Proceedings of the 41st Symposium on Ring Theory and Representation Theory

September 5 (Fri.) – 7 (Sun.), 2008
Shizuoka University, Shizuoka

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JSPS Grants-in-Aid for 2008 Scientific Research (B)
(Representative Researcher: Kiyoichi Oshiro)

February, 2009
Tsukuba, JAPAN

第 41 回 環論および表現論シンポジウム報告集

２００８年９月５日（金）－７日（日）
静岡大学

編集:　藤田尚昌 (筑波大学)

平成 20 年度科学研究費補助金 基盤研究 (B)
（代表：大城紀代市）

２００９年２月
筑波大学

Organizing Committee of The Symposium on Ring Theory and Representation Theory

The Symposium on Ring Theory and Representation Theory has been held annually in Japan and the Proceedings have been published by the organizing committee. The first Symposium was organized in 1968 by H. Tominaga, H. Tachikawa, M. Harada and S. Endo. After their retirement, a new committee was organized in 1997 for managing the Symposium. The present members of the committee are H. Asashiba (Shizuoka Univ.), Y. Hirano (Naruto Univ. of Education), S. Koshitani (Chiba Univ.), M. Sato (Yamanashi Univ.) and K. Oshiro (Yamaguchi Univ.).

The Proceedings of each Symposium is edited by program organizer. Anyone who wants these Proceedings should ask to the program organizer of each Symposium or one of the committee members.

The Symposium in 2009 will be held at Osaka City University for Sep. 11-13, and the program will be arranged by S. Kawata (Osaka City Univ.).

Concerning several information on ring theory group in Japan containing schedules of meetings and symposiums as well as the addresses of members in the group, you should refer the following homepage, which is arranged by M. Sato (Yamanashi Univ.):

http://fuji.cec.yamanashi.ac.jp/~ring/ (in Japanese)

civil2.cec.yamanashi.ac.jp/~ring/japan/ (in English)

Yasuyuki Hirano
Naruto Japan
December, 2008

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PREFACE

The 41st Symposium on Ring Theory and Representation Theory was held at Shizuoka University, Shizuoka. The symposium and the proceedings are financially supported by Kiyoichi Oshiro (Yamaguchi University) JSPS Grant-in-Aid for 2008 Scientific Research (B), No.18340011.

This volume consists of several (15) articles presented at the symposuim. We would like to thank all speakers and coauthors for their contributions.

We would also like to express our thanks to all members of the organizing commitee (Professors Hideto Asashiba, Yasuyuki Hirano, Shigeo Koshitani, Kiyoichi Oshiro and Masahisa Sato) for their helpful suggestions concerning the symposium. Finally we would like to express our gratitude to Professors Hideto Asashiba and Izuru Mori, and students of Shizuoka University who contributed in the organization of the symposium.

Hisaaki Fujita
Tsukuba, Japan
January, 2009

第41回 環論および表現論シンポジウム

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Geometric regularity of direct-sum decompositions of modules

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Tilting mutation and its application

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On filtered semi-dualizing bimodules

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The 41st Symposium on Ring Theory and Representation Theory (2008)

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DERIVED EQUIVALENCES FOR TRIANGULAR MATRIX RINGS

HIROKI ABE AND MITSUO HOSHINO

ABSTRACT. We generalize derived equivalences for triangular matrix rings induced by a certain type of classical tilting module introduced by Auslander, Platzeck and Reiten to generalize reflection functors in the representation theory of quivers due to Bernstein, Gelfand and Ponomarev.

1. NOTATION

For a ring A, we denote by Mod-A the category of right A-modules, by mod-A the full subcategory of Mod-A consisting of finitely presented modules and by $\mathcal{P}_A$ the full subcategory of Mod-A consisting of finitely generated projective modules. We denote by A^{op} the opposite ring of A and consider left A-modules as right A^{op}-modules. Sometimes, we use the notation X_A (resp., ${}_AX$) to stress that the module X considered is a right (resp., left) A-module. We denote by $\mathcal{K}(\text{Mod-}A)$ (resp., $\mathcal{D}(\text{Mod-}A)$) the homotopy (resp., derived) category of cochain complexes over Mod-A and by $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ the full triangulated subcategory of $\mathcal{K}(\text{Mod-}A)$ consisting of bounded complexes over $\mathcal{P}_A$. We consider modules as complexes concentrated in degree zero. For any integer $n \in \mathbb{Z}$ we denote by $(-)[n]$ the n-shift of complexes. Also, we use the notation $\mathrm{Hom}^\bullet(-,-)$ to denote the single complex associated with the double hom complex.

2. INTRODUCTION

Let R be a finite dimensional algebra over a field k and M a finitely generated projective right R-module. Set

$$A = \begin{pmatrix} k & M \\ 0 & R \end{pmatrix} \quad \text{and} \quad e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in A.$$

As pointed out by Brenner and Butler (see [4, p.111]), we know from [1] (cf. also [3]) that $\mathrm{Ext}^1_A(A/AeA, A) \oplus Ae \in \text{Mod-}A^{\mathrm{op}}$ is a tilting module of projective dimension at most one (see [6]) with

$$\mathrm{End}_{A^{\mathrm{op}}}(\mathrm{Ext}^1_A(A/AeA, A) \oplus Ae)^{\mathrm{op}} \cong \begin{pmatrix} R & \mathrm{Hom}_R(M, R) \\ 0 & k \end{pmatrix},$$

so that the triangular matrix rings

$$\begin{pmatrix} k & M \\ 0 & R \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} R & \mathrm{Hom}_R(M, R) \\ 0 & k \end{pmatrix}$$

are derived equivalent to each other. Our aim is to extend this type of derived equivalence to the case where M_R has finite projective dimension.

The detailed version of this paper has been submitted for publication elsewhere.

3. General case

Let A be a ring and $e \in A$ an idempotent satisfying the following conditions:

(E1) Ae admits a projective resolution $\varepsilon : P^\bullet \to Ae$ in Mod-eAe with $P^\bullet \in \mathcal{K}^b(\mathcal{P}_{eAe})$, in particular, $d = \text{proj dim } Ae_{eAe} < \infty$;

(E2) $\mu : Ae \otimes_{eAe} eA \to A, x \otimes y \mapsto xy$ is monic;

(E3) $\varphi : eA \to \text{Hom}_{eAe}(Ae, eAe), x \mapsto (y \mapsto xy)$ is monic;

(E4) if $d > 0$ then φ is an isomorphism and $\text{Ext}^i_{eAe}(Ae, eAe) = 0$ for $1 \le i < d$; and

(E5) $\text{Tor}_i^{eAe}(Ae, eA) = 0$ for $i \neq 0$.

Set $T_1^\bullet = eA[d+1]$, let $T_2^\bullet$ be the mapping cone of the composite

$$\mu \circ (\varepsilon \otimes_{eAe} eA) : P^\bullet \otimes_{eAe} eA \to Ae \otimes_{eAe} eA \to A$$

and set $T^\bullet = T_1^\bullet \oplus T_2^\bullet$. Then the following hold.

Theorem 1. *The complex $T^\bullet \in \mathcal{K}^b(\mathcal{P}_A)$ is a tilting complex with*

$$\text{End}_{\mathcal{K}(\text{Mod-}A)}(T^\bullet) \cong \begin{pmatrix} eAe & \text{Ext}_A^{d+1}(A/AeA, eA) \\ 0 & A/AeA \end{pmatrix}.$$

Remark 2. Assume $\text{Ext}^i_A(A/AeA, A) = 0$ for $i \neq d+1$. Then we have

$$\text{Hom}^\bullet_A(T^\bullet, A)[d+1] \cong \text{Ext}_A^{d+1}(A/AeA, A) \oplus Ae$$

in $\mathcal{D}(\text{Mod-}A^{\text{op}})$. Thus $\text{Ext}_A^{d+1}(A/AeA, A) \oplus Ae \in \text{Mod-}A^{\text{op}}$ is a tilting module with

$$\text{End}_{A^{\text{op}}}(\text{Ext}_A^{d+1}(A/AeA, A) \oplus Ae)^{\text{op}} \cong \begin{pmatrix} eAe & \text{Ext}_A^{d+1}(A/AeA, eA) \\ 0 & A/AeA \end{pmatrix}.$$

4. Main results

Let R and S be rings and M an S-R-bimodule satisfying the following conditions:

(M1) M admits a projective resolution $P^\bullet \to M$ in Mod-R with $P^\bullet \in \mathcal{K}^b(\mathcal{P}_R)$, in particular, $d = \text{proj dim } M_R < \infty$; and

(M2) $\text{Ext}^i_R(M, R) = 0$ for $i < d$.

Set

$$A = \begin{pmatrix} S & M \\ 0 & R \end{pmatrix} \quad \text{and} \quad e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in A.$$

Then the conditions (E1)–(E5) in the preceding section are satisfied. Also, we have $\text{Ext}_A^{d+1}(A/AeA, eA) \cong \text{Ext}^d_R(M, R)$. Note that $eAe \cong R$ and $A/AeA \cong S$ as rings. Thus by Theorem 1 the following hold.

Theorem 3. *The triangular matrix rings*

$$\begin{pmatrix} S & M \\ 0 & R \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} R & \text{Ext}^d_R(M, R) \\ 0 & S \end{pmatrix}$$

are derived equivalent to each other.

Consider next the case where R is a finite dimensional algebra over a field k and $S = k$. Then by Theorem 3 the following hold.

Proposition 4. *The triangular matrix algebras*

$$\begin{pmatrix} k & M \\ 0 & R \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} k & D\mathrm{Ext}_R^d(M,R) \\ 0 & R \end{pmatrix}$$

are derived equivalent to each other, where $D = \mathrm{Hom}_k(-,k)$.

Remark 5. Since the algebras above are trivial extensions of $\Lambda = k \times R$ by M and $D\mathrm{Ext}_R^d(M,R)$, respectively (see [5]). On the other hand, if inj dim ${}_RR =$ inj dim $R_R < \infty$, then $D\Lambda \in \text{Mod-}\Lambda$ is a tilting module with $\Lambda \cong \mathrm{End}_\Lambda(D\Lambda)$(see e.g. [7, Proposition 1.6]) and $M \otimes_\Lambda^{\mathbf{L}} D\Lambda[-d] \cong M \otimes_R^{\mathbf{L}} DR[-d] \cong \mathrm{Tor}_d^R(M, DR) \cong D\mathrm{Ext}_R^d(M,R)$ in $\mathcal{D}(\text{Mod-}\Lambda)$. Thus, if inj dim ${}_RR =$ inj dim $R_R < \infty$, Proposition 4 is due to [8, Corollary 5.4] (see also [2]).

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THE NUMBER OF COMPLETE EXCEPTIONAL SEQUENCES

TOKUJI ARAYA

ABSTRACT. A complete exceptional sequence is very useful to investigate the category of finitely generated modules over a finite dimensional algebra. The aim of this note is to show how to find the all complete exceptional sequences over the path algebra of Dynkin quiver of type $(\mathbf{A}_n)$.

1. INTRODUCTION

Let Λ be the path algebra of Dynkin quiver of type $(\mathbf{A}_n)$ over a field k. We denote by $\operatorname{mod}\Lambda$ the category of finitely generated left Λ-modules. The concept of exceptional sequences was introduced by Gorodentsev and Rudakov [1]. It is very useful to investigate $\operatorname{mod}\Lambda$. A finitely generated left Λ-module E is called *exceptional* if $\operatorname{Hom}_\Lambda(E,E) \cong k$ and $\operatorname{Ext}^1_\Lambda(E,E) = 0$. We remark that E is exceptinal if and only if it is indecomposable. Indeed Λ is the path algebra of $(\mathbf{A}_n)$. A pair (E,F) of exceptional modules is called an *exceptional pair* if $\operatorname{Hom}_\Lambda(F,E) = \operatorname{Ext}^1_\Lambda(F,E) = 0$. A sequence $\epsilon = (E_1, E_2, \cdots, E_r)$ of exceptional modules is called an *exceptional sequence* of length r if (E_i, E_j) is an exceptional pair for each $i < j$. An exceptinal sequence ϵ is called *complete* if the length of ϵ is equal to n. (Here, n is the number of simple modules in $\operatorname{mod}\Lambda$). We put $\mathfrak{E}$ the set of complete exceptional sequences. Siedel [2, Proposition 1.1] proved that the cardinarity of $\mathfrak{E}$ is equal to $(n+1)^{n-1}$. There are a number of complete exceptional sequences. But it is not easy to find all complete exceptional sequence. The main purpose is to get how to find the complete exceptinal sequences completely by using the conbinatorics.

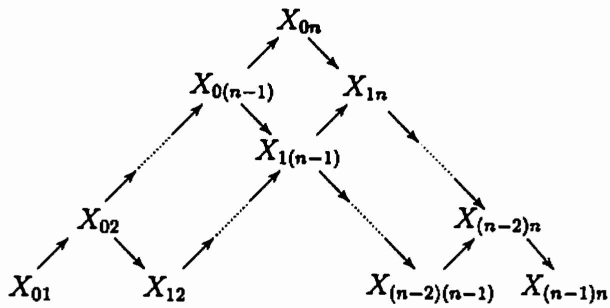
2. MAIN RESULT

First of all, we give a remark that $\mathfrak{E}$ is independent of the orientation of $(\mathbf{A}_n)$. Indeed, let Λ' be a path algebra of Dynkin quiver of type $(\mathbf{A}_n)$ whose orientation is not equal to Λ, and let $\mathfrak{E}'$ be the set of complete exceptional sequences in $\operatorname{mod}\Lambda'$. In this case, Λ and Λ' are derived equivalent and there exists a equivalence $\varphi : \mathcal{D}^b(\operatorname{mod}\Lambda) \to \mathcal{D}^b(\operatorname{mod}\Lambda')$. Therefore we can get the one to one correspondence $\psi : \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda'$ by φ and the suspention functor in $\mathcal{D}^b(\operatorname{mod}\Lambda')$. One can easily check that ψ gives the one to one correspondence between $\mathfrak{E}$ and $\mathfrak{E}'$. Thus we may assume the orientation of $(\mathbf{A}_n)$ as follows;

$$\overset{e_1}{\bullet} \to \overset{e_2}{\bullet} \to \cdots \to \overset{e_n}{\bullet}$$

Let Γ be the Auslander-Reiten quiver of $\operatorname{mod}\Lambda$. We identify the set Γ_0 of vertices in Γ with the class $\{X_{ij} \mid 0 \le i < j \le n\}$ of indecomposable Λ-modules. Then Γ is as follows;

The detailed version of this paper will be submitted for publication elsewhere.



We consider a circle with $n+1$ points labelled $0, 1, 2, \cdots, n$ counter clockwise on it. We put $c(i, j)$ the chord between the points i and j. We denote by C_{n+1} the set of chords in the circle. Since $C_{n+1} = \{c(i,j) |\ 0 \leq i < j \leq n\}$, there exists a one to one correspondence $\Phi : \Gamma_0 \to C_{n+1}$ defined by $\Phi(X_{ij}) = c(i, j)$.

For $\epsilon = (E_1, E_2, \cdots, E_n), \epsilon' = (E'_1, E'_2, \cdots, E'_n) \in \mathfrak{E}$, we define $\epsilon \sim \epsilon'$ by $\bigoplus_{i=0}^{n} E_i \cong \bigoplus_{i=0}^{n} E'_i$. Then $\sim$ is an equivalent relation on $\mathfrak{E}$. We shall prove the following theorem.

Theorem 1. *Φ gives a one to one correspondence between $\mathfrak{E}/\sim$ and the set of non crossing spanning trees by $\Phi(\epsilon) := \{\Phi(E_1), \Phi(E_2), \cdots, \Phi(E_n)\}$ for each $\epsilon = (E_1, E_2, \cdots, E_n)$.*

Here, we call a graph T a *non crossing spanning tree* if the following conditions are satisfied;

(i) the chords in T form a tree,
(ii) the chords in T meet only at endpoints.

It is known the number of noncrossing spanning trees. We get the following corollary.

Corollary 2. *The cardinarity of $\mathfrak{E}/\sim$ is equal to* $\dfrac{1}{2n+1}\dbinom{3n}{n}$.

Proof of Theorem 1. For $X \in \Gamma_0$, we consider the following four classes.

$$\begin{aligned}
\mathcal{H}_+(X) &= \{Y \in \Gamma_0 |\ \mathrm{Hom}_\Lambda(X, Y) \neq 0\}, \\
\mathcal{H}_-(X) &= \{Y \in \Gamma_0 |\ \mathrm{Hom}_\Lambda(Y, X) \neq 0\}, \\
\mathcal{E}_+(X) &= \{Y \in \Gamma_0 |\ \mathrm{Ext}^1_\Lambda(X, Y) \neq 0\}, \\
\mathcal{E}_-(X) &= \{Y \in \Gamma_0 |\ \mathrm{Ext}^1_\Lambda(Y, X) \neq 0\}.
\end{aligned}$$

Then one can check the followings by using Auslander-Reiten sequence;

$$\begin{aligned}
\mathcal{H}_+(X_{i,j}) &= \{X_{s,t} |\ i \leq s \leq j-1, j \leq t \leq n\}, \\
\mathcal{H}_-(X_{i,j}) &= \{X_{s,t} |\ 0 \leq s \leq i, i+1 \leq t \leq j\}, \\
\mathcal{E}_+(X_{i,j}) &= \{X_{s,t} |\ 0 \leq s \leq i-1, i \leq t \leq j-1\}, \\
\mathcal{E}_-(X_{i,j}) &= \{X_{s,t} |\ i+1 \leq s \leq j, j+1 \leq t \leq n\}.
\end{aligned}$$

Furthermore, we consider the following four classes for each $X \in \Gamma_0$;

$$\begin{aligned}
\mathfrak{P}(X) &= \{Y \mid \text{ Both } (X,Y) \text{ and } (Y,X) \text{ are exceptional pair.}\},\\
\mathfrak{P}_+(X) &= \left\{ \text{Y} \;\middle|\; \begin{array}{l} (X,Y) \text{ is an exceptinal pair,} \\ (Y,X) \text{ is not an exceptinal pair.} \end{array} \right\},\\
\mathfrak{P}_-(X) &= \left\{ \text{Y} \;\middle|\; \begin{array}{l} (Y,X) \text{ is an exceptinal pair,} \\ (X,Y) \text{ is not an exceptinal pair.} \end{array} \right\},\\
\overline{\mathfrak{P}(X)} &= \{Y \mid \text{ Both } (X,Y) \text{ and } (Y,X) \text{ are not exceptional pair.}\}.
\end{aligned}$$

Note that

$$\begin{aligned}
\mathfrak{P}(X) &= \Gamma_0 \setminus (\mathcal{H}_+(X) \cup \mathcal{E}_+(X) \cup \mathcal{H}_-(X) \cup \mathcal{E}_-(X)),\\
\mathfrak{P}_+(X) &= (\mathcal{H}_+(X) \cup \mathcal{E}_+(X)) \setminus (\mathcal{H}_-(X) \cup \mathcal{E}_-(X)),\\
\mathfrak{P}_-(X) &= (\mathcal{H}_-(X) \cup \mathcal{E}_-(X)) \setminus (\mathcal{H}_+(X) \cup \mathcal{E}_+(X)),\\
\overline{\mathfrak{P}(X)} &= (\mathcal{H}_+(X) \cup \mathcal{E}_+(X)) \cap (\mathcal{H}_-(X) \cup \mathcal{E}_-(X)),
\end{aligned}$$

we get the followings for each $X_{i,j} \in \Gamma_0$;

$$\begin{aligned}
\mathfrak{P}(X_{i,j}) &= \{X_{s,t} \mid 0 \le s < t \le i\} \cup \{X_{s,t} \mid i+1 \le s < t \le j-1\}\\
&\quad \cup \{X_{s,t} \mid j \le s < t \le n\} \cup \{X_{s,t} \mid 0 \le s \le i-1, j+1 \le t \le n\},\\
\mathfrak{P}_+(X_{i,j}) &= \{X_{s,i} \mid 0 \le s \le i-1\} \cup \{X_{i,t} \mid j+1 \le t \le n\} \cup \{X_{s,j} \mid i+1 \le s \le j-1\},\\
\mathfrak{P}_-(X_{i,j}) &= \{X_{i,t} \mid i+1 \le s \le j-1\} \cup \{X_{s,j} \mid 0 \le s \le j-1\} \cup \{X_{j,t} \mid j+1 \le s \le n\},\\
\overline{\mathfrak{P}(X_{i,j})} &= \{X_{s,t} \mid 0 \le s \le i-1, i+1 \le t \le j-1\}\\
&\quad \cup \{X_{s,t} \mid i+1 \le s \le j-1, j+1 \le t \le n\}.
\end{aligned}$$

We apply Φ for each above classes, we get followings;

$$\begin{aligned}
\Phi(\mathfrak{P}(X_{i,j})) &= \{c(s,t) \mid 0 \le s < t \le i\} \cup \{c(s,t) \mid i+1 \le s < t \le j-1\}\\
&\quad \cup \{c(s,t) \mid j \le s < t \le n\} \cup \{c(s,t) \mid 0 \le s \le i-1, j+1 \le t \le n\},\\
\Phi(\mathfrak{P}_+(X_{i,j})) &= \{c(s,i) \mid 0 \le s \le i-1\} \cup \{c(i,t) \mid j+1 \le t \le n\}\\
&\quad \cup \{c(s,j) \mid i+1 \le s \le j-1\},\\
\Phi(\mathfrak{P}_-(X_{i,j})) &= \{c(i,t) \mid i+1 \le s \le j-1\} \cup \{c(s,j) \mid 0 \le s \le j-1\}\\
&\quad \cup \{c(j,t) \mid j+1 \le s \le n\},\\
\Phi(\overline{\mathfrak{P}(X_{i,j})}) &= \{c(s,t) \mid 0 \le s \le i-1, i+1 \le t \le j-1\}\\
&\quad \cup \{c(s,t) \mid i+1 \le s \le j-1, j+1 \le t \le n\}.
\end{aligned}$$

Thus we have followings;

- $Y \in \mathfrak{P}(X) \Leftrightarrow \Phi(Y)$ does not meet to $\Phi(X)$.
- $Y \in \mathfrak{P}_+(X) \Leftrightarrow \Phi(Y)$ meets $\Phi(X)$ for some vertex i and $\Phi(Y)$ is the chord moved $\Phi(X)$ around a vertex i counterclockwise across the interior of the circle.
- $Y \in \mathfrak{P}_-(X) \Leftrightarrow \Phi(Y)$ meets $\Phi(X)$ for some vertex i and $\Phi(Y)$ is the chord moved $\Phi(X)$ around a vertex i clockwise across the interior of the circle.

- $Y \in \overline{\mathfrak{P}(X)} \Leftrightarrow \Phi(Y)$ meets to $\Phi(X)$ at interior of the circle.

Therefore for any $\epsilon \in \mathfrak{E}$, each chords in $\Phi(\epsilon)$ do not meet each other at interior of the circle.

For $X_1, X_2, \cdots, X_r \in \Gamma_0$, suppose $\{\Phi(X_1), \Phi(X_2), \cdots, \phi(X_r)\}$ makes a cycle. We may assume $\Phi(X_\ell)$ meets $\Phi(X_{\ell+1})$ at a vertex i_ℓ for each $\ell = 1, 2, \cdots, r$ (where $X_{r+1} = X_1$) and $i_1 > i_2 > \cdots > i_r$. Then, $(X_1, X_2), (X_2, X_3), \cdots, (X_{r-1}, X_r)$ and (X_r, X_1) are exceptional pairs but $(X_2, X_1), (X_3, X_2), \cdots, (X_r, X_{r-1})$ and (X_1, X_r) are not exceptional pairs. Therefore any permutatin of $(X_1, X_2, \cdots, X_r)$ is not an exceptional sequence.

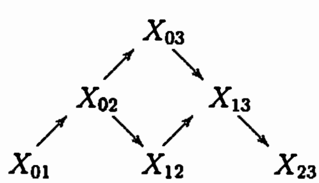
Thus we get $\Phi(\epsilon)$ is a non crossing spanning tree for any $\epsilon \in \mathfrak{E}$.

For $\epsilon = (E_1, E_2, \cdots, E_n), \epsilon' = (E'_1, E'_2, \cdots, E'_n) \in \mathfrak{E}$ suppose $\Phi(\epsilon) = \Phi(\epsilon')$. Then $\{\Phi(E_1), \Phi(E_2), \cdots, \phi(E_r)\} = \{\Phi(E_1), \Phi(E_2), \cdots, \phi(E_r)\}$. Since $\Phi : \Gamma_0 \to C_{n+1}$ is one to one, we get $\epsilon \sim \epsilon'$.

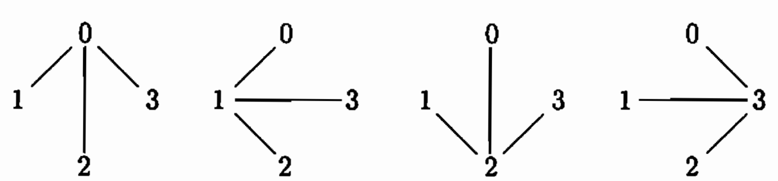
Conversely, suppose $T = \{c_1, c_2, \cdots, c_n\} \subset C_{n+1}$ is a non crossing spanning tree. We put $X_i := \Phi^{-1}(c_i)$ for each i. If there exists a pair (X_i, X_j) $(i \neq j)$ such that both (X_i, X_j) and (X_j, X_i) are not exceptional pair, then c_i crosses c_j at interior. Thus, there does not exist a such pair.

If there exists a subsequence $\{X_{a_1}, X_{a_2}, \cdots X_{a_r}\}$ such that (X_{a_1}, X_{a_2}), (X_{a_2}, X_{a_3}), $\cdots$, $(X_{a_{r-1}}, X_{a_r})$, and (X_{a_r}, X_{a_1}) are exceptional pairs but (X_{a_2}, X_{a_1}), (X_{a_3}, X_{a_2}), $\cdots$, $(X_{a_r}, X_{a_{r-1}})$, and (X_{a_1}, X_{a_r}) are not exceptional pairs, then $\{c_{a_1}, c_{a_2}, \cdots, c_{a_r}\}$ makes a cycle. Therefore there exists a parmutation σ such that $(X_{\sigma(1)}, X_{\sigma(2)}, \cdots X_{\sigma(n)})$ is a complete exceptional sequence. □

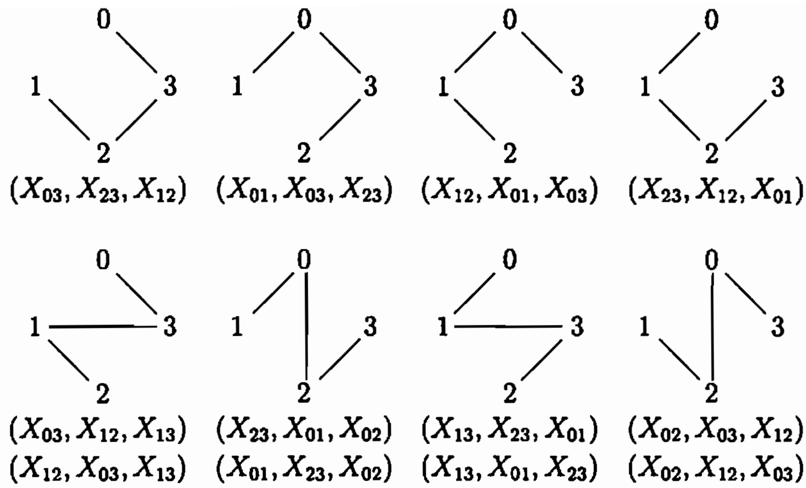
Example 3. If $n = 3$, the following quiver is the Auslander-Reiten quiver of $\operatorname{mod} \Lambda$.



In this case, there are 16 complete exceptional sequences and 12 non crossing spanning trees. The followings are the complete exceptional sequences and corresponding non crossing spanning trees.



(X_{01}, X_{02}, X_{03}) (X_{12}, X_{13}, X_{01}) (X_{23}, X_{02}, X_{12}) (X_{03}, X_{13}, X_{23})



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SOME REGULAR DIRECT-SUM DECOMPOSITIONS IN MODULE THEORY

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ABSTRACT. We review recent results about a weak form of the Krull-Schmidt Theorem that holds in some classes of modules.

1. INTRODUCTION

This is a survey about some direct-sum decompositions of modules with regular and interesting behaviors presented in two talks given in Shizuoka at the "Fortyfirst Symposium on Ring Theory and Representation Theory" (September 5-7, 2008). In particular, the first half of the paper will be devoted to describing some notions that have proved to be useful in the study of direct-sum decompositions. The symbol R will always denote an arbitrary associative ring with identity $1_R \neq 0_R$, and modules will be unital right R-modules unless otherwise stated explicitly.

Our aim is to describe the direct-sum decompositions $M_R = M_1 \oplus \cdots \oplus M_n$ of a fixed module M_R into a direct sum of finitely many direct summands $M_1, \ldots, M_n$. Several behaviors can take place. The best case we can have is when we have uniqueness up to isomorphism, as in the case of the celebrated *Krull-Schmidt Theorem*, which we all know:

Theorem 1. [Krull-Schmidt Theorem] *Every module M of finite composition length is a direct sum of indecomposable modules. If*

$$M = M_1 \oplus \cdots \oplus M_t = N_1 \oplus \cdots \oplus N_s$$

are two decompositions of M into direct sums of indecomposables, then $t = s$ and there is a permutation σ of $\{1, 2, \ldots, t\}$ such that $M_i \cong N_{\sigma(i)}$ for every $i = 1, 2, \ldots, t$.

A theorem of this kind appeared for the first time in a paper of Frobenius and Stickelberger [19], who proved the structure theorem of finite abelian groups (finite abelian groups are direct sums of cyclic subgroups whose orders are powers of primes, and these powers of primes are uniquely determined by the group). The Krull-Schmidt Theorem was later generalized by Azumaya in 1950 to infinite direct sums of modules with local endomorphism ring [3]. Important work on the Krull-Schmidt-Azumaya Theorem can be found in Harada [20], who introduced the use of factor categories in this setting. For an interesting survey on these results and their relation with the exchange property and extending modules, see [24].

Partially supported by Ministero dell'Istruzione, dell'Università e della Ricerca, Italy (Prin 2007 "Rings, algebras, modules and categories") and by Università di Padova (Progetto di Ricerca di Ateneo CPDA071244/07).
The paper is in the final form and no version of it will be submitted for publication elsewhere.

Uniqueness of direct-sum decomposition is an exception in Module Theory, and we will give in §4.1 an easy example of failure of the Krull-Schmidt Theorem for finitely generated modules over a noetherian commutative integral domain. A different possibility we can have decomposing a module M_R is that the module M_R possesses only finitely many direct-sum decompositions up to isomorphism. This is the case of finite-rank torsion-free abelian groups [23].

Another possible case we can meet is that of the modules M_R that do not decompose in a unique way up to isomorphism, but their direct sums enjoy some kind of regularity. We see in §4.2 that this happens for modules with a semilocal endomorphism ring, for instance for artinian modules. Several other possibilities can occur: a module can be a direct sum of indecomposables or not, can be decomposable but with no indecomposable direct summands, and so on.

2. Commutative monoids, order-units, and the Bergman-Dicks Theorem

Fix a class $\mathcal{C}$ of right R-modules. We want to study the direct-sum decompositions of the modules belonging to $\mathcal{C}$. We will assume that $\mathcal{C}$ is closed under isomorphism, direct summands and finite direct sums. For every module A_R, let $\langle A_R \rangle := \{ B_R \mid B_R \cong A_R \}$ denote the *isomorphism class* of the module A_R. Set $V(\mathcal{C}) := \{ \langle A_R \rangle \mid A_R \in \mathcal{C} \}$. Assume that $V(\mathcal{C})$ is a set[1]. Define $\langle A_R \rangle + \langle B_R \rangle := \langle A_R \oplus B_R \rangle$ for every $A_R, B_R \in \mathcal{C}$. Then $V(\mathcal{C})$ becomes an additive commutative monoid, which is clearly the algebraic structure that describes the direct-sum decompositions in $\mathcal{C}$.

In our first example (the Krull-Schmidt Theorem, that is, Theorem 1), $\mathcal{C}$ is the class of all right R-modules M_R of finite composition length, and $V(\mathcal{C})$ turns out to be a *free* commutative monoid, that is, a monoid isomorphic to $\mathbb{N}_0^{(X)}$ for some set X. In this example, X can be any set of representatives, up to isomorphism, of the indecomposable R-modules of finite composition length.

All the monoids we will consider in this paper are commutative, and the operation will be denoted as addition. Thus our monoids will be commutative additive semigroups with a zero element 0. For such a monoid M, $U(M)$ will denote the set of all invertible elements with respect to the addition, that is, all elements $a \in M$ with an opposite $-a$ in M. A commutative monoid M is said to be *reduced* if $U(M) = \{0\}$. For every monoid M, the quotient monoid $M/U(M) = \{ x + U(M) \mid x \in M \}$ is a reduced monoid. For any class $\mathcal{C}$ of modules, the commutative monoid $V(\mathcal{C})$ is reduced. The converse appears in the following wonderful theorem, due to Bergman [4, Theorems 6.2 and 6.4] and Bergman-Dicks [5, p. 315]. See [12, Corollary 5].

[1]This is an odd assumption, because in Axiomatic Set Theory, where elements of sets are sets, $V(\mathcal{C})$ can never be a set. More precisely, $V(\mathcal{C})$ cannot be a set by Zermelo's Sum Axiom (Union Axiom) of General Set Theory, which is the axiom that guarantees that the union of a set of sets is still a set ("for any set S there exists the set whose elements are the elements of the elements of S"). This set theoretical difficulty can be avoided fixing once for all a set of representatives of $V(\mathcal{C})$ up to isomorphism. Hence, when we say "assume that $V(\mathcal{C})$ is a set" we mean "assume that $V(\mathcal{C})$ can be put in one-to-one correspondence with a set", that is, a class whose cardinality can be measured with a cardinal number.

Theorem 2. *Let k be a field and M a reduced commutative monoid. Then there exist a right and left hereditary k-algebra R and a class $\mathcal{C}$ of finitely generated projective right R-modules with $\mathcal{C}$ closed under isomorphism, direct summands and finite direct sums, and $V(\mathcal{C}) \cong M$.*

This theorem gives, in a sense, a complete answer to what can be done with our description of direct-sum decompositions in a class $\mathcal{C}$ of modules making use of the monoid $V(\mathcal{C})$.

If, instead of the direct-sum decompositions of the modules in a class $\mathcal{C}$, we want to study the direct-sum decompositions of one fixed module A_R, t he following refinement of the construction of $V(\mathcal{C})$ is sufficient. Given a fixed module A_R, we can construct the class $\mathrm{add}(A_R)$ whose elements are all modules B_R isomorphic to a direct summand of A_R^n for some integer $n \geq 0$. This is the smallest class of right R-modules containing A_R and closed under isomorphism, direct summands and finite direct sums. For instance, if A_R is the right module R_R, then $\mathrm{add}(R_R)$ is the class proj-R of all finitely generated projective right R-modules. For any ring R and module A_R, we will denote with $V(R)$ and $V(A_R)$ the monoids $V(\text{proj-}R)$ and $V(\mathrm{add}(A_R))$, respectively. Clearly, for every module A_R, the element $\langle A_R \rangle$ of the monoid $V(A_R)$ is a special element: it is an *order-unit* in the commutative monoid. Let us briefly present order-units, monoids with order-unit, and the category of commutative monoids with order-unit.

An element u of a commutative additive monoid M is an *order-unit* if, for every $x \in M$, there exist $y \in M$ and an integer $n \geq 0$ with $x + y = nu$. For instance, the element $\langle R_R \rangle$ of the commutative additive reduced monoid $V(R)$ is an order-unit. More generally, as we have said above, $\langle A_R \rangle$ is an order-unit in the monoid $V(A_R)$. The category of commutative monoids with order-unit has as its objects the pairs (M, u), where M is a commutative monoid and $u \in M$ is an order-unit, and as morphisms $f\colon (M, u) \to (M', u')$ the monoid homomorphisms $f\colon M \to M'$ that preserve the order-units, that is, such that $f(u) = u'$. Notice that V is a functor of the category of associative rings with identity into the category of commutative monoids with order-unit.

Clearly, as the commutative monoid $V(\mathcal{C})$ describes the direct-sum decompositions of the modules in a fixed class $\mathcal{C}$, so the commutative monoid with order-unit $(V(\mathrm{add}A_R), \langle A_R \rangle)$ describes the direct-sum decompositions of a fixed module A_R.

For any given module A_R, we can consider the endomorphism ring $E := \mathrm{End}(A_R)$ and the covariant functor

$$\mathrm{Hom}_R(A_R, -)\colon \text{Mod-}R \to \text{Mod-}E.$$

By restriction, the functor $\mathrm{Hom}_R(A_R, -)$ induces a categorical equivalence between the full subcategory of Mod-R whose class of objects is $\mathrm{add}(A_R)$ and the full subcategory of Mod-E whose class of objects is proj-E [10, Theorem 4.7]. This equivalence induces an isomorphism $(V(\mathrm{add}(A_R)), \langle A_R \rangle) \cong (V(E), \langle E_E \rangle)$ of monoids with order-unit. Therefore, in the study of "pathologies" of direct-sums, we can suppose $A_R = R_R$, that is, it suffices to study direct-sum decompositions of finitely generated projective modules.

Similarly, notice that the contravariant functor

$$\mathrm{Hom}_R(-, R)\colon \text{Mod-}R \to R\text{-Mod}$$

induces by restriction a duality between the full subcategory of Mod-R whose class of objects is proj-R and the full subcategory of R-Mod whose class of objects R-proj consists of all finitely generated projective *left* R-modules. This duality induces an isomorphism of monoids with order-unit $(V(\text{proj-}R), \langle R_R \rangle) \cong (V(R\text{-proj}), \langle {}_R R \rangle)$. In other words, in the definition of the monoid $V(R)$ there is no difference considering right or left finitely generated projective modules. The monoid $V(R)$ is the object of study of Non-Stable Algebraic K-Theory, as the Grothendieck group $K_0(R)$ is the object of study of (classical) Algebraic K-Theory. Here the Grothendieck group $K_0(R)$ is the enveloping group of $V(R)$, and its elements are the *stable isomorphism classes* $[P_R]$ of the finitely generated projective R-modules P_R. There is a pre-order (=reflexive, transitive and translation-invariant relation) on $K_0(R)$, for which the positive cone (=set of non-negative elements of $K_0(R)$) is the image of the universal mapping $\psi_R\colon V(R) \to K_0(R)$. If $J(R)$ denotes the Jacobson radical of R, the canonical projection $p\colon R \to R/J(R)$ induces a pullback diagram

$$\begin{array}{ccc} V(R) & \overset{V(p)}{\longrightarrow} & V(R/J(R)) \\ \psi_R \downarrow & & \downarrow \psi_{R/J(R)} \\ K_0(R) & \overset{K_0(p)}{\longrightarrow} & K_0(R/J(R)) \end{array}$$

in the category of commutative monoids [2].

We can adapt the Bergman-Dicks Theorem (Theorem 2) to monoids with order-units as follows.

Theorem 3. *Let k be a field and let M be a commutative reduced monoid with order-unit u. Then there exists a right and left hereditary k-algebra R such that (M,u) and $(V(R), \langle R_R \rangle)$ are isomorphic as monoids with order-unit.*

3. LOCAL MORPHISMS AND SEMILOCAL RINGS

3.1. **Local morphisms.** In Algebraic Geometry and Commutative Algebra, local morphisms are defined as the ring morphisms $\varphi\colon R \to S$, between local commutative rings $(R, \mathcal{M})$ and $(S, \mathcal{N})$, for which $\varphi(\mathcal{M}) \subseteq \mathcal{N}$. Here $\mathcal{M}$ and $\mathcal{N}$ denote the maximal ideals of R and S respectively. More generally, let R and S be arbitrary associative rings with identity (not necessarily commutative and not necessarily local). We will say that a ring morphism $\varphi\colon R \to S$ is *local* if, for every $r \in R$, $\varphi(r)$ invertible in S implies r invertible in R. These two definitions coincide in the case of R and S local commutative rings. The notion of local morphism for non-commutative rings was introduced, in the case in which S was a division ring, by Cohn [8].

Here is a list of trivial properties of local morphisms. Their proofs follow immediately from the definition. Let $\varphi\colon R \to S$, $\psi\colon S \to T$ be ring morphisms.

(1) If φ is a local morphsm, then $\ker(\varphi) \subseteq J(R)$.

(2) If φ is onto and is a local morphism, then $\varphi(J(R)) = J(S)$, and the induced morphism $\mathrm{M}_n(\varphi)\colon \mathrm{M}_n(R) \to \mathrm{M}_n(S)$ between the $n \times n$ matrix rings is local for every $n > 1$.

(3) If φ and ψ are local morphisms, then so is $\psi \circ \varphi$.

(4) If the composite morpisms $\psi \circ \varphi$ is local, then φ local.

(5) If I is any two-sided ideal of R contained in the Jacobson radical $J(R)$, the canonical projection $R \to R/I$ is a local morphism.

3.2. Semilocal rings, dual Goldie dimension. A ring R is a *semilocal* ring if $R/J(R)$ is a semisimple artinian ring. (In Commutative Algebra a commutative ring is semilocal if it has only finitely many maximal ideals. The two definitions coincide in the case of commutative rings, but notice that a semilocal non-commutative ring can have infinitely many maximal right ideals, as the example of the ring $\mathrm{M}_n(k)$ of $n \times n$ matrices over an infinite field k shows.)

The relation between the notions of semilocal ring and local morphism is given by the following theorem, due to Camps and Dicks [6].

Theorem 4. *A ring R is semilocal if and only if there exists a local morphism of R into a semilocal ring, if and only if there exists a local morphism of R into a semisimple artinian ring.*

The notion of semilocal ring is also related to the notion of dual Goldie dimension. Goldie dimension can be defined not only for modules M_R, but more generally for any modular lattice L with a greatest element 1 and a least element 0 [10, § 2.6]. If $\mathcal{L}(M_R)$ denotes the lattice of all submodules of a module M_R, the Goldie dimension $\dim(M_R)$ of the module M_R coincides with the Goldie dimension $\dim(\mathcal{L}(M_R))$ of the lattice $\mathcal{L}(M_R)$. The *dual Goldie dimension* $\mathrm{codim}(M_R)$ of a module M_R is by definition the Goldie dimension of the dual (=opposite) lattice of the lattice $\mathcal{L}(M_R)$. The next result describes the relation between the notions of semilocal ring and dual Goldie dimension of a ring.

