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Edited by
Kiyoichi Oshiro
Yamaguchi University

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Organizing Committee of The Symposium on Ring Theory and Representation Theory

The Symposium in 2007 was held at National Olympic Memorial Youth Center, Tokyo as the joint Symposium with the 5th China-Japan-Korea International Symposium on Ring Theory for Sep. 10 – 15.

The Symposium on Ring Theory and Representation Theory has been held annually in Japan and the Proceedings have been published by the organizing committee. The first Symposium was organized in 1968 by H. Tominaga, H. Tachikawa, M. Harada and S. Endo. After their retirement, a new committee was organized in 1997 for managing the Symposium. The present members of the committee are H. Asashiba (Shizuoka Univ.), Y. Hirano (Naruto Univ. of Education), S. Koshitani (Chiba Univ.), M. Sato (Yamanashi Univ.) and K. Oshiro (Yamaguchi University).

The Proceedings of each Symposium is edited by program organizer. Anyone who wants these Proceedings should ask to the program organizer of each Symposium or one of the committee members.

The Symposium in 2008 will be held at Shizuoka University for Sep. 4 – 6.

Concerning several information on ring theory group in Japan containing schedules of meetings and symposiums as well as the addresses of members in the group, you should refer the following homepage, which is arranged by M. Sato (Yamanashi Univ.):

http://fuji.cec.yamanashi.ac.jp/~ring/ (in Japanese)

civil2.cec.yamanashi.ac.jp/~ring/japan/ (in English)

Kiyoichi Oshiro
Yamaguchi Japan
December, 2007

CONTENTS

PREFACE

The 40th Symposium on Ring Theory and Representation Theory was held at National Olympic Memorial Youth Center, Tokyo. The symposium and the proceedings are financially supported by Masahisa Sato (Yamanashi University) JSPS Grants-in-Aid for 2007 International Scientific Meetings in Japan and Kiyoichi Oshiro (Yamaguchi University) JSPS Grant-in-Aid for 2007 Scientific Research (B), No.18340011.

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Kiyoichi Oshiro
Yamaguchi, Japan
January, 2008

DERIVED EQUIVALENCES AND SERRE DUALITY FOR GORENSTEIN ALGEBRAS

HIROKI ABE AND MITSUO HOSHINO

ABSTRACT. We introduce a notion of Gorenstein algebras of codimension c and demonstrate that Serre duality theory plays an essential role in the theory of derived equivalences for Gorenstein algebras.

1. INTRODUCTION

Let R be a commutative noetherian ring and A a Noether R-algebra, i.e., A is a ring endowed with a ring homomorphism $R \to A$ whose image is contained in the center of A and A is finitely generated as an R-module. Let $c \geq 0$ be an integer. Assume $\mathrm{Ext}^i_R(A, R) = 0$ for $i \neq c$ and set

$$\Omega = \mathrm{Ext}^c_R(A, R).$$

We call A a Gorenstein R-algebra of codimension c if $R_\mathfrak{p}$ is Gorenstein for all $\mathfrak{p} \in \mathrm{Supp}_R(A)$ and Ω is a projective generator for right A-modules. If A is a Gorenstein R-algebra of codimension c, then we will show that Ω lies in the center of the Picard group of A (Proposition 12), that Ω is a dualizing complex for A if $\sup\{\dim R_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Supp}_R(A)\} < \infty$ (Proposition 9), and that $\mathrm{Ann}_R(A)$ contains an R-regular sequence $x_1, \cdots, x_c$ and A is a Gorenstein S-algebra of codimension 0, where S is the residue ring of R over the ideal generated by $x_1, \cdots, x_c$ (Proposition 13). Also, we will see that our Gorenstein algebras are Gorenstein in the sense of [11] (Proposition 7). In particular, commutative Gorenstein algebras are Gorenstein rings. We refer to [11] for properties enjoyed by Gorenstein algebras and for the relationship of the notion of Gorenstein algebras to the theory of commutative Gorenstein rings.

Our main aim of this note is to demonstrate that Serre duality theory plays an essential role in the theory of derived equivalences for Gorenstein algebras. In Section 3, we will extend Serre duality theory (cf. [8]) to Noether algebras. We will see that for an arbitrary Noether R-algebra A there exists a bifunctorial isomorphism in Mod-R

$$\mathrm{Hom}_{\mathcal{D}(\text{Mod-}A)}(Y^\bullet, X^\bullet \otimes^{\mathbf{L}}_A V^\bullet) \cong \mathrm{RHom}^\bullet_A(X^\bullet, Y^\bullet)^*$$

for $X^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}$ and $Y^\bullet \in \mathcal{D}(\text{Mod-}A)$, where $V^\bullet = \mathrm{Hom}^\bullet_R(A, I^\bullet)$ with $I^\bullet$ a minimal injective resolution of R and $(-)^* = \mathrm{Hom}_{\mathcal{D}(\text{Mod-}R)}(-, R)$ (Proposition 16). On the other hand, we know from [1, Theorem 4.7] that if $V^\bullet$ is a dualizing complex for A and if inj dim ${}_AA$ = inj dim $A_A < \infty$ then $- \otimes^{\mathbf{L}}_A V^\bullet$ induces a self-equivalence of $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)$.

Assume A is a Gorenstein R-algebra of codimension c. Let $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ be a tilting complex and $B = \mathrm{End}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet)$. In Section 4, we will ask when B is also a Gorenstein

The detailed version of this paper will be submitted for publication elsewhere.

R-algebra of codimension c. Set $\nu = - \otimes^{\mathbf{L}}_A \Omega$. Then by Serre duality theory we have an isomorphism of B-bimodules

$$\mathrm{Hom}_{\mathcal{D}(\mathrm{Mod}\text{-}A)}(P^\bullet, \nu P^\bullet[i]) \cong \mathrm{Ext}^{i+c}_R(B, R)$$

for all $i \in \mathbb{Z}$. On the other hand, denoting by $\mathcal{S}$ the full subcategory of $\mathcal{D}^-(\mathrm{Mod}\text{-}A)$ consisting of complexes $X^\bullet$ with $\mathrm{Hom}_{\mathcal{D}(\mathrm{Mod}\text{-}A)}(P^\bullet, X^\bullet[i]) = 0$ for $i \neq 0$, we have an equivalence $\mathrm{Hom}_{\mathcal{D}(\mathrm{Mod}\text{-}A)}(P^\bullet, -) : \mathcal{S} \to \mathrm{Mod}\text{-}B$ (see [18, Section 4]). Thus B is a Gorenstein R-algebra of codimension c if and only if $\mathrm{add}(\nu P^\bullet) = \mathrm{add}(P^\bullet)$ (Corollary 21). Unfortunately, this is not the case in general (Example 23). However, B is a Gorenstein R-algebra of codimension c with $\mathrm{Ext}^c_R(B, R) \cong B$ as B-bimodules if and only if A is a Gorenstein R-algebra of codimension c with $\Omega \cong A$ as A-bimodules (Corollary 22).

We refer to [7], [12] and [20] for basic results in the theory of derived categories and to [18], [19] for definitions and basic properties of tilting complexes and derived equivalences. Also, we refer to [10] for standard homological algebra in module categories and to [16] for standard commutative ring theory.

2. Preliminaries

Notation. For a ring A we denote by Mod-A the category of right A-modules and by mod-A the full subcategory of Mod-A consisting of finitely presented modules. We denote by Proj-A (resp., Inj-A) the full subcategory of Mod-A consisting of projective (resp., injective) modules and by $\mathcal{P}_A$ the full subcategory of Proj-A consisting of finitely generated projective modules. We denote by A^{op} the opposite ring of A and consider left A-modules as right A^{op}-modules. Sometimes, we use the notation X_A (resp., ${}_AX$) to stress that the module X considered is a right (resp., left) A-module. In particular, we denote by proj dim X_A (resp., proj dim ${}_AX$) the projective dimension of a right (resp., left) A-module X. Similar notation is used to denote the injective dimension.

In this note, complexes are cochain complexes of modules and, as usual, modules are considered as complexes concentrated in degree zero. For any $n \in \mathbb{Z}$ we denote by $\mathrm{H}^n(-)$ the n-th homology of a complex. For an additive category $\mathcal{B}$, we denote by $\mathcal{K}(\mathcal{B})$ (resp., $\mathcal{K}^+(\mathcal{B})$, $\mathcal{K}^-(\mathcal{B})$, $\mathcal{K}^{\mathrm{b}}(\mathcal{B})$) the homotopy category of complexes (resp., bounded below complexes, bounded above complexes, bounded complexes) over $\mathcal{B}$. For an abelian category $\mathcal{A}$, we denote by $\mathcal{D}(\mathcal{A})$ (resp., $\mathcal{D}^-(\mathcal{A})$, $\mathcal{D}^+(\mathcal{A})$, $\mathcal{D}^{\mathrm{b}}(\mathcal{A})$) the derived category of complexes (resp., complexes with bounded above homology, complexes with bounded below homology, complexes with bounded homology) over $\mathcal{A}$. We always consider $\mathcal{K}^*(\mathcal{B})$ (resp., $\mathcal{D}^*(\mathcal{A})$) as a full triangulated subcategory of $\mathcal{K}(\mathcal{B})$ (resp., $\mathcal{D}(\mathcal{A})$) closed under isomorphism classes, where $* = +$, $-$ or b. In particular, for a noetherian ring A, we identify $\mathcal{D}^*(\mathrm{mod}\text{-}A)$ with $\mathcal{D}^*_{\mathrm{mod}\text{-}A}(\mathrm{Mod}\text{-}A)$, the full triangulated subcategory of $\mathcal{D}^*(\mathrm{Mod}\text{-}A)$ consisting of complexes $X^\bullet$ with $\mathrm{H}^n(X^\bullet) \in \mathrm{mod}\text{-}A$ for all $n \in \mathbb{Z}$, where $* = -$ or b. We denote by $\mathrm{Hom}^\bullet(-,-)$ (resp., $- \otimes^\bullet -$) the single complex associated with the double hom (resp., tensor) complex and by $\mathrm{RHom}^\bullet(-,-)$ (resp., $- \otimes^{\mathbf{L}} -$) the right (resp., left) derived functor of $\mathrm{Hom}^\bullet(-,-)$ (resp., $- \otimes^\bullet -$).

Finally, for an object X in an additive category $\mathcal{B}$, we denote by $\mathrm{add}(X)$ the full subcategory of $\mathcal{B}$ whose objects are direct summands of finite direct sums of copies of X.

Preliminaries. Throughout this note, R is a commutative noetherian ring. We denote by $\dim R$ the Krull dimension of R, by $\mathrm{Spec}(R)$ the set of prime ideals of R and by $(-)_\mathfrak{p}$ the localization at $\mathfrak{p} \in \mathrm{Spec}(R)$. For an R-module M, we set $\mathrm{Supp}_R(M) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid M_\mathfrak{p} \neq 0\}$ and $\mathrm{Ann}_R(M) = \{x \in R \mid xM = 0\}$ and we denote by $E_R(M)$ an injective envelope of M in Mod-R. We set

$$D = \mathrm{RHom}^\bullet_R(-, R) : \mathcal{D}(\text{Mod-}R) \to \mathcal{D}(\text{Mod-}R).$$

Then for any $X^\bullet, Y^\bullet \in \mathcal{D}(\text{Mod-}R)$ we have a bifunctorial isomorphism

$$\theta_{X^\bullet, Y^\bullet} : \mathrm{Hom}_{\mathcal{D}(\text{Mod-}R)}(X^\bullet, DY^\bullet) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}(\text{Mod-}R)}(Y^\bullet, DX^\bullet).$$

For any $X^\bullet \in \mathcal{D}(\text{Mod-}R)$ we set

$$\xi_{X^\bullet} = \theta^{-1}_{X^\bullet, DX^\bullet}(\mathrm{id}_{DX^\bullet}) : X^\bullet \to D^2 X^\bullet.$$

We recall several basic results on Gorenstein dimension for finitely generated R-modules and bounded complexes of finitely generated R-modules (see e.g. [9] for details).

Definition 1 ([3]). A module $M \in$ mod-R is said to have Gorenstein dimension zero if the canonical homomorphism

$$M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R), x \mapsto (f \mapsto f(x))$$

is an isomorphism and $\mathrm{Ext}^i_R(M, R) = \mathrm{Ext}^i_R(\mathrm{Hom}_R(M, R), R) = 0$ for $i > 0$. We denote by $\mathcal{G}_R$ the full additive subcategory of mod-R consisting of modules which have Gorenstein dimension zero. Note that $\mathcal{P}_R \subset \mathcal{G}_R$. Next, a module $M \in$ mod-R is said to have finite Gorenstein dimension if M has a left resolution $P^\bullet \to M$ with $P^\bullet \in \mathcal{K}^\mathrm{b}(\mathcal{G}_R)$.

Definition 2. A complex $X^\bullet \in \mathcal{D}^\mathrm{b}(\text{mod-}R)$ is said to have finite Gorenstein dimension if $X^\bullet \cong Y^\bullet$ in $\mathcal{D}(\text{Mod-}R)$ for some $Y^\bullet \in \mathcal{K}^\mathrm{b}(\mathcal{G}_R)$.

Remark 3. For any $M \in$ mod-R the following are equivalent.

(1) M has finite Gorenstein dimension as a module.
(2) M has finite Gorenstein dimension as a complex.

Lemma 4 ([14, Proposition 2.10]). *For any $X^\bullet \in \mathcal{D}^\mathrm{b}(\text{mod-}R)$ the following are equivalent.*

(1) *$X^\bullet$ has finite Gorenstein dimension.*
(2) *$\mathrm{H}^i(DX^\bullet) = 0$ for $i \gg 0$ and $\xi_{X^\bullet}$ is an isomorphism.*

Throughout the rest of this note, A is a Noether R-algebra, i.e., A is a ring endowed with a ring homomorphism $R \to A$ whose image is contained in the center of A and A is finitely generated as an R-module. Note that $\mathrm{Ann}_R(A)$ coincides with the kernel of the structure ring homomorphism $R \to A$ and that $\mathrm{Supp}_R(A)$ coincides with the set of prime ideals of R containing $\mathrm{Ann}_R(A)$. We fix a minimal injective resolution $R \to I^\bullet$ in Mod-R and set $V^\bullet = \mathrm{Hom}^\bullet_R(A, I^\bullet) \in \mathcal{K}^+(\text{Mod-}A^\mathrm{e})$, where $A^\mathrm{e} = A^\mathrm{op} \otimes_R A$. Note that $V^\bullet \in \mathcal{K}^+(\text{Inj-}A)$ and $V^\bullet \in \mathcal{K}^+(\text{Inj-}A^\mathrm{op})$. We refer to [12] for the definition and basic properties of dualizing complexes.

Lemma 5 ([1, Propositions 3.7 and 3.8]). *The following are equivalent.*

(1) *$V^\bullet$ is a dualizing complex for A.*
(2) *$R_\mathfrak{p}$ is Gorenstein for all $\mathfrak{p} \in \mathrm{Supp}(A)$ and $\sup\{\dim R_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Supp}(A)\} < \infty$.*

3. Gorenstein algebras

Throughout the rest of this note, $c \geq 0$ is an integer.

Definition 6. We say that A is a Gorenstein R-algebra of codimension c if the following conditions are satisfied:

(1) $R_\mathfrak{p}$ is Gorenstein for all $\mathfrak{p} \in \mathrm{Supp}_R(A)$;
(2) $\mathrm{Ext}^i_R(A, R) = 0$ for $i \neq c$; and
(3) $\mathrm{Ext}^c_R(A, R)$ is a projective generator in Mod-A.

Throughout this section, we assume $\mathrm{Ext}^i_R(A, R) = 0$ for $i \neq c$ and set

$$\Omega = \mathrm{Ext}^c_R(A, R).$$

Note that $V^\bullet \cong \Omega[-c]$ in $\mathcal{D}(\text{Mod-}A^e)$. Also, $\mathrm{H}^i(V^\bullet) \cong \mathrm{Ext}^i_R(A, R)$ for all $i \in \mathbb{Z}$.

We know from the following proposition that our Gorenstein algebras are Gorenstein in the sense of [11]. So we refer to [11] for properties enjoyed by Gorenstein algebras and for the relationship of the notion of Gorenstein algebras to the theory of commutative Gorenstein rings.

Proposition 7. *For any* $\mathfrak{p} \in \mathrm{Supp}_R(A)$ *with* $R_\mathfrak{p}$ *Gorenstein the following hold.*

(1) $\Omega_\mathfrak{p} \neq 0$ *and hence* $\dim R_\mathfrak{p} \geq c$.
(2) $\mathrm{Ext}^i_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}, \Omega_\mathfrak{p}) = 0$ *for* $i < \dim R_\mathfrak{p} - c$.
(3) $\mathrm{inj\ dim}\ \Omega_{\mathfrak{p}A_\mathfrak{p}} = \dim R_\mathfrak{p} - c$.

If $R_\mathfrak{p}$ is Gorenstein for all $\mathfrak{p} \in \mathrm{Supp}_R(A)$, then A has finite Gorenstein dimension as an R-module.

Lemma 8. *The following are equivalent.*

(1) *A has finite Gorenstein dimension as an R-module.*
(2) $\mathrm{Ext}^i_A(\Omega, \Omega) = 0$ *for* $i > 0$ *and* $A \xrightarrow{\sim} \mathrm{End}_A(\Omega), a \mapsto (w \mapsto aw)$.
(3) $\mathrm{Ext}^i_{A^{\mathrm{op}}}(\Omega, \Omega) = 0$ *for* $i > 0$ *and* $A \xrightarrow{\sim} \mathrm{End}_{A^{\mathrm{op}}}(\Omega)^{\mathrm{op}}, a \mapsto (w \mapsto wa)$.

Throughout the rest of this section, we assume further that $R_\mathfrak{p}$ is Gorenstein for all $\mathfrak{p} \in \mathrm{Supp}_R(A)$. Then by Lemma 8 and we have $\mathrm{Ext}^i_A(\Omega, \Omega) = \mathrm{Ext}^i_{A^{\mathrm{op}}}(\Omega, \Omega) = 0$ for $i > 0$ and $\Omega \in \text{mod-}A^e$ is faithfully balanced, i.e., $A \xrightarrow{\sim} \mathrm{End}_A(\Omega), a \mapsto (w \mapsto aw)$ and $A \xrightarrow{\sim} \mathrm{End}_{A^{\mathrm{op}}}(\Omega)^{\mathrm{op}}, a \mapsto (w \mapsto wa)$.

Proposition 9. *The following are equivalent.*

(1) $\sup\{\dim R_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Supp}_R(A)\} < \infty$.
(2) $V^i = 0$ *for* $i \gg 0$.
(3) $\mathrm{inj\ dim}\ \Omega_A < \infty$.
(4) $\mathrm{inj\ dim}\ {}_A\Omega < \infty$.

We refer to [17] for tilting modules. Note however that a module is a tilting module if and only if it is isomorphic to a tilting complex in the derived category (see e.g. [2, Proposition 3.9]).

There is another notion of Gorenstein algebras. Consider the case where R is an artinian Gorenstein ring. Then an Artin R-algebra A is sometimes called Gorenstein if inj dim ${}_AA$ = inj dim $A_A < \infty$ (see e.g. [4]). It follows by [17, Proposition 1.6] that an Artin R-algebra A is Gorenstein in this sense if and only if $\mathrm{Hom}_R(A, R) \in \text{mod-}A^e$ is a tilting module. We extend this fact to Noether algebras.

Proposition 10. *Assume* $\sup\{\dim R_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Supp}_R(A)\} < \infty$. *Then the following are equivalent.*

(1) $\Omega \in \text{mod-}A^e$ *is a tilting module.*
(2) proj dim ${}_A\Omega$ = proj dim $\Omega_A < \infty$.
(3) inj dim ${}_AA$ = inj dim $A_A < \infty$.

Proposition 11. *The following are equivalent.*

(1) $\Omega \in \mathcal{P}_A$ *and* $\Omega \in \mathcal{P}_{A^{op}}$.
(2) $\mathrm{add}(\Omega) = \mathcal{P}_A$ *in* Mod-A.
(3) $\mathrm{add}(\Omega) = \mathcal{P}_{A^{op}}$ *in* Mod-A^{op}.

Proposition 12. *Assume A is a Gorenstein R-algebra of codimension c. Then Ω lies in the center of the Picard group of A.*

As for the ring structure of a Gorenstein R-algebra A, we may restrict ourselves to the case where $c = 0$.

Proposition 13. *There exists an R-regular sequence $x_1, \cdots, x_c$ in* $\mathrm{Ann}_R(A)$. *Set $S = R/(x_1, \cdots, x_c)$ with $(x_1, \cdots, x_c)$ the ideal of R generated by $x_1, \cdots, x_c$. Then the following hold.*

(1) *A has Gorenstein dimension zero as an S-module.*
(2) $\mathrm{Hom}_S(A, S) \cong \Omega$ *in* Mod-A^e.
(3) *$S_\mathfrak{q}$ is Gorenstein for all* $\mathfrak{q} \in \mathrm{Supp}_S(A)$.

4. Serre duality

In this section, we will extend Serre duality theory (cf. [8]) to Noether algebras. We set

$$(-)^* = \mathrm{Hom}_{\mathcal{D}(\text{Mod-}R)}(-, R) : \mathcal{D}(\text{Mod-}R) \to \text{Mod-}R.$$

Note that $(-)^* \cong \mathrm{H}^0(D(-))$.

Recall that a complex $X^\bullet \in \mathcal{D}^b(\text{mod-}A)$ is said to have finite projective dimension if $\mathrm{Hom}_{\mathcal{D}(\text{Mod-}A)}(X^\bullet[-i], -)$ vanishes on mod-A for $i \gg 0$. We denote by $\mathcal{D}^b(\text{mod-}A)_{\text{fpd}}$ the full triangulated subcategory of $\mathcal{D}^b(\text{mod-}A)$ consisting of complexes which have finite projective dimension. Note that $\mathcal{K}^b(\mathcal{P}_A) \xrightarrow{\sim} \mathcal{D}^b(\text{mod-}A)_{\text{fpd}}$ canonically. Similarly, a complex $X^\bullet \in \mathcal{D}^b(\text{mod-}A)$ is said to have finite injective dimension if $\mathrm{Hom}_{\mathcal{D}(\text{Mod-}A)}(-, X^\bullet[i])$ vanishes on mod-A for $i \gg 0$. We denote by $\mathcal{D}^b(\text{mod-}A)_{\text{fid}}$ the full triangulated subcategory of $\mathcal{D}^b(\text{mod-}A)$ consisting of complexes which have finite injective dimension.

Definition 14. We say that A has Serre duality if there exist a self-equivalence of a triangulated category $F : \mathcal{D}^b(\text{mod-}A) \xrightarrow{\sim} \mathcal{D}^b(\text{mod-}A)$ and a bifunctorial isomorphism in Mod-R

$$\mathrm{Hom}_{\mathcal{D}(\text{Mod-}A)}(Y^\bullet, FX^\bullet) \cong \mathrm{RHom}^\bullet_A(X^\bullet, Y^\bullet)^*$$

for $X^\bullet \in \mathcal{D}^b(\text{mod-}A)_{\text{fpd}}$ and $Y^\bullet \in \mathcal{D}^b(\text{mod-}A)$. If this is the case, we call F a Serre functor for A.

Note that if A has finite global dimension then $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}} = \mathcal{D}^{\mathrm{b}}(\text{mod-}A)$ and that if R is selfinjective then we have bifunctorial isomorphisms in Mod-R

$$\begin{aligned}\mathrm{RHom}^\bullet_A(X^\bullet, Y^\bullet)^* &\cong \mathrm{H}^0(D\mathrm{RHom}^\bullet_A(X^\bullet, Y^\bullet)) \\ &\cong D\mathrm{H}^0(\mathrm{RHom}^\bullet_A(X^\bullet, Y^\bullet)) \\ &\cong \mathrm{Hom}_{\mathcal{D}(\text{Mod-}A)}(X^\bullet, Y^\bullet)^*\end{aligned}$$

for $X^\bullet, Y^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{mod-}A)$. These facts would justify the definition above.

Remark 15. Assume there exists a Serre functor $F : \mathcal{D}^{\mathrm{b}}(\text{mod-}A) \xrightarrow{\sim} \mathcal{D}^{\mathrm{b}}(\text{mod-}A)$ for A. Then the restriction of F to $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}$ is unique up to isomorphism and the following hold.

(1) F induces a self-equivalence of $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}$ and there exists a tilting complex $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ such that $FA \cong P^\bullet$ in $\mathcal{D}(\text{Mod-}A)$ and $A \cong \mathrm{End}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet)$.

(2) For any $i \in \mathbb{Z}$ we have a functorial isomorphism in Mod-A^{op}

$$\mathrm{Hom}_{\mathcal{D}(\text{Mod-}A)}(M, FA[i]) \cong \mathrm{Ext}^i_R(M, R)$$

for $M \in$ mod-A. In particular, $\mathrm{H}^i(FA) \cong \mathrm{Ext}^i_R(A, R)$ in Mod-A^{e} for all $i \in \mathbb{Z}$.

(3) Assume $\mathrm{Ext}^i_R(A, R) = 0$ for $i \neq c$ and set $\Omega = \mathrm{Ext}^c_R(A, R)$. Then $FA \cong \Omega[-c]$ in $\mathcal{D}(\text{Mod-}A)$ and $\Omega \in$ mod-A^{e} is a tilting module.

Proposition 16. *We have a bifunctorial isomorphism in* Mod-R

$$\mathrm{Hom}_{\mathcal{D}(\text{Mod-}A)}(Y^\bullet, X^\bullet \otimes^{\mathbf{L}}_A V^\bullet) \cong \mathrm{RHom}^\bullet_A(X^\bullet, Y^\bullet)^*$$

for $X^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}$ *and* $Y^\bullet \in \mathcal{D}(\text{Mod-}A)$.

Corollary 17. *Assume A is a Gorenstein R-algebra of codimension c. Then A has Serre duality with a Seere functor*

$$- \otimes^{\mathbf{L}}_A V^\bullet : \mathcal{D}^{\mathrm{b}}(\text{mod-}A) \xrightarrow{\sim} \mathcal{D}^{\mathrm{b}}(\text{mod-}A).$$

Theorem 18. *Assume that $R_{\mathfrak{p}}$ is Gorenstein for all* $\mathfrak{p} \in \mathrm{Supp}_R(A)$ *and that* $\sup\{\dim R_{\mathfrak{p}} \mid \mathfrak{p} \in \mathrm{Supp}_R(A)\} < \infty$. *Then $V^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{mod-}A^{\mathrm{e}})$ and the following are equivalent.*

(1) *A has Serre duality with a Serre functor*

$$- \otimes^{\mathbf{L}}_A V^\bullet : \mathcal{D}^{\mathrm{b}}(\text{mod-}A) \xrightarrow{\sim} \mathcal{D}^{\mathrm{b}}(\text{mod-}A).$$

(2) *A and A^{op} have Serre duality.*

(3) $\mathrm{inj\,dim}\ {}_AA = \mathrm{inj\,dim}\ A_A < \infty$.

5. Derived equivalences

Throughout this section, we fix a tilting complex $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ and set $B = \mathrm{End}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet)$. Note that B is a Noether R-algebra and that there exists a tilting complex $Q^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_B)$ such that $A \cong \mathrm{End}_{\mathcal{K}(\text{Mod-}B)}(Q^\bullet)$.

Proposition 19. *The following hold.*

(1) $\mathrm{Ann}_R(A) = \mathrm{Ann}_R(B)$ *and hence* $\mathrm{Supp}_R(A) = \mathrm{Supp}_R(B)$.

(2) *If A has finite Gorenstein dimension as an R-module, then so does B.*

(3) *If* $\mathrm{inj\,dim}\ {}_AA = \mathrm{inj\,dim}\ A_A < \infty$, *then* $\mathrm{inj\,dim}\ {}_BB = \mathrm{inj\,dim}\ B_B < \infty$.

Throughout the rest of this section, we assume $\mathrm{Ext}^i_R(A,R)=0$ for $i\neq c$. We set $\Omega=\mathrm{Ext}^c_R(A,R)$ and

$$\nu = -\otimes^{\mathbf{L}}_A \Omega : \mathcal{D}^-(\text{mod-}A)\to\mathcal{D}^-(\text{mod-}A).$$

We denote by $\mathcal{S}$ the full subcategory of $\mathcal{D}^-(\text{Mod-}A)$ consisting of complexes $X^\bullet$ with $\mathrm{Hom}_{\mathcal{D}(\text{Mod-}A)}(P^\bullet, X^\bullet[i])=0$ for $i\neq 0$. In the following, we define $\mathrm{add}(P^\bullet)$ as a full subcategory of $\mathcal{D}^-(\text{Mod-}A)$. However, the canonical functor $\mathcal{K}(\text{Mod-}A)\to\mathcal{D}(\text{Mod-}A)$ induces an equivalence between $\mathrm{add}(P^\bullet)$ defined in $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ and $\mathrm{add}(P^\bullet)$ defined in $\mathcal{D}^-(\text{Mod-}A)$ (cf. [13, Remark 1.7]).

Remark 20. Assume $R_\mathfrak{p}$ is Gorenstein for all $\mathfrak{p}\in\mathrm{Supp}_R(A)$ and $\mathrm{add}(\Omega)=\mathcal{P}_A$ in Mod-A. Then by Proposition 11 we have a self-equivalence $\nu:\mathcal{P}_A\xrightarrow{\sim}\mathcal{P}_A$.

