

It is clear R is a finite dimensional graded K-algebra. In fact, we denote by R_i the subset of R consisting of all homogeneous elements with degree i, then we have $R=\bigoplus_{i=0}^{s+t-2}R_i$. And we immediately know $\dim_K R_i$ are as follows $(1, 2, \ldots, s, s, \ldots, s, s-1, \ldots, 1)$ for $0\leqslant i\leqslant s+t-2$.

The subalgebra $K[\theta]$ of R is uniserial, and hence is a quasi-Frobenius. We denote by n the *nilpotency* of θ (i.e. $\theta^n\neq 0$, but $\theta^{n+1}=0$), and then we can choose $(1, \theta, \cdots, \theta^n)$ as a K-basis of $K[\theta]$. By easy calculation, we have the following inequality on n:

Lemma 2. *We have $t-1\leqslant n\leqslant s+t-2$. In particular, $n=s+t-2$ if $p=0$.*

Since the algebra $K[\theta]$ is uniserial, any indecomposable summand M of $R_{K[\theta]}$ can be of written as $K[\theta]\omega$ for some element ω in R. Hence we can write the indecomposable decomposition of $R_{K[\theta]}$ such as:

$$R=\bigoplus_{i=1}^{r}K[\theta]\omega_i\quad(\omega_i\in R). \tag{2.1}$$

We shall call each element ω_i a *generator* (for an indecomposable summand of $R_{K[\theta]}$), and the set $\{\omega_1, \dots, \omega_r\}$, which consists of the generators in (2.1), a *generating set* (for the indecomposable decomposition of $R_{K[\theta]}$). Although a generating set is not unique, we can choose some generating set that helps us to consider our problem:

Theorem 3. *There exists a generating set* $\{\omega_0, \omega_1, \dots, \omega_{s-1}\}$ *whose generator* ω_i *is an* i*-th degree homogeneous element. Hence,*

$$R = \bigoplus_{i=0}^{s-1} K[\theta]\omega_i \quad (\omega_i \in R_i).$$

We prepare some lemmas and notation for the proof of Theorem 3.

For a uniserial $K[\theta]$-submodule M of R generated by some homogeneous elements of R, we denote by $\sigma(M)$ the socle degree of M as a $K[\theta]$-module; i.e. $\sigma(M) = d$ if $\mathrm{soc}_{K[\theta]}(M) \subseteq R_d$. For example, $\sigma(K[\theta]) = n$, and $\sigma(K[\theta]x) = n+1$ if $\theta^n x \neq 0$. The following lemmas are easily checked:

Lemma 4. *Let* α, β *be homogeneous elements of* R. *If* $\sigma(K[\theta]\alpha) \neq \sigma(K[\theta]\beta)$, *then* $K[\theta]\alpha \cap K[\theta]\beta = \{0\}$ *holds. Hence* $K[\theta]\alpha + K[\theta]\beta = K[\theta]\alpha \oplus K[\theta]\beta$.

Lemma 5. *Let* κ *be a homogeneous element of* R. *If* $d := \sigma(K[\theta]\kappa) < s+t-2$, *then* $\kappa x^{s+t-2-d} \neq 0$ *holds. Hence,*

$$\sum_{i=0}^{s+t-2-d} K[\theta]\kappa x^i = \bigoplus_{i=0}^{s+t-2-d} K[\theta]\kappa x^i.$$

The multiplication map $\times\theta^j : R_i \to R_{i+j}$ is a K-linear map. We denote by $K(i, i+j)$ the kernel of this map.

Lemma 6. *For each* $0 \leqslant i \leqslant s-1$, *we have the following:*

(1) *The map* $\times\theta^{t-1-i} : R_i \to R_{t-1}$ *is injective.*

(2) *The map* $\times\theta^{s+t-1-2i} : R_i \to R_{s+t-1-i}$ *is not injective.*

Hence, for an elemant κ_i *in* $K(i, s+t-1-i) \subseteq R_i$, *we have*

$$\theta^{s+t-2-1-2i}\kappa_i = 0, \quad \textit{but} \quad \theta^{t-1-i}\kappa_i \neq 0.$$

We now prove Theorem 3:

The proof of Theorem 3. We put $n_0 = n$ and $m_0 = s+t-2-n_0$. If $m_0 > 0$, then we have

$$\sum_{i_0=0}^{m_0} K[\theta]x^{i_0} = \bigoplus_{i_0=0}^{m_0} K[\theta]x^{i_0} \subseteq R$$

by Lemma 5. If this direct sum coincides with R, then we finish the proof. Suppose not. By Lemma 6, we can take an element $\kappa_{(1)} \in K(m_0+1, n_0)$ and then we have $t-1 \leqslant \sigma(K[\theta]\kappa_{(1)}) \leqslant n_0 - 1$. We put $n_1 = \sigma(K[\theta]\kappa_{(1)})$ and $m_1 = (n_0-1) - n_1$. If $m_1 > 0$, then we have

$$(\bigoplus_{i_0=0}^{m_0} K[\theta]x^{i_0}) + (\sum_{i_1=0}^{m_1} K[\theta]\kappa_{(1)}x^{i_1}) = \bigoplus_{i_0=0}^{m_0} K[\theta]x^{i_0} \oplus \bigoplus_{i_1=0}^{m_1} K[\theta]\kappa_{(1)}x^{i_1} \subseteq R$$

from Lemma 5. Thus, we can construct the direct sum of $K[\theta]$-submodules of R. However, since R is finite dimensional, this construction will be over in finite steps. And it is clear that this construction finishes just when s-th direct summand is constructed. By Krull-Schmidt theorem, this decomposition is the indecomposable decomposition of $R_{K[\theta]}$. (And this argument does work when some m_i is zero.) □

Remark 7. (1) This proof gives concretely the indecomposable summands of $R_{K[\theta]}$ such as:

$$K[\theta],\ K[\theta]x,\ \ldots,\ K[\theta]x^{m_0},$$
$$K[\theta]\kappa_{(1)},\ K[\theta]\kappa_{(1)}x,\ \ldots,\ K[\theta]\kappa_{(1)}x^{m_1},$$
$$\cdots\cdots$$
$$K[\theta]\kappa_{(r-1)},\ K[\theta]\kappa_{(r-1)}x,\ \ldots,\ K[\theta]\kappa_{(r-1)}x^{m_{r-1}},$$

where $\kappa_{(i)}$ means some elnement in $K(m_{i-1}+1,\ n_{i-1})$ and $m_i = (n_{i-1}-1) - n_i$, $n_i = \sigma(K[\theta]\kappa_{(i)})$. Thus, these $\kappa_{(i)}$, m_i, n_i are determined by the following order:

$$n = n_0 \to m_0 \to \kappa_{(1)} \to n_1 \to m_1 \to \kappa_{(2)} \to \cdots \to n_{i-1} \to m_{i-1} \to \kappa_{(i)} \to \cdots.$$

(Then we define $n_{-1} = s + t - 1$, $m_{-1} = 0$, and $\kappa_{(0)} = 1_R$ for convenience).

(2) We have to discuss on whether the value of $n_i = \sigma(K[\theta]\kappa_{(i)})$ varies by the choice of an element $\kappa_{(i)} \in K(m_{i-1}+1,\ n_{i-1})$. However, we immediately find that the sequence $(n_0, n_1, \ldots, n_{r-1})$ is unique by the uniqueness of the indecomposable decomposition of $R_{K[\theta]}$. Therefore we can choose κ_i free.

(3) Theorem 3 declares the number of Jordan blocks of $J(a, s) \otimes J(b, t)$ is s if $ab \neq 0$.

Definition 8. Thus, the particular indecomposable summands

$$(K[\theta] =)\, K[\theta]\kappa_{(0)},\ K[\theta]\kappa_{(1)},\ \ldots,\ K[\theta]\kappa_{(r-1)}$$

of $R_{K[\theta]}$ characterize the indecomposable decomposition of $R_{K[\theta]}$. So, we shall call each $K[\theta]\kappa_{(i)}$ a *leading module* (of $R_{K[\theta]}$). And we call the number of the indecomposable summands of $R_{K[\theta]}$ whose lengths are equal to that of $K[\theta]\kappa_{(i)}$ the *leading degree* of $K[\theta]\kappa_{(i)}$.

By this result, if there are r leading modules $K[\theta]\kappa_{(0)}, K[\theta]\kappa_{(1)}, \ldots, K[\theta]\kappa_{(r-1)}$, then we have

$$J(a, s) \otimes J(b, t) \equiv \bigoplus_{i=0}^{r-1} J(ab, \ell_i)^{\oplus d_i},$$

where ℓ_i and d_i mean the length and leading degree of $K[\theta]\kappa_{(i)}$ respectively.

In the case of $ab = 0$, the algebra $K[\theta]$ is also uniserial. Hence we can apply a similar argument of the proof of Theorem 3.

Theorem 9. *If $ab = 0$. Then, for any characteristic p, we have the following:*

(1) $a = 0, b \neq 0$: *By taking $\{1, y, \ldots, y^{t-1}\}$ as a generating set;*

$$J(0, s) \otimes J(b, t) \equiv J(0, s)^{\oplus t}.$$

(2) $a \neq 0, b = 0$: *By taking $\{1, x, \ldots, x^{s-1}\}$;*

$$J(a, s) \otimes J(0, t) \equiv J(0, t)^{\oplus s}.$$

(3) $a = 0 = b$: *By taking* $\{1, x, \dots, x^{s-1}, y, y^2, \dots, y^{t-1}\}$;

$$J(0, s) \otimes J(0, t) \equiv J(0, s)^{\oplus t-s+1} \oplus \bigoplus_{i=1}^{s-1} J(0, s-i)^{\oplus 2}.$$

2.2. An algorithm for computing a JCF of $J(a, s) \otimes J(b, t)$. Next, we show there exists a good way to compute a JCF of $J(a, s) \otimes J(b, t)$. To compute it, we find the lengths and the leading degrees of the leading modules.

For each $0 \leqslant i \leqslant s-1$, we define a function such as

$$D_p(i) = \begin{cases} 0 & (\text{if the map } \times\theta^{s+t-2-2i} : R_i \to R_{s+t-2-i} \text{ is bijective}) \\ 1 & (\text{if the map } \times\theta^{s+t-2-2i} : R_i \to R_{s+t-2-i} \text{ is not bijective}) \end{cases}.$$

And we put

$$\Delta_p = (D_p(0), D_p(1), \dots, D_p(s-1)).$$

Remark 10. By Lemma 6 (1), we have known the map $\times\theta^{t-s} : R_{s-1} \to R_{t-1}$ is always injective (hence, bijective) independently of the value of characteristic p. So $D_p(s-1) = 0$ holds.

By Theorem 3, we may assume that R is of the form of $\bigoplus_{i=0}^{s-1} K[\theta]\omega_i$, i.e. any base of R is of the form of $\theta^j\omega_i$. This procedures the following lemmas:

Lemma 11. *If an indecomposable summand $K[\theta]\omega_i$ is a leading module and $D_p(i) = 0$. Then we have the following:*

(1) $\sigma(K[\theta]\omega_i) = s + t - 2 - i$. *Hence the length and the leading degree of $K[\theta]\omega_i$ are $s + t - 1 - 2i$ and one respectively.*

(2) *The next indecomposable summand $K[\theta]\omega_{i+1}$ is a leading module if $i + 1 < s$.*

Lemma 12. *If an indecomposable summand $K[\theta]\omega_i$ is a leading module, $D_p(i) = D_p(i+1) = \cdots = D_p(i+f-1) = 1$, and $D_p(i+f) = 0\,(f > 0)$. Then we have the following:*

(1) $\sigma(K[\theta]\omega_i) = s + t - 2 - i - f$. *Hence the length and the leading degree of $K[\theta]\omega_i$ are $s + t - 1 - 2i - f$ and $f + 1$ respectively.*

(2) *The indecomposable summand $K[\theta]\omega_{i+f+1}$ is a leading module if $i + f + 1 < s$.*

Since the indecomposable summand $K[\theta]\omega_0$ is a leading module, we can apply Lemma 11 and 12 to the components of an arbitrary Δ_p inductively. Thus, via the sequence Δ_p, we can compute the lengths and the leading degrees of the leading modules concretely:

Theorem 13. *We can compute a JCF of $J(a, s) \otimes J(b, t)$ by using the sequence Δ_p.*

We can compute the determinant $D(i)$ of the linear map $\times\theta^{s+t-2-2i} : R_i \to R_{s+t-2-i}$ by using elementary techniques of linear algebra:

Theorem 14. *For each $0 \leqslant i \leqslant s-1$, we have*

$$D(i) = \prod_{k=0}^{i} \frac{\binom{s+t-2-2i+k}{t-1-i}}{\binom{t-1-i+k}{t-1-i}}.$$

By Theorem 13 and 14, we get an algorithm for computing a JCF of $J(a, s) \otimes J(b, t)$:

Theorem 15. *We can compute a JCF of $J(a, s) \otimes J(b, t)$ by taking the following steps:*
Step 1: Computing $D(i)$ for each $0 \leqslant i \leqslant s-1$.
Step 2: Computing the sequence Δ_p. $D_p(i) = 0$ iff $D(i) \not\equiv 0 \pmod p$.
Step 3: Applying Theorem 13.