Proposition 5. *A ring R is semilocal if and only if the dual Goldie dimension of the right R-module R_R is finite, if and only if the dual Goldie dimension of the left R-module ${}_RR$ is finite. Moreover, if these equivalent conditions hold, then*

$$\mathrm{codim}(R_R) = \mathrm{codim}({}_RR) = \dim(R/J(R)).$$

In this proposition, note that R is semilocal exactly when $R/J(R)$ is semisimple artinian, that is, when $R/J(R)$ is a direct sum of simple modules, and in this case the Goldie dimension $\dim(R/J(R))$ of $R/J(R)$ is simply the number of direct summands in a direct-sum decomposition of $R/J(R)$ into simple submodules, that is, into simple right ideals of $R/J(R)$. The next theorem is related to Theorem 4 and Proposition 5.

Theorem 6. [6] *If $R \to S$ is a local morphism between two rings R and S, then* $\mathrm{codim}(R) \leq \mathrm{codim}(S)$.

3.3. Modules with semilocal endomorphism rings. The reason why we are interested in semilocal rings is that we want to study modules whose endomorphism ring is semilocal. Having a semilocal endomorphism ring is a finiteness condition on modules. For instance, a module with semilocal endomorphism ring is always a direct sum of finitely many indecomposable modules, it is not a direct sum of infinitely many non-zero modules, and it is directly finite. The class of the modules with semilocal endomorphism rings is closed under direct summands and finite direct sums. We will see in §4.2 that direct-sum decompositions of modules with semilocal endomorphism rings are described by reduced Krull monoids, and this implies a regularity in the behavior of direct-sum decompositions.

We begin with a proposition that shows how the property of having a semilocal endomorphism ring is related to restriction of scalars.

Proposition 7. [15] *Let $R \to S$ be a ring morphism, and let M_S be an S-module with* $\mathrm{End}(M_R)$ *semilocal. Then* $\mathrm{End}(M_S)$ *is semilocal.*

The proof is incredibly easy. The embedding

$$\mathrm{End}(M_S) \to \mathrm{End}(M_R)$$

is a local morphism, because an S-endomorphism is an S-automorphism if and only if it is an R-automorphism. Hence Theorem 4 applies.

4. Examples. Krull monoids.

This Section 4 is devoted to analyzing some examples of modules with semilocal endomorphism rings.

4.1. **Noetherian modules, artinian modules.** Our first example of class of modules with semilocal endomorphism rings is the class of all artinian right modules over a fixed ring R. Recall that the Krull-Schmidt Theorem (Theorem 1) holds for modules of finite composition lenght. Now a module has finite composition length if and only if it is both noetherian and artinian. A very natural question is therefore whether "noetherian" or "artinian" are sufficient conditions for the Krull-Schmidt Theorem to hold.

It is very easy to construct examples of noetherian modules for which the Krull-Schmidt Theorem does not hold. For instance, take a non-local noetherian commutative integral domain of Krull dimension ≥ 2, for example $R = k[x, y]$ (the ring of polynomials in two indeterminates x and y with coefficients in a field k). Then R has two distinct maximal ideals M_1, M_2, necessarily non-principal. Thus $R_R = M_1 + M_2$. The exact sequence $0 \to M_1 \cap M_2 \to M_1 \oplus M_2 \to R_R \to 0$ splits, so that

$$M_1 \oplus M_2 \cong R_R \oplus (M_1 \cap M_2). \tag{4.1}$$

But M_1 and M_2 are non-cyclic modules, and R_R is a cyclic module, so that the two direct-sum decompositions (4.1) are not isomorphic.

It was Krull who first asked in 1932 whether "the Krull-Schmidt Theorem holds for artinian modules" [22]. That is, any artinian module is a direct sum of indecomposables, but is such a direct-sum decomposition unique up to isomorphism? The first examples showing that there exist artinian modules with non-isomorphic direct-sum decompositions were given by Facchini, Herbera, Levy and Vámos in [16]. Nevertheless, direct-sum decompositions of artinian modules, and more generally of any class of modules with semilocal endomorphism rings are regular, because their behavior is described by a Krull monoids.

4.2. **Krull monoids and regular decompositions.** Krull monoids are the analogue for commutative monoids of what Krull domains are in Commutative Algebra. They were introduced by Chouinard in [7]. In Commutative Algebra we can fix a field F, take a family of valuations on F, consider the corresponding valutation subrings, and their intersection, when it is of finite character, is called a *Krull domain.* Then we can consider the fractional ideals, construct the divisor class semigroup, and so on. We have a perfectly similar case when we deal with commutative monoids instead of commutative integral domains. We can fix an abelian group G, take a family of valuations on G, consider the corresponding valutation submonoids, and their intersection, when it is of finite character, is called a *Krull monoid.* Then we can consider the fractional ideals, construct the divisor class semigroup of the Krull monoid, and so on. For the details, see [7]. For us, now, it is sufficient to know that the finitely generated reduced Krull monoids are the monoids isomorphic to monoids of the form $G \cap \mathbb{N}_0^t$, where $t \geq 0$ is an integer and G is a *subgroup* of the free abelian group $\mathbb{Z}^t$.

Theorem 8. [14, 26] *For every artinian module A_R, the monoid $V(A_R)$ is a finitely generated reduced Krull monoid with order-unit $\langle A_R \rangle$. Conversely, for every finitely generated reduced Krull monoid V with an order-unit u there exists an artinian module A_R with*

$$(V(A_R), \langle A_R \rangle) \cong (V, u).$$

More generally, for any class $\mathcal{C}$ of modules with semilocal endomorphism rings with $\mathcal{C}$ closed under isomorphism, direct summands and finite direct sums, the monoid $V(\mathcal{C})$ turns out to be a reduced Krull monoid [11]. Notice the geometric regularity implied by Krull monoids. In the language of Minkowski's Geometry of Numbers, a subgroup G of $\mathbb{Z}^t$ is represented by a "lattice", that is, a structure with a very regular geometric pattern (Here we are using the word *lattice* with a meaning completely different from the meaning employed until now in this paper.) If V is a reduced Krull monoid, then $V \cong \mathbb{N}_0^t \cap G$ is the intersection of the lattice $G \subseteq \mathbb{Z}^t$ with the positive cone $\mathbb{N}_0^t$. The failure of the Krull-Schmidt Theorem is minimal in this case, due only to the presence of the border of $\mathbb{N}_0^t \cap G$. Hence, when $V(A_R)$ is a Krull monoid that is not free, Krull-Schmidt uniqueness fails, but direct-sum decompositions still have a very regular geometric pattern.

4.3. **Further examples.** Let us pass to present other examples of modules with semilocal endomorphism rings. The following result is well known. For a proof, see [15, Proposition 3.1].

Proposition 9. *Every finitely generated module over a commutative semilocal ring has a semilocal endomorphism ring.*

Here is an extension of the previous proposition.

Proposition 10. [15, Theorem 3.3] *Every finitely presented module over a semilocal ring has a semilocal endomorphism ring.*

Notice that we have extended the class of rings (from commutative semilocal rings to arbitrary semilocal rings), but we have to restrict the class of modules (from finitely generated modules to finitely presented modules). Proposition 10 cannot be extended to finitely generated modules over non-commutative rings: there exist finitely generated modules over non-commutative semilocal rings whose endomorphism rings are not semilocal [15, Example 3.5].

Here are further examples of modules with semilocal endomorphism rings. We say that a module M is *quotient finite dimensional* if every homomorphic image of M has finite Goldie dimension.

Corollary 11. [15, Corollary 5.8] *Every submodule of a quotient finite dimensional injective module has a semilocal endomorphism ring.*

Recall that a module M is *uniserial* if, for any submodules A and B of M, either $A \subseteq B$ or $B \subseteq A$. Thus a module M is uniserial if and only if the lattice $\mathcal{L}(M)$ of its submodules is linearly ordered under set inclusion. Clearly, uniserial modules are quotient finite dimensional. A module is *serial* if it is a direct sum of uniserial submodules. Hence a module is serial and has finite Goldie dimension if and only if it is a direct sum of finitely many uniserial submodules.

Corollary 12. [15, Corollary 5.10] *Let E be an injective serial right module of finite Goldie dimension. Then the endomorphism ring of every submodule of E is semilocal.*

For further examples of modules with semilocal endomorphism rings, see [15] and [21].

5. Monogeny class, epigeny class

5.1. **Biuniform modules.** We say that two right R-modules A_R and B_R *belong to the same monogeny class*, and write $[A_R]_m = [B_R]_m$, if there exist a monomorphism $A_R \to B_R$ and a monomorphism $B_R \to A_R$. Similarly, we say that A_R and B_R *belong to the same epigeny class*, and write $[A_R]_e = [B_R]_e$, if there exist an epimorphism $A_R \to B_R$ and an epimorphism $B_R \to A_R$.

Recall that a module A_R is said to be: *uniform* if it has Goldie dimension 1, that is, it is non-zero and the intersection of any two non-zero submodules is a non-zero submodule; *couniform* if it has dual Goldie dimension 1, that is, it is non-zero and the sum of any two proper submodules is a proper submodule; *biuniform* if it uniform and couniform. For instance, uniserial non-zero modules are biuniform modules.

Theorem 13. [10, Theorem 9.1] *Let A_R be a biuniform module over an arbitrary ring R and let $E = \mathrm{End}(A_R)$ be its endomorphism ring. Let $I = \{ f \in E \mid f$ is not injective$\}$ and $K = \{ f \in E \mid f$ is not surjective$\}$. Then I and K are two-sided completely prime ideals of E, and every proper right ideal of E and every proper left ideal of E is contained either in I or in K. Moreover, exactly one of the following two conditions hold:*

(a) *Either E is a local ring, or*

(b) *$E/J(E) \cong E/I \times E/K$, where E/I and E/K are division rings.*

From Theorem 13 we get the following weak form of the Krull-Schmidt Theorem, proved by the author in [9, Theorem 1.9].

Theorem 14. *Let $U_1, \ldots, U_n, V_1, \ldots, V_t$ be biuniform right modules over an arbitrary ring R. Then the direct sums $U_1 \oplus \cdots \oplus U_n$ and $V_1 \oplus \cdots \oplus V_t$ are isomorphic if and only if $n = t$ and there are two permutations σ, τ of $\{1, 2, \ldots, n\}$ such that $[U_i]_m = [V_{\sigma(i)}]_m$ and $[U_i]_e = [V_{\tau(i)}]_e$ for every $i = 1, 2, \ldots, n$.*

This theorem allowed us to solve a problem posed by Warfield in [25].

5.2. **Cyclically presented modules over local rings.** We will now present some results proved in [1]. Recall that a right module over a ring R is said to be *cyclically presented* if it is isomorphic to R/aR for some $a \in R$. For any ring R with identity, $U(R)$ will denote the group of all invertible elements of R.

If R/aR and R/bR are cyclically presented modules over a local ring R, we say that R/aR and R/bR *have the same lower part*, and write $[R/aR]_l = [R/bR]_l$, if there exist $u, v \in U(R)$ and $r, s \in R$ with $au = rb$ and $bv = sa$. (The reason why we give this definition is that in this way two cyclically presented modules over a local ring turn out to have the same lower part exacly when their Auslander-Bridger transposes have the same epigeny class; cf. [1].)

We will now describe the endomorphism ring of a cyclically presented module. Clearly, the endomorphism ring $\mathrm{End}_R(R/aR)$ of a non-zero cyclically presented module R/aR is isomorphic to E/aR, where $E := \{ r \in R \mid ra \in aR \}$ is the *idealizer* of aR.

Theorem 15. *Let a be a non-zero non-invertible element of a local ring R, let E be the idealizer of aR, and let E/aR be the endomorphism ring of the cyclically presented right R-module R/aR. Set $I := \{ r \in R \mid ra \in aJ(R) \}$ and $K := J(R) \cap E$. Then I and K are completely prime two-sided ideals of E containing aR, the union $(I/aR) \cup (K/aR)$ is the set of all non-invertible elements of E/aR, and every proper right ideal of E/aR and every proper left ideal of E/aR is contained either in I/aR or in K/aR. Moreover, exactly one of the following two conditions hold:*

(a) *Either E/aR is a local ring, or*

(b) *I and K are not comparable, $J(E/aR) = (I \cap K)/aR$, and $(E/aR)/J(E/aR)$ is canonically isomorphic to the direct product of the two division rings E/I and E/K.*

Theorem 16. (Weak Krull-Schmidt Theorem) *Let $a_1, \dots, a_n, b_1, \dots, b_t$ be non-invertible elements of a local ring R. Then*

$$R/a_1R \oplus \cdots \oplus R/a_nR \quad \text{and} \quad R/b_1R \oplus \cdots \oplus R/b_tR$$

are isomorphic right R-modules if and only if $n = t$ and there are two permutations σ, τ of $\{1, 2, \dots, n\}$ such that $[R/a_iR]_l = [R/b_{\sigma(i)}R]_l$ and $[R/a_iR]_e = [R/b_{\tau(i)}R]_e$ for every $i = 1, 2, \dots, n$.

This has an immediate consequence as far as equivalence of matrices is concerned. Recall that two $m \times n$ matrices A, B with entries in a ring R are *equivalent*, denoted $A \sim B$, if there exist an $m \times m$ invertible matrix P and an $n \times n$ invertible matrix Q with $B = PAQ$. We denote by $\mathrm{diag}(a_1, \dots, a_n)$ the $n \times n$ diagonal matrix whose (i, i) entry is a_i and whose other entries are zero.

Corollary 17. *Let $a_1, \dots, a_n, b_1, \dots, b_n$ be elements of a local ring R. Then* $\mathrm{diag}(a_1, \dots, a_n) \sim \mathrm{diag}(b_1, \dots, b_n)$ *if and only if there exist two permutations σ, τ of $\{1, 2, \dots, n\}$ with*

$$[R/a_iR]_l = [R/b_{\sigma(i)}R]_l \quad \text{and} \quad [R/a_iR]_e = [R/b_{\tau(i)}R]_e$$

for every $i = 1, 2, \dots, n$.

6. Kernels of morphisms, couniformly presented modules

6.1. Kernels of morphisms between indecomposable injective modules. The next results are taken from [17]. We say that two modules A_R and B_R *have the same upper part*, and write $[A_R]_u = [B_R]_u$, if there exist a homomorphism $\varphi\colon E(A_R) \to E(B_R)$ and a homomorphism $\psi\colon E(B_R) \to E(A_R)$ such that $\varphi^{-1}(B_R) = A_R$ and $\psi^{-1}(A_R) = B_R$. Here $E(-)$ denotes the injective envelope.

We need some further notation for the statement of the next theorem. Let E_1, E_2, E'_1, E'_2 be indecomposable injective right modules over an arbitrary ring R, and let $\varphi\colon E_1 \to E_2, \varphi'\colon E'_1 \to E'_2$ be two non-injective morphisms. Any morphism $f\colon \ker\varphi \to \ker\varphi'$ extends to a morphism $f_1\colon E_1 \to E'_1$. Hence f_1 induces a morphism $\widetilde{f}_1\colon E_1/\ker\varphi \to E'_1/\ker\varphi'$, which extends to a morphism $f_2\colon E_2 \to E'_2$. Thus we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \to & \ker\varphi & \to & E_1 & \xrightarrow{\varphi} & E_2 \\ & & \downarrow f & & \downarrow f_1 & & \downarrow f_2 \\ 0 & \to & \ker\varphi' & \to & E'_1 & \xrightarrow{\varphi'} & E'_2. \end{array}$$

Notice that f_1 and f_2 are not uniquely determined by f.

Theorem 18. *Let E_1 and E_2 be two indecomposable injective right modules over an arbitrary ring R, and let $\varphi\colon E_1 \to E_2$ be a non-zero non-injective morphism. Set $S := \mathrm{End}_R(\ker\varphi)$, $I\colon = \{\, f \in S \mid f \text{ is not injective}\,\} = \{\, f \in S \mid f_1 \text{ is not injective}\,\}$ and $K\colon = \{\, f \in S \mid f_2 \text{ is not injective}\,\} = \{\, f \in S \mid f_1^{-1}(\ker\varphi) \text{ properly contains } \ker\varphi\,\}$. Then I and K are two completely prime two-sided ideals of S, and one of the following two conditions hold:*

(a) *Either S is a local ring, or*

(b) *$S/J(S)) \cong S/I \times S/K$, where S/I and S/K are division rings.*

Theorem 19. (Weak Krull-Schmidt Theorem) *Let $\varphi_i\colon E_{i,1} \to E_{i,2}$ $(i = 1, 2, \dots, n)$ and $\varphi'_j\colon E'_{j,1} \to E'_{j,2}$ $(i = 1, 2, \dots, t)$ be $n + t$ non-injective morphisms between indecomposable injective modules $E_{i,1}, E_{i,2}, E'_{j,1}, E'_{j,2}$ over an arbitrary ring R. Then $\oplus_{i=1}^n \ker\varphi_i \cong \oplus_{j=1}^t \ker\varphi'_j$ if and only if $n = t$ and there exist two permutations σ, τ of $\{1, 2, \dots, n\}$ such that $[\ker\varphi_i]_m = [\ker\varphi'_{\sigma(i)}]_m$ and $[\ker\varphi_i]_u = [\ker\varphi'_{\tau(i)}]_u$ for every $i = 1, 2, \dots, n$.*

Hence, also in this case we find the same behavior: at most two maximal ideals and the same weak form of the Krull-Schmidt Theorem. Now we will present a further class of modules over arbitrary rings with exactly the same behavior. It extends the class of cyclically presented modules over local rings we have met with in §5.2.

6.2. **Couniformly presented modules.** These modules have been introduced and studied in [13].

It is easily seen that a projective right module P_R is couniform, that is, has dual Goldie dimension one (cf. §5.1) if and only if P_R is the projective cover of a simple module, if and only if $\mathrm{End}(P_R)$ is a local ring, if and only if there exists an idempotent $e \in R$ with $P_R \cong eR$ and eRe a local ring, if and only if P_R is a finitely generated module with a unique maximal submodule [1, Lemma 8.7].

We say that a module M_R is *couniformly presented* if it is non-zero and there exists an exact sequence

$$0 \to C_R \xrightarrow{\iota} P_R \to M_R \to 0 \tag{6.1}$$

with P_R projective and both C_R and P_R couniform modules. Under these hypotheses, (6.1) will be called a *couniform presentation* of the couniformly presented module M_R.

For such a module M_R, every endomorphism f of M_R lifts to an endomorphism f_0 of the projective cover P_R of M_R, and we will denote by f_1 the restriction of f_0 to C_R. Hence we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & C_R & \xrightarrow{\iota} & P_R & \to & M_R & \to & 0 \\ & & f_1 \downarrow & & \downarrow f_0 & & \downarrow f & & \\ 0 & \to & C_R & \xrightarrow{\iota} & P_R & \to & M_R & \to & 0. \end{array}$$

Theorem 20. *Let $0 \to C_R \to P_R \to M_R \to 0$ be a couniform presentation of a couniformly presented module M_R. Let $K := \{ f \in \mathrm{End}(M_R) \mid f$ is not surjective$\}$ and $I := \{ f \in \mathrm{End}(M_R) \mid f_1 \colon C_R \to C_R$ is not surjective$\}$. Then K and I are completely prime two-sided ideals of $\mathrm{End}(M_R)$, and the union $K \cup I$ is the set of all non-invertible elements of $\mathrm{End}(M_R)$. Moreover, exactly one of the following two conditions hold:*
(a) *Either $\mathrm{End}(M_R)$ is a local ring, or*
(b) *$J(\mathrm{End}(M_R)) = K \cap I$, and $\mathrm{End}(M_R)/J(\mathrm{End}(M_R))$ is canonically isomorphic to the direct product of the two division rings $\mathrm{End}(M_R)/K$ and $\mathrm{End}(M_R)/I$.*

If M_R and M'_R are two couniformly presented modules with couniform presentations $0 \to C_R \to P_R \to M_R \to 0$ and $0 \to C'_R \to P'_R \to M'_R \to 0$ respectively, we say that M_R and M'_R *have the same lower part,* and write $[M_R]_\ell = [M'_R]_\ell$, if there are two homomorphisms $f_0 \colon P_R \to P'_R$ and $f'_0 \colon P'_R \to P_R$ such that $f_0(C_R) = C'_R$ and $f'_0(C'_R) = C_R$. (The definition of "having the same lower part" had been given in §5.2 only for cyclically presented modules over local rings. Here we are giving it for arbitrary couniformly presented modules over arbitrary rings.)

Theorem 21. (Weak Krull-Schmidt Theorem for couniformly presented modules) *Let $M_1, \dots, M_n, N_1, \dots, N_t$ be couniformly presented right R-modules. Then the modules $M_1 \oplus \dots \oplus M_n$ and $N_1 \oplus \dots \oplus N_t$ are isomorphic if and only if $n = t$ and there are two permutations σ, τ of $\{1, 2, \dots, n\}$ with $[M_i]_\ell = [N_{\sigma(i)}]_\ell$ and $[M_i]_e = [N_{\tau(i)}]_e$ for every $i = 1, \dots, n$.*

6.3. **Relation between upper part and lower part.** We have seen that kernels of morphisms between indecomposable injective modules are described by their monogeny class and their upper part. Couniformly presented modules are described by their epigeny class and their lower part. Let us explain the reason of this symmetry.

Let R be a fixed ring. Let $\{ E_\lambda \mid \lambda \in \Lambda \}$ be a set of representatives up to isomorphism of all indecomposable injective right R-modules. Set $E_R := E(\oplus_{\lambda \in \Lambda} E_\lambda)$ and $S := \mathrm{End}(E_R)$, so that ${}_SE_R$ turns out to be an S-R-bimodule and $H := \mathrm{Hom}(-, {}_SE_R) \colon \text{Mod-}R \to S\text{-Mod}$ is an additive contravariant exact functor.

If $\mathcal{K}$ is the full subcategory of Mod-R whose objects are finite direct sums of kernels of morphisms between uniform (equivalently, indecomposable) injective right R-modules, and $\mathcal{C}$ is the full subcategory of S-Mod whose objects are finite direct sums of cokernels of morphisms between couniform projective left S-modules, then the restriction $H = \mathrm{Hom}(-, {}_SE_R) \colon \mathcal{K} \to \mathcal{S}$ is a duality. It exchanges monogeny and epigeny (and upper part and lower part) as stated in the next proposition.

Proposition 22. *Let K_R and K'_R be the kernels of two non-zero non-injective morphisms between uniform injective right R-modules. Then:*

(a) $[K_R]_m = [K'_R]_m$ *if and only if* $[H(K_R)]_e = [H(K'_R)]_e$.
(b) $[K_R]_u = [K'_R]_u$ *if and only if* $[H(K_R)]_\ell = [H(K'_R)]_\ell$.

7. Seeking a general theory

We have seen three pair-wise incomparable classes of modules with the same behavior: (1) The class of biuniform modules. It contains the class of uniserial modules. These modules are described by their monogeny classes and their epigeny classes. (2) The class of all couniformly presented modules. It contains the class of all cokernels of morphisms between projective couniform modules, which in turn contains the class of all cyclically presented modules when the base ring R is local. These modules are described by their lower parts and their epigeny classes. (3) The class of all kernels of morphisms between uniform injective right R-modules. They are described by the monogeny classes and the upper parts, and there is a duality between this class and the class of all cokernels of morphisms between projective couniform modules. It would be easy to construct further examples of classes of modules with exactly the same behavior. For instance, fix two simple non-isomorphic right R-modules S_1 and S_2. Then the class of all artinian right R-modules with socle isomorphic to $S_1 \oplus S_2$ has this kind of behavior.

P. Příhoda and the author have found a general theory, a general setting able to describe all these particular classes [18]. We say that a ring S *has type* n if the factor ring $S/J(S)$ is a direct product of n division rings, and we say that a right module M_R over a ring R *has type* n if its endomorphism ring $\mathrm{End}(M_R)$ is a ring of type n. A ring R has type 1 if and only if it is a local ring, if and only if there is a local morphism of R into a division ring.

Lemma 23. *The following conditions are equivalent for a ring S with Jacobson radical $J(S)$ and a positive integer n.*

(i) *n is the smallest of the integers m such that there exists a local morphism of the ring S into a direct product of m division rings.*

(ii) *S has exactly n distinct maximal right ideals, and they are all two-sided ideals in S.*

(iii) *The ring S has type n.*

The natural question is: if $\mathcal{T}$ is the full subcategory of Mod-R whose class of objects consists of all indecomposable right R-modules of type 2, does a weak Krull-Schmidt Theorem hold for $\mathcal{T}$?

Let $\mathcal{C}$ be a full subcategory of Mod-R whose objects are indecomposable modules. A *completely prime ideal* $\mathcal{P}$ of $\mathcal{C}$ consists of a subgroup $\mathcal{P}(A,B)$ of $\mathrm{Hom}_R(A,B)$ for every pair of objects $A, B \in \mathrm{Ob}\,\mathcal{C}$ such that for every $A, B, C \in \mathrm{Ob}\,\mathcal{C}$, every $f\colon A \to B$ and every $g\colon B \to C$ one has that $gf \in \mathcal{P}(A,C)$ if and only if either $f \in \mathcal{P}(A,B)$ or $g \in \mathcal{P}(B,C)$. In all the previous situations, we have a pair of completely prime ideals $\mathcal{P}, \mathcal{Q}$ of $\mathcal{C}$ with the property that, for every object $A \in \mathrm{Ob}\,\mathcal{C}$, and endomorphism $f \in \mathrm{End}(A)$ of A is an automorphism of A if and only if $f \notin \mathcal{P}(A,A) \cup \mathcal{Q}(A,A)$.

If $\mathcal{C}$ is a full subcategory of $\mathcal{T}$, M is an object of $\mathcal{I}$, and I is a fixed ideal of $\mathrm{End}_R(M)$, let $\mathcal{I}$ be the ideal of the category $\mathcal{C}$ defined as follows: a morphism $f\colon X \to Y$ is in $\mathcal{I}(X,Y)$ if and only if $\beta f \alpha \in I$ for every $\alpha\colon M \to X$ and every $\beta\colon Y \to M$. We call $\mathcal{I}$ the *ideal of $\mathcal{C}$ associated to I.* It is the greatest among the ideals $\mathcal{I}'$ of $\mathcal{C}$ with $\mathcal{I}'(M,M) \subseteq I$, and in this case, as it is easily seen, $\mathcal{I}(M,M) = I$.

We can associate to the category $\mathcal{C}$ a graph $G(\mathcal{C})$. The edges of $G(\mathcal{C})$ are the isomorphisms classes $\langle M \rangle := \{ Y \in \mathrm{Ob}(\mathcal{C}) \mid Y \cong M \text{ in Mod-}R \}$, where M ranges in $\mathrm{Ob}(\mathcal{C})$; the vertices of $G(\mathcal{C})$ are the ideals $\mathcal{I}$ in the category $\mathcal{C}$ associated to a maximal ideal I of $\mathrm{End}(M_R)$ for some $M \in \mathrm{Ob}(\mathcal{C})$; for every $M \in \mathrm{Ob}(\mathcal{C})$, the endomorphism ring $\mathrm{End}(M_R)$ has exactly two maximal ideals I_1, I_2, and the edge $\langle M \rangle$ connects the vertices $\mathcal{I}_1$ and $\mathcal{I}_2$.

Theorem 24. *Let $\mathcal{C}$ be a full subcategory of $\mathcal{T}$. A weak Krull-Schmidt Theorem holds for $\mathcal{C}$ if and only if the graph $G(\mathcal{C})$ does not contain a subgraph isomorphic to the complete graph K_4.*

For suitable rings R, the graph $G(\mathcal{T})$ contains a copy of the complete graph K_4, so that a weak Krull-Schmidt Theorem does not hold for $\mathcal{T}$.

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PICARD GROUPS OF ADDITIVE FULL SUBCATEGORIES

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1. Introduction

Let k be a commutative ring and let A be a commutative k-algebra. We denote by A-Mod the category of all A-modules and all A-homomorphisms. Let $\mathfrak{C}$ be an additive full subcategory of A-Mod. Since A is a k-algebra, every additive full subcategory $\mathfrak{C}$ is a k-category. A covariant functor $\mathfrak{C} \to \mathfrak{C}$ is called a k-linear automorphism of $\mathfrak{C}$ if it is a k-linear functor giving an auto-equivalence of the category $\mathfrak{C}$. We denote the set of all the isomorphism classes of k-linear automorphisms of $\mathfrak{C}$ by $\mathrm{Aut}_k(\mathfrak{C})$, which forms a group by defining the multiplication to be the composition of functors.

Our study was motivated by the following computational result. Recall that a local ring $(A, \mathfrak{m})$ is said to have only an isolated singularity if $A_{\mathfrak{p}}$ is a regular local ring for all prime ideals $\mathfrak{p}$ except $\mathfrak{m}$.

Theorem 1. *Let A be a Cohen-Macaulay local k-algebra with dimension d. Suppose that A has only an isolated singularity. Then,*

$$\mathrm{Aut}_k(\mathrm{CM}(A)) \cong \begin{cases} \mathrm{Aut}_{k\text{-}alg}(A) & (d \neq 2) \\ \mathrm{Aut}_{k\text{-}alg}(A) \ltimes C\ell(A) & (d = 2), \end{cases}$$

where $\mathrm{CM}(A)$ *is the additive full subcategory consisting of all maximal Cohen-Macaulay modules and* $C\ell(A)$ *denotes the divisor class group of* A.

In this note we generalize this computation to much wider classes of additive full subcategories $\mathfrak{C}$ of A-Mod, and we shall show a certain structure theorem for $\mathrm{Aut}_k(\mathfrak{C})$.

2. Automorphism groups

Throughout the paper, k is a commutative ring and A is a commutative k-algebra. When we say that $\mathfrak{C}$ is a full subcategory of A-Mod, we always assume that $\mathfrak{C}$ is closed under isomorphisms, and we simply write $X \in \mathfrak{C}$ to indicate that X is an object of $\mathfrak{C}$. Suppose that we are given an additive full subcategory $\mathfrak{C}$ of A-Mod and an additive covariant functor $F : \mathfrak{C} \to \mathfrak{C}$. Recall that F is a k-linear functor if it induces k-linear mappings $\mathrm{Hom}_A(X, Y) \to \mathrm{Hom}_A(F(X), F(Y))$ for all $X, Y \in \mathfrak{C}$.

Definition 2. $\mathrm{Aut}_k(\mathfrak{C})$ is the group of all the isomorphism classes of k-linear automorphisms of $\mathfrak{C}$, i.e.

$$\mathrm{Aut}_k(\mathfrak{C}) = \{\, F : \mathfrak{C} \to \mathfrak{C} \mid \begin{array}{l} F \text{ is a } k\text{-linear covariant functor that} \\ \text{gives an equivalence of the category } \mathfrak{C} \end{array} \}/\cong .$$

The detailed version of this paper will be submitted for publication elsewhere.

Note that the multiplication in $\mathrm{Aut}_k(\mathfrak{C})$ is defined to be the composition of functors, hence the identity element of $\mathrm{Aut}_k(\mathfrak{C})$ is represented by the class of the identity functor on $\mathfrak{C}$.

We denote by $\mathrm{Aut}_{k\text{-alg}}(A)$ the group of all the k-algebra automorphisms of A. For $\sigma \in \mathrm{Aut}_{k\text{-alg}}(A)$, we can define a covariant k-linear functor $\sigma_* : A\text{-Mod} \to A\text{-Mod}$ as in the following manner. For each A-module M, we define $\sigma_* M$ to be M as an abelian group on which the A-module structure is defined by $a \circ m = \sigma^{-1}(a)m$ for $a \in A$, $m \in M$. For an A-homomorphism $f : M \to N$, we define $\sigma_* f : \sigma_* M \to \sigma_* N$ to be the same mapping as f. Note that $\sigma_* f$ is an A-homomorphism, since $(\sigma_* f)(a \circ m) = f(\sigma^{-1}(a)m) = \sigma^{-1}(a)f(m) = a \circ (\sigma_*)f(m)$ for all $a \in A$ and $m \in M$. Notice that σ_* is a k-automorphism of the category A-Mod.

Definition 3. Let $\mathfrak{C}$ be an additive full subcategory of A-Mod. Then $\mathfrak{C}$ is said to be stable under $\mathrm{Aut}_{k\text{-alg}}(A)$ if $\sigma_*(\mathfrak{C}) \subseteq \mathfrak{C}$ for all $\sigma \in \mathrm{Aut}_{k\text{-alg}}(A)$.

Note that if $\mathfrak{C}$ is stable under $\mathrm{Aut}_{k\text{-alg}}(A)$ then $\sigma_*|_{\mathfrak{C}}$ gives a k-automorphism of $\mathfrak{C}$ for all $\sigma \in \mathrm{Aut}_{k\text{-alg}}(A)$. Therefore we have a natural group homomorphism $\Psi : \mathrm{Aut}_{k\text{-alg}}(A) \to \mathrm{Aut}_k(\mathfrak{C})$ which maps σ to the class of $\sigma_*|_{\mathfrak{C}}$. It is easy to verify the following lemma.

Lemma 4. *Assume that $\mathfrak{C}$ is stable under $\mathrm{Aut}_{k\text{-alg}}(A)$ and that $A \in \mathfrak{C}$. Then the natural group homomorphism $\Psi : \mathrm{Aut}_{k\text{-alg}}(A) \to \mathrm{Aut}_k(\mathfrak{C})$ is an injection.*

By this lemma, we can regard $\mathrm{Aut}_{k\text{-alg}}(A)$ as a subgroup of $\mathrm{Aut}_k(\mathfrak{C})$.

Definition 5. Let N be an A-module. Given a k-algebra homomorphism $\sigma : A \to A$, we define an $(A \otimes_k A)$-module N_σ by $N_\sigma = N$ as an abelian group on which the ring action is defined by $(a \otimes b) \cdot n = a\sigma(b)n$ for $a \otimes b \in A \otimes_k A$ and $n \in N$. In such a case, we can define a k-linear functor $\mathrm{Hom}_A(N_\sigma, -) : A\text{-Mod} \to A\text{-Mod}$, for which the A-module structure on $\mathrm{Hom}_A(N_\sigma, X)$ $(X \in A\text{-Mod})$ is defined by $(b \cdot f)(n) = f((1 \otimes b) \cdot n)$ for $f \in \mathrm{Hom}_A(N_\sigma, X)$, $b \in A$ and $n \in N$.

If σ is a k-algebra automorphism of A, then it is easy to see the following equality of functors holds:

$$(\sigma^{-1})_* \circ \mathrm{Hom}_A(N, \ ) = \mathrm{Hom}_A(N_\sigma, \ ).$$

The following theorem is one of the main results of this note.

Theorem 6 ([2, Theorem 2.5]). *Let A be a commutative k-algebra and let $\mathfrak{C}$ be an additive full subcategory of A-Mod such that $A \in \mathfrak{C}$. For a given k-linear automorphism $F \in \mathrm{Aut}_k(\mathfrak{C})$, there is a k-algebra automorphism $\sigma \in \mathrm{Aut}_{k\text{-alg}}(A)$ such that F is isomorphic to the composition of functors $\sigma_* \circ \mathrm{Hom}_A(N, -)|_{\mathfrak{C}}$, where N is any object in $\mathfrak{C}$ satisfying $F(N) \cong A$ in $\mathfrak{C}$.*

Proof. We give below an outline of the proof. See [2, Theorem 2.5] for the detail.

Since A is commutative, the multiplication map $a_X : X \to X$ by an element $a \in A$ is an A-homomorphism for all objects $X \in \mathfrak{C}$. Thus we can define a natural transformation $\alpha(a) : F \to F$ by $\alpha(a)(X) = F(a_X) : F(X) \to F(X)$. Denote by $\mathrm{End}(F)$ the set of all the natural transformations $F \to F$, and this induces the mapping

$$\alpha : A \to \mathrm{End}(F) \ ; \quad a \mapsto F(a_{(\ )}).$$

Note that End(F) is a ring by defining the composition of natural transformations as the multiplication and it is also a k-algebra, since F is a k-linear functor. By using the fact that F is an auto-equivalence, it is straightforward to see that α is a k-algebra isomorphism.

Since F is a dense functor and $A \in \mathfrak{C}$, there is an object $N \in \mathfrak{C}$ such that $F(N) \cong A$. For such an object N, we can identify $\mathrm{End}_A(F(N))$ with A as k-algebra through the mapping $A \to \mathrm{End}_A(F(N))$ which sends $a \in A$ to the multiplication mapping $a_{F(N)}$ by a on $F(N)$. Thus we have a k-algebra homomorphism

$$\beta : \mathrm{End}(F) \to \mathrm{End}_A(F(N)) \cong A\,; \quad \varphi \mapsto \varphi(N).$$

We easily see that β is a k-algebra isomorphism.

Now define a k-algebra automorphism $\sigma : A \to A$ as the composition of α and β;

$$\begin{array}{ccccccc} A & \xrightarrow{\alpha} & \mathrm{End}(F) & \xrightarrow{\beta} & \mathrm{End}_A(F(N)) & \xrightarrow{\cong} & A \\ a & \longrightarrow & F(a_{(\ )}) & \longrightarrow & F(a_{(N)}) & \longrightarrow & \sigma(a). \end{array}$$

Then, for each object $X \in \mathfrak{C}$, we have isomorphisms of k-modules;

$$\begin{array}{ccccc} F(X) & \xrightarrow{\cong} & \mathrm{Hom}_A(F(N), F(X)) & \xrightarrow{\cong} & \mathrm{Hom}_A(N_{\sigma^{-1}}, X) \\ x & \longrightarrow & (x_{F(N)} : 1 \mapsto x) & \longrightarrow & F^{-1}(x_{F(N)}), \end{array}$$

whose composition we denote by φ_X. Since $F^{-1}(\sigma(a)_{F(N)}) = a_{(N)}$ holds for $a \in A$, we can show that φ_X is an A-module isomorphism for all $X \in \mathfrak{C}$. Since it is easily verified that φ_X is functorial in X, we have the isomorphism of functors $F \cong \mathrm{Hom}_A(N_{\sigma^{-1}}, \ )$, and the proof is completed. □

3. Picard groups

In this section, we study the group of all the A-linear automorphisms of an additive full subcategory of A-Mod. As in the previous section $\mathfrak{C}$ is an additive full subcategory of A-Mod. We always assume that $\mathfrak{C}$ contains A as an object.

By virtue of Theorem 6, we have the following corollary.

Corollary 7 ([2, Corollary 3.1]). *For any element* $[F] \in \mathrm{Aut}_A(\mathfrak{C})$, *there is an isomorphism of functors* $F \cong \mathrm{Hom}_A(N, -)|_{\mathfrak{C}}$ *for some* $N \in \mathfrak{C}$.

Taking this corollary into consideration, we make the following definition.

Definition 8. We define Pic($\mathfrak{C}$) to be the set of all the isomorphism classes of A-modules $M \in \mathfrak{C}$ such that $\mathrm{Hom}_A(M, -)|_{\mathfrak{C}}$ gives an auto-equivalence of the category $\mathfrak{C}$. That is,

$$\mathrm{Pic}(\mathfrak{C}) = \{M \in \mathfrak{C} \mid \mathrm{Hom}_A(M, -)|_{\mathfrak{C}} \text{ gives an } (A\text{-linear}) \text{ equivalence } \mathfrak{C} \to \mathfrak{C}\}/\cong.$$

We define the group structure on Pic($\mathfrak{C}$) as follows: Let $[M]$ and $[N]$ be in Pic($\mathfrak{C}$). Since the composition $\mathrm{Hom}_A(M, -)|_{\mathfrak{C}} \circ \mathrm{Hom}_A(N, -)|_{\mathfrak{C}}$ is also an A-linear equivalence, it follows from Corollary 7 that there exists an $L \in \mathfrak{C}$ such that

$$\mathrm{Hom}_A(L, -)|_{\mathfrak{C}} \cong \mathrm{Hom}_A(M, -)|_{\mathfrak{C}} \circ \mathrm{Hom}_A(N, -)|_{\mathfrak{C}}.$$

We define the multiplication in $\mathrm{Pic}(\mathfrak{C})$ by $[M]\cdot[N]=[L]$. Note that

$$\begin{aligned}\mathrm{Hom}_A(M,-)|_{\mathfrak{C}} \circ \mathrm{Hom}_A(N,-)|_{\mathfrak{C}} &\cong \mathrm{Hom}_A(M\otimes_A N,-)|_{\mathfrak{C}}\\ &\cong \mathrm{Hom}_A(N,-)|_{\mathfrak{C}} \circ \mathrm{Hom}_A(M,-)|_{\mathfrak{C}},\end{aligned}$$

and hence $[M]\cdot[N]=[N]\cdot[M]$. In such a way $\mathrm{Pic}(\mathfrak{C})$ is an abelian group with the identity element $[A]$. We call $\mathrm{Pic}(\mathfrak{C})$ the Picard group of $\mathfrak{C}$.

Note from Yoneda's lemma that the multiplication in $\mathrm{Pic}(\mathfrak{C})$ is well-defined. Furthermore, the mapping $\mathrm{Pic}(\mathfrak{C})\to \mathrm{Aut}_A(\mathfrak{C})$ which sends $[M]$ to $\mathrm{Hom}_A(M,-)|_{\mathfrak{C}}$ is an isomorphism of groups by Corollary 7. Since $\mathrm{Aut}_A(\mathfrak{C})$ is naturally a subgroup of $\mathrm{Aut}_k(\mathfrak{C})$, we can regard $\mathrm{Pic}(\mathfrak{C})$ as a subgroup $\mathrm{Aut}_k(\mathfrak{C})$ through the isomorphism $\mathrm{Pic}(\mathfrak{C})\cong \mathrm{Aut}_A(\mathfrak{C})$.

Assume furthermore that an additive full subcategory $\mathfrak{C}$ is stable under $\mathrm{Aut}_{k\text{-alg}}(A)$. Then we have shown by the above argument together with Lemma 4 that $\mathrm{Aut}_k(\mathfrak{C})$ contains two subgroups, $\mathrm{Pic}(\mathfrak{C})$ and $\mathrm{Aut}_{k\text{-alg}}(A)$. Moreover, Theorem 6 implies that these two subgroups generate the group $\mathrm{Aut}_k(\mathfrak{C})$. Thus it is straightforward to see that the following theorem holds.