Theorem 21. *The following hold.*

(1) $\mathrm{Ext}^i_R(B,R)=0$ *for* $i\neq c$ *if and only if* $\nu P^\bullet\in\mathcal{S}$.
(2) *Assume* $\nu P^\bullet\in\mathcal{S}$. *Then* $\mathrm{Ext}^c_R(B,R)$ *is a projective generator in* Mod-B *if and only if* $\mathrm{add}(\nu P^\bullet)=\mathrm{add}(P^\bullet)$.
(3) *If* $\Omega\cong A$ *in* Mod-A^{e}, *then* $\mathrm{Ext}^i_R(B,R)=0$ *for* $i\neq c$ *and* $\mathrm{Ext}^c_R(B,R)\cong B$ *in* Mod-B^{e}.

Corollary 22. *Assume A is a Gorenstein R-algebra of codimension c. Then B is a Gorenstein R-algebra of codimension c if and only if* $\mathrm{add}(\nu P^\bullet)=\mathrm{add}(P^\bullet)$.

Corollary 23. *The following are equivalent.*

(1) *A is a Gorenstein R-algebra of codimension c with* $\mathrm{Ext}^c_R(A,R)\cong A$ *in* Mod-A^{e}.
(2) *B is a Gorenstein R-algebra of codimension c with* $\mathrm{Ext}^c_R(B,R)\cong B$ *in* Mod-B^{e}.

Example 24. Assume R is a Gorenstein ring containing an R-regular sequence $x_1,\cdots,x_c,x$. Set $S=R/(x_1,\cdots,x_c)$ with $(x_1,\cdots,x_c)$ the ideal of R generated by $x_1,\cdots,x_c$ and define Noether R-algebras A,B as follows:

$$A=\begin{pmatrix} S & S\\ xS & S\end{pmatrix}\quad\text{and}\quad B=\begin{pmatrix} S & S/xS\\ 0 & S/xS\end{pmatrix}.$$

In [2, Example 4.7], we have constructed a tilting complex $P^\bullet\in\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ such that $B\cong\mathrm{End}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet)$. Also, we have seen that A is a Gorenstein S-algebra of codimension 0. Thus A is a Gorenstein R-algebra of codimension c. On the other hand, $\mathrm{Ext}^i_R(B,R)\neq 0$ for $i=c$ and $c+1$, so that $\nu P^\bullet\notin\mathcal{S}$.

Consider the case where A is a Gorenstein R-algebra of codimension c and $\mathrm{Ext}^i_R(B,R)=0$ for $i\neq c$. At present, we do not know whether or not B is a Gorenstein R-algebra of codimension c. The example above does not tell us anything about this question.

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Institute of Mathematics
University of Tsukuba
Ibaraki 305-8571 JAPAN
E-mail address: abeh@math.tsukuba.ac.jp

Institute of Mathematics
University of Tsukuba
Ibaraki 305-8571 JAPAN
E-mail address: hoshino@math.tsukuba.ac.jp

AUSLANDER-REITEN CONJECTURE ON GORENSTEIN RINGS

TOKUJI ARAYA

ABSTRACT. The Auslander-Reiten conjecture is related closely to the Nakayama conjecture. In this lecture, we consider the Auslander-Reiten conjecture for a Gorenstein rings.

1. INTRODUCTION

The Nakayama's 1958 conjecture (NC) is a one of most famous and important conjecture in ring theory.

(NC) Let $0 \to {}_\Lambda\Lambda \to I^0 \to I^1 \to \cdots$ be a minimal injective resolution of an artin algebra Λ. If all I^j are projective, then Λ is self-injective.

Auslander and Reiten conjectured the generalized Nakayama conjecture (GNC) in [3]

(GNC) Let $0 \to {}_\Lambda\Lambda \to I^0 \to I^1 \to \cdots$ be a minimal injective resolution of an Artin algebra Λ. For any indecomposable injective Λ-module I, I is a direct summand of some I^j.

They showed that (GNC) holds for all artin algebras if and only if the following conjecture (ARC') holds for all artin algebras.

(ARC') For an Artin algebra Λ, if M is a finitely generated Λ-module and $\mathrm{Ext}^i_\Lambda(M, M \oplus \Lambda) = 0$ $(\forall i > 0)$, then M is projective.

M. Auslander, S. Ding, and Ø. Solberg widened the context to algebras over commutative local rings [2].

(ARC) For a commutative Noetherian local ring R, if M is a finitely generated R-module and $\mathrm{Ext}^i_R(M, M \oplus R) = 0$ $(\forall i > 0)$, then M is free.

They showed in [2] that if R is a complete intersection, then R satisfies (ARC). We shall show the following main theorem.

Theorem 1. *Let R be a Gorenstein ring. If R_p satisfies* (ARC) *for all $p \in \mathrm{Spec}R$ with* $\mathrm{ht}\, p \leq 1$, *then R_p satisfies* (ARC) *for all $p \in \mathrm{Spec}R$.*

The detailed version of this paper will be submitted for publication elsewhere.

2. Main Results

Through in this paper, we denote by R the d-dimensional commutative Gorenstein local ring with the unique maximal ideal $\mathfrak{m}$. We also denote by $\operatorname{mod} R$ the category of finitely generated R-modules and by $\operatorname{CM} R$ the full subcategory of $\operatorname{mod} R$ consisting of all maximal Cohen-Macaulay modules.

We give a following condition to consider the Auslander-Reiten conjecture.

(ARC) For $M \in \operatorname{mod} R$, suppose $\operatorname{Ext}^i_R(M, M \oplus R) = 0$ $(i > 0)$, then M is free.

The main theorem of this paper is following;

Theorem 1. *If R_p satisfies* (ARC) *for all $p \in \operatorname{Spec} R$ with $\operatorname{ht} p \leq 1$, then R_p satisfies* (ARC) *for all $p \in \operatorname{Spec} R$.*

It is difficult to check the freeness of modules in general. We give a following theorem to check the freeness of vector bundles.

Theorem 2. *We assume $\dim R = d \geq 2$. Let $M \in \operatorname{CM} R$ be a vector bundle. Suppose $\operatorname{Ext}^{d-1}_R(M, M) = 0$, then M is free.*

We say M is a *vector bundle* if M_p is a free R_p-module for all prime ideal p which is not maximal ideal $\mathfrak{m}$. We want to omit the assumption M is a vector bundle in Theorem 2. But there is a counterexample if M is not a vector bundle.

Example 3. Let k be a field. We set $R = k[x, y, z]/(xy)$ be a 2-dimensional hypersurface and $M = R/(x)$. In this case, we can check that $\operatorname{Ext}^i_R(M, M) = 0$ if and only if i is odd. In particular, we see that $\operatorname{Ext}^{2-1}_R(M, M) = 0$ even if M is not free.

We prepare a lemma to show Theorem 2.

Lemma 4. [9, Lemma 3.10.] *Let R be a d-dimensional Cohen-Macaulay local ring and ω be a canonical module. We denote by $(-)^\vee$ the canonical dual $\operatorname{Hom}_R(-, \omega)$. For vector bundles M and $N \in \operatorname{CM} R$, we have a following isomorphism;*

$$\operatorname{Ext}^d_R(\underline{\operatorname{Hom}}(N, M), \omega) \cong \operatorname{Ext}^{d+1}_R(M, (\operatorname{tr} N)^\vee)$$

Here, $\underline{\operatorname{Hom}}(N, M)$ is the set of stable homomorphisms.

Proof of Theorem 2. Let $M \in \operatorname{CM} R$ be a vector bundle and we assume $\operatorname{Ext}^{d-1}_R(M, M) = 0$. We take a minimal free resolution of M;

$$F_\bullet : \cdots \to F_1 \to F_0 \to M \to 0.$$

Apply $(-)^* := \operatorname{Hom}_R(-, R)$, we get exact sequence;

$$0 \to M^* \to F_0^* \to F_1^* \to \operatorname{tr} M \to 0.$$

Since R is Gorenstein and M is maximal Cohen-Macaulay, we have $\Omega^2 M \cong (\operatorname{tr} M)^*(\cong (\operatorname{tr} M)^\vee)$. Therefore, we have

$$\begin{aligned}\operatorname{Ext}_R^{d+1}(M,(\operatorname{tr}N)^\vee) &\cong \operatorname{Ext}_R^{d+1}(M,(\operatorname{tr}N)^*)\\ &\cong \operatorname{Ext}_R^{d+1}(M,\Omega^2 M)\\ &\cong \operatorname{Ext}_R^{d-1}(M,M)=0.\end{aligned}$$

Since M is vector bundle,

$$\underline{\operatorname{Hom}}_R(M,M)_p \cong \underline{\operatorname{Hom}}_{R_p}(M_p,M_p)=0\ (\forall p\neq \mathfrak{m}).$$

Thus we have $\underline{\operatorname{Hom}}_R(M,M)$ has finite length and we have

$$\begin{aligned}\underline{\operatorname{Hom}}_R(M,M) &\cong \operatorname{Ext}_R^d(\operatorname{Ext}_R^d(\underline{\operatorname{Hom}}_R(M,M),R),R)\\ &\cong \operatorname{Ext}_R^d(\operatorname{Ext}_R^{d+1}(M,(\operatorname{tr}M)^\vee),R)=0\end{aligned}$$

Thus we get M is free. □

Proof of Theorem 1. We put $\mathfrak{P} := \{\, p\in \operatorname{Spec}R \mid R_p$ does not satisfy (ARC) $\}$ and assume $\mathfrak{P}\neq\phi$. Let q be a minimal element in $\mathfrak{P}$ and replace R with R_q. By the minimalty, R is a $d(\geq 2)$-dimrnsional Gorenstein local ring which does not satisfy (ARC) but R_p satisfy (ARC) for all prime $p\neq \mathfrak{m}$. There exists $M\in \operatorname{mod}R$ s.t. $\operatorname{Ext}_R^i(M,M\oplus R)=0\ (\forall i>0)$ but M is not free. Since $\operatorname{Ext}_R^i(M,R)=0\ (i>0)$, M is maximal Cohen-Macaulay. For any $p\neq\mathfrak{m}$, $\operatorname{Ext}_{R_p}^i(M_p,M_p\oplus R_p)=0\ (\forall i>0)$ and R_p satisfies (ARC), we have M_p is a free R_p-module. Thus we get M is vector bundle. Furthermore, $\operatorname{Ext}_R^{d-1}(M,M)=0$ implies M is free. ($\because$Theorem 2.) Therefore we get contradiction and we have $\mathfrak{P}=\phi$. □

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Nara University of Education

E-mail address: `araya@math.okayama-u.ac.jp`

ON COLOCAL PAIRS

YOSHITOMO BABA

ABSTRACT. In [9, Theorem 3.1] K. R. Fuller characterized indecomposable injective projective modules over artinian rings using i-pairs. In [3] the author generalized this theorem to indecomposable projective quasi-injective modules and indecomposable quasi-projective injective modules over artiniain rings. In [2] the author and K. Oshiro studied the above Fuller's theorem minutely. Further in [12] M. Hoshino and T. Sumioka extended these results to perfect rings. In this paper we studies the results in [3] from the point of view of [2], [12].

1. ON FULLER'S THEOREM AND PAST RESULTS

Throughout this paper, we let R be a semiperfect ring. By M_R (resp. ${}_RM$) we stress that M is a unitary right (resp. left) R-module. For an R-module M, we denote the injective hull, the Jacobson radical, the socle, the top $M/J(M)$, and the composition length of M by $E(M)$, $J(M)$, $S(M)$, $T(M)$, and $|M|$, respectively. Further we denote the right (resp. left) annihilator of T in S by $r_S(T)$ (resp. $l_S(T)$).

Definition 1. Let M, N be R-modules. We say that M is N-*injective* if, for any submodule X of N and any R-homomorphism $\varphi : X \to M$, there exists $\tilde{\varphi} : N \to M$ with $\tilde{\varphi}|_X = \varphi$. And we say that M is N-*simple-injective* if, for any submodule X of N and any R-homomorphism $\varphi : X \to M$ with $\mathrm{Im}\varphi$ simple, there exists $\tilde{\varphi} : N \to M$ with $\tilde{\varphi}|_X = \varphi$.

Definition 2. Let e, f be primitive idempotents in R and let g be an idempotent in R. We say that R satisfies $\alpha_r[e, g, f]$ if $r_{gRf}l_{eRg}(X) = X$ for any right fRf-submodule X of gRf with $r_{gRf}(eRg) \subseteq X$. And we say that R satisfies $\alpha_l[e, g, f]$ if $l_{eRg}r_{gRf}(Y) = Y$ for any left eRe-submodule Y of eRg with $l_{eRg}(gRf) \subseteq Y$. Further we say that (eR, Rf) is an *injective pair* (abbreviated *i-pair*) if $S(eR_R) \cong T(fR_R)$ and $S({}_RRf) \cong T({}_RRe)$.

The following theorem is given by K. R. Fuller in [9]. By this theorem, indecomposable projective injective right R-modules over right artinian rings are characterized using i-pairs.

Theorem 3. (**Fuller**) *Let R be a right artinian ring and let e, f be primitive idempotents in R. Then the following are equivalent.*

The detailed version of this paper will be submitted for publication elsewhere.

(a) eR_R *is injective with* $S(eR_R) \cong T(fR_R)$.
(b) (eR, Rf) *is an i-pair.*
(c) R *satisfies* $\alpha_r[e,1,f]$ *and* $\alpha_l[e,1,f]$.

In [2] Theorem 3 is minutely studied by the author and K. Oshiro over semiprimary rings as follows.

Theorem 4. (**Baba, Oshiro**) *Let R be a semiprimary ring and let e, f be primitive idempotents in R.*

(I) *The following are equivalent.*
   (a) eR_R *is injective.*
   (b) (i) *There exists a primitive idempotent f in R with (eR, Rf) an i-pair.*
      (ii) R *satisfies* $\alpha_r[e,1,f]$.
(II) *Suppose that (eR, Rf) is an i-pair.*
   (1) *If ACC holds on right annihilator ideals, then*
      (i) $\alpha_r[e,1,f]$ *holds,*
      (ii) *the equivalent conditions in the following* (2) *hold.*
   (2) *The following are equivalent.*
      (a) $|{}_{eRe}eR| < \infty$.
      (b) $|Rf_{fRf}| < \infty$.
      (c) *Both* eR_R *and* ${}_RRf$ *are injective.*

Theorem 4 is further considered over perfect rings by M. Hoshino and T. Sumioka in [12]. And the following theorem is given.

Theorem 5. (**Hoshino, Sumioka**) *Let R be a left perfect ring and let e, f be primitive idempotents in R.*

(I) *The following are equivalent.*
   (a) eR_R *is R-simple-injective.*
   (b) *There exists a primitive idempotent f in R such that*
      (i) (eR, Rf) *is an i-pair.*
      (ii) R *satisfies* $\alpha_r[e,1,f]$.
(II) *Suppose that (eR, Rf) is an i-pair. Then the following are equivalent.*
   (a) $|{}_{eRe}eR| < \infty$.
   (b) $|Rf_{fRf}| < \infty$.
   (c) *Both* eR_R *and* ${}_RRf$ *are injective.*

On the other hand, in [3] the author generalized Theorem 3 to indecomposable projective quasi-injective modules and indecomposable quasi-projective injective modules over artiniain rings as follows.

Theorem 6. (**Baba**) *Let R be a semiprimary ring and let e, f be primitive idempotents in R. Suppose that DCC holds on $\{r_{Rf}(I) \mid {}_{eRe}I \subseteq eR\}$. Then the following are equivalent.*
   (a) eR_R *is quasi-injective with* $S(eR_R) \cong T(fR_R)$.

(b) $E(T({}_RRe))$ *is quasi-projective of the form* ${}_RRf/r_{Rf}(eR)$.
(c) $S(eR_R) \cong T(fR_R)$ *and* $S({}_{eRe}eRf)$ *is simple.*

2. ON COLOCAL PAIRS

Definition 7. Let M be an R-module and let e, f be primitive idempotents in R. We say that M is *colocal* if $S(M)$ is simple and essential in M. And we say that (eR, Rf) is a *colocal pair* (abbreviated *c-pair*) if both eRf_{fRf} and ${}_{eRe}eRf$ are colocal.

First we characterize $\alpha_r[e, g, f]$ using a quasi-projective right R-module $eR/l_{eR}(Rf)_R$ in case that (eR, Rf) is a c-pair.

Proposition 8. *Let e, f be primitive idempotents in R and let g be an idempotent in R. Suppose that (eR, Rf) is a c-pair.*

(1) *Consider the following two conditions:*
 (a) *R satisfies $\alpha_r[e, g, f]$.*
 (b) *$eR/l_{eR}(Rf)_R$ is $gR/r_{gR}(eRg)$-simple-injective.*
 Then (a) $\Rightarrow$ (b) holds. And, if fRf is a right or left perfect ring, then the converse also holds.
(2) *The following are equivalent.*
 (c) *$eR/l_{eR}(Rf)_R$ is $gR/l_{gR}(Rf)$-simple-injective.*
 (d) (i) *The condition (b) holds.*
  (ii) $r_{gRf}(eRg) = 0$.

Definition 9. Let M be an R-module. We say that M is *simple-quasi-injective* if M is M-simple-injective.

Next we give an equivalent condition of a quasi-projective module $eR/l_{eR}(Rf)$ to be simple-quasi-injective. This proposition will give more important successive results.

Theorem 10. *Let R be a left perfect ring and let e, f be primitive idempotents in R with $eRf \neq 0$. The following are equivalent.*

(a) *$eR/l_{eR}(Rf)_R$ is simple-quasi-injective.*
(b) (i) *(eR, Rf) is a c-pair.*
 (ii) *R satisfies $\alpha_r[e, e, f]$.*

As a corollary we have the following interesting result.

Corollary 11. *Let R be a semiprimary ring, let e, f be primitive idempotents in R with $eRf \neq 0$. Suppose that ACC holds on right annihilator ideals. Then the following are equivalent.*

(a) *${}_RRf/r_{Rf}(eR)$ is quasi-injective.*

(b) $eR/l_{eR}(Rf)_R$ *is quasi-injective.*
(c) (eR, Rf) *is a c-pair.*

Next we characterize indecomposable projective simple-quasi-injective modules and indecomposable quasi-projective R-simple-injective modules, which is a generalized result of Theorem 5 (I).

Theorem 12. (1) *Let R be a right perfect ring and let f be a primitive idempotent in R. The following are equivalent.*

(a) ${}_RRf$ *is simple-quasi-injective.*
(b) *There exists a primitive idempotent e in R such that*
  (i) $S({}_RRf) \cong T({}_RRe)$,
  (ii) eRf_{fRf} *is colocal,*
  (iii) R *satisfies* $\alpha_l[e, f, f]$.

(2) *Let R be a left perfect ring and let e, f be primitive idempotents in R. The following are equivalent.*

(a) $eR/l_{eR}(Rf)_R$ *is R-simple-injective.*
(b) (i) $S({}_RRf)$ *is simple essential with* $S({}_RRf) \cong T({}_RRe)$,
  (ii) eRf_{fRf} *is colocal,*
  (iii) R *satisfies* $\alpha_r[e, e, f]$.

Further we generalize Theorem 5 (II) to c-pairs. We note that, in the following theorem, the equivalence between (c) and (d) is already given by Hoshino and Sumioka in [13].

Theorem 13. *Let e, f be primitive idempotents in R and let g be an idempotent of R. Suppose that (eR, Rf) is a c-pair and fRf is a left perfect ring. Then the following are equivalent.*

(a) (i) $eR/l_{eR}(Rf)_R$ *is* $gR/r_{gR}(eRg)$*-injective.*
  (ii) ${}_RRf/r_{Rf}(eR)$ *is* $Rg/l_{Rg}(gRf)$*-injective.*
(b) (i) $eR/l_{eR}(Rf)_R$ *is* $gR/r_{gR}(eRg)$*-simple-injective.*
  (ii) ${}_RRf/r_{Rf}(eR)$ *is* $Rg/l_{Rg}(gRf)$*-simple-injective.*
(c) $|(gRf/r_{gRf}(eRg))_{fRf}| < \infty$.
(d) $|{}_{eRe}(eRg/l_{eRg}(gRf))| < \infty$.
(e) *ACC holds on* $\{ r_{gRf}(I) \mid {}_{eRe}I \subseteq eRg \}$
  ($\Leftrightarrow$ *DCC holds on* $\{ l_{eRg}(I') \mid I'_{fRf} \subseteq gRf \}$).

As a corollary we obtain the following corollary. We note that, in the following theorem, the equivalence between (c) and (d) is already given by Hoshino and Sumioka in [13].

Corollary 14. *Let e, f be primitive idempotents in R. Suppose that (eR, Rf) is a c-pair and fRf is a left perfect ring. Then the following are equivalent.*

(a) *Both* $eR/l_{eR}(Rf)_R$ *and* ${}_RRf/r_{Rf}(eR)$ *are injective.*
(b) *Both* $eR/l_{eR}(Rf)_R$ *and* ${}_RRf/r_{Rf}(eR)$ *are R-simple-injective.*
(c) $|\, Rf/r_{Rf}(eR)_{fRf} \,| < \infty$.
(d) $|\, {}_{eRe}eR/l_{eR}(Rf) \,| < \infty$.

(e) ACC *holds on* $\{\, r_{Rf}(I) \mid {}_{eRe}I \subseteq eR \,\}$.

Last we give another corollary.

Corollary 15. *Let e, f be primitive idempotents in R. Suppose that (eR, Rf) is an i-pair and fRf is a left perfect ring. Then the following are equivalent.*

(a) *Both eR_R and ${}_RRf$ are injective.*
(b) *Both eR_R and ${}_RRf$ are R-simple-injective.*
(c) $|\, Rf_{fRf} \,| < \infty$.
(d) $|\, {}_{eRe}eR \,| < \infty$.
(e) ACC *holds on* $\{\, r_{Rf}(I) \mid {}_{eRe}I \subseteq eR \,\}$.

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Department of Mathematics
Osaka Kyoiku University
Kashiwara, Osaka 582-8582 JAPAN
E-mail address: ybaba@cc.osaka-kyoiku.ac.jp

ON RINGS ALL OF WHOSE IDEALS ARE N-PRIMARY

Yasuyuki Hirano and Hisaya Tsutsui

Abstract: *Let k be a positive integer. The structure of rings all of whose ideals are n-primary for some positive integer $n \leq k$ is studied and several examples of such rings are constructed. Rings all of whose nonzero ideals are n-primary for some positive integer $n \leq k$ is also considered.*

Throughout this paper, we assume that a ring R is associative with an identity element but not necessarily commutative.

__Definition.__ An ideal P of a ring R will be called right k-primary if there exists an integer $k \geq 1$ minimum with respect to the following condition: for any ideals I, J of R, $IJ \subseteq P$ implies $I \subseteq P$, or $J^k \subseteq P$. An ideal P of a ring R will be called left k-primary if there exists an integer $k \geq 1$ minimum with respect to the following condition: for any ideals I, J of R, $IJ \subseteq P$ implies $J \subseteq P$, or $I^k \subseteq P$. A ring R will be called right (left) k-primary if 0 is a right (left) k-primary ideal. A ring R will be called fully right (left) k-primary if there exists an integer $k \geq 1$ minimum with respect to the following property: every ideal of R is right (left) n-primary for some positive integer $n \leq k$. A fully right and left k-primary ring will be called a fully k-primary ring.

The properties of commutative rings in which all ideals are primary were studied by __Satyanarayana [9]__ and __Chaudhuri [3]__. Let R be a commutative Noetherian ring. An ideal A of R is called irreducible if there are no ideals B, C properly containing A such that $A = B \cap C$. It is well-known that an irreducible ideal is primary. Hence if the set of ideals is linearly ordered, then every ideal I of R is primary. However, we should not that a ring in which every ideal is primary is not necessarily a fully k-primary ring. The formal power series ring $R = Z_2[[x]]$ is an example of a ring in which every ideal is primary but not a fully k-primary ring.

A commutative fully 1-primary ring is a field. The ring Z_n of integer modulo n is fully k-primary if and only if $n = p^k$ for some prime p. If a ring R has a unique maximal ideal M and $M^t = 0$ for some integer $t \geq 1$, then it is clear that R is fully k-primary for some integer $k \leq t$, and M is the only prime ideal in R. A fully right(left) k-primary ring has a unique maximal ideal but in general the maximal ideal of a fully

[1] The detailed version of this paper has been submitted elsewhere.

right(left) k-primary ring does not have to be nilpotent (Example 1). However, if the center of a ring R is not a field, then R is fully right (left) k-primary if and only if $M^k = 0$ where M is the unique maximal ideal of R (Theorem 2). A commutative ring R is fully k-primary if and only if R has a unique maximal ideal M and $M^k = 0$. (Theorem 3). This result can be extended to PI- rings (Theorem 4) and FBN rings (Theorem 5). We will show a necessary and sufficient condition for a ring to be fully right (left) k-primary (Theorem1). Using this condition, one can show that if a ring R is a fully right (left) k-primary ring, so are any n by n matrix rings over R, and eRe for any idempotent element e in R. Hence, fully right (left) primary is a Morita invariant property. Among other observations, we will point out that if a ring R is not n-primary but all nonzero ideals of R are n-primary for some positive integer $n \le k$, then R has either one or two minimal nonzero ideals.

Recently, Gorton-Heatherly [7] investigated some characteristics of a k-primary rings and ideals. They started their paper by showing any powers of a maximal ideal of a ring are right and left k-primary for some integer k, while not all right k- primary ideal is left primary. We begin our paper by a few examples that show a fully left (right) k-primary ring is not necessarily a fully right (left) k-primary ring. These examples also show that the maximal ideal of a fully left (right) k-primary ring is not necessarily nilpotent.

Example 1. Let $S = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \middle| a, b \in F \right\}$, a subring (without identity) of $M_{2\times 2}(F)$ where F is a field. Let $P = \left\{ \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \middle| a \in F \right\}$, the only nonzero proper ideal of S. Consider $R = \{(s,t) \mid s \in S, t \in F\}$ with component-wise addition and the multiplication defined by $(s_1,t_1)(s_2,t_2) = (s_1s_2 + s_1t_2 + s_2t_1,\ t_1t_2)$. $\left(S \times F \to S \text{ is defined by} \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \cdot t = \begin{bmatrix} at & bt \\ 0 & 0 \end{bmatrix} . \right)$
Then R is a ring (with identity) and it has two nonzero proper ideals $M = \{(s,0) \mid s \in S\}$ and $I = \{(p,0) \mid p \in P\}$. Since $M^2 = M$, $I^2 = 0$, $MI = I$, and $IM = 0$, R is a fully left 2-primary ring but not a fully right 2-primary ring. If we use $S' = \left\{ \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix} \middle| a, b \in F \right\}$ for the same construction, we can obtain a right fully 2-primary ring that is not fully left 2-primary. ☐

If we slightly modify the example above, we can obtain other examples of our interest. The following is an example of a ring that is not fully 2-primary, not fully right k-primary for any k, but it is fully left 3-primary.

Example 2. Let $S=\left\{\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \middle| a,b\in T\right\}$, a subring (without identity) of $M_{2\times 2}(T)$ where T is a ring with unique nonzero proper ideal P and $P^2=0$ (e.g., Z_4).

Consider $R=\{(s,r)\mid s\in S, r\in Z_2\}$ with component-wise addition and the multiplication defined by $(s_1,t_1)(s_2,t_2)=(s_1s_2+s_1t_2+s_2t_1,\ t_1t_2)$. (Give Z_2-module structure on S by the obvious manner.)

Let $I_0=\{(s,0)\mid s\in S\}$, $I_1=\{(p,0)\mid p\in P_1\}$ where $P_1=\left\{\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \middle| a\in P,\ b\in T\right\}$, $I_2=\{(p,0)\mid p\in P_2\}$ where $P_2=\left\{\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \middle| a,b\in P\right\}$, $I_3=\{(p,0)\mid p\in P_3\}$ where $P_3=\left\{\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \middle| b\in T\right\}$, and $I_4=\{(p,0)\mid p\in P_4\}$ where $P_4=\left\{\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \middle| b\in P\right\}$.

Since $I_4I_1=0$ and I_1 is idempotent, R is not a right fully k-primary ring for any integer k. Since $I_1I_t=I_t$ for $t=1,2,3,4$, $I_2I_4=0$, $I_2^2=I_3\neq 0$, and $I_2^3=I_3^2=I_4^2=0$, R is not fully left 2-primary but a fully left 3-primary ring. □

A right ideal I of a ring R is called eventually idempotent if there is a natural number n, in general depending on I such that $I^n=I^{n+1}$. Clark [4] studied the structure of rings in which every right ideal is eventually idempotent (called eventually idempotent rings.) If, for a ring R, there is a natural number n such that $I^n=I^{n+1}$ for all ideals I of R, then the least such n is called the idempotent bound for R. By Proposition 1 of Clark [4], Proposition 1 below, and Example 3 below, the class of eventually idempotent rings with idempotent bound k strictly contains the class of fully right k-primary rings.

Proposition 1. If a ring R is fully right k-primary then $I^k=I^{k+1}$ for any ideal I.

Example 3.

(A) Let $R=\{(a,b)\mid a,b\in F\}$ where F is a field, with component-wise addition and multiplication. Then R is an eventually idempotent ring with idempotent bound 1, known in literature as a fully idempotent, but R is not a fully right (left) k-primary ring for any positive integer k. □

(B) Let $R=\left\{\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \middle| a,b,c\in F\right\}$, where F is a field. Then R has three nonzero proper ideals $I_1=\left\{\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \middle| a\in F\right\}$, $I_2=\left\{\begin{bmatrix} b & a \\ 0 & 0 \end{bmatrix} \middle| a,b\in F\right\}$, and $I_3=\left\{\begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} \middle| a,b\in F\right\}$.
Since I_2 and I_3 are idempotent and $I_1^2=0$, R is eventually idempotent with idempotent bound 2. However, since I_2 and I_3 are idempotent and $I_3I_2=0$, R is not fully right k-primary for any positive integer k. □

Notice that the ring in Example 3 has two maximal ideals. It is an immediate consequence of the next proposition that a fully right k-primary ring has a unique maximal ideal.