Example 16. Let us compute a JCF of $J(a, 4) \otimes J(b, 5)\,(ab \neq 0)$. The determinants $D(i)$ are

$$D(0) = \frac{\binom{7}{4}}{\binom{4}{4}} = 5 \cdot 7,\ \ D(1) = \frac{\binom{5}{3}\binom{6}{3}}{\binom{3}{3}\binom{4}{3}} = 2 \cdot 5^2,\ \ D(2) = \frac{\binom{3}{2}\binom{4}{2}\binom{5}{2}}{\binom{2}{2}\binom{3}{2}\binom{4}{2}} = 2 \cdot 5,\ \ D(3) = 1.$$

So the sequence Δ_p is

$$\begin{aligned} \Delta_p &= (0, 0, 0, 0) \ (p \neq 2, 5, 7), \\ \Delta_2 &= (0, 1, 1, 0), \\ \Delta_5 &= (1, 1, 1, 0), \\ \Delta_7 &= (1, 0, 0, 0). \end{aligned}$$

Therefore

$$J(a, 4) \otimes J(b, 5) \equiv \begin{cases} J(ab, 8) \oplus J(ab, 6) \oplus J(ab, 4) \oplus J(ab, 2) & (p \neq 2, 5, 7) \\ J(ab, 8) \oplus J(ab, 4)^{\oplus 3} & (p = 2) \\ J(ab, 5)^{\oplus 4} & (p = 5) \\ J(ab, 7)^{\oplus 2} \oplus J(ab, 4) \oplus J(ab, 2) & (p = 7) \end{cases}.$$

If $p = 0$ or $p > s + t - 2$, then the determinants $D(i)$ are clearly all non-zero. Hence:

Corollary 17. *If $p = 0$ or $p > s + t - 2$, then*

$$J(a, s) \otimes J(b, t) \equiv \bigoplus_{i=0}^{s-1} J(ab, s + t - 1 - 2i).$$

References

[1] A. Martsinkovsky and A. Vlassov, *The representation ring of $K[X]$*, (preprint) http://www.math.neu.edu/ martsinkovsky/GreenExcerpt.pdf. (2004).

[2] T. Wakamatsu, *On graded Frobenius algebras*, Journal of Algebra. **267** (2003), 377–395.

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FINITE GROUPS HAVING EXACTLY ONE NON-LINEAR IRREDUCIBLE CHARACTER

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Recently, A.S. Muktibodh [12, 11, 10] considered a 2-Con-Cos group G defined as follows: the commutator subgroup G' of a finite group G consist of two conjugate classes C_a and $C_1 = \{1\}$, and cosets $G'x$ are conjugate classes C_x of $x \in G \setminus G'$. In this paper, we replace "2-Con-Cos" by "concos".

These groups are just groups having exactly one non-linear irreducible character because the number of irreducible characters is equal to the number of conjugate classes, $G' \neq \{1\}$ contains at least two conjugate classes and a coset $G'x$ of G' contains at least one conjugate class C_x.

In §1, we shall prove these groups are isomorphic to affine groups over finite fields or central products of some dihedral groups D of order 8 and quaternion groups Q, and conversely.

After my talk, Professor Y. Ninomiya informed me that this characterization was known in some papers [14, 13, 2]. Further the paper [2] stated that more general information was considered in [8]. However I have arranged this characterization for some reasons that our proof is slight different from others, rather self contained and necessary for §2 and §3.

In §2, we determine $\mathbb{C}$-irreducible $\mathbb{R}$-representations of concos groups.

In §3, we shall show concos groups appear in the proof of Hurwitz theorem concerning quadratic forms. We also determine $\mathbb{C}$-irreducible $\mathbb{R}$-representations of slight different groups in the proof of this theorem.

All representations and characters are considered over $\mathbb{C}$.

1. Characterization of concos groups

First we show elementary properties of concos groups from the definition.

Lemma 1. Let G be concos. Then we have

(1) *If N is a normal subgroup of G then $N = \{1\}$ or $N \supset G'$.*
(2) *G' is an elementary abelian p-group.*
(3) *Exactly one non-linear irreducible character η of G has the next values and we can see from these values that η is faithful.*

$$\eta(1)^2 = |G/G'|(|G'| - 1), \quad \eta(x) = -\frac{|G/G'|}{\eta(1)} \ \text{ for } x \in G' \setminus \{1\},$$

and $\eta(x) = 0$ for $x \in G \setminus G'$.

The detailed version of this paper will be submitted for publication elsewhere.

Proof. (1) If N contains $b \neq 1$, then N contains C_b. In case $b \in G'$, $N \supset C_b = C_a$ and so $N \supset G'$. In case $b \notin G'$, $N \supset C_b = G'b$ and so $N = Nb^{-1} \supset G'$.

(2) G' is a p-group because $G' = \{1\} \cup C_a$ and G' contains an element of prime order p. Thus G'' is a normal subgroup of G properly contained in a p-group G' and so $G'' = 1$ from (1).

(3) Let ρ_G and $\rho_{G/G'}$ are regular characters of G and G/G', respectively. Then we have

$$\rho_G = \rho_{G/G'} + \eta(1)\eta.$$

Using this we obtain our assertion.

The next theorem follows from Lemma 1 (1) and (2).

Theorem 2 ([14, 13, 2, 12]). Let G be concos. Then we have the next groups and conversely.

1. G is the central product QD^{r-1} or D^r where D is the dihedral group of order 8 and Q is the quaternion group of order 8.
2. G is the next permutation group over a finite field $\mathbb{F}_q$ of order $q > 2$.

$$G = \{x \to \alpha x + \beta \mid \alpha \in \mathbb{F}_q^* \text{ and } \beta \in \mathbb{F}_q\}.$$

2. Real representations of concos groups

Let Ψ be a $\mathbb{C}$-irreducible representation of a finite group G and let χ be a character afforded by Ψ. We set

$$\nu(\chi) = \frac{1}{|G|}\sum_{x \in G} \chi(x^2).$$

If χ is linear, then $\nu(\chi) = (\chi, \bar{\chi})$, where $(\chi, \bar{\chi})$ is the inner product of χ and $\bar{\chi}$ is the complex conjugate of χ. Thus it is easy to see that $\nu(\chi) = 1, 0$ and also that (1) $\nu(\chi) = 1$ if and only if $\chi = \bar{\chi}$ and (2) $\nu(\chi) = 0$ if and only if $\chi \neq \bar{\chi}$.

Frobenius and Schur proved in [4] (see [3]) that $\nu(\chi) = 1, 0, -1$ and

(1) $\nu(\chi) = 1$ if and only if Ψ is equivalent to an $\mathbb{R}$-representation.

(2) $\nu(\chi) = 0$ if and only if $\chi \neq \bar{\chi}$.

(3) $\nu(\chi) = -1$ if and only if $\chi = \bar{\chi}$ but Ψ is not equivalent to an $\mathbb{R}$-representation.

Let d be a fixed element of a finite group G and let s_d be the number of elements $x \in G$ such that $x^2 = d$. There is the formula [3, p. 22 (3.6)] about s_d as follows:

$$s_d = \sum_{\lambda \in \Lambda} \nu(\lambda)\lambda(d)$$

where Λ is the set of irreducible characters of G.

The next lemma is useful on $\mathbb{C}$-irreducible $\mathbb{R}$-representations of concos groups.

Lemma 3. Let G be a concos group, $G' = \{1\} \cup C_a$ and let η be exactly one non-linear irreducible character. Then we have

$$s_1 - s_a = \frac{|G|}{\eta(1)}\nu(\eta).$$

In the next proposition we can see $\mathbb{C}$-irreducible $\mathbb{R}$-representations of concos 2-groups. We also can see the numbers of elements of orders 4, 2 in these groups. Our counting method is different from [5, pp. 205-207]. Therefore this gives a different proof about that D^r and QD^{r-1} are not isomorphic (see Remark. (2)).

Proposition 4. Let G be an extra special 2-group D^r or QD^{r-1} of order 2^n, where $n = 2r+1$. Then elements in G are of order 1 or 2 or 4. Let R be the $\mathbb{C}$-irreducible representation of degree 2^r and η is a character afforded by R. Let s be the number of elements of order 2 or 1 and let t be the number of elements of order 4. Then we have

(1) In case $G = D^r$, R is equivalent to an $\mathbb{R}$-representation, $s = 2^{n-1} + 2^r$ and $t = 2^{n-1} - 2^r$.

(2) In case $G = QD^{r-1}$, R is not equivalent to an $\mathbb{R}$-representation but $\eta = \bar{\eta}$, $s = 2^{n-1} - 2^r$ and $t = 2^{n-1} + 2^r$.

Remark.

(1) The groups D and Q have the same character table. Hence group algebras $\mathbb{C}D$ and $\mathbb{C}Q$ over $\mathbb{C}$ are isomorphic. But two group algebras over $\mathbb{R}$ are not isomorphic. In fact,

$$\mathbb{R}D \cong \mathbb{R}^{(4)} \oplus (\mathbb{R})_2 \text{ and } \mathbb{R}Q \cong \mathbb{R}^{(4)} \oplus \mathbb{H}$$

where $\mathbb{H}$ is the quaternion algebra over $\mathbb{R}$.

(2) D^r is not isomorphic to QD^{r-1} because $\mathbb{C}$-irreducible $\mathbb{R}$-representations of degree 2^r are different (see [5, pp. 205-206]).

Here we state about $\mathbb{R}$-representations of affine groups over finite fields.

Proposition 5. Let G be a permutation group on finite field $\mathbb{F}_q$, where q is a power of a prime p, defined by

$$G = \{x \to ax + b \mid a \in \mathbb{F}_q^*,\ b \in \mathbb{F}_q\}.$$

Let s be the number of elements x with $x^2 = 1$ and let t be the number of elements $x \in G$ such that $x^2 = u_1$ where $u_1 : x \to x+1$. Then in case $p \neq 2$, $s = |G'| + 1$ and $t = 1$ and in case $p = 2$, $s = |G'|$ and $t = 0$. The $\mathbb{C}$-irreducible representation of degree $|G'| - 1$ is equivalent to an $\mathbb{R}$-representation.

3. Theorem of Hurwitz

The converse of the next theorem is well known. In case $n = 1$, it is trivial. In case $n = 2, 4$, we have

1. $(x_1^2 + x_2^2)(y_1^2 + y_2^2) = (x_1y_1 + x_2y_2)^2 + (x_1y_2 - x_2y_1)^2$.
2. $(x_1^2 + x_2^2 + x_3^2 + x_4^2)(y_1^2 + y_2^2 + y_3^2 + y_4^2) =$
   $(x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4)^2 + (x_1y_2 + x_2y_1 + x_3y_4 - x_4y_3)^2 +$
   $(x_1y_3 - x_2y_4 + x_3y_1 + x_4y_2)^2 + (x_1y_4 + x_2y_3 - x_3y_2 + x_4y_1)^2$.

In case $n = 8$, it is also known. The next theorem is very interested to suggest that algebras over the real number field can be constructed.

Theorem 6 (Hurwitz [7,1,6]). In polynomial ring $\mathbb{C}[x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n]$, if the next equation is satisfied for $z_s = \sum_{kt} c_{st}^{(k)} x_k y_t$, then $n = 1, 2, 4, 8$

$$(x_1^2 + x_2^2 + \cdots + x_n^2)(y_1^2 + y_2^2 + \cdots + y_n^2) = z_1^2 + z_2^2 + \cdots + z_n^2.$$

A key point in the above theorem is to prove $n = 1, 2, 4, 8$ if the next group H_n has a faithful representation of degree n, namely, there is such a group H_n in $GL(n, \mathbb{C})$.

$$H_n = \{(-I)^{s_0} B_1^{s_1} B_2^{s_2} \cdots B_{n-1}^{s_{n-1}} \mid s_k = 0, 1\}$$

where $B_k^2 = -I$, $B_k B_\ell = -B_\ell B_k$ for $k \neq \ell$.

However we can show that 2-groups H_n are realized in $GL(m, \mathbb{C})$ for some m. Therefore we shall state about $\mathbb{C}$-irreducible $\mathbb{R}$-representations of 2-groups H_n.

Lemma 7. The group H_n has two irreducible characters η_1 and η_2 for an even integer n. Let s be the number of elements $x \in H_n$ with $x^2 = 1$ and let t be the number of elements $x \in H_n$ of order 4. Then we obtain.

(1) $s + t = 2^n$.

(2) $\nu(\eta_1) = \nu(\eta_2) = 2^{-\frac{n+2}{2}}(s - t)$.

(3)

$$\nu(\eta_1) = \begin{cases} 1 & \text{for } s > 2^{n-1}, \\ -1 & \text{for } s < 2^{n-1}, \\ 0 & \text{for } s = 2^{n-1}. \end{cases}$$

(4) $s = 2^{n-1} + 2^{\frac{n}{2}}\nu(\eta_1)$ and $t = 2^{n-1} - 2^{\frac{n}{2}}\nu(\eta_1)$.

(5) $\dfrac{s}{2} = \displaystyle\sum_{k \equiv 0,3 \bmod 4}^{n-1} \binom{n-1}{k}$ and $\dfrac{t}{2} = \displaystyle\sum_{k \equiv 1,2 \bmod 4}^{n-1} \binom{n-1}{k}$.

Proof. (1) is clear since every element of H_n is of order 1, 2, 4.