Theorem 9 ([2, Theorem 4.9]). *Assume that an additive full subcategory $\mathfrak{C}$ is stable under $\mathrm{Aut}_{k\text{-alg}}(A)$ and assume that $A\in\mathfrak{C}$. Then there is an isomorphism of groups*

$$\mathrm{Aut}_k(\mathfrak{C})\cong \mathrm{Aut}_{k\text{-alg}}(A)\ltimes \mathrm{Pic}(\mathfrak{C}).$$

Now we give several examples for $\mathrm{Pic}(\mathfrak{C})$.

Example 10 ([2, Example 3.8, 3.11]). We denote by A-mod the full subcategory consisting of all finitely generated A-modules. We also denote by $\mathrm{Proj}(A)$ (resp. $\mathrm{proj}(A)$) the full subcategory consisting of all projective A-modules (resp. all finitely generated projective A-modules). If A is an integral domain, we denote by $\mathrm{Tf}(A)$ (resp. $\mathrm{tf}(A)$) the full subcategory consisting of all torsion free A-modules (resp. all finitely generated torsion free A-modules). Let $\mathfrak{C}$ be one of the full subcategories A-Mod, A-mod, $\mathrm{Proj}(A)$, $\mathrm{proj}(A)$, $\mathrm{Tf}(A)$ and $\mathrm{tf}(A)$. Then we have an isomorphism $\mathrm{Pic}(\mathfrak{C})\cong \mathrm{Pic}\ A$, where $\mathrm{Pic}\ A$ denotes the (classical) Picard group of the ring A, i.e. $\mathrm{Pic}\ A=\{\text{invertible } A\text{-modules}\}/\cong$. See also [2, Proposition 3.7].

Example 11 ([2, Example 3.9, 3.10]). Let A be a Krull domain and let $\mathrm{Ref}(A)$ be the full subcategory consisting of all reflexive A-lattices. (Respectively, let A be a Noetherian normal domain and let $\mathrm{ref}(A)$ be the full subcategory consisting of all finitely generated reflexive A-modules.) Then there is an isomorphism $\mathrm{Pic}(\mathrm{Ref}(A))\cong C\ell(A)$ (resp. $\mathrm{Pic}(\mathrm{ref}(A))\cong C\ell(A)$), where $C\ell(A)$ denotes the divisor class group of A.

Example 12 ([2, Example 3.12]). Let $(A,\mathfrak{m})$ be a Noetherian local ring. We consider the full subcategory $d^{\geq 1}(A)$ of A-Mod which consists of all the finitely generated A-modules M satisfying depth $M\geq 1$. If depth $A\geq 1$, then $\mathrm{Pic}(d^{\geq 1}(A))$ is a trivial group.

4. Picard group of CM(A)

In this section, let $(A,\mathfrak{m})$ be a Cohen-Macaulay local k-algebra, i.e. A is a Noetherian local k-algebra with maximal ideal $\mathfrak{m}$ and satisfies the equality depth $A=\dim A$. We focus on the additive full subcategory $\mathrm{CM}(A)$ consisting of all the maximal Cohen-Macaulay modules over A and we give the reason why Theorem 1 holds. See [3] for the details of $\mathrm{CM}(A)$.

For the Picard group of $\mathrm{CM}(A)$, we have the following result.

Theorem 13 ([2, Theorem 5.2]). *Let A be a Cohen-Macaulay local k-algebra of any dimension. Suppose that A is regular in codimension two, i.e. $A_{\mathfrak{p}}$ is a regular local ring for any prime ideal $\mathfrak{p}$ with $\mathrm{ht}(\mathfrak{p}) = 2$. Then $\mathrm{Pic}(\mathrm{CM}(A))$ is a trivial group.*

Proof. If $\dim A = 0$, then $\mathrm{CM}(A) = A\text{-mod}$ and hence $\mathrm{Pic}(\mathrm{CM}(A)) = \mathrm{Pic}\, A$ is a trivial group by Example 10. If $\dim A = 1$, then $\mathrm{CM}(A) = d^{\geq 1}(A)$ and we have shown in Example 12 that $\mathrm{Pic}(\mathrm{CM}(A))$ is again a trivial group. If $\dim A = 2$, then our assumption means that A is a regular local ring hence a UFD. Note that $\mathrm{CM}(A) = \mathrm{ref}(A)$ in this case. Therefore $\mathrm{Pic}(\mathrm{CM}(A)) \cong C\ell(A)$ is a trivial group.

In the rest we assume $d = \dim A \geq 3$. Let $[M] \in \mathrm{Pic}(\mathrm{CM}(A))$. Assuming that M is not free, we shall show a contradiction. Take a free cover F of M and we obtain an exact sequence $0 \longrightarrow \Omega(M) \longrightarrow F \longrightarrow M \longrightarrow 0$. Note that the first syzygy module $\Omega(M)$ belongs to $\mathrm{CM}(A)$. Apply $\mathrm{Hom}_A(M, -)$ to the sequence, and we get an exact sequence

$$0 \to \mathrm{Hom}_A(M, \Omega(M)) \to \mathrm{Hom}_A(M, F) \to \mathrm{Hom}_A(M, M) \xrightarrow{f} \mathrm{Ext}^1_A(M, \Omega(M)) .$$

Notice that $f \neq 0$, since we have assumed that M is not free. Because of the assumption, we see that $\mathrm{Ext}^1_A(M, \Omega(M))_{\mathfrak{p}} = 0$ for all prime ideals $\mathfrak{p}$ with $\mathrm{ht}(\mathfrak{p}) = 2$. This implies that $\dim \mathrm{Ext}^1_A(M, \Omega(M)) \leq d - 3$, hence the image $\mathrm{Im}(f)$ is a nontrivial A-module of dimension at most $d - 3$. In particular, we have depth $\mathrm{Im}(f) \leq d - 3$.

On the other hand, since $\mathrm{Hom}_A(M, -)|_{\mathrm{CM}(A)}$ is a functor from $\mathrm{CM}(A)$ to itself, the modules $\mathrm{Hom}_A(M, \Omega(M))$, $\mathrm{Hom}_A(M, F)$ and $\mathrm{Hom}_A(M, M)$ have depth d. Hence we conclude from the depth argument [1, Proposition 1.2.9] that depth $\mathrm{Im}(f) \geq d - 2$. This is a contradiction, and the proof is completed. □

As in Theorem 1, let A be a Cohen-Macaulay local k-algebra of dimension d that has only an isolated singularity. We give a proof for the equalities in Theorem 1. If $d \neq 2$, then we see from Theorem 13 that $\mathrm{Pic}(\mathrm{CM}(A))$ is a trivial group, hence $\mathrm{Aut}_k(\mathrm{CM}(A)) \cong \mathrm{Aut}_{k\text{-alg}}(A)$ by Theorem 9. On the other hand, if $d = 2$ then A is a normal domain and we have $\mathrm{CM}(A) = \mathrm{ref}(A)$, hence $\mathrm{Pic}(\mathrm{CM}(A)) \cong C\ell(A)$ by Example 11. Therefore $\mathrm{Aut}_k(\mathrm{CM}(A)) \cong \mathrm{Aut}_{k\text{-alg}}(A) \ltimes C\ell(A)$ by Theorem 9.

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LEFT DIFFERENTIAL OPERATORS OF MODULES OVER RINGS

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ABSTRACT. We define left differential operators of modules which several algebras act and study their fundamental properties. We also characterize separable algebras by making use of our left differential operators.

1. 序

Osborn [7] および Heyneman-Sweedler [2] に始まる可換多元環の微分演算子の理論は, Sweedler [8] によって非可換多元環へ拡張されたのだが, それは片側加群に値をとる片側 derivation の概念を拡張したものであり, 両側加群に値をとる derivation には遠い概念であった. とは言っても, 極めて特殊な derivation とならば関係付けられることが [6] に示されている. 関連の研究が [3], [4], [5] にある.

本論文では, derivation を包含するような微分演算子を導入する. また, 微分演算子を用いた分離多元環の特徴付けを考察する.

本論文を通じて, K は常に単位元を有する可換環を表す. また, §4 を除いて, 多元環はすべて単位元を有し, 多元環上の加群はすべて単位元が恒等的に作用するものとする. 多元環 R の双対多元環を R° で表す.

2. 左微分演算子

$\mathcal{S} = \{R_1, \ldots, R_n\}$ を K 多元環の有限族とする. K 加群 M が, 各 $i = 1, \ldots, n$ について左 R_i 加群構造をもち, $i \neq j$ のとき R_i と R_j の作用が可換ならば, M を左 $\mathcal{S}$ 加群と呼ぶ. 左 $\mathcal{S}$ 加群の間の写像 f がどの $i = 1, \ldots, n$ についても R_i 準同型写像であるとき, f を $\mathcal{S}$ 準同型写像と呼ぶ. 右 $\mathcal{S}$ 加群や両側 $\mathcal{S}$ 加群についても同様に定義する. $\mathcal{S}$ 上の右加群 M と左加群 N に対して, $M \times N$ から K 加群への双線形写像 φ で $\varphi(ua, v) = \varphi(u, av)$ ($u \in M, v \in N, a \in R_i, i = 1, \ldots, n$) を満たすもののすべてを表現する K 加群を $M \otimes_{\mathcal{S}} N$ で表し, $\mathcal{S}$ 上のテンソル積と呼ぶ.

これらの概念は新しいものではない. $R = R_1 \otimes_K \cdots \otimes_K R_n$ とおけば, 左 $\mathcal{S}$ 加群は左 R 加群に他ならず, $\mathcal{S}$ 準同型写像は R 準同型写像に他ならず, $\mathcal{S}$ 上のテンソル積は R 上のテンソル積に他ならない. どうして $\mathcal{S}$ 加群を持ち出したのかと言うと, §4 で扱う単位元が存在しない多元環でも利用可能だからです.

M, N を左 $\mathcal{S}$ 加群とする. $f \in \mathrm{Hom}_K(M, N)$ と $a \in R_i$ に対して $\{f, a\} \in \mathrm{Hom}_K(M, N)$ を次のように定める.

$$\{f, a\}(u) = f(au) - af(u) \quad (u \in M)$$

The detailed version of this paper will be submitted for publication elsewhere.

また，$F \subseteq \mathrm{Hom}_K(M,N)$ と $A \subseteq R_i$ に対し，$\{\{f,a\} \mid f \in F,\ a \in A\}$ で生成された $\mathrm{Hom}_K(M,N)$ の K 部分加群を $\{F,A\}$ で表し，整数 $p \geq 0$ に対する $\{F,A\}_p$ を次のように帰納的に定める．

$$\{F,A\}_0 = F, \quad \{F,A\}_{p+1} = \{\{F,A\}_p,\ A\}$$

Definition 1. $\mathcal{S} = \{R_1, \ldots, R_n\}$ を K 多元環の有限族とし，M, N を左 $\mathcal{S}$ 加群とする．また，$\mathbf{p} = (p_1, \ldots, p_n)$ を非負整数の順序対とする．$f \in \mathrm{Hom}_K(M,N)$ で

$$\{\cdots\{\{f,\ R_1\}_{p_1},\ R_2\}_{p_2}, \cdots,\ R_n\}_{p_n} = 0$$

を満たすものを型 $\mathbf{p}$ の左微分演算子と呼び，M から N への型 $\mathbf{p}$ の左微分演算子の全体を $\mathcal{D}_{\mathcal{S}}^{\mathbf{p}}(M,N)$ で表す．

Example 2. $n=1$ の場合，型 (p) の左微分演算子は Sweedler [8] の $p-1$ 次左微分演算子のことである．

Example 3. $\mathcal{S} = \{R,\ R^o\}$ とする．R^o は R の双対多元環である．$f \in \mathcal{D}_{\mathcal{S}}^{(1,1)}(R,M)$ で $f(1) = 0$ を満たすものは R から R 両側加群 M への derivation に他ならない．このことから，$\mathcal{S}$ 上の左微分演算子は derivation を一般化したものと見ることができる．

$\mathcal{S} = \{R_1, \ldots, R_n\}$ を K 多元環の有限族とし，$R = R_1 \otimes_K \cdots \otimes_K R_n$ とおく．$\mathbf{p} = (p_1, \ldots, p_n)$ を非負整数の順序対とする．$u \in R_i \otimes_K R_i$ と $a \in R_i$ に対して $[u,a] = ua - au$ とおく．$U \subseteq R_i \otimes_K R_i$ と $A \subseteq R_i$ に対し，$\{[u,a] \mid u \in U,\ a \in A\}$ で生成された $R_i \otimes_K R_i$ の K 部分加群を $[U,A]$ で表し，整数 $p \geq 0$ に対する $[U,A]_p$ を次のように帰納的に定める．

$$[U,A]_0 = U, \quad [U,A]_{p+1} = [[U,A]_p,\ A]$$

$R_i \otimes_K R_i$ において $[1 \otimes 1,\ R_i]_{p_i}$ で生成された R_i 両側部分加群を $I_i^{p_i}$ で表す．特に $I_i^0 = R_i \otimes_K R_i$ である．

$$I_{\mathcal{S}}^{\mathbf{p}} = I_1^{p_1} \otimes_K \cdots \otimes_K I_n^{p_n}$$

とおき，包含写像 $\iota_i^{p_i} : I_i^{p_i} \to I_i^0$ から得られる

$$\iota_{\mathcal{S}}^{\mathbf{p}} = \iota_1^{p_1} \otimes \cdots \otimes \iota_n^{p_n} : I_{\mathcal{S}}^{\mathbf{p}} \to I_{\mathcal{S}}^{\mathbf{0}}$$

の余核を

$$\pi_{\mathcal{S}}^{\mathbf{p}} : I_{\mathcal{S}}^{\mathbf{0}} \to \mathcal{J}_{\mathcal{S}}^{\mathbf{p}}$$

とする．ここで，$\mathbf{0} = (0, \ldots, 0)$ である．$\omega_{\mathcal{S}}^{\mathbf{p}} = \pi_{\mathcal{S}}^{\mathbf{p}}\big((1 \otimes 1) \otimes \cdots \otimes (1 \otimes 1)\big)$ とおき，写像

$$d_{\mathcal{S}}^{\mathbf{p}} : R \to \mathcal{J}_{\mathcal{S}}^{\mathbf{p}}$$

を，$d_{\mathcal{S}}^{\mathbf{p}}(x) = \omega_{\mathcal{S}}^{\mathbf{p}} x\ (x \in R)$ で定める．次の定理に示すように，$d_{\mathcal{S}}^{\mathbf{p}} \in \mathcal{D}_{\mathcal{S}}^{\mathbf{p}}(R,\ \mathcal{J}_{\mathcal{S}}^{\mathbf{p}})$ であり，型 $\mathbf{p}$ のすべての左微分演算子は $d_{\mathcal{S}}^{\mathbf{p}}$ と $\mathcal{S}$ 準同型写像との合成写像として得られる．

M, N を左 $\mathcal{S}$ 加群とする．自然な同型写像 $\sigma_M : M \to R \otimes_{\mathcal{S}} M$ と，M の恒等写像 1_M を用いて，任意の $\varphi \in \mathrm{Hom}_{\mathcal{S}}(\mathcal{J}_{\mathcal{S}}^{\mathbf{p}} \otimes_{\mathcal{S}} M,\ N)$ に対して

$$\eta_{\mathcal{S}}^{\mathbf{p}}(M,N)(\varphi) = \varphi(d_{\mathcal{S}}^{\mathbf{p}} \otimes 1_M)\sigma_M \in \mathrm{Hom}_K(M,N)$$

と定義する．即ち，$\eta_S^{\mathrm{p}}(M,N)(\varphi)(u) = \varphi(\omega_S^{\mathrm{p}} \otimes u)$ である．

Theorem 4. 上の記号の下に，$d_S^{\mathrm{p}} \in \mathcal{D}_S^{\mathrm{p}}(R, \mathcal{J}_S^{\mathrm{p}})$ であり，自然な同型写像

$$\eta_S^{\mathrm{p}}(M,N) : \mathrm{Hom}_S(\mathcal{J}_S^{\mathrm{p}} \otimes_S M,\ N) \to \mathcal{D}_S^{\mathrm{p}}(M,N)$$

を得る．

3. 分離性

分離多元環の概念の拡張として，多元環の有限族の分離性を考察する．K 多元環 R の乗法から得られる自然な R 両側準同型写像 $R \otimes_K R \to R$ が分裂するとき，R は分離多元環と呼ばれる．これは，R から R 両側加群への derivation がすべて内部 derivation であることに同値である．従って，$\mathcal{S} = \{R, R^{\circ}\}$ とおけば，すべての左 $\mathcal{S}$ 加群 M に対して

$$\mathcal{D}_S^{(1,1)}(R,M) = \mathrm{Hom}_R(R,M) + \mathrm{Hom}_{R^{\circ}}(R,M)$$

が成り立つことにも同値である．この事実から，次の定義は自然であろう．

Definition 5. $\mathcal{S} = \{R_1, \ldots, R_n\}$ を K 多元環の有限族とし，$\mathbf{1} = (1, \ldots, 1)$ とする．ただし，$n > 1$ である．すべての左 $\mathcal{S}$ 加群 M, N に対して

$$\mathcal{D}_S^{\mathbf{1}}(M,N) = \sum_{i=1}^{n} \mathrm{Hom}_{R_i}(M,N)$$

が成り立つとき，$\mathcal{S}$ は分離的であるという．

次の定理は，有限族の分離性と分離多元環とのかかわりを示している．

Theorem 6. $\mathcal{S} = \{R_1, \ldots, R_n\}$ の中の $n-1$ 個が分離多元環ならば，$\mathcal{S}$ は分離的である．

特別な場合として次の強力な事実が得られる．

Theorem 7. $\mathcal{S} = \{R, R^{\circ}\}$ が分離的であるためには，R が分離多元環であることが必要十分である．

4. 単位元が存在しない場合

本節では単位元の存在しない多元環について考察する．単位元を付加するだけで得られる結果もあるのだが，分離性についてはそう簡単ではない．

K 多元環の有限族 $\mathcal{S} = \{R_1, \ldots, R_n\}$ を考える．$\mathcal{S}$ 上の加群や準同型写像の定義は自明である．まず，単位元を付加して考える．K 加群 $R_i^1 = K \oplus R_i$ の乗法を

$$(\alpha, x)(\beta, y) = (\alpha\beta,\ \alpha y + \beta x + xy)$$

と定めれば，R_i^1 は単位元 $(1,0)$ を有する K 多元環であり，R_i を部分多元環として含んでいる．$\mathcal{S}^1 = \{R_1^1, \ldots, R_n^1\}$，$R^1 = R_1^1 \otimes_K \cdots \otimes_K R_n^1$ とおく．こうすることによって Theorem 4 を利用することができる．

Theorem 8. 上の記号の下で，任意の左 $\mathcal{S}$ 加群 M, N と非負整数の順序対 $\mathbf{p} = (p_1, \ldots, p_n)$ に対して，

$$\eta^{\mathbf{p}}_{\mathcal{S}^1}(M,N) : \mathrm{Hom}_{\mathcal{S}}(\mathcal{J}^{\mathbf{p}}_{\mathcal{S}^1} \otimes_{\mathcal{S}} M,\ N) \to \mathcal{D}^{\mathbf{p}}_{\mathcal{S}}(M,N)$$

は自然な同型写像である．

次に，分離性について考察する．単位元が存在しない場合，Theorem 7 は成立しない．そこで条件を緩めて次の定義を与える．

Definition 9. K 多元環 R に対して $\mathcal{S} = \{R, R^{\circ}\}$ とおく．すべての左 $\mathcal{S}$ 加群 M に対して

$$\mathcal{D}^{(1,1)}_{\mathcal{S}}(R,M) = \mathrm{Hom}_R(R,M) + \mathrm{Hom}_{R^{\circ}}(R,M)$$

が成り立つとき，R を弱分離多元環と呼ぶ．

§3 で指摘したように，単位元を有する場合は弱分離多元環と分離多元環は同じ概念である．単位元が存在しない場合はそうはいかない．次の事実と例から，分離性については単位元を付加する技法が利用できないことがわかる．

Theorem 10 ([1, Theorem II.2.5])**.** 体 K 上の単位元を有する分離多元環は半単純であり，K 上のベクトル空間として有限次元である．

Example 11. 体 K 上の多元環 $\begin{pmatrix} K & K & 0 \\ 0 & 0 & 0 \\ K & K & 0 \end{pmatrix}$ は弱分離多元環であるが半単純ではない．

Example 12. 体 K 上の無限次正方行列で 0 とは異なる成分が有限個であるものの全体は弱分離多元環を成すが K 上有限次元ではない．

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ON FILTERED SEMI-DUALIZING BIMODULES

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ABSTRACT. In this paper, we study the homological property of Rees modules of finitely generated filtered modules. In particular we state on Gorenstein dimension (more generally G_C-dimension in the sense of T. Araya, R. Takahashi, and Y. Yoshino [1]) of Rees modules.

1. Introduction

Semi-dualizing bimodule was introduced by T. Araya, R. Takahashi and Y. Yoshino in [1], which is a generalization of semi-dualizing module in commutative ring theory. For a semi-dualizing bimodule C and a finitely generated module M, they also introduced G_C-dim M, which is a generalization of Gorenstein dimension of M, and extended the notion of Cohen-Macaulay dimension for modules over commutative Noetherian local rings to that for bounded complexes over non-commutative Noetherian rings. On the other hand, in [3] with K. Nishida, we showed the following:

Theorem A. *Let Λ be a filtered ring, and M a finitely generated filtered Λ-module with good filtration. Then the Gorenstein dimension of M is less than or equal to the Gorenstein dimension of associated graded module of M.*

In Section 2, we study the filtered semi-dualizing bimodules and give a generalization of Theorem A without proof.

In Section 3, we state on Gorenstein-dimension of Rees modules. For a filtered (Λ, Λ')-bimodule C, we show that if the associated graded bimodule $\mathrm{gr}C$ of C is semi-dualizing, then Rees bimodule $\widetilde{C}$ of C is semi-dualzing (proposition 23), and we compare $G_{\mathrm{gr}C}\text{-}\dim \mathrm{gr}M$ with $G_{\widetilde{C}}\text{-}\dim \widetilde{M}$ for a finitely generated filtered Λ-module M.

In the rest of this section, we shall recall some definitions and properties on filtered ring theory.

Definition 1. Let Λ be a ring. A family $\mathcal{F}\Lambda = \{\mathcal{F}_p\Lambda \mid p \in \mathbb{Z}\}$ of additive subgroups of Λ is called a *(positive) filtration* of Λ, if

(1) $\mathcal{F}_p\Lambda \subset \mathcal{F}_{p+1}\Lambda$ for all $p \in \mathbb{Z}$,
(2) $\mathcal{F}_p\Lambda = 0$, if $p < 0$
(3) $1 \in \mathcal{F}_0\Lambda$,
(4) $(\mathcal{F}_p\Lambda)(\mathcal{F}_q\Lambda) \subset \mathcal{F}_{p+q}\Lambda$ for all $p, q \in \mathbb{Z}$, and
(5) $\Lambda = \bigcup_{p\in\mathbb{Z}} \mathcal{F}_p\Lambda$.

The detailed version of this paper will be submitted for publication elsewhere.

A ring Λ is called a *(positive) filtered ring*, if it has a filtration. If a ring Λ has a filtration $\mathcal{F}\Lambda$, then $\bigoplus_{p\in\mathbb{Z}}\mathcal{F}_p\Lambda/\mathcal{F}_{p-1}\Lambda$ is a *graded ring* with multiplication $\sigma_p(a)\sigma_q(b) = \sigma_{p+q}(ab)$ where $\sigma_p : \mathcal{F}_p\Lambda \longrightarrow \mathcal{F}_p\Lambda/\mathcal{F}_{p-1}\Lambda$ is a canonical map, and $a \in \mathcal{F}_p\Lambda$, $b \in \mathcal{F}_q\Lambda$. We denote by $\mathrm{gr}\Lambda$ the above associated graded ring of Λ.

Definition 2. Let Λ be a filtered ring with a filtration $\mathcal{F}\Lambda$, and M a Λ-module. A family $\mathcal{F}M = \{\mathcal{F}_pM \mid p \in \mathbb{Z}\}$ of additive subgroups of M is called a *filtration* of M, if

(1) $\mathcal{F}_pM \subset \mathcal{F}_{p+1}M$ for all $p \in \mathbb{Z}$,
(2) $\mathcal{F}_pM = 0$ for $p \ll 0$,
(3) $(\mathcal{F}_p\Lambda)(\mathcal{F}_qM) \subset \mathcal{F}_{p+q}M$ for all $p, q \in \mathbb{Z}$, and
(4) $M = \bigcup_{p\in\mathbb{Z}}\mathcal{F}_pM$.

A Λ-module M is called a *filtered Λ-module* if M has a filtration. If a left Λ-module M has a filtration $\mathcal{F}M$, then $\bigoplus_{p\in\mathbb{Z}}\mathcal{F}_pM/\mathcal{F}_{p-1}M$ is a graded left $\mathrm{gr}\Lambda$-module with action $\sigma_p(a)\tau_q(x) = \tau_{p+q}(ax)$ where $a \in \mathcal{F}_p\Lambda$, $x \in \mathcal{F}_qM$, and $\tau_q : \mathcal{F}_qM \longrightarrow \mathcal{F}_qM/\mathcal{F}_{q-1}M$ is a canonical map. We denote by $\mathrm{gr}M$ the above associated graded $\mathrm{gr}\Lambda$-module of M.

Let Λ, Λ' be filtered rings. A (Λ, Λ')-bimodule M is called a *filtered bimodule* if there exists a family $\mathcal{F}M$ of subgroups of M such that $({}_\Lambda M, \mathcal{F}M)$ and $(M_{\Lambda'}, \mathcal{F}M)$ are filtered modules.

Definition 3. Let Λ be a filtered ring with a filtration $\mathcal{F}\Lambda$. Then the graded ring $\oplus_{p\in\mathbb{Z}}\mathcal{F}_p\Lambda$ is called the *Rees ring* of $(\Lambda, \mathcal{F})$, and denoted by $\widetilde{\Lambda}$. Similarly, for a filtered left module $(M, \mathcal{F}M)$ over a filtered ring $(\Lambda, \mathcal{F}\Lambda)$, the graded left $\widetilde{\Lambda}$-module $\oplus_{p\in\mathbb{Z}}\mathcal{F}_pM$ is called the *Rees module* of M, and denoted by $\widetilde{M}$.

Let Λ be a filtered ring. Then the Rees ring $\widetilde{\Lambda}$ has the canonical central regular element $X = (\delta_{1p})_{p\in\mathbb{Z}} \in \widetilde{\Lambda}$ where δ_{ij} is the Kronecker's delta. Suppose that $(M, \mathcal{F}M)$ is a filtered Λ-module. Then,

(1) $\widetilde{\Lambda}/X\widetilde{\Lambda} \cong \mathrm{gr}\Lambda$ (as graded ring) and $\widetilde{M}/X\widetilde{M} \cong \mathrm{gr}_{\mathcal{F}}M$ (as graded module).
(2) $\widetilde{M}/(1-X)\widetilde{M} \cong M$

Definition 4. Let $(\Lambda, \mathcal{F}\Lambda)$ be a filtered ring. A filtration $\mathcal{F}M$ of a Λ-module M is called *good*, if there exist $p_1, \cdots, p_r \in \mathbb{Z}$ and $m_1, \cdots, m_r \in M$ such that for all $p \in \mathbb{Z}$

$$\mathcal{F}_pM = \sum_{i=1}^{r}(\mathcal{F}_{p-p_i}\Lambda)m_i.$$

From the above definition, we can easily check the following:

(1) For a filtered Λ-module $(M, \mathcal{F}M)$, $\mathcal{F}M$ is good if and only if $\mathrm{gr}_{\mathcal{F}}M$ is a finitely generated $\mathrm{gr}\Lambda$-module if and only if $\widetilde{M}$ is a finitely generated $\widetilde{\Lambda}$-module.
(2) Suppose that $(\Lambda, \mathcal{F}\Lambda)$ is a filtered ring. If M is a finitely generated Λ-module, then M has a good filtration.

Definition 5. Let $(M, \mathcal{F}M)$, $(N, \mathcal{F}N)$ be filtered Λ-modules. A Λ-homomorphism $f : M \longrightarrow N$ is called a *filtered homomorphism*, if $f(\mathcal{F}_pM) \subset \mathcal{F}_pN$ for all $p \in \mathbb{Z}$. Further, f is called *strict*, if $f(\mathcal{F}_pM) = \mathrm{Im}f \cap \mathcal{F}_pN$ for all $p \in \mathbb{Z}$.

Remark 6. (1) The composition of two filtered homomorphisms is also a filtered homomorphism, but it need not be strict even if both of them are strict.

(2) Let $f : M \longrightarrow N$ be a filtered homomorphism, then f indeuces canonical additive maps $f_p : \mathcal{F}_pM/\mathcal{F}_{p-1}M \longrightarrow \mathcal{F}_pN/\mathcal{F}_{p-1}N$ given by $x + \mathcal{F}_{p-1}M \longmapsto f(x) + \mathcal{F}_{p-1}N$. It is clear that $\mathrm{gr} f = \oplus_{p\in\mathbf{Z}} f_p$ defines a graded homomorphism from $\mathrm{gr}M$ to $\mathrm{gr}N$. Note that $(\mathrm{gr}g)(\mathrm{gr}f) = \mathrm{gr}(gf)$ for any filtered homomorphisms $f : M \longrightarrow N$, $g : N \longrightarrow L$. Similarly, $\tilde{f} = \oplus_{p\in\mathbf{Z}} f|_{\mathcal{F}_pM}$ defines a graded homomorphism from $\widetilde{M}$ to $\widetilde{N}$, and $\tilde{g}\tilde{f} = \widetilde{gf}$ holds.

Lemma 7. *Let* $(*)\ :\ L \xrightarrow{f} M \xrightarrow{g} N$ *be a sequence of filtered modules and filtered homomorphisms such that* $gf = 0$. *Then*

(1) *The sequence*

$$\mathrm{gr}(*)\ :\ \mathrm{gr}L \xrightarrow{\mathrm{gr}f} \mathrm{gr}M \xrightarrow{\mathrm{gr}g} \mathrm{gr}N$$

is exact if and only if $(*)$ *is exact and* f, g *are strict.*

(2) *The sequence*

$$(\tilde{*})\ :\ \widetilde{L} \xrightarrow{\tilde{f}} \widetilde{M} \xrightarrow{\tilde{g}} \widetilde{N}$$

is exact if and only if $(*)$ *is exact and* f *is strict.*

Lemma 8. *([2] Chapter III Proposition 2.2.4) Let* M *and* N *be filtered* Λ*-modules. Then* $\mathrm{gr}\,\mathrm{Ext}^i_\Lambda(M,N)$ *is a subfactor of* $\mathrm{Ext}^i_\Lambda(\mathrm{gr}\,M, \mathrm{gr}\,N)$ *for each* $i \geq 0$.

2. Semi-dualizing Filtered Modules

First, we recall the definition of semi-dualizing bimodules.

Definition 9. ([1] Definition 2.1) Let R, R' be Noetherian rings. An (R,R')-bimodule C is called a *semi-dualizing bimodule* if the following conditions hold:

(1) The right homothety R'-bimodule morphism $R' \longrightarrow \mathrm{Hom}_R(C,C)$ is a bijection,
(2) The left homothety R-bimodule morphism $R \longrightarrow \mathrm{Hom}_{R'}(C,C)$ is a bijection,
(3) $\mathrm{Ext}^i_R(C,C) = 0$ for all $i > 0$, and
(4) $\mathrm{Ext}^i_{R'}(C,C) = 0$ for all $i > 0$.

Definition 10. ([1] Definition 2.2) Let R, R' be Noetherian rings and C a semi-dualizing (R,R')-bimodule. An R-module M is called *C-reflexive* if the following conditions hold:

(1) $\mathrm{Ext}^i_R(M,C) = 0$ for all $i > 0$,
(2) $\mathrm{Ext}^i_{R'}(\mathrm{Hom}_R(M,C),C) = 0$ for all $i > 0$, and
(3) The natural morphism

$$M \longrightarrow \mathrm{Hom}_{R'}(\mathrm{Hom}_R(M,C),C)$$

is a bijection.

Definition 11. ([1] Definition 2.3) Let C be a semi-dualizing (R, R')-bimodule and M an R-module. If there exists an exact sequence

$$0 \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_0 \longrightarrow M \longrightarrow 0$$

where each X_i is a C-reflexive R-module, M is called that G_C-dimension is less than or equal to n (denoted by $G_C\text{-dim}M \leq n$). If $G_C\text{-dim}M \leq n$ and $G_C\text{-dim}M \not\leq n-1$, then we say G_C-dimension of M is equal to n (denoted by $G_C\text{-dim}\, M = n$).

Remark 12. (1) In [1], a semi-dualizing bimodule was defined over a left Noetherian ring R and a right Noetherian ring R'. In this paper we assume that both R and R' are (left and right) Noetherian rings.

(2) The ring R itself is a semidualizing (R, R)-bimodule and the R-reflexive modules coincide with the modules whose Gorenstein dimension are equal to 0. Moreover, in the case of $C = {}_R R_R$, we have $G_C\text{-dim}\, M = G\text{-dim}\, M$.

The following lemma is indispensable for the study of filtered semi-dualizing bimodules.

Lemma 13. *Let $(C, \mathcal{F})$ be a filtered (Λ, Λ')-bimodule such that ${}_{\mathrm{gr}\Lambda}\mathrm{gr}_{\mathcal{F}} C$ and $\mathrm{gr}_{\mathcal{F}} C_{\mathrm{gr}\Lambda}$ are finitely generated. If $\mathrm{gr}_{\mathcal{F}} C$ is a semi-dualizing $(\mathrm{gr}\,\Lambda, \mathrm{gr}\,\Lambda')$-bimodule, then C is a semi-dualizing bimodule.*

Proof. Assume that $f : \Lambda' \longrightarrow \mathrm{Hom}_\Lambda(C, C)$ is the right homothety Λ'-bimodule morphism, and $\varphi : \mathrm{gr}\,\Lambda' \longrightarrow \mathrm{Hom}_{\mathrm{gr}\Lambda}(\mathrm{gr}\, C, \mathrm{gr}\, C)$ is the right homothety $\mathrm{gr}\,\Lambda'$-bimodule morphism. Since there is a natural graded monomorphism

$$\psi : \mathrm{gr}\,\mathrm{Hom}_\Lambda(C, C) \longrightarrow \mathrm{Hom}_{\mathrm{gr}\Lambda}(\mathrm{gr}\, C, \mathrm{gr}\, C),$$

we get the following commutative diagram:

$$\begin{array}{ccc} \mathrm{gr}\,\Lambda' & \xrightarrow{\varphi} & \mathrm{Hom}_{\mathrm{gr}\Lambda}(\mathrm{gr}\, C, \mathrm{gr}\, C) \\ {\scriptstyle \mathrm{gr} f}\downarrow & & \| \\ \mathrm{gr}\,\mathrm{Hom}_\Lambda(C, C) & \xrightarrow{\psi} & \mathrm{Hom}_{\mathrm{gr}\Lambda}(\mathrm{gr}\, C, \mathrm{gr}\, C) \end{array}$$

Since $\varphi = \psi \circ \mathrm{gr} f$ is an isomorphism from the assumption, ψ is an epimorphism. Thus f is a Λ'-isomorphism. It follows from the lemma 8 that $\mathrm{gr}\,\mathrm{Ext}^i_\Lambda(C, C)$ is a subfactor of $\mathrm{Ext}^i_\Lambda(\mathrm{gr}\, C, \mathrm{gr}\, C)$ for each $i \geq 0$. Therefore $\mathrm{Ext}^i_\Lambda(C, C) = 0$ for all $i > 0$. Similarly, we can prove that the left homothety morphhism $g : \Lambda \longrightarrow \mathrm{Hom}_{\Lambda'}(C, C)$ is a bijection and $\mathrm{Ext}^i_{\Lambda'}(C, C) = 0$ for all $i > 0$. Therefore C is a semi-dualizing bimodule. □

Definition 14. We say that a filtered (Λ, Λ')-bimodule C is a *semi-dualizing filtered bimodule* if $\mathrm{gr}\, C$ is a semi-dualizing $(\mathrm{gr}\,\Lambda, \mathrm{gr}\,\Lambda')$-bimodule.

All semi-dualizing filtered bimodules are semi-dualizing bimodules by lemma 13. In the rest of this section, C is a semi-dualizing filtered (Λ, Λ')-bimodule.

In [1], it is proved that G_C-dim$M \leq k$ if and only if G_C-dim$\Omega^k M = 0$, where $\Omega^k M$ is the k-th syzygy of M (Lemma 2.7). Applying this lemma, we can prove the following result in a completely similar way to the proof of Theorem A. So we give only the result without proof.

Proposition 15. *Let M be a filtered Λ-module. Then the following inequality holds:*

$$G_C\text{-dim}M \leq G_{\mathrm{gr}C}\text{-dim}\,\mathrm{gr}M$$

3. G_C-DIMENSION FOR REES MODULES

Throughout this section, we denote by X (resp. X') the canonical central regular element $(\delta_{1p})_{p\in\mathbf{Z}} \in \widetilde{\Lambda}$ (resp. $(\delta_{1p})_{p\in\mathbf{Z}} \in \widetilde{\Lambda'}$) where δ_{ij} is the Kronecker's delta. First of all, we shall recall the Rees theorem, that is

Theorem 16 (Rees theorem). *([4] Theorem 9.37) Let R be a ring, $T \in R$ a central regular element, and M a T-torsionfree left R-module (T-torsionfree means that left multiplication by T is injectoin). Then*

$$\mathrm{Ext}^n_{R/TR}(A, M/TM) \cong \mathrm{Ext}^{n+1}_R(A, M)$$

for any left R/TR-module A and $n \geq 0$.

Since $\widetilde{M}/X\widetilde{M} \cong \mathrm{gr}M$, $\widetilde{M}$ is X-torsionfree for any $M \in \mathrm{filt}\Lambda$, and $\widetilde{\Lambda}/X\widetilde{\Lambda} \cong \mathrm{gr}_{\mathcal{F}}\Lambda$, we can get the following:

$$\mathrm{Ext}^n_{\mathrm{gr}\Lambda}(\mathrm{gr}M, \mathrm{gr}C) \cong \mathrm{Ext}^{n+1}_{\widetilde{\Lambda}}(\widetilde{M}/X\widetilde{M}, \widetilde{C})$$

In order to prove our main theorem, we give some easy lemmata without proofs.

Lemma 17. *Assume that M, N are filtered Λ-modules. Then there exists a natural isomorphism* $\widetilde{\mathrm{Hom}_\Lambda(M, N)} \cong \mathrm{Hom}_{\widetilde{\Lambda}}(\widetilde{M}, \widetilde{N})$

Lemma 18. *Assume that $M \in \mathrm{filt}\Lambda$ and*

$$\mathrm{Ext}^{i>0}_{\mathrm{gr}\Lambda}(\mathrm{gr}M, \mathrm{gr}C) = \mathrm{Ext}^{i>0}_{\mathrm{gr}\Lambda'}(\mathrm{Hom}_{\mathrm{gr}\Lambda}(\mathrm{gr}M, \mathrm{gr}C), \mathrm{gr}C) = 0.$$

Then, the natural map $\varphi : M \longrightarrow \mathrm{Hom}_{\Lambda'}(\mathrm{Hom}_\Lambda(M, C), C)$ is bijective if and only if the natural map $\Phi : \mathrm{gr}M \longrightarrow \mathrm{Hom}_{\mathrm{gr}\Lambda'}(\mathrm{Hom}_{\mathrm{gr}\Lambda}(\mathrm{gr}M, \mathrm{gr}C), \mathrm{gr}C)$ is bijective.

Lemma 19. *Let $M \in \mathrm{filt}\Lambda$. Then, the natural map*

$$\varphi : M \longrightarrow \mathrm{Hom}_{\Lambda'}(\mathrm{Hom}_\Lambda(M, C), C)$$

is strict isomorphism if and only if the natural map

$$\Phi : \widetilde{M} \longrightarrow \mathrm{Hom}_{\widetilde{\Lambda'}}(\mathrm{Hom}_{\widetilde{\Lambda}}(\widetilde{M}, \widetilde{C}), \widetilde{C})$$

is isomorphism.

Remark 20. Since X is an $\widetilde{M}$-regular element for all $M \in \text{filt}\Lambda$, there exists an exact sequence:

$$0 \longrightarrow \widetilde{M} \xrightarrow{\mu_X} \widetilde{M} \longrightarrow \widetilde{M}/X\widetilde{M} \longrightarrow 0$$

where μ_X is the left multiplication by X. Applying $\text{Hom}_{\widetilde{\Lambda}}(-,\widetilde{C})$, we get a long exact sequence:

$$(\dagger) \quad \cdots \longrightarrow \text{Ext}^i_{\widetilde{\Lambda}}(\widetilde{M}/X\widetilde{M},\widetilde{C}) \longrightarrow \text{Ext}^i_{\widetilde{\Lambda}}(\widetilde{M},\widetilde{C}) \xrightarrow{\overline{\mu_X}} \text{Ext}^i_{\widetilde{\Lambda}}(\widetilde{M},\widetilde{C}) \longrightarrow \text{Ext}^{i+1}_{\widetilde{\Lambda}}(\widetilde{M}/X\widetilde{M},\widetilde{C}) \longrightarrow \cdots$$

Since $\overline{\mu_X}$ is a right multiplication by X and $\text{Ext}^i_{\widetilde{\Lambda}}(\widetilde{M},\widetilde{C})$ is graded $\widetilde{\Lambda}'$-module such that $(\text{Ext}^i_{\widetilde{\Lambda}}(\widetilde{M},\widetilde{C}))_{(p)} = 0$ for $p \ll 0$, we get the following:

Lemma 21. *With the same notations as above, if* $\text{Ext}^i_{\widetilde{\Lambda}}(\widetilde{M},\widetilde{C}) \neq 0$, *then*

$$\text{Ext}^i_{\widetilde{\Lambda}}(\widetilde{M},\widetilde{C}) \xrightarrow{\overline{\mu_X}} \text{Ext}^i_{\widetilde{\Lambda}}(\widetilde{M},\widetilde{C})$$

is not surjective.