Proposition 2. If a ring R is fully right k-primary then for any ideals I, J of R, one of the following conditions must hold:

1. $I\subseteq J$
2. $J\subseteq I$
3. $I^k=J^k$

Corollary 1. The set of prime ideals in a fully right k-primary ring is linearly ordered.

By Example 2 and 3, we see that the conditions given in Proposition 2 are not sufficient for a ring to be fully right (left) k-primary.

Theorem 1. A ring R is fully right k-primary if and only if R is eventually idempotent with idempotent bound k, and for any ideals I and J of R, $I=IJ$, $J=JI$, or $I^k=J^k$. A ring S is fully left k-primary if and only S is eventually idempotent with idempotent bound k, and for any ideals I and J of S, $J=IJ$, $I=JI$, or $I^k=J^k$.

Theorem 2. Let R be a ring whose center $Z(R)$ is not a field. Then R is a fully right k-primary ring if and only if R has a unique maximal ideal M and $M^k=0$.

Theorem 3. Let R be a commutative-ring. Then R is fully k-primary ring if and only if R has a unique maximal ideal M and $M^k=0$.

As a natural generalization of commutative rings, we consider rings that satisfy a polynomial identity.

Theorem 4. Let R be a PI-ring. Then the following are equivalent:

(1) R is a fully right k-primary ring.
(2) R is a fully left k-primary ring.
(3) R has a unique maximal ideal M and $M^k = 0$.

Recall that a PI-ring is fully right and left fully bounded.

Theorem 5: Let R be a FBN (fully right bounded right Noetherian) ring. Then the following are equivalent:

(1) R is a fully right k-primary ring.
(2) R is a fully left k-primary ring.
(3) R has a unique maximal ideal M and $M^k = 0$.

We now consider a subring S of a fully k-primary ring R that might help in studying the structure of R.

Lemma 1: Let R be a fully k-primary ring with idempotent maximal ideal M. Let L be an ideal of M when we consider M as a ring without identity. Then L is an ideal of R.

An ideal of a maximal ideal of a fully k-primary ring R (when I is considered as a ring without identity) is in general, not an ideal of R as the following examples shows.

Example 4. Let $R = \{(r_1, r_2) \mid r_1 \in P, r_2 \in T\}$ where T is a ring with unique nonzero proper ideal P and $P^2 = 0$; with component-wise addition and multiplication defined by $(a, b)(c, d) = (ac, ad + bc)$. Then R is a fully 2- primary ring whose only nonzero proper ideals is the maximal ideal $M = 0 \oplus \mathbb{R}$. Let $I = 0 \oplus \mathbb{Q}(\sqrt{2})$. Then I is an ideal of M when M is considered as a ring without identity but not an ideal of R. □

Lemma 1 above yields the following Theorem.

Theorem 6. Let R be a fully k- primary ring whose maximal ideal M is idempotent. Let $Z(R)$ be the center of R. Then $S = M + Z(R)$ is a fully k- primary ring. Further R and S have the same set of proper ideals.

Theorem 7. Let R and $S = M + Z(R)$ be as stated in Theorem 4. Then

(1) R is semiprime if and only if S is semiprime
(2) R is prime if and only if S is prime
(3) S is semiprimitive if and only if R is semiprimitive.
(4) S is right primitive if and only if R is right primitive.
(5) S is right Artinian if and only if R is right Artinian..

Definition. A ring R will be called almost fully right (left) k-primary if there exists an integer $k \geq 1$ minimum with respect to the following property: every nonzero ideal of R is right (left) n-primary for some positive integer $n \leq k$. A almost fully right and left k-primary ring will be called an almost k-primary ring.

Theorem 8. An almost fully right k-primary ring R has one or two maximal ideals. Further, R has two maximal ideals if and only if R is a direct sum of two simple rings.

Theorem 9. An almost fully right k-primary ring R that is not fully right k-primary has one or two minimal non-zero ideals. If R has exactly one minimal nonzero ideal, then every nonzero ideal contains the minimal ideal. Further, if R has two minimal ideals, then R is a direct sum of two fully right k-primary rings.

Note that a direct sum of two fully right k-primary rings is not necessarily almost fully right k-primary. For example, Z_4 is a fully 2-primary ring but $Z_4 \oplus Z_4$ contains nonzero ideals that are not 2-primary.

Theorem 10. If a ring R has a unique minimal nonzero ideal L and $J(R) = 0$, then R is almost fully right k-primary if and only if R is fully right k-primary R.

If an almost fully right k-primary ring R has two maximal ideals M_1 and M_2, then $M_1 \cap M_2 = 0$. Thus, if R has a unique minimal nonzero ideal, then since every nonzero ideal contains the minimal ideal, R must have a unique maximal ideal. The example below shows that the converse of the statement is false.

Example 5. Let R be a simple domain but not a division ring, and let $0 \neq a \in R$. Consider $S = \{(r_1, r_2) \mid r_1 \in aR, r_2 \in aR + Z(R)\}$ with component-wise addition and multiplication defined by $(a, b)(c, d) = (ac, ad + bc + bd)$. Let $I_1 = \{(r,s) \mid r,s \in aR\}, I_2 = \{(r,0) \mid r \in aR\}$, and $I_3 = \{(r,-r) \mid r \in aR\}$. Then since

I_1, I_2, I_3 are all idempotent but $I_2 \cdot I_3 = 0$, S is an almost 2-fully primary but not a 2-primary ring, with unique maximal ideal I_1 and two minimal ideals I_2 and I_3. $\square$

Theorem 11. A PI ring R is almost fully right k-primary if and only if

(1) R has a unique maximal ideal M and $M^k = 0$,
(2) R is a direct sum of two simple Artinian rings,
(3) R has a unique maximal ideal M and M^k is the unique minimal nonzero ideal of R and every nonzero ideal contains M^k, or
(4) R has a unique maximal ideal M and $M^k = M^{k+1}$ is a minimal nonzero ideal of R and there exists exactly one nonzero ideal that does not contain M^k.

Theorem 12. A FBN ring R is almost fully right k-primary if and only if

(1) R has a unique maximal ideal M and $M^k = 0$,
(2) R is a direct sum of two simple Artinian rings,
(3) R has a unique maximal ideal M and M^k is the unique minimal nonzero ideal of R and every nonzero ideal contains M^k, or
(4) R has a unique maximal ideal M and $M^k = M^{k+1}$ is a minimal nonzero ideal of R and there exists exactly one nonzero ideal that does not contain M^k.

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Department of Mathematics
Naruto University
Tokushima, JAPAN
(yhirano@naruto-u.ac.jp)

Department of Mathematics
ERAU
Prescott, AZ, USA
(hisaya.tsutsui@erau.edu)

A TILED ORDER OF FINITE GLOBAL DIMENSION WITH NO NEAT PRIMITIVE IDEMPOTENT

HISAAKI FUJITA AND AKIRA OSHIMA

Let R be a discrete valuation ring with a unique maximal ideal πR and a quotient field K, and let $F = R/\pi R$ be the residue class field. Let $n \geq 2$ be an integer and $\{\lambda_{ij} \mid 1 \leq i, j \leq n\}$ a set of n^2 integers satisfying

$$\lambda_{ii} = 0, \quad \lambda_{ik} + \lambda_{kj} \geq \lambda_{ij}, \quad \lambda_{ij} + \lambda_{ji} > 0 \quad (\text{if } i \neq j)$$

for all $1 \leq i, j, k \leq n$. Then $\Lambda = (\pi^{\lambda_{ij}} R)$ is a basic semiperfect Noetherian R-subalgebra of the full $n \times n$ matrix algebra $\mathrm{M}_n(K)$. We call such Λ a *tiled R-order* in $\mathrm{M}_n(K)$.

Let S be a semiperfect Noetherian ring and e a primitive idempotent of S. Following Ágoston, Dlab and Wakamatsu [1], we call e a *neat* primitive idempotent if $\mathrm{Ext}^i_S(V, V) = 0$ for all $i \geq 1$, where V is a simple right S-module with $Ve \neq 0$ (see [5], too).

It was proved by Jategaonkar [7] that for a fixed integer $n \geq 2$, there are, up to isomorphism, only finitely many tiled R-orders of finite global dimension in $\mathrm{M}_n(K)$. The literature contains a number of papers concerned with determining tiled R-orders of finite global dimension. Tiled R-orders of global dimension two were studied by Roggenkamp and Wiedemann in connection with the interest of orders of finite lattice type (see [2], [11], [12], [20]). As for the problem to determine the maximum finite global dimension among tiled R-orders in $\mathrm{M}_n(K)$ for a fixed n, some authors studied tiled R-orders having large global dimension, but it is not known what is the maximum (see [4], [5], [6], [7], [8], [9], [14], [17], [18]). In such examples, neat primitive idempotents play an essential role when we compute global dimension inductively. Then in [5], we posed a question "Does any tiled R-order of finite global dimension have a neat primitive idempotent?", which can be considered as an improved version of Jategaonkar's conjecture disproved by Kirkman and Kuzmanovich [9] and [4] for all $n \geq 6$.

We notice that in those studies, almost all known results hold if R is an arbitrary discrete valuation ring. However, among other things, Rump [14] proved that global dimension gl.dim Λ of a tiled R-order $\Lambda = (\pi^{\lambda_{ij}} R)$ is determined by the set $\{\lambda_{ij} \mid 1 \leq i, j \leq n\}$ and char F (characteristic of F), and that if gl.dim $\Lambda \leq 2$ then gl.dim Λ does not depend on char F, by using matroid theory (see Tutte [19]). Moreover, he provided an example of a tiled R-order Λ in $\mathrm{M}_n(K)$ such that gl.dim $\Lambda = 3$ if char $F \neq 2$, and gl.dim $\Lambda = 4$ if char $F = 2$, where $n = 14$. In accordance with matroid theory, Rump calls a tiled R-order *regular* if its global dimension does not depend on char F, and he added the following sentence: "For the present, at least, we have demonstrated that the problem to determine the tiled orders of finite global dimension can hardly be solved without a careful inspection of regularity."

In this report, we announce a new example of non-regular tiled R-orders. Namely, for an arbitrary prime p, we construct a tiled R-order Λ in $\mathrm{M}_n(K)$ such that gl.dim $\Lambda = 5$ if char $F \neq p$ and gl.dim $\Lambda = \infty$ if char $F = p$, where $n = 4p + 5$. Moreover, in the

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computation of gl.dim Λ, we see that Λ has no neat primitive idempotent. Thus, if char $F \neq p$, Λ is a counterexample to the question mentioned above.

1. Example

Let $(\mathcal{P}, \leq)$ be a finite poset. We can consider $\mathcal{P}$ a finite quiver $\mathcal{P} = (\mathcal{P}_0, \mathcal{P}_1)$ as follows. $\mathcal{P}_0$ is the set of vertices in $\mathcal{P}$, that is, the set $\mathcal{P}$ itself. $\mathcal{P}_1$ is the set of arrows of $\mathcal{P}$ defined by $a \to b \in \mathcal{P}_1$ provided $a < b$ and there is no $x \in \mathcal{P}_0$ with $a < x < b$. Note that the order of $\mathcal{P}$ is generated by $\mathcal{P}_1$. That is, for $a, b \in \Omega$, $a < b$ if and only if there is a path from a to b in the quiver $\mathcal{P}$.

From a given finite poset $\mathcal{P}$ with n vertices, we can construct a tiled R-order $\Lambda = (\pi^{\lambda_{xy}} R)$ in $\mathrm{M}_n(K)$ by defining $\lambda_{xy} = 0$ if $x \leq y$, and $\lambda_{xy} = 1$ otherwise.

The following is the example of our tiled R-order.

Example. Let p be an arbitrary prime, and put $l := p + 1$. Then we define a finite quiver $\mathcal{P} = (\mathcal{P}_0, \mathcal{P}_1)$ as follows. The set $\mathcal{P}_0$ has the following $4l + 1$ $(= 4p + 5)$ vertices.

$$\mathcal{P}_0 := \{a_i, b_i, c_i, d_i \mid 1 \leq i \leq l\} \cup \{d\}$$

The set $\mathcal{P}_1$ has the following $5l + l^2$ $(= p^2 + 7p + 6)$ arrows.

$$\begin{cases} b_i \to a_i \quad (1 \leq i \leq l) & b_i \to a_{i+1} \quad (1 \leq i \leq l) \\ c_i \to a_i \quad (1 \leq i \leq l) & c_i \to a_{i+1} \quad (1 \leq i \leq l) \\ d_i \to c_i \quad (1 \leq i \leq l) & d_i \to b_{i+k} \quad (1 \leq i \leq l,\ 1 \leq k \leq p) \\ d \to c_i \quad (1 \leq i \leq l) & \end{cases}$$

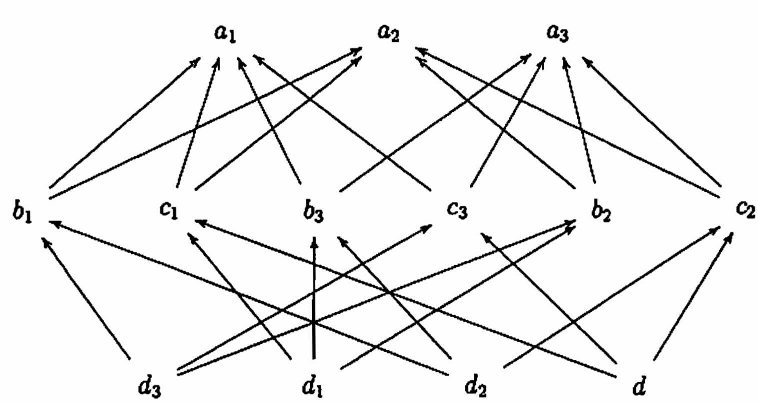
where we consider the indices i of a_i, b_i modulo l. Let Λ be the tiled R-order in $\mathrm{M}_n(K)$ corresponding to $\mathcal{P}$, where $n = 4p + 5$. Then

$$\text{gl.dim}\,\Lambda = \begin{cases} 5 & \text{if} \quad \text{char}\,F \neq p \\ \infty & \text{if} \quad \text{char}\,F = p. \end{cases}$$

Moreover, all primitive idempotents e_i $(1 \leq i \leq n)$ of Λ are not neat.

In the case of $p = 2$, the quiver $\mathcal{P}$ and its tiled R-order Λ in $\mathrm{M}_{13}(K)$ are as follows.

$\mathcal{P}$:



$$
\Lambda := \begin{pmatrix}
R & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} \\
\boldsymbol{\pi} & R & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} \\
\boldsymbol{\pi} & \boldsymbol{\pi} & R & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} \\
R & R & \boldsymbol{\pi} & R & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} \\
R & R & \boldsymbol{\pi} & \boldsymbol{\pi} & R & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} \\
\boldsymbol{\pi} & R & R & \boldsymbol{\pi} & \boldsymbol{\pi} & R & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} \\
\boldsymbol{\pi} & R & R & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & R & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} \\
R & \boldsymbol{\pi} & R & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & R & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} \\
R & \boldsymbol{\pi} & R & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & R & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} \\
R & R & R & R & \boldsymbol{\pi} & R & \boldsymbol{\pi} & \boldsymbol{\pi} & R & R & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} \\
R & R & R & \boldsymbol{\pi} & R & R & \boldsymbol{\pi} & R & \boldsymbol{\pi} & \boldsymbol{\pi} & R & \boldsymbol{\pi} & \boldsymbol{\pi} \\
R & R & R & R & \boldsymbol{\pi} & \boldsymbol{\pi} & R & R & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & R & \boldsymbol{\pi} \\
R & R & R & \boldsymbol{\pi} & R & \boldsymbol{\pi} & R & \boldsymbol{\pi} & R & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & R
\end{pmatrix}
$$

where $\boldsymbol{\pi} = \pi R$.

Let $J(\Lambda)$ be the Jacobson radical of Λ. We compute minimal projective resolutions of $J(\Lambda)e_i$ $(1 \le i \le n)$ by using Rump's theory [14], which is slightly modified in the detailed version of this paper.

The exponent matrix (λ_{ij}) of a tiled R-order $\Lambda = (\pi^{\lambda_{ij}} R)$ defines an infinite poset Ω_Λ (called *σ-poset* in [14] with an automorphism σ of Ω_Λ). If R is the formal power series ring $F[[t]]$ in the indeterminate t, then there is a correspondence between left Λ-lattices and bounded finite dimensional Ω_Λ-representations over F (see Zavadskij and Kirichenko [21], [22], Roggenkamp and Wiedemann [13], de la Peña and Raggi-Cárdenas [3], and Simson [15]). Using this correspondence, in [14], Rump develops an axiomatic theory to compute global dimension of arbitrary tiled R-orders.

REMARK. In [14], Rump provided an example of a non-regular tiled R-order in $\mathrm{M}_{14}(K)$ with $\mathrm{char}\, F = 2$, which is constructed from a finite poset. A similar finite poset can be found in [16]. Oshima [10] extended Rump's example to the case of an arbitrary prime p, that is, he constructed a tiled R-order Λ in $\mathrm{M}_n(K)$ such that $\mathrm{gl.dim}\, \Lambda = 3$ if $\mathrm{char}\, F \ne p$, and $\mathrm{gl.dim}\, \Lambda = 4$ if $\mathrm{char}\, F = p$, where $n = 8p - 2$.

As suggested in [14], it may be an interesting problem to find smaller size n which admits non-regular tiled R-orders. When $p = 2$, our example provides a non-regular tiled R-order in $\mathrm{M}_n(K)$ such that $n = 13 < 14$, at present, that is the minimum among known examples.

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Institute of Mathematics
University of Tsukuba
Tsukuba, Ibaraki 305-8571 JAPAN
E-mail: fujita@math.tsukuba.ac.jp

ON HOCHSCHILD COHOMOLOGY RING OF AN ORDER OF A QUATERNION ALGEBRA

TAKAO HAYAMI

ABSTRACT. We will give an efficient bimodule projective resolution of an order Γ, where Γ is an order of a simple component of the rational group ring $\mathbb{Q}Q_{2^r}$ of the generalized quaternion 2-group Q_{2^r} of order 2^{r+2}. Moreover we will determine the ring structure of the Hochschild cohomology $HH^*(\Gamma)$ by calculating the Yoneda products using this bimodule projective resolution.

1. INTRODUCTION

The cohomology theory of associative algebras was initiated by Hochschild [6], Cartan and Eilenberg [1] and MacLane [7]. Let R be a commutative ring with identity and Λ an R-algebra which is a finitely generated projective R-module. If M is a Λ-bimodule (i.e., a $\Lambda^e = \Lambda \otimes_R \Lambda^{op}$-module), then the nth Hochschild cohomology of Λ with coefficients in M is defined by $H^n(\Lambda, M) := \mathrm{Ext}^n_{\Lambda^e}(\Lambda, M)$. We set $HH^n(\Lambda) = H^n(\Lambda, \Lambda)$. The Yoneda product gives $HH^*(\Lambda) := \bigoplus_{n \geq 0} HH^n(\Lambda)$ a graded ring structure with $1 \in Z\Lambda \simeq HH^0(\Lambda)$ where $Z\Lambda$ denotes the center of Λ. $HH^*(\Lambda)$ is called the Hochschild cohomology ring of Λ. The Hochschild cohomology ring $HH^*(\Lambda)$ is graded-commutative, that is, for $\alpha \in HH^p(\Lambda)$ and $\beta \in HH^q(\Lambda)$ we have $\alpha\beta = (-1)^{pq}\beta\alpha$. The Hochschild cohomology is an important invariant of algebras. However the Hochschild cohomology ring is difficult to compute in general.

Let G be a finite group and e a centrally primitive idempotent of the rational group ring $\mathbb{Q}G$. Then $\mathbb{Q}Ge$ is a central simple algebra over the center K. We set $\Gamma = \mathbb{Z}Ge$. Then Γ is an R-order of $\mathbb{Q}Ge$, where R denotes the ring of integers of K. The author is interested in the Hochschild cohomology ring $HH^*(\Gamma)$ of an R-algebra Γ, which is an invariant of the finite group G and the central idempotent e. On the other hand, a ring homomorphism $\phi : \mathbb{Z}G \to \Gamma; x \mapsto xe$ induces a ring homomorphism $HH^*(\Gamma) \to H^*(G, {}_\psi\Gamma)$, where ${}_\psi\Gamma$ denotes Γ regarded as a G-module by conjugation and $H^*(G, {}_\psi\Gamma)$ denotes the ordinary cohomology ring of G with coefficients in ${}_\psi\Gamma$. In fact, we consider that the study of the ring structure of $H^*(G, {}_\psi\Gamma)$ and the ring homomorphism gives us much helpful information about $HH^*(\Gamma)$. So there are some examples of the ring structure of $H^*(G, {}_\psi\Gamma)$ and the ring homomorphism $HH^*(\Gamma) \to H^*(G, {}_\psi\Gamma)$ ([4], [5]). The Hochschild cohomology ring $HH^*(\Gamma)$ is in general hard to compute, however it is theoretically possible to calculate if an efficient Γ^e-projective resolution is given. In this paper, as an example of it, we will give the ring structure of the Hochschild cohomology $HH^*(\Gamma)$, where Γ is an order of a simple component of the rational group ring of the generalized quaternion 2-group of order 2^{r+2}.

The detailed version of this paper will be submitted for publication elsewhere.

Let G be the generalized quaternion 2-group of order 2^{r+2} for $r \geq 1$:

$$Q_{2^r} = \langle x, y \mid x^{2^{r+1}} = 1, x^{2^r} = y^2, yxy^{-1} = x^{-1} \rangle.$$

We set $e = (1 - x^{2^r})/2 \in \mathbb{Q}G$ and denote xe by ζ, a primitive 2^{r+1}-th root of e. Then e is a centrally primitive idempotent of $\mathbb{Q}G$ and $\mathbb{Q}Ge$ is the (ordinary) quaternion algebra over the field $K := \mathbb{Q}(\zeta + \zeta^{-1})$ with identity e, that is, $\mathbb{Q}Ge = K \oplus Ki \oplus Kj \oplus Kij$ where we set $i = x^{2^{r-1}}e$ and $j = ye$ (see [2, (7.40)]). Note that $i^2 = j^2 = -e, ij = -ji$ hold. In the following we set $R = \mathbb{Z}[\zeta + \zeta^{-1}]$, the ring of integers of K, and we set $\Gamma = \mathbb{Z}Ge = R \oplus R\zeta \oplus Rj \oplus R\zeta j$. Note that R is a commuting parameter ring, because y commutes with $x + x^{-1}$. Then Γ is an R-order of $\mathbb{Q}Ge$. In particular if $r = 1$, $\Gamma = \mathbb{Z}e \oplus \mathbb{Z}i \oplus \mathbb{Z}j \oplus \mathbb{Z}ij$ is just the (ordinary) quaternion algebra over $\mathbb{Z}$ with identity e.

We will give an efficient bimodule projective resolution of Γ, and we will determine the ring structure of the Hochschild cohomology $HH^*(\Gamma)$ by calculating the Yoneda products using this bimodule projective resolution. This paper is a summary of [3].

2. A bimodule projective resolution of Γ

In this section, we state a Γ^e-projective resolution of Γ. For each $q \geq 0$, let Y_q be a direct sum of $q+1$ copies of $\Gamma \otimes \Gamma$. As elements of Y_q, we set

$$c_q^s = \begin{cases} (0, \ldots, 0, \underbrace{e \otimes e}_{s}, 0, \ldots, 0) & (\text{if } 1 \leq s \leq q+1), \\ 0 & (\text{otherwise}). \end{cases}$$

Then we have $Y_q = \bigoplus_{k=1}^{q+1} \Gamma c_q^k \Gamma$. Let $t = 2^r$. Define left Γ^e-homomorphisms $\pi : Y_0 \to \Gamma; c_0^1 \mapsto e$ and $\delta_q : Y_q \to Y_{q-1}$ $(q > 0)$ given by

$$\delta_q(c_q^s) = \begin{cases} -\zeta c_{q-1}^s + c_{q-1}^s \zeta + (-1)^{(q-s)/2} \zeta j c_{q-1}^{s-1} j \zeta - c_{q-1}^{s-1} & \text{for } q \text{ even, } s \text{ even,} \\ \sum_{l=0}^{t-1} \zeta^{t-1-l} c_{q-1}^s \zeta^l + (-1)^{(q-s-1)/2} j c_{q-1}^{s-1} j + c_{q-1}^{s-1} & \text{for } q \text{ even, } s \text{ odd,} \\ -\sum_{l=0}^{t-1} \zeta^{t-1-l} c_{q-1}^s \zeta^l + (-1)^{(q-s-1)/2} j c_{q-1}^{s-1} j - c_{q-1}^{s-1} & \text{for } q \text{ odd, } s \text{ even,} \\ \zeta c_{q-1}^s - c_{q-1}^s \zeta + (-1)^{(q-s)/2} \zeta j c_{q-1}^{s-1} j \zeta + c_{q-1}^{s-1} & \text{for } q \text{ odd, } s \text{ odd.} \end{cases}$$

Theorem 1. *The above (Y, π, δ) is a Γ^e-projective resolution of Γ.*

Proof. By the direct calculations, we have $\pi \cdot \delta_1 = 0$ and $\delta_q \cdot \delta_{q+1} = 0$ $(q \geq 1)$.

To see that the complex (Y, π, δ) is acyclic, we state a contracting homotopy. In general, it suffices to define the homotopy as an abelian group homomorphism. However, we can see that there exists a homotopy as a right Γ-module, which permits us to cut down the number of cases. We define right Γ-homomorphisms $T_{-1} : \Gamma \to Y_0$ and $T_q : Y_q \to Y_{q+1}$ $(q \geq 0)$ as follows:

$$T_{-1}(\gamma) = c_0^1 \gamma \qquad (\text{for } \gamma \in \Gamma).$$

If $q(\geq 0)$ is even, then

$$T_q(\zeta^k c_q^s) = \begin{cases} 0 & (k=0,\ s=1), \\ \displaystyle\sum_{l=0}^{k-1} \zeta^{k-1-l} c_{q+1}^1 \zeta^l & (1 \leq k < t,\ s=1), \\ 0 & (s(\geq 2) \text{ even}), \\ -\zeta^k c_{q+1}^{s+1} & (s(\geq 3) \text{ odd}), \end{cases}$$

$$T_q(\zeta^k j c_q^s) = \begin{cases} (-1)^{q/2} c_{q+1}^2 j & (k=0,\ s=1), \\ (-1)^{q/2} \left(\displaystyle\sum_{l=0}^{k-1} \zeta^{k-1-l} c_{q+1}^1 \zeta^l j + \zeta^k c_{q+1}^2 j \right) & (1 \leq k < t,\ s=1), \\ \zeta^k j c_{q+1}^{s+1} & (s(\geq 2) \text{ even}), \\ 0 & (s(\geq 3) \text{ odd}). \end{cases}$$

If $q(\geq 1)$ is odd, then

$$T_q(\zeta^k c_q^s) = \begin{cases} 0 & (0 \leq k \leq t-2,\ s=1), \\ c_{q+1}^1 & (k=t-1,\ s=1), \\ 0 & (s(\geq 2) \text{ even}), \\ -\zeta^k c_{q+1}^{s+1} & (s(\geq 3) \text{ odd}), \end{cases}$$

$$T_q(\zeta^k j c_q^s) = \begin{cases} (-1)^{(q-1)/2} \left(c_{q+1}^1 j\zeta + \zeta^{t-1} c_{q+1}^2 j \zeta \right) & (k=0,\ s=1), \\ (-1)^{(q+1)/2} \zeta^{k-1} c_{q+1}^2 j \zeta & (1 \leq k < t,\ s=1), \\ \zeta^k j c_{q+1}^{s+1} & (s(\geq 2) \text{ even}), \\ 0 & (s(\geq 3) \text{ odd}). \end{cases}$$

Then by the direct calculations, we have

$$\delta_{q+1} T_q + T_{q-1} \delta_q = \mathrm{id}_{Y_q}$$

for $q \geq 0$. Hence (Y, π, δ) is a Γ^e-projective resolution of Γ. □

3. Hochschild cohomology $HH^*(\Gamma)$

In this section, we will determine the ring structure of the Hochschild cohomology $HH^*(\Gamma)$. This is obtained by using the Γ^e-projective resolution (Y, π, δ) of Γ stated in Theorem 1. In the following we denote a direct sum of q copies of a module M by M^q.

3.1. Module structure. In this subsection, we give the module structure of $HH^*(\Gamma)$.

As elements of Γ^{q+1}, we set

$$\iota_q^s = \begin{cases} (0, \ldots, 0, \overset{s}{\check{e}}, 0, \ldots, 0) & (\text{if } 1 \leq s \leq q+1), \\ 0 & (\text{otherwise}). \end{cases}$$

Then we have $\Gamma^{q+1} = \bigoplus_{k=1}^{q+1} \Gamma \iota_q^k$.