(2) follows from $\eta_1(1) = \eta_2(1) = 2^{\frac{n-2}{2}}$, $\eta_1(-1) = \eta_2(-1) = -2^{\frac{n-2}{2}}$ and the next equations

$$\nu(\eta_1) = \frac{1}{|H_n|}(s\eta_1(1) + t\eta_1(-1)) = \frac{\eta_1(1)}{|H_n|}(s - t) = \nu(\eta_2).$$

(3) and (4) follow easily from (1) and (2).

(5) follows from the equations

$$(B_{t_1} B_{t_2} \cdots B_{t_k})^2 = \begin{cases} I & \text{for } k \equiv 0, 3 \bmod 4, \\ -I & \text{for } k \equiv 1, 2 \bmod 4. \end{cases}$$

We proved our assertion

Using (5) in the above lemma, we can find value of $\nu(\eta_1)$. For this purpose, we consider the next equation

$$(1 + i)^m = \{\sqrt{2}(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4})\}^m = 2^{\frac{m}{2}}(\cos\frac{m\pi}{4} + i \sin\frac{m\pi}{4})$$

where $i = \sqrt{-1}$. Comparing imaginary parts between left and right sides in the above equation, we have the next formula

$$\sum_{r=0}^{[\frac{m-1}{2}]} (-1)^r \binom{m}{2r+1} = 2^{\frac{m}{2}} \sin\frac{m\pi}{4}$$

where $[\ ]$ is the Gauss symbol. In particular, we have for $m = 4k+2$.

$$\sum_{r=0}^{[\frac{m-1}{2}]} (-1)^r \binom{m}{2r+1} = (-1)^{\frac{m-2}{4}} 2^{\frac{m}{2}} \text{ (see [9, p.11]).} \qquad (¶)$$

The next proposition state about $\mathbb{C}$-irreducible $\mathbb{R}$-representations of groups H_n.

Proposition 8. Let η be a non-linear irreducible character of H_n. Then in case n is odd, H_n is concos and so we already know about $\mathbb{C}$-irreducible $\mathbb{R}$-representations of groups H_n. In case n is even, we have

$$\nu(\eta) = \begin{cases} 0 & \text{for } n \equiv 2 \bmod 4, \\ -1 & \text{for } n \equiv 4 \bmod 8, \\ 1 & \text{for } n \equiv 0 \bmod 8. \end{cases}$$

Proof. In case $n \equiv 2 \bmod 4$, noting that $k \equiv 0,3 \bmod 4$ is equivalent to $n-1-k \equiv 1,2 \bmod 4$ for $0 \le k \le n-1$, we obtain easily

$$\begin{aligned} \frac{s}{2} &= \sum_{k\equiv 0,3 \bmod 4}^{n-1} \binom{n-1}{k} = \sum_{k\equiv 0,3 \bmod 4}^{n-1} \binom{n-1}{n-1-k} \\ &= \sum_{\ell\equiv 1,2 \bmod 4}^{n-1} \binom{n-1}{\ell} = \frac{t}{2}. \end{aligned}$$

In the another cases, using the above formula (¶), we have the our assertions.

$$\begin{aligned} \frac{s}{2} &= \sum_{k\equiv 0,3 \bmod 4}^{n-1} \binom{n-1}{k} = \sum_{k\equiv 0,3 \bmod 4}^{n-2} \left\{ \binom{n-2}{k-1} + \binom{n-2}{k} \right\} \\ &= 2^{n-2} - \sum_{\ell=0}^{[\frac{n-3}{2}]} (-1)^{\ell} \binom{n-2}{2\ell+1} = \begin{cases} 2^{n-2} + 2^{\frac{n-2}{2}} & \text{for } n \equiv 0 \bmod 8, \\ 2^{n-2} - 2^{\frac{n-2}{2}} & \text{for } n \equiv 4 \bmod 8. \end{cases} \end{aligned}$$

where $0 \le k \le n-1$ and $\binom{n-2}{-1} = \binom{n-2}{n-1} = 0$.

References

[1] B. Eckmann, Gruppen Theoretisher Beweis des Satzes von Hurwitz -Radon über die Komposition quadratisher Formen, Comment. Helv., 15 (1942), 358-366.

[2] Y. Egawa and N. Iwahori, Finite group which have only one non-linear irreducible character, J. Fac. Sci. Univ. Tokyo Sect. IA Math., 24 (1977), 247-253.

[3] W. Feit, Characters of finite groups, Benjamin, 1967.

[4] G. Frobenius and I. Schur, Über die reellen Darstellungen der endlichen Gruppen, Sitzungsber, Preuss, Akad. d. Wiss, 1906, 186-208.

[5] D. Gorenstein, Finite groups, Chelsea, 1980.

[6] I. N. Herstein, Noncommutative rings, The Curaus Mathematical Monographs, M.A.A., 1965.

[7] A. Hurwitz, Über die Komposition quadratischer Formen von beliebig vielen Variablen, Nachr. Ges. d. Wiss. Göttingen (1898), 309-316 (Math. Werke II, 565-571).

[8] L. M. Issacs, Character theory of finite groups, Academic Press, 1976.

[9] S. Moriguchi, K. Udagawa, S. Hitotumatu, Mathematical formulas II (in Japanese), Iwanami shoten, 1957

[10] A.S. Muktibodh, Con-Cos groups, Review Bulletin of Calcutta Mathematical Society, **11**(2003), 53-56.
[11] A.S. Muktibodh, Generalized Con-Cos groups, Advances in Algebra, Proceedings of the ICM Satelite Conference in Algebra and related topics, 2003, 434-441.
[12] A.S. Muktibodh, Characterization of Frobenius Groups of Special type, to appear.
[13] J. Pelikán, Finite groups with few non-linear irreducible characters, Acta Math. Acad. Sci. Hungar., **25** (1974), 223-226.
[14] G. Seitz, Finite groups having only one irreducible representation of degree greater than one, Proc. Amer. Math. Soc., **19** (1968), 459-461.

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INTEGRALITY OF EIGENVALUES OF CARTAN MATRICES IN FINITE GROUPS

TOMOYUKI WADA

ABSTRACT. Let C_B be the Cartan matrix of a p-block B of a finite group G. We show that there is a unimodular eigenvector matrix U_B of C_B over a discrete valuation ring R, if all eigenvalues of C_B are integers when B is a cyclic block, a tame block, a p-block of a p-solvable group, the principal 2-block with abelian defect group or the principal 3-block with elementary abelian defect group of order 9.

Keywords: Cartan matrix; Eigenvalue; Eigenvector matrix; Block; Finite group

1. Introduction

Let G be a finite group, let F be an algebraically closed field of characteristic $p > 0$, and let B be a block of the group algebra FG with defect group D. Let C_B be the Cartan matrix of B and $\rho(B)$ the Frobenius-Perron eigenvalue (i.e. the largest eigenvalue) of C_B. Let (K, R, F) be a p-modular system, where R is a complete discrete valuation ring of rank one with $R/(\pi) \simeq F$ for a unique maximal ideal (π) and K is the quotient field of R with characteristic 0. Let us denote the number $l(B)$ of irreducible Brauer characters in B simply by l.

We studied on integrality of eigenvalues of the Cartan matrix of a finite group in [3], [9],[10]. Recently C.C.Xi and D.Xiang showed that integrality of all eigenvalues of the Cartan matrix of a cellular algebra is closely related to its semisimplicity in Theorem 1.1 of [11]. Let R_B and E_B be the set of all eigenvalues (i.e. the spectrum) and of $\mathbb{Z}$-elementary divisors of C_B, respectively.

First we show some known properties of the Cartan matrix C_B of a finite group (e.g. see [6]).

(C1) $C_B = (D_B)^T D_B$, where D_B is the *decomposition matrix* of B.
(C2) C_B is nonnegative integral, indecomposable and symmetric.
(C3) C_B is positive definite (this comes from (1)).
(C4) $\det C_B = p^r \geq |D|$.

Secondly we show some known properties of $E_B = \{e_1, \ldots, e_l\}$ (e.g. see [6]).

(E1) Every e_i is a power of p, there is a unique largest $e_1 = |D| \in E_B$ and others $e_i < |D|$ for all $i > 1$.

The detailed version of this paper will be submitted for publication elsewhere.

(E2) Every $e_i = |C_G(x_i)|_p$ for some p-regular element $x_i \in G$.

(E3) $\prod_{i=1}^{l} e_i = \det C_B$.

(E4) If two blocks B and B' are derived equivalent, then there is a perfect isometry between the set of $\mathbb{Z}$-linear combination of ordinary irreducible characters of B and that of B'. Therefore $C_{B'} = V^T C_B V$ for some $V \in \mathrm{GL}(l, \mathbb{Z})$ and so we have $E_B = E_{B'}$ (see [2, 4.2 Proposition]).

Comparing with elementary divisors, properties of eigenvalues are not well known and they seem complicated and sensitive. We show some known properties of $R_B = \{\rho_1, \dots, \rho_l\}$.

(R1) ρ_i's need not be integers. But there is a unique largest eigenvalue $\rho_1 = \rho(B) \in R_B$ such that $\rho_i < \rho(B)$ for all $i > 1$. It can occur both cases $\rho(B) < |D|$ and $\rho(B) > |D|$ (see Examples 1 and 2 below).

(R1) For any $\rho \in R_B$ there is an algebraic integer λ such that $\rho\lambda = |D|$. In particular, if $\rho \in R_B$ is a rational integer, then $\rho = p^s \mid |D|$.

(R3) $\prod_{i=1}^{l} \rho_i = \det C_B$.

(R4) For two blocks B and B', R_B and $R_{B'}$ are not necessarily equal even if B and B' are derived equivalent (see Examples 1 and 2 below. It is known that the principal 2-blocks of S_4 and S_5 are derived equivalent). But of course, if B and B' are Morita equivalent, then $C_B = C_{B'}$ and so $R_B = R_{B'}$.

We show some examples of the Cartan matrices C for symmetric groups of small degree.

Example 1 S_4, $p = 2$, $C = \begin{pmatrix} 4 & 2 \\ 2 & 3 \end{pmatrix}$. There is only one block B_1 and $l(B_1) = 2$. Then $R_{B_1} = \{\rho_1 = \frac{7+\sqrt{17}}{2} < |D| = 8,\ \rho_2\}$, $E_{B_1} = \{8, 1\}$.

Example 2 S_5, $p = 2$, $C = \begin{pmatrix} 8 & 4 & \\ 4 & 3 & \\ & & 2 \end{pmatrix}$. There are two blocks B_1, B_2, and $l(B_1) = 2, l(B_2) = 1$. Then $R_{B_1} = \{\rho_1 = \frac{11+\sqrt{89}}{2} > |D| = 8, \rho_2\}$, $R_{B_2} = \{2\}$, and $E_{B_1} = \{8, 1\}$, $E_{B_2} = \{2\}$

Example 3 S_4, $p = 3$, $C = \begin{pmatrix} 2 & 1 & & \\ 1 & 2 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$. There are three blocks B_1, B_2, B_3 and $l(B_1) = 2$, $l(B_2) = l(B_3) = 1$. Then $R_{B_1} = \{3, 1\}, R_{B_2} = \{1\}, R_{B_3} = \{1\}$ and $E_{B_1} =$

$\{3,1\}, E_{B_2} = \{1\}, E_{B_3} = \{1\}$. In this case, all eigenvalues are rational integers (see Conjecture below).

2. Questions and Conjecture

It is fundamental to ask the following about integrality of eigenvalues of the Cartan matrix of a finite group G.

- When are eigenvalues of C_B of G rational integers?
- What relations are there between eigenvalues and elementary divisors?
- What do eigenvalues and eigenvectors represent?

We had the following very strong conjecture studying many examples and some typical blocks.

Conjecture. Let C_B be the Cartan matrix of a block B of FG with defect group D for a finite group G. Let $\rho(B)$ be the Frobenius-Perron eigenvalue. Then the following are equivalent.

(a) $\rho(B) \in \mathbb{Z}$.
(b) $\rho(B) = |D|$.
(c) $R_B = E_B$.
(d) All eigenvalues are rational integers.

Considering the condition (d) ((d) itself does not have so deep meanings), we had the notion U_B an *eigenvector matrix* of C_B whose rows consist of linearly independent l eigenvectors of C_B over the field of real numbers $\mathbb{R}$. We have the following question for U_B.

Question. When all eigenvalues are rational integers, can we take a unimodular eigenvector matrix U_B over a complete discrete valuation ring R?

If Question is answered affirmatively, then for example, we find that the matrix $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ cannot be the Cartan matrix of a finite group, because this matrix never has a unimodular eigenvector matrix when $p = 2$. In fact, this is the Cartan matrix of a Brauer tree algebra whose tree consists of three vertices such that both end points are exceptional with multiplicity 2. So Question is not true for the Cartan matrix of a general algebra.

3. Some results

We show some evidences for Conjecture and Question. The following proposition is the most fundamental result and a starting point for this research.

Proposition 1 ([3, Proposition 2]). Assume that a defect group D of B is a normal subgroup of G. Then $R_B = E_B$. In fact, the following condition $(*)$ holds.

$$(*)\ C_B\Phi_B = \Phi_B\ \mathrm{diag}\{|C_D(x_1)|, \ldots, |C_D(x_l)|\}$$

where $\Phi_B = (\varphi_i(x_j))$ is the Brauer character table of B, and $\{x_1, \ldots, x_l\}$ is a complete set of representatives of p-regular classes associated with B.