Form the lemma 21, we can immediately get the following:

Lemma 22. *With the same notations as avobe,* $\text{Ext}^{i>0}_{\widetilde{\Lambda}}(\widetilde{M},\widetilde{C}) = 0$ *if and only if* $\text{Ext}^{i>0}_{\text{gr}\Lambda}(\text{gr}M,\text{gr}C) = 0$.

Proposition 23. *Let M be a filnitely generated filtered (Λ,Λ')-bimodule. If M is a semi-dualizing filtered bimodule then $\widetilde{M}$ is a semi-dualizing $(\widetilde{\Lambda},\widetilde{\Lambda}')$-bimodule.*

Proof. By the lemma 22, we have

$$\text{Ext}^{i>0}_{\widetilde{\Lambda}}(\widetilde{C},\widetilde{C}) = 0 \text{ if and only if } \text{Ext}^{i>0}_{\text{gr}\Lambda}(\text{gr}C,\text{gr}C) = 0, \text{ and}$$

$$\text{Ext}^{i\geq 0}_{\widetilde{\Lambda}'}(\widetilde{C},\widetilde{C}) = 0 \text{ if and only if } \text{Ext}^{i\geq 0}_{\text{gr}\Lambda'}(\text{gr}C,\text{gr}C) = 0.$$

Assume that $f : \Lambda' \longrightarrow \text{Hom}_\Lambda(C,C)$ is the right homothety Λ'-bimodule morphism, $g : \text{gr}\,\Lambda' \longrightarrow \text{Hom}_{\text{gr}\Lambda}(\text{gr}\,C,\text{gr}\,C)$ is the right homothety $\text{gr}\,\Lambda'$-bimodule morphism, and $h : \widetilde{\Lambda}' \longrightarrow \text{Hom}_{\widetilde{\Lambda}}(\widetilde{C},\widetilde{C})$ is the right homothety $\widetilde{\Lambda}'$-bimodule morphism. Since there is a natural graded monomorphisms

$$\varphi : \text{gr}\,\text{Hom}_\Lambda(C,C) \longrightarrow \text{Hom}_{\text{gr}\Lambda}(\text{gr}\,C,\text{gr}\,C),$$

we get the following two commutative diagrams:

$$\begin{array}{ccc} \text{gr}\,\Lambda' & \xrightarrow[g]{} & \text{Hom}_{\text{gr}\Lambda}(\text{gr}\,C,\text{gr}\,C) \\ {\scriptstyle \text{gr}f}\downarrow & & \| \\ \text{gr}\,\text{Hom}_\Lambda(C,C) & \xrightarrow[\varphi]{} & \text{Hom}_{\text{gr}\Lambda}(\text{gr}\,C,\text{gr}\,C) \end{array}$$

$$\begin{array}{ccc} \widetilde{\Lambda}' & \xrightarrow[h]{} & \text{Hom}_{\widetilde{\Lambda}}(\widetilde{C},\widetilde{C}) \\ {\scriptstyle \tilde{f}}\downarrow & & \| \\ \widetilde{\text{Hom}_\Lambda(C,C)} & \xrightarrow[\psi]{} & \text{Hom}_{\widetilde{\Lambda}}(\widetilde{C},\widetilde{C}) \end{array}$$

Note that f is a strict Λ'-isomorphism if g is a bijection as in the proof of the lemma 13. Hence $\tilde{f}$ is an isomorphism. Since ψ is also an isomorphism from the remark 17, h is an isomorphism. Similarly, we can show that the left homothety $\widetilde{\Lambda}$-bimodule morphism is bijective. Therefore $\widetilde{C}$ is semi-dualizing.

□

Now we can show the main theorem of this paper.

Theorem 24. *Let M be a filtered Λ-module. Then $G_{\widetilde{C}}$-dim$\widetilde{M} = 0$ if and only if $G_{\mathrm{gr}C}$-dimgr$M = 0$.*

Proof. Assume that $G_{\widetilde{C}}$-dim$\widetilde{M} = 0$. Since $\mathrm{Ext}^{i>0}_{\widetilde{\Lambda}}(\widetilde{M}, \widetilde{C}) = 0$, we have $\mathrm{Ext}^{i>0}_{\mathrm{gr}\Lambda}(\mathrm{gr}M, \mathrm{gr}C) = 0$ from the lemma 22. Moreover we get the following short exact seqence from the (†) in the remark 20:

$$0 \longrightarrow (\widetilde{M})^* \longrightarrow (\widetilde{M})^* \longrightarrow \mathrm{Ext}^1_{\widetilde{\Lambda}}(\widetilde{M}/X\widetilde{M}, \widetilde{C}) \longrightarrow 0$$

By the remark 17, we get the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\widetilde{M})^* & \longrightarrow & (\widetilde{M})^* & \longrightarrow & \mathrm{Ext}^1_{\widetilde{\Lambda}}(\widetilde{M}/X\widetilde{M}, \widetilde{C}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \widetilde{M^*} & \longrightarrow & \widetilde{M^*} & \longrightarrow & \widetilde{M^*}/X\widetilde{M^*} & \longrightarrow & 0 \quad (\dagger)^* \end{array}$$

Thus, $\mathrm{gr}M^* \cong \widetilde{M^*}/X\widetilde{M^*} \cong \mathrm{Ext}^1_{\widetilde{\Lambda}}(\widetilde{M}/X\widetilde{M}, \widetilde{C}) \cong (\mathrm{gr}M)^*$ by Rees theorem. By taking the long exact sequence of $(\dagger)^*$, we get $\mathrm{Ext}^{i>0}_{\mathrm{gr}\Lambda'}((\mathrm{gr}M)^*, \mathrm{gr}\Lambda) = 0$. Hence it follows from the lemma 18 and 19 that the natural map

$$\Phi : \mathrm{gr}M \longrightarrow \mathrm{Hom}_{\mathrm{gr}\Lambda'}(\mathrm{Hom}_{\mathrm{gr}\Lambda}(\mathrm{gr}M, \mathrm{gr}C), \mathrm{gr}C)$$

is an isomorphism. Therefore $G_{\mathrm{gr}C}$-dim gr$M = 0$.

Conversely, assume that $G_{\mathrm{gr}C}$-dim gr$M = 0$. By the remark 20, the lemma 22 and the Rees theorem, we can show

$$\mathrm{Ext}^{i>0}_{\widetilde{\Lambda}}(\widetilde{M}, \widetilde{C}) = \mathrm{Ext}^{i>0}_{\widetilde{\Lambda'}}((\widetilde{M})^*, \widetilde{C}) = 0.$$

Therefore $G_{\widetilde{C}}$-dim $\widetilde{M} = 0$. Since $G_{\mathrm{gr}C}$-dim gr$M = 0$,

$$\mathrm{Ext}^{i>0}_{\mathrm{gr}\Lambda}(\mathrm{gr}M, \mathrm{gr}C) = \mathrm{Ext}^{i>0}_{\mathrm{gr}\Lambda'}(\mathrm{Hom}_{\mathrm{gr}\Lambda}(\mathrm{gr}M, \mathrm{gr}C), \mathrm{gr}C) = 0$$

and the natural map $\Phi : \mathrm{gr}M \longrightarrow \mathrm{Hom}_{\mathrm{gr}\Lambda'}(\mathrm{Hom}_{\mathrm{gr}\Lambda}(\mathrm{gr}M, \mathrm{gr}C), \mathrm{gr}C)$ is an isomorphism. Therefore the natural map

$$\Psi : \widetilde{M} \longrightarrow \mathrm{Hom}_{\widetilde{\Lambda'}}(\mathrm{Hom}_{\widetilde{\Lambda}}(\widetilde{M}, \widetilde{C}), \widetilde{C})$$

is an isomorphism from the lemma 18 and 19. Hence, $G_{\widetilde{C}}$-dim$\widetilde{M} = 0$.

□

We can show the following by induction on $G_{\mathrm{gr}C}$-dim gr$M$.

Corollary 25. *For any finitely generated filtered Λ-module M with good filtration,*

$$G_{\mathrm{gr}C}\text{-dim}\,\mathrm{gr}M = G_{\widetilde{C}}\text{-dim}\,\widetilde{M}$$

holds.

In particular, in the case of $C = {}_\Lambda\Lambda_\Lambda$, we can get the following.

Corollary 26. G-dim gr$M$ = G-dim $\widetilde{M}$ *for all finitely generated filtered Λ-module M with good filtration.*

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NOTES ON THE FEIT-THOMPSON CONJECTURE

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ABSTRACT. In this paper, we present partial solutions about Feit Thompson Conjecture.

Key Words: primes, divisor, greatest common divisor, relatively prime,

2000 *Mathematics Subject Classification:* Primary 11A15 ; Secondary 20D05, 20D10.

Feit and Thompson [2] conjectured that $F = (q^p - 1)/(q-1)$ does not divide $T = (p^q - 1)/(q-1)$ for distinct odd primes $p < q$ (see also [6]).

In the paper [1, p.1], it was mentioned that if it could be proved, Odd paper [3] could be shortened by nearly 50 pages (see also [4, p.125]).

Stephan [6] conjectured that F and T are relatively prime. However, using computer, he found a common divisor $r = 112643 = 2pq + 1$ for a pair $p = 17, q = 3313$.

This is a rare example by the equation $q^{\frac{p-1}{2}} \equiv 1 \bmod p^2$ for this pair (see [5]).

He also confirmed that r is the greatest common divisor of F and T by computer, so this example leaves Feit-Thompson conjecture unresolved.

At the present, it is known by computer that no other such pairs exist for $p < q < 10^7$ and $p = 3 < q < 10^{14}$ (see [4]).

The next is easily proved as in my talk.

Proposition. *In either case of the next conditions, F does not divide T.*

(1) $q \equiv 1 \bmod p$.
(2) $p = 3 < q$ *and F is composite.*
(3) $p \equiv 3$ *and* $q \equiv 1 \bmod 4$.
(4) $r = 2p + 1$ *is prime, Legendre symbol* $\left(\frac{p}{r}\right) = 1$, *and* $q \not\equiv 1 \bmod r$.

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The detailed version of this paper will be submitted for publication elsewhere.

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Errata to " Some congruences concerning finite groups "
appeared in this Proceedings of the 40th Symposium.

In Lemma 10, the assumption $|\Delta| = p + s$ with $s < p$ should be corrected to $|\Delta|$ is p or $p + 1$.

HOCHSCHILD COHOMOLOGY OF BRAUER ALGEBRAS

HIROSHI NAGASE

ABSTRACT. Suppose B is an algebra with a stratifying ideal BeB generated by an idempotent e. We will establish long exact sequences relating the Hochschild cohomology groups of the three algebras B, B/BeB and eBe. This provides a common generalization of various known results, all of which extend Happel's long exact sequence for one-point extensions. Applying one of these sequences to Hochschild cohomology algebras modulo the ideal generated by homogeneous nilpotent elements, we obtain, in some cases, that these algebras are finitely generated.

1. INTRODUCTION

Let B be an algebra with a stratifying ideal BeB generated by an idempotent e and $\mathrm{HH}^n(B)$ the nth Hochschild cohomology group of B. In [11], we obtain a long exact sequence

$$\cdots \to \mathrm{Ext}^n_{B^e}(B/BeB, BeB) \to \mathrm{HH}^n(B) \to \mathrm{HH}^n(B/BeB) \oplus \mathrm{HH}^n(eBe) \to \cdots,$$

which is a generalization of Happel's long exact sequence in [9]. Moreover this is a generalization of various known long exact sequences in the case of triangular matrix algebras by Michelena and Platzeck in [13], Green and Solberg in [8] and Cibils, Marcos, Redondo and Solotar in [2], and in the case of algebras with heredity ideals by de la Peña and Xi in [14].

For any finite dimensional algebra B with a stratifying ideal BeB, we will apply our long exact sequence to the quotient of the Hochschild cohomology algebra $\mathrm{HH}^*(B)$ modulo the ideal $\mathcal{N}_B$ generated by homogeneous nilpotent elements. We denote by $\overline{\mathrm{HH}}^*(B)$ the graded factor algebra $\mathrm{HH}^*(B)/\mathcal{N}_B$.

In [16], for any finite dimensional algebra A, Snashall and Solberg studied support variety by using Hocschild cohomology algebra $\mathrm{HH}^*(A)$ and conjectured that $\overline{\mathrm{HH}}^*(A)$ is a finitely generated algebra. Green, Snashall and Solberg have shown the conjecture to hold true for self-injective algebras of finite representation type [6] and for monomial algebras [7]. Recently Xu has shown that there exists a counter example to the conjecture in [17]. We are, however, interested in the condition when $\overline{\mathrm{HH}}^*(A)$ is finitely generated.

Applying the long exact sequence above to Brauer algebra $B_k(n,\delta)$, we obtain an embedding

$$\overline{\mathrm{HH}}^*(B_k(n,\delta)) \hookrightarrow \overline{\mathrm{HH}}^*(k\Sigma_n) \times \overline{\mathrm{HH}}^*(k\Sigma_{n-2}) \times \cdots \times \overline{\mathrm{HH}}^*(k\Sigma_t)$$

where Σ_m is the symmetric group on m letters, $k\Sigma_0 = k$ and t is 0 or 1 depending on whether n is even or odd (see Proposition 5). By using this embedding, we obtain the result that $\overline{\mathrm{HH}}^*(B_k(n,\delta))$ is finitely generated in some cases.

The detailed version of this paper will be submitted for publication elsewhere.

2. STRATIFYING IDEALS

In this section we recall some results about Hochschild cohomology groups of algebras with stratifying ideals in [11]. The following definition is due to Cline, Parshall and Scott ([3], 2.1.1 and 2.1.2), who work with finite dimensional algebras over fields. We keep our general setup of algebras projective over a commutative noetherian ring.

Definition 1. Let B be an algebra and e an idempotent. The two-sided ideal BeB generated by e is called a *stratifying ideal* if the following equivalent conditions (A) and (B) are satisfied:
(A) (a) The multiplication map $Be \otimes_{eBe} eB \to BeB$ is an isomorphism.
(b) For all $n > 0$, $\mathrm{Tor}_n^{eBe}(Be, eB) = 0$.
(B) The epimorphism $B \to A := B/BeB$ induces isomorphisms

$$\mathrm{Ext}_A^\bullet(X, Y) \simeq \mathrm{Ext}_B^\bullet(X, Y)$$

for all A-modules X and Y.

The following remark can be used to check if an ideal is stratifying.

Remark 2. Let e be an idempotent element in B. Then BeB is projective as a left (resp. right) B-module if and only if eB (resp. Be) is projective as a left (respectively right) eBe-module and the multiplication map $Be \otimes_{eBe} eB \to BeB$ is an isomorphism.

Heredity ideals are examples of stratifying ideals, thus our results will extend results obtained in [14]. On the other hand, for any triangulated algebra B has an idempotent e such that BeB is projective. By Remark 2, BeB is a stratifying ideal. Thus our results also will extend results of [2, 8, 13]. There are, however, plenty of other examples. Stratifying ideals and stratified algebras occur frequently in applications, for example in algebraic Lie theory in the context of Schur algebras and of blocks of the Bernstein-Gelfand-Gelfand category of a semisimple complex Lie algebra.

From now on, we assume that BeB is a stratifying ideal of B and we put $A := B/BeB$.

Theorem 3. *There are long exact sequences as follows:*
(1) $\cdots \to \mathrm{Ext}^n_{B^e}(B, BeB) \to \mathrm{HH}^n(B) \to \mathrm{HH}^n(A) \to \cdots$;
(2) $\cdots \to \mathrm{Ext}^n_{B^e}(A, B) \to \mathrm{HH}^n(B) \to \mathrm{HH}^n(eBe) \to \cdots$; *and*
(3) $\cdots \to \mathrm{Ext}^n_{B^e}(A, BeB) \to \mathrm{HH}^n(B) \xrightarrow{f} \mathrm{HH}^n(A) \oplus \mathrm{HH}^n(eBe) \to \cdots$.

We remark that by using the partial recollement of bounded below derived categories

$$D^+(\mathrm{mod}\, A) \rightleftarrows D^+(\mathrm{mod}\, B) \rightleftarrows D^+(\mathrm{mod}\, eBe),$$

we also can obtain the long exact sequence (3).

We also note that Suarez-Alvarez [15] independently has obtained the first long exact sequence in Theorem 3 above by using different methods based on spectral sequences.

Recall the notation that $\mathcal{N}_B$ is the ideal of $\mathrm{HH}^\bullet(B)$ which is generated by homogeneous nilpotent elements, and $\overline{\mathrm{HH}}^\bullet(B)$ is the factor algebra $\mathrm{HH}^\bullet(B)/\mathcal{N}_B$.

Corollary 4.
(1) *Let* $f : \mathrm{HH}^\bullet(B) \to \mathrm{HH}^\bullet(A) \times \mathrm{HH}^\bullet(eBe)$ *be the graded algebra homomorphism in sequence (3) above. Then* $(\mathrm{Ker}\, f)^2$ *vanishes.*

(2) *The induced homomorphism* $\overline{f}: \overline{\mathrm{HH}}^{*}(B) \to \overline{\mathrm{HH}}^{*}(A) \times \overline{\mathrm{HH}}^{*}(eBe)$ *is injective.*

We note that the the graded algebra homomorphism in Corollary above was studied in the case of a one point extension by Green, Marcos and Snashall [5].

3. Brauer algebras

Finally we give an example of an algebra occurring in algebraic Lie theory, see for instance [12] or [10] for the properties of Brauer algebras used in this example. We denote by Σ_n the symmetric group on n letters and k an algebraically closed field. For any natural number n and any δ in k, we denote by $B_k(n,\delta)$ the Brauer algebra.

Proposition 5. *If δ is not 0 or n is odd, then there is an injective graded algebra homomorhism*

$$\overline{\mathrm{HH}}^{*}(B_k(n,\delta)) \hookrightarrow \overline{\mathrm{HH}}^{*}(k\Sigma_n) \times \overline{\mathrm{HH}}^{*}(k\Sigma_{n-2}) \times \cdots \times \overline{\mathrm{HH}}^{*}(k\Sigma_t)$$

where $k\Sigma_0 = k$ and t is 0 or 1 depending on whether n is even or odd.

Proof. For any Brauer algebra $B_k(n,\delta)$, if δ is not 0 or n is odd, then there is a filtration

$$0 < I_t < I_{t+2} < \cdots < I_{n-2} < I_n = B_k(n,\delta)$$

such that the subquotient I_s/I_{s-2} is a stratifying ideal of $B_s = B_k(n,\delta)/I_{s-2}$,where t is 0 or 1 depending on whether n is even or odd and $I_s = 0$ if $s < 0$ (see [10]). Moreover there is an idempotent e_s in B_s such that $I_s/I_{s-2} = B_s e_s B_s$, $e_s B_s e_s \cong k\Sigma_s$ and e_n is the identity of B_n (see [4]). By Corollary 4, there exists an injective graded algebra homomorphism

$$\overline{\mathrm{HH}}^{*}(B_s) \hookrightarrow \overline{\mathrm{HH}}^{*}(B_{s+2}) \times \overline{\mathrm{HH}}^{*}(k\Sigma_s).$$

Since $B_t = B_k(n,\delta)$ and $B_n \cong k\Sigma_n$, the claim follows. □

Corollary 6. *Suppose that δ is not 0 or n is odd. If the characteristic of k is either zero or bigger than n, then $\overline{\mathrm{HH}}^{*}(B_k(n,\delta))$ is a finitely generated algebra.*

Proof. If the characteristic of k is either zero or bigger than n, then for any $s < n$, $k\Sigma_s$ is semisimple and $\overline{\mathrm{HH}}^{*}(k\Sigma_s) \cong k^m$ where m is the number of the blocks of $k\Sigma_s$. By Proposition 5, $\overline{\mathrm{HH}}^{*}(B_k(n,\delta))$ is a finitely generated algebra. □

Corollary 7. $\overline{\mathrm{HH}}^{*}(B_k(2,\delta))$ *and* $\overline{\mathrm{HH}}^{*}(B_k(3,\delta))$ *are finitely generated algebras.*

Proof. By Proposition 5, there exists an embedding

$$\overline{\mathrm{HH}}^{*}(B_k(3,\delta)) \hookrightarrow \overline{\mathrm{HH}}^{*}(k\Sigma_3) \times \overline{\mathrm{HH}}^{*}(k\Sigma_1)$$

as a graded algebra homomorphism. Since $k\Sigma_1 \cong k$ and $k\Sigma_3$ is a self-injective algebra of finite representation type, $\overline{\mathrm{HH}}^{*}(k\Sigma_3) \times \overline{\mathrm{HH}}^{*}(k\Sigma_1)$ is isomorphic to a product of some polynomial algebras in one variable $k[x]$ and some copies of the ground field k (see [6]). Because any graded subalgebra of a product of some polynomial algebras with one variable $k[x]$ is a finitely generated algebra, we obtain the result that $\overline{\mathrm{HH}}^{*}(B_k(3,\delta))$ is a finitely generated algebra.

By Proposition 5, if δ is not zero, then there exists an embedding

$$\overline{\mathrm{HH}}^{*}(B_k(2,\delta)) \hookrightarrow \overline{\mathrm{HH}}^{*}(k\Sigma_2) \times \overline{\mathrm{HH}}^{*}(k\Sigma_0)$$

as a graded algebra homomorphism. Since $k\Sigma_0 = k$ and $k\Sigma_2$ is a self-injective algebra of finite representation type, $\overline{\mathrm{HH}}^*(B_k(2,\delta))$ is a finitely generated algebra by the same argument above. If $\delta = 0$, then $B_k(2,\delta)$ is isomorphic to

$$k \times k[x]/x^2 \text{ (char} k \neq 2) \text{ or } k[x,y]/(x^2, xy, y^2) \text{ (char} k = 2).$$

Since both are radical square zero algebras, $\overline{\mathrm{HH}}^*(B_k(2,\delta))$ is a finitely generated algebra (see [1] or [7]). □

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DESCENT OF DIVISOR CLASS GROUPS OF KRULL DOMAINS

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ABSTRACT. For a pair of Krull domains (A, B) such that $Q(B)/Q(A)$ is a field extension of their quotient fields and $Q(A) \cap B = A$, we study on the relation between the divisor class groups $\mathrm{Cl}(A)$ and $\mathrm{Cl}(B)$ by A. Magid's diagram showing finite generation of class groups of rings of invariants (cf. [4]). We define the descent properties with the existence of the canonical morphism of class groups in the sense of Magid and obtain a ladder property of these descents. This can be applied to regular actions of algebraic tori on affine normal varieties and characterizes freeness of monomials of prime relative invariants on these varieties. Furthermore we define certain subgroups of class groups of normal domains and their invariant subrings which determine a class of modules of relative invariants to be free.

Key Words: Krull Domain, Class Group, Relative Invariant, Algebraic Torus.

2000 *Mathematics Subject Classification:* Primary 13A50, 14L24; Secondary 13C20, 14L30.

1. INTRODUCTION

In this paper, we denote by (A, B) a pair of Krull domains such that the quotient field $Q(B)$ of B is an extension of $Q(A)$ satisfying $A = Q(A) \cap B$, which is called a *generic dominant Krull pair*. This is related to invariant theory of normal varieties as follows: Let (X, G) be a regular action of affine algebraic group G on an affine normal variety over an algebraically closed field K. Then, putting $A = \mathcal{O}(X)^G$ and $B = \mathcal{O}(X)$, we obtain a generic dominant Krull pair (A, B), where $\mathcal{O}(X)$ denotes the affine algebra of regular functions on X.

We now introduce the notations which are used throughout in this paper (cf. [1] for a general reference). For a Krull domain R, let $\mathrm{Ht}_1(R) := \{\mathfrak{P} \in \mathrm{Spec}\, R \mid \mathrm{ht}(\mathfrak{P}) = 1\}$, $\mathrm{Div}(R) :=$ the divisor group of R, $\mathrm{Prin}(R) :=$ the group of principal divisors of R and $\mathrm{Cl}(R) :=$ the divisor class group of R. Let $I_R(D)$ denote the divisorial fractional ideal of R defined by a divisor $D \in \mathrm{Div}(R)$. For a non-empty $Y \subseteq Q(R)$ such that $R \cdot Y$ is a fractional ideal of R, let $(R \cdot Y)^{\sim}$ denote the divisorialization of $R \cdot Y$ in R and put $\mathrm{div}_R(Y) :=$ the divisor defined by $(R \cdot Y)^{\sim}$. Let $\mathrm{v}_{R,\mathfrak{P}}$ stand for the discrete valuation defined by $\mathfrak{P} \in \mathrm{Ht}_1(R)$.

At first we review A. Magid's descent (cf. [4, 5]) of a generic dominant Krull pair (A, B). Let $X_{\mathfrak{q}}(B)$ be the set $\{\mathfrak{P} \in \mathrm{Ht}_1(B) \mid \mathfrak{P} \cap A = \mathfrak{q}\}$ which is non-empty for $\mathfrak{q} \in \mathrm{Ht}_1(A)$.

The detailed version of this paper will be submitted for publication elsewhere.

Moreover put

$$x_{\mathfrak{q}} := \sum_{\mathfrak{P}\in X_{\mathfrak{q}}(B)} \mathrm{e}(\mathfrak{P},\mathfrak{q})\cdot \mathrm{div}_B(\mathfrak{P}) \in \mathrm{Div}(B)$$

where $\mathrm{e}(\mathfrak{P},\mathfrak{q})$ denotes the reduced ramification index of $\mathfrak{P}$ over $\mathfrak{q}$. Set $\mathrm{Ht}_1(A,B) := \{\mathfrak{P} \in \mathrm{Ht}_1(B) \mid \mathfrak{P}\cap A \in \mathrm{Ht}_1(A)\}$. We define the subgroup

$$E^*(A,B) := \left(\bigoplus_{\mathfrak{q}\in\mathrm{Ht}_1(A)} Z\cdot x_{\mathfrak{q}}\right) \oplus \left(\bigoplus_{\mathfrak{P}\in\mathrm{Ht}_1(B),\mathrm{ht}(\mathfrak{P}\cap A)\geqq 2} Z\cdot \mathrm{div}_B(\mathfrak{P})\right)$$

of $\mathrm{Div}(B)$ and the homomorphism

$$\Phi^*_{A,B} : E^*(A,B) \xrightarrow{\mathrm{pr.}} \bigoplus_{\mathfrak{q}\in\mathrm{Ht}_1(A)} Z\cdot x_{\mathfrak{q}} \longrightarrow \mathrm{Div}(A)$$

induced by $\Phi^*_{A,B}(x_{\mathfrak{q}}) = \mathrm{div}_A(\mathfrak{q}) \in \mathrm{Div}(A)$. Moreover we set

$$F(A,B) := (\mathrm{Prin}(B)\cap E^*(A,B))/\mathrm{div}_B(\mathrm{U}(Q(A))),$$

$$E(A,B) := E^*(A,B))/\mathrm{div}_B(\mathrm{U}(Q(A))$$

respectively. Then one has the following commutative diagram with exact rows and columns (e.g., [5]) which is called the Magid diagram of (A,B):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F(A,B) & \longrightarrow & E(A,B) & \longrightarrow & \mathrm{Cl}(B) & & \\
 & & \downarrow & & \downarrow{\scriptstyle \Phi_{A,B}} & & & & \\
 & & W(A,B) & \longrightarrow & \mathrm{Cl}(A) & \longrightarrow & Y(A,B) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where $\Phi_{A,B}$ is the homomorphism induced by $\Phi^*_{A,B}$ and the groups $W(A,B)$ and $Y(A,B)$ are naturally defined.

Definition 1.1. A generic dominant Krull pair (A,B) has the *(MDP)*, if the Magid diagram induces the following diagram with exact rows

$$\begin{array}{ccccc}
E(A,B) & \xrightarrow{\mathrm{can.}} & \mathrm{Cl}(B) & \longrightarrow & 0 \\
{\scriptstyle =}\downarrow & & {\scriptstyle \cong}\downarrow & & \\
E(A,B) & \xrightarrow{\Phi_{A,B}} & \mathrm{Cl}(A) & \longrightarrow & 0
\end{array}.$$

On the other hand, define $\phi_{B,A} : \mathrm{Div}(B) \longrightarrow \mathrm{Div}(A)$ by

$$\phi_{B,A}(D) = \sum_{\mathfrak{q}\in\mathrm{Ht}_1(A)} \left(\max_{\mathfrak{P}\in X_{\mathfrak{q}}(B)} \left(-\left[-\frac{a_{\mathfrak{P}}}{\mathrm{e}(\mathfrak{P},\mathfrak{q})}\right]\right)\right)\cdot \mathrm{div}_A(\mathfrak{q}) \in \mathrm{Div}(A)$$

where $D = \sum_{\mathfrak{P}\in\mathrm{Ht}_1(B)} a_{\mathfrak{P}}\cdot \mathrm{div}_B(\mathfrak{P}) \in \mathrm{Div}(B)$ and $[\,\cdot\,]$ denotes the Gauss symbol.

Put $\mathrm{BU}(A,B) := \bigoplus_{\mathrm{ht}(\mathfrak{P}\cap A)\geqq 2} Z\cdot\mathrm{div}_B(\mathfrak{P})$ and $\mathrm{Div}(A,B) := \bigoplus_{\mathfrak{P}\cap A\neq\{0\}} Z\cdot \mathrm{div}_B(\mathfrak{P})$.

Definition 1.2. The following conditions are considered for the pair (A, B):

(1) $\mathrm{BU}(A,B) \subseteq \ker(E^*(A,B) \xrightarrow{\text{can.}} \mathrm{Cl}(B))$

(2) For $\mathrm{Div}(A,B) \ni D_0 \geqq 0$ s.t.

$$\mathrm{supp}_B(D_0) := \{\mathfrak{P} \mid v_{B,\mathfrak{P}}(I_B(D_0)) \neq 0\} \not\ni \mathfrak{P}: \forall \text{ principal prime}$$

and for $E^*(A,B) \ni D \geqq 0$ s.t. $D + D_0$: principal, we require that $\phi_{B,A}(D + D_0)$ is principal.

(3) The canonical morphism $E^*(A,B) \longrightarrow \mathrm{Cl}(B)$ is surjective.

We say that a generic dominant Krull pair (A, B) has the *(TDP)*, if these three conditions hold for (A, B).

In the next section we summarize our results on the (MDP) and (TDP) for generic dominant Krull pairs. In Sect. 3 we apply the results to the case where A is obtained as a subring of invariants in B under the action of an algebraic torus and characterize freeness of monomials of prime relative invariants. In Sect. 4 we study on the relation of certain class groups and freeness of a class of modules of relative invariants. Consequently we see a numerical criterion of obstructions of an algebraic torus of equidimensional actions. The detailed account of this part can be found in [8].

2. Descent property in abstract case

At first we point out the elementary relation between (MDP) and (TDP) in a general situation.

Proposition 2.1. For a generic dominant Krull pair (A, B), the (TDP) holds if and only if the (MDP) and (2) of Definition 1.2 hold.

Then we must have the following criterion that (TDP) holds for (A, B) which is useful in invariant theory of algebraic tori.

Theorem 2.2. For a generic dominant Krull pair (A, B), the following conditions (i) and (ii) are equivalent:

(i) (A, B) has the (TDP).

(ii) The following three conditions hold:

(a) (A, B) has the (MDP).

(b) $|\{\mathfrak{P} \in X_{\mathfrak{q}}(B) \mid \mathrm{div}_B(\mathfrak{P}) \in \mathrm{Prin}(B)\}| \geqq |X_{\mathfrak{q}}(B)| - 1$ for any $\mathfrak{q} \in \mathrm{Ht}_1(A)$.

(c) $\mathfrak{P} \in X_{\mathfrak{q}}(B)$ s.t. $e(\mathfrak{P}, \mathfrak{q}) > 1 \Longrightarrow \mathrm{div}_B(\mathfrak{P}) \in \mathrm{Prin}(R)$, for any $\mathfrak{q} \in \mathrm{Ht}_1(A)$.

The next result is another version of Theorem 2.2 which is useful in showing the ladder type induction of descents of class groups of a sequence of generic dominant Krull pairs.

Theorem 2.3. The following conditions (i), (ii) are equivalent:

(i) (A, B) has the (TDP).

(ii) The following four conditions hold:

(a) $E^*(A,B) \longrightarrow \mathrm{Cl}(B)$ is surjective.

(b) $\mathrm{BU}(A,B) \subseteq \ker(E^*(A,B) \longrightarrow \mathrm{Cl}(B))$.

(c) $|\{\mathfrak{P} \in X_{\mathfrak{q}}(B) \mid \mathrm{div}_B(\mathfrak{P}) \in \mathrm{Prin}(B)\}| \geqq |X_{\mathfrak{q}}(B)| - 1$ for any $\mathfrak{q} \in \mathrm{Ht}_1(A)$.

(d) $\mathfrak{P} \in X_{\mathfrak{q}}(B)$ s.t. $e(\mathfrak{P}, \mathfrak{q}) > 1 \Longrightarrow \mathrm{div}_B(\mathfrak{P}) \in \mathrm{Prin}(R)$, for any $\mathfrak{q} \in \mathrm{Ht}_1(A)$.

We now consider an intermediate subring M of the extension B/A of rings as follows.

Notation 2.4. Let M be a subring of B containing A as a subring such that $M = \mathcal{Q}(M) \cap B$. Then there exist the Krull pairs as follows; i.e., (A, B), (M, B) and (A, M).

From now on to the end of this section, we use Notation 2.4 and describe how the descent properties of (A, M) and (M, B) induce one of (A, B).

Proposition 2.5. Suppose $\mathrm{Ht}_1(A, B) \subseteq \mathrm{Ht}_1(M, B)$. Then

$$\phi_{B,M}(E^{\bullet}(A, B)) \subseteq E^{\bullet}(A, M)$$

and the following diagram is commutative:

$$\begin{array}{ccc} E^{\bullet}(A,B) & \xrightarrow{\phi_{B,M}} & E^{\bullet}(A,M) \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle \Phi^{\bullet}_{A,M}} \\ E^{\bullet}(A,B) & \xrightarrow{\Phi^{\bullet}_{A,B}} & \mathrm{Div}(A) \end{array} .$$

This proposition is only a technical assertion, however from this we deduce the next two propositions.

Proposition 2.6. Suppose that $\mathrm{Ht}_1(A, B) \subseteq \mathrm{Ht}_1(M, B)$. If (A, M) and (M, B) have the (MDP), then the canonical morphism $E^{\bullet}(A, B) \longrightarrow \mathrm{Cl}(B)$ is surjective.

Proposition 2.7. Suppose that $\mathrm{Ht}_1(A, B) \subseteq \mathrm{Ht}_1(M, B)$. If (A, M) has the (MDP) and (M, B) has the (TDP), then (A, B) has the (MDP).

Consequently we must have the following theorem which gives an inductive examination on the descent properties of a sequence of generic dominant Krull pairs. In fact consider a descending chain of normal series of subgroups

$$G = G_0 \rhd G_1 \rhd G_2 \rhd \cdots \rhd G_n = \{e\}$$

and a homomorphism $G \to \mathrm{Aut}(B)$. We have a chain of generic dominant Krull pairs $(B^{G_{n-1}}, B^{G_n})$, $(B^{G_{n-2}}, B^{G_{n-1}}), \ldots, (B^{G_0}, B^{G_1})$ and the study on the descent property of (B^G, B) can be reduced to the one on the sequence.

Theorem 2.8. Suppose that $\mathrm{Ht}_1(A, B) \subseteq \mathrm{Ht}_1(M, B)$. If both (A, M) and (M, B) have the (TDP), then (A, B) has the (TDP).

We have studied on the implication concerning the descent property of $(A, B) \Longrightarrow$ ones of (A, M) and (M, B) under some conditions which is the converse of the assertions in the results as above, however we omit to state the results in this paper.

3. Free monomials of prime semi-invariants and descent property

Let R be a Krull domain on which a group G acts as automorphisms and let $Z^1(G, \mathrm{U}(R))$ denote the (additive) group of 1-cocycles of G on $\mathrm{U}(R)$. For any $\chi \in Z^1(G, \mathrm{U}(R))$, put

$$R_\chi := \{a \in R \mid \sigma(a) = \chi(\sigma) \cdot a\}$$

whose elements are known as invariants of G in R relative to χ and is regarded as an R^G-module.

Since (R^G, R) is a generic Krull pair, we immediately have its Magid diagram with P. Samuel's diagram (cf. [2]) in the Galois descent method

$$
\begin{array}{ccccccccc}
 & & \mathrm{Prin}(R)^G & \longrightarrow & \mathrm{Div}(R)^G & \longrightarrow & \mathrm{Cl}(R) & & \\
 & & \uparrow & & \uparrow & & \uparrow \, = & & \\
 & & \mathrm{Prin}(R) \cap E^{*}(R^G, R) & \longrightarrow & E^{*}(R^G, R) & \longrightarrow & \mathrm{Cl}(R) & & \\
 & & \downarrow & & \downarrow & & = \, \downarrow & & \\
0 & \longrightarrow & F(R^G, R) & \longrightarrow & E(R^G, R) & \longrightarrow & \mathrm{Cl}(R) & & \\
 & & \downarrow & & \Phi_{R^G,R} \downarrow & & & & \\
 & & W(R^G, R) & \longrightarrow & \mathrm{Cl}(R^G) & \longrightarrow & Y(R^G, R) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

where $F(R^G, R)$ can be regarded as a subgroup of the first cohomology group $H^1(G, \mathrm{U}(R))$.

In this section we apply the results in Sect. 2 to the generic Krull pair induced by the action of an algebraic torus defined over an algebraically closed field K of characteristic zero. Let $\mathfrak{X}(H)$ be the rational character (additive) group of an algebraic group H.

Notation 3.1. Let G be an affine algebraic group over K whose identity component G^0 is an algebraic torus and let (X, G) be a *faithful* regular action of G on an affine normal variety X over K. Put $R := \mathcal{O}(X)$ on which G acts naturally.

Recall that (X, G) is said to be stable, if X contains a non-empty open set consisting of closed G-orbits.

Definition 3.2. For $\{f_1, \ldots, f_n\} \subseteq R$ such that f_i are prime in R; the set $\{f_1, \cdots, f_n\}$ is defined to be (R, G)-*free*, if there exist rational characters $\chi_k \in \mathfrak{X}(G)$ $(1 \leq k \leq n)$ such that

$$R_{\sum_{k=1}^{n} i_k \cdot \chi_k} = R^G \cdot \prod_{k=1}^{n} f_k^{i_k} \quad (\forall i_k \in Z_0)$$

where Z_0 denote the additive monoid of all nonnegative integers.

As in the statement preceding to the ladder property in Sect. 2, from Theorem 2.2, Theorem 2.3 and Theorem 2.8 we deduce the following characterization of (R, G)-freeness of prime relative invariants on X in the sense of the descent property defined in this paper:

Theorem 3.3. Under the circumstances as in Notation 3.1, suppose that $Z_G(G^0) = G$, H is a closed normal subgroup such that the induced action $(X/\!/H, G/H)$ is stable. Suppose that one can choose prime semi-invariants f_i $(1 \leqq i \leqq n)$ of G on R in such a way that $H = \cap_{i=1}^{n} G_{f_i}$. If $\mathrm{rank}(G/H) = n$, then the following conditions are equivalent:

(i) The generic Krull pair (R^G, R^H) has the (TDP).

(ii) There exists a finite normal subgroup N of G generated by a part of the union of inertia groups at principal ideals in $\mathrm{Ht}_1(R^G, R)$ under the action of G such that there exists an (R, G)-free prime set $\{g_1, \ldots, g_n\}$ contained in R^N satisfying $HN = \cap_{i=1}^{n} G_{g_i}$.

Remark 3.4. The equivalence in Theorem 3.3 does not hold without assumption that $\{f_1, \dots, f_n\}$ consists of prime elements. There are counter-examples for a set $\{f_1, \dots, f_n\}$ containing a non-prime element. One might generalize this in the case where f_i's may not be prime, although the conditions should be complicated.

4. Subgroups of class groups and modules of relative invariants

We now return to the general case where R is a Krull domain acted by a group G as automorphisms which is treated in the former half in Sect. 3 and introduce some subsets of the group of the 1-cocyles of G. From now on to the end of Proposition 4.12, without specifying we suppose that the equality $Q(R^G) = Q(R)^G$ holds.

Definition 4.1. Put $Z^1(G, \mathrm{U}(R))^R := \{\chi \in Z^1(G, \mathrm{U}(R)) \mid R_\chi \neq \{0\}\}$ and

$$Z^1_R(G, \mathrm{U}(R))_e := \{\chi \in Z^1(G, \mathrm{U}(R))^R \mid \exists f_{\mathfrak{P}} \in R_\chi \setminus \{0\} \text{ such that } \mathrm{v}_{R,\mathfrak{P}}(f_{\mathfrak{P}}) \equiv 0 \bmod (\mathrm{e}(\mathfrak{P}, \mathfrak{P} \cap \mathfrak{q}))\ (\forall \mathfrak{P} \in \mathrm{Ht}_1(R^G, R))\}.$$

Let $Z^1_R(G, \mathrm{U}(R))_{(2)}$ denote the set of all $\chi \in Z^1(G, \mathrm{U}(R))$ such that $\{0\} \neq R_{-\chi} \not\subseteq \mathfrak{P}$ for all $\mathfrak{P} \in \mathrm{Ht}_1(R)$ satisfying $\mathrm{ht}(\mathfrak{P} \cap R^G) \geqq 2$ and put

$$\widetilde{Z^1_R}(G, \mathrm{U}(R)) := Z^1_R(G, \mathrm{U}(R))_{(2)} \cap (-Z^1_R(G, \mathrm{U}(R))_{(2)}).$$

Definition 4.2. An effective divisor $D \in \mathrm{Div}(R)$ is said to be *minimal effective relative to* (R^G, R), if D has a decomposition $D = D_1 + D_2$ for $0 \leqq D_1 \in E^*(R^G, R)$ and $0 \leqq D_2 \in \mathrm{Div}(R)$, then the divisor D_1 must be equal to zero.