Applying the functor $\mathrm{Hom}_{\Gamma^e}(-,\Gamma)$ to the resolution (Y,π,δ), we have the following complex, where we identify $\mathrm{Hom}_{\Gamma^e}(Y_q,\Gamma)$ with Γ^{q+1} using an isomorphism $\mathrm{Hom}_{\Gamma^e}(Y_q,\Gamma) \to \Gamma^{q+1}; f \mapsto \sum_{k=1}^{q+1} f(c_q^k)\iota_q^k$:

$$(\mathrm{Hom}_{\Gamma^e}(Y,\Gamma),\delta^\#): \quad 0 \to \Gamma \xrightarrow{\delta_1^\#} \Gamma^2 \xrightarrow{\delta_2^\#} \Gamma^3 \xrightarrow{\delta_3^\#} \Gamma^4 \xrightarrow{\delta_4^\#} \Gamma^5 \to \cdots,$$

$$\delta_{q+1}^\#(\gamma\iota_q^s) = \begin{cases} -\sum_{l=0}^{t-1} \zeta^{t-1-l}\gamma\zeta^l\iota_{q+1}^s + ((-1)^{(q-s)/2}\zeta j\gamma j\zeta + \gamma)\iota_{q+1}^{s+1} & \text{for } q \text{ even, } s \text{ even,} \\ (\zeta\gamma - \gamma\zeta)\iota_{q+1}^s + ((-1)^{(q-s-1)/2} j\gamma j - \gamma)\iota_{q+1}^{s+1} & \text{for } q \text{ even, } s \text{ odd,} \\ -(\zeta\gamma - \gamma\zeta)\iota_{q+1}^s + ((-1)^{(q-s-1)/2} j\gamma j + \gamma)\iota_{q+1}^{s+1} & \text{for } q \text{ odd, } s \text{ even,} \\ \sum_{l=0}^{t-1} \zeta^{t-1-l}\gamma\zeta^l\iota_{q+1}^s + ((-1)^{(q-s)/2}\zeta j\gamma j\zeta - \gamma)\iota_{q+1}^{s+1} & \text{for } q \text{ odd, } s \text{ odd.} \end{cases}$$

In the above, note that

$$\gamma\iota_q^s = \begin{cases} (0,\dots,0,\overset{s}{\check{\gamma}},0,\dots,0) & (\text{if } 1 \le s \le q+1), \\ 0 & (\text{otherwise}), \end{cases}$$

for $\gamma \in \Gamma$, and so on. By the direct calculations, we have the following theorem:

Theorem 2. (1) *If $r = 1$, then we have*

$$HH^n(\Gamma) = \begin{cases} \mathbb{Z} & (n = 0), \\ (\mathbb{Z}/2\mathbb{Z})^{2n+1} & (n \ge 1). \end{cases}$$

(2) *If $r \ge 2$, then we have*

$$HH^n(\Gamma) = \begin{cases} R & (n = 0), \\ (R/(\zeta+\zeta^{-1})R)^{2n+1} & (n \text{ odd}), \\ R/2^r R \oplus (R/(\zeta+\zeta^{-1})R)^{2n} & (n (\neq 0) \text{ even}). \end{cases}$$

3.2. Ring structure. Recall the Yoneda product in $HH^*(\Gamma)$. Let $\alpha \in HH^n(\Gamma)$ and $\beta \in HH^m(\Gamma)$, where α and β are represented by cocycles $f_\alpha : Y_n \to \Gamma$ and $f_\beta : Y_m \to \Gamma$, respectively. There exists the commutative diagram of Γ^e-modules:

$$\begin{array}{ccccccccccccc} \cdots & \xrightarrow{\delta_{n+m+1}} & Y_{n+m} & \xrightarrow{\delta_{n+m}} & \cdots & \xrightarrow{\delta_{m+2}} & Y_{m+1} & \xrightarrow{\delta_{m+1}} & Y_m & \xrightarrow{f_\beta} & \Gamma & & \\ & & \downarrow{\scriptstyle \mu_n} & & & & \downarrow{\scriptstyle \mu_1} & & \downarrow{\scriptstyle \mu_0} & & \| & & \\ \cdots & \xrightarrow[\delta_{n+1}]{} & Y_n & \xrightarrow[\delta_n]{} & \cdots & \xrightarrow[\delta_2]{} & Y_1 & \xrightarrow[\delta_1]{} & Y_0 & \xrightarrow[\pi]{} & \Gamma & \longrightarrow & 0, \end{array}$$

where μ_l $(0 \le l \le n)$ are liftings of f_β. We define the product $\alpha \cdot \beta \in HH^{n+m}(\Gamma)$ by the cohomology class of $f_\alpha\mu_n$. This product is independent of the choice of representatives f_α and f_β, and liftings μ_l $(0 \le l \le n)$.

First, we consider the case $r = 1$. Note the Hochschild cohomology ring $HH^*(\Gamma)$ is graded-commutative. From Theorem 2 (1), $HH^*(\Gamma)$ is a commutative ring in this case.

We take generators of $HH^1(\Gamma)$ as follows (see [3, Theorem 2 (1)]):

$$A = \zeta\iota_1^2,\ B = \zeta j\iota_1^1,\ C = j\iota_1^1 + \zeta j\iota_1^2.$$

Then we have $2A = 2B = 2C = 0$. We calculate the Yoneda products. Then $HH^n(\Gamma)$ ($n \geq 2$) is multiplicatively generated by A, B and C, and the equation $A^2+B^2+C^2 = 0$ holds. Moreover the relations are enough. Thus we can determine the ring structure of $HH^*(\Gamma)$ in the case $r = 1$ (see [3, Section 3.1] for details).

Next, we consider the case $r \geq 2$. The computation is similar to the case where $r = 1$, however it is more complicated. By [3, Theorem 2 (2)], we take generators of $HH^1(\Gamma)$ as follows:

$$A = (e - \eta\zeta)\iota_1^2,\ B = (j - \eta\zeta j)\iota_1^1,\ C = (\zeta j - \eta j)\iota_1^1 + (j - \eta\zeta j)\iota_1^2.$$

In the above η denotes $2e/(\zeta + \zeta^{-1}) \in R$ (see also [3, Lemma 2.1]). Then we have $(\zeta+\zeta^{-1})A = (\zeta+\zeta^{-1})B = (\zeta+\zeta^{-1})C = 0$.

Note that products of A, B, C and $X \in HH^n(\Gamma)$ ($n \geq 0$) are commutative, because $HH^*(\Gamma)$ is graded-commutative and the equations $2A = 2B = 2C = 0$ hold. We calculate the Yoneda products. Then the following equations hold in $HH^2(\Gamma)$:

$$A^2 = \iota_2^3,\ AB = j\iota_2^2,\ AC = \zeta j\iota_2^2 - j\iota_2^3,\ B^2 = 2^{r-1}\eta\zeta\iota_2^1 + \zeta\iota_2^2,$$
$$BC = 2^{r-1}\eta(e - \eta\zeta)\iota_2^1,\ C^2 = 2^{r-1}\eta\zeta\iota_2^1 + \zeta\iota_2^2 + \iota_2^3.$$

In particular, generators of $HH^2(\Gamma)$ except $(e - \eta\zeta)\iota_2^1$ are generated by the products of A, B and C, and the equation $A^2 + B^2 + C^2 = 0$ holds.

In the following, we put $D = (e - \eta\zeta)\iota_2^1$ which is a generator of $HH^2(\Gamma)$, and then we have $2^rD = 0$ and $BC = 2^{r-1}\eta D$. Similarly, we calculate the Yoneda products. Then $HH^n(\Gamma)$ ($n \geq 3$) is multiplicatively generated by A, B, C and D, and the relations are enough. Thus we can determine the ring structure of $HH^*(\Gamma)$ in the case $r \geq 2$ (see [3, Section 3.2] for details).

Finally we state the ring structure of the Hochschild cohomology ring $HH^*(\Gamma)$:

Theorem 3. (1) *If $r = 1$, then the Hochschild cohomology ring $HH^*(\Gamma)$ is isomorphic to*

$$\mathbb{Z}[A, B, C]/(2A, 2B, 2C, A^2 + B^2 + C^2),$$

where $\deg A = \deg B = \deg C = 1$.

(2) *If $r \geq 2$, then the Hochschild cohomology ring $HH^*(\Gamma)$ is isomorphic to*

$$R[A, B, C, D]/((\zeta+\zeta^{-1})A, (\zeta+\zeta^{-1})B, (\zeta+\zeta^{-1})C, 2^rD,$$
$$A^2 + B^2 + C^2, BC - 2^{r-1}\eta D),$$

where $R = \mathbb{Z}[\zeta + \zeta^{-1}]$, $\deg A = \deg B = \deg C = 1$ *and* $\deg D = 2$.

Remark 4. In the case $r = 1$, this cohomology ring is already known by Sanada [8, Section 3.4]. In [8], he treats the Hochschild cohomology of crossed products over a commutative ring and its product structure using a spectral sequence of a double complex. As a special case, he determines the Hochschild cohomology ring of the quaternion algebra over $\mathbb{Z}$.

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Department of Mathematics
Faculty of Science
Science University of Tokyo
Wakamiya-cho 26, Shinjuku-ku, Tokyo 162-0827, JAPAN
E-mail address: hayami@ma.kagu.tus.ac.jp

REMARKS ON QF-2 RINGS, QF-3 RINGS AND HARADA RINGS

KEN-ICHI IWASE

1. Introduction

In the procedings of the 1978 antwerp conference, M.Harada studied those rings whose non-small left modules contains non-zero injective submodules. K.Oshiro called perfect rings with this condition "*left Harada* rings". These rings are two sided artinian, right QF-2, and right and left QF-3 rings containing *QF* rings and *Nakayama* rings, and moreover, these rings have left and also right ideal theoretic characterizations.

The purpose of this paper is to study the following well known theorems (*see Anderson-Fuller* [1]) :

> Theorem I. Right or left artinian QF-2 rings are QF-3.
> Theorem II. For a right or left artinian ring R, R is QF-3 if and only if its injective hull $E(R_R)$ is projective.
> Theorem III. Every Nakayama ring R with a simple projective right ideal is expressed as a factor ring of an upper triangular matrix ring over a division ring.

In Theorems I, II, we are little anxious whether the assumption "right or left artinian" is natural or not. This assumption also appears in the following well known theorem due to Fuller [6] :

> Let R be a right or left artinian ring and let e be a primitive idempotent in R. Then eR_R is injective if and only if there exists a primitive idempotent f in R such that $S(eR) \cong fR/fJ$ and $S(Rf) \cong Re/Je$, where $S(X)$ and J mean the socle of X and the Jacobson radical of R, respectively.

In Baba-Oshiro [2], this theorem is improved for a semiprimary ring with "ACC or DCC" for right annihilator ideals, where ACC and DCC mean the ascending chain condition and the descending chain condition, respectively. As the condition ACC or DCC for right annihilator ideals is equivalent to the condition ACC or DCC for left annihilator ideals, the replacement of "right or left artinian" with "semiprimary ring with ACC or DCC for annihilator right ideals" is quite natural.

In this paper, from this view point, we improve Theorem I as follows: Semiprimary QF-2 rings with ACC or DCC for right annihilator ideals are QF-3. For Theorem II, we show that, for a left perfect ring R with ACC or DCC for right annihilator ideals, R is QF-3 if its injective hull $E(R_R)$ is projective. For Theorem III, using the structure theorem of left Harada rings, we improve the theorem as follows: Left Harada rings with a simple projective right ideal is expressed as a factor ring of an upper triangular matrix ring over a division ring.

The detailed version of this paper will be submitted for publication elsewhere.

2. Improve versions of Theorem I and Theorem II

Recall that a right R-module M is called *uniform* if every non-zero submodule of M is essential. We note that, if R is left perfect, M_R is uniform if and only if M_R is colocal.

The *uniform dimension*of a module M is the infimum of those cardinal numbers c such that $\#I \leq c$ for every independent set $\{N_i\}_{i\in I}$ of non-zero submodules of M. We denote the uniform dimension of M by $unif.dimM$, where $\#I$ means the number of elements of I.

PROPOSITION 1. (*c.f.*[3, Proposition.3.1.2]) Let R be a ring. We consider the following four conditions.

(a) R is right *QF-3*.
(b) R contains a faithful injective right ideal.
(c) For any projective right R module P_R, $E(P_R)$ is projective.
(d) $E(R_R)$ is projective.

Then the following hold.

(1) (a) $\Rightarrow$ (b) holds. Further, if R is a left perfect ring, then (b) $\Rightarrow$ (a) also holds.
(2) If R is left perfect, then (b) $\Leftrightarrow$ (c) holds.
(3) If ACC or DCC holds on right annihilator ideals, then (d) $\Rightarrow$ (b) does.
(4) (c) $\Rightarrow$ (d) holds in general.

Remark : By Proposition 1, when R is a left perfect ring, we have (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) and (a) $\Rightarrow$ (d), but in general (d) $\Rightarrow$ (a) is not true.

For example, for the set $\mathbb{Q}$ of rational numbers and the set $\mathbb{Z}$ of integers, we consider $R = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Z} \end{pmatrix}$. Then ${}_RR$ is noetherian and has a faithful injective right ideal, so that $E(R_R)$ is projective, but R does not have minimal faithful right R module. (c.f. [18, Theorem 6.2 (Vinsinhaler)])

The following Theorem is due to K.R.Fuller.

THEOREM A. ([1, Theorem 31.3]) Let R be a right or left artinian ring and let $f \in Pi(R)$. Then ${}_RRf$ is injective if and only if there is a primitive idempotent e in R such that $S({}_RRf) \cong T({}_RRe)$ and $S(eR_R) \cong T(fR_R)$.

Using this Theorem A, he showed that every right or left artinian QF-2 ring is QF-3.

Y. Baba and K. Oshiro improved Theorem A in [2] as follows:

THEOREM B. ([2]) Let R be a semiprimary ring which satisfies ACC or DCC for right annihilator ideais and let $e, f \in Pi(R)$. Then the following conditions are equivalent:

(1) ${}_RRf$ is injective with $S({}_RRf) \cong T({}_RRe)$.
(2) eR_R is injective with $S(eR_R) \cong T(fR_R)$.

Now we show the following.

THEOREM 2. If R is a semiprimary *QF-2* ring with *ACC* or *DCC* for right annihilator ideals, then R is *QF-3*.

3. An improve version of Theorem III

For our purpose, we need the following structure theorem due to Oshiro ([15]-[17]):

THEOREM C. Let R be a basic left Harada ring. Then R can be constructed as an upper staircase factor ring of a block extension of its frame QF-subring $F(R)$.

In order to understand this structure theorem, we must review the sketch of the proof of Theorem C (for details, see Baba-Oshiro's Lecture Note).

Let F be a basic QF-ring with $Pi(F) = \{e_1, \ldots, e_y\}$. We put $A_{ij} := e_iFe_j$ for any i, j, and, in particular, put $Q_i := A_{ii}$ for any i. Then we may represent F as

$$F = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1y} \\ A_{21} & A_{22} & \cdots & A_{2y} \\ \cdots & \cdots & \cdots & \cdots \\ A_{y1} & A_{y2} & \cdots & A_{yy} \end{pmatrix} = \begin{pmatrix} Q_1 & A_{12} & \cdots & A_{1y} \\ A_{21} & Q_2 & \cdots & A_{2y} \\ \cdots & \cdots & \cdots & \cdots \\ A_{y1} & \cdots & A_{y,y-1} & Q_y \end{pmatrix}.$$

For $k(1), \ldots, k(y) \in \mathbb{N}$, the *block extension* $F(k(1), \ldots, k(y))$ of F is defined as follows: For each $i,\ s \in \{1, \ldots, y\},\ j \in \{1, \ldots, k(i)\},\ t \in \{1, \ldots, k(s)\}$, let

$$P_{ij,st} = \begin{cases} Q_i & \text{if } i = s,\ j \le t, \\ J(Q_i) & \text{if } i = s,\ j > t, \\ A_{is} & \text{if } i \ne s, \end{cases}$$

and

$$P(i,s) = \begin{pmatrix} P_{i1,s1} & P_{i1,s2} & \cdots & P_{i1,sk(s)} \\ P_{i2,s1} & P_{i2,s2} & \cdots & P_{i2,sk(s)} \\ \vdots & \vdots & \ddots & \vdots \\ P_{ik(i),s1} & P_{ik(i),s2} & \cdots & P_{ik(i),sk(s)} \end{pmatrix}.$$

Consequently, when $i = s$, we have the $k(i) \times k(i)$ matrix

$$P(i,i) = \begin{pmatrix} Q_i & \cdots & \cdots & Q_i \\ J(Q_i) & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ J(Q_i) & \cdots & J(Q_i) & Q_i \end{pmatrix}$$

which we denote by $Q(i)$, and, when $i \ne s$, we have the $k(i) \times k(s)$ matrix

$$P(i,s) = \begin{pmatrix} A_{is} & \cdots & A_{is} \\ & \cdots & \\ A_{is} & \cdots & A_{is} \end{pmatrix}$$

Furthermore, we set

$$P = F(k(1),\dots,k(y)) = \begin{pmatrix} P(1,1) & P(1,2) & \cdots & P(1,y) \\ P(2,1) & P(2,2) & \cdots & P(2,y) \\ \cdots & \cdots & \cdots & \cdots \\ P(y,1) & P(y,2) & \cdots & P(y,y) \end{pmatrix}$$

$$= \begin{pmatrix} Q(1) & P(1,2) & \cdots & P(1,y) \\ P(2,1) & Q(2) & \cdots & P(2,y) \\ \cdots & \cdots & \cdots & \cdots \\ P(y,1) & P(y,2) & \cdots & Q(y) \end{pmatrix}.$$

Since F is a basic QF-ring, we see that P is a basic left *Harada* ring with matrix size $k(1)+\cdots+k(y)$. We say that $F(k(1),\dots,k(y))$ is a *block extension* of F for $\{k(1),\dots,k(y)\}$. In more detail, this matrix representation is given by

$$P = F(k(1),\dots,k(y)) = \begin{pmatrix} P_{11,11} & \cdots & P_{11,1k(1)} & \cdots & P_{11,y1} & \cdots & P_{11,yk(y)} \\ \vdots & & \vdots & & \vdots & & \vdots \\ P_{1k(1),11} & \cdots & P_{1k(1),1k(1)} & \cdots & P_{1k(1),y1} & \cdots & P_{1k(1),yk(y)} \\ \vdots & & \vdots & & \vdots & & \vdots \\ P_{y1,11} & \cdots & P_{y1,1k(1)} & \cdots & P_{y1,y1} & \cdots & P_{y1,yk(y)} \\ \vdots & & \vdots & & \vdots & & \vdots \\ P_{yk(y),11} & \cdots & P_{yk(y),1k(1)} & \cdots & P_{yk(y),y1} & \cdots & P_{yk(y),yk(y)} \end{pmatrix}.$$

If we set

$$p_{ij} = \langle 1 \rangle_{ij,ij} ,$$

where this means an element of P which the $(ij,\ ij)$-position $= 1$, and another positions are 0.
For each $i = 1,\dots,y,\ j = 1,\dots,k(i)$, then $\{p_{ij}\}_{i=1,j=1}^{y\ \ k(i)}$ is a well-indexed set of a complete set of orthogonal primitive idempotents of $P = F(k(1),\dots,k(y))$.

For $Pi(P)$, we note that

$$p_{ij}P_P \cong p_{i1}J(P)_P^{j-1}$$

for any $i = 1,\dots,y$ and $j = 1,\dots,k(i)$.

Given the situation above, the following are equivalent:

(1) F is a QF ring with a Nakayama permutation:

$$\begin{pmatrix} e_1 & \cdots & e_y \\ e_{\sigma(1)} & \cdots & e_{\sigma(y)} \end{pmatrix}.$$

(2) $P = F(k(1),\dots,k(y))$ is a basic left *Harada* ring of type $(*)$ with a well-indexed set $Pi(P) = \{p_{ij}\}_{i=1,j=1}^{y\ \ k(i)}$.

Let R be a basic *Harada* ring. We call R a basic *Harada* ring of *type* $(*)$ if there is a permutation σ of $\{1,2,\dots,m\}$ such that $(e_{j1}R, Re_{\sigma(j)n(\sigma(j))})$ is an i-pair for every $j \in \{1,2,\dots,m\}$.

From now on, we assume that the Nakayama permutation of F is

$$\begin{pmatrix} e_1 & \cdots & e_y \\ e_{\sigma(1)} & \cdots & e_{\sigma(y)} \end{pmatrix},$$

and

we take the block extension $P = F(k(1), \ldots, k(y))$ of F. Let $i \in \{1, \ldots, y\}$ and consider the i-pair $(e_iF; Fe_{\sigma(i)})$. Put $S(A_{ij}) := S({}_{Q_i}A_{ij}) = S(A_{ij\,Q_j})$. Then we define an upper staircase left $Q(i)$- right $Q(\sigma(i))$-subbimodule $S(i,\sigma(i))$ of $P(i,\sigma(i))$ with tiles $S(A_{ij})$ as follows:

(I) Suppose that $i = \sigma(i)$: Then we see from above argument that $S(A_{ij})$ is simple as both a left and a right ideal of $Q_i = A_{ii}$. Put $Q := Q_i$, $J := J(Q_i)$ and $S := S(Q_i)$. Then, in the $k(i) \times k(i)$ matrix ring,

$$Q(i) = P(i,i) = \begin{pmatrix} Q & \cdots & \cdots & Q \\ J & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ J & \cdots & J & Q \end{pmatrix},$$

we define an upper staircase left $Q(i)$- right $Q(i)$-subbimodule $S(i,i) = S(i,\sigma(i))$ of $Q(i)$ as follows:

$$S(i,i) = \begin{pmatrix} 0\cdots 0 & & S \\ & & \\ 0 & & \end{pmatrix}. \qquad (\text{ the } (1,1)\text{-position} = 0),$$

where, for the form of $S(i,i)$, we assume that

(1) the $(1,1)$-position $= 0$,

(2) when Q is a division ring, that is, $Q = S$,

$$S(i,i) = \begin{pmatrix} 0\cdots 0 & & & S \\ \ddots & & & \\ & \ddots & & \\ & & \ddots & \\ 0 & & & 0 \end{pmatrix}$$

Then, since S is an ideal of Q, we see that $S(i,i) = S(i,\sigma(i))$ is an ideal of $Q(i)$.

We let $\overline{Q(i)} = \overline{P(i,\sigma)} = P(i,\sigma)/S(i,\sigma(i))$ for the subbimodule $S(i,\sigma(i))$. In $Q(i)$, we replace Q or J of the (p,q)-position by $\overline{Q} = Q/S$ or $\overline{J} = J/S$, respectively, when the (p,q)-position of $S(i,i)$ is S. Then we may represent $\overline{Q(i)}$ with the matrix ring which is made by these replacements.

For example,

$$Q(i)=\begin{pmatrix} Q & \overline{Q} & \overline{Q} & \overline{Q} & \overline{Q} & \overline{Q}\\ J & \overline{Q} & \overline{Q} & \overline{Q} & \overline{Q} & \overline{Q}\\ J & \overline{J} & \overline{Q} & \overline{Q} & \overline{Q} & \overline{Q}\\ J & J & J & Q & \overline{Q} & \overline{Q}\\ J & J & J & J & \overline{Q} & \overline{Q}\\ J & J & J & J & \overline{J} & \overline{Q} \end{pmatrix} = \begin{pmatrix} Q & Q & Q & Q & Q & Q\\ J & Q & Q & Q & Q & Q\\ J & J & Q & Q & Q & Q\\ J & J & J & Q & Q & Q\\ J & J & J & J & Q & Q\\ J & J & J & J & J & Q \end{pmatrix} \Big/ \begin{pmatrix} 0 & S & S & S & S & S\\ 0 & S & S & S & S & S\\ 0 & S & S & S & S & S\\ 0 & 0 & 0 & 0 & S & S\\ 0 & 0 & 0 & 0 & S & S\\ 0 & 0 & 0 & 0 & S & S \end{pmatrix}$$

(II) Now suppose that $i \neq \sigma(i)$: Put $S := S_{i\sigma(i)} = S({}_{Q_i}A_{i\sigma(i)}) = S(A_{i\sigma(i)Q_{\sigma(i)}})$. Then S is a left Q_i- right $Q_{\sigma(i)}$-subbimodule of $A = A_{i\sigma(i)}$. In the left $Q(i)$- right $Q(\sigma(i))$-bimodule

$$P(i,\sigma(i)) = \begin{pmatrix} A & \cdots & A\\ & \cdots & \\ A & \cdots & A \end{pmatrix} \qquad (\,k(i)\times k(\sigma(i))\text{-matrix}\,),$$

we define an upper staircase subbimodule $S(i,\sigma(i))$ of $P(i,\sigma(i))$ with tiles S of $P(i,\sigma(i))$ as follows:

$$S(i,\sigma(i)) = \begin{pmatrix} 0\cdots 0 & & S\\ & & \\ 0 & & \end{pmatrix} \qquad (\text{ the } (1,1)\text{-position} = 0)$$

and put $\overline{P(i,\sigma)} := P(i,\sigma(i))/S(i,\sigma(i))$. We may represent $\overline{P(i,\sigma)}$ as

$$\overline{P(i,\sigma)} = \begin{pmatrix} A\cdots A & & \overline{A}\\ & & \\ A & & \end{pmatrix}.$$

Next we define a subset X of $P = F(k(1),\ldots,k(y))$ by

$$X = \begin{pmatrix} X(1,1) & X(1,2) & \cdots & X(1,y)\\ X(2,1) & X(2,2) & \cdots & X(2,y)\\ \cdots & \cdots & \cdots & \cdots\\ X(y,1) & X(y,2) & \cdots & X(y,y) \end{pmatrix},$$

where $X(i,j)$ $(\subseteq Q_i)$ and $X(i,j)$ $(\subseteq P(i,j))$ are defined by

$$X(i,i) = \begin{cases} 0 & \text{if } i \neq \sigma(i),\\ S(i,i) & \text{if } i = \sigma(i), \end{cases}$$

$$X(i,j) = \begin{cases} 0 & \text{if } j \neq \sigma(i),\\ S(i,j) & \text{if } j = \sigma(i). \end{cases}$$

Then we see that X is an ideal of $P = F(k(1), \ldots, k(y))$. The factor ring $F(k(1), \ldots, k(y))/X$ is then called an *upper staircase factor ring* of $P = F(k(1), \ldots, k(y))$. If, in the representation

$$P = F(k(1), \ldots, k(y)) = \begin{pmatrix} P(1,1) & P(1,2) & \cdots & P(1,y) \\ P(2,1) & P(2,2) & \cdots & P(2,y) \\ \cdots & \cdots & \cdots & \cdots \\ P(y,1) & P(y,2) & \cdots & P(y,y) \end{pmatrix},$$

we replace $P(i, \sigma(i))$ with $\overline{P(i, \sigma(i))}$ and put $\overline{P} := F(k(1), \ldots, k(y))/X$, then it is convenient to represent $\overline{P}$ as follows:

$$\overline{P} = \begin{pmatrix} P(1,1) & \cdots & \overline{P(1,\sigma(1))} & \cdots & \cdots & \cdots & P(1,y) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ P(i,1) & \cdots & \cdots & \overline{P(i,\sigma(i))} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ P(y,1) & \cdots & \cdots & \cdots & \overline{P(y,\sigma(y))} & \cdots & P(y,y) \end{pmatrix}$$

From the form of $\overline{P}$ together with $k \geq 1$, where the k appears in the matrices above (I), (II), we can see that $\overline{P} = F(k(1), \ldots, k(y))/X$ is a basic left *Harada* ring. Moreover, by the upper staircase form of $S(i, \sigma(i))$, we have left *Harada* rings $P = P_1 = F(k(1), \ldots, k(y))$, P_2, $P_3, \ldots, P_{l-1}$, $P_l = \overline{P}$ and canonical surjective ring homomorphisms $\varphi_i : P_i \to P_{i+1}$ with $\ker \varphi_i$ a simple ideal of P_i as follows:

$$P_1 \xrightarrow{\varphi_1} P_2 \xrightarrow{\varphi_2} P_3 \xrightarrow{\varphi_3} \cdots \xrightarrow{\varphi_{l-2}} P_{l-1} \xrightarrow{\varphi_{l-1}} P_l = \overline{P} = F(k(1), \ldots, k(y))/X.$$

The following is the fundamental structure theorem (see Oshiro [17]).

THEOREM D. For a given basic QF-ring F, every upper staircase factor ring P/X of a block extension $P = F(k(1), \ldots, k(y))$ is a basic left *Harada* ring, and, for any basic left Harada ring R, there is a basic QF-subring $F(R)$ which is called the frame QF-subring, R is represented in this form by $F(R)$.

Using this theorem, we show the following

THEOREM 3. Let R be a basic indecomposable left *Harada* ring. If R has a simple projective right R-module, then R can be represented as an upper triangular matrix ring over a division ring as follows:

$$R \cong \begin{pmatrix} \mathrm{D} & & & & \\ & \ddots & \mathrm{D} & 0 & \\ & & \ddots & & \\ & 0 & & \ddots & \\ & & & & \mathrm{D} \end{pmatrix}$$

By *THEOREM 3* we have the following corollary.

COROLLARY 4. (*c.f.*[1, Theorem 32.8]) Let R be a basic indecomposable Nakayama ring. If R has a simple projective right R-module, then R can be represented as a factor ring of an upper triangular matrix ring over a division ring.

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Tennoji Senior High-School attached to Osaka-kyoiku University
Tennoji, Osaka 543-0054 JAPAN
E-mail address: iwase@cc.osaka-kyoiku.ac.jp

SOME CONGRUENCES CONCERNING FINITE GROUPS

KAORU MOTOSE

ABSTRACT. In this paper, we present a lemma about orders of normal subgroups in a transitive group of prime degree. This lemma has an application to prove simplicity of the alternative group A_5 of degree 5, and 4-transitive Mathieu groups $M_{11}, M_{12}, M_{23}, M_{24}$. Please use this lemma for your lecture to your students about group theory or Galois theory. I present some comments to Feit-Thompson conjecture. I think also it is not so popular, to mathematician, even to finite group theorists and number theorists.