Remark of Proof. We consider a block decomposition of the formula in [1, p.419, ↑ 7]. At this time, we associate a complete set of p-regular classes to B, furthemore we should arrange the first l_1 classes to $\overline{B}_1, \ldots$, the last l_r classes to $\overline{B}_r$, where $\overline{B} = \overline{B}_1 + \cdots + \overline{B}_r$ is a block decomposition of $\overline{B}$ which is the homomorphic image of B by the canonical algebra epimorphism $\tau : FG \to F\overline{G}$, for a normal p-subgroup Q and $\overline{G} := G/Q$. In our case, $Q = D$. This means each $\overline{B}_i$ is of defect 0 and so $l_1 = \cdots = l_r = 1$. Thus $C_{\overline{B}} = I_l$ is the identity matrix. So we have the formula $(*)$ above. As a consequence we may admit any choice of block association of p-regular classes with B.

It is known that $\det \Phi_B \not\equiv 0 \pmod{\pi}$ and then Φ_B is a unimodular matrix over R (see [6, Theorem V 11.6]). $(*)$ implies each $|C_D(x_i)|$ is an eigenvalue of C_B and $\varphi^{(i)} = (\varphi_1(x_i), \ldots, \varphi_l(x_i))^T$ is its eigenvector, when $D \lhd G$. Furthermore, we can take Φ_B as a unimodular eigenvector matrix U_B of C_B.

Then we have the following lemma as a direct corollary of Proposition.

Lemma. Assume that a block B of FG is Morita equivalent to the Brauer correspondent b of B which is a block of $FN_G(D)$. Then we can take Φ_b as a unimodular eigenvector matrix U_B of C_B.

In the following we state some theorems about integrality of $\rho(B)$ most of which satisfy the condition mentioned in above Lemma.

Theorem 1 ([3],[10]). If D is cyclic (i.e. B is a finite type), then the following are equivalent.

(1) $\rho(B) \in \mathbb{Z}$
(2) $\rho(B) = |D|$
(3) $R_B = E_B$
(4) $B \sim b$ (Morita equivalent), where b is the Brauer correspondent block in $FN_G(D)$
(5) The Brauer tree of B is the star with the exceptional vertex at the center if it exists.
In this case, we can take Φ_b as a unimodular eigenvector matrix U_B of C_B.

Theorem 2 ([3],[10]). If B is a tame block (not finite type, i.e. $p = 2$ and $D \simeq$ a dihedral, a generalized quaternion or a semidihedral 2-group), then the following are equivalent.

(1) $\rho(B) \in \mathbb{Z}$
(2) $\rho(B) = |D|$
(3) $R_B = E_B$

(4) $B \sim b$ (Morita equivalent), where b is the Brauer correspondent block in $FN_G(D)$
(5) One of the following holds.
(i) $l = 1, C_B = (|D|)$

(ii) $l = 3, D \simeq E_4, C_B = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

(iii) $l = 3, D \simeq Q_8, C_B = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$.

In this case, we can take Φ_b as a unimodular eigenvector matrix U_B of C_B.

Theorem 3 ([3],[10]). If B is a p-block of a p-solvable group G, then the following are equivalent.
(1) $\rho(B) = |D|$
(2) $R_B = E_B$

In this case, B and its Brauer correspondent b are not necessarily Morita equivalent. For example, let $G = \mathrm{SL}(2,3) \cdot E_{27}$, $p = 3$ and let B be a unique non-principal block. Then $l(B) = 1$, and the number of ordinary irreducible characters $k(B) = 13$, but $k(b) = 17$, respectively. So B and b are not Morita equivalent. However, we can take Φ_β as a unimodular eigenvector matrix U_B of C_B, where β is a block of a subgroup of G or of a factor group of a central extension of a subgroup of G.

We cannot prove yet that if $\rho(B) \in \mathbb{Z}$, then $\rho(B) = |D|$ for a block B of a p-solvable group. In the following two results we are inspired by many author's results proving Broué's abelian defect group conjecture for $p = 2$ and 3 to be true (see e.g. [2, 4, 7, 8]). In abelian defect group case, our question yields a special case of Broué's abelian defect group conjecture.

Theorem 4 ([5], [10]). If $p = 2$, $\tilde{B}$ and $\tilde{b}$ are the principal blocks of $\tilde{G}$ and $N_{\tilde{G}}(D)$ respectively, with abelian defect group D, then the following are equivalent.
(1) $\rho(\tilde{B}) \in \mathbb{Z}$
(2) $\rho(\tilde{B}) = |D|$
(3) $R_{\tilde{B}} = E_{\tilde{B}}$
(4) $\tilde{B} \sim \tilde{b}$ Morita equivalent (even stronger Puig equivalent)
(5) For a finite group $\tilde{G}$ with an abelian Sylow 2-subgroup D and $O(\tilde{G}) = 1$, the following holds. Let $G := O'(\tilde{G})$. Then

$$G = G_1 \times \ldots \times G_r \times S,$$

where $G_i \simeq \mathrm{PSL}(2, q_i)$, $3 < q_i \equiv 3 \pmod 8$ for $1 \leq i \leq r$ and S is an abelian 2-group.
In this case, we can take Φ_b as a unimodular eigenvector matrix U_B of C_B.

Theorem 5 ([10]). If $p = 3$, $\tilde{B}$ and $\tilde{b}$ are the principal blocks of $\tilde{G}$ and $N_{\tilde{G}}(D)$ respectively, with elementary abelian defect group D of order 9, then the following are equivalent.

(1) $\rho(\tilde{B}) \in \mathbb{Z}$

(2) $\rho(\tilde{B}) = |D|$

(3) $R_{\tilde{B}} = E_{\tilde{B}}$

(4) $\tilde{B} \sim \tilde{b}$ Morita equivalent (even stronger Puig equivalent)

(5) Let $\tilde{G}$ be a finite group with an elementary abelian Sylow 3-subgroup D of order 9 and $O_{3'}(\tilde{G}) = 1$. Let $G := O^{3'}(\tilde{G})$. Then G satisfies the following (i) or (ii).

(i) $G = X \times Y$ for simple groups X, Y with a cyclic Sylow 3-subgroup of order 3, respectively.

(ii) G is one of the following simple groups.

(a) $\mathrm{PSU}(3, q^2)$, $2 < q \equiv 2$ or $5 \pmod 9$

(b) $\mathrm{PSp}(4, q)$, $q \equiv 4$ or $7 \pmod 9$

(c) $\mathrm{PSL}(5, q)$, $q \equiv 2$ or $5 \pmod 9$

(d) $\mathrm{PSU}(4, q^2)$, $q \equiv 4$ or $7 \pmod 9$

(e) $\mathrm{PSU}(5, q^2)$, $q \equiv 4$ or $7 \pmod 9$

In this case, we can take Φ_b as a unimodular eigenvector matrix U_B of C_B.

We use Koshitani-Kunugi's method in [4] to prove (5) $\rightarrow$ (4) in Theorems 4 and 5. Also we use the following fundamental Proposition to prove (1) $\rightarrow$ (5) in Theorems 4 and 5. The last statements of Theorems 4 and 5 are clear from Lemma.

Proposition 2 ([10]). Assume $H \lhd G$ and $|G : H| = q$ (a prime $\neq p$). Let b be a p-block of H. Let B be any p-block of G covering b. Then $\rho(B) = \rho(b)$.

References

[1] J.L. Alperin, M.J. Collins and D.A. Sibley, *Projective modules, filtrations and Cartan invariants*, Bull. London Math. Soc., **16** (1984), 416–420.

[2] M. Broué, *Equivalences of blocks of group algebras*, in Finite Dimensional Algebras and Related Topics, (edited by V. Dlab and L.L. Scott) Kluwer Acad. Pub., Dordrecht, 1994, pp.1–26.

[3] M. Kiyota, M. Murai and T. Wada, *Rationality of eigenvalues of Cartan matrices in finite groups*, Jour. Algebra **249** (2002), 110–119.

[4] S. Koshitani and N. Kunugi, *Broué's conjecture holds for principal 3-blocks with elementary abelian defect group of order 9*, Jour. Algebra **248** (2002), 575–604.

[5] N. Kunugi and T.Wada, *Eigenvalues of Cartan matrices of principal 2-blocks with abelian defect groups*, submitted (2006).

[6] H. Nagao and Y. Tsushima, *Representations of Finite Groups*, Academic Press, Tokyo, New York, 1988.

[7] T. Okuyama, *Some examples of derived equivalent blocks of finite groups*, preprint, 1997.

[8] R. Rouquier, *Block theory via stable and Rickard equivalences*, in "Modular representation theory of finite groups" (Charlottesville, VA, 1998), 101–146, de Gruyter, Berlin, 2001.

[9] T. Wada, *Eigenvalues and elementary divisors of Cartan matrices of cyclic blocks with $l(B) \leq 5$ and tame blocks*, Jour. Algebra **281**, 306–331 (2004).

[10] T. Wada, *Eigenvector matrices of Cartan matrices for finite groups*, In press in Jour. Algebra (2006).

[11] C.C. Xi and D. Xiang, *Cellular algebras and Cartan matrices*, Linear. Algebra Appl. **365**, 369–388 (2003).

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MACKEY FUNCTOR AND COHOMOLOGY OF FINITE GROUPS

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ABSTRACT. Let G be a finite group and H a subgroup. We consider an algebraic proof of Mislin's theorem which states that the restriction map from G to H on mod-p cohomology is an isomorphism if and only if H controls p-fusion in G. We follow the approach of P. Symonds (Bull. London Math. Soc. 36 (2004) 623-632) using cohomological Mackey functor for G. We will consider the structure of cohomology as a Mackey functor.

1. INTRODUCTION

G を有限群, k を標数 $p>0$ の (代数的閉) 体とする. 一次元の自明な kG-加群を k で表すと, k 係数の cohomology は代数的には,

$$H^n(G,k)=\mathrm{Ext}^n_{kG}(k,k)$$

と定義される. H を G の部分群, $x\in G$ とするとき次の3つの線形写像がある.

$$\mathrm{res}^G_H : H^n(G,k)\longrightarrow H^n(H,k)$$

$$c_x : H^n(H,k)\longrightarrow H^n(H^x,k)$$

$$\mathrm{tr}^G_H : H^n(H,k)\longrightarrow H^n(G,k)$$

これらについて, 次の2つが成り立つ. $H,K\le G$ とする.

$$\mathrm{tr}^G_H\mathrm{res}^G_H=|G:H| \tag{1.1}$$

$$\mathrm{res}^G_K\mathrm{tr}^G_H=\sum_{x\in K\backslash G/H}\mathrm{tr}^K_{H^x\cap K}\mathrm{res}^{H^x}_{H^x\cap K}c_x \tag{1.2}$$

2番目のものは Mackey formula と呼ばれている. H が G の Sylow p-部分群を含むとすると, (1.1) より

$$\mathrm{res}^G_H : H^n(G,k)\longrightarrow H^n(H,k)$$

は単射であることがわかる. そこでこれがいつ同型になるか, ということが問題となる.

Theorem 1 ([5]). 次は同値である.
(1) 任意の $n\ge 0$ に対して, 制限写像

$$\mathrm{res}^G_S : H^n(G,k)\longrightarrow H^n(H,k)$$

は同型写像である.
(2) H の p-部分群 Q と $x\in G$ について $Q^x\subseteq H$ となるならば, $x\in C_G(Q)H$ である.

The detailed version of this paper will be submitted for publication elsewhere.

すなわち cohomology が同型であることと p-部分群達の構造が同じであることが同値となる. (2) から (1), すなわち p-部分群達の構造が cohomology を決めるということは, (1.2) と [2] の stable element に関する議論からわかることであり古くから知られている. 一方, 逆の (1) から (2), すなわち cohomology が G の p-部分群達の状況を決定するということは, 長い空白の後 1990 年に Mislin によって証明された.

位相幾何学的な立場から見ると, G の cohomology $H^n(G,k)$ は, G の分類空間 BG の cohomology である. [5] における証明はこの立場からのものでありホモトピー論における手法を用いている.

しかし Theorem 1 において $H^n(G,k)$ は代数的にも定義されるものであり, (2) の主張は群の内部構造に関するものであるので, 代数的な, あるいはモヂュラー表現を用いた証明ができないかというのが, ここでの問題である ([1]).

2. Mackey functor and cohomology of trivial source modules

Mislin の定理 Theorem 1 の代数的な証明に関して, Symonds は Mackey functor を応用することを考えている. M が G 上の cohomological Mackey functor であるとは, 各 $H \leq G$ に対して k-vector spaces $M(H)$ が与えられ, $H \leq K \leq G$, $x \in G$, に対して k-線形写像

$$R_H^K : M(K) \longrightarrow M(H)$$
$$I_H^K : M(H) \longrightarrow M(K)$$
$$c_x : M(H) \longrightarrow M(H^x)$$

が定義され ((1.1)(1.2) に類似した) いくつかの公理をみたすものである. 例えば部分群 H に cohomology $H^n(H,k)$ を対応させる対応

$$H^n(-,k)$$

は cohomological Mackey functor である. G 上の cohomological Mackey functor の概念はある有限次元多元環上の加群と同じことであり, 既約 (単純) な functor や組成因子などを考えることができる ([8], [9], [10]).