With each $\chi \in Z^1_R(G, \mathrm{U}(R))^R$ we can associate the divisor $D(\chi)$ minimal effective relative to (R^G, R) as follows:

Lemma 4.3. Let χ be a cocycle in $Z^1_R(G, \mathrm{U}(R))^R$. Then:

(i) There exists a unique minimal effective divisor $D(\chi)$ on R relative to (R^G, R) such that, for a nonzero element $f \in R_\chi$,

$$E^*(R^G, R) \ni \mathrm{div}_R(f) - D(\chi) \geqq 0.$$

Moreover $D(\chi)$ does not depend on the choice of a nonzero element $f \in R_\chi$.

(ii) If $\chi \in Z^1_R(G, \mathrm{U}(R))_e$, then the divisor $D(\chi)$ and $D(m\chi)$ defined in (i) for χ and $m\chi$ satisfy $m \cdot D(\chi) = D(m\chi)$ in $\mathrm{Div}(R)$ for any $m \in \boldsymbol{N}$.

The next criterion for the individual R^G-module R_χ to be R^G-free can be easily shown in [7].

Proposition 4.4 ([7]). Without the assumption that $Q(R^G) = Q(R)^G$, for any cocycle $\chi \in Z^1(G, \mathrm{U}(R))^R$, R_χ is R^G-free of rank one if and only if the following conditions are satisfied:

(i) $\dim Q(R^G) \otimes_{R^G} R_\chi = 1$.

(ii) There exists a nonzero element $f \in R_\chi$ satisfying

$$\forall \mathfrak{q} \in \mathrm{Ht}_1(R^G) \Rightarrow \exists \mathfrak{P} \in X_Q(R) \text{ such that } \mathrm{v}_{R,\mathfrak{P}}(f) < \mathrm{e}(\mathfrak{P}, \mathfrak{q}) \tag{4.1}$$

If these equivalent conditions are satisfied, $R_\chi = R^G \cdot f$ for any nonzero element $f \in R_\chi$ such that (4.1) holds.

We apply Proposition 4.4 to some restricted χ and obtain the corollary which shall be needed.

Corollary 4.5. Let χ be a cocycle in $Z^1(G, \mathrm{U}(R))^R$. Then R_χ is R^G-free if and only if $D(\chi) + \mathrm{BU}(R^G, R) \ni \mathrm{div}_R(f)$ for some nonzero $f \in R_\chi$. In the case where $\chi \in (-Z^1_R(G, \mathrm{U}(R))_{(2)})$, $R_\chi \cong R^G$ as R^G-modules if and only if $D(\chi) = \mathrm{div}_R(f)$ for some nonzero $f \in R_\chi$.

Moreover the equality $R_\chi = R^G \cdot f$ holds, in both the cases where these equivalent conditions are satisfied.

By the choice of χ, Lemma 4.3 and Corollary 4.5, we see

Proposition 4.6. Let $\chi \in Z^1_R(G, \mathrm{U}(R))_e \cap (-Z^1_R(G, \mathrm{U}(R))_{(2)})$. Suppose that there exists a nonzero element $g \in R_\chi$ satisfying the condition as follows; for any $l \in N$ and G-invariant principal ideal $R \cdot h$ in R containing g^l such that $\mathrm{div}_R(h) \in E^*(R^G, R)$,

$$\exists n \in N \text{ such that } (h^n \cdot \mathrm{U}(R)) \cap R^G \neq \emptyset \Rightarrow (h \cdot \mathrm{U}(R)) \cap R^G \neq \emptyset.$$

Then the following conditions are equivalent:

(i) $D(\chi)$ is a principal divisor and there exists a number $m \in N$ such that $R_{m\chi} \cong R^G$ as R^G-modules.

(ii) For any $m \in N$, $R_{m\chi} \cong R^G$ as R^G-modules.

(iii) $R_\chi \cong R^G$ as R^G-modules.

Corollary 4.7. Under the same circumstances as in Proposition 4.6, suppose that there is a number $m \in N$ satisfying $R_{m\chi} \cong R^G$ as R^G-modules. Then the divisor class $[D(\chi)]$ in $\mathrm{Cl}(R)$ has a finite order and the following equality holds;

$$\mathrm{ord}([D(\chi)]) = \min\{q \in N \mid R_{q\chi} \cong R^G \text{ as } R^G\text{-modules}\}.$$

Proof. Since $p \cdot [D(\chi)] = [D(p\chi)]$ in $\mathrm{Cl}(R)$ for any $p \in N$ as in Proposition 4.6 and $n \cdot [D(\chi)] = 0$ (cf. Corollary 4.5), the former assertion is obvious and the latter one follows from Proposition 4.6. □

Definition 4.8. For $\chi \in Z^1(G, \mathrm{U}(R))^R$, the R^G-module R_χ is R^G-isomorphic to a nonzero integral ideal I of R^G and the divisor class of the divisorialization $\bar{I}$ in $\mathrm{Cl}(R^G)$ is denoted to $[R_\chi]$.

Proposition 4.9. Let χ be a cocycle in $Z^1_R(G, \mathrm{U}(R))_e \cap Z^1_R(G, \mathrm{U}(R))_{(2)}$. Then, for a number $n \in N$, $R_{n\chi} \cong R^G$ as R^G-modules if and only if $n \cdot [R_\chi] = 0$ in $\mathrm{Cl}(R^G)$.

Combining Corollary 4.7 with Proposition 4.9, we immediately have

Theorem 4.10. Let χ be a cocycle in $Z^1_R(G, \mathrm{U}(R))_e \cap \widetilde{Z^1_R}(G, \mathrm{U}(R))$ and suppose that there exists a nonzero element $g \in R_\chi$ satisfying the condition as follows; for any $l \in N$ and G-invariant principal ideal $R \cdot h$ in R containing g^l such that $\mathrm{div}_R(h) \in E^*(G, R)$,

$$\exists n \in N \text{ such that } (h^n \cdot \mathrm{U}(R)) \cap R^G \neq \emptyset \Rightarrow (h \cdot \mathrm{U}(R)) \cap R^G \neq \emptyset.$$

If $[R_\chi] \in \mathrm{tor}(\mathrm{Cl}(R^G))$, then

$$\mathrm{ord}([R_\chi]) \text{ in } \mathrm{Cl}(R^G) = \mathrm{ord}([D(\chi)]) \text{ in } \mathrm{Cl}(R),$$

which is equal to $\min\{q \in N \mid R_{q\chi} \cong R^G \text{ as } R^G\text{-modules}\}$. □

Definition 4.11. Let $\mathrm{UrCl}(R,G)$ denote the subgroup of $\mathrm{CL}(R)$ generated by

$$\{[D(\chi)] \mid \chi \in Z^1_R(G,\mathrm{U}(R))_e \cap \widetilde{Z^1_R}(G,\mathrm{U}(R))\},$$

where $[D(\chi)]$ denotes the divisor class of $D(\chi) \in \mathrm{Div}(R)$. Define $\widetilde{\mathrm{Cl}}(R,G)$ to be the subgroup $\left\langle \{[R_\chi] \mid \chi \in Z^1_R(G,\mathrm{U}(R))_e \cap \widetilde{Z^1_R}(G,\mathrm{U}(R))\} \right\rangle$ of $\mathrm{Cl}(R^G)$.

The next result follows easily from Theorem 4.10.

Proposition 4.12. Suppose that the canonical image of the semigroup $Z^1_R(G,\mathrm{U}(R))_e \cap (-(Z^1_R(G,\mathrm{U}(R))_{(2)})$ in $H^1(G,\mathrm{U}(R))$ does not contain a non-trivial torsion element. Suppose that $\widetilde{\mathrm{Cl}}(R,G)$ is a torsion group. If one of $\exp(\mathrm{UrCl}(R,G))$ and $\exp(\widetilde{\mathrm{Cl}}(R,G))$ is finite, then

$$\exp(\mathrm{UrCl}(R,G)) = \exp(\widetilde{\mathrm{Cl}}(R,G)),$$

which are equal to

$$\max\left\{\min\{q \in N \mid R_{q\chi} \cong R^G\} \mid \chi \in Z^1_R(G,\mathrm{U}(R))_e \cap \widetilde{Z^1_R}(G,\mathrm{U}(R))\right\}. \quad \square$$

Hereafter let (X,G) be a regular faithful stable action of an algebraic torus G on an affine normal variety X defined over an algebraically closed field K of characteristic zero whose coordinate ring $\mathcal{O}(X)$ denoted to R. We have the canonical pairing $G \times \mathfrak{X}(G) \to \mathrm{U}(K)$.

Definition 4.13. Let $\widetilde{\mathfrak{R}}(R,G)$ be the subgroup of G generated by the set consisting of $\mathcal{I}_{\mathfrak{P}}(G)$'s for all $\mathfrak{P} \in \mathrm{Ht}_1(R^G,R)$ such that $\mathfrak{P}$ are not principal which is called the *maximal non-principal pseudo-reflection subgroups of the action* (X,G). Here $\mathcal{I}_{\mathfrak{P}}(G)$ stands for the inertia group of $\mathfrak{P}$ under the action of G. Put $\mathfrak{R}(R,G) := \langle \cup_{\mathfrak{P}\in \mathrm{Ht}_1(R^G,R)} \mathcal{I}_{\mathfrak{P}}(G) \rangle$. Clearly both $\mathfrak{R}(R,G)$ and $\widetilde{\mathfrak{R}}(R,G)$ are finite (normal) subgroups of G. In the case where $\exp(\mathrm{UrCl}(R,G))$ is finite, define

$$\mathrm{Obs}(R,G) := \{\sigma \in G \mid \sigma^{\exp(\mathrm{UrCl}(R,G))} \in \widetilde{\mathfrak{R}}(R,G)\},$$

which is called the *obstruction subgroup for cofreeness of* (X,G).

Lemma 4.14. We have $\mathfrak{X}(G)^{\perp \mathfrak{R}(R,G)} = Z^1_R(G,\mathrm{U}(R))_e$.

With the aid of [10], the following proposition is shown in [6].

Proposition 4.15. Suppose that both X and (X,G) are conical. If the action (X,G) is equidimensional, then $\widetilde{\mathrm{Cl}}(R,G)$ is a torsion group.

Applying Theorem 4.10 to this, we must have

Theorem 4.16. Suppose that both X and (X,G) are conical. Then the following conditions are equivalent:

(i) The action (X,G) is equidimensional.
(ii) The exponent $\exp(\mathrm{UrCl}(R,G))$ is finite and the action $(X/\!/\mathrm{Obs}(R,G), G/\mathrm{Obs}(R,G))$ induced naturally is cofree.

Especially if R is factorial, the obstruction subgroup $\mathrm{Obs}(R, G)$ should be a trivial group by its definition. It is not hard to formally generalize Theorem 4.16 to in the case where (X, G) may not be stable. For linear representations of connected algebraic groups with affine rings of invariants, V. G. Kac and V. L. Popov have conjectured that equidimensionality of these actions implies cofreeness, which is known as the Russian conjecture (cf. [3, 9]) and is partially related to this theorem.

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THE MODULI SPACES OF NON-THICK IRREDUCIBLE REPRESENTATIONS FOR THE FREE GROUP OF RANK 2

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ABSTRACT. There are several types among irreducible representations. Considering such types as "thick" and "dense" gives us rich problems on representation theories, and the first step to describe the moduli spaces of representations. The moduli of irreducible representations for the free group F_2 is very big, and difficult to be investigated. However, we can describe some parts of the moduli of irreducible representations by using the notion of thickness. In this talk, we describe the moduli of 4-dimensional non-thick irreducible representations for the free group of rank 2.

1. INTRODUCTION

群の既約表現の中でも，さらにいくつかのクラスの表現に分類され，それが表現論的な興味深い問題提供と，表現のモジュライを記述する際の足がかりを与えてくれる。自由群 F_2 の表現は，2つの生成元の行き先である行列2個を指定すれば作ることが出来るのであるが，そのため自由群 F_2 の既約表現は多くありすぎて，のっぺらぼうのような感じで，どう調べてよいのかわからない状況であった。だが，thick や dense という新しい概念により，既約表現はいくつかの "層" に分離されることがわかる。次の節から具体的な定義を説明するが，ここではどうしてわれわれがこのような概念を考え，現時点でどのようなことを考えているのかを、"安直" に "ざっくばらん" に日本語で述べておきたいと思う。

thick という概念は，本来 $\mathrm{SL}(2,\mathbb{Z})$ という離散群の既約表現を調べる際に考え付いたものである。既約表現が持つべき性質であろうと思っていたものが，実はいつも成立するわけではないと気づき，thick という名前をつけ，いつ既約表現が thick であるかを調べることとなった。thick という名前の由来は，群の表現 $\rho: G \to \mathrm{GL}(V)$ の像 $\rho(G)$ が $\mathrm{GL}(V)$ の中で，既約表現よりも多く元を持ち，"thick である" というイメージから来る。thick よりも強い概念である dense も同様で，$\mathrm{GL}(V)$ の中での像 $\rho(G)$ が非常に多くあるというイメージと既約表現が持つべき期待できる "最大限の性質" という理由から名前を付けた。既約表現なら当然 thick と呼ばれる性質を持つに違いないと当初思っていたが，実はそうではなく，irreducible, thick, dense の間にギャップがあることに気づいたのが，本講演の出発点になる。

自由群の（既約）表現のモジュライがどうなっているか，そして関連して行列の不変式環がどういう構造をしているかを調べるのが長年付き合ってきた研究テーマなのであるが，その原点ともいうべき結果が次である。

Theorem 1 (Teranishi [4]). 不変式環 $\mathbb{Q}[\mathrm{M}_4 \times \mathrm{M}_4]^{\mathrm{PGL}_4}$ の *Poincaré* 級数 $P_{4,2}(s,t)$ は次で与えられる。

The detailed version of this paper will be submitted for publication elsewhere.

$$P_{4,2}(s,t) := \sum_{i,j\geq 0} \dim \mathbb{Q}[\mathrm{M}_4 \times \mathrm{M}_4]^{\mathrm{PGL}_4}_{(i,j)} \cdot s^i t^j = \frac{R_{4,2}(s,t)}{Q_{4,2}(s,t)},$$

ここで

$$\begin{aligned}Q_{4,2}(s,t) = (1-s)(1-s^2)(1-s^3)(1-s^4)(1-t)(1-t^2)(1-t^3)(1-t^4)\\ \times (1-st)(1-s^2t^2)^2(1-st^2)(1-s^2t)\\ \times (1-st^3)(1-s^3t)(1-s^2t^4)(1-s^4t^2)\end{aligned}$$

$$\begin{aligned}R_{4,2}(s,t) = 1 + s^2t^3 + s^3t^2 + 2s^3t^3 + s^3t^4 + s^4t^3 + 2s^4t^4 + s^3t^5 + s^5t^3\\ + s^3t^6 + s^4t^5 + s^5t^4 + s^6t^3 + 2s^5t^5 + s^4t^6 + s^6t^4 + 2s^5t^6\\ + 2s^6t^5 + 2s^6t^6 + 2s^6t^7 + 2s^7t^6 + 2s^7t^7 + s^6t^8 + s^8t^6\\ + s^6t^9 + s^7t^8 + s^8t^7 + s^9t^6 + 2s^8t^8 + s^7t^9 + s^9t^7 + s^8t^9\\ + s^9t^8 + 2s^9t^9 + s^9t^{10} + s^{10}t^9 + s^{12}t^{12}.\end{aligned}$$

4×4 行列 2 個の不変式環ですら，その構造はわかる気がしない。非常に複雑であるという理由からどう良い問題を取り出していいのかわからない，といった難しさもあろう。だけど，なんとかしてよい問題を切り出して，面白い構造を取り出そうというのは一つの行動原理として，認められることであろう。

その良い問題を切り出す一つのきっかけとして，thick, dense といった概念を使わない手はない。自由群の既約表現のモジュライの中に，thick 表現から成る開集合や dense 表現から成る開集合がある。既約表現のモジュライがどうなっているかを調べたいのであるが，まずは薄い皮を剥がすが如く，non-thick な既約表現がどうなっているかを調べようというのが本講演の主たるアイデアである。実際に調べてみると確かに薄い皮なのだが，非常に面白いのである。ただ，現時点では薄い皮しか調べていないので，Theorem 1 が成り立つ背景は当然わからないままである。

この後，non-dense な既約表現がどうなっているのか，最後に dense 表現がどうなっているのか，と続くのであるが，それはまたの機会にしよう。また，具体的な群の既約表現が thick であるかどうかすらよくわかっていない。いっぱいやることがある。

2. m-THICK AND m-DENSE

以下，k を体，V を k 上の n 次元ベクトル空間，G を群とする。また，$\rho : G \to \mathrm{GL}(V)$ を G の表現とする。

Definition 2. $\rho : G \to \mathrm{GL}(V)$ が m-thick であるとは，$\dim V_1 = m$ なる V の任意の部分ベクトル空間 V_1 と，$\dim V_2 = n - m$ なる V の任意の部分ベクトル空間 V_2 に対して，ある $g \in G$ が存在して，$(\rho(g)V_1) \oplus V_2 = V$ が成り立つときをいう。また，ρ が thick であるとは，$0 < m < n$ なる任意の整数 m に対して，ρ が m-thick であるときをいう。

Definition 3. $\rho : G \to \mathrm{GL}(V)$ が m-dense であるとは，ρ から誘導される G の外積表現 $\wedge^m \rho : G \to \mathrm{GL}(\Lambda^m V)$ が既約であるときをいう。また，ρ が dense であるとは，$0 < m < n$ なる任意の整数 m に対して，ρ が m-dense であるときをいう。

Definition 4. 表現 $\rho: G \to \mathrm{GL}(V)$ に対して，G 同変 perfect pairing

$$\begin{array}{ccc} \Lambda^m V \times \Lambda^{n-m} V & \to & \Lambda^n V \cong k \\ (x,y) & \mapsto & x \wedge y \end{array}$$

を考える。部分空間 $W \subseteq \Lambda^m V$ に対して，$W^\perp \subseteq \Lambda^{n-m} V$ を

$$W^\perp := \{y \in \Lambda^{n-m} V \mid x \wedge y = 0 \text{ for } \forall x \in W\}$$

と定める。W が G 不変部分空間なら，$W^\perp$ も G 不変部分空間となる。

Proposition 5. $\rho: G \to \mathrm{GL}(V)$ について，*m-thick* であることと *$(n-m)$-thick* であることは同値である。また，*m-dense* であることと，*$(n-m)$-dense* であることは同値である。

Proof. m-thick と $(n-m)$-thick が同値であることは，定義より明らか。また，Proposition 4より，$\Lambda^m V$ が既約であることと，$\Lambda^{n-m} V$ が既約であることは同値である。このことから，m-dense と $(n-m)$-dense は同値であることがわかる。 □

Proposition 6.

$$m\text{-}dense \Rightarrow m\text{-}thick \Rightarrow 1\text{-}dense \Leftrightarrow 1\text{-}thick \Leftrightarrow irreducible$$

Proof. まず「m-dense $\Rightarrow$ m-thick」を示す。V_1, V_2 をそれぞれ V の m 次元，$(n-m)$ 次元部分空間とする。$V_1 = \langle e_1, e_2, \ldots, e_m \rangle$, $V_2 = \langle f_1, f_2, \ldots, f_{n-m} \rangle$ とおく。ρ が m-dense であれば，$\{\wedge^m \rho(g)(e_1 \wedge e_2 \wedge \cdots \wedge e_m) \mid g \in G\}$ が $\Lambda^m V$ を span するので，$\wedge^m \rho(g)(e_1 \wedge e_2 \wedge \cdots \wedge e_m) \wedge f_1 \wedge \cdots \wedge f_{n-m} \neq 0$ となる $g \in G$ がとれる。これは，$(\rho(g)V_1) \oplus V_2 = V$ を意味し，m-thick であることがいえた。また,「1-dense $\Rightarrow$ 1-thick」も言えたことになる。

「irreducible $\Rightarrow$ 1-dense」であることは定義より明らか。後は「m-thick $\Rightarrow$ irreducible」を示せばよい。ρ が既約でないと仮定すると，V の自明でない G 不変部分空間 V' が存在する。$\ell := \dim V'$ とせよ。もし $\ell \leq \min(m, n-m)$ なら，$V' \subseteq V_1$, $V' \subseteq V_2$ となるように，V の m 次元部分空間 V_1 と $(n-m)$ 次元部分空間 V_2 を適当にとれば，$\rho(g)V_1$ と V_2 は必ず V' を含むので，$(\rho(g)V_1) \oplus V_2 = V$ となり得ない。このときは，m-thick でないことがわかる。よって，$\ell > m$ または $\ell > n-m$ のときを考えればよい。対称性より $m \leq n-m$ としても一般性を失わない。$m < \ell \leq n-m$ のときは，$V_1 \subset V' \subseteq V_2$ となるよう V_1, V_2 をとれば，どんな $g \in G$ に対しても $(\rho(g)V_1) \oplus V_2 = V$ となり得ない。また，$m \leq n-m < \ell$ なら，$V_1 \subseteq V_2 \subset V'$ となるよう V_1, V_2 をとれば，$(\rho(g)V_1) + V_2 \subseteq V' \neq V$ となる。いずれにしても m-thick ではないので,「m-thick $\Rightarrow$ irreducible」がいえた。 □

上の Proposition から容易に次の Corollary がわかる。

Corollary 7.

$$dense \Rightarrow thick \Rightarrow irreducible$$

Corollary 8. $n \leq 3$ とする。このとき，

$$dense \Leftrightarrow thick \Leftrightarrow irreducible$$

上の Corollary から，dense, thick, irreducible の概念がずれるのが，4次元以上の表現からになることがわかる。そして，実際に dense, thick, irreducible の間にはギャップがある。

Example 9. G を有限群とする。G の既約表現のうち，最大の次数を与える表現 $\rho : G \to \mathrm{GL}(V)$ の次数 n が 4 以上であるとする。このとき，ρ は dense ではない。

事実，$\wedge^n \rho$ は n より大きな次数の表現を与えるが，仮定により既約とならない。この例は，既約であるが dense でない例を与える。

thick であるという条件は，非常にわかりずらい。これは，線型代数で捕らえきることが出来ない，むしろ外積代数やグラスマン多様体といった，線型を超えた (?) 範疇の条件だからだと思われる。しかしながら，ここでは便宜上の thick と等価な条件を提示しておく。

Definition 10. 部分ベクトル空間 $W \subseteq \Lambda^m V$ について，W が realizalble（実現可能）であるとは，V のある部分ベクトル空間 $V' = \langle e_1, e_2, \ldots, e_m \rangle$ が存在して，$e_1 \wedge e_2 \wedge \cdots \wedge e_m \in W$ となるときをいう。以後，$e_1 \wedge e_2 \wedge \cdots \wedge e_m \in W$ のことを $\Lambda^m V' \in W$ と書くことにする。

Proposition 11. ρ が "*m-thick* でない" ということと，ある G-不変 *realizable* 部分空間 $W_1 \subseteq \Lambda^m V$ とある G-不変 *realizable* 部分空間 $W_2 \subseteq \Lambda^{n-m} V$ が存在して，$W_1 \wedge W_2 := \{w_1 \wedge w_2 \in \Lambda^n V \mid w_1 \in W_1, w_2 \in W_2\} = 0$ が成り立つことは同値である。

Proof. 条件をみたす W_1, W_2 がとれたとしよう。W_1, W_2 は realizable なので，V の m 次元部分空間 V_1 および $(n-m)$ 次元部分空間 V_2 が存在して，$\Lambda^m V_1 \in W_1$, $\Lambda^{n-m} V_2 \in W_2$ となる。ところが，条件によりどんな $g \in G$ に対しても，$\rho(g)V_1 \oplus V_2 = V$ となり得ない。これは m-thick でないことを意味する。

逆に m-thick でないと仮定する。このとき，V の m 次元部分空間 V_1 および $(n-m)$ 次元部分空間 V_2 が存在して，どんな $g \in G$ に対しても，$\rho(g)V_1 \oplus V_2 = V$ となり得ない。$\{(\wedge^m \rho)(g)(\Lambda^m V_1) \mid g \in G\}$ で生成される $\Lambda^m V$ の部分空間を W_1，$\{(\wedge^{n-m} \rho)(g)(\Lambda^{n-m} V_2) \mid g \in G\}$ で生成される $\Lambda^{n-m} V$ の部分空間を W_2 とする。W_1, W_2 は G-不変 realizable 部分空間であり，$W_1 \wedge W_2 = 0$ をみたす。以上より，主張がいえた。 □

Remark 12. "dense でない" という条件を上の Proposition と対比させて表現すると，"thick でない" 条件と "dense でない" 条件が比較しやすくなる。

「ρ が "m-dense でない" ということと，ある G-不変部分空間 $W_1 \subseteq \Lambda^m V$ とある G-不変部分空間 $W_2 \subseteq \Lambda^{n-m} V$ が存在して，$W_1 \wedge W_2 := \{w_1 \wedge w_2 \in \Lambda^n V \mid w_1 \in W_1, w_2 \in W_2\} = 0$ が成り立つことは同値である。」

以下の Proposition は証明なしで紹介する。4 次元表現，5 次元表現が thick であるための必要十分条件を与える。

Proposition 13. k を代数的閉体とする。4 次元表現 $\rho : G \to \mathrm{GL}(V)$ に対して，次は同値である。

(1) ρ : *thick*
(2) ρ : *2-thick*
(3) ρ : *irreducible* かつ $\wedge^2 \rho : G \to \mathrm{GL}(\Lambda^2 V)$ が 2 次元ないし 3 次元 G-不変部分空間 $W \subset \Lambda^2 V$ を持たない。

Proposition 14. k を代数的閉体とする。5 次元表現 $\rho : G \to \mathrm{GL}(V)$ に対して，次は同値である。

(1) ρ : *thick*

(2) ρ : *2-thick*
(3) ρ : *irreducible* かつ $\wedge^2\rho : G \to \mathrm{GL}(\Lambda^2 V)$ が G-不変部分空間 $W \subset \Lambda^2 V$ で $4 \le \dim W \le 6$ なるものを持たない。

次は，thick ではあるが dense でない例である。

Example 15. $k = \mathbb{C}$, $G = \mathrm{GL}(2, \mathbb{C})$ とする。$V_{(a+b,b)}$ を G の highest weight $(a+b, b)$ なる既約表現とする。このとき，

$$\begin{aligned} a \ge 3 \quad &\Rightarrow \quad V_{(a+b,b)} \text{ is not dense} \\ a = 3, 4 \quad &\Rightarrow \quad V_{(a+b,b)} \text{ is thick} \end{aligned}$$

特に，

$$a = 3, 4 \quad \Rightarrow \quad V_{(a+b,b)} \text{ is not dense, but thick}$$

3. Moduli of representations

この節では，表現のモジュライ，とくに階数 2 の自由群 F_2 の 4 次元既約表現のモジュライについて言及する。以下では，便宜上 k は標数が 2 でない代数的閉体であると仮定する。

記号として，

$$\mathrm{Rep}_n(G)_{air} := \{\rho \mid \rho : n\text{-dimensional (absolutely) irreducible representation of } G\}$$

とおく。また，2 つの n 次元表現 ρ, ρ' が同値であるとは，ある $P \in \mathrm{GL}_n(k)$ が存在して，$P\rho P^{-1} = \rho'$ が成り立つときをいう。ρ を含む表現の同値類を $[\rho]$ と表す。このとき，

$$\mathrm{Ch}_n(G)_{air} := \{[\rho] \mid \rho : n\text{-dimensional (absolutely) irreducible representation of } G\}$$

とおく。

$\mathrm{Rep}_n(G)_{air}$ および $\mathrm{Ch}_n(G)_{air}$ は，体 k 上の scheme となる。平たく言うと（そして多少の語弊があることを承知で言うと），いくつかの代数方程式で定義される代数多様体になる。$\mathrm{Ch}_n(G)_{air}$ のことを，G の n 次元既約表現のモジュライと呼ぶ。既約表現のモジュライは，各点が既約表現の同値類に対応しており，それらの点が集まって一つの幾何学的対象になったものである。

群 $\mathrm{PGL}_n(k) := \mathrm{GL}_n(k)/k^*$ の $\mathrm{Rep}_n(G)_{air}$ への作用を $\rho \mapsto P\rho P^{-1}$ で定めると，$\mathrm{Ch}_n(G)_{air} = \mathrm{Rep}_n(G)_{air}/PGL_n$ である。

$F_2 = \langle \alpha, \beta \rangle$ を階数 2 の自由群とする。以下，$G = F_2$ として自由群 F_2 の 4 次元表現に限って表現のモジュライ $\mathrm{Ch}_4(F_2)_{air}$ を調べることにする。一般に自由群の既約表現は無数にあるので，このようにモジュライという幾何学的対象を使って，既約表現がどのくらいあるのかを視覚的にとらえようとするわけである。

Proposition 16. $\mathrm{Ch}_4(F_2)_{air}$ は 17 次元非特異代数多様体である。

Outline of Proof. 非特異多様体であることは認めて，なぜ 17 次元なのかをラフに説明する。F_2 の表現 ρ を与えることと，F_2 の生成元 α, β の行き先 $\rho(\alpha), \rho(\beta)$ を与えることは同値である。よって，F_2 の 4 次元表現を与えるためには，4 次正則行列を 2 個与えればよ

い。なお，表現のモジュライの中で既約表現であることはopenな条件であることに注意すると，既約であるという条件に目くじらを立てず次元を安直に計算出来て，

$$\dim \mathrm{Rep}_4(F_2)_{air} = \dim \mathrm{GL}_4(k) \times \mathrm{GL}_4(k) = 4^2 + 4^2 = 32$$

および

$$\dim \mathrm{PGL}_4(k) = \dim \mathrm{GL}_4(k) - \dim k^* = 4^2 - 1 = 15$$

より，

$$\dim \mathrm{Ch}_4(F_2)_{air} = \dim \mathrm{Rep}_4(F_2)_{air} - \dim \mathrm{PGL}_4(k) = 32 - 15 = 17$$

となる。 □

既約表現のモジュライの中で，既約表現がdenseである，もしくはthickであるという条件はopenな条件である。つまり，既約表現のモジュライのある点でdenseもしくはthickであれば，その近傍もdenseもしくはthickであることがいえる。そこで，つぎのような定義をしておく。

Definition 17.

$$\mathrm{Ch}_n(G)_{dense} := \{[\rho] \mid \rho : n\text{-dimensional (absolutely) dense representation of } G\}$$

$$\mathrm{Ch}_n(G)_{thick} := \{[\rho] \mid \rho : n\text{-dimensional (absolutely) thick representation of } G\}$$

とおく。このとき，

$$\mathrm{Ch}_n(G)_{air} \supseteq \mathrm{Ch}_n(G)_{thick} \supseteq \mathrm{Ch}_n(G)_{dense}$$

がいえる。

本講演の主題は，$\mathrm{Ch}_4(F_2)_{non-thick} := \mathrm{Ch}_4(F_2)_{air} \setminus \mathrm{Ch}_4(F_2)_{thick}$ である。non-thick 4次元既約表現のモジュライ $\mathrm{Ch}_4(F_2)_{non-thick}$ は，既約表現のモジュライ $\mathrm{Ch}_4(F_2)_{air}$ の中で閉集合を成すが，その形はどうなっているのかを調べるのが，今回の主題である。

$[\rho] \in \mathrm{Ch}_4(F_2)_{non-thick}$ を考える。$\wedge^2\rho : F_2 \to \mathrm{GL}(\Lambda^2 V)$ について，Proposition 13より，F_2 不変部分空間 $W \subseteq \Lambda^2 V$ として，$\dim W = 2$ または $\dim W = 3$ となるものがとれる。

まずは，$\dim W = 2$ のときを考える。次の命題は $\dim W = 2$ の場合の表現の正規化定理というべきものである。証明なしで紹介する。

Proposition 18. $\rho : G \to \mathrm{GL}(V)$ を4次元 *non-thick* 既約表現とする。G 不変部分空間 $W \subseteq \Lambda^2 V$ として，$\dim W = 2$ となるものがとれると仮定せよ。このとき，V の基底 e_1, e_2, e_3, e_4 が存在して，$W = \langle e_1 \wedge e_2, e_3 \wedge e_4 \rangle$ かつ任意の $g \in G$ に対して

$$\rho(g) = \begin{pmatrix} A_1 & 0_2 \\ 0_2 & A_2 \end{pmatrix} \ or \ \begin{pmatrix} 0_2 & A_1 \\ A_2 & 0_2 \end{pmatrix}$$

となるものがとれる。ここで，A_1, A_2 は 2×2 行列である。

Definition 19. 上の正規化定理において，

$$\rho(g)=\begin{pmatrix} A_1 & 0_2 \\ 0_2 & A_2 \end{pmatrix}$$

の形の行列を type $+$,

$$\rho(g)=\begin{pmatrix} 0_2 & A_1 \\ A_2 & 0_2 \end{pmatrix}$$

の形の行列を type $-$ と呼ぶことにする。

ここで，$\dim W=2$ の場合の $\mathrm{Ch}_4(F_2)_{non-thick}$ の既約成分を定義しよう。

Definition 20.

$$S_4(F_2)_{(+,-)}:=\{\rho\in\mathrm{Rep}_4(F_2)_{air}\mid\rho(\alpha):\text{type}+,\ \rho(\beta):\text{type}-\}$$

$$S_4(F_2)_{(-,+)}:=\{\rho\in\mathrm{Rep}_4(F_2)_{air}\mid\rho(\alpha):\text{type}-,\ \rho(\beta):\text{type}+\}$$

$$S_4(F_2)_{(-,-)}:=\{\rho\in\mathrm{Rep}_4(F_2)_{air}\mid\rho(\alpha):\text{type}-,\ \rho(\beta):\text{type}-\}$$

とおく。また，標準的な射 $\phi_{(+,-)}:S_4(F_2)_{(+,-)}\to\mathrm{Ch}_4(F_2)_{non-thick}$ を考え，同様に $\phi_{(-,+)}$, $\phi_{(-,-)}$ も考える。このとき，

$$\mathrm{Ch}(+,-):=\overline{\mathrm{Im}\phi_{(+,-)}}$$

$$\mathrm{Ch}(-,+):=\overline{\mathrm{Im}\phi_{(-,+)}}$$

$$\mathrm{Ch}(-,-):=\overline{\mathrm{Im}\phi_{(-,-)}}$$

と定義する。

次に $\dim W=3$ のときを考える。次の命題は $\dim W=3$ の場合の表現の正規化定理である。これも証明なしで紹介する。

Proposition 21. $\rho:G\to\mathrm{GL}(V)$ を 4 次元 *non-thick* 既約表現とする。G 不変部分空間 $W\subseteq\Lambda^2V$ として，$\dim W=2$ となるものは存在せず，$\dim W=3$ となるものがとれると仮定せよ。このとき，ある 2 つの 2 次元既約表現 ρ_1,ρ_2 が存在して，ρ と $\rho_1\otimes\rho_2$ は同値である。

$\dim W=3$ の場合に対応する既約成分を定義しておこう。

Definition 22. 有理射

$$\begin{array}{cccc}\phi_{\dim=3} & : & \mathrm{Ch}_2(F_2)_{air}\times\mathrm{Ch}_2(F_2)_{air} & \dashrightarrow & \mathrm{Ch}_4(F_2)_{non-thick}\\ & & (\rho_1,\rho_2) & \mapsto & \rho_1\otimes\rho_2\end{array}$$

の像の閉包を $\mathrm{Ch}(\dim=3)$ とおく。

これでようやく主定理を述べることができる。

Theorem 23. $\mathrm{Ch}_4(F_2)_{non-thick}$ の既約分解は

$$\mathrm{Ch}_4(F_2)_{non-thick} = \mathrm{Ch}(+,-) \cup \mathrm{Ch}(-,+) \cup \mathrm{Ch}(-,-) \cup \mathrm{Ch}(\dim = 3)$$

で与えられる。また，$\dim \mathrm{Ch}(+,-) = \dim \mathrm{Ch}(-,+) = \dim \mathrm{Ch}(-,-) = 9$ かつ $\mathrm{Ch}(+,-) \cap \mathrm{Ch}(-,+) \cap \mathrm{Ch}(-,-) \neq \emptyset$ である。

では，具体的に $\mathrm{Ch}_4(F_2)_{non-thick}$ の形はどうなっているのか。特異点はどうなっているのか。さらに non-dense 既約表現は？　問いは尽きないようで。続きは次回の講釈にて。

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IWASAWA ALGEBRAS, CROSSED PRODUCTS AND FILTERED RINGS

KENJI NISHIDA

ABSTRACT. We apply the theory of crossed product to Iwasawa algebra $\Lambda(G) = \Lambda(H) * (G/H)$. A J-adic filtration of $\Lambda(H)$ can be extended to that of $\Lambda(G)$. We study Gorenstein dimension of a graded module over $\Lambda(G)$.

1. Introduction: Iwasawa Algebras

Let p be a prime integer and $\mathbb{Z}_p$ denote the ring of p-adic integers. A topological group G is a compact p-adic analytic group if and only if G has an open normal uniform pro-p subgroup H of finite index [6]. The *Iwasawa algebra* of G is defined by

$$\Lambda(G) := \varprojlim \mathbb{Z}_p[G/N],$$

where N ranges over the open normal subgroups of G.

The ring theoretical survey of Iwasawa algebras is given by K. Ardakov and K.A. Brown [1]. In this paper, we address crossed products and filtered rings arising from Iwasawa algebras. Therefore, we direct our attension to the fact that a ring $\Lambda(G)$ is a crossed product of a finite group G/H over a ring $\Lambda(H)$ (Iwasawa algebra of H): $\Lambda(G) \cong \Lambda(H) * (G/H)$. Since the topological group H has good conditions, a ring $\Lambda(H)$ has good properties among them, we need:

(1) local with the radical $J := \mathrm{rad}\Lambda(H)$ and $\Lambda(H)/J \cong \mathbb{F}_p$, a field of p-elements,

(2) a filtered ring with the J-adic filtration whose associated graded ring is isomorphic to a polynomial ring $\mathbb{F}_p[x_0, \cdots, x_d]$, where $d = \mathrm{dim}G$ is a minimal number of generators of G as a topological group.

(3) a left and right Noetherian domain,

(4) Auslander regular with $\mathrm{gldim}\Lambda(H) = d + 1$.

(cf. [1], [4], [5], [6], [12])

2. A crossed product and a filtered ring

2.1. Let R be a ring and A a finite group. A *crossed product* S ([8], [11]) of a group A over a ring R, denoted by $S := R * A$, is a ring such that:

1) R is a subring of $R * A$

2) $\bar{A} = \{\bar{a} : a \in A\}$ is a subset of $R * A$ consisting of units of $R * A$

3) $R * A$ is a free right R-module with basis $\bar{A}$

The detailed version of this paper will be submitted for publication elsewhere.

4) For all $a, b \in A$, the equalities $\bar{a}R = R\bar{a}$ and $\bar{a}\bar{b}R = \overline{ab}R$ hold.

Remarks. (see [11]) (1) We may assume $\bar{1}_A = 1_S$. A left R-module $R * A$ is also free with basis $\bar{A}$. Usually, we write

$$R * A = \bigoplus_{a \in A} \bar{a}R.$$

(2) There exists a map $\sigma : A \to \mathrm{Aut}R$ such that $r\bar{a} = \bar{a}r^{\sigma(a)}$, $r \in R$, $a \in A$. In what follows, we shortly write $r\bar{a} = \bar{a}r^{a}$. There exists a map $\tau : A \times A \to U(R)$ such that $\bar{a}\bar{b} = \overline{ab}\tau(a,b)$. In order to assure the associativity of $R * A$, maps σ, τ satisfy some conditions (see [11]).

We start with the theorem which implies that the Iwasawa algebra is Auslander Gorenstein.

A ring R is said to satisfy *Auslander condition*, if, for all finitely generated left R-module M, for all $i \geq 0$ and for all right R-submodules N of $\mathrm{Ext}^i_R(M, R)$, grade of N is greater than or equal to i, where grade of an R-module X is $\inf\{j \geq 0 : \mathrm{Ext}^j_R(X, R) \neq 0\}$.

Theorem 1. *Let $S = R * A$ be a crossed product. Then* id$R$ = id$S$ *holds, where* id *stands for injective dimension. Moreover, if R satisfies Auslander condition, then S satisfies it, too.*

Proof. It follows from [2] that, for all finitely generated left S-modules M and for all $i \geq 0$,

$$\mathrm{Ext}^i_S(M, S) \cong \mathrm{Ext}^i_R(M, R).$$

The statement is an easy consequence of this formula. □

Since $\mathrm{gldim}\Lambda(H) = d + 1$, we see $\mathrm{id}\Lambda(H) = d + 1$. Hence $\Lambda(G)$ is Auslander Gorenstein of $\mathrm{id}\Lambda(G) = d + 1$.

2.2. A ring R is called a *filtered ring* with a filtration $\mathcal{F} = \{\mathcal{F}_i R\}_{i \in \mathbb{Z}}$ if

i) $\mathcal{F}_i R$ is an additive subgroup of R for all $i \in \mathbb{Z}$ and $1 \in \mathcal{F}_0 R$,

ii) $\mathcal{F}_i R \subset \mathcal{F}_{i+1} R$ $(i \in \mathbb{Z})$,

iii) $(\mathcal{F}_i R)(\mathcal{F}_j R) \subset \mathcal{F}_{i+j} R$ $(i, j \in \mathbb{Z})$,

iv) $\cup_{i \in \mathbb{Z}} \mathcal{F}_i R = R$.

([7])

Let $S = R * A$ be a crossed product and further, assume that R is a filtered ring. Then a filtration $\mathcal{F}$ is called *A-stable*, if

v) $(\mathcal{F}_i R)^a \subset \mathcal{F}_i R$ for all $a \in A$ and $i \in \mathbb{Z}$.

Let $\mathrm{gr}R = \oplus_{p \in \mathbb{Z}} \mathcal{F}_p R / \mathcal{F}_{p-1} R$ an associated graded ring of R. Then forming a crossed product '$*$' and an associated graded ring 'gr' commutes each other.