Key Words: Sylow theorem, Alternative groups, Mathieu groups,

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Sylow theorem states that the number of distinct p-Sylow subgroups is congruent to 1 modulo p. In this paper, we call it Sylow congruence and using this, we shall present a lemma to prove the simplicity of the alternative group A_5 and Mathieu groups M_{11}, M_{12}, M_{23}, M_{24}. Moreover, we shall give some comments to Feit-Thompson Conjecture. The part up to Theorem 9 was written in considering for education to students.

Congruences in finite groups are important between group theory and number theory. It is the most important congruence in group theory that the order $|H|$ of subgroup H of a group G is a divisor of $|G|$. For the proof of this, we use all conditions in the definition of the group. Apply this to the unit group of the residue ring $\mathbb{Z}/n\mathbb{Z}$, we have Fermat little theorem and Euler theorem.

Let Γ_n be the set of complex numbers of order n and we define cyclotomic polynomial $\Phi_n(x) = \prod_{\eta\in\Gamma_n}(x-\eta)$. Then formula $x^n - 1 = \prod_{d|n}\Phi_d(x)$ yields form classifying orders in the group of roots of $x^n - 1 = 0$, which is equivalent to the definition of the cyclotomic polynomial $\Phi_m(x)$. It follows from this formula that the group $\mathbb{F}_q^*$ is cyclic, where $\mathbb{F}_q$ is a finite field of order q and $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$. Usually, we prove this using orders of two elements in the abelian group $\mathbb{F}_q^*$.

From Sylow congruence, non cyclic groups of order pq with primes $p < q$ have a normal q-Sylow subgroup and q distinct p-Sylow subgroups. Groups of this kind are there infinite many for a fixed prime p by the Dirichlet theorem which can be proved by cyclotomic polynomials in case $q \equiv 1 \bmod p$. These groups suggest Burnside's p^sq^t theorem.

1. Some notations and elementary results

In this section we shall give some notations and elementary well known results. Let G be a finite group and let Δ be a finite set. We say Δ is a G-set if satisfying the following

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The detailed version of the rest from Lemma 10 will be submitted for publication elsewhere.

conditions

$$\alpha^h \in \Delta,\ (\alpha^g)^h = \alpha^{gh} \text{ and } \alpha^1 = \alpha \text{ for } \alpha \in \Delta \text{ and } g, h \in G.$$

We set $G_\Gamma = \{g \in G \mid \alpha^g = \alpha$ for all $\alpha \in \Gamma\}$ for a subset Γ of Δ. In case $G_\Delta = \{1\}$, we say Δ is a faithful G-set or G is a permutation group on Δ. We can classify elements in G-set Δ by orbits $\alpha^G = \{\alpha^g \mid g \in G\}$ for $\alpha \in \Delta$ and we obtain

$$\Delta = \bigcup_k \alpha_k^G, \text{ and } |\Delta| = \sum_k |\alpha_k^G|.$$

We set $G_\alpha = \{g \in G \mid \alpha^g = \alpha\}$ for $\alpha \in \Delta$. Then G_α is a subgroup of G. Since $\alpha^g = \alpha^h$ is equivalent to $G_\alpha g = G_\alpha h$, we have

$$|G| = |\alpha^G||G_\alpha|.$$

Let G be a permutation group on Δ. G is transitive if there exists $g \in G$ with $\alpha^g = \beta$ for arbitrary $\alpha, \beta \in \Delta$. G is k-transitive ($k \geq 2$) if G_α is $(k-1)$-transitive on $\Delta \setminus \{\alpha\}$.

Lemma 1. *Let G be 2-transitive on a finite set Δ and let $\{1\} \neq N$ be a normal subgroup of G. Then we have*

(1) *$G = G_\alpha \cup G_\alpha x G_\alpha$ for $x \notin G_\alpha$.*
(2) *$G = G_\alpha N$ and N is transitive on Δ.*

Proof. (1) Let $g \in G \setminus G_\alpha$ then $\alpha \neq \alpha^g$ and $\alpha \neq \alpha^x$. Since G is 2-transitive, there exists $h \in G_\alpha$ such that $\alpha^g = \alpha^{xh}$. Thus $g(xh)^{-1} \in G_\alpha$ and so $g \in G_\alpha xh \subset G_\alpha x G_\alpha$.

(2) If $N \subset G_\alpha$, then we have a contradiction

$$N \subset \bigcap_{g\in G} g^{-1}G_\alpha g = \bigcap_{g \in G} G_{\alpha^g} = G_\Delta = \{1\}.$$

Hence we have $N \not\subset G_\alpha$ and there exists $n \in N \setminus G_\alpha$. Since G is 2-transitive,

$$G = G_\alpha \bigcup G_\alpha n G_\alpha \subset G_\alpha N G_\alpha = G_\alpha N \text{ and } \Delta = \alpha^G = \alpha^{G_\alpha N} = \alpha^N.$$

A transitive group G is regular on Δ if $G_\alpha = 1$ for some $\alpha \in \Delta$. Moreover, for subset T of G, we set normalizer $\mathcal{N}_G(T) = \{g \in G \mid g^{-1}Tg = T\}$ of T and centralizer $\mathcal{C}_G(T) = \{g \in G \mid gt = tg$ for all $t \in T\}$ of T.

Lemma 2. *Let G be 2-transitive on a finite set Δ and let $N \neq \{1\}$ be a regular normal subgroup of G.*

(1) *N is elementary abelian and $|\Delta| = |N|$ is a power of a prime.*
(2) *If G_α is simple, then G_α is a subgroup of $\mathrm{Aut}(N) = \mathrm{GL}(s, \mathbb{F}_p)$ where $\mathrm{Aut}(N)$ is a automorphism group of N, $|\Delta| = |N| = p^s$, and $\mathrm{GL}(s, \mathbb{F}_p)$ is the general linear group over a prime field $\mathbb{F}_p$.*

Proof. (1) We prove that G_α is transitive on $N \setminus \{1\}$ by the action $n^g = g^{-1}ng$ for $n \in N \setminus \{1\}$ and $g \in G_\alpha$. Let $s \neq t$ be arbitrary elements of $N \setminus \{1\}$. Then $\alpha^s, \alpha^t \in \Delta \setminus \{\alpha\}$ and there exists $g \in G_\alpha$ such that $\alpha^{sg} = \alpha^t$ because G_α is transitive on $\Delta \setminus \{\alpha\}$. Hence we have $gtg^{-1} = s$ from $\alpha^{sg} = \alpha^t = \alpha^{gt}$ and $gtg^{-1}s^{-1} \in N_\alpha = \{1\}$.

Thus $x^p = 1$ for all $x \in N$ since N contains an element of a prime order p and G_α is transitive on $N \setminus \{1\}$ by the action $a^g = gag^{-1}$ for $a \in N$ and $g \in G_\alpha$.

Thus N is a p-group and the center $Z \neq 1$ of N is normal in G (see the paragraph before Lemma 8). Hence it follow from the next that $N = Z$, namely, N is elementary.

$$G_\alpha Z = G = G_\alpha N,\ G_\alpha \cap Z = \{1\} = G_\alpha \cap N$$

On the other hand,

$$p^e = |N| = |N_\alpha||\alpha^N| = |\alpha^N| = |\Delta|.$$

(2) If $G_\alpha \cap \mathcal{C}_G(N) \neq \{1\}$, then $G_\alpha = G_\alpha \cap \mathcal{C}_G(N)$ since G_α is simple and $\mathcal{C}_G(N)$ is normal. Thus $G_\alpha \subset \mathcal{C}_G(N)$ which is a contradiction to that G_α is transitive on $N \setminus \{1\}$. Thus it follows from the above that

$$|\mathcal{C}_G(N)| = |\alpha^{\mathcal{C}_G(N)}| = |\Delta| = |\alpha^N| = |N|$$

Thus this implies $N = \mathcal{C}_G(N)$ from $N \subset \mathcal{C}_G(N)$. Hence we have

$$G_\alpha \cong G_\alpha N/N = G/N = \mathcal{N}_G(N)/\mathcal{C}_G(N)$$

is a subgroup of the automorphism $\mathrm{Aut}(N)$ of N by considering map $n \to g^{-1}ng$ for $n \in N$ and $g \in G$.

In the next well known theorem concerning the simplicity of groups, (1) is useful for multiply transitive groups. (2) is useful for linear groups. As the corollary of (2), (3) is useful for A_5 and $\mathrm{PSL}(2, K)$, where K is a field with $|K| \geq 4$. In this theorem, it is unnecessary to assume Δ is finite and G is finite.

Theorem 3. *Let G be 2-transitive on a set Δ. Then we have*

(1) (see [7, p.22] and [8, p. 263]) *If G_α is simple and G has no regular normal subgroups $\neq \{1\}$, then G is simple.*
(2) (Iwasawa, 1941, see [8, p. 263]) *If $G = G'$ and G_α has a normal solvable subgroup H such that $G = \langle x^{-1}Hx \mid x \in G\rangle$, then G is simple.*
(3) (Corollary of (2)) *If $G = G'$ and $G_\alpha \neq \{1\}$ is solvable, then G is simple.*

Proof. Let $N \neq \{1\}$ be a normal subgroup of G.

(1) We have $G = G_\alpha N$ from Lemma 1 (2) and $G_\alpha \bigcap N \neq \{1\}$ since G has no regular normal subgroups. $G_\alpha \bigcap N \neq \{1\}$ is normal in G_α and so $G_\alpha = G_\alpha \bigcap N \subset N$ from the assumption. Hence we have $G = G_\alpha N = N$ because G is 2-transitive.

(2) HN is normal in G by $G = G_\alpha N$. Hence we have

$$G = \langle x^{-1}Hx \mid x \in G\rangle \subset HN \text{ and } G = HN$$

Thus we have a contradiction such that a non solvable group $(G/N)' = G/N = HN/N$ and a solvable group $H/H \cap N$ are isomorphic.

(3) We set $L = \langle G_\alpha \mid \alpha \in \Delta\rangle$. If $L = G_\beta$ for some $\beta \in \Delta$, then $L = G_\alpha$ for all $\alpha \in \Delta$ because these are conjugate and L is normal. Hence we have a contradiction $L = \bigcap_{\alpha\in\Delta} G_\alpha = \{1\}$. Since G_α is maximal from Lemma 1 (2), we have

$$L = \langle x^{-1}G_\alpha x \mid x \in G\rangle = G$$

Thus G is simple from (2).

Another proof. We have $G = G_\alpha N$ from Lemma 1 (2). G_α has a normal subgroup H such that G_α/H is abelian. Noting HN is normal in G, Hence $G/HN = G_\alpha N/HN$ is abelian because this is a homomorphic image of abelian group G_α/H. Thus $G = G' \subset HN$ and $G = HN$. H has a normal subgroup K such that H/K is abelian and $G/KN = HN/KN$ is abelian. Thus $G = G' \subset KN$ and $G = KN$. We continue this process and we have $G = N$.

The next (1) is trivial and is needless to prove. However it is very important to obtain all conjugate classes of the symmetric group S_n. If students don't know (1), then it needs much calculations to prove (2).

Remark 4. (1) $(k^\tau)^{\tau^{-1}\sigma\tau} = k^{\sigma\tau}$ *for* $k \in \Delta$, *namely, we have*

$$\tau^{-1}(i_1 i_2 \cdots i_r)(j_1 j_2 \cdots j_s)\cdots(k_1 k_2 \cdots k_t)\tau = (i_1^\tau i_2^\tau \cdots i_r^\tau)(j_1^\tau j_2^\tau \cdots j_s^\tau)\cdots(k_1^\tau k_2^\tau \cdots k_t^\tau)$$

and

$$\tau^{-1}\sigma\tau = \begin{pmatrix} 1^\tau & 2^\tau & \cdots & n^\tau \\ 1^{\sigma\tau} & 2^{\sigma\tau} & \cdots & n^{\sigma\tau} \end{pmatrix}.$$

(2) *Using the above, we have* $\tau^{-1}(12)(34)\tau = (1^\tau 2^\tau)(3^\tau 4^\tau)$ *for* $\sigma = (12)(34)$. *Thus subgroup* $V = \{(1), (12)(34), (13)(24), (14)(23)\}$ *is normal in the symmetric group* S_4 *of degree* 4.

The next are well known and appears in many text books for students. Note that product of permutation should be left hand in this paper because actions on Δ is right hand.

Remark 5. *We may write here* $1, 2, 3, 4, 5$ *instead of arbitrary* $k_1, k_2, k_3, k_4, k_5 \in \Delta$, *respectively, in* (2) *and* (3).

(1) $A_n (n \geq 3)$ *is (n-2)-transitive on* $\Delta = \{1, 2, \ldots, n\} = \{a_1, a_2, \ldots, a_n\}$ *since either* σ *or* $\tau = \sigma(a_{n-1}a_n)$ *is an even permutation for* $k^\sigma = a_k$ *for all k.*

(2) $A_n (n \geq 3)$ *is generated by 3-cycles in virtue of* $(12)(23) = (132)$ *and* $(12)(34) = (12)(23)(23)(34) = (132)(243)$.

(3) $A_n (n \geq 5)$ *is perfect, namely* $A_n = A_n'$ *by* (2) *and*

$$(123) = (23)(45)(123)\{(23)(45)\}^{-1}(123)^{-1}$$

2. Some proofs of simplicity of A_5

The simplicity of the alternative group A_5 is important for history of mathematics and education on students studying group theory and Galois theory. There are many proofs about this.

Method 1: A_5 is 3-transitive and generated by 3-cycles. Non trivial normal subgroup contains a 3-cycles.

Method 2: The numbers of elements in five conjugate classes are $1, 12, 12, 20, 15$ and any partial sums of these containing 1 is not a divisor of 60.

We shall give another two proofs using the above lemmas and theorem.

Theorem 6. *A_5 is simple.*

Proof 1. Stabilizer A_4 of 5 is solvable because A_4 has a normal subgroup $V = \{(1), (12)(34), (13)(24), (14)(23)\}$ such that A_4/V and V are abelian (see Remark 1 (2)). A_5 is perfect by Remark 2 (3) and 3-transitive by Remark 2 (1). Thus A_5 is simple by Theorem 1 (3).

Proof 2. Let $\{1\} \neq N$ be a normal subgroup of A_5.

If $5 \mid |N|$ then N contains all 5-cycles from Sylow theorem and so $|N| \geq 4! = 24$. If N is regular then $|N| = 5$ and so we have a contradiction from the above.

Thus N is not regular, namely, $M = A_4 \cap N \neq \{1\}$. Since A_5 and A_4 are 2-transitive, $A_5 = A_4N$ and $A_4 = A_3M$, and so $A_5 = A_4N = A_3MN = A_3N$. In case $A_3 \cap N = \{1\}$, we have $|N| = 20$ contradicts to the first statement in this proof. Hence $A_3 = A_3 \cap N \neq \{1\}$ from $|A_3| = 3$ and $A_5 = N$.

Theorem 7. *$A_n (n \geq 5)$ is simple.*

Proof. We may assume $n \geq 6$. Let $\{1\} \neq N$ be a normal subgroup of A_n. In case N is regular, $n = p^s$ from Lemma 2 (1) where p is prime, and A_{n-1} is a subgroup of $\mathrm{GL}(s,p)$ from Lemma 2 (2). Thus we have the next contradiction from $p^s = n > 4$.

$$\tfrac{(p^s-1)!}{2} = |A_{n-1}| \leq |\mathrm{GL}(s,p)| = \textstyle\prod_{k=0}^{s-1}(p^s - p^k) < \tfrac{(p^s-1)!}{2}$$

Thus N is not regular, namely, $N \cap A_{n-1} \neq \{1\}$. We may assume inductively A_{n-1} is simple. Hence $A_{n-1} = A_{n-1} \cap N \subset N$ and so $A_n = A_{n-1}N = N$
(see also Theorem 1 (1)).

3. Transitive groups of prime degrees

We set $\Delta_T = \{\alpha \in \Delta \mid \alpha^t = \alpha \text{ for all } t \in T\}$ for a subset T of G. Considering G-set G for p-group G by conjugation, (1) in the next shows that the center G_G of p-group G is non trivial. (2) is also proved by elementary number theory or as the special case to cyclic group of order $p^e r$ in the following proof of Sylow theorem. However, the next proof is very simple.

Lemma 8. (1) $|\Delta| \equiv |\Delta_G| \bmod p$ *for a p-group G and G-set Δ.*
(2) $\binom{p^e r}{p^e} \equiv r \bmod p$ *for a prime p.*

Proof. (1) It follows from $|G| = |G_{\alpha_k}||\alpha_k^G|$ that

$$|\Delta| = |\Delta_G| + \sum_{|\alpha_k^G|>1} |\alpha_k^G| \equiv |\Delta_G| \bmod p.$$

(2) Compare coefficients of x^{p^e} in both sides of the next equation.

$$(x+1)^{p^e r} = (x^{p^e} + 1)^r \text{ in } \mathbb{F}_p[x].$$

The following is the proof of Sylow theorem by H. Wielandt. This is useful to the order of non trivial normal subgroup of a transitive group of a prime degree.

Theorem 9 (Sylow). *We set $|G| = p^e r$ with $(p,r) = 1$, and $n_p \geq 0$ is the number of distinct p-Sylow subgroups. Then $n_p \equiv 1 \bmod p$, in particular, there exists a p-Sylow subgroup, and a p-subgroup is contained in $t^{-1}St$ for $t \in G$ and a p-Sylow subgroup S. In particular, p-Sylow subgroups are mutually conjugate.*

Proof. We set $\Delta = \{S \subset G \mid |S| = p^e\}$. Then Δ is G-set by $S^g = Sg$ for $g \in G$. We also consider $S \in \Delta$ is $G_{\{S\}}$-set by $s^h = sh$ for $s \in S$ and $h \in G_{\{S\}}$. We can see that $G_{\{S\}}$ is a p-subgroup because $|s^{G_{\{S\}}}| = |sG_{\{S\}}| = |G_{\{S\}}|$ for all $s \in S$ and so $|G_{\{S\}}|$ is a divisor of $|S|$. Using $|G| = |G : G_{\{S\}}||G_{\{S\}}|$, we can see $G_{\{S\}}$ is a p-Sylow subgroup if and only if $p \not| \, |G : G_{\{S\}}|$. Hence we have

$$0 \not\equiv r \equiv \binom{p^e r}{p^e} = |\Delta| = \sum_{S \in \Delta} |G : G_{\{S\}}| \equiv n_p r \bmod p.$$

From this congruence, we have $n_p \equiv 1 \bmod p$.

Let H be a p-subgroup and let G/S be the set of right cosets of a p-Sylow subgroup S. G/S is H-set by $(Sg)^h = Sgh$.

$$0 \not\equiv r = |G/S| \equiv |(G/S)_H| \bmod p.$$

Hence $|(G/S)_H| \neq 0$ implies there exists St with $StH = St$ and so $tH \subset StH = St$.

In the next lemma, $|\mathcal{N}_G(P)| = pr$ is the foundation on the proof of simplicity of multiply transitive groups $A_5, M_{11}, M_{12}, M_{23}, M_{24}$.

Lemma 10. *Let p be a prime and let G be a transitive group on a set Δ, where $|\Delta| = p+s$ with $s < p$. We set $|G|/p \equiv r \bmod p$, where $0 < r < p$. Then for a p-Sylow subgroup P of G, we have*

$$C_G(P) = P, \ r \mid p-1 \text{ and } |\mathcal{N}_G(P)| = pr.$$

The following lemma gives structure of normal subgroups in a transitive group of prime degree. The assertion $|G/G'| \mid r$ follows from Lemma 10.

Lemma 11 (5, p. 607). *Let G be a transitive group of odd prime degree p on a set Δ and let $G' \neq \{1\}$ be the commutator group of G. We set $|G|/p \equiv r \bmod p$, where $0 < r < p$. Then we have*

(1) *G' is contained in all non trivial normal subgroups.*
(2) *G/G' is cyclic, $|G/G'| \mid r$ and $r \mid p-1$.*

Corollary 12. (1) *Let G be transitive on a set Δ of a prime degree $p > 3$ and $|G|/p \equiv r \bmod p$, where $0 < r < p$. Then $r \mid p-1$.*
If N is a non trivial normal subgroup of G, then $|G|/r \mid |N|$.
If G be 3-transitive and $(r, \frac{|G|}{p-1}) = 1$, then G is simple.

(2) *Let G be 2-transitive on a set Δ of a degree $p+1$, where $p > 3$ is prime but not a Mersenne prime and $|G|/p \equiv r \bmod p$, where $0 < r < p$. Then $r \mid p-1$.*
If N is a non trivial normal subgroup of G, then $|G|/r \mid |N|$.
If G is 4-transitive and $(r, \frac{|G|}{p-1}) = 1$, then G is simple.

Example 13. (1) Simplicity of groups $A_5, M_{11}, M_{12}, M_{23}, M_{24}$ follows from Corollary 12, founded on Lemma 10, because A_5, M_{11}, M_{23} are 3-transitive and these orders are $60,\ 11\cdot 10\cdot 9\cdot 8,\ 23\cdot 22\cdot 21\cdot 20\cdot 48,$ respectively and because M_{12}, M_{24} are 4-transitive and these orders are $12\cdot|M_{11}|,\ 24\cdot|M_{23}|,$ respectively (see [6, p.303], [7, p. 298] and [8, p. 292]).

(2) If M_{12} has a transitive extension $G = M_{13}$, then we set $p = 13$ and $\dfrac{|G|}{13} = 12\cdot 11\cdot 10\cdot 9\cdot 8 \equiv (-1)^5 5! \equiv 10 \bmod 13$. Thus $r = 10$ is not a divisor of $12 = p-1$. Hence there does not exists M_{13} (see [6, p. 302] and [8, p. 298]).

The next is well known and shows that transitive groups of odd prime degrees are closed to simple groups.

Theorem 14. *Let G be a transitive group of odd prime degree p on a set Δ and let G' be the commutator group of G. Then we have*

(1) *If $G' = 1$, then $|G| = p$.*
(2) (Galois) *If $G' \neq 1$ and $G'' = 1$, then G is an affine group over a prime field $\mathbb{F}_p$.*
(3) *If $G'' \neq 1$, then G' is simple. In particular, $G = G'$ implies G is simple.*

Proof. (1) We have $G = C_G(P) = P$ from Lemma 10 (1).

(2) G' is transitive, abelian and of degree p. Hence $G' = P$ from (1) and P can be identified to the additive group $(\mathbb{F}_p, +)$ of $\mathbb{F}_p$. A subgroup $G_\alpha P$ has the order $p|G_\alpha| = |G|$ since $G_\alpha \cap P = P_\alpha = 1$. Hence $G = G_\alpha P$ Since $G' = P = C_G(P),\ G = \mathcal{N}_G(P)$, we have $G_\alpha \cong G/P$ is a cyclic subgroup of

$$\mathrm{Aut}((\mathbb{F}_p, +)) = \{x \to sx \mid s \in \mathbb{F}_p^*\}$$

and the action of G_α to P by conjugation is the same with the multiplication in $\mathbb{F}_p$.

(3) We have $G' = G''$ from $G \rhd G'' \neq \{1\}$ and Lemma 11 (1). Let $H \neq \{1\}$ be normal in G'. Then $H \supset G'' = G'$ from Lemma 11 (1) since G' is transitive and of degree p.

4. Some comments to Feit-Thompson Conjecture

In this paper we shall give some comments to Feit-Thompson Conjecture (see below Conjectures 1 [2] and 2 [9]). For distinct primes p and q, we set

$$A = \Phi_p(q) = (q^p - 1)/(q-1) \text{ and } B = \Phi_q(p) = (p^q - 1)/(q-1).$$

Conjecture 1. *A does not divide B for $A < B$ (see [2]).*

In the paper [1, p.1] and the book [4, p.125], it was mentioned that if it could be proved, it would greatly simplify the very long proof of the Feit-Thompson theorem that every group of odd order is solvable (see [3]).

(1) in the next is fundamental to consider Conjecture 1 because of $B > A$ for $q > p \geq 2$.
(2) is very easy but it is slightly useful for using computer and a starting point for Conjecture 1. As a special case of (2), we may assume p and q are odd for Conjecture 1.
In case $p = 3$, it seems to be very important from [2]. In this case, we may consider $q \equiv -1 \bmod 6$ noting (2) and q is odd. Moreover we may assume A is prime from (3).

Comment 1. (1) $\dfrac{m^n-1}{m-1} > \dfrac{n^m-1}{n-1}$ *for integers* $n > m \geq 2$.

(2) *In case* $q \equiv 1 \bmod p$, *then* A *does not divide* B.

(3) *In case* $p = 3 < q$ *and* A *is composite, then* A *does not divide* B.

(4) *In case* $p = 3, 7 < q$ *and* $q \equiv 2$ *or* $4 \bmod 7$, *then* A *does not divide* B.

Conjecture 2. *A and B are relatively prime (see [9]).*

If a prime number r divides both A and B then $r = 2\lambda pq + 1$ for some integer λ (see Comment 3 (3)). Using computer, Stephens found a counterexample $p = 17, q = 3313$ and $r = 112643 = 2pq + 1$ and confirmed that r is the greatest common divisor of A and B by computer, so this example leaves conjecture 1 unresolved (see [9]).

At the present, it is known by computer that no other such pairs exist for $p < q < 10^7$ and $p = 3 < q < 10^{14}$ (see [4]).

We don't know that Conjectures have some relations with (2) and (3).

Comment 2. *If* $p = 17$ *and* $q = 3313$, *then we have*

(1) (Stephens [9]) $(\Phi_p(q), \Phi_q(p)) = 2pq + 1$.

(2) $p^{\frac{q-1}{2}} \equiv 1 \bmod q$.

(3) $q^{\frac{p-1}{2}} \equiv 1 \bmod p^2$.

In general, there are few prime numbers p satisfying congruence $a^{\frac{p-1}{2}} \equiv 1 \bmod p^2$ for a fixed natural number $a > 1$ with $(a,p) = 1$. For example,

a	2	3	17	3313
$3 < p < 131077$	3511 ($p < 6 \times 10^9$)	11 ($p < 10^7$)	46021, 48947	7, 17

(1) and (2) in the next are not useful to the computer but may be useful to consider Conjectures. Here the notation $|c|_d$ means the order of $c \bmod d$ for natural numbers c and d with $(c,d) = 1$.

The conjecture 1 is now open in case $p \equiv 3 \bmod 4$ and $q \equiv 3 \bmod 4$ though there are another unsolved cases.

Comment 3. *Let* p, q *are distinct primes. We set* $pj + qk = 1$, $\ell = pj^2 + qk^2$, $a = (pq)^\ell$, *and* $1 < d$ *is a common divisor of* $\Phi_p(q)$ *and* $\Phi_q(p)$. *Then the following hold.*

(1) $p = |q|_d$ *and* $q = |p|_d$.

(2) $a^p \equiv p$, $a^q \equiv q \bmod d$ *and* $pq = |a|_d$ *namely,* $\Phi_{pq}(a) \equiv 0 \bmod d$.

(3) $2pq \mid \varphi(d)$.

(4) *If* $p \equiv 3 \bmod 4$, *then* $d \equiv 1 \bmod 4$.

(5) *If* $p \equiv 3$ *and* $q \equiv 1 \bmod 4$, *then* A *does not divide* B.

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EMERITUS PROFESSOR OF HIROSAKI UNIVERSITY
TORIAGE 5-13-5, HIROSAKI, 036-8171, JAPAN
E-mail address: `moka.mocha_no_kaori@snow.ocn.ne.jp`

AN INTRODUCTION TO NONCOMMUTATIVE ALGEBRAIC GEOMETRY

IZURU MORI

ABSTRACT. Since classification of low dimensional projective varieties has been active and successful in algebraic geometry for many years, one of the major projects in noncommutative algebraic geometry is to classify low dimensional noncommutative projective varieties defined by Artin and Zhang. In this note, we will survey this project. Classification of noncommutative projective curves were completed by Artin and Stafford (1995). For classification of noncommutative projective surfaces, we have the following conjecture due to Artin (1997); every noncommutative projective surface is birationally equivalent to either (1) a quantum projective plane, (2) a quantum ruled surface, or (3) a surface finite over its center. Classification of quantum projective planes were completed by Artin, Tate and Van den Bergh (1990), however, classification of the other types of surfaces together with the above conjecture are still open.

1. QUASI-SCHEMES

Throughout, let k be an algebraically closed field. In this paper, we assume that all rings, schemes and abelian categories are noetherian. First, we define the basic object of study in noncommutative algebraic geometry.

Definition 1 (Artin-Zhang 1994 [6], Van den Bergh 2001 [18]). A quasi-scheme (over k) is a pair $X = (\operatorname{mod} X, \mathcal{O}_X)$ where $\operatorname{mod} X$ is a (k-linear) abelian category, and $\mathcal{O}_X \in \operatorname{mod} X$ is an object. Two quasi-schemes X and Y are isomorphic (over k) if there exists a (k-linear) equivalence functor $F : \operatorname{mod} X \to \operatorname{mod} Y$ such that $F(\mathcal{O}_X) \cong \mathcal{O}_Y$.

The above definition was modeled by the following example.

Example 2. A (usual) scheme X is a quasi-scheme $X = (\operatorname{mod} X, \mathcal{O}_X)$ where $\mathcal{O}_X$ is the structure sheaf on X, and $\operatorname{mod} X$ is the category of coherent $\mathcal{O}_X$-modules.

Notion of quasi-scheme includes noncommutative schemes.