G 上の既約な cohomological Mackey functor は, 同型を除くと組 (P,V) (P は G の p-部分群, V は 既約な $k(N_G(P))$-加群, 共役と同型を除く) と一対一に対応している. (P,V) に対応する既約 cohomological Mackey functor を $S_{P,V}^G$ と表すことにする.

cohomology $H^n(-,k)$ の組成因子はどうなっているであろうか. p-部分群 P に対して, $H^n(P,k)$ は共役の作用により $N_G(P)$-加群となるが, 中心化群 $C_G(P)$ は自明に作用する. このことから $S_{P,V}^G$ が $H^n(-,k)$ の組成因子ならば $C_G(P)$ は V に自明に作用していなければならないことがわかる. 逆に, 次の結果は $H^n(-,k)$ が可能な組成因子を全て持つことを示している.

Theorem 2 ([7]). *P を G の p-部分群, V を既約な $k(N_G(P))$-加群とする. $C_G(P)$ が V に自明に作用しているならば, ある $n \geq 0$ に対して $S_{P,V}^G$ は $H^n(-,k)$ の組成因子である.*

Mislin の定理 Theorem 1 はこの Theorem 2 から導かれることがわかる. ただし [7] における Theorem 2 の証明も位相幾何学の深い結果を用いている ([4] 参照).

一方, Mackey functor に関するこの定理は trivial source を持つ加群の cohomology に関する命題に言い換えることができる. 直既約 kG-加群 M が, 部分群の自明な加群から誘導された加群の直和因子であるとき, M を trivial source を持つ加群という. これは同型を除き, 既約な cohomological Mackey functor と同じように組 (P,V) (P は G の p-部分

群, V は既約な $k(N_G(P))$-加群, 共役と同型を除く) と一対一に対応している. 組 (P,V) に対応する trivial source を持つ加群を $M^G_{P,V}$ で表すことにする. $S^G_{P,V}$ が $H^n(-,k)$ の組成因子であるということを言い換えると, Theorem 2 は次と同値であることがわかる.

Theorem 3. P を G の p-部分群, V を既約な $k(N_G(P))$-加群とする. $C_G(P)$ が V に自明に作用しているならば, ある $n \geq 0$ に対して $H^n(G, M^G_{P,V}) \neq 0$ となる.

結局, Mislin の定理 Theorem 1 の証明のためにはこの (モヂュラー表現の命題ともいえる) Theorem 3 を証明すればよいことになった訳である. この Theorem 3 については, [6] と [3] において独立に, 代数的な証明が得られている.

3. Indecomposable direct summands of cohomology

ここでは G 上の cohomological Mackey functor としての $H^n(-,k)$ の構造をさらに考えてみたい. Theorem 2 により組成因子については (どの n に現れるか, ということは別として) 可能な全てのものが現れることがわかったので, 次に直既約因子を考えて見たい. 特に全ての $n \geq 0$ を考え,

$$H^*(-,k) = \bigoplus_{n=0}^{\infty} H^n(-,k)$$

の構造を調べたい. まずある意味で, これは有限生成である.

Proposition 4. G に対してある n があり,

$$M_n = \bigoplus_{i=0}^{n} H^i(-,k)$$

とおくと, 任意の $m \geq 0$ に対して H_n の有限個のコピーの直和からの *Mackey functor* としての全射

$$M_n \oplus M_n \oplus \cdots \oplus M_n \longrightarrow H^m(-,k)$$

がある.

一般に $H^n(-,k)$ の直和因子として, 直既約なものがどれくらい現れるか, 同型を除いて有限かどうかなどを調べたいのであるが, G 自身の cohomology だけでなく全ての部分群とその間の制限や transfer 写像を同時に考えていかなければならないためかなり複雑な状況となる. 一般的にはあまりよく分かっていないが, 次の例はかなり分かりやすい状況となっている.

Example 5. $p=2$ とし G を位数が 8 の二面体群とする.

$$H^*(G,k) = k[\alpha,\beta,\zeta]/(\alpha\beta)$$

$$\deg\alpha = \deg\beta = 1, \quad \deg\zeta = 2$$

である. $H^n = H^n(-,k)$ とおく. α をかけることにより引き起こされる Mackey functor の準同型写像

$$H^1 \longrightarrow H^2$$

の像を αH^1, ζ の生成する H^2 の部分 Mackey functor を $<\zeta>$ などのように表すことにすると, $n \geq 1$ に対して,

$$H^{2n} \cong \alpha H^1 \oplus (n-1)\alpha\zeta H^1 \oplus <\zeta> \oplus (n-1)\beta\zeta H^1 \oplus \beta H^1$$

$$H^{2n+1} \cong \alpha H^1 \oplus (n-1)\alpha\zeta H^1 \oplus \zeta H^1 \oplus (n-1)\beta\zeta H^1 \oplus \beta H^1$$

となっている. ($(n-1)\alpha\zeta H^1$ は $n-1$ 個の直和である.)

特に H^n の直和因子として現れる直既約な functor はすべて $n=0$ から $n=4$ までに現れて, 同型なものを除くと有限個であることがわかる.

REFERENCES

[1] J. L. Alperin, *On a theorem of Mislin*, J. Pure Appl. Algebra, 206 (2006) 55-58.

[2] H. Cartan and S. Eilenberg, Homological algebra, Princeton University Press, 1956.

[3] A. Hida, *Control of fusion and cohomology of trivial source modules*, preprint (2006).

[4] M. Kameko, Modular representation theory and stable decompostion of classifying spaces, 数理解析研究所講究録 1466, 有限群のコホモロジー論とその周辺, (2006) 9-20.

[5] G. Mislin, *On group homomorphisms inducing mod-p cohomology isomorphisms*, Comment. Math. Helvetici 65 (1990) 454-461.

[6] T. Okuyama, *On a theorem of Mislin on cohomology isomorphism and control of fusion*, 数理解析研究所講究録 1466, 有限群のコホモロジー論とその周辺, (2006) 86-92.

[7] P. Symonds, *Mackey functors and control of fusion*, Bull. London Math. Soc. 36 (2004) 623-632.

[8] J. Thévenaz and P. J. Webb, *Simple Mackey functors*, Proc. of the Second International Group Theory Conference (Bressanone, 1989), Rend. Circ. Mat. Palermo (2) Suppl. 23 (1990) 299-319.

[9] J. Thévenaz and P. J. Webb, *The structure of Mackey functors*, Trans. Amer. Math. Soc. 347 (1995) 1865-1961.

[10] T. Yoshida, *On G-functors (II): Hecke operators and G-functors*, J. Math. Soc. Japan 35 (1983) 179-190.

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INVARIANTS OF COMPLEX REDUCTIVE ALGEBRAIC GROUPS WITH SIMPLE COMMUTATOR SUBGROUPS

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ABSTRACT. Let G be a complex connected reductive algebraic group with simple commutator subgroup G'. Consider a finite dimensional representation $\rho : G \to GL(V)$ and denote by $\mathbb{C}[V]^G$ the $\mathbb{C}$-algebra consisting of polynomial functions on V which are invariant under the action of G. The purpose of this paper is to discuss on our partial classification of ρ's such that $\mathbb{C}[V]^G$ are polynomial rings over $\mathbb{C}$. Our method is based on the property of PLH defined and studied in [9] and our study on relative equidimensionality of representations under the assumption that G' is simple.

Key Words: coregular representations; relative invariants; reductive algebraic groups; algebraic tori; simple commutator subgroups; equidimensional actions

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1. INTRODUCTION

This is a worked out version of the author's talk in the 39-th Symposium on Ring Theory and Representation Theory held at Hiroshima University in 2006.

Let $\mathbb{C}$ be the complex number field (or an algebraically closed field of characteristic zero) and suppose that algebraic groups are defined over $\mathbb{C}$. We denote by G a reductive algbraic group with its identity connnected component G^0 whose commutator subgoup is denoted to G'. and, without specifying, we may assume that $G = G^0$. We use the following notations:

$\mathfrak{X}(G)$: $\mathfrak{X}(G)$ denote the group of rational characters of G whose composition is represented as an addition.

(V, G) : for a finite-dimensional representation $\rho : G \to GL(V)$, we denote ρ by (V, G).

R_χ : $R_\chi = \{f \in R \mid \sigma(f) = \chi(\sigma)f \ (\forall\sigma \in G)\}$ for a rational G-module R, which is called χ-invariants of G in R.

$\mathbb{C}[Y]$: the affine $\mathbb{C}$-algebra of polynomial functions on an affine variety Y over $\mathbb{C}$.

$Y//G$: the affine variety defined by $\mathbb{C}[Y]^G$ for a regular action (Y, G) of G on Y.

In this paper, we will study on the following classical problem:

Problem 1. For a (not necessarily connected) G, determine all finite dimensional representations (V, G) such that $\mathbb{C}[V]^G$ are polynomial rings.

This paper is based on the author's talk and the detailed proof of some results in this paper will be published elsewhere.

The following case for this problem are studied:
(1) (G. C. Shephard-J. A. Todd [14, 3]). Suppose that G is finite. Then

$$\mathbb{C}[V]^G \text{ is a polynomial ring} \iff G|_V \text{ is generated by pseudo-reflections.}$$

(2) (G.W. Schwarz-O.M. Adamovich-E.O. Golovina [2, 12]). Suppose that $G = G^0 = G'$ is simple. Then

$$\mathbb{C}[V]^G \text{ is a polynomial ring} \iff$$
$$(V, G) \text{ is listed in the Tables given by [2, 12].}$$

(3) (P. Littelmann [5]). For $G = G^0 = G'$, the problem has been solved for irreducible representations.
(4) (D. A. Shmel'kin [11]). Suppose that $G^0 = G'$ is simple. Then

$$\mathbb{C}[V]^G \text{ is a polynomial ring} \iff$$
$$(V, G) \text{ is listed in the Tables given by [11].}$$

(5) (well known?). Suppose that $G = G^0$ and $G' = \{1\}$. Then

$$\mathbb{C}[V]^G \text{ is a polynomial ring} \iff \mathbb{C}[V]^G \text{ is factorial.}$$

We will study on Problem 1 in the restricted case as follows:

Problem 2 (The Small Problem). Suppose that G' is simple. Determine all representations (V, G) of a connected non-semisimple G such that $\mathbb{C}[V]^G$ are polynomial rings. Precisely, we raise the following *problem:*

$$\mathbb{C}[V]^G \text{ is a polynomial ring (+some unknown conditions)}$$
$$\implies \mathbb{C}[V]^{G'} \text{ is a polynomial ring ?}$$

By Theorem 5 & 12, we obtain this implication for some groups *without conditions*. Then, in this case, the Small Problem can be reduced to Schwarz-Adamovich-Golovina classification (cf. (2)) and coregular toric representation (cf. (5)),

2. Blowing-up representations of codimension one

Recall that

$$(X, G) \text{ is a stable action of } G \text{ on an affine variety } X$$
$$\iff \exists\Omega \subseteq X : \text{a non-empty open subset consisiting of closed } G\text{-orbits}$$
$$\iff (X, G^0) : \text{stable}$$
$$\iff \text{both } (X, (G^0)'),\ (X/\!/(G^0)', G^0/(G^0)') \text{ are stable (cf. [10]).}$$

Definition 1. We define a generalization of stability as follows:

$$(X, G) \text{ is relatively stable} \iff (X/\!/(G^0)', G^0/(G^0)') \text{ is stable.}$$

Hereafter we suppose that G is connected reductive, and then we have an epimorphism $Z \times G' \to G$ with a finite kernel, where Z is a connected torus. So, we may assume that $G = Z \times G'$.

Remark 2. For a representation (V, G),

$$\text{it is relatively stable} \iff \mathbb{C}[V]_\chi \neq \{0\} \text{ implies } \mathbb{C}[V]_{-\chi} \neq \{0\} \text{ for } \chi \in \mathfrak{X}(Z).$$

Especially in the case where $\mathbb{C}[V_i]^{G'} \neq \mathbb{C}$ for arbitrary irreducible component V_i of V, we see that

$$\exists \text{ a relatively stable subrepresentation } (V', G) \text{ of } (V, G) \text{ such that } \mathbb{C}[V]^G = \mathbb{C}[U]^G.$$

2.1. Definition and properties of PLH. Let Y be an affine variety such that $\mathbb{C}[Y]$ is a positively graded factorial domain defined over $\mathbb{C}$ with a grade preseving rational action of Z as $\mathbb{C}$-automorphisms. For an element $\sigma \in \mathrm{Aut}_{\mathbb{C}-alg}(\mathbb{C}[Y])$ is said to be a *generalized reflection*, if $\mathrm{ht}((\sigma - I_Y)(\mathbb{C}[Y]) \cdot \mathbb{C}[Y]) = 1$. Let U be a minimal finite-dimensional ratinal module of Z admitting an homogeneous Z-equivariant epimorphism $\mathbb{C}[U] \to \mathbb{C}[Y]$. In this circumstance, we consider;

$$(W, w) \text{ a pair of a } Z\text{-submodule } W \text{ of } U \text{ and a nonzero vector } w \in U \text{ such that } U \cap < Z \cdot w >_{\mathbb{C}} = \{0\}$$

and the morphism

$$(\bullet + w) : W \ni x \to x + w \in U.$$

Definition 3. Suppose that (Y, Z) is stable. We say that (W, w) is a PLH of (Y, Z) or (U, Z), if $(\bullet + w)$ induces the $\mathbb{C}$-isomorphism $\mathbb{C}[U]^Z \cong \mathbb{C}[W]^{Z_w}$, i.e.,

$$\mathbb{C}[U]^Z \hookrightarrow \mathbb{C}[U]^{Z_w} \overset{(\bullet+w)^*}{\longrightarrow} \mathbb{C}[W]^{Z_w}.$$

Lemma 4. *Suppose that (Y, Z) is stable ane let (W, w) be a PLH of (Y, Z) (or (U, Z)). Then:*

(1) *If (W', w') is a PLH of (W, w), then (W', w') is also a PLH of (U, Z).*
(2) $\mathrm{Cl}(\mathbb{C}[Y]^Z) \cong \mathrm{Cl}(\mathbb{C}[Y]^{Z_w})$.