Theorem 2. *Let R, A, S be as above. Assume that R is a filtered ring with an A-stable filtration such that every unit of R sits in $\mathcal{F}_0R \setminus \mathcal{F}_{-1}R$. Then $\mathcal{F}' := \{\mathcal{F}'_iS\}_{i\in\mathbb{Z}}$, $\mathcal{F}'_iS = \oplus_{a\in A}\bar{a}(\mathcal{F}_iR)$ $(i \in \mathbb{Z})$, is a filtration of S and there is a ring isomorphism*

$$\mathrm{gr}_{\mathcal{F}'}S \cong (\mathrm{gr}_{\mathcal{F}}R) * A.$$

We put the J-adic filtration $\mathcal{F} = \{\mathcal{F}_i\Lambda(H)\}_{i\in\mathbb{Z}}$ of $\Lambda(H)$ by

$$\mathcal{F}_i\Lambda(H) = \begin{cases} J^{-i} \ (i < 0) \\ \Lambda(H) \ (i \geq 0) \end{cases}$$

It follows from [1],[4],[6],[12] that the associated graded ring satisfies $\mathrm{gr}_{\mathcal{F}}\Lambda(H) \cong \mathbb{F}_p[x_0, \cdots, x_d]$.

Since $J^{\alpha} \subset J$ for all $\alpha \in \mathrm{Aut}\Lambda(H)$, the J-adic filtration $\mathcal{F}$ of $\Lambda(H)$ is G/H-stable. Since $\Lambda(H)$ is a local ring, all units of $\Lambda(H)$ sit in $\Lambda(H) \setminus J$, i.e., in $\mathcal{F}_0\Lambda(H) \setminus \mathcal{F}_{-1}\Lambda(H)$. Therefore, we see $\mathrm{gr}\Lambda(G) \cong \mathrm{gr}\Lambda(H) * (G/H)$.

3. Graded modules over a crossed product

Let $S = R * A$ be a crossed product of a finite group A over a ring R. We assume that R is Noetherian, so that S is , too. A left S-module with a decomposition $M = \oplus_{a\in A}M_a$ as an abelian group is called a (*strongly*) *A-graded* module, if $\bar{a}RM_b \subset M_{ab}$ $(\bar{a}RM_b = M_{ab})$ for all $a, b \in A$. By the decomposition $S = \oplus_{a\in A}\bar{a}R$, S itself is an A-graded module with $S_a = \bar{a}R$, so S is an A-graded ring ([9], [10]). Since $\bar{a}R\bar{b}R = \overline{ab}R$, S is a strongly graded ring, therefore, every graded module over S is strongly graded ([9]).

Let $f \in \mathrm{Hom}_S(M, N)$ for M, N graded S-modules. We call f a graded homomorphism of degree $a \in A$, whenever $f(M_b) \subset N_{ba}$ for all $b \in A$. We put, for $a \in A$, $\mathrm{Hom}_a(M, N) := \{f \in \mathrm{Hom}_R(M, N) : f \text{ is graded of degree } a\}$. Then $\mathrm{Hom}_R(M, N) = \oplus_{a\in A}\mathrm{Hom}_a(M, N)$ holds.

Proposition 3. *Let N be a left R-module and M a left graded S-module. Then there exists an isomorphism* $\mathrm{Hom}_R(M_1, N) \cong \mathrm{Hom}_1(M, \mathrm{Hom}_R(S, N))$.

Proof. Note that $\mathrm{Hom}_R(S, N)$ is graded by the grading $\mathrm{Hom}_R(S, N)_a = \mathrm{Hom}_R(\overline{a^{-1}}R, N)$, $a \in A$ □

Lemma 4. [2] *Define $\alpha : \mathrm{Hom}_R(S, R) \to S$, by $\alpha(f) = \sum_{a\in A}(\bar{a})^{-1}f(\bar{a})$ for $f \in \mathrm{Hom}_R(S, R)$. Then α is an S-R-bimodule isomorphism.*

Combining Proposition 3 and Lemma 4, we get

Corollary 5. *Let M be a graded S-module. Then there is an isomorphism of right R-module:* $\mathrm{Hom}_R(M_1, R) \cong \mathrm{Hom}_1(M, S)$.

We study Gorenstein dimension(cf. [3]), one of the important homological invariants of a Noetherian ring. An R-module M is said to have *Gorenstein dimension zero*, denoted by $\text{G-dim}_R M = 0$, if $M^{**} \cong M$ and $\mathrm{Ext}^k_R(M, R) = \mathrm{Ext}^k_{R^{op}}(M^*, R) = 0$ for $k > 0$, where $M^* = \mathrm{Hom}_R(M, R)$. For a positive integer k, M is said to have *Gorenstein dimension less than or equal to k*, denoted by $\text{G-dim } M \leq k$, if there exists an exact sequence $0 \to G_k \to \cdots \to G_0 \to M \to 0$ with $\text{G-dim } G_i = 0$ for $0 \leq i \leq k$. We have that G-dim

$M \leq k$ if and only if G-dim $\Omega^k M = 0$. It is also proved that if G-dim $M < \infty$ then G-dim $M = \sup\{k : \mathrm{Ext}_R^k(M, R) \neq 0\}$.

For a graded S-module, G-dimension is controlled by an R-module.

Theorem 6. *Let M be a graded S-module, then* G-dim$_S M$ = G-dim$_R M_1$

We will prove this theorem in the following.

Let $M = \oplus_{a \in A} M_a = \oplus_{a \in A} \bar{a} M_1$ be a graded S-module. Note that $M \cong S \otimes_R M_1$. Then a right S-module $M^* = \mathrm{Hom}_S(M, S) = \oplus_{a \in A} \mathrm{Hom}_a(M, S)$. We see $\mathrm{Hom}_a(M, S) = \mathrm{Hom}_1(M, S)\bar{a}$ and M^* is a graded right S-module of grading $\mathrm{Hom}_S(M, S)_a = \mathrm{Hom}_1(M, S)\bar{a}$. By Corollary 6, it holds that $\mathrm{Hom}_1(M, S) \cong \mathrm{Hom}_R(M_1, R)$, hence $M^* \cong \oplus_a M_1^* \bar{a}$, whare $M_1^* = \mathrm{Hom}_R(M_1, R)$. Similary, there is an isomorphism $M^{**} \cong \oplus_a \bar{a} M_1^{**}$.

Let $\theta : M \to M^{**}$ be a canonical evaluation map. Then θ is a graded homomorphism of degree 1. Therefore, the following holds.

Lemma 7. *M is reflexive as an S-module if and only if M_1 is reflexive as an R-module.*

Concerning extension groups, the following holds.

Lemma 8. $\mathrm{Ext}_S^i(M, S) = 0$ *if and only if* $\mathrm{Ext}_R^i(M_1, R) = 0$ *for all* $i \geq 0$.

Proof. The combination of isomorphisms:

$$\mathrm{Ext}_S^i(M, S) \cong \mathrm{Ext}_R^i(M, R) \quad ([2])$$

$$\mathrm{Ext}_R^i(S \otimes_R M_1, R) \cong \mathrm{Ext}_R^i(M_1, \mathrm{Hom}_R(S, R))$$

$$\mathrm{Hom}_R(S, R) \cong S \quad \text{(Lemma 4)}$$

induces an isomorphism

$$\mathrm{Ext}_S^i(M, S) \cong \oplus_{a \in A} \mathrm{Ext}_R^i(M_1, \bar{a} R).$$

Consequently, the assertion holds. □

We deal with the case of G-dimension zero. Note that $(M^*)_1 = M_1^*$ for a graded S-module M.

Theorem 9. *Let M be a graded S-module. Then G*-dim$_S M = 0$ *if and only if G*-dim$_R M_1 = 0$.

Let $\cdots \to P_1 \to P_0 \to M_1 \to 0$ be a projective resolution of an R-module M_1, for a graded S-module. Then $\cdots \to S \otimes_R P_1 \to S \otimes_R P_0 \to S \otimes_R M_1 \to 0$ is a projective resolution of an S-module $S \otimes_R M_1 = M$. Hence $\Omega^i M \cong S \otimes_R \Omega^i M_1$, and then $(\Omega^i M)_1 \cong \Omega^i(M_1)$. Hence G-dim$_S \Omega^i M = 0$ if and only if G-dim$_R \Omega^i M_1 = 0$ by Theorem 9. This proves Theorem 6.

3.1. **Concluding Remarks.** Let M be a graded $\Lambda(G)$-module and take a good filtration of M_1 ([7]). Then the following (in)equalities hold:

$$\text{G-dim}_{\Lambda(G)}M + \mathfrak{m}\text{-depth}(\text{gr}M_1) \leq d+1$$

$$\text{grade}_{\Lambda(G)}M + \dim_{\text{gr}\Lambda(H)}(\text{gr}M_1) = d+1,$$

where $\mathfrak{m}$ is the *maximal ideal of $\text{gr}\Lambda(H) = \mathbb{F}[x_0, \cdots, x_d]$.

These formulae will be able to apply to homological theory of modules over the Iwasawa algebra. For example:
Suppose that $\text{gr}M_1$ is Cohen-Macaulay, then M is perfect, i.e., $\text{grade}_{\Lambda(G)}M = \text{G-dim}_{\Lambda(G)}M$.

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SUPPORT VARIETIES AND THE HOCHSCHILD COHOMOLOGY RING MODULO NILPOTENCE

NICOLE SNASHALL

ABSTRACT. This paper is based on my talks given at the '41st Symposium on Ring Theory and Representation Theory' held at Shizuoka University, Japan, 5-7 September 2008. It begins with a brief introduction to the use of Hochschild cohomology in developing the theory of support varieties of [50] for a module over an artin algebra. I then describe the current status of research concerning the structure of the Hochschild cohomology ring modulo nilpotence.

ACKNOWLEDGEMENT

I would like to thank the organisers of the '41st Symposium on Ring Theory and Representation Theory' held at Shizuoka University for the invitation to speak at this conference. The cost of visiting Japan and attending the meeting in Shizuoka was supported by the Japan Society for Promotion of Science (JSPS) Grant-in Aid for Scientific Research (B) 18340011. I would particularly like to thank the head of the grant, Kiyoichi Oshiro (Yamaguchi University), for providing this financial support and invitation. I would also like to thank Shigeo Koshitani and Katsunori Sanada for their hospitality during my visit to Japan. I also thank the University of Leicester for granting me study leave.

INTRODUCTION

This survey article is based on talks given at the '41st Symposium on Ring Theory and Representation Theory', Shizuoka University in September 2008, and is organised as follows. Section 1 gives a brief introduction to the use of Hochschild cohomology in developing the theory of support varieties of [50]. Section 2 considers the Hochschild cohomology ring of Ω-periodic algebras. In [50] it had been conjectured that the Hochschild cohomology ring modulo nilpotence of a finite-dimensional algebra is always finitely generated as an algebra. Section 3 describes many classes of algebras where this holds, that is, that the Hochschild cohomology ring modulo nilpotence is finitely generated as an algebra. The final section is devoted to studying the recent counterexample of Xu ([57]) to this conjecture.

Throughout this paper, let Λ be an indecomposable finite-dimensional algebra over an algebraically closed field K, with Jacobson radical $\mathfrak{r}$. Denote by Λ^e the enveloping algebra $\Lambda^{op} \otimes_K \Lambda$ of Λ, so that right Λ^e-modules correspond to Λ,Λ-bimodules. The Hochschild cohomology ring $\mathrm{HH}^*(\Lambda)$ of Λ is given by $\mathrm{HH}^*(\Lambda) = \mathrm{Ext}^*_{\Lambda^e}(\Lambda,\Lambda) = \oplus_{i\geq 0}\mathrm{Ext}^i_{\Lambda^e}(\Lambda,\Lambda)$ with the Yoneda product. We may consider an element of $\mathrm{Ext}^n_{\Lambda^e}(\Lambda,\Lambda)$ as an exact sequence of

This paper is in final form and no version of it will be submitted for publication elsewhere.

Λ,Λ-bimodules $0 \to \Lambda \to E^n \to E^{n-1} \to \cdots \to E^1 \to \Lambda \to 0$ where the Yoneda product is the 'splicing together' of exact sequences.

The low-dimensional Hochschild cohomology groups are well-understood via the bar resolution ([41] and see [3, 39]), and may be described as follows:

- $\mathrm{HH}^0(\Lambda) = Z(\Lambda)$, the centre of Λ.
- $\mathrm{HH}^1(\Lambda)$ is the space of derivations modulo the inner derivations. A derivation is a K-linear map $f : \Lambda \to \Lambda$ such that $f(ab) = af(b) + f(a)b$ for all $a, b \in \Lambda$. A derivation $f : \Lambda \to \Lambda$ is an inner derivation if there is some $x \in \Lambda$ such that $f(a) = ax - xa$ for all $a \in \Lambda$.
- $\mathrm{HH}^2(\Lambda)$ measures the infinitesimal deformations of the algebra Λ; in particular, if $\mathrm{HH}^2(\Lambda) = 0$ then Λ is rigid, that is, Λ has no non-trivial deformations.

Recently there has been much work on the structure of the entire Hochschild cohomology ring $\mathrm{HH}^*(\Lambda)$ and its connections and applications to the representation theory of Λ. One important property of Hochschild cohomology in this situation is its invariance under derived equivalence, proved by Rickard in [48, Proposition 2.5] (see also [39, Theorem 4.2] for a special case). It is also well-known that $\mathrm{HH}^*(\Lambda)$ is a graded commutative ring, that is, for homogeneous elements $\eta \in \mathrm{HH}^n(\Lambda)$ and $\theta \in \mathrm{HH}^m(\Lambda)$, we have $\eta\theta = (-1)^{mn}\theta\eta$. Thus, when the characteristic of K is different from two, then every homogeneous element of odd degree squares to zero. Let $\mathcal{N}$ denote the ideal of $\mathrm{HH}^*(\Lambda)$ which is generated by the homogeneous nilpotent elements. Then, for $\operatorname{char} K \neq 2$, we have $\mathrm{HH}^{2k+1}(\Lambda) \subseteq \mathcal{N}$ for all $k \geq 0$. Hence (in all characteristics) the Hochschild cohomology ring modulo nilpotence, $\mathrm{HH}^*(\Lambda)/\mathcal{N}$, is a commutative K-algebra.

Support varieties for finitely generated modules over a finite-dimensional algebra Λ were introduced using Hochschild cohomology by Snashall and Solberg in [50], where it was also conjectured that the Hochschild cohomology ring modulo nilpotence is itself a finitely generated algebra. We remark that the graded commutativity of $\mathrm{HH}^*(\Lambda)$ implies that $\mathcal{N}$ is contained in every maximal ideal of $\mathrm{HH}^*(\Lambda)$ and so $\operatorname{MaxSpec} \mathrm{HH}^*(\Lambda) = \operatorname{MaxSpec} \mathrm{HH}^*(\Lambda)/\mathcal{N}$. Although the recent paper [57] provides a counterexample to the conjecture of [50], nevertheless finiteness conditions play an key role in the structure of these support varieties (see [15]), so it remains of particular importance to determine the structure of the Hochschild cohomology ring modulo nilpotence.

1. Support varieties

One of the motivations for introducing support varieties for finitely generated modules over a finite-dimensional algebra came from the rich theory of support varieties for finitely generated modules over group algebras of finite groups. For a finite group G and finitely generated KG-module M, the variety of M, $V_G(M)$, was defined by Carlson [10] to be the variety of the kernel of the homomorphism

$$- \otimes_K M \colon H^{\mathrm{ev}}(G, K) \to \operatorname{Ext}^*_{KG}(M, M).$$

This map factors through the Hochschild cohomology ring of KG, so that we have the commutative diagram

$$H^{\mathrm{ev}}(G,K) \xrightarrow{\;-\otimes_K M\;} \mathrm{Ext}^*_{KG}(M,M)$$

$$\mathrm{HH}^*(KG) \xrightarrow{\;-\otimes_{KG} M\;} \mathrm{Ext}^*_{KG}(M,M)$$

Linckelmann considered the map $-\otimes_{KG} M\colon\ \mathrm{HH}^{\mathrm{ev}}(KG) \to \mathrm{Ext}^*_{KG}(M,M)$ when studying varieties for modules for non-principal blocks ([43]).

Now, for any finite-dimensional algebra Λ and finitely generated Λ-module M, there is a ring homomorphism $\mathrm{HH}^*(\Lambda) \xrightarrow{-\otimes_\Lambda M} \mathrm{Ext}^*_\Lambda(M,M)$. This ring homomorphism turns out to provide a similarly fruitful theory of support varieties for finitely generated modules over an arbitrary finite-dimensional algebra. As usual, let $\operatorname{mod}\Lambda$ denote the category of all finitely generated left Λ-modules.

For $M \in \operatorname{mod}\Lambda$, the support variety of M, $V_{\mathrm{HH}^*(\Lambda)}(M)$, was defined by Snashall and Solberg in [50, Definition 3.3] by

$$V_{\mathrm{HH}^*(\Lambda)}(M) = \{\mathfrak{m} \in \operatorname{MaxSpec}\mathrm{HH}^*(\Lambda)/\mathcal{N} \mid \operatorname{Ann}_{\mathrm{HH}^*(\Lambda)} \mathrm{Ext}^*_\Lambda(M,M) \subseteq \mathfrak{m}'\}$$

where $\mathfrak{m}'$ is the preimage in $\mathrm{HH}^*(\Lambda)$ of the ideal $\mathfrak{m}$ in $\mathrm{HH}^*(\Lambda)/\mathcal{N}$. We recall from above that $\operatorname{MaxSpec}\mathrm{HH}^*(\Lambda) = \operatorname{MaxSpec}\mathrm{HH}^*(\Lambda)/\mathcal{N}$.

Since we assumed that Λ is indecomposable, we know that $\mathrm{HH}^0(\Lambda)$ is a local ring. Thus $\mathrm{HH}^*(\Lambda)/\mathcal{N}$ has a unique maximal graded ideal which we denote by $\mathfrak{m}_{\mathrm{gr}}$ so that $\mathfrak{m}_{\mathrm{gr}} = \langle \operatorname{rad}\mathrm{HH}^0(\Lambda), \mathrm{HH}^{\geq 1}(\Lambda)\rangle/\mathcal{N}$. From [50, Proposition 3.4(a)], we have $\mathfrak{m}_{\mathrm{gr}} \in V_{\mathrm{HH}^*(\Lambda)}(M)$ for all $M \in \operatorname{mod}\Lambda$. We say that the variety of M is *trivial* if $V_{\mathrm{HH}^*(\Lambda)}(M) = \{\mathfrak{m}_{\mathrm{gr}}\}$.

The following result collects some of the properties of varieties from [50]. For ease of notation, we write $V(M)$ for $V_{\mathrm{HH}^*(\Lambda)}(M)$. We also denote the kernel of the projective cover of $M \in \operatorname{mod}\Lambda$ by $\Omega_\Lambda(M)$.

Recall that we assume throughout this paper that K is an algebraically closed field. This assumption is a necessary assumption in many of the results in this article. However, it is not needed in all of [50], and the interested reader may refer back to [50] to see precisely what assumptions are required there at each stage.

Theorem 1.1. *([50, Propositions 3.4, 3.7]) Let $M \in \operatorname{mod}\Lambda$.*

(1) *$V(M) = V(\Omega_\Lambda(M))$ if $\Omega_\Lambda(M) \neq (0)$,*
(2) *$V(M_1 \oplus M_2) = V(M_1) \cup V(M_2)$,*
(3) *If $0 \to M_1 \to M_2 \to M_3 \to 0$ is an exact sequence, then $V(M_{i_1}) \subseteq V(M_{i_2}) \cup V(M_{i_3})$ whenever $\{i_1, i_2, i_3\} = \{1,2,3\}$,*
(4) *If $\mathrm{Ext}^i_\Lambda(M,M) = (0)$ for $i \gg 0$, or the projective or the injective dimension of M is finite, then the variety of M is trivial.*
(5) *If Λ is selfinjective then $V(M) = V(\tau M)$, where τ is the Auslander-Reiten translate. Hence all modules in a connected stable component of the Auslander-Reiten quiver have the same variety.*

For a finitely generated module M over a group algebra of a finite group G, it is well-known ([10]) that the variety of M is trivial if and only if M is a projective module. In contrast, it is still an open question as to what are the appropriate necessary and sufficient conditions on a module for it to have trivial variety in the more general case where Λ is an arbitrary finite-dimensional algebra. There are some partial results for a particular class of monomial algebras in [25] (see Section 3). Nevertheless, the converse to Theorem 1.1(4) does not hold in general, and a counterexample may be found in [52, Example 4.7].

However, this question was successfully answered by Erdmann, Holloway, Snashall, Solberg and Taillefer in [15], by placing some (reasonable) additional assumptions on Λ. (Recall that we are already assuming that the field K is algebraically closed.) Specifically, the following two finiteness conditions were introduced.

(Fg1) H is a commutative Noetherian graded subalgebra of $\mathrm{HH}^*(\Lambda)$ with $H^0 = \mathrm{HH}^0(\Lambda)$.

(Fg2) $\mathrm{Ext}^*_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$ is a finitely generated H-module.

As remarked in [15], these two conditions together imply that both $\mathrm{HH}^*(\Lambda)$ and $\mathrm{Ext}^*_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$ are finitely generated K-algebras. In particular, the properties **(Fg1)** and **(Fg2)** hold where $\Lambda = KG$, G is a finite group, and $H = \mathrm{HH}^{ev}(\Lambda)$ ([21, 54]). With conditions **(Fg1)** and **(Fg2)**, we have the following results from [15], where we define the variety using the subalgebra H of $\mathrm{HH}^*(\Lambda)$, so that $V_H(M) = \mathrm{MaxSpec}(H/\mathrm{Ann}_H \mathrm{Ext}^*_\Lambda(M, M))$.

Theorem 1.2. *([15, Theorem 2.5]) Suppose that Λ and H satisfy* **(Fg1)** *and* **(Fg2)**. *Then Λ is Gorenstein. Moreover the following are equivalent for $M \in \mathrm{mod}\,\Lambda$:*

(i) *The variety of M is trivial;*
(ii) *M has finite projective dimension;*
(iii) *M has finite injective dimension.*

Theorem 1.3. *([15, Theorem 4.4]) Suppose that Λ and H satisfy* **(Fg1)** *and* **(Fg2)**. *Given a homogeneous ideal $\mathfrak{a}$ in H, there is a module $M \in \mathrm{mod}\,\Lambda$ such that $V_H(M) = V_H(\mathfrak{a})$.*

Theorem 1.4. *([15, Theorem 2.5 and Propositions 5.2, 5.3]) Suppose that Λ and H satisfy* **(Fg1)** *and* **(Fg2)** *and that Λ is selfinjective. Let $M \in \mathrm{mod}\,\Lambda$ be indecomposable.*

(1) *$V_H(M)$ is trivial $\Leftrightarrow$ M is projective.*
(2) *$V_H(M)$ is a line $\Leftrightarrow$ M is Ω-periodic.*

Our final results in this section concern the representation type of Λ and the structure of the Auslander-Reiten quiver; for more details see [15, 52]. First we recall that Heller showed that if Λ is of finite representation type then the complexity of a finitely generated module is at most 1 ([40]), and that Rickard showed that if Λ is of tame representation type then the complexity of a finitely generated module is at most 2 ([47]). However there are selfinjective preprojective algebras of wild representation type where all indecomposable modules are either projective or periodic and so have complexity at most 1. Nevertheless, the next result uses the Hochschild cohomology ring to give some information on the representation type of an algebra.

Theorem 1.5. *([15, Proposition 6.1]) Suppose that Λ and H satisfy* **(Fg1)** *and* **(Fg2)** *and that Λ is selfinjective. Suppose also that $\dim H \geq 2$. Then Λ is of infinite representation type, and Λ has an infinite number of indecomposable periodic modules lying in infinitely many different components of the stable Auslander-Reiten quiver.*

We end this section with the statement of Webb's theorem ([55]) for group algebras of finite groups and a generalisation of this theorem from [15].

Theorem 1.6. *([55]) Let G be a finite group and suppose that* char K *divides* $|G|$. *Then the orbit graph of a connected component of the stable Auslander-Reiten quiver of KG is one of the following:*

(a) *a finite Dynkin diagram* $(\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_{6,7,8})$,
(b) *a Euclidean diagram* $(\tilde{\mathbb{A}}_n, \tilde{\mathbb{D}}_n, \tilde{\mathbb{E}}_{6,7,8}, \tilde{\mathbb{A}}_{12})$, *or*
(c) *an infinite Dynkin diagram of type* $\mathbb{A}_\infty$, $\mathbb{D}_\infty$ *or* $\mathbb{A}_\infty^\infty$.

Theorem 1.7. *([15, Theorem 5.6]) Suppose that Λ and H satisfy* (**Fg1**) *and* (**Fg2**) *and that Λ is selfinjective. Suppose that the Nakayama functor is of finite order on any indecomposable module in* mod Λ. *Then the tree class of a component of the stable Auslander-Reiten quiver of Λ is one of the following:*

(a) *a finite Dynkin diagram* $(\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_{6,7,8})$,
(b) *a Euclidean diagram* $(\tilde{\mathbb{A}}_n, \tilde{\mathbb{D}}_n, \tilde{\mathbb{E}}_{6,7,8}, \tilde{\mathbb{A}}_{12})$, *or*
(c) *an infinite Dynkin diagram of type* $\mathbb{A}_\infty$, $\mathbb{D}_\infty$ *or* $\mathbb{A}_\infty^\infty$.

We remark that the hypotheses of Theorem 1.7 are satisfied for all finite-dimensional cocommutative Hopf algebras ([15, Corollary 5.7]).

For more information, the reader should also see the survey paper on support varieties for modules and complexes by Solberg [52]. In addition, the paper by Bergh [5] introduces the concept of a twisted support variety for a finitely generated module over an artin algebra, where the twist is induced by an automorphism of the algebra, and, in [6], Bergh and Solberg study relative support varieties for finitely generated modules over a finite-dimensional algebra over a field.

2. Ω-PERIODIC ALGEBRAS

We now turn our attention to the structure of the Hochschild cohomology ring. One class of algebras where it is relatively straightforward to determine the structure of the Hochschild cohomology ring explicitly is the class of Ω-periodic algebras. We recall that Λ is said to be an Ω-periodic algebra if there exists some $n \geq 1$ such that $\Omega^n_{\Lambda^e}(\Lambda) \cong \Lambda$ as bimodules. Such an algebra Λ has a periodic minimal projective bimodule resolution, so that $\mathrm{HH}^i(\Lambda) \cong \mathrm{HH}^{n+i}(\Lambda)$ for $i \geq 1$, and is necessarily self-injective (Butler; see [34]).

There is an extensive survey of periodic algebras by Erdmann and Skowroński in [18]. Examples of such algebras include the preprojective algebras of Dynkin type where $\Omega^6_{\Lambda^e}(\Lambda) \cong \Lambda$ as bimodules ([49]; see also [19]), and the deformed mesh algebras of generalized Dynkin type of Białkowski, Erdmann and Skowroński [7, 18]. For the selfinjective algebras of finite representation type over an algebraically closed field, it is known from [34] that there is some $n \geq 1$ and automorphism σ of Λ such that $\Omega^n_{\Lambda^e}(\Lambda)$ is isomorphic as a bimodule to the twisted bimodule ${}_1\Lambda_\sigma$. It has now been shown that all selfinjective algebras of finite representation type over an algebraically closed field are Ω-periodic ([13, 16, 17, 18]).

The structure of the Hochschild cohomology ring modulo nilpotence of these algebras was determined by Green, Snashall and Solberg in [34].

Theorem 2.1. *([34, Theorem 1.6]) Let K be an algebraically closed field. Let Λ be a finite-dimensional indecomposable K-algebra such that there is some $n \geq 1$ and some automorphism σ of Λ such that $\Omega^n_{\Lambda^e}(\Lambda)$ is isomorphic to the twisted bimodule ${}_1\Lambda_\sigma$. Then*

$$\mathrm{HH}^*(\Lambda)/\mathcal{N} \cong \begin{cases} K[x] \text{ or} \\ K. \end{cases}$$

If there is some $m \geq 1$ such that $\Omega^m_{\Lambda^e}(\Lambda) \cong \Lambda$ as bimodules, then $\mathrm{HH}^*(\Lambda)/\mathcal{N} \cong K[x]$, where x is in degree m and m is minimal.

Additional information on the ring structure of the Hochschild cohomology ring of the preprojective algebras of Dynkin type A_n was determined in [19], of the preprojective algebras of Dynkin types D_n, E_6, E_7, E_8 in [20], and of the selfinjective algebras of finite representation type A_n over an algebraically closed field in [16, 17].

Given that the Hochschild cohomology ring of these algebras is understood, this naturally leads to the study of situations where the Hochschild cohomology rings of two algebras A and B can be related. This enables us to transfer information about the Hochschild cohomology ring of, say, an Ω-periodic algebra, to other algebras. Apart from periodic algebras, there are other algebras where the Hochschild cohomology ring is known, and these provide additional examples where the transfer of properties between Hochschild cohomology rings may also be studied. One such class of examples is the class of truncated quiver algebras, which has been extensively studied in the literature by many authors.

Happel showed in [39, Theorem 5.3] that if B is a one-point extension of a finite-dimensional K-algebra A by a finitely generated A-module M, then there is a long exact sequence connecting the Hochschild cohomology rings of A and B:

$$0 \to \mathrm{HH}^0(B) \to \mathrm{HH}^0(A) \to \mathrm{Hom}_A(M,M)/K \to$$
$$\mathrm{HH}^1(B) \to \mathrm{HH}^1(A) \to \mathrm{Ext}^1_A(M,M) \to \cdots$$
$$\cdots \to \mathrm{Ext}^i_A(M,M) \to \mathrm{HH}^{i+1}(B) \to \mathrm{HH}^{i+1}(A) \to \mathrm{Ext}^{i+1}_A(M,M) \to \cdots$$

It was subsequently shown by Green, Marcos and Snashall in [29, Theorem 5.1] that there is a graded ring homomorphism

$$\mathrm{HH}^*(B) \to \mathrm{HH}^*(A) \oplus \mathcal{K}$$

which induces this long exact sequence, where $\mathcal{K}$ is the graded K-module with $\mathcal{K}_0 = K$ and $\mathcal{K}_n = 0$ for all $n \neq 0$.

These results were generalized independently to arbitrary triangular matrix algebras by Cibils [12], by Green and Solberg [36], and by Michelena and Platzeck [44].

A recent result of König and Nagase, ([42, 45]), has related the Hochschild cohomology ring of B to that of B/BeB in the case where B is an algebra with idempotent e, such that BeB is a stratifying ideal of B.

Theorem 2.2. *([42]) Let B be an algebra with idempotent e such that BeB is a stratifying ideal of B and let A be the factor algebra B/BeB. Then there are long exact sequences as follows:*

1. $\cdots \to \mathrm{Ext}^n_{B^e}(B, BeB) \to \mathrm{HH}^n(B) \to \mathrm{HH}^n(A) \to \cdots$;
2. $\cdots \to \mathrm{Ext}^n_{B^e}(A, B) \to \mathrm{HH}^n(B) \to \mathrm{HH}^n(eBe) \to \cdots$; *and*
3. $\cdots \to \mathrm{Ext}^n_{B^e}(A, BeB) \to \mathrm{HH}^n(B) \to \mathrm{HH}^n(A) \oplus \mathrm{HH}^n(eBe) \to \cdots$.

3. The Hochschild cohomology ring modulo nilpotence

The definition of a support variety in [50] led us to consider the structure of $\mathrm{HH}^*(\Lambda)/\mathcal{N}$ and to conjecture that $\mathrm{HH}^*(\Lambda)/\mathcal{N}$ is always finitely generated as an algebra. A counterexample to this conjecture was recently given by Xu in [57]; nevertheless the Hochschild cohomology ring modulo nilpotence is finitely generated as an algebra for many diverse classes of algebras.

The Hochschild cohomology ring modulo nilpotence is known to be finitely generated as an algebra in the following cases.

- any block of a group ring of a finite group ([21, 54]);
- any block of a finite-dimensional cocommutative Hopf algebra ([24]);
- finite-dimensional selfinjective algebras of finite representation type over an algebraically closed field ([34]);
- finite-dimensional monomial algebras ([35] and see [32]);
- finite-dimensional algebras of finite global dimension (see [39]).

For the last class of examples, if Λ is an algebra of finite global dimension N, then $\mathrm{HH}^i(\Lambda) = \mathrm{Ext}^i_{\Lambda^e}(\Lambda, \Lambda) = (0)$ for all $i > N$. Hence $\mathrm{HH}^*(\Lambda)/\mathcal{N} \cong K$. In [39], Happel asked whether or not it was true, for a finite-dimensional algebra Γ over a field K, that if $\mathrm{HH}^n(\Gamma) = (0)$ for $n \gg 0$ then the global dimension of Γ is finite. This question has now been answered in the negative by Buchweitz, Green, Madsen and Solberg in [8] by the following example.

Example 3.1. ([8]) Let

$$\Lambda_q = K\langle x, y\rangle/(x^2, xy + qyx, y^2)$$

with $q \in K \setminus \{0\}$. If q is not a root of unity then $\dim \mathrm{HH}^i(\Lambda_q) = 0$ for $i \geq 3$. Moreover, Λ_q is a selfinjective algebra so has infinite global dimension.

We also note from [8] that $\dim \Lambda_q = 4$, $\dim \mathrm{HH}^*(\Lambda_q) = 5$ and $\mathrm{HH}^*(\Lambda)/\mathcal{N} \cong K$.

However, the situation for commutative algebras is very different, as Avramov and Iyengar have shown.

Theorem 3.2. *([1]) Let R be a commutative finite-dimensional K-algebra over a field K. If $\mathrm{HH}^n(R) = (0)$ for $n \gg 0$ then R is a (finite) product of (finite) separable field extensions of K. In particular, the global dimension of R is finite.*

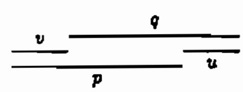
We now turn to a brief discussion of the Hochschild cohomology ring modulo nilpotence for a monomial algebra, which was studied by Green, Snashall and Solberg. Let Λ be a quotient of a path algebra so that $\Lambda = KQ/I$ for some quiver Q and admissible ideal I of KQ. Then $\Lambda = KQ/I$ is a monomial algebra if the ideal I is generated by monomials of length at least two. It should be noted that monomial algebras are very rarely selfinjective and so do not usually exhibit the same properties as group algebras. However, the

Hochschild cohomology ring modulo nilpotence of a monomial algebra turns out to have a particularly nice structure.

Theorem 3.3. *([35, Theorem 7.1]) Let $\Lambda = KQ/I$ be a finite-dimensional indecomposable monomial algebra. Then $\mathrm{HH}^*(\Lambda)/\mathcal{N}$ is a commutative finitely generated K-algebra of Krull dimension at most one.*

For some specific subclasses of monomial algebras, the structure of the Hochschild cohomology ring modulo nilpotence was explicitly determined in [32]. One of the main tools used was the minimal projective bimodule resolution of a monomial algebra of Bardzell [2]. In order to define the particular class of (D, A)-stacked monomial algebras, we require the concept of overlaps of [27, 38]; the definitions here use the notation of [32].

Definition 3.4. A path q in KQ overlaps a path p in KQ with overlap pu if there are paths u and v such that $pu = vq$ and $1 \leq \ell(u) < \ell(q)$, where $\ell(x)$ denotes the length of the path $x \in KQ$. We illustrate the definition with the following diagram. (Note that we allow $\ell(v) = 0$ here.)



A path q properly overlaps a path p with overlap pu if q overlaps p and $\ell(v) \geq 1$.

Let $\Lambda = KQ/I$ be a finite-dimensional monomial algebra where I has a minimal set of generators ρ of paths of length at least 2. We fix this set ρ and now recursively define sets $\mathcal{R}^n$ (contained in KQ). Let

$$\begin{aligned} \mathcal{R}^0 &= \text{the set of vertices of } Q, \\ \mathcal{R}^1 &= \text{the set of arrows of } Q, \\ \mathcal{R}^2 &= \rho. \end{aligned}$$

For $n \geq 3$, we say $R^2 \in \mathcal{R}^2$ maximally overlaps $R^{n-1} \in \mathcal{R}^{n-1}$ with overlap $R^n = R^{n-1}u$ if

1. $R^{n-1} = R^{n-2}p$ for some path p;
2. R^2 overlaps p with overlap pu;
3. there is no element of $\mathcal{R}^2$ which overlaps p with overlap being a proper prefix of pu.

The set $\mathcal{R}^n$ is defined to be the set of all overlaps R^n formed in this way.

Each of the elements in $\mathcal{R}^n$ is a path in the quiver Q. We follow the convention that paths are written from left to right. For an arrow α in the quiver Q, we write $\mathfrak{o}(\alpha)$ for the idempotent corresponding to the origin of α and $\mathfrak{t}(\alpha)$ for the idempotent corresponding to the tail of α so that $\alpha = \mathfrak{o}(\alpha)\alpha\mathfrak{t}(\alpha)$. For a path $p = \alpha_1\alpha_2\cdots\alpha_m$, we write $\mathfrak{o}(p) = \mathfrak{o}(\alpha_1)$ and $\mathfrak{t}(p) = \mathfrak{t}(\alpha_m)$.

The importance of these sets $\mathcal{R}^n$ lies in the fact that, for a finite-dimensional monomial algebra $\Lambda = KQ/I$, [2] uses them to give an explicit construction of a minimal projective bimodule resolution $(P^\bullet, \delta^\bullet)$ of Λ, showing that

$$P^n = \oplus_{R^n \in \mathcal{R}^n} \Lambda\mathfrak{o}(R^n) \otimes_K \mathfrak{t}(R^n)\Lambda.$$

We now use these same sets $\mathcal{R}^n$ to define a (D, A)-stacked monomial algebra.

Definition 3.5. ([32, Definition 3.1]) Let $\Lambda = KQ/I$ be a finite-dimensional monomial algebra, where I is an admissible ideal with minimal set of generators ρ. Then Λ is said to be a (D, A)-stacked monomial algebra if there is some $D \geq 2$ and $A \geq 1$ such that, for all $n \geq 2$ and $R^n \in \mathcal{R}^n$,

$$\ell(R^n) = \begin{cases} \frac{n}{2}D & \text{if } n \text{ even}, \\ \frac{(n-1)}{2}D + A & \text{if } n \text{ odd}. \end{cases}$$

In particular all relations in ρ are of length D.

The class of (D, A)-stacked monomial algebras includes the Koszul monomial algebras (equivalently, the quadratic monomial algebras) and the D-Koszul monomial algebras of Berger ([4]). Recall that the Ext algebra $E(\Lambda)$ of Λ is defined by $E(\Lambda) = \operatorname{Ext}^*_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$. It is well-known that the Ext algebra of a Koszul algebra is generated in degrees 0 and 1; moreover the Ext algebra of a D-Koszul algebra is generated in degrees 0, 1 and 2 ([28]). It was shown by Green and Snashall in [33, Theorem 3.6] that, for algebras of infinite global dimension, the class of (D, A)-stacked monomial algebras is precisely the class of monomial algebras Λ where each projective module in the minimal projective resolution of $\Lambda/\mathfrak{r}$ as a right Λ-module is generated in a single degree and where the Ext algebra of Λ is finitely generated as a K-algebra. It was also shown, for a (D, A)-stacked monomial algebra of infinite global dimension, that the Ext algebra is generated in degrees 0, 1, 2 and 3.

Theorem 3.6. [32] *Let $\Lambda = KQ/I$ be a finite-dimensional (D, A)-stacked monomial algebra, where I is an admissible ideal with minimal set of generators ρ. Suppose* char $K \neq 2$ *and* $\operatorname{gldim}\Lambda \geq 4$. *Then there is some integer $r \geq 0$ such that*

$$\mathrm{HH}^*(\Lambda)/\mathcal{N} \cong K[x_1, \ldots, x_r]/\langle x_i x_j \text{ for } i \neq j \rangle.$$

Moreover the degrees of the x_i and the value of the parameter r may be explicitly and easily calculated.

We do not give the full details of the x_i and the parameter r here; they may be found in [32]. However, it is worth remarking that, given any integer $r \geq 0$ and even integers $n_1, \ldots, n_r$, there is a finite-dimensional (D, A)-stacked monomial algebra Λ with

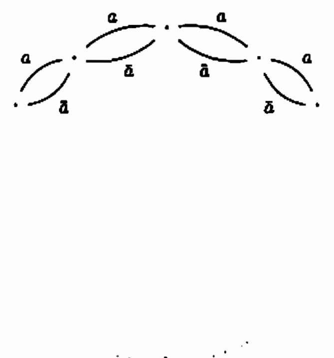
$$\mathrm{HH}^*(\Lambda)/\mathcal{N} \cong K[x_1, \ldots, x_r]/\langle x_i x_j \text{ for } i \neq j \rangle$$

where the degree of x_i is n_i, for all $i = 1, \ldots, r$.

In [25], necessary and sufficient conditions are given for a simple module over a (D, A)-stacked monomial algebra to have trivial variety. Referring back to Theorem 1.1(4), this goes part way to determining necessary and sufficient conditions on any finitely generated module for it to have trivial variety for this class of algebras.

We end this section with a class of selfinjective special biserial algebras Λ_N, for $N \geq 1$, studied by Snashall and Taillefer in [51]. The study of these algebras was motivated by the results of [14] where the algebras Λ_1 arose in the presentation by quiver and relations of the Drinfeld double $\mathcal{D}(\Lambda_{n,d})$ of the Hopf algebra $\Lambda_{n,d}$ where $d|n$. The algebra $\Lambda_{n,d}$ is given by an oriented cycle with n vertices such that all paths of length d are zero. These algebras also occur in the study of the representation theory of $U_q(\mathfrak{sl}_2)$; see work of Patra ([46]), Suter ([53]), Xiao ([56]), and also of Chin and Krop ([11]). The more general algebras Λ_N occur in work of Farnsteiner and Skowroński [22, 23], where they determine the Hopf algebras associated to infinitesimal groups whose principal blocks are tame when K is an algebraically closed field with char $K \geq 3$.

Our class of selfinjective special biserial algebras Λ_N is described as follows. Firstly, for $m \geq 1$, let Q be the quiver with m vertices, labelled $0, 1 \ldots, m-1$, and $2m$ arrows as follows:



Let a_i denote the arrow that goes from vertex i to vertex $i+1$, and let $\bar{a}_i$ denote the arrow that goes from vertex $i+1$ to vertex i, for each $i = 0, \ldots, m-1$ (with the obvious conventions modulo m). Then, for $N \geq 1$, we define Λ_N to be the algebra given by $\Lambda_N = KQ/I_N$ where I_N is the ideal of KQ generated by

$$a_i a_{i+1}, \quad \bar{a}_{i-1}\bar{a}_{i-2}, \quad (a_i\bar{a}_i)^N - (\bar{a}_{i-1}a_{i-1})^N, \quad \text{for } i = 0, 1, \ldots, m-1,$$

and where the subscripts are taken modulo m. We note that, if $N = 1$, then the algebra Λ_1 is a Koszul algebra. (We continue to write paths from left to right.)