Example 3. For a ring R, the noncommutative affine scheme associated to R is a quasi-scheme $\operatorname{Spec} R := (\operatorname{mod} R, R)$ where $\operatorname{mod} R$ is the category of finitely generated right R-modules. In fact, if R is commutative and $X = \operatorname{Spec} R$ in the usual sense, then the global section functor $\Gamma(X, -) : \operatorname{mod} X \to \operatorname{mod} R$ induces an isomorphism of quasi-schemes $(\operatorname{mod} X, \mathcal{O}_X) \to (\operatorname{mod} R, R)$.

Example 4. For a graded ring A, the noncommutative homogeneous affine scheme associated to A is a quasi-scheme $\operatorname{GrSpec} A := (\operatorname{grmod} A, A)$ where $\operatorname{grmod} A$ is the category of finitely generated graded right A-modules.

This is an expository paper. The detailed version of this paper will be submitted for publication elsewhere.

The most important example of a quasi-scheme in noncommutative algebraic geometry is the following.

Example 5 (Artin-Zhang 1994 [6]). For a graded ring A, the noncommutative projective scheme associated to A is a quasi-scheme $\operatorname{Proj} A := (\operatorname{tails} A, \mathcal{A})$ where

- $\operatorname{tors} A = \{M \in \operatorname{grmod} A \mid M_n = 0 \text{ for all } n \gg 0\}$ is the full subcategory consisting of torsion modules,
- $\operatorname{tails} A = \operatorname{grmod} A / \operatorname{tors} A$ is the quotient category,
- $\pi : \operatorname{grmod} A \to \operatorname{tails} A$ is the quotient functor, and
- $\mathcal{A} = \pi A \in \operatorname{tails} A$.

Note that $\mathcal{M} \cong \mathcal{N}$ in $\operatorname{tails} A$ if and only if $M_{\geq n} \cong N_{\geq n}$ in $\operatorname{grmod} A$ for some n.

The above definition was inspired by the following classical result.

Theorem 6 (Serre 1955 [15]). *If A is a commutative graded algebra finitely generated in degree 1 over k and $X = \operatorname{Proj} A$ in the usual sense, then the composition of functors*

$$\begin{array}{ccccc} \operatorname{mod} X & \longrightarrow & \operatorname{grmod} A & \xrightarrow{\pi} & \operatorname{tails} A \\ \mathcal{F} & \longrightarrow & \Gamma_*(X, \mathcal{F}) := \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F}(n)) & & \end{array}$$

induces an isomorphism of quasi-schemes $(\operatorname{mod} X, \mathcal{O}_X) \to (\operatorname{tails} A, \mathcal{A})$.

If A is a graded domain finitely generated in degree 1 over k of $\operatorname{GKdim} A = d+1$, then it is reasonable to call $\operatorname{Proj} A$ a noncommutative projective variety of dimension d. In particular, we call $\operatorname{Proj} A$ a noncommutative projective curve (resp. surface) if $\operatorname{GKdim} A = 2$ (resp. $\operatorname{GKdim} A = 3$). Since noncommutative projective curves were classified by Artin and Stafford (1995) [3], the next project is to classify noncommutative projective surfaces. This project is still wide open. We only have the conjecture below.

If A is a graded domain over k and $X = \operatorname{Proj} A$, then we define the function field of X by

$$k(X) := \{a/b \mid a, b \in A \text{ are homogeneous elements of the same degree}\}.$$

We say that two noncommutative projective varieties X and Y are birationally equivalent if $k(X) \cong k(Y)$ as k-algebras.

Conjecture (Artin 1997 [1]) Every noncommutative projective surface is birationally equivalent to one of the following:

(1) a quantum projective plane.
(2) a quantum ruled surface.
(3) a surface finite over its center.

Classification of quantum projective planes were completed by Artin, Tate and Van den Bergh (1990)[4], however, classification of the other types of surfaces together with the above conjecture are still open.

2. Weak Divisors

Definition 7. [8] Let X be a quasi-scheme over k. A weak divisor on X is a k-linear autoequivalence $D : \operatorname{mod} X \to \operatorname{mod} X$.

We denote by $\operatorname{WPic} X$ the group of weak divisors on X. For $D \in \operatorname{WPic} X$ and $n \in \mathbb{Z}$, we denote the n-fold composition of D by

$$\begin{aligned} D^n : \operatorname{mod} X &\to \operatorname{mod} X \\ \mathcal{M} &\mapsto \mathcal{M}(nD). \end{aligned}$$

If X is a (usual) scheme over k and D is a Cartier divisor on X, then

$$\begin{aligned} - \otimes_X \mathcal{O}_X(D) : \operatorname{mod} X &\to \operatorname{mod} X \\ \mathcal{F} &\mapsto \mathcal{F}(D) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D) \end{aligned}$$

is a weak divisor. More generally, the pair $(\sigma, \mathcal{L})$ where $\sigma \in \operatorname{Aut} X$ is an automorphism of X and $\mathcal{L} \in \operatorname{Pic} X$ is an invertible sheaf on X defines a weak divisor by

$$\begin{aligned} D = (\sigma, \mathcal{L}) : \operatorname{mod} X &\to \operatorname{mod} X \\ \mathcal{F} &\mapsto \mathcal{F}(D) := \sigma_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}). \end{aligned}$$

In fact, if X is a smooth projective variety with an ample or anti-ample canonical divisor, then every weak divisor is given by the pair as above [7].

If X is a quasi-scheme over k and $D \in \operatorname{WPic} X$, then we can construct a graded algebra over k by

$$B(X, D) := \bigoplus_{n \in \mathbb{Z}} \operatorname{Hom}_X(\mathcal{O}_X, \mathcal{O}_X(nD))$$

with the multiplication defined as follows:

$$\begin{array}{ccc} \operatorname{Hom}_X(\mathcal{O}_X, \mathcal{O}_X(mD)) \times \operatorname{Hom}_X(\mathcal{O}_X, \mathcal{O}_X(nD)) & & (a, b) \\ \downarrow & & \downarrow \\ \operatorname{Hom}_X(\mathcal{O}_X, \mathcal{O}_X((m+n)D)) & & ab := a(nD) \circ b \end{array}$$

$$\begin{tikzcd} & \mathcal{O}_X \arrow[r, "a"] \arrow[d, "nD"'] & \mathcal{O}_X(mD) \arrow[d, "nD"] \\ \mathcal{O}_X \arrow[r, "b"] & \mathcal{O}_X(nD) \arrow[r, "a(nD)"] & \mathcal{O}_X(mD)(nD) = \mathcal{O}_X((m+n)D). \end{tikzcd}$$

Example 8. If A is a graded algebra and $X = \operatorname{GrSpec} A$, then

$$\begin{aligned} (1) : \operatorname{grmod} A &\to \operatorname{grmod} A \\ M &\mapsto M(1) \end{aligned}$$

where $M(1)_i = M_{i+1}$ is a weak divisor on X, and

$$B(X, (1)) = \bigoplus_{n \in \mathbb{Z}} \operatorname{Hom}_{\operatorname{grmod} A}(A, A(n)) \cong A.$$

Example 9. Let X be a projective scheme over k. If D is a very ample divisor on X, then $B(X, D)$ is a homogeneous coordinate ring of X so that $X \cong \operatorname{Proj} B(X, D)$.

3. Noncommutative Projective Curves

We have a nice characterization of a quasi-scheme to be a noncommutative projective scheme as in the commutative case.

Definition 10 (Artin-Zhang 1994 [6]). Let X be a quasi-scheme over k and $D \in \operatorname{WPic} X$ a weak divisor. We say that D is ample if

- $\{\mathcal{O}_X(-nD)\}_{n\in\mathbb{N}}$ is a set of generators for $\operatorname{mod} X$, and
- for every epimorphism $\mathcal{M} \to \mathcal{N}$ in $\operatorname{mod} X$,
$$\operatorname{Hom}_X(\mathcal{O}_X(-nD), \mathcal{M}) \to \operatorname{Hom}_X(\mathcal{O}_X(-nD), \mathcal{N})$$
is surjective for all $n \gg 0$.

Roughly speaking, $D \in \operatorname{WPic} X$ is ample if and only if $\mathcal{O}_X(-nD)$ is a projective generator for $\operatorname{mod} X$ for $n \gg 0$.

Definition 11 (Artin-Zhang 1994 [6]). We say that a graded algebra A satisfies χ_1 if $\dim_k \operatorname{Ext}^1_A(A/A_{\geq 1}, M) < \infty$ for all $M \in \operatorname{grmod} A$.

The following result, analogous to the commutative case, is a characterization of a quasi-scheme to be projective.

Theorem 12 (Artin-Zhang 1994 [6]). *Let X be a* Hom*-finite quasi-scheme over k. Then $X \cong \operatorname{Proj} A$ for some graded algebra A satisfying χ_1 if and only if X has an ample weak divisor. In fact, if D is an ample weak divisor on X, then $X \cong \operatorname{Proj} B(X, D)$.*

Let X be a (usual) scheme over k. Recall that the pair $(\sigma, \mathcal{L}) \in \operatorname{Aut} X \times \operatorname{Pic} X$ defines a weak divisor $D = (\sigma, \mathcal{L}) \in \operatorname{WPic} X$. We denote $B(X, \sigma, \mathcal{L}) := B(X, D)$. If D is ample, then $X \cong \operatorname{Proj} B(X, \sigma, \mathcal{L})$, so we call $B(X, \sigma, \mathcal{L})$ a twisted homogeneous coordinate ring of X. Note that if $\sigma \neq \mathrm{id}$, then $B(X, \sigma, \mathcal{L})$ is typically a noncommutative graded algebra over k.

The following results says that every noncommutative projective curve is isomorphic to a commutative one, which completes the classification of noncommutative projective curves.

Theorem 13 (Artin-Stafford 1995 [3]). *If A is a graded domain finitely generated in degree 1 over k of* $\operatorname{GKdim} A = 2$, *so that* $\operatorname{Proj} A$ *is a noncommutative projective curve, then there exist a commutative projective curve X and an ample weak divisor $D = (\sigma, \mathcal{L})$ on X such that $A_{\geq n} \cong B(X, \sigma, \mathcal{L})_{\geq n}$ for some n. In particular,* $\operatorname{Proj} A \cong \operatorname{Proj} B(X, \sigma, \mathcal{L}) \cong X$.

4. Quantum Projective Planes

Next, we will define quantum projective planes and explain their classification.

Definition 14 (Artin-Schelter 1987 [2]). A graded algebra A is called a quantum polynomial algebra if

- $\operatorname{gldim} A = d < \infty$,
- $H_A(t) := \sum_{i\in\mathbb{N}} (\dim_k A_i) t^i = (1-t)^{-d}$, and
- $\operatorname{Ext}^i_A(k, A) = \begin{cases} k & \text{if } i = d \\ 0 & \text{if } i \neq d. \end{cases}$

Since the only commutative quantum polynomial algebra is a commutative polynomial algebra generated in degree 1 over k, if A is a quantum polynomial algebra of $\operatorname{gldim} A = d+1$, then it is reasonable to call $\operatorname{Proj} A$ a quantum projective space of dimension d. In particular, we call $\operatorname{Proj} A$ a quantum projective plane if $\operatorname{gldim} A = 3$.

Let X be a (usual) scheme over k. Recall that every very ample invertible sheaf $\mathcal{L} \in \operatorname{Pic} X$ on X defines an embedding into a projective space $X \to \mathbb{P}(V^*)$ where $V = \Gamma(X, \mathcal{L})$ and V^* is the vector space dual of V. If $\sigma \in \operatorname{Aut} X$ is an automorphism of X, then we can construct a quadratic algebra

$$A(X, \sigma, \mathcal{L}) := T(V)/(\{f \in V \otimes_k V \mid f|_{\Gamma_\sigma} = 0\}$$

where

$$\Gamma_\sigma := \{(p, \sigma(p)) \mid p \in X\} \subset \mathbb{P}(V^*) \times \mathbb{P}(V^*)$$

is the graph of X under σ.

There is a natural graded algebra homomorphism (often surjective) $A(X, \sigma, \mathcal{L}) \to B(X, \sigma, \mathcal{L})$, which induces a map of quasi-schemes (often an embedding) $\operatorname{Proj} B(X, \sigma, \mathcal{L}) \to \operatorname{Proj} A(X, \sigma, \mathcal{L})$.

The following result completes the classification of quantum projective planes.

Theorem 15 (Artin-Tate-Van den Bergh 1990 [4]). *A graded algebra A is a quantum polynomial algebra of* $\operatorname{gldim} A = 3$*, so that* $\operatorname{Proj} A$ *is a quantum projective plane, if and only if $A \cong A(X, \sigma, \mathcal{L})$ where either*

(1) $X = \mathbb{P}^2$, $\mathcal{L} = \mathcal{O}_X(1)$, *and* $\sigma \in \operatorname{Aut} \mathbb{P}^2$, *or*

(2) $X \subset \mathbb{P}^2$ *is a cubic divisor*, $\mathcal{L} = \mathcal{O}_X(1)$, *and* $\sigma \in \operatorname{Aut} X$ *such that*

$$\sigma^*(\mathcal{L}) \not\cong \mathcal{L}, \text{ but } (\sigma^2)^*(\mathcal{L}) \otimes_{\mathcal{O}_X} \mathcal{L} \cong \sigma^*(\mathcal{L}) \otimes_{\mathcal{O}_X} \sigma^*(\mathcal{L}).$$

Example 16. For a generic choice of $(a, b, c) \in \mathbb{P}^2$,

$$A := k\langle x, y, z\rangle/(cx^2 + bzy + ayz, azx + cy^2 + bxz, byx + axy + cz^2) \cong A(X, \sigma, \mathcal{O}_X(1))$$

is a quantum polynomial algebra of $\operatorname{gldim} A = 3$ where

$$X = \mathcal{V}((a^3 + b^3 + c^3)xyz - abc(x^3 + y^3 + z^3)) \subset \mathbb{P}^2$$

is a smooth elliptic curve and $\sigma \in \operatorname{Aut} X$ is the translation by the point $(a, b, c) \in X$ in the group law of X. The above algebra A is called a 3-dimensional Sklyanin algebra.

5. Quantum Ruled Surfaces

Let X be a smooth projective curve over k. We will define a quantum ruled surface over X. First, we recall a commutative ruled surface over X.

One of the characterizations of a ruled surface over X is a scheme defined by $\mathbb{P}(\mathcal{E}) := \operatorname{Proj} S(\mathcal{E})$ where

- $\mathcal{E}$ is a locally free $\mathcal{O}_X$-module of rank 2, and
- $S(\mathcal{E})$ is the symmetric algebra of $\mathcal{E}$ over $\mathcal{O}_X$.

Note that $S(\mathcal{E}) \cong T(\mathcal{E})/(\mathcal{Q})$ where

- $T(\mathcal{E})$ is the tensor algebra of $\mathcal{E}$ over $\mathcal{O}_X$, and
- $\mathcal{Q} \subset \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E}$ is an invertible $\mathcal{O}_X$-subbimodule locally generated by the sections of the form $xy - yx$.

We will extend this construction.

Recall that if R is a commutative ring, then R-R bimodules can be identified with $R \otimes R$-modules. If $X = \operatorname{Spec} R$, then $\operatorname{Spec}(R \otimes R) = X \times X$, so X-X bimodules can be identified with $X \times X$-modules.

Definition 17. Let X be a smooth projective variety over k. A coherent $\mathcal{O}_X$-bimodule is a coherent sheaf $\mathcal{M}$ on $X \times X$ such that

$$pr_i : \operatorname{Supp} \mathcal{M} \subset X \times X \to X$$

are finite for $i = 1, 2$ where $pr_i(x_1, x_2) = x_i$ are projection maps.

We say that a coherent $\mathcal{O}_X$-bimodule $\mathcal{E}$ is locally free of rank r if $pr_{i*}\mathcal{E}$ are locally free of rank r on X for $i = 1, 2$.

If X is a smooth projective variety over k, then every coherent locally free $\mathcal{O}_X$-bimodule $\mathcal{E}$ of rank r has a right adjoint $\mathcal{E}^*$ which is also a locally free $\mathcal{O}_X$-bimodule of rank r, that is,

$$\operatorname{Hom}_X(- \otimes_{\mathcal{O}_X} \mathcal{E}, -) \cong \operatorname{Hom}_X(-, - \otimes_{\mathcal{O}_X} \mathcal{E}^*).$$

We say that an invertible $\mathcal{O}_X$-subbimodule $\mathcal{Q} \subset \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E}$ is non-degenerate if the composition

$$\mathcal{E}^* \otimes_{\mathcal{O}_X} \mathcal{Q} \to \mathcal{E}^* \otimes_{\mathcal{O}_X} \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E} \to \mathcal{E}$$

is an isomorphism.

For the rest of this section, let X be a smooth projective curve over k.

Definition 18 (Van den Bergh 1996 [17]). A quantum ruled surface over X is a quasi-scheme $\mathbb{P}(\mathcal{E}) := (\operatorname{mod} \mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})})$ where

- $\mathcal{E}$ is a locally free $\mathcal{O}_X$-bimodule of rank 2,
- $\mathcal{Q} \subset \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E}$ is a non-degenerate invertible $\mathcal{O}_X$-subbimodule,
- $\mathcal{A} = T(\mathcal{E})/(\mathcal{Q})$ is the graded $\mathcal{O}_X$-algebra,
- $\operatorname{mod} \mathbb{P}(\mathcal{E}) = \operatorname{tails} \mathcal{A}$, and
- $\mathcal{O}_{\mathbb{P}(\mathcal{E})} = \pi(\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{A}) \in \operatorname{mod} \mathbb{P}(\mathcal{E})$, called the structure sheaf on $\mathbb{P}(\mathcal{E})$.

It is known that $\mathbb{P}(\mathcal{E})$ is independent of the choice of a non-degenerate $\mathcal{Q}$. In fact, $\mathcal{Q}$ is not even needed to define $\mathbb{P}(\mathcal{E})$ [19].

Although quantum ruled surfaces have been studied intensively (e.g. [9], [11], [12], [13], [14], [19]), classification of them is still wide open. We will end this paper by showing a recent progress on it.

Theorem 19. [10] *If $\mathcal{E}$ is a locally free $\mathcal{O}_X$-bimodule of rank 2, and $\mathcal{L}, \mathcal{M}$ are invertible $\mathcal{O}_X$-bimodules, then*

$$\mathbb{P}(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{M}) \cong \mathbb{P}(\mathcal{E}).$$

Corollary 20. [10] *If $\mathcal{E}$ is a decomposable locally free $\mathcal{O}_X$-bimodule of rank 2, then $\mathbb{P}(\mathcal{E}) \cong \mathbb{P}(\mathcal{O}_X \oplus \mathcal{L})$ for some invertible $\mathcal{O}_X$-bimodule $\mathcal{L}$.*

Every invertible $\mathcal{O}_X$-bimodule is isomorphic to

$$\mathcal{L}_\sigma := pr_2^*\mathcal{L} \otimes_{\mathcal{O}_{X \times X}} \mathcal{O}_{\Gamma_\sigma}$$

where $(\sigma, \mathcal{L}) \in \operatorname{Aut} X \times \operatorname{Pic} X$ [5], so quantum ruled surfaces $\mathbb{P}(\mathcal{E})$ such that $\mathcal{E}$ are decomposable are also classified by the triples $(X, \sigma, \mathcal{L})$.

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Department of Mathematics
Faculty of Science
Shizuoka University
Shizuoka 422-8529 JAPAN

E-mail address: `simouri@ipc.shizuoka.ac.jp`

PRIMITIVITY OF GROUP RINGS OF ASCENDING HNN EXTENSIONS OF FREE GROUPS

TSUNEKAZU NISHINAKA

ABSTRACT. Let H be a group, and let $\varphi : H \longrightarrow H$ be a monomorphism. The ascending HNN extension corresponding to φ is the group $H_\varphi = \langle H, t | t^{-1}ht = \varphi(h)\rangle$. A ring is (right) primitive if it has a faithful irreducible (right) module. Let F be a free group and K a field. We give a necessary and sufficient condition for the group ring KF_φ to be primitive.

1. INTRODUCTION

Let H be a group, and let $\varphi : H \longrightarrow H$ be a monomorphism. The ascending HNN extension corresponding to φ is the group $H_\varphi = \langle H, t | t^{-1}ht = \varphi(h)\rangle$. A ring is right primitive if it has a faithful irreducible right module. One can analogously define left primitive and generally two propreties are not equivalent. For our purpose, the two concepts are equivalent, for the group ring possesses a nice involution. Let F be a free group and K a field. Our purpose of this paper is the study of primitivity of the group ring KF_φ.

If $H \neq 1$ is a finite group or an abelian group, then the group ring KH can never be primitive. In fact, the only primitive commutative rings are fields, and in the case of finite $H \neq 1$, the density theorem would imply that primitive KH be simple, but the augmentention ideal belies that. The first nontrivial example of primitive group ring was offered by Formanek and Snider [7] in 1972. After that, many examples which include the complete solution for primitivity of group rings of polycyclic groups settled by Domanov [3], Farkas-Passman [4] and Roseblade [14] were constructed. Perhaps one of the most interesting result is the one on free products obtained by Formanek:

Theorem 1. ([6, Theorem 5]) *Let K be a field and $G = A * B$ a free product non-trivial groups (except $G = \mathbb{Z}_2 \times \mathbb{Z}_2$). Then KG is primitive.*

As a special case of the theorem, says KF is primitive for every field K provided that F is a nonabelian free group. Moreover, in the same paper, he remarks

Theorem 2. ([6]) *Let $G = \langle t\rangle \times F$ be the direct product of a free group F and the infinite cyclic group $\langle t\rangle$. Then KG is primitive if and only if $|K| \leq |F|$ (the cardinality of K is not larger than that of F).*

It is not difficult to see the result applying to the case of the cyclic extension of F by $\langle t\rangle$ (see Theorem 3 (i) below).

Now, the ascending HNN extension H_φ of a group H, which is a generalization of the cyclic extension of H by $\langle t\rangle$, is a well-studied class of groups. For example, Feighn and

The detailed version of this paper has been submitted for publication elsewhere.

Handel [5] described all subgroups of ascending HNN extensions of free groups and showed that ascending HNN extensions of free groups are coherent (that is, their f.g. subgroups are finitely presented). Hsu and Wise [9] have recently shown that ascending HNN extensions of polycyclic-by-finite groups are residually finite (that is, each nontrivial element of those groups can be mapped to a non-identity element in some homomorphism onto a finite group). They also study on residual finiteness for ascending HNN extensions of finitely generated free groups, which is conjectured in [8] (the condition 'finitely generated' cannot be dropped [1]). More recently, Borisov and Sapir, in their paper [2], have shown that the conjecture has a positive solution. That is, the ascending HNN extension F_φ of a finitely generated free group F is residually finite [2, Theorem 1.2]. Then by reduction method based on residual properties and on series in groups, we can see that KF_φ is semiprimitive (that is, the Jacobson radical is trivial) if the characteristic of K is zero. One might therefore hope that KF_φ is semiprimitive for any field K. In this paper, we shall show that KF_φ is semiprimitive for all K even if the rank of F is countably infinite (Corollary 10). In fact, KF_φ is often primitive, which is our main result:

Theorem 3: ([11, Theorem 1.1]) *Let F be a nonabelian free group, and F_φ the ascending HNN extension of F determined by φ.*
(i) *In case $\varphi(F) = F$, the group ring KF_φ is primitive for a field K if and only if either $|K| \leq |F|$ or F_φ is not virtually the direct product $F \times \mathbb{Z}$.*
(ii) *In case $\varphi(F) \neq F$, if the rank of F is at most countably infinite, then the group ring KF_φ is primitive for any field K.*

2. Ascending HNN Extensions of Free Groups

Throughout this paper, F denotes the nonabelian free group with the basis X, and $F_\varphi = \langle F, t | t^{-1} f t = \varphi(f)\rangle$ denotes the ascending HNN extension of F determined by φ. Let H be a group and N a subgroup of H. We denote by $[H : N]$ the index of N in H. For a group N', H is said to be virtually N' if N' is isomorphic to N and $[H : N] < \infty$. If h is an element of H, we let $C_N(h)$ denote the centralizer of h in N. Let $C(H)$ be the center of H and $\Delta(H)$ the FC center of H, that is $\Delta(H) = \{h \in H \mid [H : C_H(h)] < \infty\}$.

If f is a non-trivial element in F then $C_F(f)$ is infinite cyclic, and so $\Delta(F) = C(F)$ is trivial. On the other hand, $\Delta(F_\varphi)$ is not trivial in general. However, if $\Delta(F_\varphi)$ is non-trivial then $\Delta(F_\varphi) = C(F_\varphi)$ and F_φ is virtually the direct product $F \times \mathbb{Z}$:

Lemma 4. *Let F be a nonabelian free group.*
(i) $\Delta(F_\varphi) = C(F_\varphi)$.
(ii) $C(F_\varphi) \neq 1$ *if and only if F_φ is virtually the direct product $F \times \mathbb{Z}$. When this is the case, φ is an automorphism of F and there exist $n > 0$ and $f \in F$ such that $C(F_\varphi) = \langle t^n f\rangle$.*

Proof. Since $\Delta(F_\varphi) \supseteq C(F_\varphi)$, we may assume $\Delta(F_\varphi) \neq 1$. Let $1 \neq g \in \Delta(F_\varphi)$. We shall show $C(F_\varphi) = \langle g\rangle$. Since $[F_\varphi : C_{F_\varphi}(g)] < \infty$, we have $[F : C_F(g)] < \infty$, which implies $g \notin F$ because of $\Delta(F) = 1$. By the normal form theorem, there exist $n, l \geq 0$ and $f \in F$ such that $g = t^n f t^{-l}$, where $f \notin \varphi(F)$ if $n \neq 0$ and $l \neq 0$. Then replacing g by g^{-1} if necessary, we may assume that $n \geq l \geq 0$, and then $f \notin \varphi(F)$

unless $l = 0$. Since $[F_\varphi : C_{F_\varphi}(g)] < \infty$, there exists $m \geq 1$ such that $t^m g t^{-m} = g$, and so $t^{m+n} f t^{-l-m} = t^n f t^{-l}$; thus $f = \varphi^m(f) \in \varphi(F)$. Hence we get $l = 0$, that is $g = t^n f$ with $n > 0$. Then we may assume that n is minimal in $\{n' > 0 \mid t^{n'} f' \in \Delta(F_\varphi) \text{ with } f' \in F\}$. Again by $[F_\varphi : C_{F_\varphi}(g)] < \infty$, there exists $k > 0$ such that for each $x \in F$, $x^k g x^{-k} = g$, and so $x^k g x^{-k} = x^k t^n f x^{-k} = t^n \varphi^n(x)^k f x^{-k} = t^n f$; thus $\varphi^n(x)^k = (f x f^{-1})^k$. This implies $\varphi^n(x) = f x f^{-1}$ because F is a free group (c.f. [10]). In particular, $\varphi^n(f) = f$ and also $x \in C_{F_\varphi}(g)$. Furthermore we see that φ^n is an automorphism and so is φ.

Now, if $f = 1$, then $g = t^n \in C(F_\varphi)$, which completes the proof, and therefore we may assume $f \neq 1$. Since F is free, as is well known, $C_F(f)$ is cyclic, and thus $C_F(f) = \langle h \rangle$ for some $1 \neq h \in F$ and $f = h^m$ for some $m \neq 0$. Then $h^m = f = \varphi^n(f) = \varphi^n(h)^m$, and so $\varphi^n(h) = h$. Moreover, $\varphi(h) = \varphi^n(\varphi(h)) = f\varphi(h)f^{-1}$, which implies $\varphi(h) \in C_F(f)$ and thus $\varphi(h) = h^l$ for some $l \neq 0$. Since $h = \varphi^n(h) = h^{l^n}$, we have that $l = 1$, that is, $\varphi(h) = h$. Hence we get that $\varphi(f) = f$ which means $g \in C(F_\varphi)$. We have thus seen that the assertion of (i) holds and $C(F_\varphi) \supseteq \langle g \rangle$. Conversely, $C(F_\varphi) \subseteq \langle g \rangle$. In fact, if $g_1 \in C(F_\varphi)$, then we may assume that $g_1 = t^{n_1} f_1$ for some n_1 with $n_1 \geq n$ and for some $f_1 \in F$. It is obvious that $g_1 \in C(F_\varphi)$ if and only if $\varphi(f_1) = f_1$ and $\varphi^{n_1}(x) = f_1 x f_1^{-1}$ for every $x \in F$. Let $n_1 = mn + k$, where $m > 0$ and $0 \leq k < n$. For each $x \in F$, $f_1 x f_1^{-1} = \varphi^{n_1}(x) = \varphi^k(\varphi^{nm}(x)) = \varphi^k(f^m x f^{-m}) = f^m \varphi^k(x) f^{-m}$, and therefore, if we put $f_2 = f^{-m} f_1$, then $\varphi(f_2) = f_2$ and $\varphi^k(x) = f_2 x f_2^{-1}$ for every $x \in F$; thus $t^k f_2 \in C(F_\varphi)$. By the minimality of n, we get $k = 0$. That is, $f_2 \in C(F) = 1$, and so $f_1 = f^m$. Hence we conclude that $g_1 = t^{mn} f^m = (t^n f)^m = g^m \in \langle g \rangle$.

Since $FC(F_\varphi) = F\langle g \rangle \simeq F \times \mathbb{Z}$ and $[F_\varphi : FC(F_\varphi)] = [F\langle t \rangle : F\langle t^n \rangle] < \infty$, we see that F_φ is virtually $F \times \mathbb{Z}$. Conversely, if F_φ is virtually $F \times \mathbb{Z}$, then there exists $1 \neq g \in F_\varphi$ such that $g \in \Delta(F_\varphi)$, and so $g \in C(F_\varphi)$ by (i); thus $C(F_\varphi) \neq 1$. This completes the proof. $\square$

In what follows, for $f \in F$ and $i \geq 0$, we denote by $f^{[i]}$ the element $t^i f t^{-i}$ of $t^i F t^{-i}$. The next assertions are elementary and some of them can be found in [5].