For a ring monomorphism $S \to R$ of Krull domains such that $S = Q(S) \cap R$ and $\mathfrak{q} \in \mathrm{Ht}_1(S)$, let $X_{\mathfrak{q}}(R) = \{\mathfrak{P} \in \mathrm{Ht}_1(R) \mid P \cap S = \mathfrak{q}\}$, where $\mathrm{Ht}_1(\circ)$ stands for the set consisting of all prime ideals of $\circ$ of height one. In the case where $S = R^L$ for a subgroup L of $\mathrm{Aut}(R)$, the sets $X_{\mathfrak{q}}(R)$ $(\forall \mathfrak{q} \in (S))$ are not empty (cf. [7]).

2.2. Notations for representations. Suppose that (V, G) is relatively stable. Let $V_{G'}$ denote the G-submodule of V satisfying

$$V = V_{G'} \oplus V^{G'}.$$

We express

$$\mathbb{C}[V]^{G'} = \mathbb{C}[V^{G'}] \otimes_{\mathbb{C}} \mathbb{C}[f_i, g_j, h_k],$$

where a homogeneous generating system

$$\{f_1, \ldots, f_l, g_1, \ldots, g_m, h_1, \ldots, h_n\}$$

of $\mathbb{C}[V_{G'}]^{G'}$ as a $\mathbb{C}$-algebra can be choosed in such a way that the following conditions are satisfied:

$$\begin{aligned}
&\mathrm{ht}(f\mathbb{C}[V] \cap \mathbb{C}[V]^G) = \mathrm{ht}(g\mathbb{C}[V] \cap \mathbb{C}[V]^G) = 1,\\
&\mathrm{ht}(h\mathbb{C}[V] \cap \mathbb{C}[V]^G) \geq 2,\\
&|X_{f\mathbb{C}[V]\cap\mathbb{C}[V]^G}(\mathbb{C}[V]^{G'})| = 1,\\
&|X_{g\mathbb{C}[V]\cap\mathbb{C}[V]^G}(\mathbb{C}[V]^{G'})| \geq 2.
\end{aligned}$$

We apply the concept of PHL for the pair of Y and U to the affine variety Y defined by $\mathbb{C}[V]^{G'}$ and

$$U = V^{G'} \oplus < f_i, g_j, h_k >_{\mathbb{C}}^{\vee}$$

with the natural $\mathbb{C}$-epimorphism

$$\mathbb{C}[U] \to \mathbb{C}[Y] = \mathbb{C}[V]^{G'}$$

having grade preserving Z-actions.

2.3. The Main Therem for blowing-up representations. The main result of this Section is

Theorem 5. *Suppose that G' is simple and (V, G) is relatively stable. Suppose that*

$$\{\mathfrak{q} \in \mathrm{Ht}_1(\mathbb{C}[V_{G'}]^{G'}) \ \mid\ |X_{\mathfrak{q}}(\mathbb{C}[V]^{G'})| \geq 2\}$$

is nonempty and the action of Z on the $\mathbb{C}$-subalgebra $\mathbb{C}[V_{G'}]^{G'}$ is non-trivial. Then the following conditions are equivalent:

1. *$\mathbb{C}[V]^G$ is factorial.*
2. *The stabilizer $Z_w|_{\mathbb{C}[V]^{G'}}$ is a finite group generated by generalized reflections in $\mathrm{Aut}_{\mathbb{C}-alg}(\mathbb{C}[V]^{G'})$, where w is a point in U such that (W, w) is a minimal PLH of (Y, Z) for some $W \subseteq U$.*
3. *$\mathbb{C}[V]$ is a free $\mathbb{C}[V]^{G'}$-module and the condition (2) holds.*
4. *$\mathbb{C}[V]^G$ is a polynomial ring.*

The assumption of this theorem is characterized by

Proposition 6. *$m > 0$ if and only if the following condition is satisfied:*

$$\{\mathfrak{q} \in \mathrm{Ht}_1(\mathbb{C}[V_{G'}]^{G'}) \ \mid\ |X_{\mathfrak{q}}(\mathbb{C}[V]^{G'})| \geq 2\} \text{ is nonempty.}$$

Remark 7. Since the condition (2) in Theorem 5 implies that $\mathbb{C}[V]^{G'}$ is a polynomial ring. Hence, under the condition that $m > 0$, we see that

$$\mathbb{C}[V]^G \text{ is a polynomial ring} \Longrightarrow \mathbb{C}[V]^{G'} \text{ is a polynomial ring},$$

i.e., the Small Problem is affirmative in this case.

2.4. **A Sketch of the Proof of Theorem 5.** By Lemma 4, we see that

$$\mathbb{C}[V]^G \text{ is factorial} \Longrightarrow \mathbb{C}[Y]^{Z_w} \text{ is factorial}$$
$$\Longrightarrow (Y, Z_w) : \text{cofree}$$
$$\Downarrow$$
$$(Y, Z_w) : \text{equidimensional}$$
$$\Longrightarrow (Y, (Z_w)^0) : \text{equidimensional}$$
$$\Downarrow$$

Theorem 5 is reduced to the next theorem (cf. Theorem 8). □

Theorem 8. *Suppose that G' is simple and (V, G) is relatively stable. Suppose that the action of Z on $\mathbb{C}[V_{G'}]^{G'}$ is non-trivial and on Y is equidimensional. Then*

(1) *$\mathbb{C}[V]$ is a free $\mathbb{C}[V]^G$-module.*
(2) *$\mathbb{C}[V]$ is a free $\mathbb{C}[V]^{G'}$-module.*

The proof of this theorem, which is regarded as a generalization of a part of [1] and [13], is not ring theoretical but is a consequence of case-by-case arguments on invariants of representatons of simple algebraic groups.

3. NO-BLOWING-UP REPRESENTATIONS OF CODIMENSION ONE

We will, now, treat the case where $m = 0$ and $n > 0$ (we suppose that (V, G) is relatively stable).

Lemma 9. *If $\mathbb{C}[V]^G$ a polynomial ring, then the localization $((\mathbb{C}[V_{G'}]^{G'})^{Z_w})_{h_1 \cdots h_n}$ is a regular ring. Here w is a nonzero point of U such that (W, w) is a minimal PLH of (Y, Z).*

Express $V_{G'} = \bigoplus_i V_i$, where each V_i is irreducible component of $V_{G'}$ as a representation of G. For any polynomial $h \in \mathbb{C}[V]$, put

$$\operatorname{supp}(h) = \{i \mid (\mathbb{C}[V] \xrightarrow{\text{can.}} \mathbb{C}[V/V_i])(h) \neq 0\}.$$

Consider the following condition:

$$\mathbb{C}[V_i]^{G'} \neq \mathbb{C} \quad (\forall i \text{ such that} V_i \subseteq V_{G'}). \tag{3.1}$$

Lemma 10. *Suppose that the condition (3.1) holds. For any $1 \leq k \leq n$, we see that $\operatorname{supp}(h_k)$ is a set of singleton.*

Lemma 11. *Suppose that the condition (3.1) holds. Put $J := \cup_{k=1}^n \operatorname{supp}(h_k)$. Then*

$$\mathbb{C}[\bigoplus_{i \in J} V_i]^{G'} \cong \bigotimes_{i \in J} \mathbb{C}[V_i]^{G'}$$

and this is a $|J|$-dimensional polynomial ring.

3.1. **The Main Theorem for no-blowing-up representaitons.** We use the following notation: Let $\Phi_1, \ldots, \Phi_r$ be the fundamental irreducible representations of a simple group of rank r whose numbering are standard (e.g. [15]) and $\phi\psi$ denote the highest weight irreducible representation in $\phi \otimes \psi$ of irreducible ϕ and ψ.

We say that, for example, "(V, G') contains the irreducible representation *quasi-equivalent to* $(\Phi_4, \mathbf{D}_5)$" if $(V, \tilde{G}')$ contains $(\tilde{\Phi}_4, \tilde{G}')$ for a universal covering group $\tilde{G}'$ of G'. Precisely, consider an isomorphism $\nu : \tilde{G}' \xrightarrow{\sim} \mathbf{D}_5$ and regard

$$\tilde{G}' \xrightarrow{\nu} \mathbf{D}_5 \xrightarrow{\Phi_4} GL(\Phi_4)$$

as $(\tilde{\Phi}_4, \tilde{G}')$.

Theorem 12. *Suppose that G' is simple and that (V, G') contains none of the irreducible representations quasi-equivalent to the following list:*

$$(\Phi_1, \mathbf{A}_r), (\Phi_2, \mathbf{A}_r), (\Phi_1, \mathbf{C}_r), (\Phi_4, \mathbf{D}_5).$$

Then if $\mathbb{C}[V]^G$ is a polynomial ring, so is $\mathbb{C}[V]^{G'}$.

3.2. **Some auxiliary results.** The following criterion on stability of semisimple group actions on factorial varieties is well known:

Proposition 13 (V.L. Popov [10]). *Let (V, G') be any representation of G'. The following conditions (1) and (2) are equivalent and in the case where (V, G') is irreducible and G' is simple, the following four conditions are equivalent:*

(1) *The generic stabilizer of (V, G) is reductive.*
(2) *(V, G') is a stable action.*
(3) $\mathbb{C}[V]^{G'} \neq \mathbb{C}$.
(4) *V contains non-trivial closed orbit.*

For a torus T_1 of rank one and a representation R of T_1, put

$$q_{T_1}(R) := \min\{\dim R_-, \dim R_+\},$$

where $R_- = \oplus_{j \in \mathbb{N}} R_{-j\chi}$ and $R_+ = \oplus_{j \in \mathbb{N}} R_{j\chi}$ and $\mathfrak{X}(T_1) = <\chi>$. The number $q_{T_1}(R)$ does not depend on χ.

Theorem 14 (V.G. Kac-V.L. Popov-E.B. Vinberg [4]). *Suppose that G' is simple and (R, G') is any irreducible representation which is quasi-equivalent to neighter $(\Phi_1\Phi_2, \mathbf{A}_3)$ nor $(\Phi_1^3, \mathbf{A}_3)$. If $\mathbb{C}[R]^{G'}$ is not a polynomial ring, then there exists a subtorus T_1 of rank one of G' such that the following conditions hold:*

(1) *$(R^{T_1}, N_{G'}(T_1)/T_1)$ is stable and its generic stabilizer is finite.*
(2) $q_{T_1}(R) - q_{T_1}((\mathrm{Ad}, G')) \geq 3$.

In order to study the Small Problem, we need a refinement of Theorem 14 as follows (some results related to this can be found in [8]) :

Lemma 15. *Let N be a reductive algebraic group with its commutator subgroup N' and φ ane ψ be non-trivial finite dimensional representations of N. Suppose that the action $(\varphi//N', N/N')$ is stable (i.e., (φ, N) is relatively stable) and the generic stabilizer is finite. Then $(\varphi \oplus \psi//N', N/N')$ is stable.*

Proposition 16. *Let N be a reductive algebraic group and φ and ψ be non-trivial finite dimensional representations of N. Suppose that the action $(\varphi//N', N)$ is stable (i.e., (φ, N) is relatively stable) and the generic stabilizer is finite. Then $(\varphi \oplus \psi, N)$ is stable.*

Lemma 17. *Let φ and ψ be non-trivial finite dimensional representations of G'. Suppose that (φ, G') has a closed G'-orbit whose stabilizer is denoted to H_1. Let H_2 be a reductive closed subgroup of H_1 such that $(\psi^{H_2}, N_{G'}(H_2))$ is stable and its generic stabilizer is finite. Then, for any nonzero homogeneous $h \in \mathbb{C}[\varphi]^{G'}$, there exists a nonzero $x \in (\varphi \oplus \psi)^{H_2}$ such that*

(1) *Gx is closed in $\varphi \oplus \psi$.*
(2) *$h(x) \neq 0$.*
(3) *The stabilizer $N_{G'}(H_2)_x$ is finite.*

Lemma 18. *Let (φ, G') be an irreducible representattion of a simple G' such that* $\dim(\mathbb{C}[\varphi]^{G'}) = 1$. *For arbitrary irreducible representation ψ of G' and arbitrary nonzero homogeneous polynomial function $h \in \mathbb{C}[\psi]^{G'}$, if $\mathbb{C}[\psi]^{G'}$ is not a polynomial ring, then there exist a subtorus T_1 of rank one of G' and a nonzero vector $x \in \varphi \oplus \psi$ such that*

(1) *$h(x) \neq 0$.*
(2) $q_{T_1}(\varphi \oplus \psi) - q_{T_1}((\mathrm{Ad}, G')) \geq 3$.
(3) *$N_{G'}(T)_x$ is finite.*

Lemma 19. *Suppose that G' is simple, simply connected and let ψ_i be irreducible representations of G' such that* $\dim \mathbb{C}[\psi_i]^{G'} = 1$ $(1 \leq i \leq u)$. *Suppose that $u \geq 2$. Then* $\mathbb{C}[\oplus_{i=1}^{u} \psi_i]^{G'} = \otimes_{i=1}^{u} \mathbb{C}[\psi_i]^{G'}$ *if and only if $u = 2$ and $(\oplus_{i=1}^{2} \psi_i, G')$ is quasi-equivalent to* $(\Phi_1 \oplus \Phi_3, \mathbf{B}_3)$ *or* $(\Phi_1 \oplus \Phi_3, \mathbf{D}_4)$.