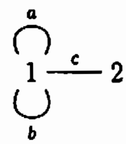
Theorem 3.7. ([51, Theorem 8.1]) *For $m \geq 1$ and $N \geq 1$, let Λ_N be as defined above. Then* $\mathrm{HH}^*(\Lambda_N)$ *is a finitely generated K-algebra. Moreover* $\mathrm{HH}^*(\Lambda_N)/\mathcal{N}$ *is a commutative finitely generated K-algebra of Krull dimension two.*

Furthermore, if $N = 1$ then [51] also showed that the conditions **(Fg1)** and **(Fg2)** hold with $H = \mathrm{HH}^{\mathrm{ev}}(\Lambda_1)$.

4. Counterexample to the conjecture of [50]

The previous section concerned algebras where the conjecture of [50] concerning the finite generation of the Hochschild cohomology ring modulo nilpotence has been shown to hold. In this section we present a counterexample to the conjecture of [50]. In [57], Xu gave a counterexample in the case where the field K has characteristic 2. It can easily be seen, for char $K = 2$, that the category algebra he presented in [57] is isomorphic to the following algebra $\mathcal{A}$ given as a quotient of a path algebra. Moreover, we will show that this algebra $\mathcal{A}$ provides a counterexample to the conjecture irrespective of the characteristic of the field.

Example 4.1. Let K be any field and let $\mathcal{A} = KQ/I$ where Q is the quiver



and $I = \langle a^2, b^2, ab - ba, ac \rangle$.

The rest of this section is devoted to studying this algebra $\mathcal{A}$ and to showing that $\mathrm{HH}^*(\mathcal{A})/\mathcal{N}$ is not finitely generated as an algebra.

We begin by giving an explicit minimal projective resolution $(P^\bullet, d^\bullet)$ for $\mathcal{A}$ as an $\mathcal{A},\mathcal{A}$-bimodule. The description of the resolution given here is motivated by [31] where the first terms of a minimal projective bimodule resolution of a finite-dimensional quotient of a path algebra were determined explicitly from the minimal projective resolution of $\Lambda/\mathfrak{r}$ as a right Λ-module of Green, Solberg and Zacharia in [37]. This same technique for constructing a minimal projective bimodule resolution was used in [26] for any Koszul algebra, and in [51] for the algebras Λ_N which were discussed at the end of Section 3.

From Happel [39], we know that the multiplicity of $\Lambda e_i \otimes_K e_j \Lambda$ as a direct summand of P^n is equal to $\dim \mathrm{Ext}^n_\Lambda(S_i, S_j)$, where S_i is the simple $\mathcal{A}$-module corresponding to the vertex i of Q. Following [31, 37], we start by defining sets g^n in KQ inductively, and then labelling the summands of P^n by the elements of g^n. The set g^0 is determined by the vertices of Q, the set g^1 by the arrows of Q, and the set g^2 by a minimal generating set of the ideal I.

Let

$$g^0 = \{g^0_0 = e_1,\ g^0_1 = e_2\},$$
$$g^1 = \{g^1_0 = a,\ g^1_1 = -b,\ g^1_2 = c\},$$
$$g^2 = \{g^2_0 = a^2,\ g^2_1 = ab - ba,\ g^2_2 = -b^2,\ g^2_3 = ac\}.$$

For $n \geq 3$ and $r = 0, 1, \ldots, n$, let

$$g^n_r = \sum_p (-1)^s p$$

where the sum is over all paths p of length n, written $p = \alpha_1\alpha_2\cdots\alpha_n$ where the α_i are arrows in Q, such that

(i) p contains $n - r$ arrows equal to a and r arrows equal to b, and
(ii) $s = \sum_{\alpha_j = b} j$.

In addition, for $n \geq 3$, define

$$g^n_{n+1} = a^{n-1}c.$$

For $r = 0, 1, \ldots, n$, we have that $g^n_r = e_1 g^n_r e_1$ so we define $\mathfrak{o}(g^n_r) = e_1 = \mathfrak{t}(g^n_r)$. Moreover $\mathfrak{o}(g^n_{n+1}) = e_1$ and $\mathfrak{t}(g^n_{n+1}) = e_2$. Thus

$$P^n = \oplus_{r=0}^{n+1} \mathcal{A}\mathfrak{o}(g^n_r) \otimes_K \mathfrak{t}(g^n_r)\mathcal{A}.$$

To describe the map $d^n\colon P^n \to P^{n-1}$, we first need to write each of the elements g^n_r in terms of the elements of the set g^{n-1}, that is, in terms of $g^{n-1}_0, \dots, g^{n-1}_n$. The following result is straightforward to verify.

Proposition 4.2. *Suppose $n \geq 2$. Then, keeping the above notation,*

$$\begin{array}{llll} g^n_0 & = g^{n-1}_0 a & = a g^{n-1}_0 & \\ g^n_r & = g^{n-1}_r a + (-1)^n g^{n-1}_{r-1} b & = (-1)^r (a g^{n-1}_r + b g^{n-1}_{r-1}) & \text{for } 1 \leq r \leq n-1 \\ g^n_n & = (-1)^n g^{n-1}_{n-1} b & = (-1)^n b g^{n-1}_{n-1} & \\ g^n_{n+1} & = g^{n-1}_0 c & = a g^{n-1}_n. & \end{array}$$

We define the map $d^0\colon P^0 \to \mathcal{A}$ to be the multiplication map. To define d^n for $n \geq 1$, we need one further piece of notation. In describing the image $d^n(\mathfrak{o}(g^n_r) \otimes \mathfrak{t}(g^n_r))$ in the projective module P^{n-1}, we use a subscript under $\otimes$ to indicate the appropriate summand of the projective module P^{n-1}. Specifically, let $- \otimes_r -$ denote a term in the summand of P^{n-1} corresponding to g^{n-1}_r. Nonetheless all tensors are over K; however, to simplify notation, we omit the subscript K. The maps $d^n\colon P^n \to P^{n-1}$ for $n \geq 1$ may now be defined. The proof of the following result is omitted and is similar to those in [31, Proposition 2.8] and [51, Theorem 1.6].

Theorem 4.3. *For the algebra $\mathcal{A}$ of Example 4.1, the sequence $(P^\bullet, d^\bullet)$ is a minimal projective resolution of $\mathcal{A}$ as an $\mathcal{A},\mathcal{A}$-bimodule, where, for $n \geq 0$,*

$$P^n = \bigoplus_{r=0}^{n+1} \mathcal{A}\mathfrak{o}(g^n_r) \otimes \mathfrak{t}(g^n_r)\mathcal{A},$$

the map $d^0 : P^0 \to \mathcal{A}$ is the multiplication map, the map $d^1 : P^1 \to P^0$ is given by $d^1(\mathfrak{o}(g^1_r) \otimes \mathfrak{t}(g^1_r)) =$

$$\begin{cases} \mathfrak{o}(g^1_0) \otimes_0 a - a \otimes_0 \mathfrak{t}(g^1_0) & \text{for } r = 0; \\ -\mathfrak{o}(g^1_1) \otimes_0 b + b \otimes_0 \mathfrak{t}(g^1_1) & \text{for } r = 1; \\ \mathfrak{o}(g^1_2) \otimes_0 c - c \otimes_1 \mathfrak{t}(g^1_2) & \text{for } r = 2; \end{cases}$$

and, for $n \geq 2$, the map $d^n : P^n \to P^{n-1}$ is given by $d^n(\mathfrak{o}(g^n_r) \otimes \mathfrak{t}(g^n_r)) =$

$$\begin{cases} \mathfrak{o}(g^n_0) \otimes_0 a + (-1)^n a \otimes_0 \mathfrak{t}(g^n_0) & \text{for } r = 0; \\ \mathfrak{o}(g^n_r) \otimes_r a + (-1)^n \mathfrak{o}(g^n_r) \otimes_{r-1} b & \\ \qquad + (-1)^{r+n}(a \otimes_r \mathfrak{t}(g^n_r) + b \otimes_{r-1} \mathfrak{t}(g^n_r)) & \text{for } 1 \leq r \leq n-1; \\ (-1)^n \mathfrak{o}(g^n_n) \otimes_{n-1} b + b \otimes_{n-1} \mathfrak{t}(g^n_n) & \text{for } r = n; \\ \mathfrak{o}(g^n_{n+1}) \otimes_0 c + (-1)^n a \otimes_n \mathfrak{t}(g^n_{n+1}) & \text{for } r = n+1. \end{cases}$$

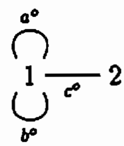
Proposition 4.4. *The algebra $\mathcal{A}$ of Example 4.1 is a Koszul algebra.*

Proof. We apply the functor $\mathcal{A}/\mathfrak{r} \otimes_{\mathcal{A}} -$ to the resolution $(P^\bullet, d^\bullet)$ of Theorem 4.3 to give a (minimal) projective resolution of $\mathcal{A}/\mathfrak{r}$ as a right $\mathcal{A}$-module. Thus $\mathcal{A}/\mathfrak{r}$ has a linear projective resolution, and so $\mathcal{A}$ is a Koszul algebra. □

Since, $\mathcal{A}$ is a Koszul algebra, it now follows from [9] that the image of the ring homomorphism $\phi_{\mathcal{A}/\mathfrak{r}} = \mathcal{A}/\mathfrak{r} \otimes_{\mathcal{A}} - : \mathrm{HH}^*(\mathcal{A}) \to E(\mathcal{A})$ is the graded centre $Z_{\mathrm{gr}}(E(\mathcal{A}))$ of $E(\mathcal{A})$, where $Z_{\mathrm{gr}}(E(\mathcal{A}))$ is the subalgebra generated by all homogeneous elements z such that $zg = (-1)^{|g||z|} gz$ for all $g \in E(\mathcal{A})$. Thus $\phi_{\mathcal{A}/\mathfrak{r}}$ induces an isomorphism

$\mathrm{HH}^*(\mathcal{A})/\mathcal{N} \cong Z_{gr}(E(\mathcal{A}))/\mathcal{N}_Z$, where $\mathcal{N}_Z$ denotes the ideal of $Z_{gr}(E(\mathcal{A}))$ generated by all homogeneous nilpotent elements.

From [30, Theorem 2.2], $E(\mathcal{A})$ is the Koszul dual of $\mathcal{A}$ and is given explicitly by quiver and relations as $E(\mathcal{A}) \cong KQ^{\mathrm{op}}/I^\perp$, where Q is the quiver of $\mathcal{A}$ and $I^\perp$ is the ideal generated by the orthogonal relations to those of I. Specifically, for this example, $E(\mathcal{A})$ has quiver



and $I^\perp = \langle a^o b^o + b^o a^o, b^o c^o \rangle$, where, for an arrow $\alpha \in Q$, we denote by α^o the corresponding arrow in Q^{op}. Moreover, the left modules over $E(\mathcal{A})$ are the right modules over $KQ/\langle ab+ba, bc\rangle$.

It is now easy to calculate $Z_{gr}(E(\mathcal{A}))$ to give the following theorem. The structure of $\mathrm{HH}^*(\mathcal{A})/\mathcal{N}$ for $\operatorname{char} K = 2$ was given by Xu in [57].

Theorem 4.5. *Let $\mathcal{A}$ be the algebra of Example 4.1.*

1.
$$Z_{gr}(E(\mathcal{A})) \cong \begin{cases} K \oplus K[a,b]b & \text{if } \operatorname{char} K = 2 \\ K \oplus K[a^2,b^2]b^2 & \text{if } \operatorname{char} K \neq 2, \end{cases}$$
where b is in degree 1 and ab is in degree 2.

2.
$$\mathrm{HH}^*(\mathcal{A})/\mathcal{N} \cong \begin{cases} K \oplus K[a,b]b & \text{if } \operatorname{char} K = 2 \\ K \oplus K[a^2,b^2]b^2 & \text{if } \operatorname{char} K \neq 2, \end{cases}$$
where b is in degree 1 and ab is in degree 2.

3. $\mathrm{HH}^*(\mathcal{A})/\mathcal{N}$ *is not finitely generated as an algebra.*

This example now raises the new question as to whether we can give necessary and sufficient conditions on a finite-dimensional algebra for its Hochschild cohomology ring modulo nilpotence to be finitely generated as an algebra.

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QF-3' MODULES RELATIVE TO TORSION THEORIES AND OTHERS

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Let R be a ring with identity, and let Mod-R be the category of right R-modules. Let M be a right R-module. We denote by $E(M)$ the injective hull of M. M is called *QF-3' module*, if $E(M)$ is M-torsionless, that is, $E(M)$ is isomorphic to a submodule of a direct product ΠM of some copies of M.

A subfunctor of the identity functor of Mod-R is called a *preradical*. For a preradical σ, $\mathcal{T}_\sigma := \{M \in \text{Mod-}R \ ; \ \sigma(M) = M\}$ is the class of σ-torsion right R-modules, and $\mathcal{F}_\sigma := \{M \in \text{Mod-}R \ ; \ \sigma(M) = 0\}$ is the class of σ-torsionfree right R-modules. A right R-module M is called *σ-injective* (resp. *σ-projective*) if the functor $\text{Hom}_R(\ \ , M)$ (resp. $\text{Hom}_R(M, \ \ ))$ preserves the exactness for any exact sequence $0 \to A \to B \to C \to 0$ with $C \in \mathcal{T}_\sigma$ (resp. $A \in \mathcal{F}_\sigma$). A right R-module M is called *σ-QF-3' module* if $E_\sigma(M)$ is M-torsionless, where $E_\sigma(M)$ is defined by $E_\sigma(M)/M := \sigma(E(M)/M)$.

In this note, we characterize σ-QF-3' modules and give some related facts.

1. QF-3' MODULES RELATIVE TO HEREDITARY TORSION THEORIES

In [1], Y.Kurata and H.Katayama characterized QF-3' modules by using torsion theories. In this section we generarize QF-3' modules by using an idempotent radical. A preradical σ is *idempotent* (resp. *radical*) if $\sigma(\sigma(M)) = \sigma(M)$ (resp. $\sigma(M/\sigma(M)) = 0$) for any module M. For modules M and N, $k_N(M)$ denotes $\cap\{\ker f \ ; \ f \in \text{Hom}_R(M, N)\}$. It is well known that k_A is a radical for any module A and that $\mathcal{T}_{k_A} = \{M \in \text{Mod-}R \, ; \, \text{Hom}_R(M, A) = 0\}$ and $\mathcal{F}_{k_A} = \{M \in \text{Mod-}R \ ; \ M \subseteq \Pi A\}$.

Theorem 1. *Let A be a module and σ a preradical. Then the following conditions* (1), (2) *and* (3) *are equivalent. If σ is an idempotent radical, then* (1), (2), (3) *and* (4) *are equivalent. Moreover if σ is a left exact radical and A is σ-torsion, then all conditions are equivalent.*

(1) *A is a σ-QF-3' module.*

(2) $k_A(E_\sigma(A)) = 0$

(3) $k_A(\_) = k_{E_\sigma(A)}(\_)$

(4) *$k_A(N) = N \cap k_A(M)$ holds for any module M and any submodule N such that M/N is σ-torsion.*

(5) *Let M be a module and N a submodule of M such that M/N is σ-torsion. Then for any nonzero $f \in \text{Hom}_R(N, A)$, there exists $p \in \text{Hom}_R(A, A)$ and $\bar{f} \in \text{Hom}_R(M, A)$ such that $p \cdot f = \bar{f} \cdot i \neq 0$.*

(6) *Let $0 \to N \xrightarrow{f} M \to L \to 0$ be an exact sequence such that L is σ-torsion. If $\text{Hom}_R(f, A) = 0$, then $\text{Hom}_R(N, A) = 0$.*

(7) *For any module M and a submodule N of M,*

The detailed version of this paper will be submitted for publication elsewhere.

(*i*) *If* $M \in \mathcal{T}_{k_A}$ *and* $M/N \in \mathcal{T}_\sigma$, *then* $N \in \mathcal{T}_{k_A}$.

(*ii*) *If* $N \in \mathcal{F}_{k_A}$ *and* $M/N \in \mathcal{F}_{k_A} \cap \mathcal{T}_\sigma$, *then* $M \in \mathcal{F}_{k_A}$.

(8) *If* $M \in \mathcal{F}_{k_A}$, *then* $E_\sigma(M) \in \mathcal{F}_{k_A}$.

(9) *If* N *is an essential submodule of a module* M *such that* $M/N \in \mathcal{T}_\sigma$ *and* $N \in \mathcal{F}_{k_A}$, *then* $M \in \mathcal{F}_{k_A}$.

As an application of Theorem 1, we give a characterization of the ring having the property that a right maximal quotient ring Q is torsionless.

Corollary 2. *Let* Q *be a maximal right quotient ring of* R. *Then the following conditions are equivalent.*

(1) Q *is torsionless (i.e.,* $Q \subset \Pi R$*).*

(2) $k_R(Q) = 0$

(3) $k_R(\_) = k_Q(\_)$

(4) $k_R(N) = N \cap k_R(M)$ *holds for a module* M *and any submodule* N *of* M *such that* $\mathrm{Hom}_R(M/N, E(R)) = 0$.

Proposition 3. *If* σ *is a left exact radical,* (7) *of (i) is equivalent to the condition* (10) $\mathcal{T}_{k_A} = \mathcal{T}_{k_{E_\sigma(A)}}$.

For a module M, $Z(M)$ denote the singular submodule of M, that is ,$Z(M) := \{m \in M \ ; \ (0:m)$ is essential in $R\}$, where $(0:m) = \{r \in R \ ; \ mr = 0\}$.

Proposition 4. *If* σ *is a left exact radical and* $A \in \mathcal{T}_\sigma \cap \mathcal{F}_Z$, *then* (7) *of (i) is equivalent to the condition* (1), *that is,* $E_\sigma(A) \subseteq \Pi A$ *is equivalent to the condition that* $\mathcal{T}_{k_A}$ *is closed under taking* σ*-dense submodules.*

A module N is called a *σ-essential extension* of M if N is an essential submodule of M such that M/N is σ-torsion.

Lemma 5. *Let* σ *be an idempotent radical and* M *a* σ*-essential extension of a module* N. *Then* $E_\sigma(M) = E_\sigma(N)$ *holds.*

Proposition 6. *Let* σ *be an idempotent radical. Then the class of* σ*-QF-3' modules is closed under taking* σ*-essential extensions.*

2. σ-LEFT EXACT PRERADICALS AND σ-HEREDITARY TORSION THEORIES

A preradical t is *left exact* if $t(N) = N \cap t(M)$ holds for any module M and any submodule N of M. In this section we generalize left exact preradicals by using torsion theories.

Let σ be a preradical. We call a preradical t *σ-left exact* if $t(N) = N \cap t(M)$ holds for any module M and any submodule N of M with $M/N \in \mathcal{T}_\sigma$. If a module A is σ-QF-3' and $t = k_A$, then t is a σ-left exact radical. Now we characterize σ-left exact preradicals.

Lemma 7. *For a preradical* t *and* σ, *let* $t_\sigma(M)$ *denote* $M \cap t(E_\sigma(M))$ *for any module* M. *Then* $t_\sigma(M)$ *is uniquely determined for any choice of* $E(M)$.

Lemma 8. *Let* t *be a preradical and* σ *an idempotent radical. Then* t_σ *is a* σ*-left exact preradical.*

Theorem 9. *Let σ be an idempotent radical. We consider the following conditions on a preradical t. Then the implications (5) $\leftarrow$ (1) $\Leftrightarrow$ (2) $\rightarrow$ (3) $\Leftrightarrow$ (4) hold. If t is a radical, then (4) $\rightarrow$ (1) holds. If t is an idempotent preradical and σ is left exact, then (5)(i) $\rightarrow$ (1) holds. Thus if t is an idempotent radical and σ is a left exact radical, then all conditions are equivalent.*

(1) *t is a σ-left exact preradical.*

(2) *$t(M) = M \cap t(E_\sigma(M))$ holds for any module M.*

(3) *$\mathcal{F}_t$ is closed under taking σ-essential extension, that is, if M is an essential extension of a module $N \in \mathcal{F}_t$ with $M/N \in \mathcal{T}_\sigma$, then $M \in \mathcal{F}_t$.*

(4) *$\mathcal{F}_t$ is closed under taking σ-injective hulls, that is, if $M \in \mathcal{F}_t$, then $E_\sigma(M) \in \mathcal{F}_t$.*

(5) *For any module M and a submodule N of M,*

(*i*) *$\mathcal{T}_t$ is closed under taking σ-dense submodules, that is, if $M \in \mathcal{T}_t$ and $M/N \in \mathcal{T}_\sigma$, then $N \in \mathcal{T}_t$.*

(*ii*) *$\mathcal{F}_t$ is closed under taking σ-extensions, that is, if $N \in \mathcal{F}_t$ and $M/N \in \mathcal{F}_t \cap \mathcal{T}_\sigma$, then $M \in \mathcal{F}_t$.*

A *torsion theory* for Mod-R is a pair $(\mathcal{T}, \mathcal{F})$ of classes of objects of Mod-R satisfying the following three conditions.

(*i*) $\mathrm{Hom}_R(T, F) = 0$ for all $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

(*ii*) If $\mathrm{Hom}_R(M, F) = 0$ for all $F \in \mathcal{F}$, then $M \in \mathcal{T}$.

(*iii*) If $\mathrm{Hom}_R(T, N) = 0$ for all $T \in \mathcal{T}$, then $N \in \mathcal{F}$.

We put $t(M) = \sum_{\mathcal{T} \ni N \subset M} N \ (= \bigcap_{M/N \in \mathcal{F}} N)$, then $\mathcal{T} = \mathcal{T}_t$ and $\mathcal{F} = \mathcal{F}_t$ hold.

For a torsion theory $(\mathcal{T}, \mathcal{F})$, if $\mathcal{T}$ is closed under taking submodules, then $(\mathcal{T}, \mathcal{F})$ is called a *hereditary torsion theory.* $\mathcal{T}$ is closed under taking submodules if and only if $\mathcal{F}$ is closed under taking injective hulls.

Now we call $(\mathcal{T}, \mathcal{F})$ a *σ-hereditary torsion theory* if $\mathcal{T}$ is closed under taking σ-dense submodules. If σ is a left exact radical, $\mathcal{T}$ is closed under taking σ-dense submodules if and only if $\mathcal{F}$ is closed under taking σ-injective hulls by Theorem 9.

Proposition 10. *Let t be an idempotent preradical and σ a radical such that $\mathcal{F}_\sigma$ is included $\mathcal{F}_t$. If $\mathcal{F}_t$ is closed under taking σ-injective hulls, then $\mathcal{F}_t$ is closed under taking injective hulls.*

Thus if σ is a left exact radical, $\mathcal{T}_\sigma \supseteq \mathcal{T}_t$ and $(\mathcal{T}_t, \mathcal{F}_t)$ is a σ-hereditary torsion theory, then $(\mathcal{T}_t, \mathcal{F}_t)$ is a hereditary torsion theory.

Proposition 11. *If $\sigma(M)$ contains the singlar submodule $Z(M)$ for any module M, then a σ-left exact preradical is a left exact preradical.*

Theorem 12. *Let σ be a left exact radical. Then $(\mathcal{T}, \mathcal{F})$ is σ-hereditary if and only if there exists a σ-injective (σ-QF-3') module Q such that $\mathcal{T} = \{M \in$ Mod-R ; $\mathrm{Hom}_R(M, Q) = 0\}$.*

Proposition 13. *Let σ be an idempotent radical and $(\mathcal{T}, \mathcal{F})$ a σ-hereditary torsion theory, where $\mathcal{T} = \{M \in$ Mod-R ; $\mathrm{Hom}_R(M, Q) = 0\}$ for some σ QF-3' module Q in $\mathcal{F}$. Let M be a σ-torsion module. Then M is in $\mathcal{F}$ if and only if M is contained in a direct product of some copies of Q.*

3. CQF-3' MODULES RELATIVE TO TORSION THEORIES

A preradical t is called *epi-preserving* if $t(M/N) = (t(M) + N)/N$ for any module M and any submodule N of M. A short exact sequence $0 \to K(M) \to P(M) \to M \to 0$ is a *projective cover* of a module M if $P(M)$ is projective and $K(M)$ is small in $P(M)$.

In [2], F.F. Mbuntum and K. Varadarajan dualized QF-3' modules and characterized them. Let M be a module with a projective cover. M is called a *CQF-3' module* if $P(M)$ is M-generated, that is, $P(M)$ is isomorphic to a homomorphic image of a direct sum $\oplus M$ of some copies of M. In this section we generalize CQF-3' modules and characterize them.

A short exact sequence $0 \to K_\sigma(M) \to P_\sigma(M) \to M \to 0$ is called *σ-projective cover* of a module M if $P_\sigma(M)$ is σ-projective and $K_\sigma(M)$ is σ-torsion and small in $P_\sigma(M)$. If σ is an idempotent radical and a module M has a projective cover, then M has a σ-projective cover and it is given $K_\sigma(M) = k(M)/\sigma(K(M)), P_\sigma(M) = P(M)/\sigma(K(M))$. Now we call a module M with a projective cover a *σ-CQF-3'module* if $P_\sigma(M)$ is M-generated. Let $t_M(N)$ denote the sums of images of all homomorphisms from M to N for a module M and a module N.It is well known that t_A is an idempotent preradical for any module A and $\mathcal{T}_{t_A} = \{M \in \text{Mod-}R\ ;\ \oplus\text{A} \to \text{M} \to 0\}$ and $\mathcal{F}_{t_A} = \{M \in \textit{Mod-R}\ ;\ \text{Hom}(A, M) = 0\}$.

Theorem 14. *Let σ be a preradical, and suppose that a module A has a σ-projective cover $0 \to K_\sigma(A) \to P_\sigma(A) \to A \to 0$. Consider the following conditions.*

(1) *$P_\sigma(A)$ is a σ-CQF-3' module.*

(2) $t_A(P_\sigma(A)) = P_\sigma(A)$

(3) $t_A(\_) = t_{P_\sigma(A)}(\_)$

(4) *$t_A(\_)$ is a σ-epi-preserving preradical, that is, $t_A(M/N) = (t_A(M) + N)/N$ holds for any module M and any submodule $N \in \mathcal{F}_\sigma$.*

(5) (*i*) *$\mathcal{T}_{t_A}$ is closed under taking $\mathcal{F}_\sigma$-extensions, that is, $t_A(M) = M$ holds for any module M and any submodule N of M such that $M/N \in \mathcal{T}_{t_A}$ and $N \in \mathcal{F}_\sigma \cap \mathcal{T}_{t_A}$.*

(*ii*) *$\mathcal{F}_{t_A}$ is closed under taking $\mathcal{F}_\sigma$-factor modules, that is, $M/N \in \mathcal{F}_{t_A}$ holds for any module $M \in \mathcal{F}_{t_A}$ and any submodule $N \in \mathcal{F}_\sigma$ of M.*

(6) *$\mathcal{T}_{t_A}$ is closed under taking σ-projective covers, that is, $P_\sigma(M) \in \mathcal{T}_{t_A}$ holds for any module $M \in \mathcal{T}_{t_A}$.*

(7) *$\mathcal{T}_{t_A}$ is closed under taking σ-coessential extensions, that is, for any module M if there exists a small submodule N in $\mathcal{F}_\sigma$ such that $M/N \in \mathcal{T}_{t_A}$ then M is in $\mathcal{T}_{t_A}$.*

(8) *If* $\text{Hom}_R(A, f) = 0$, *then* $\text{Hom}_R(A, M/N) = 0$ *holds for any module M and any submodule $N \in \mathcal{F}_\sigma$.*

Then (1) $\to$ (2) $\to$ (3) $\to$ (1) *and* (4) $\to$ (1) *hold. If σ is idempotent, then* (3) $\to$ (4), (1) $\to$ (8) *and* (6) $\to$ (5), (7) *hold. If σ is a radical, then* (7) $\to$ (6), (4) $\to$ (2), (6) *hold. If σ is an epi-preserving radical and A is in $\mathcal{F}_\sigma$, then* (8) $\to$ (5) *holds, moreover if σ is idempotent then* (5) $\to$ (2) *holds.*

Thus if σ is an epi-preserving idempotent radical and A is in $\mathcal{F}_\sigma$, all conditions are equivalent.

Proposition 15. *Let σ be an epi-preserving idempotent radical. Then the following conditions on a module A are equivalent.*

(1) $\mathcal{F}_{t_A}$ *is closed under taking* $\mathcal{F}_\sigma$*-factor modules.*
(2) $\mathcal{F}_{t_A} = \mathcal{F}_{t_{P_\sigma(A)}}$

Lemma 16. *Let* σ *be an idempotent radical. If* N *is in* $\mathcal{F}_\sigma$ *and is a small submodule of* M*, then* $P_\sigma(M/N) \cong P_\sigma(M)$ *holds.*

Proposition 17. *Let* σ *be an idempotent radical. The class of* σ *CQF-3' modules is closed under taking* σ*-coessntial extensions, that is, if a module* M *has a small submodule* $N \in \mathcal{F}_\sigma$ *such that* M/N *is a* σ*-CQF-3' module, then* M *is also a* σ*-CQF-3' module.*

4. σ-EPI-PRESERVING PRERADICALS AND σ-COHEREDITARY TORSION THEORIES

In this section we characterize σ-epi-preserving preradicals when R is a right perfect ring.

Theorem 18. *Let* R *be a right perfect ring and* σ *an idempotent radical. Consider the following conditions on a preradical* t.

(1) t *is an* σ*-epi-preserving preradical, that is,* $t(M/N) = (t(M) + N)/N$ *holds for a module* M *and any submodule* $N \in \mathcal{F}_\sigma$ *of* M.

(2) $\mathcal{T}_t$ *is closed under taking* σ*-coessential extensions, that is, for any module* M *if there exists a small submodule* N *in* $\mathcal{F}_\sigma$ *such that* $M/N \in \mathcal{T}_t$ *then* M *is in* $\mathcal{T}_t$.

(3) $\mathcal{T}_t$ *is closed under taking* σ*-projective covers, that is,* $P_\sigma(M) \in \mathcal{T}_t$ *holds for any module* $M \in \mathcal{T}_t$.

(4) (*i*) $\mathcal{F}_t$ *is closed under taking* $\mathcal{F}_\sigma$*-factor modules, that is,* $M/N \in \mathcal{F}_t$ *holds for any module* $M \in \mathcal{F}_t$ *and any submodule* $N \in \mathcal{F}_\sigma$ *of* M.

(*ii*) $\mathcal{T}_t$ *is closed under taking* $\mathcal{F}_\sigma$*-extensions, that is,* $t(M) = M$ *holds for any module* M *and any submodule* N *of* M *such that* $M/N \in \mathcal{T}_t$ *and* $N \in \mathcal{F}_\sigma \cap \mathcal{T}_t$.

Then (4) $\leftarrow$ (1) $\rightarrow$ (2) $\Leftrightarrow$ (3) *hold. If* t *is an idempotent preradical, then* (3) $\rightarrow$ (1) *holds. If* σ *is an epi-preserving preradical and* t *is a radical, then* (4) $\rightarrow$ (1) *holds. Thus if* σ *is an epi-preserving idempotent radical and* t *is an idempotent radical, then all conditions are equivalent.*

We call a torsion theory $(\mathcal{T}, \mathcal{F})$ *σ-cohereditary torsion theory* if $\mathcal{F}$ is cosed under taking $\mathcal{F}_\sigma$-factor modules for an idempotent radical σ.

Theorem 19. *Let* R *be a right perfect ring and* σ *an epi-preserving idempotent radical. Then a torsion theory* $(\mathcal{T}, \mathcal{F})$ *is* σ*-cohereditary if and only if there exists an* σ*-projective (*σ*-CQF-3') module* Q *such that* $\mathcal{F} = \{M \in \text{Mod-}R\ ;\ \text{Hom}_R(Q, M) = 0\}$.

Proposition 20. *Let* R *be a right perfect ring,* σ *be an idempotent radical and* $(\mathcal{T}, \mathcal{F})$ *be a* σ*-cohereditary torsion theory, where* $\mathcal{F} = \{M \in \text{Mod-}R\ ;\ \text{Hom}_R(Q, M) = 0\}$ *for some* σ*-CQF-3' module* $Q \in \mathcal{T}$*. Let* M *be a* σ*-torsionfree module. Then* $M \in \mathcal{T}$ *if and only if* M *is generated by* Q.

5. σ-STABLE TORSION THEORY AND σ-COSTABLE TORSION THEORY

A torsion theory $(\mathcal{T}_t, \mathcal{F}_t)$ is called *stable* if $\mathcal{T}_t$ is closed under taking injective hulls. In this section we generalize stable torsion theory by using torsion theories.

Proposition 21. *Let σ be an idempotent radical and L a submodule of a module M. Then the implications (1) $\to$ (2) $\to$ (3) hold. Moreover, if σ is a left exact radical, then (3) $\to$ (1) holds.*

(1) *L is σ-complemented in M, that is, there exists a submodule K of M such that L is maximal in $\Gamma_K = \{L_i \ ; \ L_i \subseteq M, L_i \cap K = 0, M/(L_i + K) \in \mathcal{T}_\sigma\}$*

(2) *$L = E_\sigma(L) \cap M$.*

(3) *L is σ-essentially closed in M, that is, there is no σ-essential extension of L in M.*

We call a preradical t *σ-stable* if $\mathcal{T}_t$ is closed under taking σ-injective hulls. We put $\mathcal{X}_t(M) := \{X \ ; \ M/X \in \mathcal{T}_t\}$ and $N \cap \mathcal{X}_t(M) := \{N \cap X \ ; \ X \in \mathcal{X}_t(M)\}$.

Theorem 22. *Let t be an idempotent preradical and σ an idempotent radical. Then the following conditions (1), (2) and (3) are equivalent. Moreover, if σ is left exact and $\mathcal{T}_t$ is closed under taking σ-dense submodules, then all the following conditions are equivalent.*

(1) *t is σ-stable, that is, $\mathcal{T}_t$ is closed under taking σ-injective hulls.*

(2) *The class of σ-injective modules are closed under taking the unique maximal t-torsion submodules, that is, $t(M)$ is σ-injective for any σ-injective module M.*

(3) *$E_\sigma(t(M)) \subset t(E_\sigma(M))$ holds for any module M.*

(4) *$\mathcal{T}_t$ is closed under taking σ-essential extensions.*

(5) *If M/N is σ-torsion, then $N \cap \mathcal{X}_t(M) = \mathcal{X}_t(N)$ holds.*

(6) *For any module M, $t(M)$ is σ-complemented in M.*

(7) *For any module M, $t(M) = E_\sigma(t(M)) \cap M$ holds.*

(8) *For any module M, $t(M)$ is σ-essentially closed in M.*

(9) *For any σ-injective module E with $E/t(E) \in \mathcal{T}_\sigma$, $t(E)$ is a direct summand of E.*

(10) *$E_\sigma(t(M)) = t(E_\sigma(M))$ holds for any module M.*

If $\mathcal{T}_t$ is closed under taking σ-dense submodules, then (1) $\to$ (4) $\to$ (5) hold. Moreover, if σ is left exact, then (1) $\to$ (6) and (3) $\to$ (10) hold.

It is well known that if R is right noetherian, t is stable if and only if every indecomposable injective module is t-torsion or t-torsionfree. By using Theorem 1 in [3], we generalized this as follows.

Theorem 23. *Let t be an idempotent radical and σ a left exact radical. Then*

(1) *If t is σ-stable, then $(*)$ every indecomposable σ-injective module E with $E/T(E) \in \mathcal{T}_\sigma$ is either t-torsion or t-torsionfree.*

(2) *If the ring R satisfies the condition $(*)$ and the ascending chain conditions on σ-dense ideals of R, then $\mathcal{T}_t \cap \mathcal{T}_\sigma$ is closed under taking σ-injective hulls.*

We now dualize σ-stable torsion theory. Let R be a right perfect ring. We call a preradical t *σ-costable* if $\mathcal{F}_t$ is closed under taking σ-projective covers.

Theorem 24. *Let σ be an idempotent radical. Then a radical t is σ-costable if and only if the class of σ-projective modules is closed under taking the unique maximal t-torsionfree factor modules, that is, $P/t(P)$ is σ-projective for any σ-projective module P.*

6. σ-SINGULAR SUBMODULES

Let σ be a left exact radical. For a module M we put $Z_\sigma(M) := \{m \in M \ ; \ (0 : m)$ is σ-essential in $R\}$ and call it *σ-singular submodule of M*. Since $R/(0 : m) \in \mathcal{T}_Z \cap \mathcal{T}_\sigma$, then

$Z_\sigma(M) \subseteq Z(M) \cap \sigma(M) = Z(\sigma(M)) = \sigma(Z(M))$, and so $Z_\sigma(M) = \{m \in M \ ; \ mR \in \mathcal{T}_Z \cap \mathcal{T}_\sigma\}$. Since Z and σ is left exact, Z_σ is also left exact. We will call M *σ-singular* (resp. *σ-nonsingular*) if $Z_\sigma(M) = M$ (resp. $Z_\sigma(M) = 0$).

Proposition 25. *Let σ be an idempotent radical and E a σ-nonsingular module and $\mathcal{T} = \{M \in \text{Mod-}R \ ; \ \text{Hom}_R(M, E) = 0\}$. Then $\mathcal{T}$ is closed under taking σ-essential extensions. Therefore a torsion theory $(\mathcal{T}, \mathcal{F})$ is σ-stable, where $\mathcal{F} = \{M \in \text{Mod-}R \ ; \ \text{Hom}_R(X, M) = 0$ for any $X \in \mathcal{T}\}$.*

Proposition 26. *Let σ a left exact radical. Then the following facts hold.*

(1) *If N is σ-essential in M, then $Z_\sigma(M/N) = M/N$.*

(2) *A right ideal of R is σ-essential in R if and only if $Z_\sigma(R/I) = R/I$.*

(3) *Let M be a σ-nonsingular module and N a submodule of M. Then N is σ-essential in M if and only if $Z_\sigma(M/N) = M/N$.*

(4) *For a module M, $Z_\sigma(M/Z_\sigma(M)) = M/Z_\sigma(M)$ holds if and only if $Z_\sigma(M)$ is σ-essential in M.*

(5) *For a simple right R-module S, S is σ-nonsingular if and only if S is σ-torsionfree or projective.*

(6) *If R is σ-nonsingular, then Z_σ is left exact radical.*

(7) *If M/N is σ-nonsingular for a module M and a submodule N of M, then N is σ-complemented in M. If M is σ-nonsingular, then the converse holds.*

7. σ-SMALL AND σ-RADICAL

Let σ be a left exact radical. A submodule N of a module M is called *σ-dense* in M if M/N is σ-torsion. A module M is called *σ-cocritical* if M is σ-torsionfree and L is σ-dense in M for any nonzero submodule L of M. It is well known that nonzero submodule of σ-cocritical module M is essential in M. A module M is called *σ-noetherian* if for every ascending chain $I_1 \subseteq I_2 \subseteq I_3 \subseteq I_4 \subseteq \cdots\cdots \subseteq M$, (where $\cup I_j$ is σ-dense in M), there exists a positive integer k such that I_k is σ-dense in M. Let $J_\sigma(M)$ denote $\cap N_i(M/N_i$ is σ-cocritical).

Now we define σ-small submodule as follows. A submodule N of a module M is called *σ-small* in M if $M/(N + X) \in \mathcal{T}_\sigma$ implies $M/X \in \mathcal{T}_\sigma$ for any submodule X of M.

Theorem 27. *$J_\sigma(M)$ contains $\sum N(N$ is σ-small in $M)$. Conversely if M be a σ-noetherian module, then $J_\sigma(M)$ coincides with $\sum N(N$ is σ-small in $M)$.*

Remark 28. We can see in [4] that the definition of σ-small is different from ours.

(B.A.Benander's definition). N is σ-small in M if $M/(N' + X) \in \mathcal{T}_\sigma$ and $M/X \in \mathcal{F}_\sigma$, then $M = X$ for any X of M, where $\sigma(M/N) = N'/N$.

Benander's definition of σ-small is a stronger condition than ours.

In fact, if $M/(N + X) \in \mathcal{T}_\sigma$, then $M/(N' + X) \in \mathcal{T}_\sigma$. We put $X'/X := \sigma(M/X)$. Then $M/X' \in \mathcal{F}_\sigma$. Since $M/(N' + X) \in \mathcal{T}_\sigma$, $M/(N' + X') \in \mathcal{T}_\sigma$. Thus $M = X'$, and so $M/X \in \mathcal{T}_\sigma$, as desired.

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ABOUT DECOMPOSITION NUMBERS OF J_4

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ABSTRACT. The decomposition numbers with some unknown parameters of non-principal blocks of the largest Janko group J_4 [5] for characteristic 3 are determined. We also concerned with the decomposition numbers of maximal 2-local subgroups [6] of J_4 in odd characteristics. We used the character table library in GAP[4]

Key Words: Sporadic group J_4, Green correspondence, Modular representation.

1. NOTATION

Let G be a finite group. Let p be an odd prime such that p devides the order of G. We denote by $Bl_p^+(G)$ a set of p-block of G with positive defect. For $A \in Bl_p^+(G)$, we denote by $\mathrm{Irr}(A)$ a set of irreducible ordinary characters in A and by $\mathrm{IBr}(A)$ a set of irreducible Brauer characters in A. Let $k(A)$ and $l(A)$ be numbers of irrecucible characters in $\mathrm{Irr}(A)$ and $\mathrm{IBr}(A)$, respectively. Let I_G be the trivial character of G. Let $b_0(G)$ be the principal block of G i.e. $b_0(G) \in Bl_p^+(G)$ and $I_G \in \mathrm{Irr}(b_0(G))$. We denote by $D(A)$ the decomposition matrix of A with respect to $\mathrm{Irr}(A) = \{\chi_1, \cdots, \chi_{k(A)}\}$ and $\mathrm{IBr}(A) = \{\varphi_1, \cdots, \varphi_{l(A)}\}$. So $D(A)$ is the $k(A) \times l(A)$-matrix $\{d_{ij}\}$ such that $\chi_i = \sum_{j=1}^{l(A)} d_{ij}\varphi_j$ for $i = 1, \ldots, k(A)$ on p'-elements in G.