Lemma 5. *Let N_0 be a subgroup of F with $\varphi(N_0) \subseteq N_0$. For each non-negative integer i, let $N_i = t^i N_0 t^{-i}$ and $N = \cup_{i=0}^{\infty} N_i$.*
(i) *$N_i \simeq N_0$ and $N_i \subseteq N_{i+1}$, where the equality holds if and only if $\varphi(N_0) = N_0$.*
(ii) *If N_0 is a normal subgroup of F, then N is a normal subgroup of F_φ.*
(iii) *If $[N_i, N_i]$ is the derived subgroup of N_i, then $[N, N] = \cup_{i=0}^{\infty}[N_i, N_i]$.*
(vi) *If the rank of N_0 is finite and $\varphi(N_0) \subset N_0$, then $\varphi([N_0, N_0]) \subset [N_0, N_0]$.*

3. Primitivity of Group Rings of F_φ

We will start this section with presenting the next two lemmas which are basic results on group rings (c.f. [13]).

Lemma 6. *Let K be a field, H a group and N a subgroup of H.*
(i)([16, Theorem 1]) *Suppose that N is normal. If $\Delta(H) = 1$ and $\Delta(H/N) = H/N$, then KN is primitive implies KH is primitive.*

(ii)([15, Theorem 3]) *If $\Delta(H)$ is torsion free abelian and $[H : N]$ is finite, then KN is primitive implies KH is primitive.*

Lemma 7. ([12, Theorem 2]) *Let K be a field and H a group. If $\Delta(H) = 1$ and KH is primitive, then for any field extension K' of K, $K'H$ is primitive.*

In view of Lemma 4, 6 (ii) and Theorem 2, we have immediately

Corollary 8. ([11, Corollary 2.6]) *Let K be a field, and suppose that $C(F_\varphi) \neq 1$. Then the group ring KF_φ is primitive if and only if K is any field with $|K| \leq |F|$.*

In what follows, let F be a free group with a countably infinite basis X, and $F_\varphi = \langle F, t | t^{-1}ft = \varphi(f)\rangle$ the ascending HNN extension of F determined by φ. For an element w in F, $\mathcal{R}(w)$ denotes the reduced word equivalent to w on X, and we set $\mathfrak{L}(w) = \{x^{\pm 1} \in X^{\pm 1} \mid x \text{ is a letter contained in } \mathcal{R}(w)\}$. For a non-negative integer i, let G_i be the subgroup of F_φ generated by $\{t^i f t^{-i} \mid f \in F\}$, and $G = \cup_{i=0}^{\infty} G_i$. Moreover, let K be a field with $|K| \leq |G|$, and KG denotes the group ring of G over K.

Let $\mathbb{N}$ be the set of positive integers. Since $|KG| = |\mathbb{N}|$, there exists bijection s from $\mathbb{N}$ to the elements of KG except for the zero element. Let $s(i) = s_i = \sum_{j=1}^{m_i} \alpha_{ij} f_{ij}^{[l_{ij}]}$, where $\alpha_{ij} \in K$, $f_{ij} \in F$, $m_i > 0$, $l_{ij} \geq 0$ and $f_{ij}^{[l_{ij}]} = t^{l_{ij}} f_{ij} t^{-l_{ij}} \in G_i$ satisfying

$$(3.1)\qquad f_{ij}^{[l_{ij}]} \neq f_{ij'}^{[l_{ij'}]} \text{ if } j \neq j', \text{ and } f_{ij} \notin \varphi(F) \text{ if } l_{ij} \neq 0.$$

For s_i above, we set $q_1 = max\{l_{1j} \mid 1 \leq j \leq m_1\}$, $S_1 = \mathfrak{L}(\varphi^{q_1 - l_{1j}}(f_{1j}) \mid 1 \leq j \leq m_1)$, and for $i > 1$, inductively $q_i = max\{q_{i-1}+1,\ l_{ij} \mid 1 \leq j \leq m_i\}$ and $S_i = \mathfrak{L}(\varphi^{q_i - l_{ij}}(f_{ij}),\ \varphi^{q_i - q_{i-1}}(x) \mid 1 \leq j \leq m_i,\ x \in S_{i-1})$. We choose three elements x_{11}, x_{12} and x_{13} in $X \setminus S_1$ which are different from each other, and set $B_1 = \widehat{B_1} = \{x_{11}, x_{12}, x_{13}\}$ and $S_{B_1} = \mathfrak{L}(\varphi^{q_2 - q_1}(x) \mid x \in B_1)$. Moreover, for $i > 1$, we set inductively $B_i = \{x_{i1}, x_{i2}, x_{i3}\}$, $\widehat{B_i} = \widehat{B_{i-1}} \cup B_i$, where $x_{i1}, x_{i2}, x_{i3} \in X \setminus (S_i \cup S_{B_{i-1}} \cup \widehat{B_{i-1}})$ with $x_{ik} \neq x_{ik'}$ ($k \neq k'$), and $S_{B_i} = \mathfrak{L}(\varphi^{q_{i+1} - q_i}(x) \mid x \in S_{B_{i-1}} \cup B_i)$. Because $|X|$ is countably infinite, $X \setminus (S_i \cup S_{B_{i-1}} \cup \widehat{B_{i-1}})$ is non-empty for every $i > 0$, in fact, it is an infinite set, and thereby the above argument is valid. Then we have that

$$(3.2)\qquad i \neq i' \Longrightarrow B_i \cap B_{i'} = \emptyset,$$

$$(3.3)\qquad i' \geq i \Longrightarrow \{x_{i'1}, x_{i'2}, x_{i'3}\} \cap \mathfrak{L}(\varphi^{q_{i'} - l_{ij}}(f_{ij}) \mid 1 \leq j \leq m_i) = \emptyset,$$

$$(3.4)\qquad i' > i \Longrightarrow \{x_{i'1}, x_{i'2}, x_{i'3}\} \cap \mathfrak{L}(\varphi^{q_{i'} - q_i}(x_{ik}) \mid 1 \leq k \leq 3) = \emptyset.$$

Here we define the element $\varepsilon(s_i)$ in KG for each s_i as follows;

$$(3.5)\qquad \varepsilon(s_i) = z_i^{[q_i]} s_i z_i^{[q_i]^{-1}} + x_{i1}^{[q_i]} z_i^{[q_i]} s_i z_i^{[q_i]^{-1}} + \alpha_{i1} x_{i2}^{[q_i]} z_i^{[q_i]} f_{i1}^{[q_i]} z_i^{[q_i]^{-1}},$$

where $z_i = x_{i2}^{-1} x_{i3}$ and $\{x_{i1}, x_{i2}, x_{i3}\} = B_i$.

The next lemma plays an essential role in the proof of our main result Theorem 3.

Lemma 9. ([11, Lemma 3.3]) *Let $\varepsilon(s_i)$ be as defined by (3.5) and let $\rho = \sum_{i=1}^{\infty} \varepsilon(s_i)KG$ be the right ideal of KG. Then ρ is a proper right ideal of KG.*

The proof of the above lemma is not short and so we omit it. The reader should refer to the paper [11]. By making use of the above lemma, we can prove Theorem 3:

Proof of Theorem 3 (i): If $\varphi(F) = F$ then F_φ/F is isomorphic to $\langle t\rangle$, and so $\Delta(F_\varphi/F) = F_\varphi/F$. In addition, if $C(F_\varphi) = 1$ then $\Delta(F_\varphi) = 1$ by Lemma 4 (i). By [6, Theorem 2], KF is primitive for any field K, and therefore it follows from Lemma 6 (i) that KF_φ is primitive. By virtue of Lemma 4, $C(F_\varphi) \neq 1$ if and only if F_φ is virtually $F \times \mathbb{Z}$, and hence the result follows from Corollary 8.

(ii): If $\varphi(F) \neq F$, then $\Delta(F_\varphi) = C(F_\varphi) = 1$ by Lemma 4. By virtue of Lemma 7, we may assume that K is a prime field. For each non-negative integer i, let $G_i = t^iFt^{-i}$, and $G = \cup_{i=0}^{\infty} G_i$. Moreover, let $D_i = [G_i, G_i] = t^i[F,F]t^{-i}$, the derived subgroup of G_i, and $D = \cup_{i=0}^{\infty} D_i$. If we put $N_0 = F$ in Lemma 5 (ii), then the lemma asserts that G is a normal subgroup of F_φ. It is obvious that F_φ/G is isomorphic to $\langle t\rangle$, and thereby, by virtue of Lemma 6 (i), it suffices to show that KG is primitive. If the rank of F is finite, then $D_0 = [F,F]$ is a free group of countably infinite rank. If we put $N_0 = D_0$ in Lemma 5 (i), then the lemma asserts that D_i is isomorphic to D_0 and $D_i \subseteq D_{i+1}$ (in fact, $\varphi(D_0) \neq D_0$; thus $D_i \subset D_{i+1}$ by (iv)) for every $i \geq 0$. Since G is locally free by lemma 5 (i), we see that the finite conjugate center of G is trivial. Moreover, G/D is abelian by Lemma 5 (iii), and therefore, again by Lemma 6 (i), it suffices to show that KD is primitive. In other words, we may further assume that the rank of F is countably infinite. Then KG satisfies all of the conditions supposed in Lemma 9.

Let $\varepsilon(s_i)$ be the element in KG defined by (3.5), and let $\rho = \sum_{i=1}^{\infty} \varepsilon(s_i)KG$ be the right ideal of KG. By Lemma 9, ρ is a proper right ideal of KG, and therefore, ρ is extended to a maximal right ideal ρ_m of KG. To complete the proof, we shall show that KG acts faithfully on the irreducible module KG/ρ_m. Let κ be the kernel of the action of KG on KG/ρ_m so that, certainly, $\kappa \subseteq \rho_m$. Now, if $\kappa \neq 0$, then κ contains the element s_i for some $i \in \mathbb{N}$, and therefore, by (3.5) the definition of $\varepsilon(s_i)$, we see that $\varepsilon(s_i) - g_i \in \kappa \subseteq \rho_m$, where $g_i = \alpha_{i1}x_{i2}^{[q_i]}z_i^{[q_i]}f_{i1}^{[q_i]}{z_i^{[q_i]}}^{-1}$ is a trivial unit in KG. On the other hand, $\varepsilon(s_i)$ is also contained in ρ_m; thus we conclude that $g_i \in \rho_m$, a contradiction. Hence the action is faithful, and KG is primitive. □

As a corollary of Theorem 3, we finally state the semiprimitivity of F_φ.

Corollary 10. ([11, Corollary 3.7]) *Let F be a nonabelian free group of at most countably infinite rank, and F_φ the ascending HNN extension of F determined by φ. If K is any field then the group ring KF_φ is semiprimitive.*

Proof. Let K_0 be the prime field of K. Since $|K_0| \leq |F|$, by virtue of Theorem 3, K_0F_φ is primitive and so semiprimitive. As is well known, semiprimitive group rings are separable algebras, thus semiprimitivity of group rings close under extensions of coefficient fields, and therefore KF_φ is semiprimitive. □

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OKAYAMA SHOKA UNIVERSITY
TSUSHIMA KYOUMACHI OKAYAMA 700-8601 JAPAN
E-mail address: nishinak@po.osu.ac.jp

HOCHSCHILD COHOMOLOGY AND STRATIFYING IDEALS

HIROSHI NAGASE

ABSTRACT. Suppose B is an algebra with a stratifying ideal BeB generated by an idempotent e. We will establish long exact sequences relating the Hochschild cohomology groups of the three algebras B, B/BeB and eBe. This provides a common generalization of various known results, all of which are extending Happel's long exact sequence for one-point extensions. Applying one of these sequences to Hochschild cohomology algebras modulo nilpotent shows, in some cases, that these algebras are finitely generated.

1. INTRODUCTION

Hochschild cohomology is not functorial. Thus there is no natural way to relate Hochschild cohomology of an algebra to that of its quotient or subalgebras. Still it is natural to try to find a way relating cohomology of an algebra B to that of an 'easier' or 'smaller' algebra A, such as a quotient modulo an idempotent ideal or a centralizer subalgebra. One such situation is that of B being a one-point extension of A, which has been studied by Happel in [7]. More recently, Happel's long exact sequence has been generalized to the case of triangular matrix algebras, for example by Michelena and Platzeck in [10], Green and Solberg in [6] and Cibils, Marcos, Redondo and Solotar in [2]. On the other hand, in [11], de la Peña and Xi have generalized Happel's long exact sequence to the case of algebras with heredity ideals.

Here, these results will be extended further. A natural common generalization of both triangular algebras and algebras with heredity ideals are algebras with stratifying ideals; indeed, heredity ideals are stratifying and any triangular matrix algebra has stratifying ideals such that the quotients are the respective triangular parts. A stratifying ideal of a finite dimensional algebra B is generated by an idempotent e in B. By one of our long exact sequences in Theorem 6

$$\cdots \to \mathrm{Ext}^n_{B^e}(B/BeB, BeB) \to \mathrm{HH}^n(B) \to \mathrm{HH}^n(B/BeB) \oplus \mathrm{HH}^n(eBe) \to \cdots,$$

we can compare Hochschild cohomology groups of three algebras B, B/BeB and eBe. We will get this long exact sequence, and another two long exact sequences (Theorem 6), by using elementary homological methods based on a key observation in triangulated categories, see Lemma 1.

For any finite dimensional algebra B, we will apply our long exact sequence to the quotient of the Hochschild cohomology algebra $\mathrm{HH}^*(B)$ modulo the ideal $\mathcal{N}_B$ generated by homogeneous nilpotent elements. We denote by $\overline{\mathrm{HH}}^*(B)$ the graded factor algebra $\mathrm{HH}^*(B)/\mathcal{N}_B$.

In [13], Snashall and Solberg conjectured that $\overline{\mathrm{HH}}^*(A)$ is a finitely generated algebra for any finite dimensional algebra A. Green, Snashall and Solberg have shown the conjecture

The detailed version of this paper will be submitted for publication elsewhere.

to hold true for self-injective algebras of finite representation type [4] and for monomial algebras [5].

In Corollary 7, for any algebra B with a stratifying ideal BeB, we get an injective graded algebra homomorphism

$$\overline{\mathrm{HH}}^*(B) \hookrightarrow \overline{\mathrm{HH}}^*(B/BeB) \times \overline{\mathrm{HH}}^*(eBe)$$

Applying this embedding, we verify the conjecture for the case of Brauer algebra $B_k(3,\delta)$ in Example 13. Moreover, we consider a condition when the above embedding induces an isomorphism $\overline{\mathrm{HH}}^*(B) \cong \overline{\mathrm{HH}}^*(eBe)$ of graded algebras. By using this isomorphism, we can produce many examples of finite dimensional algebras for which the conjecture holds, including an algebra which is neither self-injective nor monomial (Example 12).

2. A GENERAL LEMMA

Throughout this paper we assume that k is a commutative noetherian ring and algebras are associative unital k-algebras that are projective as k-modules. For any algebra A, $\operatorname{mod} A$ denotes the category of finitely generated left A-modules, A^e the enveloping algebra $A \otimes_k A^{\mathrm{op}}$, $\mathrm{HH}^n(A)$ the n-th Hochschild cohomology group of A with coefficients in A itself and $\mathrm{HH}^*(A)$ the Hochschild cohomology algebra $\oplus_n \mathrm{HH}^n(A)$. It is known that $\mathrm{HH}^n(A) \cong \mathrm{Ext}^n_{A^e}(A,A)$ as groups and $\mathrm{HH}^*(A) \cong \oplus_n \mathrm{Ext}^n_{A^e}(A,A)$ as graded algebras.

For simplicity we will use the language of triangulated categories only in Lemma 1 below. Let $\mathcal{T}$ be a triangulated category with a shift functor T. For any X and Y in $\mathcal{T}$ and any $n \in \mathbb{Z}$, we denote by $\mathcal{T}^n(X,Y)$ the morphism group $\mathcal{T}(X, T^nY)$ and by $\mathcal{T}^*(X,X)$ the graded ring $\oplus_{n\in\mathbb{Z}}\mathcal{T}^n(X,X)$. If $\mathcal{T}$ is a derived category $D(\operatorname{Mod} A^e)$ for some algebra A, then $\mathcal{T}^n(A,A) \cong \mathrm{Ext}^n_{A^e}(A,A) \cong \mathrm{HH}^n(A)$ as groups and $\mathcal{T}^*(A,A) \cong \mathrm{HH}^*(A)$ as graded algebras. The following is the key lemma of this paper.

Lemma 1. *Let $\mathcal{T}$ be a triangulated category. Suppose there is a triangle $X \to Y \to Z \to$ in $\mathcal{T}$ such that $\mathcal{T}^n(X,Z) = 0$ for all $n \in \mathbb{Z}$.*

(1) *We have the following three long exact sequences:*
$\cdots \to \mathcal{T}^n(Y,X) \to \mathcal{T}^n(Y,Y) \to \mathcal{T}^n(Z,Z) \to \mathcal{T}^{n+1}(Y,X) \to \cdots$;
$\cdots \to \mathcal{T}^n(Z,Y) \to \mathcal{T}^n(Y,Y) \to \mathcal{T}^n(X,X) \to \mathcal{T}^{n+1}(Z,Y) \to \cdots$; *and*
$\cdots \to \mathcal{T}^n(Z,X) \xrightarrow{u_n} \mathcal{T}^n(Y,Y) \to \mathcal{T}^n(Z,Z) \oplus \mathcal{T}^n(X,X) \to \mathcal{T}^{n+1}(Z,X) \to \cdots$.

(2) *Let $u : \mathcal{T}^*(Y,Y) \to \mathcal{T}^*(Z,Z) \times \mathcal{T}^*(X,X)$ be the graded ring homomorphism induced from the third long exact sequence. Then $(\operatorname{Ker} u)^2$ vanishes.*

The following facts are well-known (see [1]).

Lemma 2. *Let X be an A-B-bimodule, Y a B-C-bimodule and Z an A-C-bimodule. Then there are the following isomorphisms:*

(1) *If $\mathrm{Tor}^B_i(X,Y) = 0$ and $\mathrm{Ext}^i_C(Y,Z) = 0$ for all $i \geq 1$ then, for any $n \geq 0$,*

$$\mathrm{Ext}^n_{A-C}(X \otimes_B Y, Z) \cong \mathrm{Ext}^n_{A-B}(X, \mathrm{Hom}_C(Y,Z)).$$

(2) *If $\mathrm{Tor}^B_i(X,Y) = 0$ and $\mathrm{Ext}^i_A(X,Z) = 0$ for all $i \geq 1$ then, for any $n \geq 0$,*

$$\mathrm{Ext}^n_{A-C}(X \otimes_B Y, Z) \cong \mathrm{Ext}^n_{B-C}(Y, \mathrm{Hom}_A(X,Z)).$$

3. Stratifying ideals

In this section we study Hochschild cohomology groups of algebras with stratifying ideals. The following definition is due to Cline, Parshall and Scott ([3], 2.1.1 and 2.1.2), who work with finite dimensional algebras over fields. We keep our general setup of algebras projective over a commutative noetherian ring.

Definition 3. Let B be an algebra and $e = e^2$ an idempotent. The two-sided ideal BeB generated by e is called a *stratifying ideal* if the following equivalent conditions (A) and (B) are satisfied:
(A) (a) The multiplication map $Be \otimes_{eBe} eB \to BeB$ is an isomorphism.
(b) For all $n > 0$: $\mathrm{Tor}_n^{eBe}(Be, eB) = 0$.
(B) The epimorphism $B \to A := B/BeB$ induces isomorphisms

$$\mathrm{Ext}^*_A(X, Y) \simeq \mathrm{Ext}^*_B(X, Y)$$

for all A-modules X and Y

The following remark can be used to check if an ideal is stratifying.

Remark 4. Let e be an idempotent element in B. Then BeB is projective as a left (resp. right) B-module if and only if eB (resp. Be) is projective as a left (respectively right) eBe-module and the multiplication map $Be \otimes_{eBe} eB \to BeB$ is an isomorphism.

Proof. Suppose that BeB is a projective left B-module. Then $Be \otimes_k eB \twoheadrightarrow BeB$ splits in $\mathrm{mod}\, B$. Multiplying by e on the left hand side, $eBe \otimes_k eB \twoheadrightarrow eB$ splits in $\mathrm{mod}\, eBe$. Thus eB is a projective left eBe-module. Let X be the kernel of the multiplication map $Be \otimes_{eBe} eB \twoheadrightarrow BeB$. Multiplying by e on the left hand side, eX is the kernel of the multiplication map $eBe \otimes_{eBe} eB \twoheadrightarrow eB$. But the latter multiplication map is an isomorphism, and therefore $eX = 0$. Applying the functor $\mathrm{Hom}_B(-, X)$ to the short exact sequence $0 \to X \to Be \otimes_{eBe} eB \to BeB \to 0$, yields a short exact sequence

$$0 \to \mathrm{Hom}_B(BeB, X) \to \mathrm{Hom}_B(Be \otimes_{eBe} eB, X) \to \mathrm{Hom}_B(X, X) \to 0,$$

because BeB is a projective left B-module. Since the middle term $\mathrm{Hom}_B(Be \otimes_{eBe} eB, X) \cong \mathrm{Hom}_{eBe}(eB, eX) = 0$, we get $End_B(X) = 0$ and thus $X = 0$, so that the multiplication map $Be \otimes_{eBe} eB \twoheadrightarrow BeB$ is an isomorphism.
The converse is shown by using the isomorphism

$$\mathrm{Hom}_B(BeB, -) \cong \mathrm{Hom}_B(Be \otimes_{eBe} eB, -) \cong \mathrm{Hom}_{eBe}(eB, \mathrm{Hom}_B(Be, -)).$$

□

Heredity ideals are examples of stratifying ideals, thus our results will extend results obtained in [11]. On the other hand, for any triangulated algebra

$$B = \begin{pmatrix} A & 0 \\ M & C \end{pmatrix}$$

and the idempotent

$$e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

we have $BeB = eB$, so that BeB is a stratifying ideal. Thus our results also will extend results of [2, 6, 10]. There are, however, plenty of other examples. Stratifying ideals and stratified algebras occur frequently in applications, for example in algebraic Lie theory in the context of Schur algebras and of blocks of the Bernstein-Gelfand-Gelfand category of a semisimple complex Lie algebra.

From now on, we assume that BeB is a stratifying ideal of B and we put $A := B/BeB$.

Proposition 5. *For any $i \geq 0$, the following hold:*

(1) $\mathrm{Ext}^i_{B^e}(BeB, A) = 0$.
(2) $\mathrm{Ext}^i_{B^e}(BeB, BeB) \cong \mathrm{Ext}^i_{(eBe)^e}(eBe, eBe)$.
(3) $\mathrm{Ext}^i_{A^e}(A, A) \cong \mathrm{Ext}^i_{B^e}(A, A)$.
(4) *The isomorphisms in (2) and (3) preserve Yoneda products.*

Theorem 6. *There are long exact sequences as follows:*

(1) $\cdots \to \mathrm{Ext}^n_{B^e}(B, BeB) \to \mathrm{HH}^n(B) \to \mathrm{HH}^n(A) \to \cdots$;
(2) $\cdots \to \mathrm{Ext}^n_{B^e}(A, B) \to \mathrm{HH}^n(B) \to \mathrm{HH}^n(eBe) \to \cdots$; *and*
(3) $\cdots \to \mathrm{Ext}^n_{B^e}(A, BeB) \to \mathrm{HH}^n(B) \xrightarrow{f} \mathrm{HH}^n(A) \oplus \mathrm{HH}^n(eBe) \to \cdots$.

Proof. By Lemma 1 and Proposition 5 . □

We remark that by using the partial recollement of bounded below derived categories

$$D^+(\mathrm{mod}\, B/BeB) \;\equiv\; D^+(\mathrm{mod}\, B) \;\equiv\; D^+(\mathrm{mod}\, eBe),$$

we also can get the long exact sequence (3).

We also note that Suarez-Alvarez [12] independently has obtained the first long exact sequence in Theorem 6 above by using different methods based on spectral sequences.

Recall the notation that $\mathcal{N}_B$ is the ideal of $\mathrm{HH}^*(B)$ which is generated by homogeneous nilpotent elements, and $\overline{\mathrm{HH}}^*(B)$ is the factor algebra $\mathrm{HH}^*(B)/\mathcal{N}_B$.

Corollary 7.

(1) *Let $f : \mathrm{HH}^*(B) \to \mathrm{HH}^*(A) \times \mathrm{HH}^*(eBe)$ be the graded algebra homomorphism in sequence (3) above. Then $(\mathrm{Ker}\, f)^2$ vanishes.*
(2) *The induced homomorphism $\overline{f} : \overline{\mathrm{HH}}^*(B) \to \overline{\mathrm{HH}}^*(A) \times \overline{\mathrm{HH}}^*(eBe)$ is injective.*

Proof. By Lemma 1 and statement (4) of Proposition 5.

□

4. Examples

By adapting the well-known recursive constructions of quasi-hereditary algebras, we construct for any algebra C a new algebra B which is an extension of C and has a stratifying ideal. We will compare the Hochschild cohomology algebras of C and of B. For simplicity we will assume all algebras to be finite dimensional and split over a field k.

Let A and C be algebras, M a C-A-bimodule and N an A-C-bimodule. For any morphism $\mu : M \otimes_A N \to \mathrm{rad}\, C$ of C-C-bimodules, we can define a split extension $\tilde{A}$ of A by $N \otimes_C M$ (where $N \otimes_C M$ multiplies trivially with itself) so that we get an algebra (with multiplication induced by μ)

$$B = \begin{pmatrix} \tilde{A} & N \\ M & C \end{pmatrix}.$$

For the idempotent

$$e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

we observe that $A \cong B/BeB$, that $C \cong eBe$ and that the multiplication map $Be \otimes_{eBe} eB \to BeB$ is an isomorphism. We keep the notation above in this section.

Lemma 8. *If $\mathrm{Tor}^n_C(N, M) = 0$ for any $n \geq 1$, then BeB is a stratifying ideal.*

Lemma 9. *Let A be the ground field k. If ${}_CM$ and N_C are projective C-modules, then $\mathrm{pd}_{B^e}A \leq 2$.*

Lemma 10. *Let D be a finite dimensional algebra, split over the field k.*

(1) *Let n be the number of blocks of D. Then $\overline{\mathrm{HH}}^0(D) \cong k^n$ as an algebra.*

(2) *If $\mathrm{char} k \neq 2$, then $\overline{\mathrm{HH}}^*(D) \cong \overline{\mathrm{HH}}^{even}(D) := \oplus_{n \geq 0} \overline{\mathrm{HH}}^{2n}(D)$.*

Proposition 11. *Let A be the ground field k. If ${}_CM$ and N_C are non-zero projective C-modules, the number of blocks of C is the same as that of B. If $\mathrm{char} k \neq 2$, then $\overline{\mathrm{HH}}^*(B) \cong \overline{\mathrm{HH}}^*(C)$ as graded algebras.*

Proof. By Lemma 10, it is enough to show that $\overline{\mathrm{HH}}^{even}(B) \cong \overline{\mathrm{HH}}^{even}(C)$. By Lemma 8, BeB is stratifying. By Lemma 9, Theorem 6 and $\mathrm{HH}^*(A) \cong k$, we have that $\mathrm{HH}^n(B) \cong \mathrm{HH}^n(C)$ for any $n \geq 3$ and $\mathrm{HH}^2(B) \to \mathrm{HH}^2(C)$ is surjective. Hence, by Corollary 7, $\overline{\mathrm{HH}}^n(B) \cong \overline{\mathrm{HH}}^n(C)$ for any $n \geq 2$. By Lemma 10, $\dim_k \overline{\mathrm{HH}}^0(B)=$ the number of blocks of B equals the number of blocks of $C=\dim_k \overline{\mathrm{HH}}^0(C)$. Therefore $\overline{\mathrm{HH}}^{even}(B) \cong \overline{\mathrm{HH}}^{even}(C)$ as graded algebras. $\square$

The following example shows that we cannot drop the condition $\mathrm{char} k \neq 2$ in Proposion 11 above.

Example 12. Keep the notation in the previous section. Let A be the ground field k, C a truncated polynomial algebra $k[x]/(x^p)$. If $M = C$, $N = C$, $\mu : M \otimes_A N \to \mathrm{rad}\, C$ is defined by $\mu(1 \otimes 1) = x^q$ and $1 \leq q < p$, then B is given by the following quiver

$$\begin{tikzcd} 1 \arrow[r, shift left, "a"] & 2 \arrow[l, shift left, "b"] \arrow[loop right, "c"] \end{tikzcd}$$

with two relations $c^p = 0$ and $ab = c^q$. Note that B is neither self-injective nor monomial unless $q = 1$. By Proposition 11, if $\mathrm{char} k \neq 2$, then $\overline{\mathrm{HH}}^*(B) \cong \overline{\mathrm{HH}}^*(C)$. Since $\overline{\mathrm{HH}}^*(C)$ is a finitely generated algebra (see [4]), so is $\overline{\mathrm{HH}}^*(B)$.