Lemma 20. *Suppose that G' is simple, simply connected and of type $\mathbf{B}_3$ or $\mathbf{D}_3$. Let ψ be an irreducible representation of G' such that $\mathbb{C}[\psi]^{G'}$ is not a polynomial ring. Suppose that $(V, G') = (\psi \oplus \Phi_1 \oplus \Phi_3, \mathbf{B}_3)$ or $(\psi \oplus \Phi_1 \oplus \Phi_3, \mathbf{D}_3)$. Moreover, let h_1 and h_2 be homogeneous polynomial fuctions satisfying $\mathbb{C}[\Phi_1]^{G'} = \mathbb{C}[h_1]$, $\mathbb{C}[\Phi_3]^{G'} = \mathbb{C}[h_2]$. Then there exists a subtorus T_1 of rank one of G' and a nonzero vector $x \in V^{T_1}$ such that*

(1) *$h_i(x) \neq 0$ $(i = 1, 2)$.*
(2) $q_{T_1}(\varphi \oplus \psi) - q_{T_1}((\mathrm{Ad}, G')) \geq 3$.
(3) *The stabilizer $N_{G'}(T)_x$ is finite.*

3.3. **A Sketch of the Proof of Theorem 12.** By Lemma 9, 18, 19, 20 and the Slice Étale [6], we see

$\mathbb{C}[V]^{G}$ is a polynomial ring $\Longrightarrow$

(V, G') does not contain non-coregular irreducible components.

computation $\Downarrow$ C.I.T.

$\mathbb{C}[V]^{G'}$ is a polynomial ring. □

Consequently we conclude that the Small Problem is affirmative also in this case.

References

[1] O.M. Adamovich, *Equidimensional representations of simple algebraic groups*, Transl. II. Ser., Amer. Math. Soc. **128** (1986), 25–29.

[2] O.M. Adamovich, E.O. Golovina, *Simple linear Lie groups having a free algebra of invariants*, Selecta Math. Soviet **3** (1983/1984), 183–220.

[3] C. Chevalley, *Invariants of finite groups generated by reflections*, Amer. J. Math. **77** (1955), 778–782.

[4] V.G. Kac, V.L. Popov, E.B. Vinberg, *Sur les groupes linéaires algébriques dont l'algèbre des invariants est libre*, C. R. Acad. Sci. Paris **283** (1976), 865–878.

[5] P. Littelmann, *Koreguläre und äquidimensionale Darstellungen*, J. Algebra **123** (1989), 193–222.

[6] D. Luna, *Slices Étales*, Bull. Soc. Math. France **33** (1973), 81–105.

[7] H. Nakajima, *Class groups of localities of rings of invariants of reductive algebraic groups*, Math. Zeit. **182** (1983), 1–15.

[8] ———, *Representations of a reductive algebraic group whose algebras of invariants are complete intersections*, J. für die Reine und Angew. Math. **367** (1986), 115–138.

[9] ———, *Minimal presentations of torus invariants in paralleled linear hulls* (to appear).

[10] V.L. Popov, Groups, Generators Syzygies, and Orbits in Invariant Theory, Transl. Math. Monographs **100**, Providence: Amer. Math. Soc. 1992.

[11] D. A. Shmel'kin, *On non-connected simple linear groups with a free algebra of invariants*, Izvestiya Math. **60** (1996), 811–856.

[12] G.W. Schwarz, *Representations of simple Lie groups with regular rings of invariants*, Invent. Math. **49** (1978), 167-191.

[13] ———, *Representations of simple Lie groups with a free module of covariants*, Invent. Math. **50** (1978), 1–12.

[14] G. C. Shephard, J. A. Todd, *Finite unitary reflection groups*, Canad. J. Math. **6** (1954), 274–304.

[15] J. Tits, Tabellen zu den einfachen Lie Gruppen und ihren Darstellungen, Lecture Notes in Math. **40**, Berlin/Heidelberg/New York: Springer-Verlag 1970.

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VON NEUMANN REGULAR RINGS WITH COMPARABILITY

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ABSTRACT. In this paper, we report some known recent results for regular rings with comparability including general comparability, 1-comparability, s-comparability and weak comparability. In Section 1, we introduce the history and results for regular rings with comparability, with respect to finiteness conditions. In Sections 2 and 3, we tell some results for regular rings with weak comparability.

Key Words: Regular Ring, Comparability.

1. 正則環における比較可能性と有限性

環 R は単位元を持つ環とし、R-加群は右 R-加群を意味する。最初に、この論文の中で使用される記号を準備する。

記号 1. R-加群 M と濃度 n に対して、n 個の M の直和を nM で表す。N が R-加群 M の部分加群であるとき $N \leq M$ で表し、特に N が M の直和因子であるとき $N \leq_\oplus M$ で表す。R-加群 M, N に対して、単射 $f : M \longrightarrow N$ が存在するとき $M \lesssim N$ と書く。特に、$f(M) \leq_\oplus N$ のとき $M \lesssim_\oplus N$ で表す。また、$f(M) < N$ を満たす単射 $f : M \longrightarrow N$ が存在するとき $M \prec N$ で表す。

まず正則環及び正則環に関連する定義を与える。

定義 2. 環 R が（ノイマン）正則環であるとは、R の各元 x に対して $xyx = x$ を満たす R の元 y が存在するときに言う。環 R がユニット正則環であるとは、R の各元 x に対して $xux = x$ を満たす R の可逆元 u が存在するときに言う。環 R がダイレクト・ファイナイトであるとは、R の元 x, y に対して「$xy = 1$ ならば $yx = 1$」を満たすときに言う。即ち、R の任意の右又は左可逆元は可逆元であることを意味する。

正則環は 1936 年ノイマンによって連続幾何学の研究から見出された環であり、1950 年代から 1960 年代にかけての内海による商環の存在性の考察により、多数の正則環が存在することが知られるようになった。そして、1960 年代後半に入り、有限条件と呼ばれるダイレクト・ファイナイト性やユニット正則性の研究が始められるようになった。ダイレクト・ファイナイト性はノイマン有限性或いはデデキント有限性とも呼ばれており、可換環やネーター環及びアルチン環がダイレクト・ファイナイト環であることはよく知られている。ユニット正則性は 1968 年 G.Ehrich によって与えられた概念である。ユニット正則性やダイレクト・ファイナイト性は、正則環研究における重要な有限条件と呼ばれている。何故これらの概念が有限性と呼ばれるかは、次の定理 3 の性質を持つからであると推察される。

The detailed version of this paper has been submitted for publication elsewhere ever.

定理 3 ([5]). 次が成立する。

(1) 環 R がダイレクト・ファイナイト環であることは、「$A_R \leq_{\oplus} R_R$ かつ $A_R \cong R_R$ ならば $A = R_R$」を満たすことと同値である。即ち、R_R は自分自身と同型な真の直和因子を持たないことを意味する。

(2) 正則環 R がユニット正則環であることは、有限生成射影 R-加群 A, B, C に対して「$A \oplus C \cong B \oplus C$ ならば $A \cong B$」を満たすことと同値である。

上記有限性に関して、歴史的には 1973 年 M. Henriksen により「ユニット正則環はダイレクト・ファイナイト環である」ことが証明された。その後、この結果の逆の成立性が問題となったが、1977 年 G. Bergman により「ユニット正則環でないダイレクト・ファイナイト正則環の存在」が保証された。そして、正則環における有限性研究の流れは、次の問題 A, B の研究へと進展していった。

問題 A. どの種のダイレクト・ファイナイト正則環はユニット正則環か?
問題 B. 全てのダイレクト・ファイナイト単純正則環はユニット正則環か? ([5, Open Problem 3])

次に比較可能性について述べる。正則環における比較可能性には、主なものとして次の 4 つの比較可能性がある：

(1) 一般比較可能性
(2) 1 -比較可能性
(3) s-比較可能性 (但し、s は自然数)
(4) 弱比較可能性

今、上記 (1)〜(3) の定義を与える。

定義 4. 正則環 R が一般比較可能性を満たすとは、R の各元 x, y に対して「$exR \lesssim eyR$ かつ $(1-e)yR \lesssim (1-e)xR$」を満たす R の中心べき等元 e が存在するときに言う。

定義 5. R を正則環とし、s を自然数をする。R が s-比較可能性を満たすとは、R の各元 x, y に対して「$xR \lesssim s(yR)$ 又は $yR \lesssim s(xR)$」を満たすときに言う。特に、1- 比較可能性は比較可能公理とも呼ばれており、上記 4 つの比較可能性の中で最も基本的なものである。

ここで、比較可能性を持つ正則環に関する上記問題 A に対する歴史を話す。一般比較可能性の概念は、1960 年代 Operator Algebra や Baer 環の研究から見出された概念であり、右自己入射正則環（例えば、内海の商環）はこの性質を持つ。又、1 -比較可能性は次元関数の存在に関連して 1975 年 K.Goodearl と D.Handelman によって見出された。素環である右自己入射正則環はこの性質を持つ。これらの比較可能性を持つダイレクト・ファイナイト正則環がユニット正則環であることはよく知られている ([5])。その後、1-比較可能性の一般化として、1976 年 D.Handelman や K.Goodearl により s-比較可能性という概念が見出されたが、この研究はすぐには進まず、1990 年代に入り、弱比較可能性の研究に刺激される形で研究が進められた。s-比較可能性を持つ正則環に関しては、上記問題 A, B について次の結果が知られている。

定理 6 ([9]). s-比較可能性を持つダイレクト・ファイナイト単純正則環は、ユニット正則環である。

定理 7 ([3]). 2-比較可能性を持つダイレクト・ファイナイト正則環で、単純環でもユニット正則環でもない環が存在する。

次に、「全てのダイレクト・ファイナイト単純正則環はユニット正則環か?」という上記問題Bに関しては、1991年K.C. O'Mearaにより弱比較可能性という概念が見出され、この問題に対する大きな貢献（下述の定理9）が彼によって与えられた。ここで、正則環に対する弱比較可能性の定義を与え、K.C. O'Mearaによる結果を紹介する。

定義 8 ([9]). 正則環Rが弱比較可能性を満たすとは、Rの各元$x(\neq 0)$に対して、「$n(yR) \lesssim R_R$ $(y \in R)$ ならば $yR \lesssim xR$」を満たす自然数nが存在するときに言う。この自然数nは、Rの元xに依存することに注意する。

定理 9 ([9]). 弱比較可能性を満たすダイレクト・ファイナイト単純正則環は、ユニット正則環である。

その後、弱比較可能性を満たす単純正則環の研究が論文[1], [2], [4]で続けられ、単純正則環が弱比較可能性を持つための特徴付けが次のように与えられた。

定理 10 ([2]). Rを単純正則環とする。このとき、次は同値である。

(a) Rが弱比較可能性を満たす。

(b) Rは、有限生成射影R-加群の集まりに関する「Strict Unperforation Property」を満たす。即ち、$nA \prec nB$を満たす自然数nと有限生成射影R-加群A, Bが存在するならば、$A \prec B$を満たす。

注意 11. 弱比較可能性を持たないダイレクト・ファイナイト単純正則環の存在は、現在のところ知られていない。

以後§2, §3では、単純環とは限らない弱比較可能性を満たす正則環について、上記定理10の結果を発展させることを目的に、この環の持つ性質を紹介する。

2. 弱比較可能正則環上のダイレクト・ファイナイト射影加群

最初に、弱比較可能性を満たす正則環の典型的な例を紹介する。このために必要な定義を与える。

定義 12. 環Rのべき零元xのべき零指数を、$x^n = 0$を満たす自然数nの最小数と定める。環Rのべき零指数を、Rの全てのべき零元のべき零指数の上限と定める。

補題 13 ([5]). 正則環Rのべき零指数がnであることと、Rが$(n+1)$個の同型な零でない右イデアルの直和を含まないことは同値である。

定理 14 ([9]). 次が成立する。

(1) 有限べき零指数を持つ正則環は、弱比較可能性を満たす。

(2) 弱比較可能性を満たす正則環は、素環又は有限べき零指数を持つ環である。

上記の定理14(1)は、弱比較可能性の定義8と補題13から容易に導かれることに注意する。ここで、弱比較可能性を満たす正則環の典型的な例を与える。

例 15 ([9]). 単純環でない弱比較可能性を満たすユニット正則環は存在する。

[証明] Sを、アルチン環でないが比較可能公理を満たす単純ユニット正則環とする。例えば、$S = \varinjlim M_{2^n}(K)$ (但しKは体)を考えればよい。そして、FをSの中心とする。Sはアルチン環でないので、Sの中に互いに直交する零でないべき等元の無限列$\{e_1, e_2, \cdots\}$を選ぶことが出来る。各自然数nに対して$f_n := e_1 + \cdots + e_n$と定め、$J := \cup_{n=1}^{\infty} f_n S f_n$とおく。このとき、$R := F + J$は求めるものである。□