Let k be an algebraically closed field. Let H be a subgroup of G. We called a kG-module M is a trivial source module if M is a direct summand of the induced module of the trivial kH-module. Since trivial source modules have some good property, it is important to find may trivial source modules. In particular, simple trivial source modules are very important.

2. FONG'S THEOREM

Let X be a normal p'-subgroup of G. Let b be a p-block of X. Since X is p'-group, $\mathrm{Irr}(b)$ has only one irreducible character ξ. Let $T = T(b)$ be an inertial group of b in G. If a p-block B of T is a direct summand of $e_b kT$ as a k-algebra, we call that B covers b. We denote by $Bl(T\,|\,b)$ the set of all p-blocks of T which cover b. In [3], Fong showed the following two theorems.

Theorem 1. *(2B in* [3]*) Let A be a p-block in $Bl(G\,|\,b)$. Then there is a p-block B in $Bl(T\,|\,b)$, such that the following are true:*

(i) *A and B have a defect group in common.*

The detailed version of this paper will be submitted for publication elsewhere.

(ii) *There is a 1-1 height-preserving correspondence between the irreducible ordinary characters of A and B.*
(iii) *There is a 1-1 correspondence between the irreducible modular characters of A and B.*
(iv) *With respect to these correspondences of characters, the matrices of decomposition numbers and Cartan invariants of A and B are same.*

Let s be the Schur multiplier of T/X.

Theorem 2. *(2D in [3]) Let B be a p–block in $Bl(T\,|\,b)$. Then there is a group $\widehat{T}$ with a cyclic normal p'-subgroup Z and p-block $\widehat{B}$ in $Bl(\widehat{T}\,|\,\widehat{b})$ where $\widehat{b}$ is a p–block of Z such that the following are true:*

(i) *B and $\widehat{B}$ have isomorphic defect groups.*
(ii) *There is a 1-1 height-preserving correspondence between the irreducible ordinary characters of B and $\widehat{B}$.*
(iii) *There is a 1-1 correspondence between the irreducible modular characters of B and $\widehat{B}$.*
(iv) *With respect to these correspondences of characters, the matrices of decomposition numbers and Cartan invariants of B and $\widehat{B}$ are same.*

The group $\widehat{T}$ has the following structure:

(a) *Z is the center of $\widehat{T}$.*
(b) *$\widehat{T}/Z \cong T/X$.*
(c) *The order of Z is s.*

In case that the irreducible character ξ is linear and T is a semidirect product of X with T/X. It is easy to see that the p–block $\widehat{b}$ is the principal block. Thus we can identify p–blocks in $Bl(\widehat{T}\,|\,\widehat{b})$ with p–blocks in $Bl(\widehat{T}/Z) = Bl(T/X)$. So the next corollary follows.

Corollary 3. *If ξ is a linear character and T is a semidirect product of X with T/X, then there is a bijection between p–blocks in $Bl(T\,|\,b)$ and $Bl(T/X)$ such that the same statements in Theorem 2 hold.*

3. Decomposition Matrix of J_4

Let G be the largest Janko group J_4. The order of G is $2^{31}\cdot 3^3\cdot 5\cdot 7\cdot 11^3\cdot 23\cdot 29\cdot 31\cdot 37\cdot 43$. There are non-conjugate involutions s, t in G such that $K := C_G(s) \cong 2_+^{1+12}.3M_{22}{:}2$, $C_G(t) \cong 2^{11}{:}M_{22}{:}2$. The centralizer $C_G(t)$ is contained in a subgroup $H \cong 2^{11}{:}M_{24}$. The subgroups H and K are the maximal subgroups of G.

In [6], B. Kleidman and R. A. Wilson investigate these groups in detail. The character tables of these groups are found by GAP[4]. We apply Fong's theorem for getting the decomposition numbers of these maximal 2-local subgroups H and K.

Proposition 4. *Let p be an odd prime. Then all $D(B)$ and $D(C)$ where $B \in Bl_p^+(H)$ and $C \in Bl_p^+(K)$ are determined.*

In case that $p = 3$, let
$Bl_3^+(H) = \{B_{3a},\ B_{3b},\ B_2,\ B_{1a}, \cdots, B_{1p}\}$

$Bl_3^+(K) = \{C_{3a},\ C_{3b},\ C_{2a},\ C_{2b}\ C_{1a}, \dots, C_{1h}\}$
$Bl_3^+(6.M_{22}{:}2) = \{X_{3a},\ X_{3b},\ X_{2a},\ X_{2b}\ X_{1a}, \dots, X_{1d}\}$
where indeces of each p-blocks indicate its defect.

(1) $D(B_{1\bullet})$ and $D(C_{1\bullet})$ are $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(2) $D(B_{3a}) = D(b_0(M_{24}))$, $D(B_{3b}) = D(b_0(M_{12}.2))$, $D(B_2) = D(b_0(2^4{:}A_8))$

(3) $D(C_{3a}) = D(X_{3a})$, $D(C_{3b}) = D(X_{3b})$, $D(C_{2a}) = D(X_{2a})$, $D(C_{2b}) = D(X_{2b})$

From above decomposition matrices, I can calculate projective indecomposable characters of H and K. By inducing these characters to G, we can get projective characters of G. Using tensor of characters and Green correspondence between G and H or G and K, we can prove the following proposition.

Proposition 5. *Let $p = 3$ then $Bl_p^+(G) = \{A_{3a}, A_{3b}, A_2, A_{1a}, A_{1b}, A_{1c}, A_{1d}\}$. The almost all decomposition numbers (=entries of decomposition matrix) of non-principal blocks are determined the followsing. The indeces which we put on the irreducible ordinary characters are same one in* [2].

(1) $D(A_{1\bullet})$ *are* $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ *or* $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

$$(2)\ D(A_2) = \begin{pmatrix} 1 & & & & \\ 0 & 1 & & & \\ 0 & 0 & 1 & & \\ 0 & 0 & 0 & 1 & \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

where $\mathrm{Irr}(A_2) = \{\chi_{14}, \chi_{21}, \chi_{25}, \chi_{27}, \chi_{28}, \chi_{30}, \chi_{31}, \chi_{35}, \chi_{41}\}$.

$$(3)\ D(A_{3b}) = \left(\begin{array}{ccccc|cccc} 1 & & & & & & & & \\ 0 & 1 & & & & & & & \\ 1 & 0 & 1 & & & & & & \\ 0 & 1 & 0 & 1 & & & & & \\ 0 & 0 & 1 & 0 & 1 & & & & \\ 0 & 0 & 0 & 1 & 1 & & & & \\ \hline 0 & 0 & 0 & 0 & \alpha & 1 & & & \\ 0 & 0 & 0 & 0 & \alpha+1 & 1 & & & \\ 0 & 0 & 0 & 0 & \alpha+1 & 1 & & & \\ 0 & 0 & 1 & 1 & \beta & 0 & 1 & & \\ 0 & 0 & 0 & 0 & \alpha+\beta+\gamma+1 & 1 & 1 & 1 & \\ 0 & 0 & 0 & 0 & \alpha+\beta+\gamma+1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & \beta+2\gamma+1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2\beta+2\gamma+2 & 0 & 2 & 1 & 1 \end{array}\right)$$

where $0 \leqq \alpha \leqq 3$, $0 \leqq \beta \leqq 7$, $0 \leqq \alpha+\beta+\gamma \leqq 15$, $0 \leqq \beta+\gamma \leqq 12$ *and* $\mathrm{Irr}(A_{3b}) = \{\chi_2, \chi_3, \chi_{12}, \chi_{13}, \chi_{17}, \chi_{18}, \chi_{22}, \chi_{23}, \chi_{24}, \chi_{26}, \chi_{38}, \chi_{39}, \chi_{44}, \chi_{50}\}$.

4. Green correspondence and Trivial Source Module

In G, there are 2 simple modules M_a and M_b with dimension 1,333. In this section, we see that these modules are trivial source modules.

Let χ_2 and χ_3 in $\mathrm{Irr}(A_{3b})$ of degree 1,333. Then these two characters are corresponding to M_a and M_b.

Let P be a Sylow 3-subgroup which is isomorphic to the extraspecial group of the order 27. The center of P denote by $Z := Z(P)$. We can get the following inclusion.

$$Z \subset P \subset N_G(P) \cong (2 \times P : 8) : 2 \subset N_G(Z) \cong 6.M_{22}{:}\, 2 \subset K \subset G$$

Let F and f be the Green correspondence with respect to $(G, P, N_G(Z))$ and $(G, P, N_G(P))$, respectively.

Proposition 6. *The simple modules M_a and M_b are trivial source.*

Proof:

Since the restriction of χ_2 to K is a direct sum of $640a \in \mathrm{Irr}(C_{3b})$ and $693c \in \mathrm{Irr}(C_{1f})$ and the restriction of $640a$ to $N_G(Z)$ is a direct sum of $10a \in \mathrm{Irr}(X_{3b})$ and $210c + 420a \in \mathrm{Irr}(X_{2b})$ by GAP, $F(M_a) = 10a$. Moreover we can check that the restriction of $10a$ to $N_G(P)$ are a direct sum of $1a$ and $9a$ by MAGMA[1]. Thus $f(M_a) = 1a$. So M_a is the direct summand of the induced module $1a^G$ and M_a is the trivial source module. For M_b, we can prove by the same way. □

There is an irreducible character θ in B_{3a} with degree 45. This character is corresponding to simple trivial source module. Since θ^G has a direct summand $\chi_2 + \chi_{44}$, there are a trivial source module M which is corresponding to $\chi_2 + \chi_{44}$. So we can prove that the top and the Socle of M are isomorphic to M_a. I hope that I can determine the unknown number β and γ by the investigation of the Loewy structure of M.

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POLYNOMIAL INVARIANTS OF FINITE-DIMENSIONAL HOPF ALGEBRAS DERIVED FROM BRAIDING STRUCTURES

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ABSTRACT. We introduce invariants of a finite-dimensional semisimple and cosemisimple Hopf algebra A over a field k by using the braiding structures of A. The invariants are given in the form of polynomials. The polynomials have integral coefficients under some condition, and become stable by taking some suitable extension of the base field. Furthermore, the polynomials give invariants of the representation category of a finite-dimensional semisimple and cosemisimple Hopf algebra under k-linear tensor equivalence. By using the polynomials, we can find some pairs of Hopf algebras, whose representation rings are same, but representation categories are different.

1. INTRODUCTION

Given a quantum group, namely, a Hopf algebra with a braiding structure, we have a topological invariant of low-dimensional manifolds, for example, (framed) knots and links. Such an invariant is so-called a quantum invariant. It is well-known that quantum invariants are not only powerful tool for investigating topologies of low-dimensional manifolds, but also closely related to mathematical physics as well as other areas, for example, number theory, gauge theory, and so on.

Although in many investigations on quantum invariants, topological problems of low-dimensional manifolds are studied under a fixed Hopf algebra, in this research, we fix a framed knot or link, and study on representation categories of Hopf algebras. In this article, by using quantum invariants of the unknot with (+1)-framing, for a finite-dimensional semisimple and cosemisimple Hopf algebra A over a field k, polynomials $P_A^{(d)}(x)$ $(d = 1, 2, \cdots)$ are introduced as invariants of A, and properties of them are studied. That polynomials are defined as in the following form thanks to some results of Etingof and Gelaki[5] (for detail see Section 2):

$$P_A^{(d)}(x) = \prod_{i=1}^{t} \prod_{R:\,\text{braidings of } A} \left(x - \frac{\underline{\dim}_R M_i}{\dim M_i} \right) \quad \in \quad k[x],$$

where $\{M_1, \cdots, M_t\}$ is a full set of non-isomorphic absolutely simple left A-modules with dimension d (so, $\dim M_i = d$ for all i), and $\underline{\dim}_R M_i$ is the quantum invariant of unknot with (+1)-framing and colored by M_i. In algebraic language, $\underline{\dim}_R M_i$ is the category-theoretic rank of M_i in the left rigid braided monoidal category $({}_A\mathbb{M}^{\text{f.d.}}, c_R)$ [10], where ${}_A\mathbb{M}^{\text{f.d.}}$ is the monoidal category of finite-dimensional left A-modules and A-homomorphisms, and c_R is the braiding of ${}_A\mathbb{M}^{\text{f.d.}}$ determined by R.

The detailed version of this paper will be submitted for publication elsewhere.

This research is partially supported by Grant-in-Aid for Scientific Research (No. 19540025), JSPS.

Each polynomial $P_A^{(d)}(x)$ has the following properties. All coefficients of the polynomial are integers if k is a finite Galois extension of the rational number field $\mathbb{Q}$, and A coincides with the scalar extension of some finite-dimensional semisimple Hopf algebra over $\mathbb{Q}$. The polynomial becomes also stable by taking some suitable extension of the base field, more precisely, there is a finite separable field extension L/k so that $P_{A^E}^{(d)}(x) = P_{A^L}^{(d)}(x)$ for any field extension E/L.

It is more interesting to note that our polynomial invariants give an invariant of representation categories of Hopf algebras, that is, if representation categories of finite-dimensional semisimple and cosemisimple Hopf algebras A and B are equivalent as k-linear tensor categories, then $P_A^{(d)}(x) = P_B^{(d)}(x)$. In general, if representation categories of two finite-dimensional semisimple Hopf algebras A and B over an algebraically field k of characteristic 0 are equivalent as k-linear tensor categories, then their representation rings are isomorphic as rings (with $*$-structure)[13, 15]. However, the converse is not true. For example, by Tambara and Yamagami[18], it was proved that three non-commutative and semisimple Hopf algebras $\mathbb{C}[D_8], \mathbb{C}[Q_8], K_8$ of dimension 8 over the complex number field $\mathbb{C}$ have the same representation ring, but their representation categories are not mutually equivalent, where D_8 is the dihedral group of order 8, Q_8 is the quaternion group, and K_8 is the Kac-Paljutkin algebra[6, 11]. This result is generalized by Masuoka[12] in the case where the base field of Hopf algebras is an algebraically closed field of characteristic 0 or $p > 2$. In this article, we give an another proof of Tambara and Yamagami's result by using our polynomial invariants, and furthermore, give other examples of pairs of Hopf algebras, whose representation rings are same, but representation categories are mutually different (see the final section).

Throughout this article, we use the notation $\otimes$ instead of $\otimes_k$, and denote by $\mathrm{ch}(k)$ the characteristic of the field k.

Acknowledgement. The author would like to thank Professor Akira Masuoka and Professor Ikuo Satake for helpful advice.

2. Definition of polynomial invariants

In this section, we introduce invariants of a semisimple and cosemisimple Hopf algebra of finite dimension over an arbitrary field. They are given by polynomials derived from the quasitriangular structures of the Hopf algebra, and become invariants under tensor equivalence of representation categories of Hopf algebras.

Let us recall the definition of a quasitriangular Hopf algebra [3]. Let A be a Hopf algebra and $R \in A \otimes A$ an invertible element. The pair (A, R) is said to be a *quasitriangular Hopf algebra*, and R is said to be a *universal R-matrix* of A, if the following three conditions are satisfied:

(i) $\Delta^{\mathrm{cop}}(a) = R \cdot \Delta(a) \cdot R^{-1}$ for all $a \in A$,

(ii) $(\Delta \otimes \mathrm{id})(R) = R_{13}R_{23}$,

(iii) $(\mathrm{id} \otimes \Delta)(R) = R_{13}R_{12}$.

Here $\Delta^{\mathrm{cop}} = T \circ \Delta, T : A \otimes A \longrightarrow A \otimes A,\ T(a \otimes b) = b \otimes a$, and $R_{ij} \in A \otimes A \otimes A$ is given by $R_{12} = R \otimes 1,\ R_{23} = 1 \otimes R,\ R_{13} = (T \otimes \mathrm{id})(R_{23}) = (\mathrm{id} \otimes T)(R_{12})$.

If $R = \sum_i \alpha_i \otimes \beta_i$ is a universal R-matrix of A, then the element $u = \sum_i S(\beta_i)\alpha_i$ of A is invertible, and has the following properties:

(i) $S^2(a) = uau^{-1}$ for all $a \in A$,

(ii) $S(u) = \sum_i \alpha_i S(\beta_i)$.

The above element u is called the *Drinfel'd element* associated to R. If the characteristic of $\boldsymbol{k}$ is 0, and A is semisimple or cosemisimple of finite dimension, then the Drinfel'd element u belongs to the center of A by the property (i) and $S^2 = \mathrm{id}_A$ [8].

Let (A, R) be a quasitriangular Hopf algebra over a field $\boldsymbol{k}$ and u the Drinfel'd element associated to R. For a finite-dimensional left A-module M, we denote by $\underline{\dim}_R M$ the trace of the left action of u on M, and call it the *R-dimension* of M.

To define polynomial invariants, we use the following result on a semisimple and cosemisimple Hopf algebra of finite dimension due to Etingof and Gelaki [5, Corollary 3.2(ii), Corollary 1.5].

Theorem 1 (Etingof-Gelaki). *Let A be a semisimple and cosemisimple Hopf algebra of finite dimension over a field $\boldsymbol{k}$. Then*

(1) $(\dim M)1_{\boldsymbol{k}} \neq 0$ *for any absolutely simple left A-module M,*

(2) the set of universal R-matrices $\underline{\mathrm{Braid}}(A)$ is finite. □

Let $A = (A, \Delta, \varepsilon, S)$ be a semisimple and cosemisimple Hopf algebra of finite dimension over a field $\boldsymbol{k}$. For a finite-dimensional left A-module M with $(\dim M)1_{\boldsymbol{k}} \neq 0$, we set

$$P_{A,M}(x) := \prod_{R \in \underline{\mathrm{Braid}}(A)} \left(x - \frac{\underline{\dim}_R M}{\dim M} \right).$$

This is a polynomial in $\boldsymbol{k}[x]$. Furthermore, for each positive integer d we define a polynomial $P_A^{(d)}(x)$ in $\boldsymbol{k}[x]$ by

$$P_A^{(d)}(x) := \prod_{i=1}^{t} P_{A,M_i}(x),$$

where $\{M_1, \cdots, M_t\}$ is a full set of non-isomorphic absolutely simple left A-modules with dimension d. If there is no absolutely simple left A-module, then we set $P_A^{(d)}(x) := 1$.

Example 2. Let G be the cyclic group of order m, and $\boldsymbol{k}$ a field of $\mathrm{ch}(\boldsymbol{k}) \nmid m$ which contains a primitive m-th root of unity. Then, the polynomial invariant $P_{\boldsymbol{k}[G]}^{(1)}(x)$ of the group Hopf algebra $\boldsymbol{k}[G]$ is given by the formula

$$P_{\boldsymbol{k}[G]}^{(1)}(x) = \prod_{d,j=0}^{m-1} (x - \omega^{dj^2}) = \prod_{j=0}^{m-1} \left(x^{\frac{m}{\gcd(j^2,m)}} - 1\right)^{\gcd(j^2,m)}.$$

For a $\boldsymbol{k}$-bialgebra A we write ${}_A\mathbb{M}$ for the $\boldsymbol{k}$-linear monoidal category whose objects are left A-modules and morphisms are left A-homomorphisms. Two bialgebras A and B

over k are called *monoidally Morita equivalent* if monoidal categories ${}_A\mathbb{M}$ and ${}_B\mathbb{M}$ are equivalent as k-linear monoidal categories.

Lemma 3. *Let A and B two Hopf algebras of finite dimension over k. If a k-linear monoidal functor $F : {}_A\mathbb{M} \longrightarrow {}_B\mathbb{M}$ gives an equivalence between monoidal categories, then* $\dim M = \dim F(M)$ *for a finite-dimensional left A-module M, and there is a bijection* $\Phi : \underline{\mathrm{Braid}}(A) \longrightarrow \underline{\mathrm{Braid}}(B)$ *such that* $\underline{\dim}_R M = \underline{\dim}_{\Phi(R)} F(M)$ *for a finite-dimensional left A-module M and a universal R-matrix $R \in \underline{\mathrm{Braid}}(A)$.* □

From the above lemma we have the following theorem immediately.

Theorem 4. *Let A and B be semisimple and cosemisimple Hopf algebras of finite dimension over k. If A and B are monoidally Morita equivalent, then $P_A^{(d)}(x) = P_B^{(d)}(x)$ for any positive integer d.* □

3. Properties of polynomial invariants

In this section, we describe properties of polynomial invariants $P_A^{(d)}(x)$ defined in Section 2.

Lemma 5. *Let (A, R) be a quasitriangular Hopf algebra over a field k and u the Drinfel'd element associated to R. If A is semisimple and cosemisimple, then $u^{(\dim A)^3} = 1$.*

Proof. Let us consider the following sub-Hopf algebras B and H of A :

$$B := \{ \, (\alpha \otimes \mathrm{id})(R) \mid \alpha \in A^* \, \},$$
$$H := \{ \, (\mathrm{id} \otimes \alpha)(R) \mid \alpha \in A^* \, \}.$$

By [14, Proposition 2], the Hopf algebra B is isomorphic to the Hopf algebra $H^{*\mathrm{cop}}$. Let $(D(H), \mathcal{R})$ be the Drinfel'd double of H. By [14, Theorem 2], there is a homomorphism $F : (D(H), \mathcal{R}) \longrightarrow (A, R)$ of quasitriangular Hopf algebras. It follows that the Drinfel'd element $\tilde{u}$ associated to $(D(H), \mathcal{R})$ satisfies $F(\tilde{u}) = u$. Since A is semisimple, sub-Hopf algebras H and $H^{*\mathrm{cop}} \cong B$ are also semisimple [9, Corollary 2.5]. Thus H is semisimple and cosemisimple. So, we have $\tilde{u}^{(\dim H)^3} = 1$ by [4, Theorem 2.5 & Theorem 4.3], and whence $u^{(\dim H)^3} = 1$. Since $\dim A$ is divided by $\dim H$ [14, Proposition 2], we have $u^{(\dim A)^3} = 1$. □

For a field K, let Z_K denote the integral closure the prime ring of K, that is, if the characteristic of K is 0, then Z_K is the ring of algebraic integers in K, and if the characteristic of K is $p > 0$, then Z_K is the algebraic closure of the prime field $\mathbb{F}_p$ in K.

From the above lemma, we have:

Proposition 6. *Let H be a semisimple and cosemisimple Hopf algebra of finite dimension over a field K. Then, for any absolutely simple left H-module M, the coefficients of the polynomial $P_{H,M}(x)$ are in Z_K. Therefore, $P_H^{(d)}(x) \in Z_K[x]$ for any positive integer d.* □

Next, we examine relationship between polynomial invariants and Galois extensions of fields. Let K/k be a field extension, and H a Hopf algebra over K. By a *k-form* of H we mean a Hopf algebra A over k such that $H \cong A^K = A \otimes K$ as K-Hopf algebras[1, p.181].

Theorem 7. *Let K/k be a finite Galois extension of fields, and H a semisimple and cosemisimple Hopf algebra of finite dimension over K. If H possesses a k-form, then $P_H^{(d)}(x) \in (k \cap Z_K)[x]$ for each positive integer d.* □

We have two corollaries as applications of the above theorem.

Corollary 8. *Let K be a finite Galois extension field of $\mathbb{Q}$, and H a semisimple Hopf algebra of finite dimension over K. If H possesses a $\mathbb{Q}$-form, then $P_H^{(d)}(x) \in \mathbb{Z}[x]$ for a positive integer d, where $\mathbb{Z}$ is the rational integral ring.*

Proof. By [7] a semisimple Hopf algebra over a field of characteristic 0 of finite dimension is cosemisimple. So, the semisimple Hopf algebra H is also cosemisimple. Since $\mathbb{Q} \cap Z_K = \mathbb{Z}$, applying Theorem 7 to H, we have $P_H^{(d)}(x) \in \mathbb{Z}[x]$. □

Corollary 9. *Let Γ be a finite group, and K a finite Galois extension field of $\mathbb{Q}$. Then, $P_{K[\Gamma]}^{(d)}(x) \in \mathbb{Z}[x]$ for a positive integer d.* □

Next, we discuss on stability of polynomial invariants under extension of fields.

Let A be a Hopf algebra over a field k, and L a commutative algebra over k. Then, $A^L = A \otimes L$ becomes a Hopf algebra over L. Furthermore, if $R = \sum_i \alpha_i \otimes \beta_i$ is a universal R-matrix of A, then

$$R^L = \sum_i (\alpha_i \otimes 1_K) \otimes_L (\beta_i \otimes 1_K) \quad \in \quad A^L \otimes_L A^L$$

is a universal R-matrix of A^L.

Let $\underline{\mathrm{alg}_k}$ denote the k-additive category whose objects are commutative algebras over k and morphisms are algebra maps between them. Let A and B be two Hopf algebras over k. For a commutative algebra $L \in \underline{\mathrm{alg}_k}$, we set

$$\mathrm{Hopf}_L(A \otimes L, B \otimes L) := \{ \text{ the } L\text{-Hopf algebra maps } A \otimes L \longrightarrow B \otimes L \ \},$$

and for an algebra map $f : L_1 \longrightarrow L_2$ between commutative algebras $L_1, L_2 \in \underline{\mathrm{alg}_k}$ and $\varphi \in \mathrm{Hopf}_{L_1}(A \otimes L_1, B \otimes L_1)$ we define a map $f_*\varphi \in \mathrm{Hopf}_{L_2}(A \otimes L_2, B \otimes L_2)$ by the composition:

$$A \otimes L_2 \xrightarrow{\mathrm{id}\otimes\eta} A \otimes (L_1 \otimes L_2) \cong (A \otimes L_1) \otimes L_2 \xrightarrow{\varphi\otimes\mathrm{id}} (B \otimes L_1) \otimes L_2$$
$$\xrightarrow{\mathrm{id}_B\otimes f} (B \otimes L_2) \otimes L_2 \cong B \otimes (L_2 \otimes L_2) \xrightarrow{\mathrm{id}\otimes\mu_{L_2}} B \otimes L_2,$$

where μ_{L_2} is the multiplication of L_2, and $\eta : L_2 \longrightarrow L_1 \otimes L_2$ is the k-algebra map defined by $\eta(y) = 1_{L_1} \otimes y$ $(y \in L_2)$. This k-linear map $f_*\varphi$ is directly defined by

$$(f_*\varphi)(a \otimes y) = \sum_i b_i \otimes f(x_i)y, \qquad \left(\varphi(a \otimes 1_{L_1}) = \sum_i b_i \otimes x_i\right)$$

for all $a \in A$, $y \in L_2$. Let Set denote the category whose objects are sets and morphisms are maps. Then we have a covariant functor $\mathbf{Hopf}(A, B) : \underline{\mathrm{alg}_k} \longrightarrow \underline{\mathrm{Set}}$ such as

$$\text{for an object}: \quad L \longmapsto \mathrm{Hopf}_L(A \otimes L, B \otimes L),$$
$$\text{for a morphism}: \quad f \longmapsto \Big(\mathbf{Hopf}(A, B)(f) : \varphi \longmapsto f_*\varphi\Big).$$

If A and B are of finite dimension over k, then the functor $\mathbf{Hopf}(A, B)$ can be represented by some finitely generated commutative algebra $Z \in \underline{\mathrm{alg}_k}$ [20, p.4–5 & p.58]. Furthermore, if A is semisimple, and B is cosemisimple, then the representing object Z is separable and of finite dimension[5, Corollary 1.3]. This fact leads to the following theorem.

Theorem 10. *Let A be a cosemisimple Hopf algebra over a field k of finite dimension. Then, there is a separable finite extension field L of k such that*

(i) *there are only finitely many universal R matrices of A^L, and*
(ii) *for any field extension E/L, the map $\underline{\mathrm{Braid}}(A^L) \longrightarrow \underline{\mathrm{Braid}}(A^E)$, $R \longmapsto R^E$ is bijective.* □

Corollary 11. *Let A be a semisimple and cosemisimple Hopf algebra over a field k of finite dimension. Then, there is a separable finite extension field L of k such that for any field extension E of L and any positive integer d, $P^{(d)}_{A^E}(x) = P^{(d)}_{A^L}(x)$ in $E[x]$.* □

4. Examples

In this section, we give computational results of polynomial invariants for several Hopf algebras. By comparing polynomial invariants one has new examples of pairs of Hopf algebras such that their representation rings are isomorphic, but they are not monoidally Morita equivalent.

Let $N \geq 1$ be an odd integer and $n \geq 2$, and consider the finite group

$$G_{Nn} = \langle h,\ t,\ w \mid t^2 = h^{2N} = 1,\ w^n = h^N,\ tw = w^{-1}t,\ ht = th,\ hw = wh \rangle.$$

The group G_{Nn} is non-commutative, and the order of it is $4Nn$. We remark that if $N = 1$, then $G_{Nn} \cong D_{4n}$, the dihedral group of order $4n$. Let k be a field of $\mathrm{ch}(k) \nmid 2Nn$ which contains a primitive $4Nn$-th root of unity. The group algebra $k[G_{Nn}]$ has a Hopf algebra structure in a usual way. At the same time, one can define another Hopf algebra structure on $k[G_{Nn}]$ as follows.

$$\Delta(h) = h \otimes h, \quad \Delta(t) = h^N wt \otimes e_1 t + t \otimes e_0 t, \quad \Delta(w) = w \otimes e_0 w + w^{-1} \otimes e_1 w,$$
$$\varepsilon(h) = 1, \quad ,\varepsilon(t) = 1, \quad \varepsilon(w) = 1,$$
$$S(h) = h^{-1}, \quad S(t) = (-e_1 w + e_0)t, \quad S(w) = e_1 w^{-1} + e_1 w,$$

where $e_0 = \frac{1+h^N}{2}$, $e_1 = \frac{1-h^N}{2}$. We denote this Hopf algebra by A_{Nn}. If we set

$$\lambda = \begin{cases} -1 & (n \text{ is even}), \\ +1 & (n \text{ is odd}), \end{cases}$$

then the Hopf algebra A_{Nn} is isomorphic to the Hopf algebra $A_{Nn}^{+\lambda}$ which is introduced by Satoshi Suzuki[16]. Properties of the Hopf algebras A_{1n} are studied in detail in [2], and A_{12} especially coincides with the Kac-Paljutkin algebra K_8 [6, 11], which is the unique non-commutative and non-cocommutative semisimple Hopf algebra of dimension 8 up to isomorphism.

Let $\omega \in \boldsymbol{k}$ be a primitive $4Nn$-th root of unity. Then, a full set of non-isomorphic (absolutely) simple left $\boldsymbol{k}[G_{Nn}]$-modules is given by

$$\begin{aligned} &\{\ V_{ijk} \mid i, j = 0, 1,\ k = 0, 2, \cdots, 2N-2\ \} \\ \cup &\{\ V_{jk} \mid k = 0, 1, \cdots, 2N-1,\ j = 1, 2, \cdots, n-1,\ j \equiv k \pmod 2\ \}, \end{aligned}$$

where the action χ_{ijk} of $\boldsymbol{k}[G_{Nn}]$ on $V_{ijk} = \boldsymbol{k}$ is given by

$$\chi_{ijk}(t) = (-1)^i,\ \chi_{ijk}(w) = (-1)^j,\ \chi_{ijk}(h) = \begin{cases} \omega^{2kn} & (n \text{ is even}), \\ \omega^{2(k+j)n} & (n \text{ is odd}), \end{cases}$$

and the left action ρ_{jk} of $\boldsymbol{k}[G_{Nn}]$ on $V_{jk} = \boldsymbol{k} \oplus \boldsymbol{k}$ is given by

$$\rho_{jk}(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \rho_{jk}(w) = \begin{pmatrix} \omega^{2jN} & 0 \\ 0 & \omega^{-2jN} \end{pmatrix},\ \rho_{jk}(h) = \begin{pmatrix} \omega^{2kn} & 0 \\ 0 & \omega^{2kn} \end{pmatrix}.$$

Since A_{Nn} is isomorphic to the dual Hopf algebra, we can compute $P^{(d)}_{A_{Nn}}(x)$ $(d = 1, 2)$ by using the data of the braidings of $A_{Nn}^{+\lambda}$ determined by S. Suzuki[16]. We set

$$\epsilon(n) = \begin{cases} 0 & (n \text{ is even}), \\ 1 & (n \text{ is odd}). \end{cases}$$

Proposition 12. *(1) In case of $n \geq 3$,*

$$P^{(1)}_{A_{Nn}}(x) = \begin{cases} \displaystyle\prod_{s=0}^{N-1} \prod_{i=0}^{N-1} (x - \omega^{-8nis^2})^{4n} (x - \omega^{-8ins^2}(-1)^{\frac{n}{2}})^{4n} & \text{if } n \text{ is even}, \\ \displaystyle\prod_{s=0}^{N-1} \prod_{i=0}^{N-1} (x - \omega^{-8nis^2})^{4n} (x^2 - \omega^{-4in(2s+1)^2})^{2n} & \text{if } n \text{ is odd}, \end{cases}$$

$$P^{(2)}_{A_{Nn}}(x) = \prod_{s=0}^{N-1} \prod_{t=1}^{\frac{n-\epsilon(n)}{2}} \prod_{i=0}^{N-1} \prod_{j=0}^{n-1} (x^2 - \omega^{-4in(2s+1)^2 - 2N(2t-1)^2(2j+1-\epsilon(n))})$$

$$\times \prod_{s=0}^{N-1} \prod_{t=1}^{\frac{n-2+\epsilon(n)}{2}} \prod_{i=0}^{N-1} \prod_{j=0}^{n-1} (x - \omega^{-8ins^2 - 4Nt^2(2j+1-\epsilon(n))})^2.$$

(2) In case of $n = 2$,

$$P^{(1)}_{A_{N2}}(x) = \prod_{s=0}^{N-1}\prod_{i=0}^{N-1}(x - \omega^{-16is^2})^{16}(x + \omega^{-8is^2})^8(x + \omega^{-16is^2})^8,$$

$$P^{(2)}_{A_{N2}}(x) = \prod_{s=0}^{N-1}\prod_{i=0}^{N-1}(x^4 + \omega^{-16i(2s+1)^2})(x^2 - \omega^{-8i(2s+1)^2})^2. \quad \square$$

On the other hand, we can determine the universal R-matrices of the group Hopf algebra $k[G_{Nn}]$ by using the method developed in [19], and compute the polynomial invariants $P^{(d)}_{k[G_{Nn}]}(x)$ $(d = 1, 2)$.

Proposition 13. *(1) In case of $n \geq 3$,*

$$P^{(1)}_{k[G_{Nn}]}(x) = \begin{cases} \prod_{s=0}^{N-1}\prod_{i=0}^{N-1}(x - \omega^{-8nis^2})^{8n} & \text{if } n \text{ is even,} \\ \prod_{s=0}^{N-1}\prod_{i=0}^{N-1}(x - \omega^{-8nis^2})^{4n}(x^2 - \omega^{-4in(2s+1)^2})^{2n} & \text{if } n \text{ is odd,} \end{cases}$$

$$P^{(2)}_{k[G_{Nn}]}(x) = \prod_{s=0}^{N-1}\prod_{t=1}^{\frac{n-\epsilon(n)}{2}}\prod_{i=0}^{N-1}\prod_{j=0}^{n-1}(x^2 - \omega^{-4in(2s+1)^2-4Nj(2t-1)^2})$$
$$\times \prod_{s=0}^{N-1}\prod_{t=1}^{\frac{n-2+\epsilon(n)}{2}}\prod_{i=0}^{N-1}\prod_{j=0}^{n-1}(x - \omega^{-8ins^2-8Njt^2})^2.$$

(2) In case of $n = 2$,

$$P^{(1)}_{k[G_{Nn}]}(x) = \prod_{s=0}^{N-1}\prod_{i=0}^{N-1}(x - \omega^{-16is^2})^{32},$$

$$P^{(2)}_{k[G_{Nn}]}(x) = \prod_{s=0}^{N-1}\prod_{i=0}^{N-1}(x^4 - \omega^{-16i(2s+1)^2})(x^2 - \omega^{-8i(2s+1)^2})^2. \quad \square$$

By comparing polynomial invariants of two Hopf algebras A_{Nn} and $k[G_{Nn}]$, we see immediately that if n is odd, then $P^{(d)}_{A_{Nn}}(x) = P^{(d)}_{k[G_{Nn}]}(x)$ for $d = 1, 2$. So, our polynomial invariants do not detect the representation categories of A_{Nn} and $k[G_{Nn}]$ for an odd integer n. However, for an even integer n our polynomial invariants are useful.

Theorem 14. *Let $N \geq 1$ be an odd integer and $n \geq 2$, and let k be a field of ch(k) $\nmid 2Nn$ which contains a primitive $4Nn$-th root of unity. If n is even, then two Hopf algebras A_{Nn} and $k[G_{Nn}]$ are not monoidally Morita equivalent.* $\square$

Example 15. For a non-negative integer h, Φ_h denotes the h-th cyclotomic polynomial. Then, by using Maple12 software, we see that the polynomial invariants of Hopf algebras $k[G_{Nn}]$ and A_{Nn} for $N = 1, 3, 5$ and $n = 2, 3, 4$ are given as in the following list.

Hopf algebra A	$P_A^{(1)}(x)$	$P_A^{(2)}(x)$
$k[G_{12}]$	Φ_1^{32}	$\Phi_4\Phi_2^3\Phi_1^3$
A_{12}	$\Phi_2^{16}\Phi_1^{16}$	$\Phi_8\Phi_2^2\Phi_1^2$
$k[G_{32}]$	$\Phi_3^{64}\Phi_1^{160}$	$\Phi_{12}^2\Phi_4^5\Phi_6^6\Phi_3^6\Phi_2^{15}\Phi_1^{15}$
A_{32}	$\Phi_6^{32}\Phi_3^{32}\Phi_2^{80}\Phi_1^{80}$	$\Phi_{24}^2\Phi_8^5\Phi_6^4\Phi_3^4\Phi_2^{10}\Phi_1^{10}$
$k[G_{52}]$	$\Phi_5^{128}\Phi_1^{288}$	$\Phi_{20}^4\Phi_{10}^{12}\Phi_5^{12}\Phi_4^9\Phi_2^{27}\Phi_1^{27}$
A_{52}	$\Phi_{10}^{64}\Phi_5^{64}\Phi_2^{144}\Phi_1^{144}$	$\Phi_{40}^4\Phi_{10}^8\Phi_5^8\Phi_8^9\Phi_2^{18}\Phi_1^{18}$
$k[G_{13}]$ A_{13}	$\Phi_2^6\Phi_1^{18}$	$\Phi_6\Phi_3^3\Phi_2\Phi_1^3$
$k[G_{33}]$ A_{33}	$\Phi_6^{12}\Phi_3^{36}\Phi_2^{30}\Phi_1^{90}$	$\Phi_6^9\Phi_3^{27}\Phi_2^9\Phi_1^{27}$
$k[G_{53}]$ A_{53}	$\Phi_{10}^{24}\Phi_5^{72}\Phi_2^{54}\Phi_1^{162}$	$\Phi_{30}^4\Phi_{15}^{12}\Phi_{10}^4\Phi_5^{12}\Phi_6^9\Phi_3^{27}\Phi_2^9\Phi_1^{27}$
$k[G_{14}]$	Φ_1^{32}	$\Phi_8^2\Phi_4^2\Phi_2^6\Phi_1^6$
A_{14}		$\Phi_{16}^2\Phi_4^4$
$k[G_{34}]$	$\Phi_3^{64}\Phi_1^{160}$	$\Phi_{24}^4\Phi_{12}^4\Phi_8^{10}\Phi_6^{12}\Phi_3^{12}\Phi_4^{10}\Phi_2^{30}\Phi_1^{30}$
A_{34}		$\Phi_{48}^4\Phi_{16}^{10}\Phi_{12}^8\Phi_4^{20}$
$k[G_{54}]$	$\Phi_5^{128}\Phi_1^{288}$	$\Phi_{40}^8\Phi_{20}^8\Phi_{10}^{24}\Phi_5^{24}\Phi_8^{18}\Phi_4^{18}\Phi_2^{54}\Phi_1^{54}$
A_{54}		$\Phi_{80}^8\Phi_{20}^{16}\Phi_{16}^{18}\Phi_4^{36}$

Remark 16. For an odd integer $N \geq 1$ and an integer $n \geq 2$, the representation rings of two Hopf algebras A_{Nn} and $k[G_{Nn}]$ are isomorphic as rings with $*$-structure. In the case of $N = 1$, this result is obtained by Masuoka[12]. By the above theorem, hence, for an even integer n, A_{Nn} and $k[G_{Nn}]$ give an example of a pair of Hopf algebras such that their representation rings are isomorphic, but their representation categories are not. Such an example was first found by Tambara and Yamagami[18]. They showed that 8-dimensional non-commutative semisimple Hopf algebras $\mathbb{C}[D_8], \mathbb{C}[Q_8], K_8$ over $\mathbb{C}$ are not mutually monoidally Morita equivalent. From the viewpoint of extension of Hopf algebras Masuoka[12] showed that their result holds in the case where the base field are algebraically closed, and its characteristic does not divide 2. By using our polynomial invariants we can also prove the Tambara and Yamagami's result mentioned above. The polynomial invariants of 8-dimensional non-commutative semisimple Hopf algebras $k[D_8], k[Q_8], K_8$ are given by

$$P^{(1)}_{k[D_8]}(x) = P^{(1)}_{k[Q_8]}(x) =(x-1)^{32}, \qquad P^{(1)}_{K_8}(x) = (x-1)^{16}(x+1)^{16},$$

$$P^{(2)}_{k[D_8]}(x) = x^8 - 2x^6 + 2x^2 - 1,$$
$$P^{(2)}_{k[Q_8]}(x) = x^8 + 2x^6 - 2x^2 - 1,$$
$$P^{(2)}_{K_8}(x) = x^8 - 2x^6 + 2x^4 - 2x^2 + 1.$$

Since polynomials $P^{(2)}_{k[D_8]}(x)$, $P^{(2)}_{k[Q_8]}(x)$, $P^{(2)}_{K_8}(x)$ are all different, we conclude that by Theorem 4 the Hopf algebras $k[D_8], k[Q_8], K_8$ are not mutually monoidally Morita equivalent.

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