On the other hand, if $\mathrm{char} k = 2$, $q = 1$ and $p = 2$, then $\overline{\mathrm{HH}}^*(B) \cong k[x, z]/(x^3 - z^2)$ with deg x=2 and deg z=3 by [13] and $\overline{\mathrm{HH}}^*(C) \cong k[x]$ with deg x=1 by [8] or [4]. Hence $\overline{\mathrm{HH}}^*(B)$ is strictly contained in $\overline{\mathrm{HH}}^*(C)$, so that we cannot drop the condition $\mathrm{char} k \neq 2$ in Proposition 11.

Finally we give an example of an algebra occuring in algebraic Lie theory, see for instance [9] for the properties of Brauer algebras used in this example.

Example 13. Let B be a Brauer algebra $B_k(3,\delta)$, where δ is in k. B has a stratifying ideal BeB such that $eBe \cong k$ and $B/BeB \cong k\Sigma_3$, where Σ_3 is the symmetric group on three letters. By Corollary 7, there exists an embedding

$$\overline{\mathrm{HH}}^*(B) \hookrightarrow \overline{\mathrm{HH}}^*(k\Sigma_3) \times \overline{\mathrm{HH}}^*(k)$$

as a graded algebra homomorphism. Since $k\Sigma_3$ is a self-injective algebra of finite representation type, $\overline{\mathrm{HH}}^*(k\Sigma_3)$ is isomorphic to a product of some polynomial algebras in one variable $k[x]$ and some copies of the ground field k (see [4]). Because any graded subalgebra of a product of some polynomial algebras with one variable $k[x]$ is a finitely generated algebra, we get that $\overline{\mathrm{HH}}^*(B_k(3,\delta))$ is a finitely generated algebra.

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Nara National College of Technology
22 Yata-cho, Yamatokoriyama,
Nara, 639-1080, JAPAN
E-mail address: nagase@libe.nara-k.ac.jp

ON CONTRAVARIANTLY FINITE SUBCATEGORIES OF FINITELY GENERATED MODULES

RYO TAKAHASHI

ABSTRACT. This paper studies contravariantly finite resolving subcategories of the category of finitely generated modules over a commutative ring. The main theorem of this paper implies that there exist only three contravariantly finite resolving subcategories over a henselian Gorenstein local ring. It also implies the theorem of Christensen, Piepmeyer, Striuli and Takahashi.

Key Words: contravariantly finite subcategory, resolving subcategory, Gorenstein ring, Cohen-Macaulay ring, maximal Cohen-Macaulay module, totally reflexive module.

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INTRODUCTION

The notion of a contravariantly finite subcategory (of the category of finitely generated modules) was first introduced over artin algebras by Auslander and Smalø [6] in connection with studying the problem of which subcategories admit almost split sequences. The notion of a resolving subcategory was introduced by Auslander and Bridger [3] in the study of modules of Gorenstein dimension zero, which are now also called totally reflexive modules. There is an application of contravariantly finite resolving subcategories to the study of the finitistic dimension conjecture [5].

This paper deals with contravariantly finite resolving subcategories over commutative rings. Let R be a commutative noetherian henselian local ring. We denote by $\operatorname{mod} R$ the category of finitely generated R-modules, by $\mathcal{F}(R)$ the full subcategory of free R-modules, and by $\mathcal{C}(R)$ the full subcategory of maximal Cohen-Macaulay R-modules. The subcategory $\mathcal{F}(R)$ is always contravariantly finite, and so is $\mathcal{C}(R)$ provided that R is Cohen-Macaulay. The latter fact is known as the Cohen-Macaulay approximation theorem, which was shown by Auslander and Buchweitz [4].

In this paper, we shall prove the following amazing theorem; the category of finitely generated modules over a henselian Gorenstein local ring possesses only three contravariantly finite resolving subcategories.

Theorem A. *If R is Gorenstein, then all the contravariantly finite resolving subcategories of* $\operatorname{mod} R$ *are* $\mathcal{F}(R)$, $\mathcal{C}(R)$ *and* $\operatorname{mod} R$.

This theorem especially says that if R is a commutative selfinjective local ring, then there are no contravariantly finite resolving subcategories other than $\mathcal{F}(R)$ and $\operatorname{mod} R$.

The main theorem of this paper asserts the following: let $\mathcal{X}$ be a resolving subcategory of $\operatorname{mod} R$ such that the residue field of R has a right $\mathcal{X}$-approximation. Assume that

The detailed version of this paper will be submitted for publication elsewhere.

there exists an R-module $G \in \mathcal{X}$ of infinite projective dimension with $\mathrm{Ext}_R^i(G,R)=0$ for $i \gg 0$. Let M be an R-module such that each $X \in \mathcal{X}$ satisfies $\mathrm{Ext}_R^i(X,M)=0$ for $i \gg 0$. Then M has finite injective dimension. From this result, we will prove the following two theorems. Theorem A will be obtained from Theorem B. The assertion of Theorem C is a main result of [10], which has been a motivation for this paper. (Our way of obtaining Theorem C is quite different from the original proof given in [10].)

Theorem B. *Let $\mathcal{X} \neq \mathrm{mod}\, R$ be a contravariantly finite resolving subcategory of* $\mathrm{mod}\, R$*. Suppose that there is an R-module $G \in \mathcal{X}$ of infinite projective dimension such that* $\mathrm{Ext}_R^i(G,R)=0$ *for $i \gg 0$. Then R is Cohen-Macaulay and $\mathcal{X} = \mathcal{C}(R)$.*

Theorem C (Christensen-Piepmeyer-Striuli-Takahashi). *Suppose that there is a nonfree R-module in $\mathcal{G}(R)$. If $\mathcal{G}(R)$ is contravariantly finite in* $\mathrm{mod}\, R$*, then R is Gorenstein.*

Here, $\mathcal{G}(R)$ denotes the full subcategory of totally reflexive R-modules. A totally reflexive module, which is also called a module of Gorenstein dimension (G-dimension) zero, was defined by Auslander [2] as a common generalization of a free module and a maximal Cohen-Macaulay module over a Gorenstein local ring. Auslander and Bridger [3] proved that the full subcategory of totally reflexive modules over a left and right noetherian ring is resolving. The other details of totally reflexive modules are stated in [3] and [9].

If R is Gorenstein, then $\mathcal{G}(R)$ coincides with $\mathcal{C}(R)$, and so $\mathcal{G}(R)$ is contravariantly finite by virtue of the Cohen-Macaulay approximation theorem. Thus, Theorem C can be viewed as the converse of this fact. Theorem C implies the following: let R be a homomorphic image of a regular local ring. Suppose that there is a nonfree totally reflexive R-module and are only finitely many nonisomorphic indecomposable totally reflexive R-modules. Then R is an isolated simple hypersurface singularity. For the details, see [10].

Conventions

In the rest of this paper, we assume that all rings are commutative and noetherian, and that all modules are finitely generated. Unless otherwise specified, let R be a henselian local ring. The unique maximal ideal of R and the residue field of R are denoted by $\mathfrak{m}$ and k, respectively. We denote by $\mathrm{mod}\, R$ the category of finitely generated R-modules. By a *subcategory* of $\mathrm{mod}\, R$, we always mean a full subcategory of $\mathrm{mod}\, R$ which is closed under isomorphisms. Namely, in this paper, a subcategory $\mathcal{X}$ of $\mathrm{mod}\, R$ means a full subcategory such that every R-module which is isomorphic to some R-module in $\mathcal{X}$ is also in $\mathcal{X}$.

1. Contravariant finiteness of totally reflexive modules

In this section, we will state background materials which motivate the main results of this paper. We start by recalling the definition of a totally reflexive module.

Definition 1. We denote by $(-)^*$ the R-dual functor $\mathrm{Hom}_R(-,R)$. An R-module M is called *totally reflexive* (or *of Gorenstein dimension zero*) if

(1) the natural homomorphism $M \to M^{**}$ is an isomorphism, and
(2) $\mathrm{Ext}_R^i(M,R) = \mathrm{Ext}_R^i(M^*,R) = 0$ for any $i > 0$.

We introduce three subcategories of $\operatorname{mod} R$ which will often appear throughout this paper.

We denote by $\mathcal{F}(R)$ the subcategory of $\operatorname{mod} R$ consisting of all free R-modules, by $\mathcal{G}(R)$ the subcategory of $\operatorname{mod} R$ consisting of all totally reflexive R-modules, and by $\mathcal{C}(R)$ the subcategory of $\operatorname{mod} R$ consisting of all maximal Cohen-Macaulay R-modules. By definition, $\mathcal{F}(R)$ is contained in $\mathcal{G}(R)$. If R is Cohen-Macaulay, then $\mathcal{G}(R)$ is contained in $\mathcal{C}(R)$. If R is Gorenstein, then $\mathcal{G}(R)$ coincides with $\mathcal{C}(R)$.

Next, we recall the notion of a right approximation over a subcategory of $\operatorname{mod} R$.

Definition 2. Let $\mathcal{X}$ be a subcategory of $\operatorname{mod} R$.

(1) Let $\phi : X \to M$ be a homomorphism of R-modules with $X \in \mathcal{X}$. We say that ϕ is a *right $\mathcal{X}$-approximation* (of M) if the induced homomorphism $\operatorname{Hom}_R(X', \phi) : \operatorname{Hom}_R(X', X) \to \operatorname{Hom}_R(X', M)$ is surjective for any $X' \in \mathcal{X}$.

(2) We say that $\mathcal{X}$ is *contravariantly finite* (in $\operatorname{mod} R$) if every R-module has a right $\mathcal{X}$-approximation.

The following result is well-known.

Theorem 3 (Auslander-Buchweitz). *Let R be a Cohen-Macaulay local ring. Then $\mathcal{C}(R)$ is contravariantly finite.*

Corollary 4. *If R is Gorenstein, then $\mathcal{G}(R)$ is contravariantly finite.*

The converse of this corollary essentially holds:

Theorem 5. [10] *Suppose that there is a nonfree totally reflexive R-module. If $\mathcal{G}(R)$ is contravariantly finite in* $\operatorname{mod} R$*, then R is Gorenstein.*

This theorem yields the following corollary, which is a generalization of [12, Theorem 1.3].

Corollary 6. *Let R be a non-Gorenstein local ring. If there is a nonfree totally reflexive R-module, then there are infinitely many nonisomorphic indecomposable totally reflexive R-modules.*

Combining this with [13, Theorems (8.15) and (8.10)] (cf. [11, Satz 1.2] and [8, Theorem B]), we obtain the following result.

Corollary 7. *Let R be a homomorphic image of a regular local ring. Suppose that there is a nonfree totally reflexive R-module but there are only finitely many nonisomorphic indecomposable totally reflexive R-modules. Then R is a simple hypersurface singularity.*

2. Contravariantly finite resolving subcategories

In this section, we will give the main theorem of this paper and several results it yields. One of them implies Theorem 5, which is the motive fact of this paper.

First of all, we recall the definition of the syzygies of a given module. Let M be an R-module and n a positive integer. Let

$$F_\bullet = (\cdots \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to 0)$$

be a minimal free resolution of M. We define the *nth syzygy* $\Omega^n M$ of M as the image of the homomorphism d_n. We set $\Omega^0 M = M$.

We recall the definition of a resolving subcategory.

Definition 8. A subcategory $\mathcal{X}$ of $\operatorname{mod} R$ is called *resolving* if it satisfies the following four conditions.

(1) $\mathcal{X}$ contains R.
(2) $\mathcal{X}$ is closed under direct summands: if M is an R-module in $\mathcal{X}$ and N is a direct summand of M, then N is also in $\mathcal{X}$.
(3) $\mathcal{X}$ is closed under extensions: for an exact sequence $0 \to L \to M \to N \to 0$ of R-modules, if L and N are in $\mathcal{X}$, then M is also in $\mathcal{X}$.
(4) $\mathcal{X}$ is closed under kernels of epimorphisms: for an exact sequence $0 \to L \to M \to N \to 0$ of R-modules, if M and N are in $\mathcal{X}$, then L is also in $\mathcal{X}$.

Now we state the main theorem in this paper.

Theorem 9. *Let $\mathcal{X}$ be a resolving subcategory of* $\operatorname{mod} R$ *such that the residue field k has a right $\mathcal{X}$-approximation. Assume that there exists an R-module $G \in \mathcal{X}$ of infinite projective dimension such that $\operatorname{Ext}_R^i(G, R) = 0$ for $i \gg 0$. Let M be an R-module such that each $X \in \mathcal{X}$ satisfies $\operatorname{Ext}_R^i(X, M) = 0$ for $i \gg 0$. Then M has finite injective dimension.*

We shall prove Theorem 9 in the next section. In the rest of this section, we will state and prove several results by using Theorem 9. We begin with two corollaries which are immediately obtained.

Corollary 10. *Let $\mathcal{X}$ be a resolving subcategory of* $\operatorname{mod} R$ *which is contained in the subcategory $\{ M \mid \operatorname{Ext}_R^i(M, R) = 0$ for $i \gg 0 \}$ of* $\operatorname{mod} R$*. Suppose that in $\mathcal{X}$ there is an R-module of infinite projective dimension. If k has a right $\mathcal{X}$-approximation, then R is Gorenstein.*

Proof. Each module X in $\mathcal{X}$ satisfies $\operatorname{Ext}_R^i(X, R) = 0$ for $i \gg 0$. Hence Theorem 9 implies that R has finite injective dimension as an R-module. $\square$

Corollary 11. *Let $\mathcal{X}$ be one of the following.*

(1) $\mathcal{G}(R)$.
(2) *The subcategory $\{ M \mid \operatorname{Ext}_R^i(M, R) = 0$ for $i > n \}$ of* $\operatorname{mod} R$*, where n is a nonnegative integer.*
(3) *The subcategory $\{ M \mid \operatorname{Ext}_R^i(M, R) = 0$ for $i \gg 0 \}$ of* $\operatorname{mod} R$.

Suppose that in $\mathcal{X}$ there is an R-module of infinite projective dimension. If k has a right $\mathcal{X}$-approximation, then R is Gorenstein.

Proof. The subcategory $\mathcal{X}$ of $\operatorname{mod} R$ is resolving. Since $\mathcal{X}$ is contained in the subcategory $\{ M \mid \operatorname{Ext}_R^i(M, R) = 0 \text{ for } i \gg 0 \}$, the assertion follows from Corollary 10. $\square$

Remark 12. Corollary 11 implies Theorem 5. Indeed, any nonfree totally reflexive module has infinite projective dimension by [9, (1.2.10)].

For a subcategory $\mathcal{X}$ of $\operatorname{mod} R$, let $\mathcal{X}^{\perp}$ (respectively, ${}^{\perp}\mathcal{X}$) denote the subcategory of $\operatorname{mod} R$ consisting of all R-modules M such that $\operatorname{Ext}_R^i(X, M) = 0$ (respectively,

$\mathrm{Ext}_R^i(M, X) = 0$) for all $X \in \mathcal{X}$ and $i > 0$. Applying Wakamatsu's lemma to a resolving subcategory, we obtain the following lemma.

Lemma 13. *Let $\mathcal{X}$ be a resolving subcategory of* $\mathrm{mod}\, R$. *If an R-module M has a right $\mathcal{X}$-approximation, then there is an exact sequence $0 \to Y \to X \to M \to 0$ of R-modules with $X \in \mathcal{X}$ and $Y \in \mathcal{X}^{\perp}$.*

By using this lemma and the theorem which was formerly called "Bass' conjecture", we obtain another corollary of Theorem 9.

Corollary 14. *Let $\mathcal{X}$ be a resolving subcategory of* $\mathrm{mod}\, R$ *such that k has a right $\mathcal{X}$-approximation and that k is not in $\mathcal{X}$. Assume that there is an R-module $G \in \mathcal{X}$ with* $\mathrm{pd}_R\, G = \infty$ *and* $\mathrm{Ext}_R^i(G, R) = 0$ *for $i \gg 0$. Then R is Cohen-Macaulay and* $\dim R > 0$.

Before giving the next corollary of Theorem 9, we establish an easy lemma without proof.

Lemma 15. (1) *Let $\mathcal{X}$ be a contravariantly finite resolving subcategory of* $\mathrm{mod}\, R$. *Then, $k \in \mathcal{X}$ if and only if $\mathcal{X} = \mathrm{mod}\, R$.*

(2) *Let $\mathcal{X}$ be a resolving subcategory of* $\mathrm{mod}\, R$. *Suppose that every R-module in ${}^{\perp}(\mathcal{X}^{\perp})$ admits a right $\mathcal{X}$-approximation. Then $\mathcal{X} = {}^{\perp}(\mathcal{X}^{\perp})$.*

(3) *Let M and N be nonzero R-modules. Assume either that M has finite projective dimension or that N has finite injective dimension. Then one has an equality*

$$\sup\{\, i \mid \mathrm{Ext}_R^i(M, N) \neq 0 \,\} = \operatorname{depth} R - \operatorname{depth}_R M.$$

Now we can show the following corollary. There are only two contravariantly finite resolving subcategories possessing such G as in the corollary.

Corollary 16. *Let $\mathcal{X}$ be a contravariantly finite resolving subcategory of* $\mathrm{mod}\, R$. *Assume that there is an R-module $G \in \mathcal{X}$ with* $\mathrm{pd}_R\, G = \infty$ *and* $\mathrm{Ext}_R^i(G, R) = 0$ *for $i \gg 0$. Then either of the following holds.*

(1) *$\mathcal{X} = \mathrm{mod}\, R$,*
(2) *R is Cohen-Macaulay and $\mathcal{X} = \mathcal{C}(R)$.*

Proof. Suppose that $\mathcal{X} \neq \mathrm{mod}\, R$. Then k is not in $\mathcal{X}$. By Corollary 14, R is Cohen-Macaulay.

First, we show that $\mathcal{C}(R)$ is contained in $\mathcal{X}$. For this, let M be a maximal Cohen-Macaulay R-module. We have only to prove that M is in ${}^{\perp}(\mathcal{X}^{\perp})$. Let N be a nonzero R-module in $\mathcal{X}^{\perp}$. Theorem 9 implies that N is of finite injective dimension. Since M is maximal Cohen-Macaulay, we have $\sup\{\, i \mid \mathrm{Ext}_R^i(M, N) \neq 0 \,\} = 0$. Therefore $\mathrm{Ext}_R^i(M, N) = 0$ for all $N \in \mathcal{X}^{\perp}$ and $i > 0$. It follows that M is in ${}^{\perp}(\mathcal{X}^{\perp})$, as desired.

Next, we show that $\mathcal{X}$ is contained in $\mathcal{C}(R)$. We have an exact sequence $0 \to Y \to X \to k \to 0$ with $X \in \mathcal{X}$ and $Y \in \mathcal{X}^{\perp}$ by Lemma 13. Since k is not in $\mathcal{X}$, the module Y is nonzero. By Theorem 9, Y has finite injective dimension. For a nonzero R-module X' in $\mathcal{X}$, we have equalities $0 \geq \sup\{\, i \mid \mathrm{Ext}_R^i(X', Y) \neq 0 \,\} = \operatorname{depth} R - \operatorname{depth}_R X' = \dim R - \operatorname{depth}_R X'$. Therefore X' is a maximal Cohen-Macaulay R-module, as desired. □

Next, we study contravariantly finite resolving subcategories all of whose objects X satisfy $\mathrm{Ext}_R^{\gg 0}(X, R) = 0$. We start by considering special ones among such subcategories.

Proposition 17. *Let $\mathcal{X}$ be a contravariantly finite resolving subcategory of* $\operatorname{mod} R$. *Suppose that every R-module in $\mathcal{X}$ has finite projective dimension. Then either of the following holds.*

(1) $\mathcal{X} = \mathcal{F}(R)$,

(2) *R is regular and* $\mathcal{X} = \operatorname{mod} R$.

Proof. If $\mathcal{X} = \operatorname{mod} R$, then our assumption says that all R-modules have finite projective dimension. Hence R is regular. Assume that $\mathcal{X} \neq \operatorname{mod} R$. Then there is an R-module M which is not in $\mathcal{X}$. There is an exact sequence $0 \to Y \to X \to M \to 0$ with $X \in \mathcal{X}$ and $Y \in \mathcal{X}^{\perp}$ by Lemma 13. Note that $Y \neq 0$ as $M \notin \mathcal{X}$. Fix a nonzero R-module $X' \in \mathcal{X}$. We have $\operatorname{Ext}_R^i(X', Y) = 0$ for all $i > 0$, and hence $\operatorname{pd}_R X' = \sup\{ i \mid \operatorname{Ext}_R^i(X', Y) \neq 0 \} = 0$ by the Auslander-Buchsbaum formula. Hence X' is free. This means that $\mathcal{X}$ is contained in $\mathcal{F}(R)$. On the other hand, $\mathcal{X}$ contains $\mathcal{F}(R)$ since $\mathcal{X}$ is resolving. Therefore $\mathcal{X} = \mathcal{F}(R)$. □

Combining Proposition 17 with Corollary 16, we can get the following.

Corollary 18. *Let $\mathcal{X}$ be a contravariantly finite resolving subcategory of* $\operatorname{mod} R$. *Suppose that every module $X \in \mathcal{X}$ is such that $\operatorname{Ext}_R^i(X, R) = 0$ for $i \gg 0$. Then one of the following holds.*

(1) $\mathcal{X} = \mathcal{F}(R)$,

(2) *R is Gorenstein and $\mathcal{X} = \mathcal{C}(R)$,*

(3) *R is Gorenstein and* $\mathcal{X} = \operatorname{mod} R$.

Proof. The corollary follows from Proposition 17 in the case where all R-modules in $\mathcal{X}$ are of finite projective dimension. So suppose that in $\mathcal{X}$ there exists an R-module of infinite projective dimension. Then Corollary 16 shows that either of the following holds.

(1) $\mathcal{X} = \operatorname{mod} R$,

(2) R is Cohen-Macaulay and $\mathcal{X} = \mathcal{C}(R)$.

By the assumption that every $X \in \mathcal{X}$ satisfies $\operatorname{Ext}_R^i(X, R) = 0$ for $i \gg 0$, we have $\operatorname{Ext}_R^i(k, R) = 0$ for $i \gg 0$ in the case (i). In the case (ii), since $\Omega^d k$ is in $\mathcal{X}$ where $d = \dim R$, we have $\operatorname{Ext}_R^{i+d}(k, R) \cong \operatorname{Ext}_R^i(\Omega^d k, R) = 0$ for $i \gg 0$. Thus, in both cases, the ring R is Gorenstein. □

Finally, we obtain the following result from Corollary 18 and Theorem 3. It says that the category of finitely generated modules over a Gorenstein local ring possesses only three contravariantly finite resolving subcategories.

Corollary 19. *Let R be a Gorenstein local ring. Then all the contravariantly finite resolving subcategories of* $\operatorname{mod} R$ *are $\mathcal{F}(R)$, $\mathcal{C}(R)$ and* $\operatorname{mod} R$.

3. Proof of the main theorem

Let M be an R-module. Take a minimal free resolution $F_\bullet = (\cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to 0)$ of M. We define the *transpose* $\operatorname{Tr} M$ of M as the cokernel of the R-dual homomorphism $d_1^* : F_0^* \to F_1^*$ of d_1. The transpose $\operatorname{Tr} M$ has no nonzero free summand.

For an R-module M, let M^*M be the ideal of R generated by the subset

$$\{ f(x) \mid f \in M^*, x \in M \}$$

of R. Note that M has a nonzero free summand if and only if $M^*M = R$.

Proposition 20. *Let $\mathcal{X}$ be a subcategory of* $\operatorname{mod} R$ *and* $0 \to Y \xrightarrow{f} X \to M \to 0$ *an exact sequence of R-modules with $X \in \mathcal{X}$ and $Y \in \mathcal{X}^{\perp}$. Let $G \in \mathcal{X}$, set $H = \operatorname{Tr}\Omega G$, and suppose that $(H^*H)M = 0$. Let $0 \to K \xrightarrow{g} F \xrightarrow{h} H \to 0$ be an exact sequence of R-modules with F free. Then the induced sequence*

$$0 \longrightarrow K \otimes_R Y \xrightarrow{g \otimes_R Y} F \otimes_R Y \xrightarrow{h \otimes_R Y} H \otimes_R Y \longrightarrow 0$$

is exact, and the map $h \otimes_R Y$ factors through the map $F \otimes_R f : F \otimes_R Y \to F \otimes_R X$.

Proof. We can show that there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H \otimes_R Y & \xrightarrow{\delta} & H \otimes_R X & \xrightarrow{\epsilon} & H \otimes_R M & \longrightarrow & 0 \\ & & {\scriptstyle\alpha}\big\downarrow{\scriptstyle\cong} & & {\scriptstyle\beta}\big\downarrow & & {\scriptstyle\gamma}\big\downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & \operatorname{Hom}_R(H^*, Y) & \xrightarrow{\zeta} & \operatorname{Hom}_R(H^*, X) & \xrightarrow{\eta} & \operatorname{Hom}_R(H^*, M) & \longrightarrow & 0 \end{array}$$

with exact rows, and see that δ is a split monomorphism. Thus, the homomorphism $h \otimes_R Y$ factors through the homomorphism $F \otimes_R f$. We have isomorphisms $\operatorname{Tor}_1^R(H, Y) = \operatorname{Tor}_1^R(\operatorname{Tr}\Omega G, Y) \cong \underline{\operatorname{Hom}}_R(\Omega G, Y) = 0$, which completes the proof of the proposition. $\square$

Now we can prove the following, which will play a key role in the proof of Theorem 9.

Proposition 21. *Let $\mathcal{X}$ be a subcategory of* $\operatorname{mod} R$ *which is closed under syzygies. Let $0 \to Y \to X \to M \to 0$ be an exact sequence of R-modules with $X \in \mathcal{X}$ and $Y \in \mathcal{X}^{\perp}$. Suppose that there is an R-module $G \in \mathcal{X}$ with $\operatorname{pd}_R G = \infty$ and $\operatorname{Ext}_R^i(G, R) = 0$ for $i \gg 0$. Put $H_i = \operatorname{Tr}\Omega(\Omega^i G)$ and assume that $((H_i)^* H_i)M = 0$ for $i \gg 0$. Let $D = (D^j)_{j \geq 0} : \operatorname{mod} R \to \operatorname{mod} R$ be a contravariant cohomological δ-functor. If $D^j(X) = 0$ for $j \gg 0$, then $D^j(Y) = D^j(M) = 0$ for $j \gg 0$.*

Proof. Replacing G with $\Omega^i G$ for $i \gg 0$, we may assume that $\operatorname{Ext}_R^i(G, R) = 0$ for all $i > 0$ and that $((H_i)^* H_i)M = 0$ for all $i \geq 0$. Let $F_\bullet = (\cdots \xrightarrow{d_{i+1}} F_i \xrightarrow{d_i} F_{i-1} \xrightarrow{d_{i-1}} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to 0)$ be a minimal free resolution of G. Dualizing this by R, we easily see that $H_i \cong (\Omega^{i+3} G)^*$ and $\Omega H_i \cong (\Omega^{i+2} G)^*$ for $i \geq 0$. By Proposition 20, for each integer $i \geq 0$ we have an exact sequence

$$0 \to (\Omega^{i+2} G)^* \otimes_R Y \to (F_{i+2})^* \otimes_R Y \xrightarrow{f_i} (\Omega^{i+3} G)^* \otimes_R Y \to 0$$

such that f_i factors through $(F_{i+2})^* \otimes_R X$. The homomorphism $D^j(f_i)$ factors through $D^j((F_{i+2})^* \otimes_R X)$, which vanishes for $j \gg 0$. Hence $D^j(f_i) = 0$ for $j \gg 0$, and we obtain an exact sequence

$$0 \to D^j((F_{i+2})^* \otimes_R Y) \to D^j((\Omega^{i+2} G)^* \otimes_R Y) \xrightarrow{\varepsilon_{i,j}} D^{j+1}((\Omega^{i+3} G)^* \otimes_R Y) \to 0$$

for $i \geq 0$ and $j \gg 0$. Thus, there is a sequence

$$D^j((\Omega^{i+2} G)^* \otimes_R Y) \xrightarrow{\varepsilon_{i,j}} D^{j+1}((\Omega^{i+3} G)^* \otimes_R Y) \xrightarrow{\varepsilon_{i+1,j+1}} D^{j+2}((\Omega^{i+4} G)^* \otimes_R Y) \xrightarrow{\varepsilon_{i+2,j+2}} \cdots$$

of surjective homomorphisms of R-modules, and $\varepsilon_{i,j}$ is an isomorphism. It follows that $D^j((F_{i+2})^* \otimes_R Y) = 0$ for $i \geq 0$ and $j \gg 0$. Thus we have $D^j(Y) = 0$ for $j \gg 0$, and $D^j(M) = 0$ for $j \gg 0$. $\square$

Now we can prove our main theorem.

Proof of Theorem 9. Since k admits a right $\mathcal{X}$-approximation, there exists an exact sequence $0 \to Y \to X \to k \to 0$ of R-modules with $X \in \mathcal{X}$ and $Y \in \mathcal{X}^{\perp}$ by Lemma 13. For an integer $i \geq 0$, put $H_i = \operatorname{Tr}\Omega(\Omega^i G)$. The module H_i has no nonzero free summand. We have $(H_i)^* H_i \neq R$. Hence $((H_i)^* H_i)k = 0$ for $i \geq 0$. Applying Proposition 21 to the contravariant cohomological δ-functor $D = (\operatorname{Ext}_R^j(\ \ , M))_{j \geq 0}$, we obtain $D^j(k) = 0$ for $j \gg 0$. Namely, we have $\operatorname{Ext}_R^j(k, M) = 0$ for $j \gg 0$, which implies that M has finite injective dimension. □

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Department of Mathematical Sciences
Faculty of Science
Shinshu University
3-1-1 Asahi, Matsumoto, Nagano 390-8621, JAPAN
E-mail address: `takahasi@math.shinshu-u.ac.jp`

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