例 16. ダイレクト・ファイナイトでない弱比較可能性を満たす単純正則環は存在する。

[証明] V を体 F 上の無限次元ベクトル空間とし、その自己準同型環を Q とする。$M = \{x \in Q | \, dim_F(xV) < dim_F(V)\}$ とおくと、M は Q の唯一の極大両側イデアルである。ここで、環 $\bar{Q} := Q/M$ を考えると、$\bar{Q}$ が求めるものである。$\bar{Q}$ の弱比較可能性を検証するには、$\bar{Q}$ の任意の元 $\bar{x}(\neq \bar{0})$ に対して $\bar{x}\bar{Q} \cong \bar{Q}$ が成立することに注意すればよい。□

定義 17. R-加群 M がダイレクト・ファイナイトであるとは、M の自己準同型環 $End_R(M)$ がダイレクト・ファイナイト環の時に言う。このことは、M が自分自身と同型な真の直和因子を持たないことと同値である。

ここで、射影加群に対するダイレクト・ファイナイト性の判定を与える。この判定は、以後の諸結果の証明に大きな役割を果たす。

定理 18. R を弱比較可能性を満たす正則環とする。
(I) P を有限生成射影 R-加群とし、その巡回加群分解を $P = \oplus_{i=1}^n P_i$ とする。このとき、次は同値である。
 (a) P はダイレクト・ファイナイトである。
 (b) 各 $i = 1, \cdots, n$ に対して、P_i はダイレクト・ファイナイトである。
 (c) どんな R-加群 $X(\neq 0)$ に対しても、$\aleph_0 X \not\lesssim P$ である。

(II) P を有限生成でない射影 R-加群とし、その巡回加群分解を $P = \oplus_{i \in I} P_i$ とする。このとき、次は同値である。
 (a) P はダイレクト・ファイナイトである。
 (b) 各 $i(\in I)$ に対して P_i はダイレクト・ファイナイトであり、かつ、どんな R-加群 $X(\neq 0)$ に対しても、$X \not\lesssim \oplus_{i \in I-F} P_i$ を満たす I の有限部分集合 F が存在する。
 (c) どんな R-加群 $X(\neq 0)$ に対しても $\aleph_0 X \not\lesssim P$ である。
 (d) 各 $i(\in I)$ に対して P_i はダイレクト・ファイナイトであり、かつ、どんな R-加群 $X(\neq 0)$ に対しても $\aleph_0 X \not\lesssim_{\oplus} P$ である。

注意 19. R を弱比較可能性を満たす正則環とする。定理 18 より、射影 R-加群 P がダイレクト・ファイナイトであるとは「$\aleph_0 X_R \lesssim P$ ならば $X = 0$」ということである。

注意 20. 射影加群がダイレクト・ファイナイトである為の注意 19 の判定条件は、一般の正則環では成立しない。何故ならば、$\aleph_0 X_R \lesssim R_R$ を満たす単項右イデアル $X(\neq 0)$ を持つ比較可能公理を満たすユニット正則環 R が存在するからである。

定理 18(I) より、次の結果は容易に導かれる。

系 21. 弱比較可能性を満たすダイレクト・ファイナイト正則環上の有限生成射影加群は、ダイレクト・ファイナイトである。このことは、弱比較可能性を満たすダイレクト・ファイナイト正則環上の全ての行列環はダイレクト・ファイナイト環であることを意味する。

系 21 の結果は、「ダイレクト・ファイナイト正則環上の全ての行列環はダイレクト・ファイナイト環か？」という問題 ([5, Open Problem 1]) に対して、弱比較可能性を満たす正則環では肯定的であるということを意味している。系 21 は更に次のように一般化される。

定理 22. R を弱比較可能性を満たす正則環とする。このとき、ダイレクト・ファイナイト射影 R-加群の同型な有限個の直和はダイレクト・ファイナイトである。

次に、弱比較可能性を満たす正則環上のダイレクト・ファイナイト射影加群の様子を考察する。一般に、非可算生成ダイレクト・ファイナイト射影加群を持つ弱比較可能性を満たすダイレクト・ファイナイト正則環は存在する。

[証明] F を体とし、$R=\prod_{i\in I}F_i$（但し $F_i=F$ で、I は 非可算集合）とする。このとき、R はべき零指数 1 である。従って、定理 14 から、R は単純環でない弱比較可能性を満たすダイレクト・ファイナイト正則環であることが分かる。ここで、$P_i:=F_i\times(\prod_{j\neq i}0)$ 及び $P:=\oplus_{i\in I}P_i$ とすると、P は非可算生成ダイレクト・ファイナイト射影 R-加群であることが定理 18 より分かる。□

特に、弱比較可能性を満たすダイレクト・ファイナイト単純正則環や、ダイレクト・ファイナイトでない弱比較可能性を満たす正則環については、次の結果が成立する。

命題 23. R を弱比較可能性を満たすダイレクト・ファイナイト単純正則環とする。このとき、ダイレクト・ファイナイト射影 R-加群は有限生成か可算生成である。

定理 24. R をダイレクト・ファイナイトでない弱比較可能性を満たす正則環とする。このとき、ダイレクト・ファイナイト射影 R-加群は零のみである。即ち、零でない射影 R-加群はすべてダイレクト・ファイナイトでない。

この定理 24 は、ダイレクト・ファイナイトでない弱比較可能性を満たす正則環を考察する時に、効果的な働きをする。

3. 弱比較可能正則環上の有限生成射影加群

§3 では、弱比較可能性を満たす正則環上の有限生成射影加群の集まりに関する「Strict Cancellation Property」及び「Strict Unperforation Property」を考察する。この為に、次の補題 25 を必要とする。

補題 25. R を正則環とし、有限生成射影 R-加群 X,B,C,Y に対して $X\oplus C'\oplus Y\leq_{\oplus} B\oplus C\oplus Y$ かつ $C\cong^{f} C'$ が成立すると仮定する。このとき、直和分解 $B=B_1\oplus B_1^{*}$, $B_n^{*}=B_{n+1}\oplus B_{n+1}^{*}$; $C=C_1\oplus C_1^{*}$, $C_n^{*}=C_{n+1}\oplus C_{n+1}^{*}$ が存在し、$X\cong B_1\oplus C_1$, $C_n\cong B_{n+1}\oplus C_{n+1}$, $B\oplus C\oplus Y=X\oplus Y\oplus fC_1\oplus\cdots\oplus fC_n\oplus B_{n+1}^{*}\oplus C_{n+1}^{*}$ を満たす。

この補題 25 から、$C_1\oplus C_2\oplus\cdots\oplus C_n\leq C$ かつ $C_1\gtrsim C_2\gtrsim\cdots\gtrsim C_n$ が成立し、$nC_n\lesssim C$ が分かる。このことから、弱比較可能性の適用が考えられ、次の結果が得られる。

定理 26 (Strict Cancellation Property). R を弱比較可能性を満たすダイレクト・ファイナイト正則環とする。有限生成射影 R-加群 A,B,C に対して $A\oplus C\prec B\oplus C$ が成り立つならば、$A\prec B$ である。

ダイレクト・ファイナイトでない弱比較可能性を満たす正則環に対しては、次の結果が成立する。

定理 27. R をダイレクト・ファイナイトでない弱比較可能性を満たす正則環とする。有限生成射影 R-加群 A,B,C に対して $A\oplus C\lesssim B\oplus C$ $(B\neq 0)$ が成り立つならば、$A\prec B$ である。

上記定理 26 と定理 27 をあわせると、次の主定理が得られる。

定理 28. R を弱比較可能性を満たす正則環とする。有限生成射影 R-加群 A,B,C に対して $A\oplus C\prec B\oplus C$ $(B\neq 0)$ が成り立つならば、$A\prec B$ である。

また、この定理28を有効に用いて次の主定理が得られる。

定理 29 (Strict Unperforation Property). Rを弱比較可能性を満たす正則環とする。$nA \prec nB$を満たす有限生成射影 R-加群 A, B と自然数 n が存在するならば、$A \prec B$ である。

注意 30. 単純正則環に対する弱比較可能性が、有限生成射影加群の集まりに関する「Strict Unperforation Property」と同値であることは、定理10で既に述べた。しかし、単純環とは限らない一般的な正則環に対する弱比較可能性は、有限生成射影加群の集まりに関する「Strict Unperforation Property」によって特徴付けることは出来ないことが分かる。何故ならば、環 $R := \prod_{n=1}^{\infty} M_n(F)$ (但し F は体)を考えると、Rは右自己入射ユニット正則環だから Strict Unperforation Property を満たすことが知られているが、Rは素環でもべき零指数が有限でもないので、定理14(2) から Rは弱比較可能性を満たさないからである。

注意 31([6]). 弱比較可能性を満たす単純ユニット正則環 R で、有限生成射影 R-加群の集まりに関する「Unperforation Property」を満たさない環が存在する。即ち、この環においては「$nA \lesssim nB$ かつ $A \not\lesssim B$」を満たす自然数 n と有限生成射影 R-加群 A, B が存在する。

正則環の弱比較可能性は剰余環に移行することはよく知られている。では、正則環上の弱比較可能性は森田同値であろうか。この問題を取り扱うために、次のように加群に対する弱比較可能性の概念を新しく導入する必要がある。

定義 32. Rを正則環とする。有限生成射影 R-加群 M が弱比較可能性を満たすとは、M の各直和因子 N に対して、「$nL \lesssim M$ (但し L は M の直和因子) ならば $L \lesssim N$」を満たす自然数 n が存在する時に言う。このとき、M のどんな直和因子も弱比較可能性を満たすことが分かる。

上記主定理29を用いて、次の結果が得られる。

命題 33. Rを正則環とする。このとき、次は同値である。
(a) Rは弱比較可能性を満たす。
(b) すべての有限生成射影 R-加群は弱比較可能性を満たす。

記号 34. Rを環とし、Mを R-加群とする。このとき、$add(M_R) := \{N_R \mid N \lesssim_{\oplus} nM$ を満たす自然数 n が存在する $\}$ と定める。

R-加群 M の自己準同型環 $End_R(M)$ を S とする。圏 $add(M_R)$ と圏 $add(S_S)$ の間の $Hom_R({}_SM_R, -)$ と $- \otimes_S {}_SM_R$ による Hom 関手とテンソル関手の圏同値性から、次の結果が得られる。

補題 35. Rを正則環とし、Pを有限生成射影 R-加群とする。このとき、次は同値である。
(a) Pが弱比較可能性を満たす。
(b) Pの自己準同型環 $End_R(P)$ が弱比較可能性を満たす。

定義32に注意し命題33と補題35を用いて、次の結果が容易に証明できる。

定理 36. Rを正則環とする。このとき、次は同値である。
(a) Rは弱比較可能性を満たす。
(b) すべての有限生成射影 R-加群 P に対して、$End_R(P)$ は弱比較可能性を満たす。
(c) Rに森田同値な環は弱比較可能性を満たす。
(d) すべての自然数 n に対して、行列環 $M_n(R)$ は弱比較可能性を満たす。
(e) ある自然数 n に対して、行列環 $M_n(R)$ は弱比較可能性を満たす。

この結果から、正則環上の弱比較可能性は森田同値であることが分かる。最後に、弱比較可能性を満たす正則環に関する有限性の問題を提起する。定理6より、弱比較可能性を満たすダイレクト・ファイナイト単純正則環はユニット正則環であることは知られている。しかし、単純環でない弱比較可能性を満たすダイレクト・ファイナイト正則環はユニット正則環となるかどうかは現在不明である。

REFERENCES

[1] P. Ara and K.R. Goodearl, *The almost isomorphism relation for simple regular rings*, Publ. Mat. UAB **36** (1992), 369–388.

[2] P. Ara, K.R. Goodearl, E. Pardo and D.V. Tyukavkin, *K-theoretically simple von Neumann regular rings*, J. Algebra **174** (1995), 659–677.

[3] P. Ara, K.C. O'Meara and D.V. Tyukavkin, *Cancellation of projective modules over regular rings with comparability*, J. Pure Appl. Algebra **107** (1996), 19–38.

[4] P. Ara and E. Pardo, *Refinement monoids with weak comparability and applications to regular rings and C^*-algebras*, Proc. Amer. Math. Soc. **124(3)** (1996), 715–720.

[5] K.R. Goodearl, *Von Neumann Regular Rings*, Krieger, Malabar, Florida (1991).

[6] K.R. Goodearl, *Torsion in K_0 of unit-regular rings*, Proc. Edinburgh Math. Soc. **38(2)** (1995), 331–341.

[7] M. Kutami, *On von Neumann regular rings with weak comparability*, J. Algebra **265** (2003), 285–298.

[8] M. Kutami, *On von Neumann regular rings with weak comparability II*, Comm. Algebra **33(9)** (2005), 3137–3147.

[9] K.C. O'Meara, *Simple regular rings satisfying weak comparability*, J. Algebra **141** (1991), 162–186.

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